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# ASYMPTOTIC EXPANSION FOR CLOSED GEODESICS IN HOMOLOGY CLASSES

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**Abstract.** In this paper we give a full asymptotic expansion for the number of closed geodesics in homology classes. Especially, we obtain formulae about the coefficients of error terms which depend on the homology class.

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**1. Introduction.** Let $M$ be a compact Riemannian manifold with first Betti number $b > 0$ equipped with negative sectional curvatures. There are a countable infinity closed geodesics on $M$, one in each non-zero conjugacy class in $\pi_1 M$. The problem of obtaining asymptotic estimates on the number of closed geodesics has been studied by many authors. Let $\Gamma$ be the set of its closed geodesics. For $\gamma \in \Gamma$, let $l(\gamma)$ denote the length of $\gamma$, then

$$\pi(T) := \#\{\gamma \in \Gamma : l(\gamma) \leq T\} \sim \frac{e^{hT}}{hT} \quad \text{as} \quad T \to \infty,$$

where $h$ is the topological entropy of the geodesic flow on the unit-tangent bundle $SM$ [**7**].

We denote by $[\gamma] \in H_1(M, \mathbb{Z})$ the homology class of $\gamma$. For a fixed homology class $\alpha \in H_1(M, \mathbb{Z})$, it is known that

$$\pi(T, \alpha) := \#\{\gamma \in \Gamma : l(\gamma) \leq T, [\gamma] = \alpha\} \sim c\frac{e^{Th}}{T^{b/2+1}} \quad \text{as} \quad T \to \infty,$$

for some constant $c > 0$, [**5**].

However the error terms of asymptotic expansion were not known until the work of Dolgopyat on Anosov flows [**2**], where he obtained strong results on the contractivity of transfer operators. These results led Anantharaman [**1**], Pollicott and Sharp [**9**] to find full expansions for $\pi(T, \alpha)$ and $\pi_\delta(T, \alpha)$ (the definition of $\pi_\delta(T, \alpha)$ follows).

In [**1**], by using direct approach based on Fourier-Laplace transform, Anantharaman obtained

$$\begin{aligned}\pi_\delta(T, \alpha) :&= \#\{\gamma \in \Gamma : |l(\gamma) - T| \leq \delta, [\gamma] = \alpha\} \\ &= \frac{e^{Th}}{T^{b/2+1}}\left(\sum_{n=0}^{N} \frac{c_n}{T^n} + O\big(T^{-(N+1)}\big)\right) \qquad \text{as} \quad T \to \infty,\end{aligned}$$

for all $N \in \mathbb{N}$, where the $c_n$ are constants.

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In fact, she obtained the result for Anosov flows whose stable and unstable characteristic foliations are of class $C^1$ and uniformly jointly non-integrable. Estimating the number of closed geodesics is a special case. Pollicott and Sharp [**9**] independently obtained the same results for $\pi(T, \alpha)$, but their expansion contained terms of the form $T^{-\frac{n}{2}}$ which should vanish when $n$ is odd.

However these authors did not describe how the constants $c_n$ depend on the homology class $\alpha$. In this article, we will consider how coefficients $c_n$ depend on $\alpha$ and give an explicit formula for $c_n$. Our main result is the following.

THEOREM. *Let $M$ be a compact Riemannian manifold with first Betti number $b > 0$ and with negative section curvatures. Furthermore suppose that either* $\dim M = 2$ *or that the sectional curvatures lie between $-4$ and $-1$. For a fixed $\alpha \in H_1(M, \mathbb{Z})$, we have*

$$\pi(T, \alpha) = \frac{e^{Th}}{T^{b/2+1}} \left( c_0 + \frac{c_1 - |||\alpha|||^2}{T} + O(T^{-2}) \right) \text{ as } T \to \infty,$$

*where $h$ is the topological entropy of the geodesic flow on the unit-tangent bundle $SM$ and $||| \cdot |||$ is a norm on $H_1(M, \mathbb{R})$.*

REMARKS.

1. The norm $||| \cdot |||$ on $H_1(M, \mathbb{R})$ is defined by $|||x||| = \langle x, Ax \rangle$, where $A$ is a positive definite matrix which will be defined in section 6.
2. We will obtain a more general form of the theorem, that is, we will give the formula of the coefficients of $T^{-n}$, for any $n \in \mathbb{N}$.
3. This method can be applied to an Anosov flow which is homologically full ( i.e., every homology class contains at least one closed orbit) and whose stable and unstable characteristic foliations are $C^1$ and uniformly jointly non-integrable [**6**]. In fact, the pinching condition on the curvatures of $M$ ( sectional curvatures lie between $-4$ and $-1$) is required to ensure that this condition holds.
4. After I obtained the results of this article, I learned that Motoko Kotani [**4**] had studied the same question and also showed how the coefficients $c_n(\alpha)$ depend on homology class $\alpha$. She followed the proof by Pollicott and Sharp in [**9**]. But my argument is by a more direct approach.

The analysis in this article is closely akin to that used by Anantharaman in [**1**], but we concentrate on how the coefficients depend on the homology class. In this article, the finite positive constants $c$ may vary at each occurrence.

**2. Counting function.** Let $M$ be the compact manifold with first Betti number $b > 0$ and with negative sectional curvature. Let $SM$ denote the unit-tangent bundle. Given $(x, v) \in SM$, we can choose a unique unit speed geodesic $\gamma : \mathbb{R} \to M$ with $\gamma(0) = x$, $\dot{\gamma}(0) = v$. We define the geodesic flow $\phi_t : SM \to SM$ by $\phi_t(x, v) = (\gamma(t), \dot{\gamma}(t))$ and let $h > 0$ denote its topological entropy. There is a natural one-to-one correspondence between closed geodesics on $M$ and closed orbits for $\phi$. Let $\Gamma$ and $\tilde{\Gamma}$ denote respectively the set of closed geodesics on $M$ and the set of closed orbits for $\phi$. For $\gamma \in \Gamma$ we denote by $\tilde{\gamma}$ the corresponding closed orbit for $\phi$. Let $l(\gamma)$ and $l(\tilde{\gamma})$ denote the length of $\gamma$ and the least period of $\tilde{\gamma}$. It is known that $l(\gamma) = l(\tilde{\gamma})$.

For computational convenience, we fix an isomorphism between $H^1(M, \mathbb{R})$ and $\mathbb{R}^b$. This also gives an isomorphism between $H_1(M, \mathbb{Z})/torsion$ and $\mathbb{Z}^b$. For simplicity, we shall consider manifolds $M$ whose first homology group is torsion free, i.e., $H_1(M,\mathbb{Z}) \cong \mathbb{Z}^b$. Otherwise, we can write $H_1(M,\mathbb{Z}) \cong \mathbb{Z}^b \oplus H$, where $H$ is the finite torsion subgroup. Our results will then only differ by a multiplicative constant.

For any closed geodesic $\gamma$ on $M$, there are $F_i : SM \to \mathbb{R}$, $i = 1, 2, \ldots, b$ such that

$$[\gamma] = \left(\int_{\tilde{\gamma}} F_1, \int_{\tilde{\gamma}} F_2, \ldots, \int_{\tilde{\gamma}} F_b\right) = \int_{\tilde{\gamma}} F.$$

Thus we may rewrite $\pi(T,\alpha)$ in the form

$$\pi(T,\alpha) = \#\left\{\tilde{\gamma} \in \tilde{\Gamma} : l(\tilde{\gamma}) \le T, \int_{\tilde{\gamma}} F = \alpha\right\}.$$

We shall now concentrate on estimating $\pi(T,\alpha)$. For convenience, we shall abuse notation by writing $\gamma$ both for a closed geodesic on $M$ and for the corresponding closed orbit for the geodesic flow.

