Identifiability of Points and Rigidity of Hypergraphs under Algebraic Constraints

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Abstract

The identifiability problem arises naturally in a number of contexts in mathematics and computer science. Specific instances include local or global rigidity of graphs and unique completability of partially-filled tensors subject to rank conditions. The identifiability of points on secant varieties has also been a topic of much research in algebraic geometry. It is often formulated as the problem of identifying a set of points satisfying a given set of algebraic relations. A key question then is to prove sufficient conditions for relations to guarantee the identifiability of the points.

This paper proposes a new general framework for capturing the identifiability problem when a set of algebraic relations has a combinatorial structure and develops tools to analyse the impact of the underlying combinatorics on the local or global identifiability of points. Our framework is built on the language of graph rigidity, where the measurements are Euclidean distances between two points, but applicable in the generality of hypergraphs with arbitrary algebraic measurements. We establish necessary and sufficient (hyper)graph theoretical conditions for identifiability by exploiting techniques from graph rigidity theory and algebraic geometry of secant varieties. In particular our work analyses combinatorially the effect of non-generic projections of secant varieties.

1 Introduction

Suppose one is given a set of points in $\mathbb{R}^d$ whose positions are unknown and a measuring device which provides relations among those points. The fundamental question arises: Can the locations of the points be uniquely identified from the measurements? This identifiability problem is ubiquitous across various applications in data science and engineering. Furthermore, it appears in several context of mathematics such as the identifiability of secant varieties in algebraic geometry. A key question in the identifiability problem is to determine a sufficient condition for the observations to guarantee the unique identification of the points. Such a question is often challenging if the observations are not sufficiently generic.

In this paper, we introduce a general framework for capturing the identifiability problem when a set of algebraic relations has a combinatorial structure as follows. Suppose there are n unknown points $p_1, p_2, \dots, p_n$ in $\mathbb{R}^d$. Let g be a polynomial map, which is a measurement function representing a measurement device, and suppose that the value of g is determined for each tuple of k points in $\mathbb{R}^d$.

To represent a set of possible observations, we utilise a k-uniform hypergraph denoted as $G = (V, E)$. In this case, V corresponds to the set $\{1, \dots, n\}$ representing the points. Essentially, the observer can obtain measurements $g(p_{v_1}, \dots, p_{v_k})$ for all $\{v_1, \dots, v_k\} \in E$. Then the identifiability problem revolves around determining whether the polynomial system given by:

$$g(x_{v_1}, \dots, x_{v_k}) = g(p_{v_1}, \dots, p_{v_k}) \qquad (\{v_1, \dots, v_k\} \in E) \tag{1}$$

has a unique solution (up to certain symmetry) in the variables $x_1, x_2, \dots, x_n$ in $\mathbb{R}^d$. This formulation is based on graph rigidity theory, which specifically addresses the case when g represents the Euclidean distance between two points. Our framework extends beyond this specific case, providing a general approach for the identifiability problem in cases involving algebraic relations with combinatorial structures.

2020 Mathematics Subject Classification: Primary 52C25, 05E14; Secondary 05C65, 68R05, 14N07

Graph rigidity theory has a long history with its roots in mathematics - arising from Cauchy and Euler's investigations of Euclidean polyhedra - and engineering - Maxwell's analysis of the stiffness of frames. The question is related to various branches of mathematics such as graph theory, matroid theory, and algebraic geometry, and it also appears in various modern engineering topics such as molecular conformation, network localisation, and multi-agent formation control, see, e.g., [65, 24]. The two central notions in rigidity theory are *global rigidity* and *local rigidity*, which concern whether the system (1) has a unique solution or a finite number of solutions (up to Euclidean isometry), respectively, when G is an ordinary graph and g is the Euclidean distance. Accordingly, one can define the notion of *local rigidity* and *global rigidity of hypergraphs* for each model of measurement map g using the system (1).

A key aim of the paper is to present a generalisation of graph rigidity theory that covers a wide range of algebraic topics by looking at general measurement maps g. While the concept of graph rigidity is intuitive, several variants of the problem have already been explored. Notable examples include rigidity in different metric spaces such as spherical space or ℓ_p-space, as well as rigidity concerning other geometric constraints (see, for instance, [12, 16, 27, 55, 58, 44]). While it is indeed possible to consider more general algebraic systems, doing so may result in a loss of the combinatorial perspective, which serves as the core of rigidity theory. In this paper, we demonstrate that this new rigidity model of hypergraphs is general enough to address the identifiability problem across various applications and it also provides new mathematical challenges from the graph rigidity viewpoint.

A natural example of our rigidity model is the uniqueness problem of low-rank tensor completions. Identifiability in tensor decompositions naturally arises in various fields, including phylogenetics, quantum information and signal processing [47]. Our rigidity-based approach to the unique low-rank tensor completion problem can be viewed as an extension of the rigidity-based analysis of low-rank matrix completions pioneered by Singer-Cucuringu [58] and Király-Theran-Tomioka [44]. However, as demonstrated in Subsection 3.7, the tensor case gives rise to substantially more challenging mathematical questions.

The key idea of combinatorial rigidity theory is the genericity assumption on point configurations. In the case of local rigidity, under this genericity assumption, the core question becomes equivalent to determining the dimension of the image of a measurement map. Notably, these images typically correspond to projections of well-studied algebraic varieties such as a secant variety of a Veronese variety or of a Grassmannian variety. Consequently, our rigidity problem can be formulated as an algebraic sensing problem, similar to the approach presented in [14]. While Noether's normalization lemma shows how the dimension changes under a generic projection [14], this general theory does not provide a solution to our specific problem. In our case, the projections are non-generic as they are consistently performed along a coordinate axis of the ambient affine space.

A particularly interesting case is when the measurement map is the sum of simpler measurement maps. In such a case, the image of the measurement map is a secant of a simpler variety. The computation of the dimension of secant varieties and the identifiability of secant points have been studied extensively in algebraic geometry (see, e.g., [8]), and results in that context give rise to rigidity theorems for complete hypergraphs in our language. For example, the Alexander-Hirschowitz theorem [4] on the dimension of secant varieties of Veronese varieties provides a characterisation for complete hypergraphs to be locally rigid, specifically when the measurement map g is the sum of copies of the product map over coordinates. These results highlight the connection between algebraic geometry and our notion of rigidity, offering new insights into the identifiability and rigidity properties of complete hypergraphs. Subsequent results [20, 30] on identifiability even give a characterisation of global rigidity for complete hypergraphs. (More details will be discussed in Subsection 3.7.) However, there seems to be few results on the projections of secant varieties. Our rigidity problem for general hypergraphs deals with two structures: the geometry of secant varieties and the combinatorics of coordinate projections. The goal is to establish a link between these two structures, shedding light on their connections and implications.

By leveraging techniques from both rigidity theory and secant varieties, we will derive several necessary and sufficient conditions for local and global rigidity. Our contributions include a graph-packing-type sufficient condition for local rigidity and a stress-matrix-type sufficient condition for global rigidity. We believe that our new rigidity model and the results of this article are the start of a larger effort to exploit the link between combinatorial rigidity and the geometry of secant varieties. This connection is likely to be instrumental in solving further rigidity and identifiability problems in the future. To justify this we provide two concrete applications of our work, one concerns random axis-aligned projections of varieties and the other is about the identifiability of a ℓ_p-analogue of the Cayley-Menger variety. Both applications provide novel connections between graph rigidity theory and the degeneracy/identifiability problem of secant varieties.

1.1 Roadmap

The paper consists of eight sections, each briefly summarised below for the reader's convenience.

Section 2. This section provides the core definition of our rigidity model. After reviewing the basics of classical rigidity theory in Section 2.1, we provide a new generalisation of graph rigidity to the setting of arbitrary algebraic constraints on hypergraphs in Section 2.2. The definition is justified in Section 2.3 by providing a range of examples of identifiability problems that can be modelled in this setting, including those arising from rigidity theory, matrix and tensor completions, and Chow decompositions.

Section 3. This section provides a number of basic results obtained by applying conventional rigidity analysis. A key notion here is the concept of infinitesimal rigidity, the first-order approximation of local rigidity. In terms of the underlying algebraic varieties, checking infinitesimal rigidity corresponds to analysing tangent spaces. However special attention has to be paid to fill a possible gap between local rigidity and infinitesimal rigidity. This aspect is carefully examined in Section 3.2.

Section 3.4 clarifies the connection between rigidity and secant defectivity/identifiability of varieties. Roughly speaking, local/global rigidity correspond to non-defectivity/identifiability, respectively, and analysing infinitesimal rigidity corresponds to analysing matrix representations of tangent spaces of secant varieties obtained by Terracini's lemma. Although this connection is available only when measurement maps can be written as sum of simpler maps and rigidity and secant defectivity/identifiability are different concepts in general, we believe this connection is fruitful and we explore it further throughout the paper.

In Section 3.5 and 3.6, we provide a matroid-theoretic foundation for our rigidity model, which offers a combinatorial upper bound for secant varieties that can be sharper than the conventional expected dimension. Subsection 3.7 provides detailed information and references for what is known in each of the concrete instances from Section 2.3, particularly illustrating the abstract theory of Section 3. A detailed discussion is provided for the symmetric tensor completion problem as a primary example. This includes Theorem 3.14 which translates the Alexander-Hirschowitz theorem to our language.

Section 4. In this section, we provide applications of combinatorial techniques from graph rigidity theory to our generalised model. Among other things, we examine 0-extension operations and packing-type sufficient conditions. The 0-extension operation is the most elementary graph operation that has been extensively used in graph rigidity research. In Section 4.2, we show that 0-extension always preserves local rigidity for a large class of measurement maps. The result enables us to construct various locally rigid examples as in the case of classical rigidity. On the other hand, we see that 0-extension may fail in general; in particular we demonstrate this failure for the determinant measurement map.

In Section 4.3, we provide a packing condition that is sufficient for local rigidity when the constraints arise from multilinear k-forms (Theorem 4.6). In classical rigidity, analysing d-dimensional rigidity in terms of simpler lower dimensional cases is a popular direction. It is also possible to recover some classical non-defectivity results in secant varieties [2] from Theorem 4.6. In this paper we focus on establishing results for general hypergraphs, and providing a more refined analysis for special classes of hypergraphs remains an interesting open problem. For example, a new result [10] on secant non-defectivity heavily uses the symmetry of the underlying varieties, which in our language corresponds to the case when the underlying hypergraph is dense and highly symmetric. Identifying a class of hypergraphs for which the techniques of [10] can be applied is an intriguing open problem.

Section 5. We move on to analyse global rigidity for the generalised rigidity model. In Section 5.1, we introduce the key concept of generic global rigidity, by noting that global rigidity is a generic property in the complex case but may not be in the real case for general measurements. One of the biggest achievements in rigidity theory is the theorem by Gortler, Healy, and Thurston [33], which shows that Euclidean global rigidity is a generic property over the reals, i.e., generic global rigidity over the reals is completely determined by the underlying graphs. Extending the result of Gortler, Healy, and Thurston [33] to a wider class of measurement maps is the most intriguing open problem in this paper. In Section 7.2, we provide a solution to a new special case of the problem.

The proof of the theorem of Gortler, Healy, and Thurston [33] is based on the theory of equilibrium stresses due to Connelly [25]. In Section 5.2 we shall explain its connection to the tangentially weak non-defectiveness of secant varieties. Our main results in Section 5 are Theorems 5.13 and 5.15 which provide sufficient conditions for global rigidity in the determinant and tensor product cases. These results are new non-trivial analogues of Connelly's well known sufficient condition in the Euclidean rigidity case.

Section 6. The definition of rigidity depends on the ambient class of hypergraphs, which can lead to different rigidity settings even when the measurement function remains unchanged. In this section, we present an example when the underlying hypergraphs are k-partite k-uniform, and demonstrate how the rectangular tensor completion problem can be captured in this rigidity model.

Section 7. Our generalised rigidity model clarifies a connection between graph rigidity theory and the analysis of secant varieties. We believe this connection is fruitful, and we can obtain further results if we explore structures of specific varieties. As motivating evidence, we briefly provide two applications.

The first application in Section 7.1 concerns random projections of varieties along coordinate axes. In Theorem 7.1, we prove a novel criterion for a random axis-parallel projection of varieties preserves dimension. We demonstrate its applicability by proving that a random axis-parallel projection of Segre varieties preserves dimension if the projected subspace has dimension $\Omega(n \log n)$ (Corollary 7.3), which is asymptotically tight. Our rigidity formulation enables us to reduce this question to the problem of analysing a property of the Erdős-Rényi random hypergraph and to do this we apply a result from spectral graph theory. The second application in Section 7.2 analyses the identifiability of the p-Cayley-Menger variety (an analogue of the Cayley-Menger variety for the ℓ_p-norm) and examines global rigidity in the ℓ_p-space. A combination of tools from algebraic geometry such as tangential weak defectiveness and graph-theoretical tools from rigidity theory leads to a precise combinatorial characterisation of the 2-identifiability of the axis-parallel projections of the p-Cayley-Menger variety. This result is also important in rigidity theory, as it affirmatively answers the question of whether global rigidity in the ℓ_p-plane is a generic property of the graph when p is even.

Section 8. We finish with remarks on future directions and potential applications, for example in algebraic statistics and persistent homology.

Notation. We use the following notation throughout the paper. For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. Throughout the paper, let $\mathbb{F}$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For a finite set V, a map $p : V \to \mathbb{F}^d$ is called a *point-configuration* in $\mathbb{F}^d$. We use $(\mathbb{F}^d)^V$ to denote the set of all point-configurations in $\mathbb{F}^d$. Obviously $(\mathbb{F}^d)^V$ is isomorphic to $\mathbb{F}^{d|V|}$, and hence we sometimes regard each $p \in (\mathbb{F}^d)^V$ as a $d|V|$-dimensional vector. If $V = [k]$, then $(\mathbb{F}^d)^{[k]}$ is simply denoted by $(\mathbb{F}^d)^k$.

We will need the following terminology related to hypergraphs. Let $\left\{ {X \atop k} \right\}$ be the set of all multisets of k elements of a finite set X and $\binom{X}{k}$ be the subset of $\left\{ {X \atop k} \right\}$ consisting of sets having no repeated elements. Throughout the paper, a *k-uniform hypergraph* G is defined as a pair $G = (V, E)$ of a finite set V and $E \subseteq \left\{ {V \atop k} \right\}$, and G is said to be *simple* if $E \subseteq \binom{V}{k}$. Note that $|\left\{ {|V| \atop k} \right\}| = \binom{|V|+k-1}{k}$. We use K_n^k to denote the complete k-uniform hypergraph with n vertices, that is, $K_n^k = ([n], \left\{ {[n] \atop k} \right\})$. Similarly, let $\tilde{K}_n^k = ([n], \binom{[n]}{k})$ be the simple complete k-uniform hypergraph on n vertices. For a hyperedge $e \in \left\{ {V \atop k} \right\}$ and $u \in e$, we use $e - u$ to denote the multiset obtained from e by reducing the multiplicity of u by one. Similarly, for

Table 1: Key notation.

Symbol	Description
$[n]$	The set $\{1,\dots,n\}$
$\mathbb{F}$	Base field, $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$
d	Ambient dimension of point configurations
k	Number of vertices in each hyperedge
$G=(V,E)$	k-uniform hypergraph with vertex set V and hyperedge set E
$p\in(\mathbb{F}^d)^V$	Point configuration
(G,p)	Hyper-framework
$g:(\mathbb{F}^d)^k\to\mathbb{F}$	Polynomial measurement map
$f_{g,G}$	Measurement map induced by g and G
Γ_g	Stabiliser of g
$\mathfrak{g}$	Lie algebra of Γ_g
$\mathrm{J}f_{g,G}(p)$	Jacobian of $f_{g,G}$ at p
$\mathrm{triv}_g(p)$	Trivial infinitesimal g-motions
d_{Γ_g}	Dimension of Γ_g
n_{Γ_g}	Minimum size for trivial motions
K_n^k	Complete k-uniform hypergraph
$\widetilde{K}_n^k$	Simple complete k-uniform hypergraph
$\mathcal{X}_k(V)$	Multisets of size k from V
$e\pm v$	Multiset obtained by adding/removing v
$m_e(v)$	Multiplicity of v in the mutiset e
$\mathrm{Sect}_r(X)$	r-th secant variety of X
$A(v)$	The v-th column of a $d\times n$ matrix A, for $v\in[n]$

$v\in V$, let $e+v$ be the multiset obtained from e by increasing the multiplicity of v by one. For $e\in\left\{{V \atop k}\right\}$ and $v\in V$, the multiplicity of v in e is denoted by $m_e(v)$. If A is a $d\times n$ matrix and $v\in[n]$, we denote by $A(v)$ its v-th column.

2 Rigidity of Hypergraphs

2.1 Euclidean rigidity

The ordinary Euclidean rigidity concerns the rigidity of graphs drawn in Euclidean space. Such a realisation of a graph is called a *(bar-and-joint) framework*. Conventionally a framework is defined as a pair (G,p) of a finite graph $G=(V,E)$ and a point-configuration $p:V\to\mathbb{R}^d$. The framework is *rigid* if the only edge-length preserving continuous motions of its vertices arise from isometries of $\mathbb{R}^d$, and otherwise it is called *flexible*. The study of the rigidity of frameworks has its origins in the work of Cauchy and Euler on Euclidean polyhedra and Maxwell on frames.

Abbot [1] showed that it is NP-hard to determine whether a given d-dimensional framework is rigid for $d\geq 2$. The problem becomes more tractable for generic frameworks (G,p) since we can linearise the problem and consider 'infinitesimal rigidity' instead. In this setting, the *measurement map* $f_G:(\mathbb{R}^d)^V\to\mathbb{R}^E$ is defined by putting $f_G(p)=(\dots,\|p(i)-p(j)\|^2,\dots)_{ij\in E}$, where $\|.\|$ denotes the Euclidean metric. Denote $\mathrm{J}f_G$ for the Jacobian matrix of f_G. The *rigidity matrix* $R_d(G,p)=\frac{1}{2}\mathrm{J}f_G$ is the $|E|\times d|V|$ matrix in which, for $e=ij\in E$, the row corresponding to e contains the d-dimensional block vector $p(i)-p(j)$ in the columns corresponding to vertex i, the block $p(j)-p(i)$ in the columns corresponding to vertex j, and zeros elsewhere. Here, g is the usual Euclidean distance. A map $\dot{p}:V\to\mathbb{R}^d$ is called an *infinitesimal motion* of (G,p) if

$$\langle p(i)-p(j),\dot{p}(i)-\dot{p}(j)\rangle=0 \text{ for all } ij\in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. An infinitesimal motion $\dot{p}$ is *trivial* if there exists a skew-symmetric matrix S and a vector t such that $\dot{p}(i) = Sp(i) + t$ for all $i \in V$. A framework (G, p) is *infinitesimally rigid* if every infinitesimal flex of (G, p) is trivial. Equivalently, (G, p) is infinitesimally rigid if G is complete on at most $d+1$ vertices or $|V| \geq d+2$ and rank $R_d(G, p) = d|V| - \binom{d+1}{2}$. Asimow and Roth [6] showed that infinitesimal rigidity is equivalent to rigidity for generic frameworks, hence generic rigidity depends only on the underlying graph. Figure 1 illustrates these concepts in dimension 2.

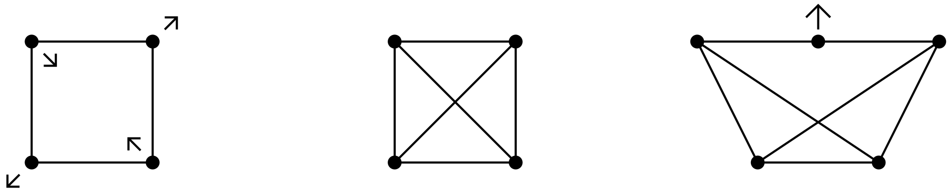

Figure 1: (Left) An infinitesimal flex of a simple flexible framework. (Middle) A generically rigid graph whose edge set is dependent in the generic 2-dimensional rigidity matroid since $|E| = 6 > 2|V| - 3$. (Right) A generically rigid graph realised as a framework that has an infinitesimal flex (indicated).

The *generic d-dimensional rigidity matroid* of a graph $G = (V, E)$ is the matroid $\mathcal{R}_d(G)$ on E in which a set of edges $F \subseteq E$ is independent if the corresponding rows of $R(G, p)$ are linearly independent, for some (or, equivalently, for every) generic p. The rank function of the generic d-dimensional rigidity matroid is given by $r(F) = \text{rank}\, R_d(G|_F, p)$ for any edge set $F \subseteq E$ and generic point configuration p.

A framework (G, p) in $\mathbb{R}^d$ is *globally rigid* if every $q \in f_G^{-1}(f_G(p))$ satisfies $t \cdot q = p$ for some isometry t of $\mathbb{R}^d$ (i.e. some composition of translations and rotations). See Figure 2 for an example.

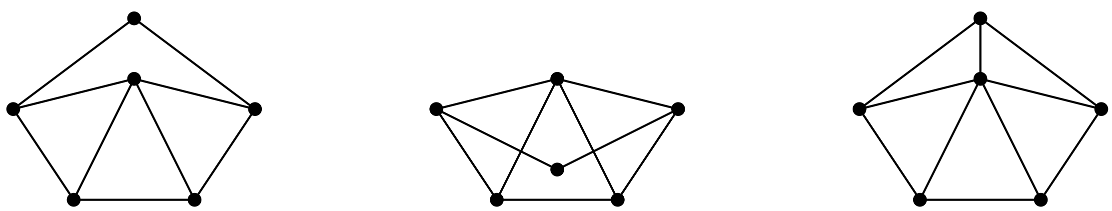

Figure 2: (Left-middle) Rigid but not globally rigid frameworks. (Right) Globally rigid in the plane.

2.2 Rigidity under algebraic constraints

The concept of rigidity is flexible in the sense that all notations can be extended by replacing the Euclidean distance function with a general smooth function. Since a measurement may involve more than two vertices, we will use hypergraphs to capture the underlying combinatorics. Extending the central object from rigidity, a pair (G, p) of a hypergraph G and a point-configuration $p \in (\mathbb{F}^d)^V$ is called a *d-dimensional hyper-framework* or a *hyper-framework in $\mathbb{F}^d$*. Throughout the paper, we use the following setup.

Setup 2.1. *Let $G = (V, E)$. Suppose we are given a k-uniform hyper-framework (G, p) and a polynomial map $g : (\mathbb{F}^d)^k \to \mathbb{F}$. The g-*measurement map *of G is a polynomial map $f_{g,G} : (\mathbb{F}^d)^V \to \mathbb{F}^E$ that sends p to the list of the g-values of the tuples $(p(v_1), p(v_2), \ldots, p(v_k))$ over the hyperedges $\{v_1, \ldots, v_k\}$ in E, i.e.,*

$$f_{g,G}(p) := (g(p(v_1), p(v_2), \ldots, p(v_k)) : e = \{v_1, \ldots, v_k\} \in E)\,.$$

In terms of the discussion in the introduction, g represents a measurement device and $f_{g,G}(p)$ is the actual list of measurements that the observer can obtain. The observer is asked to recover p from $f_{g,G}(p)$. The g-rigidity defined below captures the identifiability of p.

Note that, in order to make $f_{g,G}$ well-defined, g must be either *symmetric* or *anti-symmetric* with respect to the ordering of the points, and if g is anti-symmetric we always assume that $v_1, \dots, v_k$ are aligned in increasing order in $g(p(v_1), \dots, p(v_k))$, assuming a (fixed) total order on the vertices of G.

In most practical applications, there is a nontrivial group action that does not change the value of the g-measurement map, and rigidity has to be defined by taking care of the degree of freedom caused by such actions. Suppose the general affine group $\mathrm{Aff}(d, \mathbb{F})$ acts on $\mathbb{F}^d$ by $\gamma \cdot x = Ax + t$ for $x \in \mathbb{F}^d$ and each pair $\gamma = (A, t)$ of $A \in \mathrm{GL}(d, \mathbb{F})$ and $t \in \mathbb{F}^d$. The action of $\mathrm{Aff}(d, \mathbb{F})$ on $(\mathbb{F}^d)^V$ is also defined by $(\gamma \cdot p)(v) = Ap(v) + t$ $(v \in V)$ for any $\gamma = (A, t) \in \mathrm{Aff}(d, \mathbb{F})$ and $p \in (\mathbb{F}^d)^V$. Then the induced action on a polynomial map $g : (\mathbb{F}^d)^k \to \mathbb{F}$ is given by $\gamma \cdot g(x_1, \dots, x_k) = g(\gamma^{-1} \cdot x_1, \dots, \gamma^{-1} \cdot x_k)$ for $x_1, \dots, x_k \in \mathbb{F}^d$ and $\gamma \in \mathrm{Aff}(d, \mathbb{F})$. We say that γ *stabilises* g if g is invariant by the action of γ. The set of pairs (A, t) that stabilise g forms a subgroup of $\mathrm{Aff}(d, \mathbb{F})$, called the *stabiliser* Γ_g of g. Since the action is smooth, Γ_g is a closed subgroup of $\mathrm{Aff}(d, \mathbb{F})$, so Γ_g is also a Lie group by the closed-subgroup theorem.

