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Edge-Length Preserving Embeddings of Graphs Between Normed Spaces

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ABSTRACT

The concept of graph embeddability, initially formalized by Belk and Connelly and later expanded by Sitharam and Willoughby, extends the question of embedding finite metric spaces into a given normed space. A finite simple graph $G = (V, E)$ is said to be (X, Y)-embeddable if any set of induced edge lengths from an embedding of G into a normed space Y can also be realised by an embedding of G into a normed space X. This property, being minor-closed, can be characterized by a finite list of forbidden minors. Following the establishment of fundamental results about (X, Y)-embeddability, we identify sufficient conditions under which it implies independence with respect to the associated rigidity matroids for X and Y. We show that the spaces ℓ_2 and ℓ_∞ serve as two natural extreme spaces of embeddability and discuss (X, ℓ_p)-embeddability for varying p. We provide a complete characterization of (X, Y)-embeddable graphs for the specific case when X is 2-dimensional and Y is infinite-dimensional.

MSC: 05C10, 52A21, 52C25

1 | Introduction

A *realisation* of a (finite simple) graph $G = (V, E)$, with at least one edge, in a real normed linear space $(X, \|\cdot\|_X)$ (referred to simply as X when the context is clear) is a map $p : V \to X$. The *measurement map* of the pair (G, X) is the map

$$f_{G,X} : X^V \to \mathbb{R}^E, (x_v)_{v\in V} \mapsto (\|x_v - x_w\|_X)_{vw\in E},$$

which sends each realisation to its corresponding vector of induced edge lengths. Given another normed space Y, we say that G is $(X, Y) -$ *embeddable* if every vector of edge lengths induced by a realisation in Y can also be induced

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by a realisation in X. Equivalently, a graph is (X, Y)-embeddable if and only if $f_{G,X}(X^V) \supseteq f_{G,Y}(Y^V)$. For example, it is well-known that any n points in ℓ_2 can be isometrically embedded into ℓ_2^{n-1}. Within our framework of graph embeddability, this says that any graph with n vertices must be (ℓ_2^{n-1}, ℓ_2)-embeddable.

The concept of (X, Y)-embeddability was first considered by Belk and Connelly [1, 2] when they characterized, in our terminology, the set of (ℓ_2^d, ℓ_2)-embeddable graphs for any $d \leq 3$. Sitharam and Willoughby [3] proved several results about the set of (ℓ_p^d, ℓ_p)-embeddable graphs for all $d \geq 1$ and all $p \in [1, \infty]$. Notably, they connected the work in [1, 2] to that of Ball [4] and Witsenhausen [5], and to the theory of finite metric space embeddings in general. Note that Belk and Connelly used the term *realisability* whereas Sitharam and Willoughby used the term *flattenability*. In this article, we adopt the term *embeddability* to capture a more general context.

A *minor* of a graph G is any graph obtained via a sequence of edge deletions and edge contractions—that is, deleting two vertices connected by an edge and adding a vertex whose neighborhood is the union of the neighborhoods of the deleted vertices. It is easy to see that if G is (X, Y)-embeddable, then so are all of its minors. Hence, the famous Robertson-Seymour theorem [6] shows that, for every ordered pair (X, Y) of normed spaces, there is a finite list of forbidden minors that characterise the (X, Y)-embeddable graphs. If these forbidden minors are known, then (X, Y)-embeddability can be determined in polynomial time [7].

Previous research has primarily focused on determining the lowest dimension d for which a given graph is (ℓ_p^d, ℓ_p)-embeddable. Belk and Connelly [1, 2] showed that the forbidden minors for the set of graphs that are $(\ell_2^d, \ell_2^{d'})$-embeddable, for any $d' \geq d$, are K_3 for $d = 1$, K_4 for $d = 2$, and K_5 and $K_{2,2,2}$ for $d = 3$ (see Figure 1). Note that the set of graphs that are $(\ell_2^d, \ell_2^{d'})$-embeddable, for any $d' \geq d$, is evidently equivalent to the set of (ℓ_2^d, ℓ_2)-embeddable graphs. The forbidden minors for (ℓ_2^d, ℓ_2)-embeddability are unknown for all $d \geq 4$, but it is known that they must be a subset of the forbidden minors for a class of graphs called *partial* $d - $*trees*. Resolving a conjecture posed in [3], Fiorini et al. [8] proved that the forbidden minors for the set of $(\ell_\infty^2, \ell_\infty)$-embeddable graphs (and also for the (ℓ_1^2, ℓ_1)-embeddable graphs) are W_4 and $K_4 +_e K_4$ (see Figure 2). These known results are collected in Section 2.

In this paper we investigate (X, Y)-embeddability for general normed spaces X and Y. Basic results concerning (X, Y)-embeddability are contained in Section 3, including a full characterisation of (X, Y)-embeddable graphs when either X or Y is the real line. In Section 4, we generalize a result of Sitharam and Willoughby [3], identifying sufficient conditions under which (X, Y)-embeddability implies that the graph is *independent in* X, in that the rigidity map $f_{G,X}$ has a differentiable point where the Jacobian has rank $|E|$ (with the assumption that X is finite-dimensional). Namely, we show that if a graph G is (X, Y)-embeddable with X being a finite-dimensional normed space, and if either G is independent in Y or Y is infinite-dimensional, then G is independent in X, provided that the norm $\|\cdot\|_X$ satisfies a mild smoothness condition.