Let $G$ denote the closed subgroup of $\mathbb{R}^{b+1}$ generated by the vectors $(l(\gamma), \int_\gamma F)$ $(\gamma \in \Gamma)$. We have $G = \mathbb{R} \times \mathbb{Z}^b$. Let $M_\phi$ be the set of $\phi$-invariant probability measures on $SM$. For $u \in \mathbb{R}^b$, define function $\beta : \mathbb{R}^b \to \mathbb{R}$ by

$$\beta(u) = \sup_{m \in M_\phi}\left\{h_m(\phi) + \left\langle u, \int F dm\right\rangle\right\}.$$

$\beta(u)$ is an analytic function on $\mathbb{R}^b$ and has a positive definite Hessian at any point. It is well-known $\beta(0) = \sup_{m \in M_\phi}\{h_m\} = h$, the topological entropy of $\phi$ and $\nabla\beta(0) = 0$. The function $\beta(u)$ can be continued analytically in a complex neighbourhood of 0 in $\mathbb{C}^b$.

In order to estimate $\pi(T,\alpha)$ we shall first consider the auxiliary function

$$\pi_g(T,\alpha) = \sum_{[\gamma]=\alpha} g(l(\gamma) - T),$$

where $g : \mathbb{R} \to \mathbb{R}^+$ is a function on $\mathbb{R}$. First we assume that $g$ has a compact support and with $C^3$-regularity. Let $\hat{g}$ is the Fourier transform of $g$, by the Fourier Inversion Formula,

$$\begin{aligned}\pi_g(T,\alpha) &= \sum_{[\gamma]=\alpha} g(l(\gamma) - T) e^{\sigma(l(\gamma)-T)} e^{\sigma T} e^{-\sigma l(\gamma)} \\ &= \frac{1}{2\pi}\sum_{\gamma\in\Gamma}\int_{\mathbb{R}}\int_{\mathbb{R}^b/\mathbb{Z}^b} \hat{g}(-i\sigma + t) e^{-it(l(\gamma)-T)} e^{\sigma T} e^{-\sigma l(\gamma)} e^{2\pi i\langle v,[\gamma]\rangle} e^{-2\pi i\langle v,\alpha\rangle}\, dv\, dt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^b/\mathbb{Z}^b} Z(\sigma + it, v) e^{\sigma T + itT} \hat{g}(-i\sigma + t) e^{-2\pi i\langle v,\alpha\rangle}\, dv\, dt,\end{aligned}$$

where we have defined

$$Z(s,v) = Z(\sigma + it, v) = \sum_{\gamma\in\Gamma} e^{-sl(\gamma) + 2\pi i\langle v,[\gamma]\rangle} = \sum_{\gamma\in\Gamma} e^{-sl(\gamma)+2\pi i\langle v, \int_\gamma F\rangle}$$

for $(s,v) \in \mathbb{C} \times \mathbb{R}^b/\mathbb{Z}^b$. It is well-known that when $\mathrm{Re}\, s = \sigma > \beta(0) = h$, $Z(s,v)$ is absolutely convergent.

For the behaviour of $Z(s, v)$ in $\operatorname{Re} s < h$, we code the geodesic flow with suspended flow. Then we can determine the domain of $Z(s, v)$ by studying the norm of the transfer operator. Using the Dolgopyat's result that the spectrum of transfer operator is contracted when $\operatorname{Re} s > h - \epsilon$ and $|t| \to \infty$, we have the following proposition.

PROPOSITION 1. [**1**] *There exist $B > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and an open set $V_0$, a neighbourhood of* 0 *in $\mathbb{R}^b/\mathbb{Z}^b$ such that*

(1) *$Z(s, v)$ is analytic if $\sigma > h - \epsilon_1$, $|t| > B$ and in this domain we have $|Z(s, v)| < c|t|$;*

(2) *$Z(s, v) + \log(s - \beta(iv))$ is analytic in $\{(s, v) : \sigma > h - \epsilon_2, v \in V_0\}$;*

(3) *$Z(s, v)$ is analytic in $\{(s, v) : v \notin \bar{V}_0, \sigma > h - \epsilon_3\}$.*

**3. Analysis of counting function.** In this section we will analyse the counting function $\pi_g(T, \alpha)$. In what follows, let $g$ be of class $C^3$ with compact support. Let $\sigma > h$. Then

$$\begin{aligned}\pi_g(T, \alpha) &= \frac{1}{2\pi} \int_{\mathbb{R}\times\mathbb{R}^b/\mathbb{Z}^b} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t) e^{-2\pi i\langle v,\alpha\rangle}\, dt\, dv \\ &= \frac{1}{2\pi} \int_{V_0} e^{-2\pi i\langle v,\alpha\rangle}\, dv \int_{\mathbb{R}} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t))\, dt \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{R}^b/\mathbb{Z}^b - V_0} e^{-2\pi i\langle v,\alpha\rangle}\, dv \int_{\mathbb{R}} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t))\, dt.\end{aligned}$$

We shall consider the two integrals separately. First we consider the integral over $\mathbb{R}^b/\mathbb{Z}^b - V_0$. For $v \notin V_0$, we have the following estimate.

LEMMA 1. *For $v \notin V_0$ and for all $\sigma > h$, there exist $\epsilon > 0$ such that*

$$\left| \int_{\mathbb{R}} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t)\, dt \right| \le c \|g^{(3)}\|_{L^1} e^{T(h-\epsilon)}.$$

*Proof.* Let $\sigma$ be fixed with $\sigma > h$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$, where $\epsilon_1, \epsilon_2, \epsilon_3$ are constants in Proposition 1. By part (3) of Proposition 1, $Z(s, v)$ is analytic in $\{(s, v) : v \notin V_0, \operatorname{Re} s \ge h - \epsilon\}$. Using Cauchy's theorem,

$$\int_{\Delta} Z(s, v) e^{sT} \hat{g}(-is)\, ds = 0, \tag{1}$$

where $\Delta = \{\operatorname{Re} s = \sigma,\ |\operatorname{Im} s| \le R\} \cup \{\operatorname{Re} s = h - \epsilon,\ |\operatorname{Im} s| \le R\} \cup \{h - \epsilon \le \operatorname{Re} s \le \sigma, |\operatorname{Im} s| = R\}$.