We are now ready to give a formal definition of g-rigidity.

Definition 2.2. We say that (G, p) is *globally g-rigid* if for any $q \in f_{g,G}^{-1}(f_{g,G}(p))$ there is $\gamma \in \Gamma_g$ such that $q = \gamma \cdot p$. We say that (G, p) is *locally g-rigid* if there is an open neighbourhood N of p in $(\mathbb{F}^d)^V$ (in the Euclidean topology) such that for any $q \in f_{g,G}^{-1}(f_{g,G}(p)) \cap N$ there is $\gamma \in \Gamma_g$ such that $q = \gamma \cdot p$.

Remark 2.3. Let K_n^2 be the complete graph on n vertices. It is a well-known fact in rigidity theory that the point configuration is uniquely determined up to Euclidean isometry by measuring inter-point Euclidean distances for all edges in K_n^2, and hence the framework of a complete graph is always locally/globally rigid in the ordinary Euclidean rigidity. By this fact, we may also define the local/global rigidity of a graph G by comparing $f_{g,G}^{-1}(f_{g,G}(p))$ with $f_{g,K_n^2}^{-1}(f_{g,K_n^2}(p))$ when g is the squared Euclidean distance. However, for a general measurement g, the analogous fact is no longer true. A notable example is the case of symmetric tensor completion, where the Alexander-Hirschowitz theorem (cf. Theorem 3.14) shows that a complete hypergraph may be flexible for the corresponding polynomial map g. A more basic example is provided by rigidity theory under a smooth ℓ_p norm; as described in [29, Figure 2].

2.3 Examples of g-rigidity models

Ordinary Euclidean rigidity. A fundamental example is the case when G is 2-uniform (i.e., a graph) and $g(x, y) = (\|x - y\|_2)^2 = \sum_{i=1}^d (x_i - y_i)^2$ for $x, y \in \mathbb{R}^d$. Then the Euclidean group $E(d)$ is the stabilizer of g, and its Lie algebra is the set of pairs (S, t) of skew-symmetric matrices S and $t \in \mathbb{R}^d$. In this case, g-rigidity is nothing but the standard rigidity of bar-and-joint frameworks.

Rigidity in pseudo-Euclidean space. In the pseudo-Euclidean rigidity context G is 2-uniform (i.e., a graph) and $g(x, y) = \sum_{i=1}^{d_1} (x_i - y_i)^2 - \sum_{i=d_1+1}^{d} (x_i - y_i)^2$ for $x, y \in \mathbb{R}^d$ and $d_1 \le d$.

ℓ_p-norm rigidity. An alternative generalisation of Euclidean rigidity is to allow the distance function to be replaced by distance under another norm [45]. Specifically G is 2-uniform (i.e., a graph) and $g(x, y) = (\|x - y\|_p)^p = \sum_{i=1}^d |x_i - y_i|^p$ for $x, y \in \mathbb{R}^d$ and $1 < p < \infty$. In the case where p is a positive even integer, g is a polynomial.

Volume-constrained rigidity. For a d-dimensional pure simplicial complex realised in $\mathbb{R}^d$, volume-constrained rigidity concerns whether there is a motion of vertices keeping the (signed) volume of each d-simplex [16]. A d-dimensional pure simplicial complex can be identified with a $(d+1)$-uniform hyper-framework (G, p) with a simple hypergraph G. Hence, the volume-constrained rigidity can be captured as the g-rigidity of a simple $(d+1)$-uniform hyper-framework (G, p) with $g : (\mathbb{R}^d)^{d+1} \to \mathbb{R}$ defined by

$$g(x_1, x_2, \dots, x_{d+1}) = \det \begin{pmatrix} x_1 & x_2 & \dots & x_{d+1} \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

A rigidity-based analysis was initiated by Borcea and Streinu [12]. Recent studies can be found in [16, 59].

Positive semidefinite symmetric matrix completion. Let T be a positive semidefinite symmetric matrix of size n over $\mathbb{R}$. If the rank of T is d, then the spectral decomposition implies that

$$T = \sum_{i=1}^{d} x_i x_i^\top \tag{2}$$

for some vectors $x_1, x_2, \ldots, x_d \in \mathbb{R}^n$. Let p be a $d \times n$-matrix obtained by aligning x_i as the i-th row vector. Then $p(i) \cdot p(j)$ is equal to the (i, j)-th entry t_{ij} of T.

In the positive semidefinite symmetric matrix completion problem, we are given a partially-filled positive semidefinite symmetric matrix T and asked to recover the positive semidefinite symmetric matrix by filling missing entries. If we use a graph $G = ([n], E)$ (which may contain self-loops) to denote a set of indices $e = \{i, j\}$ of known entries t_e of T, then the problem is to find $p \in (\mathbb{R}^d)^n$ satisfying

$$\langle p(i), p(j)\rangle = t_e \qquad (e = \{i, j\} \in E), \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. We are, in particular, interested in the uniqueness of the completions rather than finding a completion. In the unique completion problem, we are given a solution $p \in (\mathbb{R}^d)^V$ of Equation (3) and are asked if it is the unique solution of (3). This uniqueness question is equivalent to asking the g-rigidity of a framework (G, p) for an appropriate choice of g.

Specifically, consider a 2-uniform hypergraph (i.e., a graph) G and $g(x, y) = \langle x, y\rangle$ for $x, y \in \mathbb{R}^d$. Then the orthogonal group $O(d)$ is the stabiliser of g, and the global g-rigidity decides if Equation (3) has the unique solution up to the action of $O(d)$, or equivalently a completion is unique. This rigidity-based formulation of matrix completion coincides with that introduced by Singer and Cucuringu [58] whose paper also describes a number of detailed examples of matrix completions from this viewpoint.

Symmetric tensor completions. The idea of converting the unique low-rank matrix completion problem to the g-rigidity of frameworks can be extended naturally to tensors as follows.

For a vector space V of dimension n over $\mathbb{C}$, let $V^{\otimes k}$ be the k-fold tensor product of V. We fix a basis of V, and assume that each $T \in V^{\otimes k}$ is represented by a k-dimensional array over $\mathbb{C}$. A tensor $T \in V^{\otimes k}$ is said to be *symmetric* if for any permutation σ on $[k]$ we have $T_{i_1,i_2,\ldots,i_k} = T_{\sigma(i_1),\sigma(i_2),\ldots,\sigma(i_k)}$. The set of symmetric tensors in $V^{\otimes k}$ is denoted by $S^k(V)$. It is known that any symmetric tensor can be written as

$$T = \sum_{i=1}^{d} x_i^{\otimes k} := \sum_{i=1}^{d} x_i \otimes x_i \otimes \cdots \otimes x_i \tag{4}$$

for some vectors $x_1, x_2, \ldots, x_d \in V$. For $T \in S^k(V)$, the smallest possible d for which we can write T in the form of Equation (4) is called the *symmetric rank* of T.

Once we introduce a notion of rank, the corresponding low-rank completion problem can be defined automatically. In the symmetric tensor completion problem, given a partially-filled tensor of order k and size n, we are asked to fill the remaining entries to obtain a symmetric tensor of symmetric rank at most d. Recall that $\left\{ {X \atop k} \right\}$ denotes the set of multisets of k elements of a finite set X. Due to the symmetry condition, each tensor's entry can be indexed by an element in $\left\{ {[n] \atop k} \right\}$. In this manner, we encode the underlying combinatorics of each instance of the completion problem using a k-uniform hypergraph $([n], E)$.

We can also convert the decomposition in Equation (4) to a form of an algebraic relation among points in $\mathbb{C}^d$. For this, let p be the $d \times n$ matrix with the i-th row equal to x_i. Then, for each $e \in \left\{ {[n] \atop k} \right\}$, the e-th entry of the right-hand-side of Equation (4) is equal to $\mathbf{1} \cdot \bigodot_{v \in e} p(v)$, where $\mathbf{1}$ denotes the all-one vector and $\odot$ denotes the Hadamard product of vectors, that is the component-wise product. Hence, the symmetric tensor completion problem can be reformulated as a hypergraph realisation problem as follows: Given a k-uniform hypergraph $G = ([n], E)$ and $a_e \in \mathbb{C}$ for each $e \in E$, find $p \in (\mathbb{C}^d)^V$ such that

$$\mathbf{1} \cdot \bigodot_{v \in e} p(v) = a_e \qquad \text{for } e \in E, \tag{5}$$

The corresponding unique completion problem is captured by g-rigidity for $g : (\mathbb{C}^d)^k \to \mathbb{C}$ with

$$g(y_1, \dots, y_k) = \mathbf{1} \cdot \bigodot_{i \in \{1,\dots,k\}} y_i. \tag{6}$$

The stabiliser Γ_g of g is the set of matrices of the form ΣD with a permutation matrix Σ and a diagonal matrix D over $\mathbb{C}$ whose diagonal entries are k-th roots of unity.

Let $h_{\text{prod}} : \mathbb{F}^k \to \mathbb{F}$ be the product map that takes k values and returns their product. Then the function g defined in Equation (6) is written as the sum of d copies of h_{prod}. This structure will be important in the analysis of g-rigidity in the subsequent discussion.

Skew-symmetric tensor completions. A tensor $T \in V^{\otimes k}$ is said to be *skew-symmetric* if for any permutation σ on $[k]$ we have $T_{i_1,i_2,\dots,i_k} = \text{sign}(\sigma) T_{\sigma(i_1),\sigma(i_2),\dots,\sigma(i_k)}$. The set of skew-symmetric tensors in $V^{\otimes k}$ is denoted by $A^k(V)$. It is known that $A^k(V)$ is linearly isomorphic to the k-th component $\bigwedge^k V$ of the exterior algebra $\bigwedge V$ and any skew-symmetric tensor can be written as

$$T = \sum_{i=1}^{r} x_1^i \wedge x_2^i \wedge \dots \wedge x_k^i \tag{7}$$

for some kr vectors x_j^i ($1 \leq i \leq r$, $1 \leq j \leq k$), where $\wedge$ denotes the tensor product of vectors. The smallest possible r for which we can write T in the form of (7) is called the *skew-symmetric rank* of T. When $V = \mathbb{F}^n$, each element in $A^k(V)$ is called a *skew-symmetric tensor of order k of size n* (over $\mathbb{F}$).

Recall that $\binom{X}{k}$ denotes the family of sets of k elements of a finite set X. Due to the skew-symmetry condition, each entry of a skew-symmetric tensor can be indexed by an element in $\binom{X}{k}$. Hence, we encode the underlying combinatorics of each instance of the skew-symmetric completion problem using a simple k-uniform hypergraph $([n], E)$ (where each hyperedge is a set unlike the symmetric case).

In the same manner as the correspondence between Equations (4) and (5) in the symmetric tensor case, we shall convert Equation (7) to a realisation problem of the underlying hypergraph. This can be done by picking column vectors in the matrix obtained by stacking x_j^i as row vectors. Specifically, consider the $k \times n$ matrix Q_i obtained by stacking x_j^i for $1 \leq j \leq k$ for each $1 \leq i \leq r$ as row vectors, and let P be the $kr \times n$ matrix obtained by stacking Q_i's. Let $q_i \in (\mathbb{C}^k)^n$ (and respectively $p \in (\mathbb{C}^{kr})^n$) be the point configuration such that $q_i(v)$ (resp., $p(v)$) is the v-th column of Q_i (resp., P). Then, for each $e = \{v_1, \dots, v_k\}$, the e-th entry of T in Equation (7) is equal to $\sum_{i=1}^{r} \det\left(q_i(v_1) \dots q_i(v_k)\right)$. This gives the following equivalent formulation of the skew-symmetric tensor completion problem: Given a simple k-uniform hypergraph $G = ([n], E)$ and $a_e \in \mathbb{C}$ for each $e \in E$, find an r-tuple $p = (q_1, \dots, q_r)$ of $q_i \in (\mathbb{C}^k)^V$ such that

$$\sum_{i=1}^{r} \det\left(q_i(v_1) \dots q_i(v_k)\right) = a_e \qquad \text{for } e = \{v_1, \dots, v_k\} \in E. \tag{8}$$

Let $h_{\det} : (\mathbb{C}^k)^k \to \mathbb{C}$ be the determinant as a multilinear k-form over $\mathbb{C}^k$. Then the corresponding unique completion problem is captured by g-rigidity for $g : (\mathbb{C}^{kr})^k \to \mathbb{C}$ defined as the sum of r copies of $h_{\det}$. Since the stabiliser of $h_{\det}$ is $\text{SL}(k, \mathbb{C})$, the stabiliser of g is

$$\{(A_1 \oplus \dots \oplus A_r)(\Sigma \otimes I_k) \mid A_i \in \text{SL}(k, \mathbb{C}), \Sigma : \text{ a permutation matrix of size } r\}.$$

Chow decompositions. Replacing the determinant with the permanent in the above discussion, we obtain the corresponding problem for Chow decompositions. Suppose f is a homogeneous polynomial in n variables of degree k. A *Chow decomposition* of f is a representation of f as the sum of r polynomials written as a product of k linear forms, i.e.,

$$f(x_1,\ldots,x_n)=\sum_{i=1}^{r}\prod_{j=1}^{k}(a_{i,j,1}x_1+\cdots+a_{i,j,n}x_n) \tag{9}$$

for some $a_{i,j,1},\ldots,a_{i,j,n}\in\mathbb{F}$ ($1\le i\le r$ and $1\le j\le d$). The smallest possible r for which f has an expression of the form given in Equation (9) is called the *Chow rank* of f. A low Chow-rank decomposition problem asks to compute a Chow decomposition of a given polynomial f that achieves its Chow rank [63]. In this paper, we consider the completion version, where we are given a partial list of the coefficients of a polynomial f of Chow rank at most r and we are asked to recover f.

Now we reformulate this decomposition or completion problem in terms of a hypergraph realisation problem. As usual, the list of indices of given coefficients of f is represented by a k-uniform hypergraph on $[n]$. Consider the right hand side of Equation (9). Let Q_i be the $k\times n$ matrix whose j-th row is $(a_{i,j,1},a_{i,j,2},\ldots,a_{i,j,n})\in\mathbb{F}^n$ for each $1\le i\le r$, and let P be the $kr\times n$ matrix obtained by stacking Q_i's. Let $q_i\in(\mathbb{F}^k)^n$ (and respectively $p\in(\mathbb{F}^{kr})^n$) be the point configuration such that $q_i(v)$ (resp., $p(v)$) is the v-th column of Q_i (resp., P). Then, the coefficient of the squarefree monomial $x_{v_1}x_{v_2}\ldots x_{v_k}$ in the right hand side of Equation (9) is equal to

$$\sum_{i=1}^{r}\operatorname{perm}\Big(q_i(v_1)\ldots q_i(v_k)\Big), \tag{10}$$

where perm denotes the permanent. Even if $x_{v_1}x_{v_2}\ldots x_{v_k}$ is not squarefree, the coefficient of $x_{v_1}\ldots x_{v_k}$ is still a scalar multiple of Equation (10). Thus, (by scaling coordinates along each axis appropriately) the low Chow-rank completion problem can be reduced to the following hypergraph embedding problem: Given a k-uniform hypergraph $G=([n],E)$ and $a_e\in\mathbb{F}$ for each $e\in E$, find an r-tuple $p=(q_1,\ldots,q_r)$ of $q_i\in(\mathbb{F}^k)^V$ such that

$$\sum_{i=1}^{r}\operatorname{perm}\Big(q_i(v_1)\ldots q_i(v_k)\Big)=a_e \qquad \text{for } e=\{v_1,\ldots,v_k\}\in E. \tag{11}$$

The corresponding unique completion problem checks if f can be uniquely recovered from a partial list of coefficients using the information that the Chow rank of f is at most r. Let $h_{\text{perm}}:(\mathbb{F}^k)^r\to\mathbb{F}$ be the permanent as a multilinear k-form. Then the unique completion problem is captured by g-rigidity for $g:(\mathbb{F}^{kr})^k\to\mathbb{F}$ defined by the sum of r copies of h_{perm}. By [13], the stabilizer $\Gamma_{h_{\text{perm}}}$ of h_{perm} is the set of matrices of the form ΣD with a permutation matrix Σ and a diagonal matrix D with $\det D=1$. The stabilizer of g is $\{(A_1\oplus\cdots\oplus A_r)(\Sigma\otimes I_k):A_i\in\Gamma_{h_{\text{perm}}},\Sigma:\text{ a permutation matrix of size } r\}$.

3 Tools for Analysing g-rigidity

In this section, we give several fundamental tools for analysing local/global g-rigidity. Applications of these tools to specific problems are explained in Subsection 3.7.

3.1 Basic facts on polynomial maps

Determining the local or global g-rigidity of a hyper-framework is a NP-hard computational problem. A formal proof of its computational hardness can be found in [1, 56], specifically addressing ordinary Euclidean rigidity. Therefore, a common approach to studying g-rigidity involves analysing the linear approximation of the concept and leveraging existing tools to fill the remaining gaps. In this strategy the following facts are often used in graph rigidity theory.

Definition 3.1 (Generic point)**.** Let $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$. A point-configuration $p\in(\mathbb{F}^d)^n$ is said to be *generic* if the set of coordinates of p is algebraically independent over the rationals. More generally, a point in an algebraic set $\mathcal{A}$ defined over $\mathbb{Q}$ is generic if its coordinates do not satisfy any $\mathbb{Q}$-polynomial besides those that are satisfied by every point on $\mathcal{A}$.

Let X be a smooth manifold and $f : X \to \mathbb{F}^m$ be a smooth map. Then $x \in X$ is said to be a *regular point* of f if the Jacobian matrix $\mathrm{J}f(x)$ of f has maximum rank. Otherwise, x is called a *critical point* of f. Also $f(x)$ is said to be a *regular value* of f if, for all $y \in f^{-1}(f(x))$, y is a regular point of f.

The following proposition summarises the basic geometric tools we shall use. All parts are known in the literature (see, e.g., [34, 33]) though may not have been stated in the full generality of g-rigidity.

Proposition 3.2. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.*

(i) *Let $f : \mathbb{F}^n \to \mathbb{F}^m$ be a polynomial map and $p \in \mathbb{F}^n$. If $f(p)$ is a regular value, then $f^{-1}(f(p))$ is a smooth manifold with codimension equal to the* rank *of $\mathrm{J}f(p)$, where $\mathrm{J}f(p)$ is the Jacobian derivative of f evaluated at p.*

(ii) *Let V, W be smooth manifolds, $f : V \to W$ a surjective $\mathbb{Q}$-polynomial map and $p \in V$. If p is generic in V, then $f(p)$ is generic in W.*

(iii) *Let $f : \mathbb{F}^n \to \mathbb{F}^m$ be a $\mathbb{Q}$-polynomial map and $p \in \mathbb{F}^n$ a generic point. Then, $f(p)$ is generic if and only if $\mathrm{J}f(p)$ is row independent.*

(iv) *Let $f : \mathbb{F}^n \to \mathbb{F}^m$ be a $\mathbb{Q}$-polynomial map and $p \in \mathbb{F}^n$ a generic point. Then $f(p)$ is a regular value of f.*

Proof. Part (i) is simply a consequence of the implicit function theorem. For (ii), suppose that there is a polynomial g that is vanishing at $f(p)$. Then $g \circ f : V \to W$ is a polynomial map. Since p is generic in V, we have that $g \circ f$ is vanishing on V. Since f is surjective, this implies that g is vanishing on W.

(iii) Suppose that $\mathrm{J}f(p)$ is row independent. Then p is a regular point of f, and hence $f(\mathbb{F}^n) \cap M$ is an m-dimensional manifold in a neighbourhood M of $f(p)$ by the inverse function theorem. This in turn implies that, for a small neighbourhood V of p in $\mathbb{F}^n$, the restriction $f|_V$ of f to V is a surjective polynomial map from V to an m-dimensional manifold $f(V)$. Applying (ii) to $f|_V$, we have that $f(p)$ is generic in $f(V)$. Since $f(V)$ is m-dimensional, we conclude that $f(p)$ is also generic in $\mathbb{F}^m$.

Conversely, if $\mathrm{J}f(p)$ is row dependent then, since the rank of $\mathrm{J}f$ is locally constant in a neighbourhood of p, it follows that $f(p)$ lies on a proper algebraic subset of $\mathbb{F}^m$. Using either the Tarski-Seidenberg Theorem (when $\mathbb{F} = \mathbb{R}$) or Chevalley's Theorem (when $\mathbb{F} = \mathbb{C}$) we can show that this algebraic subset is defined over $\mathbb{Q}$ and so $f(p)$ is not generic. (A short exposition on Chevalley's theorem is given below.)

(iv) Suppose $\mathbb{F} = \mathbb{C}$. Let X be the set of points $q \in \mathbb{F}^n$ such that $f(q)$ is not regular. Chevalley's theorem implies that any quantifier can be eliminated in an expression of a set given by boolean operations, quantifiers, and $\mathbb{Q}$-algebraic equations. This, in particular, implies that X is a $\mathbb{Q}$-algebraically constructible set. On the other hand, Bertini's theorem states that there is a dense open set $U \subseteq \overline{\mathrm{im} f}$ in which every point is regular. Since $\overline{\mathrm{im} f}$ is irreducible, U^c is a lower dimensional algebraic subset of $\overline{\mathrm{im} f}$. Since $f(X) = U^c$, X must be a lower-dimensional $\mathbb{Q}$-algebraically constructible set in $\mathbb{C}^n$, and the genericity of p implies $p \notin X$. When $\mathbb{F} = \mathbb{R}$, we can apply the same argument by using Tarski-Seidenberg quantifier elimination instead of Chevalley's theorem. □

3.2 Infinitesimal g-rigidity

Let (G, p) be a k-uniform hyper-framework with $G = (V, E)$ and $g : (\mathbb{F}^d)^k \to \mathbb{F}$ a polynomial map. Then the Jacobian $\mathrm{J}f_{g,G}(p)$ of the measurement map $f_{g,G} : (\mathbb{F}^d)^V \to \mathbb{F}^E$ is a linear map from $(\mathbb{F}^d)^V$ to $\mathbb{F}^E$. Hence the right kernel of $\mathrm{J}f_{g,G}(p)$ is a linear subspace $(\mathbb{F}^d)^V$, which is defined as the space of *infinitesimal g-motions* of the framework (G, p).

If the stabiliser Γ_g of g is positive dimensional, every framework (G, p) has a nonzero infinitesimal g-motion. This fact can be seen as follows. Recall that Γ_g is a Lie group, so let $\mathfrak{g}$ be the Lie algebra of Γ_g. Observe that Γ_g acts on $(\mathbb{F}^d)^V$ via $(\gamma \cdot p)(v) = \gamma \cdot (p(v))$ for $v \in V$. This induces a map from $\mathfrak{g} \times (\mathbb{F}^d)^V$ to $(\mathbb{F}^d)^V$, denoted $(\dot\gamma, p) \mapsto \dot\gamma \cdot p$, whose image lies in the right kernel of $\mathrm{J}f_{g,G}(p)$. Then $\dot p \in (\mathbb{F}^d)^V$ defined by $\dot{p} = \dot{\gamma} \cdot p$ for $\dot{\gamma} \in \mathfrak{g}$ is called a *trivial infinitesimal g-motion* of (G, p). For $p \in (\mathbb{F}^d)^n$, define the space of trivial infinitesimal motions of (G, p) to be

$$\mathrm{triv}_g(p) := \{\dot{\gamma} \cdot p : \dot{\gamma} \in \mathfrak{g}\} \subseteq \ker \mathrm{J} f_{g,G}(p).$$

Definition 3.3. Let (G, p) be a k-uniform hyper-framework and $g : (\mathbb{F}^d)^k \to \mathbb{F}$ a polynomial map. We say that (G, p) is *infinitesimally g-rigid* if $\mathrm{triv}_g(p) = \ker \mathrm{J} f_{g,G}(p)$.