In Section 5, we highlight that the spaces ℓ_2 and ℓ_∞ serve as two natural extreme spaces of embeddability (Theorem 5.1). Since every finite metric space can be isometrically embedded in ℓ_∞, every (X, ℓ_∞)-embeddable graph is (X, Y)-embeddable for any normed space Y. Conversely, if a graph G is (X, Y)-embeddable, where X is finite-dimensional and

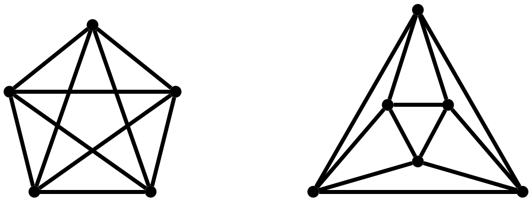

FIGURE 1 | The complete graph K_5 (left) and the complete tripartite graph $K_{2,2,2}$ (right).

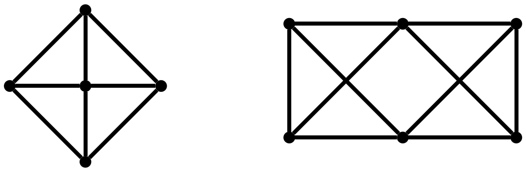

FIGURE 2 | The wheel graph W_4 (left) and the graph $K_4 +_e K_4$ formed by joining two copies of K_4 at an edge e and then removing said edge (right).

Y is infinite-dimensional, then G is also (X, ℓ_2)-embeddable. In Theorem 5.10, we show that the set of p-values such that G is (X, ℓ_p)-embeddable is closed, and is either empty or contains an interval of the form $[q, 2]$, where $1 \leq q \leq 2$.

Forbidden minors for (X, Y)-embeddability, where X is a normed plane, are discussed in Section 6. Given that X is a normed plane and Y is an infinite dimensional normed space, we prove that (X, Y)-embeddability is split into two cases depending on whether or not X is isometrically isomorphic to ℓ_∞^2 (Theorem 6.1, with Theorems 6.6 and 6.11 respectively for the individual cases). These results can all be extended to allow for X to have dimension 3 or more provided that X is strictly convex and Y is not (Theorem 6.7). Theorem 6.6 can also be improved when $X = \ell_2^2$, where we are able to fully characterise (X, Y)-embeddability for all choices of Y (Theorem 6.14).

In the last section (Section 7), we extend our definition of (X, Y)-embeddability for countable simple graphs. In Theorem 7.3, we prove that a countable graph is (X, Y)-embeddable, where X is finite-dimensional, if and only if it contains a complete tower of connected (X, Y)-embeddable subgraphs.

2 | Previously Known Results in ℓ_p Spaces

In this section, we present previously known results for (ℓ_p^d, ℓ_p)-embeddability. First, we formally define the spaces ℓ_p and ℓ_p^d. For each $p \in [1, \infty)$ and index set I, define the linear spaces

$$\ell_p(I) = \left\{ (x_i)_{i\in I} \in \mathbb{R}^I : \sum_{i\in I} |x_i|^p < \infty \right\} \quad \text{and} \quad \ell_\infty(I) = \left\{ (x_i)_{i\in I} \in \mathbb{R}^I : \sup_{i\in I} |x_i| < \infty \right\}.$$

and endow them with the respective norms

$$\|x\|_p := \left(\sum_{i\in I} |x_n|^p \right)^{1/p} \quad \text{and} \quad \|x\|_\infty = \sup_{i\in I} |x_i|.$$

It is well known that for every $p \in [1, \infty]$, the space $\ell_p(I)$ is complete with respect to the metric induced by its norm, so it is a Banach space. We denote the finite-dimensional normed space $\ell_p(\{1, \ldots, d\})$ as ℓ_p^d. This space corresponds to the space $\mathbb{R}^d$ with the $\|\cdot\|_p$ norm. We also denote the sequence space $\ell_p(\mathbb{N})$ as ℓ_p.

2.1 | Embeddability for Complete Graphs

Here we translate previous results about isometric embeddings of point sets into the language of (ℓ_p^d, ℓ_p)-embeddability, for various values of p and d.