Since $g$ is $C^3$ on $\mathbb{R}$ and has compact support, integrating by parts and only considering $g$ which has compact support contained in $(-\infty, 0]$, we obtain for $h - \epsilon \le \operatorname{Re} s \le \sigma$,

$$\hat{g}(-is) = \left| \frac{1}{s^3} \int_{\mathbb{R}} g^{(3)}(y) e^{sy}\, dy \right| \le c \frac{\|g^{(3)}\|_{L^1}}{|\operatorname{Im} s|^3},$$

where $c$ is a constant which is independent of $g$. Using Proposition 1,

$$\left| \int_{\{h-\epsilon\le\operatorname{Re} s\le\sigma, |\operatorname{Im} s|=R\}} Z(s, v) e^{sT} \hat{g}(-is)\, ds \right| \le cRe^{\sigma T} c \frac{\|g^{(3)}\|_{L^1}}{R^3} (\sigma - (h - \epsilon)) \to 0$$

as $R \to \infty$. This means that

$$\int_{\{h-\epsilon \le \operatorname{Re} s \le \sigma, |\operatorname{Im} s| = R\}} Z(s, v) e^{sT} \hat{g}(-is)\, ds \to 0 \quad \text{as} \quad R \to \infty. \qquad (*)$$

Since

$$\int_{\mathbb{R}} \left| Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t) \right| dt \le c + 2 \int_B^{+\infty} e^{\sigma T} c|t| c \frac{\|g^{(3)}\|_{L^1}}{|t|^3}\, dt < \infty,$$

we have

$$\lim_{R\to\infty} \int_{-R}^{R} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t)\, dt < \infty.$$

Similarly,

$$\lim_{R\to\infty} \int_{-R}^{R} Z(h - \epsilon + it, v) e^{(h-\epsilon+it)T} \hat{g}(-i(h-\epsilon) + t)\, dt < \infty.$$

Letting $R \to \infty$ in (1)and using (*), we have

$$\left| \int_{\mathbb{R}} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t)\, dt \right|$$
$$\le c + 2 \int_B^{\infty} e^{(h-\epsilon)T} c|t| \cdot c \frac{\|g^{(3)}\|_{L^1}}{t^3}\, dt \le c \|g^{(3)}\|_{L^1} e^{T(h-\epsilon)}. \qquad \square$$

Now we consider the integral over $V_0$. We shall prove the following lemma.

LEMMA 2. *Let $v \in V_0$, then for all $M \in \mathbb{N}$ and for all $\sigma > h$, we have*

$$\left| \int_{\mathbb{R}} Z(\sigma + it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma + t)\, dt - 2\pi \sum_{j=0}^{M} \frac{1}{T^{j+1}} \frac{d^j \hat{g}}{ds^j}(-i\beta(iv)) e^{\beta(iv)T} \right|$$
$$\le c\left( \|g^{(3)}\|_{L^1} + \|g\|_{L^1} + \cdots + \|y^M g\|_{L^1} \right) e^{T(h-\epsilon)}$$
$$+ \frac{c}{T^{M+1}} \left| \int_{C(\sigma)} \log(s - \beta(iv)) e^{sT} \frac{d^{M+1}}{ds^{M+1}} \hat{g}(-is)\, ds \right|,$$

*where $C(\sigma)$ will be defined later.*

*Proof*. When $v \in V_0$, $\sigma$ is fixed with $\sigma > h$. For $2B < R \in \mathbb{R}^+$, let
$C_0 = \{\operatorname{Re} s = h - \epsilon, -2B \le \operatorname{Im} s \le 2B\}$; $C_1 = \{\operatorname{Im} s = -2B, h - \epsilon \le \operatorname{Re} s \le \sigma\}$;
$C_2 = \{\operatorname{Re} s = \sigma, -2B \le \operatorname{Im} s \le 2B\}$; $C_3 = \{\operatorname{Im} s = 2B, h - \epsilon \le \operatorname{Re} s \le \sigma\}$;
$C_4^+ = \{\operatorname{Re} s = h - \epsilon, 2B \le \operatorname{Im} s \le R\}$; $C_4^- = \{\operatorname{Re} s = h - \epsilon, -R \le \operatorname{Im} s \le -2B\}$;
$C_5^+ = \{\operatorname{Re} s = \sigma, 2B \le \operatorname{Im} s \le R\}$; $C_5^- = \{\operatorname{Re} s = \sigma, -R \le \operatorname{Im} s \le -2B\}$;
$C_R^+ = \{h - \epsilon \le \operatorname{Re} s \le \sigma, \operatorname{Im} s = R\}$; $C_R^- = \{h - \epsilon \le \operatorname{Re} s \le \sigma, \operatorname{Im} s = -R\}$.

By part (1) of Proposition 1, $Z(s, v)$ is analytic in $\operatorname{Re} s > h - \epsilon$, $|\operatorname{Im} s| > B$,

$$\int_{\{C_3 \cup C_5^+ \cup C_R^+ \cup C_4^+\}} Z(s, v) e^{sT} \hat{g}(-is)\, ds = 0, \qquad (2)$$

$$\int_{\{C_1 \cup C_4^- \cup C_R^- \cup C_5^-\}} Z(s, v) e^{sT} \hat{g}(-is)\, ds = 0. \qquad (3)$$

By part (2) of Proposition 1, we have

$$\int_{-\{C_0\cup C_3\cup C_2\cup C_1\}} (Z(s,v)+\log(s-\beta(iv)))e^{sT}\hat{g}(-is)\,ds = 0 \tag{4}$$

where the three contours are counterclockwise and $-C_i$ means that the orientation of the path is reversed.

Letting $R\to\infty$ in (2) and using $(*)$, we have

$$\int_{\{\mathrm{Re}\,s=\sigma,2B\le \mathrm{Im}\,s\le\infty\}} Z(s,v)e^{sT}\hat{g}(-is)\,ds = \int_{-C_3\cup\{\mathrm{Re}\,s=h-\epsilon,2B\le\mathrm{Im}\,s\le\infty\}} Z(s,v)e^{sT}\hat{g}(-is)\,ds. \tag{5}$$

Letting $R\to\infty$ in (3), we have

$$\begin{aligned}\int_{\{\mathrm{Re}\,s=\sigma,-\infty\le\mathrm{Im}\,s\le-2B\}} & Z(s,v)e^{sT}\hat{g}(-is)\,ds\\ &= \int_{-C_1\cup\{\mathrm{Re}\,s=h-\epsilon,-\infty\le\mathrm{Im}\,s\le-2B\}} Z(s,v)e^{sT}\hat{g}(-is)\,ds.\end{aligned} \tag{6}$$

From (4), we have

$$\begin{aligned}\int_{-C_2} Z(s,v)e^{sT}\hat{g}(-is)\,ds &= \int_{C_1\cup C_2\cup C_3}\log(s-\beta(iv))e^{sT}\hat{g}(-is)\,ds\\ &\quad+\int_{C_0}(Z(s,v)+\log(s-\beta(iv)))e^{sT}\hat{g}(-is)\,ds\\ &\quad+\int_{C_1\cup C_3} Z(s,v)e^{sT}\hat{g}(-is)\,ds\end{aligned} \tag{7}$$

Let $C(\sigma)=C_1\cup C_2\cup C_3$. Adding three identities (5), (6), (7) we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}} Z(\sigma+it,v)e^{(\sigma+it)T}\hat{g}(-i\sigma+t)\,dt + i\int_{C(\sigma)}\log(s-\beta(iv))\hat{g}(-is)e^{sT}\,ds\right|\\ &\le\left|\int_{\{\mathrm{Re}\,s=h-\epsilon,|\mathrm{Im}\,s|>2B\}} Z(s,v)e^{sT}\hat{g}(-is)\,ds\right|\\ &\quad+\left|\int_{C_0}(Z(s,v)+\log(s-\beta(iv)))e^{sT}\hat{g}(-is)\,ds\right|\\ &\le 2e^{T(h-\epsilon)}\int_{2B}^{\infty} c|t|\frac{\|g^{(3)}\|_{L^1}}{t^3}\,dt + ce^{T(h-\epsilon)} \le c\|g^{(3)}\|_{L^1}e^{T(h-\epsilon)}.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}&\left| i\int_{C(\sigma)}\log(s-\beta(iv))\hat{g}(-is)e^{sT}\,ds + \frac{i}{T}\int_{C(\sigma)}\frac{\hat{g}(-is)}{s-\beta(iv)}e^{sT}\,ds\right.\\ &\quad\left.+\frac{i}{T}\int_{C(\sigma)}\log(s-\beta(iv))\frac{d\hat{g}}{ds}(-is)e^{sT}\,ds\right| \le c\|g\|_{L^1}e^{T(h-\epsilon)}.\end{aligned}$$