The following proposition generalises the fundamental theorem of Asimow and Roth [6] which provides the central motivation to look at infinitesimally rigidity in graph rigidity theory.

Proposition 3.4. *Let $g : (\mathbb{F}^d)^k \to \mathbb{F}$ be a $\mathbb{Q}$-polynomial map and (G, p) be a k-uniform hyper-framework.*

- *If (G, p) is infinitesimally g-rigid, then it is locally g-rigid.*
- *If p is generic, then (G, p) is infinitesimally g-rigid if and only if it is locally g-rigid.*

Proof. Let $T_p : \mathfrak{g} \to (\mathbb{F}^d)^V$ be the linear map $\dot{\gamma} \mapsto \dot{\gamma} \cdot p$. Then $\dim(\mathrm{triv}_g(p)) = \mathrm{rank}(T_p)$ is a lower semicontinuous function of p. It follows that there is a neighbourhood U of p such that, for $q \in U$, $\dim(\mathrm{triv}_g(q)) \geq \dim(\mathrm{triv}_g(p))$. Also for some neighbourhood W of p we have $\mathrm{rank}(\mathrm{J} f_{g,G}(q)) \geq \mathrm{rank}(\mathrm{J} f_{g,G}(p))$ for $q \in W$, again by the semicontinuity of rank.

If (G, p) is infinitesimally g-rigid then, by definition, $\mathrm{rank}(\mathrm{J} f_{g,G}(p)) = d|V| - \dim(\mathrm{triv}_g(p))$. Therefore, for $q \in U \cap W$, we have $\mathrm{rank}(\mathrm{J} f_{g,G}(q)) \geq \mathrm{rank}(\mathrm{J} f_{g,G}(p)) = d|V| - \dim(\mathrm{triv}_g(p)) \geq d|V| - \dim(\mathrm{triv}_g(q)) \geq \mathrm{rank}(\mathrm{J} f_{g,G}(q))$. So we have equality throughout and we can apply the constant rank theorem to deduce that, in a neighbourhood of p, $f_{g,G}$ is locally equivalent (i.e. up to a choice of coordinates) to a coordinate projection from $\mathbb{F}^{d|V|} \to \mathbb{F}^{d|V| - \dim(\mathrm{triv}_g(p))}$. It follows that (G, p) is locally g-rigid, as desired.

The second statement, that generic locally g-rigid hyper-frameworks are infinitesimally g-rigid, follows from Proposition 3.2(i). $\square$

We now compare the dimension of Γ_g and the dimension of the space of all trivial infinitesimal g-motions. Define d_{Γ_g} to be the dimension of Γ_g as a Lie group. Since Γ_g is a Lie subgroup of $\mathrm{Aff}(d, \mathbb{F})$, if p is generic and n is sufficiently large, then $\dim \mathrm{triv}_g(p) = d_{\Gamma_g}$. We use n_{Γ_g} to denote the minimum integer n such that $\dim \mathrm{triv}_g(p) = d_{\Gamma_g}$ for some $p \in (\mathbb{F}^d)^n$.

Example 3.5. The following presents two specific instances that exemplify these definitions.

- Suppose $g : (\mathbb{R}^d)^2 \to \mathbb{R}$ is the Euclidean inner product in $\mathbb{R}^d$. Then Γ_g is the orthogonal group $O(d)$, so $d_{\Gamma_g} = \binom{d}{2}$. The Lie algebra $\mathfrak{g}$ is the set of skew-symmetric matrices, and $\dim \mathrm{triv}_g(p) = \binom{d}{2} - \binom{d-n}{2}$ if p is a generic set of n points with $n \leq d$. Hence, $n_{\Gamma_g} = d - 1$.
- Suppose $g : (\mathbb{C}^d)^k \to \mathbb{C}$ is as in Equation (6). Then Γ_g is the set of matrices of the form ΓD with a permutation matrix Γ and a diagonal matrix D whose diagonal entries are k-th roots of unity. Then $d_{\Gamma_g} = 0$ and the Lie algebra is trivial. So $n_{\Gamma_g} = 0$.

Since $\mathrm{triv}_g(p)$ is the space of trivial infinitesimal g-motions, if $|V(G)| \geq n_{\Gamma_g}$ and p is generic, then Definition 3.3 implies that (G, p) is infinitesimally g-rigid if and only if $\dim \ker \mathrm{J} f_{g,G}(p) = d_{\Gamma_g}$. Combining this with Proposition 3.4, we have the following linear algebraic characterisation of generic local g-rigidity.

Proposition 3.6. *Assume Setup 2.1. Let $g : (\mathbb{F}^d)^k \to \mathbb{F}$ be a polynomial map whose stabilizer is Γ_g, and (G, p) a generic k-uniform hyper-framework with $G = (V, E)$ with $|V| \geq n_{\Gamma_g}$. Then*

$$\mathrm{rank}\, \mathrm{J} f_{g,G}(p) \leq d|V| - d_{\Gamma_g},$$

and the equality holds if and only if (G, p) is locally g-rigid.

Example 3.7. Consider the symmetric tensor completion problem of symmetric rank one and order three, that is, $d = 1$, $k = 3$ and $g = h_{\text{prod}} : (\mathbb{C}^d)^k \to \mathbb{C}$, where h_{prod} denotes the product map of k variables. Then $d_{\Gamma_g} = n_{\Gamma_g} = 0$ as seen in Example 3.5. Consider a framework (G, p), where $G = (\{a, b, c\}, \{aaa, aab, abc\})$ and $p : a \mapsto x_a, b \mapsto x_b, c \mapsto x_c$. Then,

$$\mathrm{J}f_{g,G}(p) = \begin{array}{c} \\ aaa \\ aab \\ abc \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 3x_a^2 & 0 & 0 \\ 2x_a x_b & x_a^2 & 0 \\ x_b x_c & x_c x_a & x_a x_b \end{bmatrix} \end{array}.$$

Since $\operatorname{rank} \mathrm{J}f_{g,G_i}(p) = 3 = d|V(G)| - d_{\Gamma_g}$, (G, p) is locally g-rigid.

The following geometric implication of local g-rigidity is also important.

Proposition 3.8. *Let $g : (\mathbb{F}^d)^k \to \mathbb{F}$ be a polynomial map whose stabiliser is Γ_g. Suppose p is generic and $G = (V, E)$ is a k-uniform hypergraph with $|V| \geq n_{\Gamma_g} + d_{\Gamma_g}$. Then (G, p) is locally g-rigid if and only if $f_{g,G}^{-1}(f_{g,G}(p))/\Gamma_g$ is finite.*

Proof. Suppose $n = |V|$ and $q \in f_{g,G}^{-1}(f_{g,G}(p))$. Recall that $\operatorname{triv}_g(q) = \{\dot{\gamma} \cdot p : \dot{\gamma} \in \mathfrak{g}\}$. We first show that

$$\dim \operatorname{triv}_g(q) = d_{\Gamma_g}. \tag{12}$$

For a vector x over $\mathbb{F}$, let $\mathbb{Q}(x)$ be the field extension of $\mathbb{Q}$ generated by the entries of x. By Proposition 3.2(iii), the transcendence degree of $\mathbb{Q}(f_{g,G}(p))$ over $\mathbb{Q}$ is $dn - d_{\Gamma_g}$. Since $f_{g,G}(p) = f_{g,G}(q)$, the transcendence degree of $\mathbb{Q}(f_{g,G}(q))$ over $\mathbb{Q}$ is also $dn - d_{\Gamma_g}$. Hence the transcendence degree of $\mathbb{Q}(q)$ over $\mathbb{Q}$ is at least $dn - d_{\Gamma_g}$. Therefore, since $n \geq n_{\Gamma_g} + d_{\Gamma_g}$, there is a vertex subset $X \subset V$ such that $|X| \geq n_{\Gamma_g}$ and $q|_X$ is generic. (Formally the existence of X can be seen by the following algorithm. First, set $X = \emptyset$ and repeat the following procedure from $i = 1$ through n. Check if the entries of $q(X \cup \{i\})$ are algebraically independent over $\mathbb{Q}$. If yes, update X to $X \cup \{i\}$; otherwise reject i. Since the transcendence degree of $\mathbb{Q}(q)$ over $\mathbb{Q}$ is at least $dn - d_{\Gamma_g}$, the fact that algebraic independence defines a matroid implies that the number of rejections in the above procedure is at most d_{Γ_g}. Since $n \geq n_{\Gamma_g} + d_{\Gamma_g}$, we have $|X| \geq n_{\Gamma_g}$ at the end of the procedure.) Hence, $\dim \operatorname{triv}_g(q_{|X}) = d_{\Gamma_g}$, and Equation (12) follows.

Parts (i) and (iv) of Proposition 3.2 imply that $f_{g,G}^{-1}(f_{g,G}(p))$ is a smooth manifold of dimension equal to $\dim \ker \mathrm{J}f_{g,G}(p)$. On the other hand, Equation (12) implies that, for any $q \in f_{g,G}^{-1}(f_{g,G}(p))$, $\Gamma_g \cdot q = \{\gamma \cdot q : \gamma \in \Gamma_g\}$ is a d_{Γ_g}-dimensional smooth submanifold of $f_{g,G}^{-1}(f_{g,G}(p))$. Since p is generic, there is a neighbour N of p such that $(f_{g,G}^{-1}(f_{g,G}(p)) \cap N)/\Gamma_g$ is a manifold of dimension equal to $\dim \ker \mathrm{J}f_{g,G}(p) - d_{\Gamma_g}$. By Proposition 3.4, the latter value is zero if and only if (G, p) is locally g-rigid. Hence, if (G, p) is not locally g-rigid, then $f_{g,G}^{-1}(f_{g,G}(p))/\Gamma_g$ is not finite.

Conversely, if (G, p) is locally g-rigid, then the dimension of $f_{g,G}^{-1}(f_{g,G}(p))$ is d_{Γ_g}. Since $\Gamma_g \cdot q$ is a d_{Γ_g}-dimensional smooth submanifold for any $q \in f_{g,G}^{-1}(f_{g,G}(p))$, the connected component of $f_{g,G}^{-1}(f_{g,G}(p))$ containing q coincides with the connected component of $\Gamma_g \cdot q$ containing q. Since $f_{g,G}^{-1}(f_{g,G}(p))$ is also an algebraic variety, it has finitely many components and hence $f_{g,G}^{-1}(f_{g,G}(p))/\Gamma_g$ is finite. ☐

In the context of tensor completion, g is defined as in Equation (6) and $d_{\Gamma_g} = n_{\Gamma_g} = 0$. As a consequence, Proposition 3.8 holds unconditionally, independent of the number of vertices. We believe that the assumption $|V| \geq n_{\Gamma_g} + d_{\Gamma_g}$ in Proposition 3.8 can be eliminated. Conversely, the requirement of genericity for the point configuration p is crucial. To illustrate this significance, an example showcasing non-finite $f_{g,G}^{-1}(f_{g,G}(p))/\Gamma_g$ is provided in Figure 3 within the realm of ordinary Euclidean rigidity.

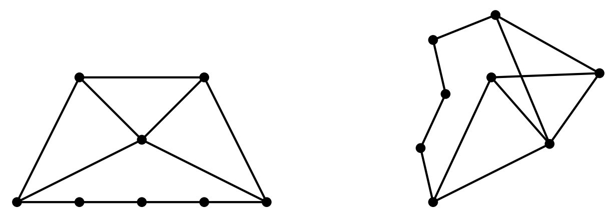

Figure 3: Consider the ordinary Euclidean rigidity in the plane. (Left) a locally g-rigid framework (G, p). (Right) the framework (G, q) satisfies $f_{g,G}(p) = f_{g,G}(q)$ but (G, q) is not locally g-rigid since the left path on five vertices in the framework can move continuously without changing the edge lengths. So $f_{g,G}^{-1}(f_{g,G}(p))/\Gamma_g$ is not finite but (G, p) is locally g-rigid.

3.3 Generic local g-rigidity and coordinate projections of affine varieties

Let $G = (V, E)$ be a k-uniform hypergraph, and consider $\overline{\operatorname{im} f_{g,G}}$, that is, the Zariski closure of the image of the g-measurement map $f_{g,G}$ in the affine space $\mathbb{F}^E$. If p is generic, then $f_{g,G}(p)$ is a generic point in $\overline{\operatorname{im} f_{g,G}}$, and hence rank $\mathrm{J}f_{g,G}(p)$ coincides with the dimension of $\overline{\operatorname{im} f_{g,G}}$. Since the latter value is determined by G (and independent of p), we can define the local g-rigidity as a property of hypergraphs (using Proposition 3.6). This extends the remarkable idea of Asimov and Roth [6] that used this property as the definition of rigidity in the Euclidean distance case. Specifically, a hypergraph G with $n = |V|$ vertices is said to be *locally g-rigid* if (G, p) is infinitesimally g-rigid for some/all generic p, and equivalently

$$\dim \overline{\operatorname{im} f_{g,G}} = dn - d_{\Gamma_g} \quad \text{if } n \geq n_{\Gamma_g}.$$

It is often convenient to analyse each hypergraph as a subgraph of the complete k-uniform hypergraph K_n^k with the same vertex set. For this, suppose $G = ([n], E)$ is a subgraph of K_n^k. Let $\pi_G : \mathbb{F}^{\binom{[n]}{k}} \to \mathbb{F}^E$ be the *coordinate projection* to the subspace indexed by the hyperedges in G. Then, $f_{g,G} = \pi_G \circ f_{g,K_n^k}$. Hence, checking the local g-rigidity of G is equivalent to checking the equality:

$$\dim \overline{\pi_G(\operatorname{im} f_{g,K_n^k})} = dn - d_{\Gamma_g}.$$

One advantage of this viewpoint is that $\overline{\operatorname{im} f_{g,K_n^k}}$ is often well-investigated in the literature, allowing us to leverage established tools. For instance, in the case where g represents the squared distance function, $\overline{\operatorname{im} f_{g,K_n^k}}$ is known as the Cayley-Menger variety [11]. Similarly, when g is the product map, $\overline{\operatorname{im} f_{g,K_n^k}}$ coincides with the affine cone of the Veronese variety. We will discuss these examples in Subsection 3.7.

3.4 Secant, non-defectivity, and identifiability of varieties

Let $\mathcal{V}$ be an affine variety defined by homogeneous polynomials in $\mathbb{C}^m$. Its *r-secant* is defined as

$$\mathrm{Sect}_r(\mathcal{V}) = \overline{\bigcup_{x_1,\ldots,x_r \in \mathcal{V}} \langle x_1, \ldots, x_r \rangle}, \tag{13}$$

where $\langle x_1, \ldots, x_r \rangle$ denotes the linear span of $x_1, \ldots, x_r$. In applications, it is often the case that a homogeneous polynomial map $g : (\mathbb{C}^d)^k \to \mathbb{C}$ is written as the sum of polynomial maps $h : (\mathbb{C}^s)^k \to \mathbb{C}$ for $s < d$. In such a case, the concept of g-rigidity is closely related to the non-defectivity and identifiability of varieties.

To formally establish this connection, let us consider a polynomial map $g : (\mathbb{C}^d)^k \to \mathbb{C}$. The argument of g is a tuple $(p_1, p_2, \ldots, p_k)$ consisting of k points p_i in $\mathbb{C}^d$. We denote the coordinates of each p_i by $p_i = (p_{i1}, p_{i2}, \ldots, p_{id})^\top$. Suppose g is separable into coordinate-wise maps, that is,

$$g(p_1, \ldots, p_k) = \sum_{i=1}^{d} h(p_{1i}, \ldots, p_{ki})$$

for some $h : (\mathbb{C}^1)^k \to \mathbb{C}$. In this case, it follows that $\overline{\mathrm{im} f_{g,G}}$ is the d-secant of $\overline{\mathrm{im} f_{h,G}}$. We need slightly more involved notation to deal with the case when $g \in (\mathbb{C}^d)^k$ is decomposed into the sum of copies of $h \in (\mathbb{C}^s)^k$. (Such situations appear in skew-symmetric completions and Chow decompositions given in Section 2.3.) Specifically, consider the identification between $(\mathbb{C}^d)^k$ and $((\mathbb{C}^s)^k)^t$ with $st = d$ such that $p \in (\mathbb{C}^d)^k$ is identified with a t-tuple $(q_1, \dots, q_t)$ of $q_i \in (\mathbb{C}^s)^k$. If for some $h : (\mathbb{C}^s)^k \to \mathbb{C}$, g is written as

$$g(p) = \sum_{i=1}^{t} h(q_i) \tag{14}$$

then $\overline{\mathrm{im} f_{g,G}}$ is the t-secant of $\overline{\mathrm{im} f_{h,G}}$. In such a case, we simply say that g *is the sum of* t *copies of* h.

Computing the dimension of secant varieties has been extensively studied in algebraic geometry (see, e.g., [8]). In particular, the algebraic notions of non-defectivity and identifiability of these varieties are closely related to the notion of g-rigidity. Although these concepts are typically defined for projective varieties, we maintain our focus on affine varieties to preserve the connection with rigidity theory.

Non-defectivity and local g-rigidity. Given an affine variety $\mathcal{V}$ in $\mathbb{C}^m$, the r-th secant of $\mathcal{V}$ has dimension at most $\min\{r \dim \mathcal{V}, m\}$. The latter number is called the *expected dimension*, and $\mathcal{V}$ is said to be *r-defective* if $\dim \mathrm{Sect}_r(\mathcal{V})$ is smaller than the expected dimension.

Assume Setup 2.1, and suppose further that g is the sum of t copies of h. If $\overline{\mathrm{im} f_{h,G}}$ is not t-defective, then $\dim(\overline{\mathrm{im} f_{g,G}}) = \min\{t \dim(\overline{\mathrm{im} f_{h,G}}), |E|\}$, and hence checking local g-rigidity is reduced to checking local h-rigidity. The defectivity of classical algebraic varieties such as Veronese, Segre, and Grassmannian varieties has been extensively studied (see, e.g., [8]). In the classical setting, the underlying hypergraphs of such varieties are always complete, and hence we can apply the existing results to prove the local g-rigidity of complete hypergraphs. In particular, when the underlying hypergraph is complete, then the notion of local g-rigidity often coincides with the classical notion of non-defectivity of the corresponding secant varieties. See Subsection 3.7 for concrete examples.

However, for the analysis of general hypergraphs, we need to understand the influence of coordinate projections on defectivity. Proposition 3.11, below, shows that the dimension for a general hypergraph can be much smaller than the expected dimension for combinatorial reasons. Therefore, we focus on the combinatorial properties of the underlying hypergraphs.

It is also important to point out that our primal applications include the case when g and h are quadratic. In such cases, $\overline{\mathrm{im} f_{h,G}}$ will always be defective whereas G can be still locally g-rigid. So the non-defectivity and local g-rigidity are different notions in general. An example is described below.

Tangent spaces. When g is the sum of t copies of h, by reordering the columns appropriately, we have

$$\mathrm{J} f_{g,G}(p) = \left(\mathrm{J} f_{h,G}(q_1) \dots \mathrm{J} f_{h,G}(q_t) \right). \tag{15}$$

Hence the tangent space of $\overline{\mathrm{im} f_{g,G}}$ at a generic point $f_{g,G}(p)$ is the linear span of those of $\overline{\mathrm{im} f_{h,G}}$ at $f_{h,G}(q_1), \dots, f_{h,G}(q_t)$. This fact corresponds to the Terracini lemma in algebraic geometry (see, e.g., [8]).

Identifiability and global g-rigidity. Suppose g is homogeneous. For a variety $\mathcal{V}$ in $\mathbb{C}^m$, a generic point $y \in \mathrm{Sect}_r(\mathcal{V})$ can be written as $y = \sum_{i=1}^{r} y_i$ for some $y_i \in \mathcal{V}$. The variety $\mathcal{V}$ is said to be *r-identifiable* if $y_1, \dots, y_r$ are uniquely determined up to permutations of indices and scaling of each y_i. As we will see in Section 5, identifiability is a useful property for checking global g-rigidity. For example, in the case of symmetric tensors, a complete hypergraph is globally g-rigid if and only if the corresponding secant variety is identifiable. However, in general, global g-rigidity and identifiability are not comparable. We now provide two examples to demonstrate this.

Example. To see the difference among local/global rigidity, defectiveness, and identifiability, we consider the symmetric rank-r matrix completion of order k. In this case $g : (\mathbb{C}^r)^k \to \mathbb{C}$ is the sum of r copies of the product map $h_{\mathrm{prod}} : \mathbb{C}^k \to \mathbb{C}$. Then $\overline{\mathrm{im} f_{g,G}}$ is the r-secant of $\overline{\mathrm{im} f_{h_{\mathrm{prod}},G}}$.

Suppose $k = 2$. This case corresponds to the rank-r matrix completion. It is known (see, e.g., [58]) that $\mathrm{J}f_{h_{\mathrm{prod}},K_n} = n$ and $\mathrm{J}f_{g,K_n} = rn - \binom{r}{2}$. So $\overline{\mathrm{im} f_{h_{\mathrm{prod}},K_n}}$ is r-defective and not r-identifiable. However, since $d_{\Gamma_g} = \binom{r}{2}$, K_n is locally g-rigid by Proposition 3.6. In fact, K_n is known to be globally g-rigid, see, e.g. [58].

Suppose $r = 1$. Then $\overline{\mathrm{im} f_{h_{\mathrm{prod}},G}}$ is 1-identifiable by definition. However, the local rigidity and the global g-rigidity of a generic framework (G, p) depends on the combinatorial structure of G. Indeed the global g-rigidity captures the uniqueness of symmetric rank-one tensor completions. So, if G is very sparse, (G, p) cannot be globally g-rigid.

3.5 Combinatorial characterisation problem

Here, we establish a connection between g-rigidity and combinatorial properties of hypergraphs.

Definition 3.9. For a $\mathbb{Q}$-polynomial map $g : (\mathbb{F}^d)^k \to \mathbb{F}$ and a positive integer n, the *g-rigidity matroid* $\mathcal{M}_{g,n}$ (or simply $\mathcal{M}_g$ if n is not important) is the row matroid of the Jacobian matrix $\mathrm{J}f(p)$.

$\mathcal{M}_{g,n}$ is an example of an algebraic matroid. See, for example, [54] which provides a detailed exposition on algebraic matroids. Given that generic g-rigidity is a property of hypergraphs, it is natural to ask whether this property can be characterised purely in terms of graph theory concepts. This pursuit of a combinatorial characterisation forms a central theme within the field of graph rigidity. The most famous example is the following Geiringer-Laman theorem [46, 52].

Theorem 3.10. *Suppose g is the squared distance function in two-dimensional Euclidean space. Then G is g-rigid if and only if G has a subgraph H satisfying the following conditions:*

- $|E(H)| = 2|V(G)| - 3$, *and*
- *for any subgraph $H' = (V', E')$ of H with $|E'| \geq 1$, $|E'| \leq 2|V'| - 3$.*

Inspired by the goal of extending the aforementioned result to encompass general polynomial maps g and hypergraphs, we review a construction of combinatorial matroids on hypergraphs.

Let $G = (V, E)$ be a k-uniform hypergraph. For nonnegative integers a and b, we say that G is (a, b)-*sparse* if every sub-hypergraph G' with $E(G') \neq \emptyset$ satisfies $|E(G')| \leq a|V(G')| - b$, and an (a, b)-sparse G is said to be (a, b)-*tight* if $|E| = a|V| - b$. It is known that, if $0 \leq b \leq a - 1$, the collection of edge sets of all (a, b)-sparse sub-hypergraphs of K_n^k forms the independent set of a matroid on E (see, e.g., [65]), which is called the (a, b)-*sparsity matroid* $\mathcal{S}_n^{a,b}$.

If $d - d_{\Gamma_g} \geq 1$ for a polynomial map g, the consideration of sparsity leads to the following "combinatorially expected dimension" of $\overline{\pi_G(\mathrm{im} f_{g,K_n^k})}$, which can be strictly smaller than the trivial expected dimension in the context of secant varieties.