Theorem 2.1 ([4, 5]). *Let G be a graph with n vertices.*

i. *G is (ℓ_1^d, ℓ_1)-embeddable for each $d \geq \binom{n}{2}$.*
ii. *If $n \geq 4$ and G is complete, then G is not (ℓ_1^d, ℓ_1)-embeddable for each $d < \binom{n-2}{2}$.*

Theorem 2.2 ([4]). *Let G be a graph with n vertices and let $p \in (1, \infty)$.*

i. *G is (ℓ_p^d, ℓ_p)-embeddable for each $d \geq \binom{n}{2}$.*
ii. *If $n \geq 3, p < 2$ and G is complete, then G is not (ℓ_p^d, ℓ_p)-embeddable for each $d < \binom{n-1}{2}$.*

Theorem 2.3 ([9]). *Let G be a graph with n vertices.*

i. *If $n \geq 4$, then G is $(\ell_\infty^d, \ell_\infty)$-embeddable for each $d \geq n - 2$.*
ii. *If G is complete, then G is not $(\ell_\infty^d, \ell_\infty)$-embeddable for each $d < \lfloor 2n/3 \rfloor$.*

By Theorem 2.3, the complete graph K_n is $(\ell_\infty^{n-2}, \ell_\infty)$-embeddable but not $(\ell_\infty^{n-3}, \ell_\infty)$-embeddable for each $n \in \{4, 5, 6\}$. In [8], it was shown that this also holds for $n = 7$; specifically, the complete graph K_7 is $(\ell_\infty^5, \ell_\infty)$-embeddable but not $(\ell_\infty^4, \ell_\infty)$-embeddable.

In [4], Ball proved that if the complete graph on n vertices is $(\ell_\infty^d, \ell_\infty)$-embeddable, then every graph with n vertices can be covered by d complete bipartite graphs. Using this, Ball showed that there exists a constant $c > 0$ such that each complete graph K_n is not $(\ell_\infty^d, \ell_\infty)$-embeddable for each $d < n - cn^{3/4}$. Fiorini et al. [8] later observed that this can be improved by using more recent results regarding bipartite graph coverings [10].

Theorem 2.4 ([4, 10]). *There exists a constant $c > 0$ such that each complete graph K_n is not $(\ell_\infty^d, \ell_\infty)$-embeddable for each $d < n - c \log n$.*

2.2 | Forbidden Minor Characterisations for Embeddability

In this subsection, we present previously known finite forbidden minor characterisations for embeddability. In the Euclidean case, we highlight the results of Belk and Connelly in Theorem 2.5 and Theorem 2.6, below.

Theorem 2.5 ([2]). *For $d \in \{1, 2\}$, a graph G is (ℓ_2^d, ℓ_2)-embeddable if and only if it contains no K_{d+2} minor.*

Theorem 2.6 ([1, 2]). *A graph G is (ℓ_2^3, ℓ_2)-embeddable if and only if it contains no K_5 or $K_{2,2,2}$ minor (see Figure 1).*

Witsenhausen showed in [5] that K_4 is $(\ell_\infty^2, \ell_\infty)$-embeddable. Sitharam and Willoughby [3] used convexity arguments on the so-called Cayley configuration space over specified non-edges of a d-dimensional framework to show that K_5 minus an edge is not $(\ell_\infty^2, \ell_\infty)$-embeddable. From this, they conjectured that the wheel graph W_4 is the only forbidden minor for the class of $(\ell_\infty^2, \ell_\infty)$-embeddable graphs on at most 5 vertices. Fiorini et al. [8] verified this conjecture, determined the complete set of forbidden minors for $(\ell_\infty^2, \ell_\infty)$-embeddability, and showed that these minors also completely characterise (ℓ_1^2, ℓ_1)-embeddability. We state these results in Theorem 2.7.

Theorem 2.7 ([8]). *For any $p \in \{1, \infty\}$, a graph G is (ℓ_p^2, ℓ_p)-embeddable if and only if it contains no W_4 or $K_4 +_e K_4$ minor (see Figure 2).*

Very little is known regarding the forbidden minors for (ℓ_1^d, ℓ_1)-embeddability and $(\ell_\infty^d, \ell_\infty)$-embeddability when $d \geq 3$. Some families of forbidden minors for $(\ell_\infty^d, \ell_\infty)$-embeddability can be found in [11].

3 | Basic Results for Embeddability Between General Normed Spaces

In this section we cover general properties of embeddability between general normed spaces. Throughout the paper we shall make the (rather trivial) assumption that every normed space has dimension higher than zero. We now begin with some easy observations.

Proposition 3.1. *If a graph G is (X, Y)-embeddable and (Y, Z)-embeddable, then G is (X, Z)-embeddable.*

Proof. This is immediate as $f_{G,X}(X^V) \supseteq f_{G,Y}(Y^V) \supseteq f_{G,Z}(Z^V)$. □

Proposition 3.2. *Let X, Y be isometrically isomorphic normed spaces. Then the following holds for every normed space Z.*

i. *Every (X, Z)-embeddable graph is (Y, Z)-embeddable.*
ii. *Every (Z, X)-embeddable graph is (Z, Y)-embeddable.*

Proof. This is immediate since $f_{G,X}(X^V) = f_{G,Y}(Y^V)$ for every graph $G = (V, E)$. □

We can also immediately characterise the (X, Y)-embeddable graphs when either $\dim X = 1$ or $\dim Y = 1$.