By using the Residue Formula,

$$\begin{aligned}\left|\int_{C(\sigma)} \frac{\hat{g}(-is)}{s-\beta(iv)} e^{sT} ds + 2\pi i \hat{g}(-i\beta(iv)) e^{\beta(iv)T}\right| &= \left|\int_{C_0} \frac{\hat{g}(-is)}{s-\beta(iv)} e^{sT}\, ds\right| \\ &\le \int_{-2B}^{2B} \left|\frac{\hat{g}(t-i(h-\epsilon))}{h-\epsilon+it-\beta(iv)} e^{((h-\epsilon)+it)T}\, dt\right| \\ &\le c\|g\|_{L^1} e^{T(h-\epsilon)}.\end{aligned}$$

So we have obtained

$$\begin{aligned}&\left|\int_{\mathbb{R}} Z(\sigma+it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma+t)\, dt - 2\pi \frac{1}{T} \hat{g}(-i\beta(iv)) e^{\beta(iv)T}\right| \\ &\quad \le c\left(\left\|g^{(3)}\right\|_{L^1} + \|g\|_{L^1}\right) e^{T(h-\epsilon)} + \frac{c}{T}\left|\int_{C(\sigma)} \log(s-\beta(iv)) e^{sT} \frac{d\hat{g}(-is)}{ds} ds\right|.\end{aligned}$$

We iterate the preceding operation $M+1$ times, and note

$$\left|\frac{d^k \hat{g}(-is)}{ds^k}\right| = \left|\int_{\mathbb{R}} y^k g(y) e^{sy}\, dy\right| \le c\|y^k g\|_{L^1}.$$

We have

$$\begin{aligned}&\left|\int_{\mathbb{R}} Z(\sigma+it, v) e^{(\sigma+it)T} \hat{g}(-i\sigma+t)\, dt - 2\pi \sum_{j=0}^{M} \frac{1}{T^{j+1}} \frac{d^j \hat{g}}{ds^j}(-i\beta(iv)) e^{\beta(iv)T}\right| \\ &\quad \le c\left(\left\|g^{(3)}\right\|_{L^1} + \|g\|_{L^1} + \cdots + \|y^M g\|_{L^1}\right) e^{T(h-\epsilon)} \\ &\qquad + \frac{c}{T^{M+1}}\left|\int_{C(\sigma)} \log(s-\beta(iv)) e^{sT} \frac{d^{M+1}}{ds^{M+1}} \hat{g}(-is)\, ds\right|.\end{aligned}$$

□

Using the Dominated Convergence Theorem, we have

$$\begin{aligned}&\lim_{\sigma\to h} \int_{V_0} e^{-2\pi i\langle v,\alpha\rangle}\, dv \int_{C(\sigma)} \log(s-\beta(iv)) e^{sT} \frac{d^{M+1}}{ds^{M+1}} \hat{g}(-is)\, ds \\ &\quad = \int_{V_0} e^{-2\pi i\langle v,\alpha\rangle}\, dv \int_{C(h)} \log(s-\beta(iv)) e^{sT} \frac{d^{M+1}}{ds^{M+1}} \hat{g}(-is)\, ds.\end{aligned}$$

By Lemma 1 and Lemma 2, we can prove the following proposition.

PROPOSITION 2. *Let $g$ be class $C^3$ with compact support. For all $M \ge 1$, we have*

$$\begin{aligned}&\left|\pi_g(T,\alpha) - \sum_{j=0}^{M} \frac{1}{T^{j+1}} \int_{V_0} e^{-2\pi i\langle v,\alpha\rangle} \frac{d^j \hat{g}}{ds^j}(-i\beta(iv)) e^{\beta(iv)T}\, dv\right| \\ &\quad \le c\frac{\|y^{M+1} g\|_{L^1}}{T^{M+1}} e^{Th} + c\left(\left\|g^{(3)}\right\|_{L^1} + \|g\|_{L^1} + \cdots + \|y^M g\|_{L^1}\right) e^{T(h-\epsilon)}. \qquad (8)\end{aligned}$$

**4. Further calculation of counting function.** By Proposition 2, in order to obtain the estimate of $\pi_g(T, \alpha)$, we only need to estimate

$$\int_{V_0} \frac{d^j \hat{g}}{ds^j}(-i\beta(iv))e^{T\beta(iv)}e^{-2\pi i\langle v,\alpha\rangle}\,dv.$$

We first have the following lemma.

LEMMA 3. *There exist polynomials $f_j^{(k)}(iv)$ in $iv_1, \dots, iv_b$ such that the total exponent of each term has the same parity as k and such that*

$$\begin{aligned}\Bigg|\int_{V_0} &\frac{d^j \hat{g}}{ds^j}(-i\beta(iv))e^{T\beta(iv)-2\pi i\langle v,\alpha\rangle}\,dv \\ &- \frac{e^{Th}}{T^{b/2}}\sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\|v\|\le\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(0)(v,v)-2\pi i\langle\frac{v}{\sqrt{T}},\alpha\rangle} f_j^{(k)}(iv)\,dv\Bigg| \\ &\le c\sup_{n\le 2N+2}\|y^{j+n}g\|_{L^1}\frac{e^{Th}}{T^{N+\frac{b}{2}+1}},\end{aligned}$$

*for some small $\rho$.*

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(1) Here $\|\cdot\|$ denotes the 2-norm, i.e., $\|v\| = (\sum_{i=1}^b v_i^2)^{1/2}$,

(2) The proof of this lemma is same as that in [**1**]. We denote $\frac{d^j\hat{g}}{ds^j}(-i\beta(iv))$ by $\bar{g}_j(iv)$, then expand $\beta(\frac{iv}{\sqrt{T}})$ and $\bar{g}_j(\frac{iv}{\sqrt{T}})$ in the neighbourhood of 0 by

$$\begin{aligned}\beta\left(\frac{iv}{\sqrt{T}}\right) &= \beta(0) + \nabla\beta(0)\cdot iv + \frac{1}{2}\beta''(0)(iv, iv) + TR_{2N+3} \\ &= h - \frac{1}{2}\beta''(0)(v, v) + TR_{2N+3}\end{aligned}$$

and

$$\bar{g}_j(iv/\sqrt{T}) = \sum_{l=0}^{2N+1}\frac{\bar{g}_j^{(l)}(0)}{l!}\cdot(iv)^l T^{-l/2} + Y_N(iv/\sqrt{T}).$$

(3) By the first condition, we mean that $f_j^{(k)}(iv)$ can be written in the form

$$f_j^{(k)}(iv) = \sum a_{l_1,l_2,\dots,l_b}(iv_1)^{l_1}(iv_2)^{l_2}\cdots(iv_b)^{l_b}.$$