Proposition 3.11. *Let $g : (\mathbb{F}^d)^k \to \mathbb{F}$ be a polynomial map, and let $r_{g,n}$ be the rank function of $\mathcal{M}_{g,n}$. Suppose that $k \geq n_{\Gamma_g}$ and $d - d_{\Gamma_g} \geq 1$. Then, for any k-uniform hypergraph $G = (V, E)$,*

$$\dim(\overline{\pi_G(\mathrm{im} f_{g,K_n^k})}) = r_{g,n}(E) \leq \min \left\{ |F_0| + \sum_{i=1}^{k} \left(d \left| \bigcup_{e \in F_i} e \right| - d_{\Gamma_g} \right) \;\middle|\; \begin{array}{c} F_0 \subseteq E \text{ and } \{F_1, \ldots, F_k\} \\ \text{is a partition of } E \setminus F_0 \end{array} \right\}.$$

Proof. By Proposition 3.6 and the assumption $d - d_{\Gamma_g} \geq 1$, if E is independent in the g-rigidity matroid $\mathcal{M}_g$, then

$$|E'| \leq d|V'| - d_{\Gamma_g} \text{ for any subgraph } G' = (V', E') \text{ with } |E'| \geq 1. \tag{16}$$

Under the assumption $d - d_{\Gamma_g} \geq 1$, the (d, d_{Γ_g})-sparsity matroid $\mathcal{S}_n^{d, d_{\Gamma_g}}$ is well-defined, and (16) implies that every independent set in $\mathcal{M}_g$ is also independent in $\mathcal{S}_n^{d, d_{\Gamma_g}}$. It is known that the rank of E in $\mathcal{S}_n^{a,b}$ is

$$\min \left\{ |F_0| + \sum_{i=1}^{k} \left(a \left| \bigcup_{e \in F_i} e \right| - b \right) \;\middle|\; F_0 \subseteq E \text{ and } \{F_1, \ldots, F_k\} \text{ is a partition of } E \setminus F_0 \right\}, \tag{17}$$

see, e.g., [57]. Hence, the rank function of $\mathcal{M}_g$ is upper bounded by (17) with $a = d$ and $b = d_{\Gamma_g}$. $\square$

The structure of (a, b)-sparsity matroids are well-understood, and there is an efficient algorithm for computing the right hand side of equation in Proposition 3.11, see [48, 60].

A natural question arises concerning whether the inequality presented above is actually an equality. While the Geiringer-Laman theorem, Theorem 3.10, establishes this equality for any graph G when g is the squared distance function in two-dimensional Euclidean space, it is important to note that, in most naturally occurring examples in applications, there exists a hypergraph G for which this inequality is strict. Nevertheless, we conjecture that the equality always holds if g is sufficiently generic.

Conjecture 3.12. *If g is generic, then $\mathcal{M}_{g,n} = \mathcal{S}_n^{d,0}$.*

3.6 Matroid Union

We now show that the decomposition property of g is inherited by the corresponding matroid $\mathcal{M}_g$.

Lemma 3.13. *Suppose a polynomial map $g : (\mathbb{F}^d)^k \to \mathbb{F}$ is the sum of t copies of $h : (\mathbb{F}^s)^k \to \mathbb{F}$ with $d = st$. If a set E of hyperedges is independent in $\mathcal{M}_{g,n}$, then E can be partitioned into $E_1, E_2, \ldots, E_t$ such that each E_i is independent in $\mathcal{M}_{h,n}$.*

Proof. For the sake of simplicity, we will provide the proof specifically for the case when $d_{\Gamma_g} = 0$. However, it is important to note that the proof can be readily adapted to the general case by considering a maximal non-singular submatrix of the Jacobian. We may further assume that E represents a basis of $\mathcal{M}_{g,n}$, as the property of decomposition holds for all bases, implying the same property for all independent sets.

For a basis E of the matroid $\mathcal{M}_{g,n}$, we denote $G[E]$ for the induced sub-hypergraph on the hyperedges E. Let p be a generic point-configuration of $G[E]$. Since E is a basis and $d_{\Gamma_g} = 0$, we have that $|E| = kn$, and so $\mathrm{J}f_{g,G[E]}(p)$ is a square matrix. Since g is the sum of t copies of h, the structure of $\mathrm{J}f_{g,G[E]}(p)$ has the form of Equation (15). Hence, the Laplace expansion tells us that

$$\det \mathrm{J}f_{g,G[E]}(p) = \sum_{\{E_1,\ldots,E_t\}} \prod_{i=1}^{t} \det \mathrm{J}f_{h,G[E_i]}(q_i), \tag{18}$$

where the sum is taken over all partitions $\{E_1, \ldots, E_t\}$ of E with $|E_i| = sn$ $(1 \leq i \leq t)$. Since E is a basis of $\mathcal{M}_{g,n}$, $\det \mathrm{J}f_{g,G[E]}(p) \neq 0$. Hence, there must be at least one non-zero term in the right hand side of Equation (18), and the corresponding partition $\{E_1, \ldots, E_t\}$ is a desired one. $\square$

Given matroids $\mathcal{M}_1, \ldots, \mathcal{M}_t$, their *union* is defined such that a set is independent if it can be partitioned into independent sets of each $\mathcal{M}_i$. The union of t copies of $\mathcal{M}$ is denoted by $t\mathcal{M}$. Lemma 3.13 implies that $\mathcal{M}_{g,n} \preceq t\mathcal{M}_{h,n}$ in the weak order poset of matroids, i.e., any independent set of $\mathcal{M}_{g,n}$ is independent in $t\mathcal{M}_{h,n}$. A natural question is whether they are indeed different, which is closely related to t-defectivity since $\mathcal{M}_{g,n} = t\mathcal{M}_{h,n}$ holds if and only if $\overline{\pi_H(\mathrm{im} f_{h,K_n^k})}$ is not t-defective for any $H \subseteq K_n^k$ having no coloop in $t\mathcal{M}_{h,n}$ (i.e., every element is covered by some circuit) by Edmonds' matroid union theorem, see, e.g., [57].

3.7 Combinatorial Analysis in Our Example Models

Euclidean rigidity. In the ordinary Euclidean rigidity, G is 2-uniform (i.e., a graph) and $g(x, y) = \sum_{i=1}^d (x_i - y_i)^2$ for $x = (x_1, \ldots, x_d)^\top, y = (y_1, \ldots, y_d)^\top \in \mathbb{R}^d$. The stabiliser Γ_g is the Euclidean group $E(d)$. We have $d_{E(d)} = \binom{d+1}{2}$ and $n_{E(d)} = d$. Then $(d, \binom{d+1}{2})$-sparsity is a necessary condition for the independence in the g-rigidity matroid. When $d = 2$, Theorem 3.10 implies that the g-rigidity matroid is equal to the $(2, 3)$-sparsity matroid. This is not true for $d \geq 3$ (see [35] for a recent discussion) and determining which $(d, \binom{d+1}{2})$-sparse graphs are independent is a fundamental open problem in rigidity theory. See [21] for a new candidate characterisation when $d = 3$.

Let $h(a,b) = (a-b)^2$ for $a,b \in \mathbb{C}$. Then $g(x,y) = \sum_{i=1}^d h(x_i, y_i)$. The affine variety defined by g is the d-secant of that of h. The h-rigidity corresponds to 1-dimensional rigidity and it is characterised by graph connectivity. However, the affine variety defined by h is defective, and currently there is no sufficient condition to guarantee the d-dimensional rigidity in terms of 1-dimensional rigidity.

Hendrickson [37] proved two natural necessary conditions for a graph to be globally g-rigid in dimension d. More precisely he proved that, when g is the usual Euclidean distance, any globally g-rigid graph G is $(d+1)$-connected and redundantly g-rigid (which means that the graph obtained from G by deleting any single edge is g-rigid). Connelly [23] proved an algebraic sufficient condition based on stress matrices (weighted Laplacians). Using these two results, and specialised to the case when $d=2$, Jackson and Jordán [38] gave a combinatorial characterisation of generic global g-rigidity. While Hendrickson's necessary conditions are known to be insufficient when $d \geq 3$ [22], Gortler, Healy and Thurston [33] proved that generic global g-rigidity does depend only on the underlying graph.

Pseudo-Euclidean rigidity. Here the situation is similar, G is 2-uniform, the stabiliser is still the Euclidean group, $d_{\Gamma_g} = \binom{d+1}{2}$ and $n_{\Gamma_g} = d$. Thus $(d, \binom{d+1}{2})$-sparsity is again a necessary condition for independence. In fact there is a direct linear algebraic transformation between independence in the Euclidean case and the pseudo-Euclidean case [55]. This translates, for example, the Geiringer-Laman theorem to pseudo-Euclidean planes.

Suppose g is the usual Euclidean distance and g' is a pseudo-Euclidean distance. Gortler and Thurston [34] proved that a generic framework (G,p) is globally g-rigid if and only if it is globally g'-rigid. Along the way they extended Connelly's sufficient condition to pseudo-Euclidean spaces. However in the case of hyperbolic space, it appears to be open to decide if Hendrickson's redundant rigidity condition applies. Earlier [25] proved that, generically, global rigidity in the Euclidean and spherical contexts coincide.

ℓ_p-rigidity. Suppose p is a positive even integer not equal to 2. Again G is 2-uniform. The stabiliser is the d-dimensional group of translations. We have $d_{\Gamma_g} = d$, $n_{\Gamma_g} = 1$ and (d,d)-sparsity is a necessary condition for independence. When $d=2$, an analogue of the Geiringer-Laman theorem was obtained in [45] showing that the g-rigidity matroid is equivalent to the $(2,2)$-sparsity matroid on a simple graph. It is conjectured that the d-dimensional case is characterised by (d,d)-sparsity (for simple graphs). This would characterise d-dimensional rigidity in terms of 1-dimensional rigidity. However, such a combinatorial description is currently only known in certain very special cases [28].

It is an open problem to analyse global g-rigidity in the ℓ_p case. Since p is even, ℓ_p-norms are analytic and hence the results of [27] apply; they showed that the natural analogue of Hendrickson's necessary conditions apply in analytic normed spaces and gave a combinatorial characterisation analogous to Jackson-Jordán when the space is an analytic normed plane.

Volume constraint rigidity. In the volume constraint rigidity, the map $g : (\mathbb{R}^d)^{d+1} \to \mathbb{R}$ is given by

$$g(x_1, x_2, \ldots, x_{d+1}) = \det \begin{pmatrix} x_1 & x_2 & \ldots & x_{d+1} \\ 1 & 1 & \ldots & 1 \end{pmatrix}.$$

for $x_1, \ldots, x_{d+1} \in \mathbb{R}^{d+1}$. The stabiliser Γ_g is the special affine group $SA(d, \mathbb{R})$. We have $d_{\Gamma_g} = d^2 + d - 1$ and $n_{\Gamma_g} = d+1$. Then (d, d^2+d-1)-sparsity is a necessary condition for the independence in the g-rigidity matroid, but it is not sufficient in general, see, e.g. [16]. Very recently, Southgate [59] has proved that global g-rigidity, in this volume case, is not a generic property of the underlying hypergraph.

Positive semidefinite matrix completion problem. In this problem, G is 2-uniform (i.e., a graph) and g is the Euclidean inner product in $\mathbb{R}^d$. The stabiliser Γ_g is $O(d)$, and $d_{E(d)} = \binom{d}{2}, n_{E(d)} = d-1$. Then $(d, \binom{d}{2})$-sparsity is a necessary condition for the independence in the g-rigidity matroid. This sparsity condition is sufficient for $d=1$, but is not in general for $d \geq 2$ [58]. It is a challenging open problem to give a good characterisation of the independence in $d=2$, see [9, 42]. The global g-rigidity is also characterised when $d=1$ in [58], but it is not well understood when $d \geq 2$. It is known that global g-rigidity is not a generic property of graphs [40], see Section 5.1 for more details.

Symmetric tensor completion problem. Recall that $h_{\rm prod} : \mathbb{C}^k \to \mathbb{C}$ denotes the product map, i.e., $h(y_1, y_2, \dots, y_k) = y_1 y_2 \dots y_k$. In the symmetric tensor completion with symmetric rank d, the measurement $g : (\mathbb{C}^d)^k \to \mathbb{C}$ is given as the sum of d copies of $h_{\rm prod}$. Hence, we denote this g by $dh_{\rm prod}$.

The g-rigidity problems when $k = 2$ and when $k \geq 3$ are very different. When $k = 2$, tensors are matrices and we have the same results as the positive semidefinite matrix completion problem discussed above. So in the subsequent discussion we focus on the case when $k \geq 3$.

The stabiliser Γ of $dh_{\rm prod}$ is the set of diagonal matrices over $\mathbb{C}$ whose diagonal entries are k-th roots of unity (up to permutation of indices). We have $d_\Gamma = 0$ and $n_\Gamma = 0$. The $(d, 0)$-sparsity is a necessary condition for the independence in the $dh_{\rm prod}$-rigidity matroid, but it is not sufficient since no k-uniform k-partite hypergraph can be $dh_{\rm prod}$-rigid. We will discuss this point in Section 6.

By definition, $G = (V, E)$ is locally $dh_{\rm prod}$-rigid if and only if $\operatorname{rank} \mathrm{J} f_{dh_{\rm prod},G}(p) = d|V|$ for a generic $p \in (\mathbb{C}^r)^k$. Let I_G be the edge-vertex incidence matrix of G, that is, the matrix of size $|E| \times |V|$ whose (e, i)-th entry is one if vertex i is contained in hyperedge e, and zero otherwise. When $d = 1$, it is easy to check that $\mathrm{J} f_{dh_{\rm prod},G}(p)$ is equivalent to I_G (in the sense of row and column operations). Hence, if $d = 1$, G is locally $dh_{\rm prod}$-rigid if and only if the edge-vertex incidence matrix is full rank. The problem becomes much harder when $d \geq 2$, and checking the locally $dh_{\rm prod}$-rigidity of K_n^k is already a nontrivial question.

Since $dh_{\rm prod}$ is the sum of d copies of $h_{\rm prod}$, $\overline{\operatorname{im} f_{dh_{\rm prod},G}}$ is the d-secant of $\overline{\operatorname{im} f_{h_{\rm prod},G}}$. When $G = K_n^k$, $f_{h_{\rm prod},K_n^k}$ is the Veronese embedding, and $\overline{\operatorname{im} f_{h^{\rm prod},K_n^k}}$ is the affine cone of the Veronese variety of degree k. It is relatively recent that Alexander and Hirschowitz [4] gave a concrete list of defective cases of Veronese varieties. Adapting this result to our language, we have the following.

Theorem 3.14. *Let k, n, d be positive integers with $k \geq 3$, and $p : [n] \to \mathbb{C}^d$ be a generic point configuration. Then*

$$\operatorname{rank} \mathrm{J} f_{dh_{\rm prod},K_n^k}(p) = \min\{|E(K_n^k)|, dn\}, \tag{19}$$

except for (k, n, d) in the set $\{(3, 5, 7), (4, 3, 5), (4, 4, 9), (4, 5, 14)\}$. In other words, K_n^k is locally $dh_{\rm prod}$-rigid if and only if $|\binom{n+k-1}{k}| \geq dn$ and $(k, n, d) \notin \{(3, 5, 7), (4, 3, 5), (4, 4, 9), (4, 5, 14)\}$.

Proof. By the Alexander–Hirschowitz Theorem [4, 8], we know that the dimension of the secant variety of the affine cone of the Veronese variety of degree k is $\min\left\{\binom{n+k-1}{k}, dn\right\}$ except for the defective case $(k, n, d) \in \{(3, 5, 7), (4, 3, 5), (4, 4, 9), (4, 5, 14)\}$. Since $\overline{\operatorname{im} f_{dh_{\rm prod},K_n^k}}$ is the d-secant of the affine cone of the Veronese variety of degree k, we have (19).

By Proposition 3.6 with $d_{\Gamma_{\mathrm{J} f_{\rm prod}}} = 0$, the local rigidity holds if and only if $\operatorname{rank} \mathrm{J} f_{dh_{\rm prod},K_n^k}(p) = dn$. By (19), the latter is equivalent that $\binom{n+k-1}{k} \geq dn$ and (k, n, d) is not the exceptional defective case. ☐

The identifiability problem for Veronese varieties has been also solved in [20, 30], which implies a characterisation of the globally $dh_{\rm prod}$-rigid complete hypergraphs in our terminology. Notably, complete hypergraphs which are both minimally locally $dh_{\rm prod}$-rigid and globally $dh_{\rm prod}$-rigid are classified in [30]. Adapting these results to our language, we have the following.

Theorem 3.15. *Let k, n, d be positive integers with $k \geq 3$.*

- *If $\binom{n+1+k}{k} > dn$, then K_n^k is globally $dh_{\rm prod}$-rigid if and only if $(k, n, d) \notin \{(6, 3, 9), (4, 4, 8), (3, 6, 9)\}$.*
- *If $\binom{n+1+k}{k} = dn$, then K_n^k is globally $dh_{\rm prod}$-rigid if and only if $(k, n, d) \in \{(3, 4, 5), (5, 3, 7)\} \cup \{(2s - 1, 2, s) : s \geq 2, s \in \mathbb{Z}\}$.*

Skew-symmetric tensor completions. Let $h_{\rm det} : (\mathbb{C}^k)^k \to \mathbb{C}$ be the determinant as a k-form over $\mathbb{C}^k$. In the skew-symmetric tensor completion with skew-symmetric rank r, the measurement map $g : (\mathbb{C}^{rk})^k \to \mathbb{C}$ is given as the sum of r copies of $h_{\rm det}$. Hence, we denote this g by $rh_{\rm det}$. The stabilizer Γ of $rh_{\rm det}$ is $\{(A_1 \oplus \cdots \oplus A_r)(\Sigma \otimes I_k) \mid A_i \in \mathrm{SL}(k, \mathbb{C}), \Sigma :$ a permutation matrix of size $r\}$. We have

$d_\Gamma = r(k^2-1)$ and $n_\Gamma = k$. The $(rk, r(k^2-1))$-sparsity is a necessary condition for the independence in the rh_{det}-rigidity matroid, but it is not sufficient even when $r=1$ (see Example 4.5 below for details).

As in the symmetric tensor case, proving the local rh_{det}-rigidity for simple complete hypergraphs $\tilde{K}_n^k$ is already challenging. Since rh_{det} is the sum of r copies of h_{det}, $\overline{\mathrm{im} f_{rh_{\text{det}},G}}$ is the r-secant of $\overline{\mathrm{im} f_{h_{\text{det}},G}}$. When $G = \tilde{K}_n^k$, $\overline{\mathrm{im} f_{h_{\text{det}},\tilde{K}_n^k}}$ is the affine cone of the Grassmannian $\mathrm{Gr}(k,n)$. Hence, the locally rh_{det}-rigid of $\tilde{K}_n^k$ is equivalent to the non-r-defectivity of $\mathrm{Gr}(k,n)$. Unlike the Veronese case, the non-defectivity problem and the identifiability problem of Grassmannian varieties have not been solved yet. See [2, 50, 49].

The same story holds for Chow decompositions if we replace the determinant function in the above discussion on skew-symmetric tensors with the permanent function. See [62, 64] for recent developments.

4 Sufficient Conditions for Local g-rigidity

The goal of this section is to extend combinatorial techniques in graph rigidity theory to k-uniform hyper-framework g-rigidity. Since g-rigidity is a rather general concept, we shall restrict our attention to polynomial maps g which are multilinear or multiaffine (defined below). Based on this assumption, we shall analyse combinatorial patterns of the Jacobians of the measurement maps g.

4.1 Multiaffine forms and combinatorial patterns of Jacobians

A map $g : (\mathbb{F}^d)^k \to \mathbb{F}$ is said to be a *multilinear (resp. multiaffine) k-form on $\mathbb{F}^d$* if it is linear (resp. affine) on each argument on the vector space $\mathbb{F}^d$. As we remarked when defining g-rigidity, g is always assumed to be symmetric or anti-symmetric with respect to permutations of indices. A symmetric (resp. anti-symmetric) multilinear k-form is nothing but a symmetric (resp. skew-symmetric) tensor of $(\mathbb{F}^d)^k$.

Suppose g is a symmetric (or anti-symmetric) multiaffine k-form. By regarding $g(x_1,\ldots,x_k)$ as an affine function in the variable x_j, each entry of the gradient of g with respect to x_j can be considered as a polynomial map in the remaining points $x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_k$. Moreover, since g is symmetric (or anti-symmetric), the resulting polynomial map is independent of the choice of j (up to the sign if g is anti-symmetric). Hence, we may consider the gradient of g with respect to a point x_j as a polynomial map from $(\mathbb{F}^d)^{k-1}$ to $\mathbb{F}^d$, and denote it by ∇g.

Based on Proposition 3.6, our analysis of the local g-rigidity of a hypergraph $G=(V,E)$ proceeds by analysing the rank of the Jacobian $\mathrm{J}f_{g,G}(p)$ at a generic p. This Jacobian has the following structure. Given that the map $f_{g,G}$ is defined from $(\mathbb{F}^d)^V$ to $\mathbb{F}^E$, each row of $\mathrm{J}f_{g,G}(p)$ is indexed by a hyperedge in E and each consecutive d-tuple of columns of $\mathrm{J}f_{g,G}(p)$ are indexed by a vertex in V. The definition of $f_{g,G}$ further tells us that the $1\times d$ block $b(e,v)$ associated with a pair (e,v) for $e\in E$ and $v \in V$ is

$$b(e,v) = \begin{cases} \pm m_e(v)\nabla g(p(e-v))^\top & \text{if } v\in e \\ 0 & \text{otherwise.}\end{cases}$$

Here, $p(e-v)$ is a tuple of $(k-1)$ points in $\{p(u) : u \in e-v\}$, and $m_e(v)$ is the multiplicity of v in e. The sign is always positive if g is symmetric, but depends on the ordering of v in e if g is anti-symmetric.

Example 4.1. A primary example arises in the symmetric tensor completion problem. As we explained in Subsection 3.7, the corresponding map g is the sum dh_{prod} of d copies of h_{prod}. Clearly h_{prod} is symmetric multilinear, and hence so is dh_{prod}. The map $\nabla(dh_{\text{prod}}) : (\mathbb{F}^d)^{k-1} \to \mathbb{F}^d$ is given by

$$\nabla(dh_{\text{prod}})(x_1,\ldots,x_{k-1}) = \begin{pmatrix} \nabla h_{\text{prod}}(x_{11},\ldots,x_{1k-1}) \\ \vdots \\ \nabla h_{\text{prod}}(x_{d1},\ldots,x_{dk-1})\end{pmatrix} = \begin{pmatrix} x_{11}\ldots x_{1k-1} \\ \vdots \\ x_{d1}\ldots x_{dk-1}\end{pmatrix}$$

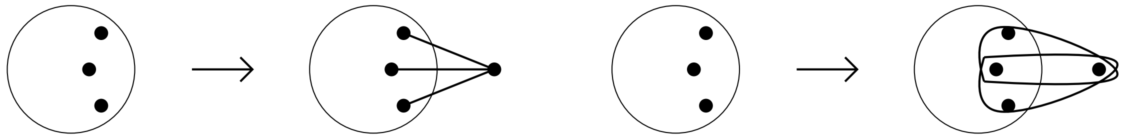

Figure 4: (Left) A 3-valent simple extension of a 2-uniform hypergraph. (Right) A 2-valent simple extension of a 3-uniform hypergraph.

for $x_i = (x_{1i}, \dots, x_{di})^\top \in \mathbb{F}^d$ for $i = 1, \dots, k-1$. Suppose $k = 3$, $d = 2$, $G = (V, E)$ with $V = \{a, b, c, d\}$ and $E = \{aaa, aab, abc, bcd\}$. Then, the Jacobian is the matrix

$$\begin{array}{c} \\ aaa \\ aab \\ abc \\ bcd \end{array} \begin{array}{cccccccc} (a,1) & (a,2) & (b,1) & (b,2) & (c,1) & (c,2) & (d,1) & (d,2) \end{array} \left[\begin{array}{cccccccc} 3x_{1a}^2 & 3x_{2a}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2x_{1a}x_{1b} & 2x_{2a}x_{2b} & x_{1a}^2 & x_{2a}^2 & 0 & 0 & 0 & 0 \\ x_{1b}x_{1c} & x_{2b}x_{2c} & x_{1c}x_{1a} & x_{2c}x_{2a} & x_{1a}x_{1b} & x_{2a}x_{2b} & 0 & 0 \\ 0 & 0 & x_{1c}x_{1d} & x_{2c}x_{2d} & x_{1d}x_{1b} & x_{2d}x_{2b} & x_{1b}x_{1c} & x_{2b}x_{2c} \end{array} \right],$$

where each row is associated with a hyperedge $e \in E$ and each column is associated with a pair (v, i) of a vertex v and a coordinate axis i.