Proposition 3.3. *If Y can be isometrically embedded into X, then every graph is (X, Y)-embeddable. In particular, every graph is (X, Y)-embeddable when $\dim Y = 1$.*

Proof. Let $G = (V, E)$ be any graph and let $T : Y \to X$ be an isometric linear map. As $f_{G,Y}(Y^V) = f_{G,X}((TY)^V) \subseteq f_{G,X}(X^V)$, G is (X, Y)-embeddable. The final part of the result now follows from the observation that all 1-dimensional normed spaces are isometrically isomorphic, and hence any 1-dimensional normed space can be isometrically embedded into any higher dimensional space. □

Proposition 3.4. *Let X, Y be normed spaces where $\dim X = 1$.*

i. *If $\dim Y = 1$, then every graph is (X, Y)-embeddable.*
ii. *If $\dim Y \geq 2$, then a graph is (X, Y)-embeddable if and only if it is a forest.*

Proof.

i. : This follows from Proposition 3.3.
ii. : Choose any graph $G = (V, E)$. It is clear that if G is a forest with specified edge lengths $(d_{vw})_{vw \in E}$, then there exists $p \in X^V$ where $\|p_v - p_w\|_X = d_{vw}$ for all $vw \in E$. Suppose G is not a forest. Since (X, Y)-embeddability is a minor-closed property, it suffices to assume that G is a complete graph with three vertices, i.e., $G \cong K_3$. Since $\dim Y \geq 2$, we can choose three points $x, y, z \in Y$ where $\|x - z\|_Y = \|y - z\|_Y = \|x - y\|_Y = 1$. For example, take $x = 0$ and let y be a unit vector in Y. Define $S_Y(y, 1)$ to be the unit sphere in Y centred at y. The function $g : S_Y(y, 1) \to \mathbb{R}, u \mapsto \|u\|_Y$, is continuous. Note that $0, 2y \in S_Y(y, 1)$ satisfy $g(0) < 1$ and $g(2y) > 1$. By the Intermediate Value Theorem, there exists $z \in S_Y(y, 1)$ such that $g(z) = 1$. We note $f_{G,Y}(x, y, z) = (1, 1, 1)$, however there exists no $p \in X^V$ with $f_{G,X}(p) = (1, 1, 1)$.

□

4 | Independence and Embeddability

In this section we explore connections between (X, Y)-embeddability and independence, a matroidal property found in rigidity theory. There are two main results which deal separately with the case where Y is finite dimensional (Theorem 4.4) and where Y is infinite dimensional (Theorem 4.8).

A realisation p of a graph $G = (V, E)$ in a finite-dimensional normed space X is *independent* if the map $f_{G,X}$ is (Fréchet) differentiable at p and $\operatorname{rank} df_{G,X}(p) = |E|$. If an independent realisation of a graph G exists in a normed space X then we say that G is *independent in* X. It was proven in [3] that any (ℓ_p^d, ℓ_p)-embeddable graph is also independent in ℓ_p^d. In this section we similarly prove that this extends to most normed spaces.

In general normed spaces, there may not be an open set of differentiable points for the norm function. For this reason, the theory of differentiable maps on open sets is not necessarily applicable and so we require the following more general theory of locally Lipschitz maps and their generalized derivatives (see [12] for further discussion). Let X, Y be finite-dimensional normed spaces and let $f : X \to Y$ be a locally Lipschitz map, i.e., a map such that for every point $x \in X$, there exists an open neighbourhood $U_x \subset X$ of x and a constant $k_x > 0$ such that $\|f(x) - f(x')\|_Y \leq k_x \|x - x'\|_X$ for all $x' \in U_x$. Let $D(f)$ denote the set of differentiable points of a locally Lipschitz map $f : X \to Y$. By Rademacher's theorem, $D(f)$ is a conull set and hence is dense in X. Let $x \in X$ and denote by $D(f; x)$ the set of sequences (x_n) in $D(f)$ which converge to x.

Denote by $L(X, Y)$ the finite dimensional space of linear maps from X to Y with the norm topology. We define the set

$$\partial f(x) := \operatorname{conv}\left\{\lim_{n\to\infty} df(x_n) : (x_n) \in D(f; x) \quad \text{and} \quad (df(x_n)) \text{ converges in } L(X, Y)\right\},$$

where $\operatorname{conv} S$ denotes the convex hull of a set S. Any linear map in $\partial f(x)$ is called a *generalised derivative* of f at x. Each set $\partial f(x)$ is a non-empty, convex and compact subset of $L(X, Y)$ ([13], Proposition 2.6.2(a)). Sets of generalised derivatives also obey the following continuity rule.

Lemma 4.1 ([13], Proposition 2.6.2(c)). *Let X, Y be finite-dimensional normed spaces, let $f : X \to Y$ be a locally Lipschitz map, and let $x_0 \in X$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x \in X$ with $\|x - x_0\|_X < \delta$, we have*

$$\partial f(x) \subset \partial f(x_0) + B_\varepsilon,$$

where B_ε is the set of all linear maps $T : X \to Y$ with $\|T\|_{\text{op}} < \varepsilon$ (with $\|\cdot\|_{\text{op}}$ being the operator norm for linear maps between X and Y).

Generalised derivatives allow for a non-smooth variant of the constant rank theorem.