In particular,

$$\begin{aligned}f_j^{(0)}(iv) &= \bar{g}_j^{(0)}(0) = \frac{d^j}{ds^j}\hat{g}(-is)\,|_{s=h}; \\ f_j^{(1)}(iv) &= \frac{1}{6}\bar{g}_j^{(0)}(0)\beta^{(3)}(0)\cdot(iv)^3 + \bar{g}_j^{(1)}(0)\cdot(iv); \\ f_j^{(2)}(iv) &= \frac{1}{72}\bar{g}_j^{(0)}(0)\big(2\big(\beta^{(3)}(0)\cdot(iv)^3\big)^2 + 3\beta^{(4)}(0)\cdot(iv)^4\big) \\ &\quad + \frac{1}{6}\bar{g}_j^{(1)}(0)\cdot(iv)\beta^{(3)}(0)\cdot(iv)^3 + \frac{1}{2}\bar{g}_j^{(2)}(0)\cdot(iv)^2.\end{aligned}$$

(4) We can see that $f_j^{(0)}(iv)$ are constants and $f_0^{(0)}(iv) > 0$ , since

$$f_j^{(0)}(iv) = \frac{d^j}{ds^j}\hat{g}(-is)\mid_{s=h} = \frac{d^j}{ds^j}\int_{\mathbb{R}} g(y)e^{sy}\,dy\mid_{s=h}$$
$$= \int_{\mathbb{R}} y^j g(y)e^{sy}\,dy\mid_{s=h} = \int_{\mathbb{R}} y^j g(y)e^{hy}\,dy.$$

Taking $M = N + b + 2$ in (8), we have proved the following proposition.

PROPOSITION 3. *For any $N \geq 1$, we have*

$$\left|\pi_g(T,\alpha) - \frac{e^{Th}}{T^{b/2+1}}\sum_{j=0}^{N+b+2}\frac{1}{T^j}\sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\|v\|\leq\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(0)(v,v)}e^{-2\pi i\langle\alpha,v/\sqrt{T}\rangle}f_j^{(k)}(iv)\,dv\right|$$
$$\leq c\sup_{n\leq 4N+b+2}\|y^n g\|_{L^1}\frac{e^{Th}}{T^{N+\frac{b}{2}+2}} + c\big(\big\|g^{(3)}\big\|_{L^1} + \|g\|_{L^1} + \cdots + \|y^{N+b+2}g\|_{L^1}\big)e^{T(h-\epsilon)},$$

*where $f_j^{(k)}(iv)$ in $iv_1,\ldots,iv_b$ are polynomials such that the total exponent of each term has the same parity as $k$.*

**5. Coefficients of error terms of $\pi_g(T,\alpha)$.** In this section, we will give the asymptotic formula for $\pi_g(T,\alpha)$ with error terms. We also give the explicit expression of coefficients of error terms.

By the Proposition 3, we need to estimate

$$\sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\|v\|\leq\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(0)(v,v)}e^{-2\pi i\langle\alpha,v/\sqrt{T}\rangle}f_j^{(k)}(iv)\,dv,$$

where $f_j^{(k)}(iv)$ are polynomials in $iv_1,\ldots,iv_b$ such that the total exponent of each term in $f_j^{(k)}$ and $k$ have the same parity. That is, we can write

$$f_j^{(k)}(iv) = \sum \bar{b}_{l_1,l_2,\ldots,l_b}(iv_1)^{l_1}(iv_2)^{l_2}\cdots(iv_b)^{l_b},$$

where $l_1 + l_2 + \cdots + l_b$ and $k$ have the same parity.

We first prove the following proposition.

PROPOSITION 4.

$$\sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\|v\|\leq\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(0)(v,v)}e^{-2\pi i\langle\alpha,v/\sqrt{T}\rangle}f_j^{(k)}(iv)\,dv$$
$$= \sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)}s_j^{(k)}(\alpha,iv)\,dv + O\left(\sup_{n\leq 4N+b+2}\|y^n g\|_{L^1}T^{-(N+1)}\right),$$

*where*

$$s_j^{(k)}(\alpha,iv) = \sum_{l=0}^{k}(-1)^{k-l}\frac{f_j^{(l)}(iv)\langle\alpha,2\pi iv\rangle^{k-l}}{(k-l)!}$$

*are polynomials in* $iv_1, \ldots, iv_b$ *and we still have that the total exponent of each term in* $s_j^{(k)}(\alpha, v)$ *has the same parity as* $k$.

*Proof.* We expand $e^{-2\pi i\langle\alpha, v/\sqrt{T}\rangle}$ in a neighborhood of 0.

$$e^{-2\pi i\langle\alpha, v/\sqrt{T}\rangle} = 1 - \frac{\langle\alpha, 2\pi iv\rangle}{\sqrt{T}} + \frac{\langle\alpha, 2\pi iv\rangle^2}{2T} - \frac{\langle\alpha, 2\pi iv\rangle^3}{3!T^{3/2}} + \cdots - \frac{\langle\alpha, 2\pi iv\rangle^{2N+1}}{(2N+1)!T^{N/2+1}} + Z_N(iv/\sqrt{T}),$$

where $|Z_N(iv/\sqrt{T})| \le c\frac{\|v\|^{2N+2}}{T^{N+1}}$. Let

$$s_j^{(k)}(\alpha, iv) = \sum_{l=0}^{k}(-1)^{k-l}\frac{f_j^{(l)}(iv)\langle\alpha, 2\pi iv\rangle^{k-l}}{(k-l)!};$$

then

$$\sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\|v\|\le\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(0)(v,v)}e^{-2\pi i\langle\alpha, v/\sqrt{T}\rangle}f_j^{(k)}(iv)\,dv = \sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\|v\|\le\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(0)(v,v)}s_j^{(k)}(\alpha, iv)\,dv + O\left(\sup_{n\le 4N+b+2}\|y^n g\|_{L^1}T^{-(N+1)}\right).$$

For $T$ is sufficiently large, for all $m \in \mathbb{N}$, we have

$$\int_{\|v\|>\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(0)(v,v)}\|v\|^m\,dv \le \int_{\|v\|>\sqrt{T}\rho} e^{-\epsilon'\|v\|^2}|v_1v_2\cdot v_d|\,dv \le \prod_{i=1}^{d}\int_{\sqrt{T}\rho}^{\infty} e^{-\epsilon' v_i^2}v_i\,dv_i \le ce^{-\epsilon' T},$$

for some $\epsilon' > 0$. So we have

$$\left|\sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\|v\|\le\sqrt{T}\rho} e^{-\frac{1}{2}\beta''(0)(v,v)}e^{-2\pi i\langle\alpha, v/\sqrt{T}\rangle}f_j^{(k)}(iv)\,dv - \sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)}s_j^{(k)}(\alpha, iv)\,dv\right| \le c\sup_{n\le 4N+b+2}\|y^n g\|_{L^1}\left(T^{-(N+1)} + e^{-\epsilon' T}\right). \qquad \square$$

In the following lemma, we will see that the coefficients of $T^{-\frac{k}{2}}$ vanish when $k$ is odd.