4.2 Extension operations

A *d-valent extension* creates a new k-uniform hypergraph from a given k-uniform hypergraph by adding a new vertex with d distinct new hyperedges containing the new vertex. We say that an extension operation is *simple* if each new hyperedge is simple. See Figure 4 for examples. It is one of the most basic and widely used theorems in rigidity theory that any d-valent simple extension preserves d-dimensional rigidity. We shall extend this to the g-rigidity setting for $g : (\mathbb{F}^d)^k \to \mathbb{F}$.

For a d-dimensional hyper-framework (G, p) and a vertex $v \in V(G)$, consider the matrix B obtained by aligning $\nabla g(p(e-v))^\top$ as row vectors over all hyperedges in G incident to v, i.e.,

$$B = \begin{pmatrix} \nabla g(p(e_1 - v))^\top \\ \vdots \\ \nabla g(p(e_j - v))^\top \end{pmatrix}, \tag{20}$$

where $e_1, \dots, e_j$ are hyperedges incident to v. We say that v is *stable* in (G, p) if B has full column rank d.

Lemma 4.2. *Assume Setup 2.1. Consider $g : (\mathbb{F}^d)^k \to \mathbb{F}$ and let $G = (V, E)$ be a k-uniform hypergraph.*

- *If (G, p) is infinitesimally g-rigid with $|V| \geq n_{\Gamma_g} + 1$, then every vertex is stable in (G, p).*
- *If G is obtained from H by 0-extension by adding a new vertex v and $|V(H)| \geq n_{\Gamma_g}$, then (G, p) is infinitesimally g-rigid if and only if $(H, p_{|V(H)})$ is infinitesimally g-rigid and v is stable in (G, p).*

Proof. Suppose that H is obtained from G by removing a vertex v. Let $e_1, \dots, e_j$ be the hyperedges in G incident to v. Then, $\mathrm{J}f_{g,G'}$ is partitioned into submatrices as follows:

$$\mathrm{J}f_{g,G} = \begin{array}{c} \\ \{e_1,\dots,e_d\} \\ E(H) \end{array} \begin{array}{cc} v & V(H) \end{array} \begin{bmatrix} B & * \\ 0 & \mathrm{J}f_{g,H} \end{bmatrix}.$$

Hence, since $|V(G)| \geq n_{\Gamma_g} + 1$, we may use Proposition 3.6 to deduce that every vertex is stable in (G, p).

The second statement follows similarly. If $|V(H)| \geq n_{\Gamma_g}$, $(H, p_{|V(H)})$ is infinitesimally g-rigid and v is stable in (G, p), then the block matrix form shown above allows us to deduce, via Proposition 3.6, that (G, p) is infinitesimally g-rigid. Conversely, if (G, p) is infinitesimally g-rigid then v is stable in (G, p) by the first statement and $(H, p_{|V(H)})$ is infinitesimally g-rigid. □

When g is symmetric, we say that g-rigidity has *the extension property* if any d-valent extension of a g-rigid graph $G = (V, E)$ with $|V| \geq n_{\Gamma_g}$ is still g-rigid. When g is anti-symmetric, the extension property is defined by checking only simple extensions since the value of g is zero for a non-simple hyperedge.

We shall now prove the extension property of g-rigidity for a class of multiaffine k-forms g. To see this, we need the following notion. A variety is *degenerate* if it is contained in a hyperplane, and $g : (\mathbb{F}^d)^k \to \mathbb{F}$ is *non-degenerate* if $\overline{\mathrm{im} f_{g,K_n^k}}$ is non-degenerate for any n. For example, h_{prod} is non-degenerate since $\overline{\mathrm{im} f_{h_{\mathrm{prod}},K_n^k}}$ is the affine cone of the Veronese variety of degree k, which is known to be non-degenerate.

Lemma 4.3. *Consider $g : (\mathbb{F}^d)^k \to \mathbb{F}$ written as the sum of d copies of a non-zero multiaffine $h : (\mathbb{F}^1)^k \to \mathbb{F}$. Suppose that ∇h is non-degenerate. Then g-rigidity has the extension property.*

Proof. By Lemma 4.2, it suffices to check that the matrix B defined by Equation (20) is non-singular for any vertex v, d distinct hyperedges $e_1, \dots, e_d$ incident to v, and a generic d-dimensional point-configuration B. Note that ∇h is a polynomial map from $(\mathbb{F}^1)^{k-1}$ to $\mathbb{F}$. In particular, ∇h is a $(k-1)$-form over $\mathbb{F}$, so the ∇h-measurement map is defined for each $(k-1)$-uniform hypergraph. Let H be the hypergraph consisting of $(k-1)$-uniform hyperedges $e_1 - v, e_2 - v, \dots, e_d - v$. Then the columns of B are $f_{\nabla h,H}(x_1), f_{\nabla h,H}(x_2), \dots, f_{\nabla h,H}(x_d)$ for some 1-dimensional generic point-configurations $x_1, \dots, x_d$.

We now show that $f_{\nabla h,H}(x_1), \dots, f_{\nabla h,H}(x_d)$ are linearly independent. Since ∇h is non-degenerate, $\overline{\mathrm{im} f_{\nabla h,K_n^{k-1}}}$ is non-degenerate. Since the non-degeneracy is preserved by projection, this in turn implies the non-degeneracy of $\overline{\mathrm{im} f_{\nabla h,H}}$. Therefore, any generic d points on $\overline{\mathrm{im} f_{\nabla h,H}}$ spans the ambient space $\mathbb{F}^{E(H)} \simeq \mathbb{F}^d$. In particular, for generic $x_1, \dots, x_d$, we have that $f_{\nabla h,H}(x_1), \dots, f_{\nabla h,H}(x_d)$ are linearly independent. This implies that B is non-singular, and hence g-rigidity has the extension property. □

An important special case is when g is the map for symmetric tensor completions.

Corollary 4.4. *Suppose g is the sum of d copies of h_{prod}. Then g-rigidity has the extension property.*

Proof. Observe that ∇h_{prod} is again the product map of $(k-1)$ variables. So $\overline{\mathrm{im} f_{\nabla h_{\mathrm{prod}},K_n^k}}$ is the affine cone of the Veronese variety of degree $k-1$, which is non-degenerate. Hence, ∇h_{prod} is non-degenerate and Lemma 4.3 can be applied. □

The following example shows that Corollary 4.4 is false in general.

Example 4.5. Let $h_{\det} : (\mathbb{F}^k)^k \to \mathbb{F}$ be the determinant as a multilinear k-form. For $h_{\det}$-rigidity to have the extension property, it is necessary that the matrix B in (20) is non-singular for any vertex v and any d distinct hyperedges $e_1, \dots, e_d$ incident to v. Denote $e_i - v = \{u_{i1}, \dots, u_{ik-1}\}$. Then $\nabla h_{\det}(p(e_i - v)) = \pm p(u_{i1}) \wedge p(u_{i2}) \wedge \cdots \wedge p(u_{ik-1})$ by using the standard basis in the exterior algebra of $\mathbb{F}$. Then B is singular if and only if $\{p(u_{i1}) \wedge p(u_{i2}) \wedge \cdots \wedge p(u_{ik-1}) : 1 \leq i \leq k\}$ is linearly dependent. The latter condition holds if and only if $\langle p(u_{i1}), p(u_{i2}), \dots, p(u_{ik-1}) \rangle$ has a common nonzero subspace for $i = 1, \dots, k$. If all hyperedges $e_1, \dots, e_k$ contain a common vertex other than v, then the condition fails. □

4.3 Packing-type sufficient condition

In this section, we aim to establish a packing-type condition that is sufficient for the local rigidity of hypergraphs G. Let us assume that g can be expressed as the sum of t copies of h. In Lemma 3.13, we demonstrated that any independent set in $\mathcal{M}_{g,n}$ can be decomposed into t independent sets in $\mathcal{M}_{h,n}$. While the converse direction does not hold in general, we will prove in this subsection that a decomposition with an additional property guarantees the validity of the converse direction.

Recall that, for a hyperedge $e \in \left\{\binom{[n]}{k}\right\}$ and $u \in e$, $e - u$ (resp., $e + u$) denotes the multiset obtained from e by reducing (resp., increasing) the multiplicity of u by one. We denote by $\mathrm{supp}(e)$ the set obtained from e by ignoring the multiplicity of each element.

Let $G = (V, E)$ be a k-uniform hypergraph. For $X \subseteq V$, let $E_G[X]$ be the set of hyperedges e of G with $\mathrm{supp}(e) \subseteq X$, and $G[X]$ be the subgraph of G induced by X, i.e., $G[X] = (X, E_G[X])$. For $E' \subseteq E$, we define the closed neighbour set $N_G(E')$ as

$$N_G(E') = \{e - u + v : e \in E', u \in e, v \in [n]\} \cap E.$$

Theorem 4.6. *Consider $g : (\mathbb{F}^d)^k \to \mathbb{F}$ written as the sum of t copies of a non-zero multilinear k-form $h : (\mathbb{F}^s)^k \to \mathbb{F}$, where $st = d$. Let $G = ([n], E)$ be a k-uniform hypergraph on $[n]$, and $\mathcal{X} = \{X_1, X_2, \dots, X_t\}$ a family of subsets X_i of $[n]$ with $|X_i| \geq n_{\Gamma_h}$. Denote $F_i = N_G(E_G[X_i])$ for each $i = 1, \dots, t$. Suppose that:*

(P1) *for every i, the hypergraph $([n], F_i)$ is locally h-rigid as a hypergraph on $[n]$;*

(P2) *for every i, $G[X_i]$ is locally h-rigid; and*

(P3) *for any i, j with $i \neq j$, and for any $e \in F_i$ and $v \in e$, $\mathrm{supp}(e - v) \not\subseteq X_j$.*

Then G is locally g-rigid.

Proof. For simplicity of notation, we present the proof in the case when $s = 1$ and $t = d$, but the extension to the general case is straightforward. Let $G' = ([n], \bigcup_i F_i)$. It is sufficient to show that G' is locally g-rigid in $\mathbb{R}^d$ as G' is a subgraph of G.

We first observe that $F_i \cap F_j = \emptyset$. Indeed, if $e \in F_i \cap F_j$, then the definition of the operator N_G implies that there are some $u \in e$ and $v \in [n]$ such that $e - u + v \in E_G[X_j]$. This, in particular, implies that $\mathrm{supp}(e - u) \subseteq X_j$, contradicting (P3). Hence, $F_1, \dots, F_d$ is a family of disjoint edge sets. We also note that $E_G[X_i] \subseteq F_i$ by definition. For each i with $1 \leq i \leq d$, we define $x_i \in \mathbb{F}^n$ by

$$x_i(v) = \begin{cases} x_{i,v} & \text{if } v \in X_i \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

where $x_{i,v}$ denotes a generic number. Let $p \in (\mathbb{F}^d)^n$ be the point configuration obtained by stacking x_i for all $1 \leq i \leq d$ as row vectors, i.e., x_i is the i-th coordinate vector of p. Consider $\mathrm{J}f_{g,G'}(p)$. Each column of $\mathrm{J}f_{g,G'}(p)$ is indexed by $(v, i) \in [n] \times [d]$ and each row is indexed by $e \in E(G')$. We use $b(e, v, i)$ to denote the entry of $\mathrm{J}f_{g,G'}(p)$ indexed by e and (v, i).

Claim 1. *Let $i, j \in [d]$, $v \in [n]$, and $e \in F_i$.*

- *If $e \in E_G[X_i]$, then $b(e, v, j) \neq 0$ if and only if $i = j$ and $v \in e$.*
- *If $e \in F_i \setminus E_G[X_i]$, then $b(e, v, j) \neq 0$ if and only if $i = j$ and $v \in \mathrm{supp}(e) \setminus X_i$.*

Proof. Since $g = \sum_{i=1}^d h(x_i)$, we have

$$b(e, v, j) = \begin{cases} m_e(v) \nabla h(p_j(e - v)) & \text{if } v \in e \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Since h is nonzero and multilinear, ∇h is also nonzero and multilinear.

We first consider a hyperedge $e \in E_G[X_i]$. Suppose $i = j$ and $v \in e$. Then, by Equation (21), $p_j(e - v)$ is a $(k-1)$-tuple of nonzero numbers, so $\nabla h(p_j(e - v))$ is nonzero. Hence, $b(e, v, j) \neq 0$ by Equation (22).

To see the converse direction, suppose next $b(e, v, j) \neq 0$. Then, by Equation (22), we have $v \in e$. The multilinearity of ∇h and Equation (21) further imply that, if $\mathrm{supp}(e - v) \not\subseteq X_j$, then $\nabla h(p_j(e - v)) = 0$. So, by Equation (22), $\mathrm{supp}(e - v) \subseteq X_j$ should hold. Hence, $i = j$ follows from (P3).

For the second statement, we consider a hyperedge $e \in F_i \setminus E_G[X_i]$. Suppose $b(e,v,j) \neq 0$. Then by Equation (21) $i = j$ clearly holds. We further have $v \in \mathrm{supp}(e) \setminus X_i$, since otherwise e would contain some vertex $w \in \mathrm{supp}(e) \setminus X_i$ other than v and $p_i(w) = 0$ would follow by $w \notin X_i$, which would imply $b(e,v,j) = 0$ by the multilinearity of ∇h. This completes the proof of the necessity for $b(e,v,j) \neq 0$. The sufficiency can be checked easily from Equations (21) and (22). ☐

Let $G_i = ([n], F_i)$. Claim 1 implies that $\mathrm{J}f_{g,G'}(p)$ is in the following block-diagonalized form:

$$\mathrm{J}f_{g,G'}(p) = \begin{array}{c} \\ F_1 \\ F_2 \\ \vdots \\ F_d \end{array} \begin{array}{c} \begin{array}{cccc} I_1 & I_2 & \dots & I_d \end{array} \\ \begin{bmatrix} \mathrm{J}f_{h,G_1}(x_1) & 0 & \dots & 0 \\ 0 & \mathrm{J}f_{h,G_2}(x_2) & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{J}f_{h,G_d}(x_d) \end{bmatrix} \end{array} \tag{23}$$

where $I_i := \{(v,i) : v \in [n]\}$ for $i = 1, \dots, d$. Let Γ_h and Γ_g be the stabilizers of h and g, respectively.

Claim 2. *We have* $\mathrm{rank}\, \mathrm{J}f_{h,G_i}(x_i) = n - d_{\Gamma_h}$ *for all* $1 \leq i \leq d$.

Proof. (For completeness we give a proof of the claim for general s. We show $\mathrm{rank}\, \mathrm{J}f_{h,G_i}(x_i) = sn - d_{\Gamma_h}$.) Let $F_{i,v}$ be the set of hyperedges incident to v in G_i, and consider the submatrix B_v of $\mathrm{J}f_{h,G_i}(x_i)$ induced by the rows indexed by $F_{i,v}$ and the corresponding s columns of v. This submatrix B_v corresponds to the matrix B in Equation (20) for v and $F_{i,v} = \{e_1, \dots, e_j\}$. By Claim 1, $\mathrm{J}f_{h,G_i}(x_i)$ has the form

$$\mathrm{J}f_{h,G_i}(p) = \begin{array}{c} \\ F_i \setminus \bigcup_{v \in [n] \setminus X_i} F_{i,v} \\ F_{i,v_1} \\ \vdots \\ F_{i,v_l} \end{array} \begin{array}{c} \begin{array}{cccc} X_i & v_1 & \dots & v_l \end{array} \\ \begin{bmatrix} \mathrm{J}f_{h,G[X_i]}(x_i) & 0 & \dots & 0 \\ 0 & B_{v_1} & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & B_{v_l} \end{bmatrix} \end{array} \tag{24}$$

where we denote $[n] \setminus X_i = \{v_1, \dots, v_l\}$. By (P2), $G[X_i]$ is locally h-rigid, and hence $\mathrm{rank}\, \mathrm{J}f_{h,G[X_i]}(x_i) = sn - d_{\Gamma_h}$. So it suffices to show that $\mathrm{rank}\, B_{v_j} = s$ for each $v_j \in [n] \setminus X_i$, i.e., v_j is stable in (G_i, x_i).

By (P1) and Lemma 4.2, every vertex v_j is stable in a generic framework of G_i (but the current x_i is not generic). Recall that every edge in F_{i,v_j} is written in the form $f + v_j$ for some $(k-1)$-multiset f with $\mathrm{supp}(f) \subseteq X_i$, and so the entries of B_{v_j} are determined by the coordinates of X_i. Since the coordinates of X_i in x_i are generic, v_j is still stable in (G_i, x_i). This completes the proof of Claim 2. ☐

Hence,
$$\begin{aligned} dn - d_{\Gamma_g} &\geq \mathrm{rank}\, \mathrm{J}f_{g,G'}(p) \qquad \text{(by Proposition 3.6)} \\ &= \sum_{i=1}^{d} \mathrm{rank}\, \mathrm{J}f_{h,G_i}(x_i) \qquad \text{(by Equation (23))} \\ &= d(n - d_{\Gamma_h}) \qquad \text{(by Claim 2)} \\ &\geq dn - d_{\Gamma_g} \qquad \text{(by the definition of } \Gamma_h \text{ and } \Gamma_g), \end{aligned}$$

implying the local g-rigidity of (G', p_i) by Proposition 3.6. This completes the proof. ☐

We give two applications of Theorem 4.6.

Example 4.7. Consider the product map h_{prod} and the sum dh_{prod} of d copies of h_{prod}. This corresponds to the case of symmetric tensor completions. Consider $k \geq 3$ and a k-uniform hypergraph $G = ([n], E)$ whose edge set is given by

$$E = \left\{ e \in \left\{ \begin{matrix} [n] \\ k \end{matrix} \right\} : m_e(v) \geq k-1 \text{ for some } v \in [n] \right\}.$$

This hypergraph is the underlying graph of a tridiagonal symmetric tensor. Suppose $d \leq n$. We choose $\mathcal{X} = \{X_1, \ldots, X_d\}$ with $X_v = \{v\}$ for each $v \in [d]$. We show $\mathcal{X}$ satisfies the hypotheses (P1)-(P3) of Theorem 4.6. We use $v_1^{i_1} \ldots v_s^{i_s}$ to denote the hyperedge consisting of $v_1, \ldots, v_s$ with $m_e(v_j) = i_j$. Then, $E = \{v^{k-1}w : v, w \in [n]\}$. For each $v \in [d]$, $E_G[X_v] = \{v^k\}$, and $F_v = \{v^{k-1}w : w \in [n]\}$. The hypergraph $G[X_v]$ is locally h_{prod}-rigid since $G[X_v]$ is a single-vertex graph with one edge. So (P2) holds. Note that F_v is obtained from $G[X_v]$ by a sequence of 1-valent extension, implying (P1) by Corollary 4.4. To see (P3), consider any $e \in F_v$ and $u \in e$. Since $e = v^{k-1}w$ for some $w \in [n]$, $e - u = v^{k-2}w$ if $v = u$ and $e - u = v^{k-1}$ if $u = w$. Hence, by $k \geq 3$, $\text{supp}(e - u) \not\subseteq X_{v'} = \{v'\}$ holds for any v' with $v \neq v'$, and (P3) follows. By Theorem 4.6, we conclude that G is locally dh_{prod}-rigid if $d \leq n$.

Example 4.8. For another (more exciting) application of Theorem 4.6, we need the following notation. For positive integers α, β, an α-uniform family $\mathcal{X} \subseteq \binom{[n]}{\alpha}$ is called β-*sparse* if $|X \cap Y| < \beta$ for any $X, Y \in \mathcal{X}$. We define the packing number $\text{packing}(n, \alpha, \beta)$ by $\text{packing}(n, \alpha, \beta) = \max\left\{|\mathcal{X}| : \mathcal{X} \subseteq \binom{[n]}{\alpha}, \mathcal{X} \text{ is } \beta\text{-sparse}\right\}$. An asymptotic bound by Rödl [53] gives $\text{packing}(n, \alpha, \beta) = (1 - o(1))\binom{n}{\beta}/\binom{\alpha}{\beta}$.

Recall that $\tilde{K}_n^k$ denotes the simple complete k-uniform hypergraph on n vertices.

Corollary 4.9. *Let $n, k \geq 3$ be positive integers. Consider $g : (\mathbb{F}^d)^k \to \mathbb{F}$ written as the sum of t copies of a non-zero multilinear k-form $h : (\mathbb{F}^s)^k \to \mathbb{F}$ with $d = st$. Suppose there is a positive integer a such that*

- *$\tilde{K}_a^k$ is locally h-rigid,*
- *$a \geq n_{\Gamma_h} + 1$, and*
- *$t \leq \text{packing}(n, a, k - 2)$.*

Then $\tilde{K}_n^k$ is locally g-rigid.

Proof. We may assume $t = \text{packing}(n, a, k - 2)$. Let $\mathcal{X} = \{X_1, \ldots, X_t\} \subseteq \binom{[n]}{a}$ be a $(k - 2)$-sparse family achieving $p(n, a, k - 2)$. By Theorem 4.6, it suffices to show that $\mathcal{X}$ satisfies (P1)-(P3). Choose $X_i \in \mathcal{X}$. Since $|X_i| = a$, the subgraph $\tilde{K}_n^k[X_i]$ is isomorphic to $\tilde{K}_a^k$, so (P2) follows from the local h-rigidity of $\tilde{K}_a^k$.

To see (P1), consider F_i as defined in the statement of Theorem 4.6 in the case when $G = \tilde{K}_n^k$, and let $G_i = ([n], F_i)$. Pick any vertex $u \in [n] \setminus X_i$. Since $a \geq n_{\Gamma_h} + 1$ and K_a^k is locally h-rigid, Lemma 4.2 implies that each vertex v in X_i is stable in a generic framework of $\tilde{K}_n^k[X_i]$. Hence each vertex v in X_i remains stable in a generic framework of $\tilde{K}_n^k[X_i \cup \{u\}]$. Since $\tilde{K}_n^k[X_i \cup \{u\}]$ is a simple complete hypergraph, the symmetry further implies that u is also stable in a generic framework of $\tilde{K}_n^k[X_i \cup \{u\}]$. Hence, by Lemma 4.2, a subgraph of $\tilde{K}_n^k[X_i \cup \{u\}]$ is obtained from $\tilde{K}_n^k[X_i]$ by an s-valent extension which preserves local g-rigidity. Note that $\tilde{K}_n^k[X_i \cup \{u\}]$ is a subgraph of G_i. Applying the same argument to each vertex $u \in [n] \setminus X_i$, we can further construct a spanning subgraph of G_i from $\tilde{K}_n^k[X_i]$ by a sequence of s-valent extensions keeping local g-rigidity. Thus G_i is locally g-rigid.

Finally, suppose, for a contradiction, that (P3) does not hold. Then $\text{supp}(e - v) \subseteq X_j$ for some $e \in F_i$, $v \in e$, and X_j with $i \neq j$. Since $e \in F_i$, there are some $u, w \in [n]$ such that $e - u + w \in E_{\tilde{K}_n^k}[X_i]$, so $\text{supp}(e - u - v) \subseteq X_i \cap X_j$. Since $|\text{supp}(e - u - v)| = k - 2$, this contradicts the fact that $\mathcal{X}$ is $(k-2)$-sparse. Thus (P3) holds, and the g-rigidity of G follows from Theorem 4.6. □

We note that when h is the determinant function, Corollary 4.9 aligns with an observation in [2].

5 Conditions for Global g-rigidity

In this section we analyse global g-rigidity.

5.1 Generic global g-rigidity

In Section 3, we observed that local g-rigidity is a generic property of the underlying hypergraphs. This notion naturally leads to the concept of local g-rigidity of hypergraphs. By applying Chevalley's theorem, similar to the approach in Proposition 3.2(iv), we can extend this result to global g-rigidity when considering constraints over the field of complex numbers.

Proposition 5.1. *Global g-rigidity is a generic property of the underlying hypergraphs over complex field.*

Additionally, in the case of Euclidean distance where g is the squared Euclidean distance function, there are notable results in the real case by Gortler, Healy, and Thurston [34, 41]. They show that global g-rigidity is a generic property in the real setting. However, it is important to note that this result does not hold for other maps. For example, if g represents the Euclidean inner product of two points over $\mathbb{R}$ [40] or if g is the map for volume rigidity [59], then global g-rigidity is not a generic property over reals.