Theorem 4.2 ([14], Theorem 3.1). *Let X, Y be finite-dimensional normed spaces, let $f : X \to Y$ be a locally Lipschitz map, and let $x \in X$. Suppose that there exists a neighbourhood $O \subset X$ of x such that every generalised derivative of every point $x' \in O$ has rank k. Then there exists open sets $U \subset \mathbb{R}^{\dim X}$, $V \subset \mathbb{R}^{\dim Y}$, $U' \subset O$, $V' \subset Y$ with $x \in U'$, $f(x) \in V'$, and there exist bilipschitz maps $\phi : U \to U'$ and $\psi : V \to V'$, so that for every $(a_1, \ldots, a_{\dim X}) \in U$ we have*

$$(\psi^{-1} \circ f \circ \phi)(a_1, \ldots, a_{\dim X}) = (a_1, \ldots, a_k, 0, \ldots, 0).$$

Lemma 4.3. *If a graph G is independent in a finite-dimensional normed space X, then the set $f_{G,X}(X^V)$ has a non-empty interior.*

Proof. Choose an independent realisation p of G in X. Then $f_{G,X}$ is differentiable at the point p and $\operatorname{rank} df_{G,X}(p) = |E|$. Given any edge $e \in E$, let $P_e : \mathbb{R}^E \to \mathbb{R}$ be the projection onto the e-component. The composition $P_e \circ f_{G,X}$ describes a convex map from X^V to $\mathbb{R}$. Note that $P_e \circ f_{G,X} : X^V \to \mathbb{R}$ is a convex function, and hence is also locally Lipschitz ([15]). Since the derivative of a convex function is continuous over its set of differentiable points (see, e.g., [16], Theorem 25.5]), it follows that $\partial f_{G,X}(p) = \{df_{G,X}(p)\}$. By Lemma 4.1, there exists an open neighbourhood O of p such that for each $q \in O$, every generalised derivative of $f_{G,X}$ at q has rank $|E|$. Since projections are open maps, it follows now from Theorem 4.2 that the set $f_{G,X}(O)$ has a non-empty interior. □

Before moving to our first main result, we describe the following concepts from differential geometry. Let X and Y be finite-dimensional normed spaces, $U \subset X$ be an open set, and $f : U \subset X \to Y$ a C^∞-differentiable map (i.e., for each positive integer k, the k-th Fréchet derivative of f exists and is continuous). We say that a point $y \in Y$ is a *critical value* if there exists a point $x \in X$ where $f(x) = y$ and $\operatorname{rank} df(x) < \dim Y$. Any point in Y that is not a critical value is said to be a *regular value*.

Theorem 4.4. *Let X be a finite-dimensional normed space where the norm is C^∞-differentiable on an open dense subset of X. If a graph G is independent in a finite-dimensional normed space Y and (X, Y)-embeddable, then G is independent in X.*

Proof. By Lemma 4.3, $f_{G,Y}(Y^V)$ has a non-empty interior. As G is (X, Y)-embeddable, it follows that $f_{G,X}(X^V)$ also has a non-empty interior. Let $U \subset X^V$ be an open dense set of realisations of G in X where $f_{G,X}$ is C^∞-differentiable. By Sard's theorem, the set of regular values of the restricted map $f_{G,X} : U \to \mathbb{R}^E$ is a dense subset of $\mathbb{R}^E$. It follows that the set $f_{G,X}(X^V)$ contains a regular value as it has a non-empty interior. Hence there exists p in U where $\operatorname{rank} df_{G,X}(p) = |E|$, and thus G is independent in X. □

In order to prove the second main result in this section (Theorem 4.8) we now turn our attention to small graphs. Our objective is to show that, in any d-dimensional space, sufficiently small graphs are independent. To facilitate the inductive arguments in the following proofs we now introduce the concept of graded independence.

Let p be a realisation of a graph $G = (V, E)$ in a finite-dimensional normed space X such that the measurement map $f_{G,X}$ is differentiable at p. For any non-zero point $x \in X$, we will denote the derivative of $\|\cdot\|_X$ at x (if it exists) by $x^* \in X^*$, where X^* is the dual space of X. For each edge $vw \in E$, fix $\varphi^X_{v,w} := (p_v - p_w)^*$; it follows from p being a differentiable point of $f_{G,X}$ that each functional $\varphi^X_{v,w}$ exists. We say p has the *graded independence property* if we can order the vertices $v_1, \ldots, v_n$ so that for each $1 < j \leq n$, the set

$$\varphi^X(G, p)_j := \left\{\varphi^X_{v_j, v_i} : 1 \leq i < j, v_i v_j \in E\right\}$$

is linearly independent; we shall refer to v_n as the *highest vertex* of G.

To provide some intuition, imagine constructing a realisation for a complete graph K_n in d-dimensional Euclidean space by successively adding the vertices $v_1, \ldots, v_n$. The graded independence property ensures that at each step of the construction, the new vertex has linearly independent adjacent edges. The following lemma makes this process more precise for general normed spaces.