LEMMA 4. *If* $k$ *is odd, then*

$$\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)}s_j^{(k)}(\alpha, iv)\,dv = 0.$$

*Proof*.

$$
\begin{aligned}
s_j^{(k)}(\alpha, iv) &= \sum_{l=0}^{k} (-1)^{k-l} \frac{f_j^{(l)}(iv)\langle \alpha, 2\pi iv\rangle^{k-l}}{(k-l)!} \\
&= \sum_{l=0}^{k} (-1)^{k-l} f_j^{(l)}(iv)(2\pi i)^{k-l} \left( \sum_{i=1}^{b} \alpha_i v_i \right)^{k-l} \\
&:= \sum a_{l_1, l_2, \ldots, l_b}(\alpha)(iv_1)^{l_1}(iv_2)^{l_2} \cdots (iv_b)^{l_b},
\end{aligned}
$$

where $l_1 + l_2 + \cdots + l_b$ is odd and $a_{l_1, L_2, \ldots, l_b}(\alpha)$ are constants which depend on $\alpha$ and $g$. Let $v' = -v$, i.e., $(v_1', v_2', \ldots, v_b') = (-v_1, -v_2, \ldots, -v_b)$; then

$$
\begin{aligned}
&\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)} s_j^{(k)}(\alpha, iv)\, dv \\
&= \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)} \sum a_{l_1, l_2, \ldots, l_b}(iv_1)^{l_1}(iv_2)^{l_2} \cdots (iv_b)^{l_b}\, dv_1\, dv_2 \cdots dv_b \\
&= -\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v',v')} s_j^{(k)}(\alpha, iv')\, dv'.
\end{aligned}
$$

Thus

$$
\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)} s_j^{(k)}(\alpha, iv)\, dv = 0. \qquad \square
$$

By Lemma 4, for $k$ odd the coefficient of $T^{-\frac{k}{2}}$ vanish. So we only need to calculate coefficients when $k$ is even. Let $b_j^{(k)}(\alpha)$ be the coefficient of $T^k$ in Proposition 4; if we write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_b)$ then

$$
\begin{aligned}
b_j^{(k)}(\alpha) &= \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)} s_j^{(2k)}(\alpha, iv)\, dv \\
&= \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)} \sum_{l=0}^{2k} (-1)^l \frac{f_j^{(l)}(iv)\langle \alpha, 2\pi iv\rangle^{2k-l}}{(2k-l)!}\, dv \\
&= \sum_{l_1+l_2+\cdots+l_b=0}^{2k} b_{l_1 l_2 \ldots l_b}^{(j)} \alpha_1^{l_1}\alpha_2^{l_2} \cdots \alpha_b^{l_b},
\end{aligned}
$$

where $b_{l_1 l_2 \cdots l_b}^{(j)}$ are constants. More precisely,

$$
\begin{aligned}
b_{l_1 l_2 \ldots l_b}^{(j)} = \frac{(l_1 + l_2 + \cdots + l_b)!}{l_1! l_2! \ldots l_b!} \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)} (2\pi i)^{l_1+l_2+\cdots+l_b} \\
\times v_1^{l_1} v_2^{l_2} \ldots v_b^{l_b} f_j^{(2k-(l_1+l_2+\cdots+l_b))}(iv)\, dv.
\end{aligned}
$$

Then we have proved the following Proposition.

PROPOSITION 5.

$$\left|\sum_{k=0}^{2N+1}\frac{1}{T^{k/2}}\int_{\|v\|\le\sqrt{T}\rho}e^{-\frac{1}{2}\beta''(0)(v,v)}e^{-2\pi i\langle\alpha,v/\sqrt{T}\rangle}f_j^{(k)}(iv)\,dv-\sum_{k=0}^{N}\frac{b_j^{(k)}(\alpha)}{T^k}\right|$$
$$\le c\sup_{n\le 4N+b+2}\|y^n g\|_{L^1}\big(T^{-(N+1)}+e^{-\epsilon' T}\big),$$

*where* $b_j^{(k)}(\alpha)=\sum_{l_1+l_2+\cdots+l_b=0}^{2k}b_{l_1l_2\ldots l_b}^{(j)}\alpha_1^{l_1}\alpha_2^{l_2}\ldots\alpha_b^{l_b}$ *are polynomials in* $\alpha_1,\alpha_2,\ldots,\alpha_b$ *and the degree of* $b_j^{(k)}$ *is* $2k$ *and* $b_{l_1l_2\ldots l_b}^{(j)}$ *are constants which depend on* $g$.

In order to obtain $\pi_g(T,\alpha)$, we shall use Proposition 4 and Proposition 5. Let

$$c_n(\alpha)=\sum_{i=0}^{n}b_i^{(n-i)}(\alpha)=\sum_{i=0}^{n}\left(\sum_{l_1+l_2+\cdots+l_b=0}^{2(n-i)}b_{l_1l_2\ldots l_b}^{(j)}\alpha_1^{l_1}\alpha_2^{l_2}\ldots\alpha_b^{l_b}\right)$$
$$:=\sum_{l_1+l_2+\cdots+l_b=0}^{2n}c_{l_1l_2\ldots l_b}\alpha_1^{l_1}\alpha_2^{l_2}\ldots\alpha_b^{l_b},$$

where $c_{l_1l_2\ldots l_b}$ are constants and

$$c_0(\alpha)=b_0^{(0)}=f_0^{(0)}\int_{\mathbb{R}^b}e^{-\frac{1}{2}\beta''(0)(v,v)}\,dv=\frac{(2\pi)^{b/2}}{\sqrt{|\det\beta''(0)|}}\hat{g}(-ih)>0,$$

so $c_0(\alpha)$ is independent of $\alpha$. On the other hand, for $n\ge 1$, $c_n(\alpha)$ is polynomial in $\alpha_1,\ldots,\alpha_b$ whose degree is $2n$ by Proposition 5. From the expression of $b_j^{(k)}(\alpha)$, we have

$$c_n(\alpha)\sim\int_{\mathbb{R}^b}e^{-\frac{1}{2}\beta''(0)(v,v)}f_0^{(0)}(iv)\frac{\langle\alpha,2\pi iv\rangle^{2n}}{(2n)!}\,dv$$
$$=(2\pi i)^{2n}\frac{f_0^{(0)}}{(2n)!}\int_{\mathbb{R}^b}e^{-\frac{1}{2}\beta''(0)(v,v)}\langle v,\alpha\rangle^{2n}\,dv$$
$$\sim(-1)^n c_g\|\alpha\|^{2n}\qquad(c_g>0\text{ and }c_g\text{ is dependent on }g),$$

where $A\sim B$ means that $\lim_{\|\alpha\|\to\infty}A/B=1$.

We have now obtained the following theorem.

THEOREM 1. *Let* $g$ *be of class* $C^3$ *with compact support. Then for all* $N\ge 1$, *we have*

$$\pi_g(T,\alpha)=\frac{e^{Th}}{T^{b/2+1}}\left(\sum_{n=0}^{N}\frac{c_{n,g}(\alpha)}{T^n}+O\big(T^{-(N+1)}\big)\right),\tag{9}$$

*where* $c_{n,g}(\alpha)$, $n=0,1,\ldots,N$ *are constants with* $c_{0,g}>0$ *independent of* $\alpha$ *but dependent on* $g$ *and for* $n\ge 1$, $c_{n,g}(\alpha)=\sum_{l_1+l_2+\cdots+l_b=0}^{2n}c_{l_1l_2\ldots l_b}\alpha_1^{l_1}\alpha_2^{l_2}\ldots\alpha_b^{l_b}$ *is a polynomial in* $\alpha_1,\alpha_2,\ldots,\alpha_b$ *whose degree is* $2n$ *with* $c_{l_1l_2\ldots l_b}$ *constants depending on* $g$. *Furthermore,* $c_{n,g}(\alpha)\sim(-1)^n c_g\|\alpha\|^{2n}$, *as* $\|\alpha\|\to\infty$.