While global g-rigidity may not be a generic property in general, it would be useful if we can guarantee the global g-rigidity for any generic realization of a given hypergraph. Therefore, we define a hypergraph G to be *globally g-rigid* (over $\mathbb{F}$) if (G, p) is globally g-rigid for any generic point-configuration p over $\mathbb{F}$.

The following proposition provides motivation for focusing on the complex case when establishing a sufficient condition for global rigidity.

Proposition 5.2. *If a hypergraph G is globally g-rigid over $\mathbb{C}$, then it is globally g-rigid over $\mathbb{R}$.*

Proof. Since G is globally g-rigid over $\mathbb{C}$, any generic hyperframework (G, p) of G over $\mathbb{C}$ is globally rigid. Since the genericity is defined in terms of algebraic independence over $\mathbb{Q}$, any generic configuration q over $\mathbb{R}$ is also generic over $\mathbb{C}$. So for any generic real configuration q, (G, q) is a generic hyperframework over $\mathbb{C}$, and it is globally rigid. □

5.2 Identifiability and global g-rigidity

The following proposition highlights the usefulness of identifiability in verifying global rigidity.

Proposition 5.3. *Suppose $g : (\mathbb{F}^d)^k \to \mathbb{F}$ is the sum of t copies of a homogeneous $h : (\mathbb{F}^s)^k \to \mathbb{F}$. Suppose further that $\overline{\operatorname{im} f_{h,G}}$ is t-identifiable. Then G is globally g-rigid if and only if G is globally h-rigid.*

Proof. Clearly global h-rigidity is necessary for global g-rigidity. To see the converse direction, we take any generic $p \in (\mathbb{F}^d)^V$ and show that, if G is globally h-rigid then (G, p) is globally g-rigid.

Since g is the sum of t copies of h, $p \in (\mathbb{F}^d)^V$ can be decomposed into $(q_1, \ldots, q_t)$ with $q_i \in (\mathbb{F}^s)^V$ such that $f_{g,G}(p) = \sum_{i=1}^t f_{h,G}(q_i)$. Since p is generic, each q_i is generic. So (G, q_i) is globally h-rigid. Hence, the t-identifiability of $\overline{\operatorname{im} f_{h,G}}$ implies the global g-rigidity of (G, p). □

We next review some sufficient conditions for checking the identifiability of varieties. A key notion is that of tangential weak defectiveness due to Chiantini and Ottaviani [18]. Let $\mathcal{V}$ be a variety over $\mathbb{C}$, and denote by $T_x\mathcal{V}$ the tangent space at $x \in \mathcal{V}$. The *t-tangential contact locus* $\mathcal{C}$ at generic t points $x_1, \ldots, x_t$ is defined as $\mathcal{C} = \overline{\{y \in \mathcal{V} : T_y\mathcal{V} \subseteq \langle T_{x_1}\mathcal{V}, \ldots, T_{x_t}\mathcal{V}\rangle\}}$. A variety $\mathcal{V}$ is said to be *t-tangentially weakly defective* if an irreducible component of $\mathcal{C}$ that contains x_i has dimension at least two for some i. (Since we are working in affine space, the tangential contact locus always contains a one-dimensional subspace of scaling.)

It was shown that t-tangential weak non-defectiveness is a sufficient condition for t-identifiability.

Theorem 5.4 (Chiantini and Ottaviani [18])**.** *Suppose an affine variety $\mathcal{V}$ over $\mathbb{C}$ is t-tangentially weakly nondefective. Then $\mathcal{V}$ is t-identifiable.*

A combination of Proposition 5.3 and Theorem 5.4 leads to the following criterion for global g-rigidity.

Corollary 5.5. *Suppose a homogeneous $g : (\mathbb{C}^d)^k \to \mathbb{C}$ is the sum of t copies of $h : (\mathbb{C}^s)^k \to \mathbb{C}$. Then G is globally g-rigid if $\overline{\operatorname{im} f_{h,G}}$ is t-tangentially weakly nondefective and G is globally h-rigid.*

The tangential contact locus can be characterised algebraically, allowing for the development of numerical algorithms to test for t-tangentially weak defectiveness. Chiantini, Ottaviani, and Vannieuwenhoven [19] proposed an algorithm based on this description, which we will briefly review as it is closely related to a well-known method in rigidity.

Let G be a k-uniform hypergraph with n vertices and m edges, and suppose $g : (\mathbb{C}^d)^k \to \mathbb{C}$ is the sum of t copies of $h : (\mathbb{C}^s)^k \to \mathbb{C}$. In view of Corollary 5.5, we want to check the t-tangentially weakly defectiveness of $\overline{\operatorname{im} f_{h,G}}$. Let $p \in (\mathbb{C}^d)^n$ be a generic point-configuration, and let $\omega_1, \dots, \omega_k \in \mathbb{C}^m$ be a basis of the orthogonal complement of the tangent space of $\overline{\operatorname{im} f_{g,G}}$ at $f_{g,G}(p)$. Since the tangent space of $\overline{\operatorname{im} f_{h,G}}$ at $f_{h,G}(q)$ is the image of $\mathrm{J} f_{h,G}(q)$, the t-tangential contact locus $\mathcal{C}$ is the closure of $\{q \in (\mathbb{C}^s)^n : \omega_i \in \ker \mathrm{J} f_{h,G}(q)^\top \ (i = 1, \dots, k)\}$. Note that $\omega_i \in \ker \mathrm{J} f_{h,G}(q)^\top$ forms a polynomial system in q (where ω_i is fixed). Therefore, there is a polynomial map $\alpha_i : (\mathbb{C}^s)^n \to (\mathbb{C}^s)^n$ such that $\omega_i \in \ker \mathrm{J} f_{h,G}(q)^\top$ if and only if $\alpha_i(q) = 0$. So, the tangential contact locus is written as the closure of $\{q \in (\mathbb{C}^s)^n : \alpha_i(q) = 0 \ (i = 1, \dots, k)\}$.

It was shown in [19] that, if an affine variety $\mathcal{V}$ is t-tangentially weakly defective, then the t-tangential contact locus formed by generic points $x_1, \dots, x_t$ contains a smooth path connecting any x_i to the locus. Hence, in our terminology above, this implies that one can determine t-tangential weak non-defectiveness by examining the rank of the Jacobian matrix of α_i.

Remark 5.6. It is worth mentioning that the resulting Jacobian matrix of α_i is the Laplacian matrix of G weighted by ω_i, in the case that g is the squared Euclidean distance. So, the numerical condition in [19] has a similarity to the stress matrix condition for global rigidity due to Connelly [23] (or more generally to the shared stress kernel condition due to Gortler-Healy-Thurston [34]). It should be noted, however, that in the case of the squared d-dimensional Euclidean distance the underlying variety is always d-tangentially weakly defective if $d > 1$.

A recent result of Massarenti and Mella [49] gives a powerful criterion for checking t-identifiability.

Theorem 5.7 ([49])**.** *Let $\mathcal{V}$ be a non-degenerate affine variety in $\mathbb{C}^m$. Suppose that $(t+1)\dim \mathcal{V} \le m$ and $\mathcal{V}$ is $(t+1)$-nondefective and 1-tangentially weakly nondefective. Then $\mathcal{V}$ is t-identifiable.*

Combining Proposition 5.3 and Theorem 5.7, we have the following criterion for global g-rigidity.

Corollary 5.8. *For a homogeneous $h : (\mathbb{C}^s)^k \to \mathbb{C}$ and a positive integer d, let dh be the sum of d copies of h, and suppose $d_{\Gamma_{(d+1)h}} = (d+1) d_{\Gamma_h}$. Then, for a positive integer d, G is globally dh-rigid if*

- *G is locally $(d+1)h$-rigid,*
- *$\overline{\operatorname{im} f_{h,G}}$ is 1-tangentially weakly nondefective, and*
- *G is globally h-rigid.*

The 1-tangential contact locus is just an ordinary contact locus, so 1-tangential weak defectiveness is equivalent to the degeneracy of the Gauss map.

5.3 Sufficient conditions

Corollary 5.5 or Corollary 5.8 provides a way to reduce the global g-rigidity problem to the global h-rigidity problem, however it is a non-trivial problem to establish global h-rigidity. In order to accomplish this, we will extend Connelly's sufficient condition [23] from graph rigidity theory.

The result is described in terms of the *adjacency matrix* of an edge weighted hypergraph. Suppose $G = (V, E)$ is a k-uniform hypergraph and $w \in \mathbb{F}^E$ is a vector representing the weight of each edge.

Consider the collection $E^{(k-1)} := \{\sigma \in \left\{ {V \atop k-1} \right\} : \sigma \subseteq e \in E\}$ of multisets of size $(k-1)$ contained in some hyperedge in G. We define the adjacency matrix $A_{G,w}$ to be an $\mathbb{F}$-matrix of size $|V| \times |E^{(k-1)}|$ such that:

$$A_{G,w}[v,\sigma] = \begin{cases} m_e(v)w(e) & \text{(if } e = \sigma + v \text{ is in } G) \\ 0 & \text{(otherwise)}, \end{cases}$$

where each row is indexed by a vertex $v \in V$ and each column is indexed by $\sigma \in E^{(k-1)}$. We will often look at the case when the edge weight w is in the left kernel of $\mathrm{J}f_{g,G}(p)$. See Example 5.9 for an example.

In order to deal with the case of anti-symmetric functions, we consider the signed variant of the adjacency matrix. For this, we assume that the elements of V are totally ordered, and consider $A^s_{G,w}$ with:

$$A^s_{G,w}[v,\sigma] = \begin{cases} \mathrm{sign}(e,v)w(e)m_e(v) & \text{(if } e = \sigma + v \text{ is in } G) \\ 0 & \text{(otherwise)} \end{cases}$$

for each $v \in V$ and $\sigma \in E^{(k-1)}$, where $\mathrm{sign}(e,v)$ denotes the standard sign function of permutations, which is positive (resp. negative) if the ordering of v in e is odd (resp. even).

Example 5.9. Consider a 4-uniform hypergraph $G_1 = (\{a,b\}, \{aaaa, abbb, bbbb\})$ with edge-weight ω_1 and a 4-uniform hypergraph $G_2 = (\{a,b\}, \{aaaa, aabb, bbbb\})$ with edge-weight ω_2. Then, A_{G_1,ω_1} and A_{G_2,ω_2} are, respectively,

$$\begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} aaa & abb & bbb \end{array} \\ \begin{bmatrix} 4\omega_1(aaaa) & 0 & \omega_1(abbb) \\ 0 & 3\omega_1(abbb) & 4\omega_1(bbbb) \end{bmatrix} \end{array} \text{ and } \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cccc} aaa & aab & abb & bbb \end{array} \\ \begin{bmatrix} 4\omega_2(aaaa) & 0 & 2\omega_2(aabb) & 0 \\ 0 & 2\omega_2(aabb) & 0 & 4\omega_2(bbbb) \end{bmatrix} \end{array}.$$

Proposition 5.10. *Suppose $g : (\mathbb{F}^d)^k \to \mathbb{F}$ is a multilinear k-form over $\mathbb{F}^d$, and (G,p) is a k-uniform hyper-framework. Let P be a matrix of size $d \times |E^{(k-1)}|$ obtained by aligning $\nabla g(p(\sigma))$ as a column vector for each $\sigma \in E^{(k-1)}$.*

(i) *If g is symmetric, then $A_{G,w}P^\top = w^\top \mathrm{J}f_{g,G}(p)$ for any edge weight w of G.*

(ii) *If g is anti-symmetric, then $A^s_{G,w}P^\top = w^\top \mathrm{J}f_{g,G}(p)$ for any edge weight w of G.*

(iii) *If (G,p) is infinitesimally g-rigid, then* $\mathrm{rank}\, P = d$.

Proof. Assume g is symmetric. By definition, the v-th entry of $w^\top \mathrm{J}f_{g,G}(p)$ is

$$\sum_{\sigma \in E^{(k-1)} : \sigma + v \in E} w(\sigma + v) m_{\sigma+v}(v) \nabla g(p(\sigma)).$$

Reshaping this in terms of $\nabla g(p(\sigma))$, we obtain $A_{G,w}P^\top$. The same argument also applies to the case when g is anti-symmetric. We thus obtain (i) and (ii).

To see (iii), assume (G,p) is infinitesimally g-rigid. By Lemma 4.2, every vertex is stable in (G,p). So the columns of P indexed by d hyperedges incident to a vertex forms an independent set, which means that P has full row-rank. □

Proposition 5.11. *Suppose $g : (\mathbb{F}^d)^k \to \mathbb{F}$ is a multilinear k-form and (G,p) is a generic locally g-rigid k-uniform hyper-framework. Suppose*

- $\dim \left(\bigcap_{\omega \in \ker \mathrm{J}f_{G,g}(p)^\top} \ker A_{G,\omega} \right) = d$ *if f is symmetric, and*
- $\dim \left(\bigcap_{\omega \in \ker \mathrm{J}f_{G,g}(p)^\top} \ker A^s_{G,\omega} \right) = d$ *if f is anti-symmetric.*

Then for each $q \in f_{g,G}^{-1}(f_{g,G}(p))$ there exists $T \in \mathbb{F}^{d\times d}$ such that

$$\nabla g(q(\sigma)) = T\nabla g(p(f)) \qquad \text{for all } \sigma \in E^{(k-1)}.$$

Proof. We give the proof only in the case when f is symmetric since the anti-symmetric case is identical.

By Proposition 3.2(iv) and the generic assumption on p, $f_{g,G}(p)$ is a regular value of $f_{g,G}$, so for any $q \in f_{g,G}^{-1}(f_{g,G}(p))$ the image of $\mathrm{J}f_{g,G}(q)$ is the tangent space of $\overline{\mathrm{im} f_{g,G}}$ at $f_{g,G}(q)$. We have that:

$$\ker \mathrm{J}f_{g,G}(q)^\top = \ker \mathrm{J}f_{g,G}(p)^\top, \tag{25}$$

since $f_{g,G}(p) = f_{g,G}(q)$. For any point-configuration q of G, let Q be a matrix of size $d\times|E^{(k-1)}|$ obtained by aligning $\nabla g(q(\sigma))$ as the columns for all $\sigma \in E^{(k-1)}$. By Proposition 5.10 and Equation (25),

$$A_{G,w}Q^\top = 0 \quad \text{for any } \omega \in \ker \mathrm{J}f_{G,g}(p)^\top. \tag{26}$$

In particular, $A_{G,w}P^\top = 0$. Hence each row of P belongs to $\bigcap_{\omega\in\ker \mathrm{J}f_{G,g}(p)^\top} \ker A_{G,\omega}$.

Note that we have assumed the dimension of $\bigcap_{\omega\in\ker \mathrm{J}f_{G,g}(p)^\top} \ker A_{G,\omega}$ is d. Moreover, we have $\operatorname{rank} P = d$ by Proposition 5.10. By combining those two facts with $A_{G,w}P^\top = 0$, we may conclude that the rows of P form a basis of $\bigcap_{\omega\in\ker \mathrm{J}f_{G,g}(p)^\top} \ker A_{G,\omega}$. Since each row of Q belongs to this space by Equation (26), there must be $T \in \mathbb{F}^{d\times d}$ such that $Q = TP$. This is equivalent to the statement. □

Proposition 5.12. *Suppose $g : (\mathbb{F}^d)^k \to \mathbb{F}$ is a multilinear k-form, (G,p) is a generic locally g-rigid k-uniform hyper-framework, and $q \in f_{g,G}^{-1}(f_{g,G}(p))$. Suppose there is $T \in \mathbb{F}^{d\times d}$ such that $\nabla g(q(\sigma)) = T\nabla g(p(\sigma))$ for any $\sigma \in E^{(k-1)}$. Then $p(v) = T^\top q(v)$ for any $v \in V$.*

Proof. Since (G,p) is generic and locally g-rigid, every vertex is stable by Lemma 4.2. So there are d hyperedges $e_1,\dots,e_d$ incident to v in G such that $\{\nabla g(p(e_i - v)) : i = 1,\dots,d\}$ is a basis of $\mathbb{F}^d$. Since $q \in f_{g,G}^{-1}(f_{g,G}(p))$, we have $f_{g,G}(q) = f_{g,G}(p)$. By the multilinearity of g, this implies that

$$\langle q(v), \nabla g(q(e_i - v))\rangle = \langle p(v), \nabla g(p(e_i - v))\rangle,$$

and hence

$$\langle T^\top q(v) - p(v), \nabla g(p(e_i - v))\rangle = 0 \quad (i = 1,\dots,d).$$

The fact that $\{\nabla g(p(e_i - v)) : i = 1,\dots,d\}$ is a basis now gives $p(v) = T^\top q(v)$ for all v. □

When g is the determinant, a combination of Propositions 5.11 and 5.12 gives the following sufficient condition for global g-rigidity.

Theorem 5.13. *Suppose $g : (\mathbb{F}^d)^d \to \mathbb{F}$ is the determinant as a multilinear d-form over $\mathbb{F}^d$. Then $G = (V,E)$ is globally g-rigid if there exists a point-configuration $p \in (\mathbb{F}^d)^V \to \mathbb{F}$ such that*

- *(G,p) is infinitesimally g-rigid, and*
- $\dim\left(\bigcap_{w\in\ker \mathrm{J}f_{g,G}(p)^\top} \ker A^s_{G,w}\right) = d.$

Proof. The infinitesimal rigidity of (G,p) implies that the rank of $\mathrm{J}f_{g,G}(p)$ takes the maximum possible value over all point-configurations. Note that $\ker \mathrm{J}f_{g,G}(p)$ changes continuously if we perturb the entries of p continuously. So any sufficiently small continuous perturbation of the entries of p does not change $\dim\left(\bigcap_{w\in\ker \mathrm{J}f_{g,G}(p)^\top} \ker A_{G,w}\right)$. Hence, we may assume that p is generic.

To see the global g-rigidity of (G,p), pick any $q \in f_{g,G}^{-1}(f_{g,G}(p))$. By Propositions 5.11 and 5.12, there is $T \in \mathbb{F}^{d\times d}$ such that $q(v) = Tp(v)$ for all $v \in V(G)$. Pick any hyperedge e in G, and assume e consists of k vertices $v_1,\dots,v_k$. Since $q \in f_{g,G}^{-1}(f_{g,G}(p))$, we have

$$\det\begin{pmatrix} p(v_1) & \dots & p(v_k)\end{pmatrix} = \det\begin{pmatrix} q(v_1) & \dots & q(v_k)\end{pmatrix} = \det T \det\begin{pmatrix} p(v_1) & \dots & p(v_k)\end{pmatrix},$$

so $\det T = 1$. This implies that T is in the stabilizer of g, and the global g-rigidity of (G,p) follows. □

By combining Theorem 5.13 with either Corollary 5.5 or Corollary 5.8, we can establish a sufficient condition for global rigidity when g is the sum of copies of the determinant map. However, when g is the sum of copies of the product map (as in the case of symmetric tensor completion), we can provide a direct sufficient condition without the need for identifiability tests. A key ingredient in establishing this condition is the following consequence of a generalised trisecant lemma given in [17].

Proposition 5.14. *Let $h := h_{\mathrm{prod}} : \mathbb{C}^k \to \mathbb{C}$ be the product map, $G = (V, E)$ a hypergraph, and $x_1, x_2, \dots, x_t$ some generic points in $\overline{\operatorname{im} f_{h,G}}$. If $t \leq |E| - \operatorname{rank} \mathrm{J} f_{h,G}$, then any point in the intersection of $\overline{\operatorname{im} f_{h,G}}$ and $\operatorname{span}\{x_1, x_2, \dots, x_t\}$ is a scalar multiple of some x_i.*

Proof. A generalised trisecant lemma [17] states that for an irreducible, nondegenerate, d-dimensional projective variety $\mathcal{X}$ in $\mathbb{P}^n$, and a set $\{y_1, y_2, \dots, y_t\}$ of generic points in $\mathcal{X}$, the condition $\operatorname{span}\{y_1, y_2, \dots, y_t\} \cap \mathcal{X} = \{y_1, y_2, \dots, y_t\}$ holds provided $t \leq n - d$. We apply this to $\overline{\operatorname{im} f_{h,G}}$. Note that $\overline{\operatorname{im} f_{h,G}}$ is an affine variety in $\mathbb{F}^{|E|}$, whose dimension is equal to $\operatorname{rank} \mathrm{J} f_{h,G}(z)$ at a generic z. So it remains to check that $\overline{\operatorname{im} f_{h,G}}$ is irreducible and nondegenerate. The irreducibility of $\overline{\operatorname{im} f_{h,G}}$ follows from the fact that it is the image of a polynomial map. To see the nondegeneracy, recall that $\overline{\operatorname{im} f_{h,K_n^k}}$ is the affine cone of the Veronese variety of degree k, which is known to be nondegenerate (and is easily checked). Since $\overline{\operatorname{im} f_{h,G}}$ is a projection of $\overline{\operatorname{im} f_{h,K_n^k}}$ and a projection preserves nondegeneracy, $\overline{\operatorname{im} f_{h,G}}$ is nondegenerate. □

Theorem 5.15. *Suppose that g is the sum of d copies of $h_{\mathrm{prod}} : \mathbb{F}^k \to \mathbb{F}$. Then $G = (V, E)$ is globally g-rigid if there exists a point-configuration $p \in (\mathbb{F}^d)^V \to \mathbb{F}$ such that*

- *(G, p) is infinitesimally g-rigid,*
- *$|E^{(k-1)}| \geq |V| + d$, and*
- *$\dim \left(\bigcap_{w \in \ker \mathrm{J} f_{g,G}(p)^\top} \ker A_{G,w} \right) = d$.*

Proof. Similar to the proof of Theorem 5.13, we may suppose, by a sufficiently small perturbation, that p is generic. Also, by Proposition 5.2, we may assume $\mathbb{F} = \mathbb{C}$. By Proposition 5.11, there is $T \in \mathbb{F}^{d \times d}$ such that $\nabla g(q(\sigma)) = T \nabla g(p(\sigma))$ for every $\sigma \in E^{(k-1)}$. Let P be a matrix of size $d \times |E^{(k-1)}|$ obtained by aligning $\nabla g(p(\sigma))$ for all $\sigma \in E^{(k-1)}$, and let Q be the corresponding matrix for q. Then $Q = TP$.

We denote the row vectors of P by $x_1, \dots, x_d$ and those of Q by $y_1, \dots, y_d$. Also, let H be a $(k-1)$-uniform hypergraph on V with the edge set $E^{(k-1)}$. Since g is the sum of d copies of the product map of k variables, each coordinate of $\nabla g(p(\sigma))$ is also an image of the product map of $(k-1)$ variables. So, each x_i is the image of $f_{h,H}$ at a generic point configuration, where h denotes the product map of $(k-1)$ variables. Since the dimension of $\overline{\operatorname{im} f_{h,H}}$ is upper bounded by n (as it is a projection of the affine cone of a Veronese variety), we can apply Proposition 5.14 to deduce that any point in the intersection of $\overline{\operatorname{im} f_{h,H}}$ and $\operatorname{span}\{x_1, \dots, x_d\}$ is a scalar multiple of some x_i. Since each y_j belongs to $\overline{\operatorname{im} f_{h,H}}$ and $Q = TP$,

$$\text{each } y_j \text{ is a scalar multiple of some } x_i. \tag{27}$$

Also, by Proposition 3.2, q is also regular, so (G, q) is infinitesimally g-rigid. Hence, by Proposition 5.10, $\operatorname{rank} Q = d$. So by $Q = TP$ and (27), T is written as $T = \Sigma D$ for some permutation matrix Σ and a diagonal matrix D. By reordering the coordinates of q if necessary, we may assume Σ is the identity matrix, so $T = D$.

Let t_i be the i-th diagonal entry T. By Proposition 5.12, $q(v) = Tp(v)$ for all $v \in V(G)$. Since $f_{g,G}(p) = f_{g,G}(q)$, comparing the e-th coordinate for each $e = \{i_1, i_2, \dots, i_k\} \in E$, we obtain

$$\left\langle (t_1^k \ t_2^k \ \dots t_k^k), \bigodot_{j=1}^{k} p(i_j) \right\rangle = \bigodot_{j=1}^{k} p(i_j), \tag{28}$$

where $(t_1^k\ t_2^k\ \ldots t_k^k)$ is the k-dimensional vector whose i-th coordinate is t_i^k. Using the same trick as the second paragraph of this proof, one can check from Proposition 5.14 that there are d linearly independent vectors among $\{\bigodot_{j=1}^k p(i_j) : e = \{i_1, i_2, \ldots, i_k\} \in E\}$. So Equation (28) implies that $t_j^k = 1$ for every $j = 1, \ldots, k$. In other words, each diagonal entry of T is the k-th root of unity. So T belongs to the stabilizer of g, and the global g-rigidity of G follows. □

Note that the theorem is analogous to [39, Theorem 6.2] which gives a sufficient condition in the matrix completion case. However the proof technique does not apply in the matrix completion case. For example, the second bullet point in the hypotheses always fails in the matrix completion context.