Lemma 4.5. *Let X be a d-dimensional normed space. Then for each $2 \leq n \leq d + 1$, there exists a realisation of the complete graph K_n with the graded independence property.*

Proof. It is immediate that any injective realisation of K_2 in X will have the graded independence property. Suppose $n > 2$ and that the result holds for each complete graph with at most $n-1$ vertices. We shall now show the result holds for K_n.

By our inductive assumption, there exists a realisation q of K_{n-1} which has the graded independence property with respect to some vertex ordering $v_1, \ldots, v_{n-1}$. Label the vertices of K_n as $v_1, \ldots, v_{n-1}, v_n$. Define $A \subset X$ to be the set of points where $\|\cdot\|_X$ is differentiable and define the subset

$$B := \bigcap_{i=1}^{n-1} \{x + q_{v_i} : x \in A\}.$$

By ([16], Theorem 25.5), A is conull with respect to the Lebesgue measure and the duality map $\psi : A \to X^*$, $\psi(x) = x^*$, is continuous. It follows that B is also conull, and hence a dense subset of X, and that the linear span of $\psi(A) = \{x^* : x \in A\}$ is X^*. Since $n \leq d+1$, there exists $y \in A$ such that the functionals,

$$\varphi^X_{v_{n-1},v_1}, \ldots, \varphi^X_{v_{n-1},v_{n-2}}, y^*$$

are linearly independent in X^*.

Let $(x_k)_{k\in\mathbb{N}}$ be the sequence in X given by $x_k := q_{v_{n-1}} + \frac{1}{k}y$, so that $x_k - q_{v_{n-1}} \in A$ and $(x_k - q_{v_{n-1}})^* = y^*$ for each $k \in \mathbb{N}$. As B is dense in X^*, we may choose for each $k \in \mathbb{N}$ some element $z_k \in B$ sufficiently close to x_k so we may suppose that

$$z_k \to q_{v_{n-1}} \quad \text{and} \quad \|(z_k - q_{v_{n-1}})^* - y^*\|_{X^*} = \|(z_k - q_{v_{n-1}})^* - (x_k - q_{v_{n-1}})^*\|_{X^*} < \frac{1}{k}.$$

Define the map $J : B \to L(X, \mathbb{R}^{n-1})$, where for each $z \in B$, $J(z)$ is the linear map

$$x \mapsto J(z)x := ((z - q_{v_1})^*(x), \ldots, (z - q_{v_{n-1}})^*(x)).$$

Since ψ is continuous on A, the map J is continuous. Now let $T \in L(X, \mathbb{R}^{n-1})$ be the linear map where for all $x \in X$,

$$T(x) := \left(\varphi^X_{v_{n-1},v_1}(x), \ldots, \varphi^X_{v_{n-1},v_{n-2}}(x), y^*(x)\right).$$

Because of our choice of y, the map T is surjective. Since the subset S of surjective linear maps in $L(X, \mathbb{R}^{n-1})$ is open and $J(z_k) \to T$ as $k \to \infty$, there exists some $N \in \mathbb{N}$ where $J(z_N) \in S$.

Define p to be the realisation of K_n with $p_{v_i} = q_{v_i}$ for $1 \leq i \leq n-1$ and $p_{v_n} = z_N$. As $J(z_N)$ is surjective, the set $\varphi^X(K_n, p)_n$ is linearly independent. For each $1 \leq j \leq n-1$, we have $\varphi^X(K_n, p)_j = \varphi^X(K_{n-1}, q)_j$, hence p has the graded independence property as required. □

Lemma 4.6. *Let X be a finite-dimensional normed space, G be a graph and p be a differentiable point of the measurement map $f_{G,X}$. If p has the graded independence property, then p is an independent realisation of G in X.*

Proof. Suppose that p is not an independent realisation of G in X, i.e., $\operatorname{rank} df_{G,X}(p) < |E|$. Then there exists a non-zero map $a : E \to \mathbb{R}$ where for each $v \in V$ we have the following equality (here $w \sim v$ denotes that $vw \in E$):

$$\sum_{w\sim v} a(vw)\varphi^X_{v,w} = 0. \tag{1}$$

We shall prove that any map a that satisfies the above conditions must be the zero map and hence obtain a contradiction.

Since p has the graded independence property with respect to some vertex ordering $v_1, \ldots, v_n$, the set $\varphi^X(G, p)_n$ is linearly independent, and so $a(v_n w) = 0$, for every $w \sim v_n$. Now suppose that for some $1 \leq k < n$, every edge $v_i v_j \in E$ with either $i \geq k + 1$ or $j \geq k + 1$ gives $a(v_i v_j) = 0$. Then Equation (1) at vertex v_k gives

$$0 = \sum_{w \sim v_k} a(v_k w)\varphi^X_{v_k, w} = \sum_{v_i \sim v_k, i<k} a(v_k v_i)\varphi^X_{v_k, v_i},$$

and so, since the set $\varphi^X(G, p)_k$ is linearly independent, we have $a(v_k w) = 0$ for every edge $v_k w \in E$. By induction it follows that $a(vw) = 0$ for every edge $vw \in E$, contradicting our initial assumption that a is a non-zero map. ☐