REMARKS.

(1) The explicit formulae for $c_{0,g}$ and $c_{1,g}(\alpha)$ are following.

$$\begin{aligned} c_{0,g} &= b_0^{(0)} = \frac{(2\pi)^{b/2}}{\sqrt{|\det \beta''(0)|}}\hat{g}(-ih) > 0; \\ c_{1,g}(\alpha) &= b_0^{(1)} + b_1^{(0)} \\ &= \int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)}\Big(\frac{1}{2}f_0^{(0)}(iv)\langle\alpha, 2\pi iv\rangle^2 - f_0^{(1)}(iv)\langle\alpha, 2\pi iv\rangle + f_0^{(2)}(iv) + f_1^{(0)}(iv)\Big)dv. \end{aligned}$$

(2) The term $O(T^{-(N+1)})$ can be bounded by $C\sup_{n\le 4N+b+2}\|y^n g\|_{L^1}T^{-(N+1)} + c\|g^{(3)}\|_{L^1}e^{-\epsilon T}$.

**6. The proof of main results.** In this section, we use approximation to estimate $\pi(T,\alpha)$.

For $N\in\mathbb{N}$, let $g = \chi_{[-T^{\frac{1}{4N+b+3}},0]}$. For all $T$ we take $g_T^-$ and $g_T^+$ of class $C^3$ with compact supports such that

(1) $g_T^- \le g \le g_T^+$;
(2) $\|g_T^+\|_\infty \le 2$ and $\|g_T^-\|_\infty \le 2$;
(3) for $0\le n\le 4N+b+2$, $\|y^n g_T^\pm\|_{L^1} \le c\|y^n g\|_{L^1}$;
(4) $\sup_{n\le 4N+b+2}\|y^n(g_T^\pm - g)\|_{L^1} \le e^{-\beta T}\sup_{n\le 4N+b+2}\|y^n g\|_{L^1}$ for some $\beta > 0$;
(5) $\|g_T^{\pm(3)}\|_{L^1} \le ce^{3\beta T}\|g\|_{L^1}$.

These can be done by a convolution argument. By (1), $\pi_{g_T^-}(T,\alpha) \le \pi_g(T,\alpha) \le \pi_{g_T^+}(T,\alpha)$. So

$$\left|\pi_g(T,\alpha) - \pi_{g_T^+}(T,\alpha)\right| \le \pi_{g_T^+}(T,\alpha) - \pi_{g_T^-}(T,\alpha).$$

By the Remark (2) of Section 5 the term $O(T^{-(N+1)})$ is bounded by

$$C\sup_{n\le 4N+b+2}\|y^n g\|_{L^1}T^{-(N+1)} + c\|g^{(3)}\|_{L^1}e^{-\epsilon T}.$$

In this case,

$$C\sup_{n\le 4N+b+2}\|y^n g\|_{L^1} = \sup_{n\le 4N+b+2}\int_{-T^{\frac{1}{4N+b+3}}}^0 |y|^n\,dy \le CT.$$

By Proposition 3, we have that

$$\begin{aligned} &\left|\pi_{g_T^\pm}(T,\alpha) - \sum_{j=0}^{N+b+2}\frac{1}{T^{j+1}}\int_{V_0} e^{-2\pi i\langle v,\alpha\rangle}\frac{d^j\widehat{g_T^\pm}}{ds^j}(-i\beta(iv))e^{\beta(iv)T}\,dv\right| \\ &\quad\le \frac{c\|y^{N+b+2}g_T^\pm\|_{L^1}}{T^{N+b+2}}e^{Th} + c\big(\big\|g_T^{\pm(3)}\big\|_{L^1} + \|g_T^\pm\|_{L^1} + \cdots + \|y^{N+b+2}g_T^\pm\|_{L^1}\big)e^{T(h-\epsilon)} \\ &\quad\le c_0\frac{e^{Th}}{T^{N+b+1}} + c_1e^{T(h-\epsilon)} + c_2e^{T(h-\epsilon+3\beta)}. \end{aligned}$$

By (4),

$$\sum_{j=0}^{N+b+2} \frac{1}{T^{j+1}} \left| \int_{V_0} e^{-2\pi i \langle v, \alpha \rangle} \left( \frac{d^j \widehat{g_T^+}}{ds^j} - \frac{d^j \widehat{g}}{ds^j} \right) (-i\beta(iv)) e^{\beta(iv)T} \, dv \right| \le c_3 e^{(h-\beta)T}.$$

Hence,

$$\begin{aligned}
&\pi_g(T, \alpha) - \sum_{j=0}^{N+b+2} \frac{1}{T^{j+1}} \int_{V_0} e^{-2\pi i \langle v, \alpha \rangle} \frac{d^j \widehat{g}}{ds^j} (-i\beta(iv)) e^{\beta(iv)T} \, dv \\
&\le \pi_{g_T^+}(T, \alpha) - \sum_{j=0}^{N+b+2} \frac{1}{T^{j+1}} \int_{V_0} e^{-2\pi i \langle v, \alpha \rangle} \frac{d^j \widehat{g}}{ds^j} (-i\beta(iv)) e^{\beta(iv)T} \, dv \\
&\le \pi_{g_T^+}(T, \alpha) - \sum_{j=0}^{N+b+2} \frac{1}{T^{j+1}} \int_{V_0} e^{-2\pi i \langle v, \alpha \rangle} \frac{d^j \widehat{g_T^+}}{ds^j} (-i\beta(iv)) e^{\beta(iv)T} \, dv \\
&\quad + \sum_{j=0}^{N+b+2} \frac{1}{T^{j+1}} \left| \int_{V_0} e^{-2\pi i \langle v, \alpha \rangle} \left( \frac{d^j \widehat{g_T^+}}{ds^j} - \frac{d^j \widehat{g}}{ds^j} \right) (-i\beta(iv)) e^{\beta(iv)T} \, dv \right| \\
&\le c_0 \frac{e^{Th}}{T^{N+b+1}} + c_1 e^{T(h-\epsilon)} + c_2 e^{T(h-\epsilon+3\beta)} + c_3 e^{T(h-\beta)}.
\end{aligned}$$

Similarly for $g_T^-$, we have

$$\begin{aligned}
&\left| \pi_g(T, \alpha) - \sum_{j=0}^{N+b+2} \frac{1}{T^{j+1}} \int_{V_0} e^{-2\pi i \langle v, \alpha \rangle} \frac{d^j \widehat{g}}{ds^j} (-i\beta(iv)) e^{\beta(iv)T} \, dv \right| \\
&\le c_0 \frac{e^{Th}}{T^{N+b+1}} + c_1 e^{T(h-\epsilon)} + c_2 e^{T(h-\epsilon+3\beta)} + c_3 e^{T(h-\beta)}.
\end{aligned}$$

Choosing $\beta < \epsilon/3$, we have

$$\left| \pi_g(T, \alpha) - \sum_{j=0}^{N+b+2} \frac{1}{T^{j+1}} \int_{V_0} e^{-2\pi i \langle v, \alpha \rangle} \frac{d^j \widehat{g}}{ds^j} (-i\beta(iv)) e^{\beta(iv)T} \, dv \right| \le c \frac{e^{Th}}{T^{N+b+1}}.$$