Example 5.16. Consider the symmetric tensor completion problem of symmetric rank one and order four, i.e., $k = 4$ and $g = h_{\text{prod}}$. We analyse two different 4-uniform hyper-frameworks (G_1, p) and (G_2, p), where $G_1 = (\{a, b\}, \{aaaa, abbb, bbbb\})$, $G_2 = (\{a, b\}, \{aaaa, aabb, bbbb\})$ and $p : a \mapsto x_a, b \mapsto x_b$. Then,

$$\mathrm{J}f_{g,G_1}(p) = \begin{array}{c} \\ aaaa \\ abbb \\ bbbb \end{array} \overset{\begin{array}{cc} a & \quad b \end{array}}{\begin{bmatrix} 4x_a^3 & 0 \\ x_b^3 & 3x_a x_b^2 \\ 0 & 4x_b^3 \end{bmatrix}}, \qquad \mathrm{J}f_{g,G_2}(p) = \begin{array}{c} \\ aaaa \\ aabb \\ bbbb \end{array} \overset{\begin{array}{cc} a & \quad b \end{array}}{\begin{bmatrix} 4x_a^3 & 0 \\ 2x_a x_b^2 & 2x_a^2 x_b \\ 0 & 4x_b^3 \end{bmatrix}}.$$

Since $\operatorname{rank} \mathrm{J}f_{g,G_i}(p) = 2$, (G_i, p) is infinitesimally g-rigid, and $\omega_i \in \mathbb{C}^E$ is in the left kernel of $\mathrm{J}f_{g,G_i}(p)$ if

$$\omega_1(aaaa) = \frac{1}{x_a^4}, \omega_1(abbb) = -\frac{4}{x_a x_b^3}, \omega_1(bbbb) = \frac{3}{x_b^4}, \text{ and } \omega_2(aaaa) = \frac{1}{x_a^4}, \omega_2(aabb) = -\frac{2}{x_a^2 x_b^2}, \omega_2(bbbb) = \frac{1}{x_b^4}.$$

The corresponding weighted adjacency matrices are

$$A_{G_1,\omega_1} = \begin{array}{c} \\ a \\ b \end{array} \overset{\begin{array}{ccc} aaa & abb & bbb \end{array}}{\begin{bmatrix} \frac{4}{x_a^4} & 0 & -\frac{4}{x_a x_b^3} \\ 0 & -\frac{12}{x_a x_b^3} & \frac{12}{x_b^4} \end{bmatrix}} \quad \text{and} \quad A_{G_2,\omega_2} = \begin{array}{c} \\ a \\ b \end{array} \overset{\begin{array}{cccc} aaa & aab & abb & bbb \end{array}}{\begin{bmatrix} \frac{4}{x_a^4} & 0 & -\frac{4}{x_a^2 x_b^2} & 0 \\ 0 & -\frac{4}{x_a^2 x_b^2} & 0 & \frac{4}{x_b^4} \end{bmatrix}}.$$

Since $\dim \ker A_{G_1,\omega_1} = 1$, we can conclude from Theorem 5.15 that G_1 is globally g-rigid, whereas we cannot apply Theorem 5.15 to G_2 since $\dim \ker A_{G_2,\omega_2} > 1$. Indeed (G_2, p) cannot be globally g-rigid since the polynomial system $x_a^4 = t_1, x_a^2 x_b^2 = t_2, x_b^4 = t_3$ with variables x_a, x_b and fixed scalars t_1, t_2, t_3 only determines the value of (x_a^2, x_b^2). (On the other hand for (G_1, p) the corresponding polynomial system $x_a^4 = t_1, x_a x_b^3 = t_2, x_b^4 = t_3$ with variables x_a, x_b and fixed scalars t_1, t_2, t_3 does determine the value of (x_a, x_b) up to simultaneous multiplication by a 4-th root of unity).

We also compute the matrix representation H_i of the Gauss map of $\overline{\operatorname{im} f_{g,G_i}}$ according to the technique by Chiantini, Ottaviani, and Vannieuwenhoven [19] for each G_i with $i = 1, 2$:

$$H_1 = \begin{array}{c} \\ a \\ b \end{array} \overset{\begin{array}{cc} a & \quad b \end{array}}{\begin{bmatrix} \frac{12}{x_a^2} & -\frac{12}{x_a x_b} \\ -\frac{12}{x_a x_b} & \frac{12}{x_b^2} \end{bmatrix}} \quad \text{and} \quad H_2 = \begin{array}{c} \\ a \\ b \end{array} \overset{\begin{array}{cc} a & \quad b \end{array}}{\begin{bmatrix} \frac{8}{x_a^2} & -\frac{8}{x_a x_b} \\ -\frac{8}{x_a x_b} & \frac{8}{x_b^2} \end{bmatrix}}.$$

Hence H_1 and H_2 are equal up to scalar multiplication.

5.4 Necessary conditions

In the context of Euclidean rigidity theory, Hendrickson [37] established two necessary conditions for a graph to be globally g-rigid in dimension d. However, we will now provide examples that demonstrate that both of these conditions fail to be necessary for global g-rigidity.

As described in [40] Hendrickson's argument does not extend to the context of matrix completion, and hence does not extend to an arbitrary polynomial map g. Hendrickson's approach involves considering a generic globally g-rigid hyper-framework (G, p) and assuming that removing an edge $e \in E(G)$ leads to a configuration $(G - e, p)$ that is not g-rigid. The key step in Hendrickson's strategy is to prove that the configuration space $f_{g,G}^{-1}(f_{g,G}(p))/\Gamma$ is compact. However, it turns out that relatively often, this configuration space can be unbounded. While it is possible that some configuration spaces might satisfy the required property to continue Hendrickson's strategy, this is not the case in several natural examples. This indicates that redundant rigidity is not necessarily a prerequisite for global g-rigidity in the multilinear case, as demonstrated by the following lemma, providing an infinite family of examples.

Lemma 5.17. *Assume Setup 2.1, and suppose g is multilinear. Then any simple d-valent extension that preserves local g-rigidity preserves global g-rigidity. In other words, if G is globally g-rigid and a simple d-valent extension H of G is locally g-rigid, then H is globally g-rigid.*

Proof. Let G be a globally g-rigid graph and suppose H is obtained from G by a simple d-valent extension by adding a new vertex v. Pick any generic $p \in (\mathbb{F}^d)^{V(H)}$. We will show that (H, p) is globally g-rigid.

For this, choose any $q \in f_{g,H}^{-1}(f_{g,H}(p))$. Since G is globally g-rigid, we may assume $q(u) = p(u)$ for all $u \in V(G)$. Since g is multilinear, $f_{g,H}(p) = f_{g,G}(q)$ leads to the equality:

$$\langle q(v) \nabla g(q(e_i - v)) \rangle = \langle p(v), \nabla g(p(e_i - v)) \rangle,$$

for all hyperedges $e_1, \ldots, e_d$ incident to v. Since the d-valent extension is simple, $e_i - v$ does not contain v, we have that $q(u) = p(u)$ for any $u \in V(G)$ and so:

$$\langle (p(v) - q(v)), \nabla g(p(e_i - v)) \rangle = 0,$$

for $i = 1, \ldots, d$. From the fact that the extension preserves local g-rigidity and Lemma 4.2, $\{\nabla g(p(e_i - v)) : i = 1, \ldots, d\}$ is a basis of $\mathbb{F}^d$. So we get $p(v) = q(v)$, implying that $p = q$ and (G, p) is globally g-rigid. $\square$

By repeatedly applying the lemma, we can construct an infinite family of hypergraphs that exhibit both minimally local g-rigidity and global g-rigidity when g is a multilinear polynomial map. There are also infinite families of hypergraphs which are locally g-rigid but not globally g-rigid.

We next consider Hendrickson's second necessary condition which is in terms of graph connectivity. We will present an analogous condition where the level of connectivity depends strongly on the map g.

Let $G = (V, E)$ be a k-uniform hypergraph. A subset $X \subset V$ is called a *separator* if $G - X$ is disconnected. We say that G is r-connected if $|V(G)| \geq r + 1$ and it has no separator of size less than r.

Lemma 5.18. *Assume Setup 2.1 and $|V| \geq n_{\Gamma_g} + 1$. If G is locally g-rigid, then G is n_{Γ_g}-connected.*

Proof. For a point configuration $q \in (\mathbb{F}^d)^n$, let $\mathrm{stab}_{\mathfrak{g}}(q) = \{\gamma \in \mathfrak{g} : \gamma \cdot q = 0\}$, where $\mathfrak{g}$ is the Lie algebra of Γ_g. Clearly $\mathrm{stab}_{\mathfrak{g}}(q)$ is a linear subspace of $\mathfrak{g}$. By definition of n_{Γ_g}, $\mathrm{stab}_{\mathfrak{g}}(q) = \{0\}$ if and only if $n \geq n_{\Gamma_g}$.

Let (G, p) be a generic framework of G, and suppose that G has a separator of size less than n_{Γ_g}. Since $|V(G)| \geq n_\Gamma + 1$, G has a separator X of size equal to $n_{\Gamma_g} - 1$. Then we can take $V_1, V_2 \subseteq V(G)$ such that $V_i \setminus X \neq \emptyset (i = 1, 2), V(G) = V_1 \cup V_2, X = V_1 \cap V_2$, and there is no hyperedge intersecting $V_1 \setminus X$ and $V_2 \setminus X$. Moreover, since $|X| \geq n_{\Gamma_g} - 1$, we have that $|V_i| \geq n_{\Gamma_g}$.

Since $|X| < n_{\Gamma_g}$, $\mathrm{stab}_{\mathfrak{g}}(p_{|X}) \neq \{0\}$. Also, since p is generic and $V_2 \setminus X \neq \emptyset$, we have $\mathrm{stab}_{\mathfrak{g}}(p_{|V_2}) \subsetneq \mathrm{stab}_{\mathfrak{g}}(p_{|X})$. Pick any $\gamma \in \mathrm{stab}_{\mathfrak{g}}(p_{|X}) \setminus \mathrm{stab}_{\mathfrak{g}}(p_{|V_2})$, and define $\dot{p} \in (\mathbb{F}^d)^{V(G)}$ by

$$\dot{p}(v) = \begin{cases} 0 & (v \in V_1 \setminus X) \\ 0 = \gamma \cdot p(v) & (v \in X) \\ \gamma \cdot p(v) & (v \in V_2 \setminus X). \end{cases}$$

From the fact that $\gamma \in \mathfrak{g}$ and there is no hyperedge intersecting $V_1 \setminus X$ and $V_2 \setminus X$, $\dot{p}$ is an infinitesimal g-motion of (G, p). From the definition, one can check that $\dot{p}$ is indeed non-trivial. $\square$

The rank 1 tensor completion problem shows that the connectivity bound may be tight in general.

6 Multipartite Rigidity Model

In some applications, we address identifiability using k-partiteness in k-partite hypergraphs. We now introduce the related rigidity concept and analyse the unique tensor completion problem as an example.

6.1 Multirigidity

A hypergraph $G = (V, E)$ is *k-partite* if there is a partition $\{V_1, \dots, V_k\}$ of V into k nonempty subsets V_i such that $|V_i \cap e| = 1$ for any $e \in E$ and $i = 1, \dots, k$. A 2-partite hypergraph is a bipartite graph.

Suppose (G, p) is a k-partite hyper-framework. When we denote a hyperedge e as $e = (v_1, \dots, v_k) \in E$, we mean that $v_i \in V_i$ for $i = 1, \dots, k$. As in Setup 2.1, for a polynomial map $g : (\mathbb{F}^d)^k \to \mathbb{F}$, we define the g-measurement map $f_{g,G} : (\mathbb{F}^d)^V \to \mathbb{F}^E$. Since G is k-partite, g may not be symmetric or anti-symmetric.

The main difference from the rigidity model given in Section 2 occurs in the action of the affine group to $(\mathbb{F}^d)^k$. Suppose the general affine group $\mathrm{Aff}(d, \mathbb{F})$ acts on $\mathbb{F}^d$ by $\gamma \cdot x = Ax + t$ for $\gamma = (A, t)$, as before. In the multipartite model, we consider the direct product $\mathrm{Aff}(d, \mathbb{F})^k$ of k copies of $\mathrm{Aff}(d, \mathbb{F})$, and consider the component-wise action of $\mathrm{Aff}(d, \mathbb{F})^k$ to $(\mathbb{F}^d)^k$. The induced action on $g : (\mathbb{F}^d)^k \to \mathbb{F}$ is given by $\gamma \cdot g(x_1, \dots, x_k) = g(\gamma_1^{-1} \cdot x_1, \dots, \gamma_k^{-1} x_k)$ for $x_1, \dots, x_k \in \mathbb{F}^d$ and $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathrm{Aff}(d, \mathbb{F})^k$. The stabilizer of g is denoted by $\Gamma_g^\times$.

Since in this k-partite model the underlying vertex set V is assumed to be partitioned into $\{V_1, \dots, V_k\}$, we can also define the action of $\mathrm{Aff}(d, \mathbb{F})^k$ to $(\mathbb{F}^d)^V$ such that $(\gamma \cdot p)(v) = \gamma_i \cdot p(v)$ for each $v \in V_i$ for $p \in (\mathbb{F}^d)^V$ and $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathrm{Aff}(d, \mathbb{F})^k$.

Definition 6.1. We say that (G, p) is *globally g-multirigid* if for any $q \in f_{g,G}^{-1}(f_{g,G}(p))$ there is $\gamma \in \Gamma_G^\times$ such that $q = \gamma \cdot p$. We say that (G, p) is *locally g-multirigid* if there is an open neighbourhood N of p in $(\mathbb{F}^d)^V$ (in the Euclidean topology) such that for any $q \in f_{g,G}^{-1}(f_{g,G}(p)) \cap N$ there is $\gamma \in \Gamma_G^\times$ such that $q = \gamma \cdot p$.

Example 6.2. Let G be 2-uniform (i.e., a graph) and $g(x, y) = \langle x, y \rangle$ for $x, y \in \mathbb{F}^d$. In the model in Section 2, the stabilizer of g is $O(d)$ and its Lie algebra is $\binom{d}{2}$-dimensional. On the other hand, in the k-partite model, the stabilizer of g is $\{(A, A^{-\top}) : A \in \mathrm{GL}(d, \mathbb{F})\}$, and its Lie algebra is d^2-dimensional.

We define the infinitesimal g-multirigidity in the same manner as described in Section 3.2. To do so, we denote $d_{\Gamma_g^\times}$ for the dimension of the variety $\Gamma_g^\times$, and let $\mathrm{triv}_{\Gamma_g^\times}(p)$ be the space of trivial infinitesimal motions of (G, p). If $|V_i|$ is sufficiently large for all i, then $\dim \mathrm{triv}_{\Gamma_g^\times}(p) = d_{\Gamma_g^\times}$. The minimum size of V_i that guarantees this property is denoted by $n_{\Gamma_g^\times}$. Specifically, let

$$n_{\Gamma_g^\times} := \min\{|V_i| : \dim \mathrm{triv}_\Gamma(p) = d_{\Gamma_g^\times} \text{ for some } p : V \to \mathbb{F}^d\}.$$

All the materials in Section 3 can be extended to the multipartite rigidity model. For example, the following proposition can be shown easily.

Proposition 6.3. *Let $g : (\mathbb{F}^d)^k \to \mathbb{F}$ be a polynomial map and $G = (V, E)$ be a k-partite hypergraph with $|V_i| \geq n_{\Gamma_g^\times}$. Then the following are equivalent:*

- *(G, p) is locally g-multirigid for some generic p.*
- *(G, p) is infinitesimally g-multirigid for some generic p.*
- *The rank of $\mathrm{J}f_{g,G}(p)$ is equal to $d|V| - d_{\Gamma_g^\times}$ for some generic p.*
- $\dim(\overline{\mathrm{im} f_{g,G}(p)}) = d|V| - d_{\Gamma_g^\times}$.

Let $K^k_{n_1,\dots,n_k}$ be the k-uniform complete hypergraph with $|V_i| = n_i$ and $n = n_1 + \cdots + n_k$. The generic g-multirigidity matroid $\mathcal{M}^{\times}_{g,n_1,\dots,n_k}$ is defined on $E(K^k_{n_1,\dots,n_k})$ whose independence is defined by the row independence in $\mathrm{J}f_{g,K^k_{n_1,\dots,n_k}}(p)$ at a generic p. If g is symmetric or anti-symmetric, $\mathcal{M}^{\times}_{g,n_1,\dots,n_k}$ is the restriction of the g-rigidity matroid $\mathcal{M}_{g,n}$ of K^k_n to $K^k_{n_1,\dots,n_k}$ by regarding $K^k_{n_1,\dots,n_k}$ as a subgraph of K^k_n.

6.2 Tensor Completion

A primary example of g-multirigidity occurs in the rectangular tensor completion problem. Let $W_1, \dots, W_k$ be a collection of finite dimensional vector spaces over $\mathbb{C}$. We denote $W_1 \otimes \cdots \otimes W_k$ for the set of all order k tensors of dimensions $n_1 = \dim(W_1), \dots, n_k = \dim(W_k)$. We fix a basis for each W_i, and assume that each $T \in W_1 \otimes \cdots \otimes W_k$ is represented by a k-dimensional array of numbers in $\mathbb{C}$. In particular, any tensor can be written as

$$T = \sum_{i=1}^{d} x_i^1 \otimes x_i^2 \otimes \cdots \otimes x_i^k \tag{29}$$

for some vectors $x_i^j \in W_j$ and $\lambda_1, \dots, \lambda_r \in \mathbb{F}$. The smallest possible d for which we can write T in the form of Equation (29) is called the *(CP) rank* of T.

As in the case of matrix completion or symmetric tensor completion, we can ask the low rank tensor completion problem as the problem of filling the missing entries of a given partially-filled matrix. Let $V_i = [n_i]$ for $1 \leq i \leq k$ and V be the disjoint union of all V_i. We can use a subset E of $V_1 \times \cdots \times V_k$ to represent the known entries in the tensor completion problem. In this manner, we encode the underlying combinatorics of each instance of the completion problem using a k-partite hypergraph (V, E).

As in the symmetric tensor completion, the decomposition in Equation (29) can be converted to a form of an algebraic relation among points in $\mathbb{C}^d$. This gives the following equivalent formulation of the tensor completion problem: Given a k-partite hypergraph $G = (V, E)$ with $E \subseteq V_1 \times \cdots \times V_k$ and $a_e \in \mathbb{C}$ for $e \in E$, find $p : V \to \mathbb{C}^r$ such that

$$\mathbf{1} \cdot \bigodot_{v \in e} p(v) = a_e \qquad \text{for } e \in E. \tag{30}$$

Note that this equation is the same as Equation (5), and we can recast the rectangular tensor completion problem as a special case of the symmetric tensor completion when the underlying hypergraph is k-partite.

The unique completion problem can be formulated as the g-multirigidity problem using the same polynomial map g as that in the symmetric tensor case, that is, the sum of d copies of the product function h_{prod}. The key difference from the symmetric tensor case is the stabilizer $\Gamma_g^{\times}$. When $k \geq 3$, $\Gamma_g^{\times}$ is the set of k-tuples $(D_1\Sigma, D_2\Sigma, \dots, D_k\Sigma)$ such that each D_i is a diagonal matrix of size d with $\prod_{i=1}^k D_i = I_d$ and Σ is a permutation matrix. Then $d_{\Gamma_g^{\times}} = d(k-1)$ and $n_{\Gamma_g^{\times}} = 1$.

The fact that $d_{\Gamma_g^{\times}} = d(k-1)$, in particular, implies that the rank of the g-multirigidity matroid of $K^r_{n_1,\dots,n_k}$ is at most $d(n_1 + \cdots + n_k) - d(r-1)$. An obvious question is whether the equality always holds (equivalently, whether $K^r_{n_1,\dots,n_k}$ is locally g-multirigid). This question is equivalent to determining the non-defectivity of Segre varieties. Indeed, since $g = dh_{\text{prod}}$ is the sum of d copies of h_{prod}, $\overline{\mathrm{im} f_{dh_{\text{prod}},G}}$ is the d-secant of $\overline{\mathrm{im} f_{h_{\text{prod}},G}}$. When $G = K^k_{n_1,\dots,n_k}$, $\overline{\mathrm{im} f_{h_{\text{prod}},K^k_{n_1,\dots,n_k}}}$ is the affine cone of the Segre variety with parameter $(n_1, \dots, n_k)$ whose dimension is $n_1 + \cdots + n_k - (k-1)$. Therefore, $K^k_{n_1,\dots,n_k}$ is locally g-multirigid if and only if $\overline{\mathrm{im} f_{h^{\text{prod}},K^k_{n_1,\dots,n_k}}}$ is not d-defective.

Unlike the Veronese case, the non-defectivity problem of Segre varieties has not been fully resolved, and it remains an open question in algebraic geometry. Although there is a long-standing conjecture that aims to provide a complete answer, a conclusive proof is still lacking, see, e.g. [8].

7 Applications

In this section, we provide two concrete applications of our theory.

7.1 Random projection of Secant varieties along coordinate axis

Let $\mathcal{V}$ be an affine variety of dimension m. By the Noether normalization theorem, projecting $\mathcal{V}$ onto a random t-dimensional space typically results in a Zariski closure with dimension $\min\{m, t\}$. Thus, $t = m$ is the threshold to preserve dimension under random projection. However, as shown in Section 3.5, this is not the case for orthogonal projections along coordinate axes. Here, we estimate the threshold dimension t for which a random axis-aligned projection to a t-dimensional subspace preserves the dimension.

Let us recall the g-rigidity formulation. Let G be a k-uniform hypergraph and $g : (\mathbb{R}^d)^k \to \mathbb{R}$ be a polynomial map. Then, our ground variety $\mathcal{V}$ is $\overline{\operatorname{im} f_{g,G}}$ and we are interested in whether the dimension is preserved under a random axis-parallel projection. This corresponds to checking local g-rigidity in the Erdös-Rényi random subgraph model $\mathbb{G}_{n,t}$ of G, i.e., a probability distribution over the subgraphs of G defined by selecting each hyperedge independently with probability t.

For a square matrix A, let $\lambda_{\min}(A)$ denote the smallest nonzero eigenvalue of A. For a matrix A, let $\|A\|_{2,\infty}$ denote the maximum Euclidean norm of the rows of A.

Following the idea of d-dimensional algebraic connectivity in [43], we define the hypergraph parameter $a_g(G)$ by

$$a_g(G) := \sup_{p \in (\mathbb{R}^d)^n} \frac{\lambda_{\min}(\mathrm{J}f_{g,G}(p)^\top \mathrm{J}f_{g,G}(p))}{\|\mathrm{J}f_{g,G}(p)\|_{2,\infty}^2}$$

for each k-uniform hypergraph G with n vertices. The next theorem gives a sufficient condition for local g-rigidity of random subgraphs.

Theorem 7.1. *Let G be a k-uniform hypergraph with n vertices, $g : (\mathbb{R}^d)^k \to \mathbb{R}$ be a polynomial map, and c be any number with $c > 1$. Let $\mathbb{G}_{n,t}$ be the random subgraph model of G, and $H \sim \mathbb{G}_{n,t}$. If*

$$1 \geq t > \frac{\log(cdn)}{a_g(G)}, \tag{31}$$

then with probability at least $1 - \frac{1}{c}$, the projection π_H preserves the dimension of $\overline{\operatorname{im} f_{g,G}}$.

Proof. This is an adaptation of the argument in [43] for proving a corresponding statement for Euclidean rigidity, and it is a direct application of the following matrix Chernoff bound.

Let E be a finite set, and for each $e \in E$, let X_e be a positive semidefinite $m \times m$ matrix. The matrix Chernoff bound implies that if

$$1 \geq t \geq \frac{h \log(rc)}{\lambda_{\min}(\sum_{e \in E} X_e)}$$

with $r = \operatorname{rank} \sum_{e \in E} X_e$ and $h = \max_{e \in E} \lambda_{\max}(X_e)$, then

$$\operatorname{rank} \sum_{e \in E(t)} X_e = \operatorname{rank} \sum_{e \in E} X_e$$

holds with probability $1 - \frac{1}{c}$ for a random subset $E(t)$ obtained by taking each $e \in E$ independently with probability t. See [43] and references therein.