Lemma 4.7. *Let G be a graph with n vertices and X be a normed space of dimension $n - 1 \leq d < \infty$. Then G is independent in X.*

Proof. By Lemma 4.5, there exists a realisation p of K_n in X with the graded independence property. It is immediate that p is also a realisation of G in X with the graded independence property. Hence by Lemma 4.6, G is independent in X as required. ☐

Theorem 4.8. *Let X be a finite-dimensional normed space where the norm is C^∞-differentiable on an open dense subset of X, and let Y be an infinite-dimensional normed space. If a graph G is (X, Y)-embeddable, then G is independent in X.*

Proof. Fix n to be the number of vertices of G. Choose any n-dimensional subspace Z of Y. Then G is (X, Z)-embeddable. By Lemma 4.7, G is independent in Z. The result now follows from applying Theorem 4.4 to the triple G, X, Z. ☐

Remark 4.9. If a graph $G = (V, E)$ is independent in ℓ_∞^d, then there exists pairwise-disjoint edge subsets $T_1, \ldots, T_d$ such that the graphs $(V, T_1), \ldots, (V, T_d)$ are forests and $E = \bigcup_{i=1}^d T_d$ (see [17] for more detail). Using this, it is immediate that Theorem 4.8 gives an alternative proof of ([8], Lemma 2.5), namely that if G is $(\ell_\infty^d, \ell_\infty)$-embeddable, then the edges of G can be partitioned into d edge-disjoint forests.

5 | Embedding From ℓ_p Spaces

In this section we focus on (X, ℓ_p)-embeddability where $p \in [1, \infty]$.

5.1 | Bounds on Embeddability

For this subsection we prove the following result.

Theorem 5.1. *Let X, Y be normed spaces and $G = (V, E)$ be a graph.*

i. *G is (ℓ_∞, Y)-embeddable.*
ii. *If G is (X, ℓ_∞)-embeddable, then it is (X, Y)-embeddable.*
iii. *Suppose X is finite-dimensional and Y is infinite-dimensional. If G is (X, Y)-embeddable, then it is (X, ℓ_2)-embeddable.*

Theorem 5.1 shows us that if $\dim X < \infty = \dim Y$ then the set of (X, Y)-embeddable graphs will: (i) contain the set of (X, ℓ_∞)-embeddable graphs, and (ii) be contained in the set of (X, ℓ_2)-embeddable graphs. This result shall be important later in Section 6.

For the proof of Theorem 5.1, we require the following three results.

Theorem 5.2 ([18]). *For every finite metric space (M, d), there exists an isometry $f : M \to \ell_\infty$.*

Theorem 5.3 ([19], Theorem 1). *Let S be a finite affinely independent subset of ℓ_2. Then there exists $n \in \mathbb{N}$ such that for any normed space X with $\dim X \geq n$, the set S can be isometrically embedded into X.*

Lemma 5.4. *Let $G = (V, E)$ be any graph, X be a finite-dimensional normed space and Y be any normed space. Suppose that there exists a dense subset $D \subset Y^V$ where for each $p \in D$ there exists $q \in X^V$ where $f_{G,X}(q) = f_{G,Y}(p)$. Then G is (X, Y)-embeddable.*

Proof. Without loss of generality, we may assume G is connected. Choose any point $p \in Y^V$. Let $(p^n)_{n\in\mathbb{N}}$ be a sequence in D that converges to p. Fix a vertex $u \in V$. For each p^n, there exists $q^n \in X^V$ such that $f_{G,X}(q^n) = f_{G,Y}(p^n)$. By applying translations, we may assume $q^n_u = 0$ for each $n \in \mathbb{N}$. As the graph G is connected and X is finite-dimensional, there exists a compact set $C \subset X^V$ such that $q^n \in C$ for all $n \in \mathbb{N}$. Hence there exists a convergent subsequence $(q^{n_i})_{i\in\mathbb{N}}$ with limit $q \in X^V$. Since both $f_{G,X}$ and $f_{G,Y}$ are continuous, we have

$$f_{G,X}(q) = \lim_{i\to\infty} f_{G,X}(q^{n_i}) = \lim_{i\to\infty} f_{G,Y}(p^{n_i}) = f_{G,Y}(p)$$

as required. □

Proof of Theorem 5.1.

i. Choose any $q \in Y^V$. For each (possibly not distinct) pair $v, w \in V$, define $d_{vw} := \|q_v - q_w\|_Y$. It follows that we can define a metric space (V, d) where $d(v, w) := d_{vw}$. By Theorem 5.2, the metric space (V, d) can be isometrically embedded into ℓ_∞, and so there exists $p' \in \ell_\infty^V$ where $f_{G,\ell_\infty}(p') = f_{G,Y}(q)$.

ii. This follows from (i) and Proposition 3.1.

iii. Suppose $\dim X < \infty$. Choose any $p \in \ell_2^V$ so that the set $\{p_v : v \in V\}$ is affinely independent. By Theorem 5.3, there exists $\tilde{p} \in Y^V$ where $f_{G,\ell_2}(p) = f_{G,Y}(\tilde{p})$. As G is (X, Y)-embeddable, it follows that there exists $q \in X^V$ so that $f_{G,X}(q) = f_{G,\ell_2}(p)$. As the set of affinely independent realisations in ℓ_2^V forms a dense subset, and X is finite-dimensional, G is (X, ℓ_2)-embeddable by Lemma 5.4.