We do the same as in Section 4 for $\int_{V_0} e^{-2\pi i \langle v, \alpha \rangle} \frac{d^j \widehat{g}}{ds^j} (-i\beta(iv)) e^{\beta(iv)T} dv$ and obtained

$$\begin{aligned}
&\sum_{j=0}^{N+b+2} \frac{1}{T^{j+1}} \int_{V_0} e^{-2\pi i \langle v, \alpha \rangle} \frac{d^j \widehat{g}}{ds^j} (-i\beta(iv)) e^{\beta(iv)T} \, dv \\
&= \frac{e^{Th}}{T^{\frac{b}{2}+1}} \left( c_0 + \sum_{k=1}^{N-1} \frac{c_{k,T}(\alpha)}{T^k} + O\left( \frac{1}{T^N} \right) \right).
\end{aligned}$$

Since for $g = \chi_{[-T^{\frac{1}{4N+b+3}}, 0]}$, $\hat{g}(-i\beta(iv)) = g(v) + O(e^{-T^{\frac{1}{4N+b+3}} h})$,

$$c_{k,T}(\alpha) = c_k(\alpha) + O\big(e^{Th - T^{\frac{1}{4N+b+3}} h}\big).$$

Hence

$$\pi_g(T,\alpha) = \frac{e^{Th}}{T^{b/2+1}}\left(c_0 + \sum_{k=1}^{N-1}\frac{c_k(\alpha)}{T^k} + O\left(\frac{1}{T^N}\right)\right).$$

Now

$$\pi\left(T - T^{\frac{1}{4N+b+3}}, \alpha\right) = O\left(e^{Th - T^{\frac{1}{4N+b+3}}h}\right),$$

and

$$\pi\left(T - T^{\frac{1}{4N+b+3}}, \alpha\right) + \pi_{\chi_{[-T^{\frac{1}{4N+b+3}},0]}}(T,\alpha) = \pi(T,\alpha).$$

However, $e^{-hT^{1/4N+b+3}} \to 0$ faster than $1/T^n$ for any $n$. So we have the following theorem.

THEOREM 2. *Let $M$ be a compact Riemannian manifold with first Betti number $b > 0$ and with negative section curvature. Furthermore suppose that either* $\dim M = 2$ *or the sectional curvatures lie between $-4$ and $-1$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_b) \in H_1(M,\mathbb{Z}) \cong \mathbb{Z}^b$, we have*

$$\pi(T,\alpha) = \frac{e^{Th}}{T^{b/2+1}}\left(c_0 + \sum_{n=1}^{N}\frac{c_n(\alpha)}{T^n} + O\left(T^{-(N+1)}\right)\right), \tag{10}$$

*where $c_0$ is a constant which is independent of $\alpha$, and*

$$c_n(\alpha) = \sum_{l_1+l_2+\cdots+l_b=0}^{2n} c_{l_1 l_2 \ldots l_b}\alpha_1^{l_1}\alpha_2^{l_2}\ldots\alpha_b^{l_b}$$

*is a polynomial in $\alpha_1,\ldots,\alpha_b$ whose degree is $2n$ . Furthermore, $c_n(\alpha) \sim (-1)^n c\|\alpha\|^{2n}$, as $\|\alpha\| \to \infty$.*

REMARK. We also can directly obtain $\pi_\delta(T,\alpha)$ by taking $g = \chi_{[-\delta,\delta]}$ then using approximation argument.

Now we consider the special case. If we take $N = 1$, then (10) reduces

$$\pi(T,\alpha) = \frac{e^{Th}}{T^{b/2+1}}\left(c_0 + \frac{c_1(\alpha)}{T} + O(T^{-2})\right).$$

By the remark following Theorem 1, we have that $c_0 > 0$, and $c_1(\alpha) = -\sum_{i,j=1}^{b} c_{ij}\alpha_i\alpha_j + \sum_{i=1}^{b} b_i\alpha_i + c$ with

$$c_{ij} = c\int_{\mathbb{R}^b} e^{-\frac{1}{2}\beta''(0)(v,v)}v_i v_j\, dv \qquad (c > 0).$$

Let $A = (c_{ij})_{b\times b}$, $B = (b_i)_{i=1}^{b}$ We have

$$\pi(T,\alpha) = \frac{e^{Th}}{T^{b/2+1}}\left(c_0 + \frac{-\langle\alpha, A\alpha\rangle + \langle B,\alpha\rangle + c}{T} + O(T^{-2})\right). \tag{11}$$

LEMMA 5. *A is positive definite.*

*Proof.* For any $x = (x_1, x_2, \cdots, x_b)$,

$$\langle x, Ax\rangle = \sum_{i,j=1}^{b} c_{ij}x_i x_j = c\sum_{i,j=1}^{b}\int_{\mathbb{R}^b} e^{-\frac{1}{2}\langle v,\beta''(0)v\rangle} x_i v_i x_j v_j\, dv \qquad (c > 0)$$

$$= c\int_{\mathbb{R}^b} e^{-\frac{1}{2}\langle v,\beta''(0)v\rangle}\sum_{i,j=1}^{b}(x_i v_i x_j v_j)\, dv = c\int_{\mathbb{R}^b} e^{-\frac{1}{2}\langle v,\beta''(0)v\rangle}\sum_{i=1}^{b}(x_i v_i)^2\, dv \geq 0.$$

For a fixed $x \neq 0$, $\{v : \sum_{i=1}^{b} x_i v_i = 0\}$ is a hyperplane in $\mathbb{R}^b$, so that the integrand is positive almost everywhere. This means that $\langle x, Ax\rangle > 0$ for $x \neq 0$, hence $A$ is positive definite. □

Now we can define $|||x||| = \langle x, Ax\rangle^{1/2}$ and $|||\cdot|||$ is a norm in $H_1(M, \mathbb{R})$ (or $\mathbb{R}^b$). So we have

$$\pi(T, \alpha) = \frac{e^{Th}}{T^{b/2+1}}\left(c_0 + \frac{-|||\alpha|||^2 + \langle B, \alpha\rangle + c}{T} + O(T^{-2})\right). \tag{12}$$

We note $\pi(T, \alpha) = \pi(T, -\alpha)$ for any $T$. In fact for any $T$, there exists a natural one-to-one correspondence in $\Gamma$ by $\gamma \leftrightarrow -\gamma$. The correspondence preserves the length but reverses the direction, so $\pi(T, \alpha) = \pi(T, -\alpha)$ which implies $B = 0$ in (12). So we have the following result.

THEOREM 3.

$$\pi(T, \alpha) = \frac{e^{Th}}{T^{b/2+1}}\left(c_0 + \frac{c_1 - |||\alpha|||^2}{T} + O(T^{-2})\right) \text{ as } T \to \infty,$$

*where $c_0$, $c_1$ are constants and $c_0 > 0$.*

It is also easy to see that the following holds.

COROLLARY 1. *For all $\alpha, \beta \in H_1(M, \mathbb{Z})$, if $|||\alpha||| > |||\beta|||$ then, for sufficiently large $T$, we have $\pi(T, \alpha) < \pi(T, \beta)$. In particular, if $\alpha \neq 0$ then, for sufficiently large $T$, $\pi(T, \alpha) < \pi(T, 0)$.*

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