We apply this in the case when $E = E(G)$ and $X_e = r_e^\top r_e$ for $e \in E(G)$, where r_e denotes the row vector of e in $\mathrm{J}f_{g,G}(p)$ with generic p. Then,

$$r \leq dn$$
$$h = \max_e \lambda_{\max}(r_e^\top r_e) = \max_e \|r_e\|_2^2 = \|\mathrm{J}f_{g,G}(p)\|_{2,\infty}^2$$
$$\lambda_{\min}\left(\sum_{e \in E} X_e\right) = \lambda_{\min}\left(\sum_{e \in E} r_e^\top r_e\right) = \lambda_{\min}\left(\mathrm{J}f_{g,G}(p)^\top \mathrm{J}f_{g,G}(p)\right).$$

Thus, by the matrix Chernoff bound, if

$$1 \geq t \geq \frac{\|\mathrm{J}f_{g,G}(p)\|_{2,\infty}^2 \log(dnc)}{\lambda_{\min}\left(\mathrm{J}f_{g,G}(p)^\top \mathrm{J}f_{g,G}(p)\right)}, \tag{32}$$

then $\operatorname{rank}\left(\sum_{e\in E} r_e^\top r_e\right) = \operatorname{rank}\left(\sum_{e\in E(t)} r_e^\top r_e\right)$ holds with probability at least $1-\frac{1}{c}$. In particular, $\operatorname{rank}\mathrm{J}f_{g,G} = \operatorname{rank}\mathrm{J}f_{g,G}^\top \mathrm{J}f_{g,G} = \operatorname{rank}\left(\sum_{e\in E} r_e^\top r_e\right)$ and $\operatorname{rank}\mathrm{J}f_{g,H} = \operatorname{rank}\mathrm{J}f_{g,H}^\top \mathrm{J}f_{g,H} = \operatorname{rank}\left(\sum_{e\in E(t)} r_e^\top r_e\right)$, imply that

$$\operatorname{rank}\mathrm{J}f_{g,G} = \operatorname{rank}\mathrm{J}f_{g,H}(p) \tag{33}$$

for $H \sim \mathbb{G}_{n,t}$ with probability at least $1-\frac{1}{c}$.

Note that this holds for any choice of p. Hence, if (31) holds, there is some p satisfying (32) and thus (33). Therefore (33) holds for all generic p by lower semicontinuity of the rank. Thus, π_H preserves the dimension of $\overline{\operatorname{im} f_{g,G}}$ with probability at least $1-\frac{1}{c}$. □

By Theorem 7.1, the question now reduces to obtaining a lower bound for $a_g(G)$. This can be done by a quantitative extension of the proof of Theorem 4.6. Since this part becomes technically more involved, we explain the idea by focusing on the case when $\mathcal{V}$ is the d-secant of the balanced Segre variety.

Let us recall the setting of Section 6.2. Let $K_{n,\dots,n}^k$ be the complete k-partite hypergraph with $|V_i| = n$, and let $h_{\text{prod}} : \mathbb{R}^k \to \mathbb{R}$ be the product function. Then $\mathcal{S}_n := \overline{\operatorname{im} f_{h_{\text{prod}},K_{n,\dots,n}^k}}$ is the affine cone of the Segre variety with balanced parameter $(n,\dots,n)$ and dimension $kn-(k-1)$. We denote by $\operatorname{Sect}_d(\mathcal{S}_n)$ the d-secant of $\mathcal{S}_n$.

Lemma 7.2. *Let n, d, k be positive integers with $k \geq 3$ and $n \geq d$. Then,*

$$a_{dh_{\text{prod}}}(K_{n,\dots,n}^k) \geq \frac{n^{k-1}}{d^{k-1}k}.$$

The proof of Lemma 7.2 is given at the end of this subsection.

Combining Theorem 7.1 and Lemma 7.2, we have the following corollary.

Corollary 7.3. *Let n, k, d be positive integers with $n \geq d$, and let c be any number with $c > 1$. Let H be taken from the random subgraph model of $K_{n,\dots,n}^k$ with probability parameter t. If*

$$1 \geq t \geq \frac{kd^{k-1}\log(cdkn)}{n^{k-1}},$$

then with probability at least $1-\frac{1}{c}$, the projection π_H preserves the dimension of $\operatorname{Sect}_d(\mathcal{S}_n)$, i.e., H is locally dh_{prod}-multirigid.

Corollary 7.3 implies that, among n^k edges in $K_{n,\dots,n}^k$, only $O(n\log n)$ hyperedges are enough to make a random sub-hypergraph dh_{prod}-multirigid. This bound is asymptotically tight (in n) since G has an isolated vertex with high probability if $t < \frac{\log n}{n^{k-1}}$.

The low-rank tensor completion problem is a major topic in machine learning, and there are various results concerning the sample complexity of tensor recovery in various settings. Corollary 7.3 implies that there is a finite number of low-rank completions if $O(n\log n)$ random entries are observed in the generic completion problem. A follow-up paper [36] by the fourth author and Hamaguchi generalizes this to the unique identifiability. To the best of our knowledge, this is a new information theoretic bound in this setting.[1]

[1]It should be noted that the sample complexity differs substantially across problem settings. In typical machine learning settings, recovery is usually only approximate. For example, in [32], it is claimed that $O(n)$ random entries suffice for recovery of low-rank tensors, whereas a simple coupon collector argument implies that $\Omega(n\log n)$ random entries are necessary for exact recovery. See, e.g., our follow-up paper [36] for more details on the existing bounds in different models.

Proof of Lemma 7.2. The proof is based on a discussion with Kota Nakagawa. For simplicity of notation, denote $V = V(K^k_{n,\ldots,n})$ and $f = f_{dh_{\mathrm{prod}},K^k_{n,\ldots,n}}$. For simplicity of exposition, we also assume that n is an integer multiple of d.

Our goal is to prove the existence of $p : V \to \mathbb{R}^d$ such that

(i) $\operatorname{rank} \mathrm{J}f(p) = d(kn - (k-1))$,

(ii) $\lambda_{\min}(\mathrm{J}f(p)^\top \mathrm{J}f(p)) \geq \frac{n^{k-1}}{d^{k-1}}$, and

(iii) each row of $\mathrm{J}f(p)$ has Euclidean norm at most $\sqrt{k}$ as a vector.

Since $K^k_{n,\ldots,n}$ consists of k disjoint vertex sets of size n, we denote them by $V_1, V_2, \ldots, V_k$. For $i \in [k]$, we partition V_i into d sets $V_{i,j}$ $(j = 1, \ldots, d)$ of size n/d. We define $p : V \to \mathbb{R}^d$ by,

$$p(v) = e_j \qquad \text{for } v \in V_{i,j} \text{ with } i \in [k] \text{ and } j \in [n],$$

where e_j is the unit vector in $\mathbb{R}^d$ whose i-th entry is one and other entries are zero. We show the properties of the statement for $\mathrm{J}f(p)$. The idea of the proof is the same as that of Theorem 4.6. Since the definition of p above is very special, the entries of $\mathrm{J}f(p)$ is well structured as in Claim 1 and we will see a block-diagonalized form in $\mathrm{J}f(p)^\top \mathrm{J}f(p)$ using the structures of the entries. Then we will analyse each block by applying a known fact from spectral graph theory.

For such an analysis, we first partition a subset of $E(K^k_{n\ldots,n})$ into $2d$ subsets as follows: for $j = 1, \ldots, d$,

$$A_j := \{e = (v_1, v_2, \ldots, v_k) \in E(K^k_{n\ldots,n}) : v_i \in V_{i,j} \text{ for every } i \in [k]\}$$
$$B_j := \{e = (v_1, v_2, \ldots, v_k) \in E(K^k_{n\ldots,n}) : v_i \in V_{i,j} \text{ for exactly } k-1 \text{ indices } i \text{ in } [k]\}.$$

Note that all sets A_j and B_j are disjoint. As in the proof of Theorem 4.6, we use $b(e, v, k)$ to denote the entry of $\mathrm{J}f(p)$ indexed by a hyperedge e, vertex $v \in V$, and a coordinate $k \in [d]$.

Claim 3. *Each entry of* $\mathrm{J}f(\boldsymbol{p})$ *is either* 0 *or* 1. *For* $e \in E(K^k_{n\ldots,n})$, $v \in V$, *and* $l \in [d]$, $b(e, v, l) = 1$ *holds if and only if either one of the followings holds:*

(i) $e \in A_j$ *for some* $j \in [d]$, $v \in e \cap V_{i,j}$ *for some* $i \in [k]$, *and* $l = j$;

(ii) $e \in B_j$ *for some* $j \in [d]$, $v \in e \cap V_{i,j'}$ *for some* $i \in [k]$ *and* $j' \in [d] \setminus \{j\}$, *and* $l = j$.

Proof. By definition of $\mathrm{J}f(p)$, the entry $b(e, v, l)$ is the product of the l-th coordinates of $p(u)$ over all $u \in e \setminus \{v\}$ if $v \in e$, and zero otherwise. Since each coordinate of $p(v)$ is either one or zero, $b(e, v, l)$ is either one or zero. The latter claim can be checked similarly by using the definition of p. ☐

Recall that each row of $\mathrm{J}f(p)$ is indexed by a hyperedge and each column of $\mathrm{J}f(p)$ is indexed by a pair (v, j) of $v \in V$ and $j \in [d]$, and hence each entry of $\mathrm{J}f(p)^\top \mathrm{J}f(p)$ is indexed by a pair $((u, j), (v, j'))$ of such pairs. For $j = 1, \ldots, d$, let

$$I_j = \{(v, j) : v \in V_{i,j} \text{ for some } i \in [k]\}, \text{ and} \tag{34}$$
$$I'_j = \{(v, j) : v \in V_i \setminus V_{i,j} \text{ for some } i \in [k]\}. \tag{35}$$

The sets I_j, I'_j $(j = 1, \ldots, d)$ are mutually disjoint, partitioning $V \times [d]$, and $|I_j| = k\frac{n}{d}$ and $|I'_j| = k(n - \frac{n}{d})$.

Claim 4. $\mathrm{J}f(p)^\top \mathrm{J}f(p)$ *has the following block-diagonalized form:*

$$\mathrm{J}f(p)^\top \mathrm{J}f(p) = \begin{array}{c} \\ I_1 \\ I'_1 \\ \vdots \\ I_k \\ I'_k \end{array} \begin{array}{c} \begin{array}{ccccc} I_1 & I'_1 & \cdots & I_k & I'_k \end{array} \\ \begin{bmatrix} Q_1 & 0 & \ldots & \ldots & 0 \\ 0 & R_1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & Q_k & 0 \\ 0 & \ldots & & 0 & R_k \end{bmatrix} \end{array}.$$

Moreover, in Q_j (resp. R_j), the value of the entry indexed by $((u,j),(v,j))$ is equal to the number of hyperedges in A_j (resp. B_j) that contain u and v.

Proof. For each $e \in E(K^k_{n,\dots n})$, let R_e be the row vector of $\mathrm{J}f(p)$ associated with e, and let $X_e = R_e^\top R_e$ by regarding R_e as a matrix of size $1 \times dn$. Note that $\mathrm{J}f(p)^\top \mathrm{J}f(p) = \sum_{e\in E(K^k_{n,\dots n})} X_e$. Hence, it is enough to check the contribution of each X_e to each entry of $\mathrm{J}f(p)^\top \mathrm{J}f(p)$.

Claim 3 implies that X_e is a $0-1$ matrix. Claim 3(i) further implies that, if $e = (v_1, v_2, \dots, v_k) \in A_j$, then R_e has nonzero entries only at the coordinates indexed by $(v_1, j), (v_2, j), \dots, (v_k, j)$. Hence, X_e contributes to increase the entry indexed by $((u,j),(v,j))$ for each $u, v \in e$ by one. This is the reason why, in Q_j the value of the entry indexed by $((u,j),(v,j))$ is equal to the number of hyperedges in A_j that contain both u and v. A similar argument using Claim 3(ii) also provides the entries of R_j. □

Observe that, for $(u,j),(v,j) \in I'_j$, the number of hyperedges in B_j that contain u and v is $\left(\frac{n}{d}\right)^{k-1}$ if $u = v$, and it is zero otherwise (i.e., when $u \neq v$). So Claim 4 implies that $R_j = \frac{n^{k-1}}{d^{k-1}} I$, where I denotes the identity matrix (of an appropriate size). Hence, by the block-diagonal form in Claim 4, in order to show that $\operatorname{rank} \mathrm{J}f(p) = d(kn - (k-1))$ and $\lambda_{\min}(\mathrm{J}f(p)^\top \mathrm{J}f(p)) \geq \frac{n^{k-1}}{d^{k-1}}$, it suffices to show that $\dim\ker Q_j = k-1$ and $\lambda_{\min}(Q_j) \geq \frac{n^{k-1}}{d^{k-1}}$ for each $j = 1, \dots, d$.

Now we analyse Q_j based on Claim 4. By counting the number of hyperedges in A_j that contain two vertices u and v, we see that Q_j is equal to the adjacency matrix of the edge-weighted complete k-partite graph on $V_{1,j} \cup V_{2,j} \cup \cdots \cup V_{k,j}$ with a loop at each vertex, where the edge-weight w is defined by

$$w(uv) = \begin{cases} \left(\frac{n}{d}\right)^{k-1} & (u = v \in V_{i,j} \text{ for some } i \in [k]) \\ \left(\frac{n}{d}\right)^{k-2} & (u \in V_{i,j}, v \in V_{i',j} \text{ for some } i, i' \in [k], i \neq j) \end{cases}$$

for each edge uv. In other words,

$$Q_j = \left(\frac{n}{d}\right)^{k-2} A_{K_{n/d,\dots,n/d}} + \left(\frac{n}{d}\right)^{k-1} I$$

where $A_{K_{n,\dots,n}}$ is the adjacency matrix of the (non-weighted) complete k-partite graph with n/d vertices on each side and I denotes the identity matrix of size $\frac{kn}{d}$. It is a well-known fact [15] from spectral graph theory that the spectrum of $A_{K_{n/d,\dots,n/d}}$ is $[(\frac{kn}{d} - \frac{n}{d})^1, 0^{k(\frac{n}{d}-1)}, (-\frac{n}{d})^{k-1}]$, where a^b means a is an eigenvalue of multiplicity b. So the spectrum of Q_j is $[\left(k\frac{n^{k-1}}{d^{k-1}}\right)^1, \left(\frac{n^{k-1}}{d^{k-1}}\right)^{k(n-1)}, (0)^{k-1}]$. Thus, $\dim\ker Q_j = k-1$ and $\lambda_{\min}(Q_j) = \frac{n^{k-1}}{d^{k-1}}$. This completes the proof. □

7.2 Identifiability of projections of p-Cayley-Menger varieties and ℓ_p-norm rigidity

The aim of this subsection is to demonstrate the impact of the link between rigidity and identifiability observed in Section 5 on graph rigidity. Since this application requires a substantially longer discussion, the proof is given in a separate paper [61].

In Section 2.3, we looked at graph rigidity in ℓ_p-space, where instead of conventional Euclidean distances we are concerned with ℓ_p-distances between points in a real vector space. An extensive study has been done for ℓ_p local rigidity, but ℓ_p global rigidity is not yet well understood [27, 29]. A major open problem in this context is whether global rigidity in ℓ_p-space is a generic property of graphs as in the case of Euclidean global rigidity by Gortler-Hearly-Thurston [33], see a discussion in Section 5.

The difficulty to extend techniques in Euclidean global rigidity is the lack of an obvious counterpart definition of stress matrices. The theory of stress matrices, developed by Connelly [22, 23], is currently the central tool in Euclidean global rigidity analysis, and it heavily relies on the fact that the squared Euclidean distance is quadratic. Interestingly, t-tangentially weakly non-defectiveness discussed in Section 5.2 provides a counterpart tool. To see this, we shall introduce the ℓ_p-analogue of the Cayley-Menger variety. For a positive integer p, define $h_p : \mathbb{C}^2 \to \mathbb{C}$ by $h_p(x_1, x_2) = (x_1 - x_2)^p$. When $p = 2$ and the domain is real, $f_{h_2,K_n}(x)$ is the list of square Euclidean distances among n points $x = (x_1, x_2, \ldots, x_n)$ on a line, and the image of f_{h_2,K_n} is called the *Cayley-Menger variety.* Motivated by this, we call the image of f_{h_p,K_n} the *p-Cayley-Menger variety* $\mathcal{CM}_n^p$.

Let g_p be the sum of d copies of h_p. If p is a positive even integer and the domain is restricted to real numbers, then f_{g_p,K_n} is the measurement map that outputs the list of p-th powered ℓ_p-distances in d-dimensional real vector space. Hence, the global g_p-rigidity of a d-dimensional framework (G, p) in our terminology is equivalent to the global rigidity of (G, p) in d-dimensional ℓ_p-space.

Since g_p is the sum of d copies of h_p, one may try to apply Corollary 5.5 to show the global g_p-rigidity of generic frameworks. To apply Corollary 5.5, we need to understand when $\overline{\pi_G(\mathcal{CM}_n^p)}$ is d-tangentially weakly nondefective and when G is globally h_p-rigid. It is a folklore fact that global rigidity in 1-dimensional ℓ_p-space is characterised by the 2-connectivity of the underlying graph, which in turn implies that G is globally h_p-rigid if and only if G is 2-connected. In [61], Sugiyama and the fourth author further gave a graph theoretical characterisation of the 2-tangentially weak non-defectiveness of $\overline{\pi_G(\mathcal{CM}_n^p)}$.

Theorem 7.4 (Sugiyama and Tanigawa [61])**.** *Let p be an even positive integer with $p \neq 2$, $n \geq 3$ a positive integer, and G a connected graph with n vertices. Then the following are equivalent:*

(i) *Some/every generic 2-dimensional framework of G is globally rigid in the ℓ_p-plane.*

(ii) *$\overline{\pi_G(\mathcal{CM}_n^p)}$ is 2-identifiable.*

(iii) *$\overline{\pi_G(\mathcal{CM}_n^p)}$ is 2-tangentially weakly nondefective.*

(iv) *G is 2-connected and $G - e$ contains two edge-disjoint spanning trees for every $e \in E(G)$.*

(v) *G is 2-connected and $G - e$ is locally rigid in the ℓ_p-plane for every $e \in E(G)$.*

To our knowledge, this purely graph-theoretical characterisation of the identifiability of an orthogonal projection of a variety is new. Also, the theorem solves the generic global rigidity problem in ℓ_p-planes.

We should remark that the proof of Theorem 7.4 relies on sophisticated graph theoretical techniques from graph rigidity theory. In particular, the proof of (iv) $\Rightarrow$ (iii) uses the inductive construction due to Dewar, Hewetson and Nixon [28], who showed that a graph satisfying the combinatorial condition (iv) can be built up from a small graph by recursively applying two local graph-operations, called K_4^--extension and generalised vertex splitting. Sugiyama and Tanigawa [61] have confirmed that each local graph-operation preserves the 2-tangentially weakly nondefective of $\overline{\pi_G(\mathcal{CM}_n^p)}$, which implies (iv) $\Rightarrow$ (iii). The proof of (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is done by applying a general tool from algebraic geometry explained in Section 5.2. Finally, (i) $\Rightarrow$ (v) and (v) $\Rightarrow$ (iv) have been already shown in [28] and [45], respectively.

We believe Theorem 7.4 is valid in d-dimensions. By the theory in Section 5, the proofs of (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (v) in Theorem 7.4 still work in dimension d. (v) $\Rightarrow$ (iv) also follows from the well-known Maxwell count argument. Hence extending the theorem to d-dimensions reduces to the following conjecture.

Conjecture 7.5. *Let p be an even positive integer with $p \neq 2$, $n \geq 3$ be a positive integer, and G be a connected graph with n vertices. Suppose G is 2-connected and $G - e$ contains d edge-disjoint spanning trees for every $e \in E(G)$. Then $\overline{\pi_G(\mathcal{CM}_n^p)}$ is d-tangentially weakly nondefective.*

8 Conclusion

We have introduced a far-reaching generalisation of graph rigidity theory to the broader setting of point identifiability, encompassing diverse applications. Since our first arXiv submission in 2023, the paper has inspired the following substantial developments:

- The connection between global rigidity and identifiability of algebraic varieties established in Section 5 broadens the applicability of Connelly's global rigidity theory. Two follow-up papers [61, 36] exploit this connection in the contexts of L_p-rigidity and tensor completion, respectively.

- Our graph-theoretical formulation can be used to express several problems in algebraic statistics in terms of random graphs or random hypergraphs. Through this formulation, follow-up work [36] applies techniques for proving rigidity of random graphs to identify the sharp sample complexity for low-rank tensor recovery.

- Expanding on the results in Subsection 3.6 for unirational varieties, [51] explores the relationship between the matroid of an embedded join $X+Y$ and the matroids of X and Y. As in the unirational case, $M(X+Y)$ is a weak image of $M(X) \vee M(Y)$. Additionally [51] characterises when equality fails in terms of defective joins of certain coordinate projections.

- The generalise model of rigidity theory we introduced in this paper turns out to be the right one to generalise the famous Erdős distinct distances problem which asks how many distinct distances must exist between a set of n points in the plane. In [26] g-rigidity is shown to be the natural setting to generalise the Erdős distinct distances problem to consider different spaces and metrics, or larger structures of points.

- Garamvölgyi [31] asks whether his technique for identifying graphs from matroids can be extended to hypergraphs via g-rigidity matroids. This question is motivated by the unlabelled rigidity problem, where the underlying graph is to be recovered from a collection of measurements. The analogous question for tensor completion seems practically useful, but to the best of our knowledge we are not aware of earlier work discussing this problem prior to Garamvölgyi.

We conclude the paper by describing a number of open problems and potential extensions of our work.

From a rigidity theoretic perspective, a natural question arises regarding the search for combinatorial characterisations of (global) g-rigidity. A notable instance is presented in Conjecture 3.12, where we propose a potential characterisation of g-rigidity in the case when g is of a generic form. Establishing such combinatorial characterisations would provide insights into the fundamental properties of g-rigidity.

A recent paper [7] by Beers et al. shows that the local identifiability of point configurations from barcodes of the persistence map associated with the Čech filtration is characterised by circumradius rigidity of hypergraphs. This is precisely g-rigidity for the circumradius measurement map. Establishing combinatorial conditions for g-rigidity in this setting would be impactful.

We expect that Theorem 5.13 applies to a wider class of polynomial maps g, beyond those covered in the paper. A notable instance is the case where g is the permanent function, as investigating its rigidity properties could shed light on its connection to other combinatorial structures.

In Section 5 we have shown a sufficient condition for global g-rigidity using weighted adjacency matrices, which is an analogue of Connelly's stress-matrix condition. An important remaining question is to establish a graph theoretical property to guarantee the applicability of the theorem. Another interesting direction is to develop sufficient conditions for (global) g-rigidity in the multipartite model from Section 6.

Corollary 4.9 also has significant application in statistics, in particular in the identifiability problem of Gaussian mixture models. This problem aims to find the minimum number of parameters needed to uniquely determine a statistical model from n sample points. The method of moments, often used for parameter recovery, reduces the identifiability problem to determining the non-defectivity of secants of the corresponding varieties. While this has been done for specific cases (e.g., zero covariance matrices), our results could provide insights into more general scenarios. Specifically, it would be interesting to capture the identifiability of Gaussian mixture models [3, 5], as specific instances of the g-rigidity model.

Acknowledgements

This project grew from discussions at the Fields Institute Thematic Program on Geometric Constraint Systems, Framework Rigidity, and Distance Geometry, and we are grateful to the organizers for bringing us together. F.M. was partially supported by the KU Leuven grant iBOF/23/064, the UiT Aurora project MASCOT, and the FWO grants G0F5921N (Odysseus) and G023721N. A.N. was partially supported by EPSRC grant numbers EP/W019698/1 and EP/X036723/1. S.T. was partially supported by JST ERATO Grant Number JPMJER1903, JST PRESTO Grant Number JPMJPR2126, and JSPS KAKENHI Grant Number 20H05961. S.T. would like to thank Bill Jackson for pointing out a simpler proof of Proposition 5.2 and Kota Nakagawa for stimulating discussion on the g-rigidity of random hypergraphs.

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