□

Remark 5.5. It is important to note that Theorem 5.1(iii) requires the assumption that Y is infinite-dimensional. For example, if $\dim X = \dim Y = 1$ then every graph is (X, Y)-embeddable by Proposition 3.4(i), but the only (X, ℓ_2)-embeddable graphs are forests by Proposition 3.4(ii). It is, however, unclear whether it is necessary for X to be finite-dimensional.

5.2 | Varying p

For a graph $G = (V, E)$ and a normed space X, define the set

$$\ell(G, X) := \{p \in [1, \infty] : G \text{ is } (X, \ell_p)\text{-embeddable}\}.$$

It follows from Theorems 2.1 to 2.3 that $p \in \ell(G, X)$ if and only if, given $d \geq \binom{|V|}{2}$, the graph G is (X, ℓ_p^d)-embeddable.

Proposition 5.6. *For every graph $G = (V, E)$ and every finite-dimensional normed space X, the set $\ell(G, X)$ is a closed subset of $[1, \infty]$ with respect to the Alexandroff topology.*

Proof. We may assume without loss of generality that G is connected. Choose a value $p \in [1, \infty]$ in the closure of $\ell(G, X)$ and fix $d \geq \binom{|V|}{2}$. As p lies in the closure of $\ell(G, X)$, there exists a sequence $(p_n)_{n\in\mathbb{N}}$ in $\ell(G, X)$ where $p_n \to p$ as $n \to \infty$ (if $p = \infty$, this means that $(p_n)_{n\in\mathbb{N}}$ tends to infinity). Choose any realisation $q \in (\ell_p^d)^V$ of G. For each $n \in \mathbb{N}$, we can consider q to be a realisation of G in $\ell_{p_n}^d$ also, since each normed space has the same underlying vector space (i.e., $\mathbb{R}^d$). Fix a vertex $u \in V$. Since G is connected, the set

$$S := \left\{ r \in X^V : r_u = 0, \max_{vw\in E} \|r_v - r_w\|_X \leq \max_{vw\in E} \|q_v - q_w\|_1 \right\}$$

is compact. For each $n \in \mathbb{N}$, there exists a realisation $r^n \in S$ such that $f_{G,X}(r^n) = f_{G,\ell_{p_n}^d}(q)$ and $r^n_u = 0$. As S is compact, there exists a convergent subsequence $(r^{n_i})_{i\in\mathbb{N}}$ with limit $r \in S$. For each $vw \in E$, we have

$$\|q_v - q_w\|_p = \lim_{i\to\infty} \|q_v - q_w\|_{p_i} = \lim_{i\to\infty} \|r_v^{n_i} - r_w^{n_i}\|_X = \|r_v - r_w\|_X,$$

hence $f_{G,X}(r) = f_{G,\ell_p^d}(q)$ and G is (X, ℓ_p^d)-embeddable. □

We recall that for any $p \in [1, \infty]$, $L_p[0, 1]$ is the normed space of measurable functions $f : [0, 1] \to \mathbb{R}$ (modulo equality on measure 1 sets) where $\|f\|_{L_p[0,1]} < \infty$, given the norm

$$\|f\|_{L_p[0,1]} := \left(\int_0^1 |f(t)|^p \, dt \right)^{1/p} \quad (p < \infty), \ \|f\|_{L_\infty[0,1]} := \operatorname{esssup}_{t \in [0,1]} |f(t)|;$$

here $\operatorname{esssup}_{t \in [0,1]} |f(t)|$ is the essential supremum of f, i.e., the smallest value $\lambda \in \mathbb{R}$ such that $|f(t)| \leq \lambda$ for almost all $t \in [0, 1]$.

Lemma 5.7. *For all $p \in [1, \infty]$, every graph is $(\ell_p, L_p[0, 1])$-embeddable.*

Proof. This follows directly from the proof of ([4], Proposition 1). ☐

The following result is due to Herz [20]. See also ([21], Theorem 2.1).

Theorem 5.8. *For all $1 \leq p \leq q \leq 2$, $L_q[0, 1]$ isometrically embeds in $L_p[0, 1]$.*

With this, we can state the following.

Lemma 5.9. *For all $1 \leq p \leq q \leq 2$, every graph is (ℓ_p, ℓ_q)-embeddable.*

Proof. Define the isometric linear map $T : \ell_q \to L_q[0, 1]$ which maps $(x_1, x_2, \ldots)$ to the function $f : [0, 1] \to \mathbb{R}$ where $f(x) = 2^{1/q} x_1$ for all $x \in [0, 1/2]$, and $f(x) = 2^{n/q} x_n$ if $x \in (\sum_{i=1}^{n-1} (1/2)^i, \sum_{i=1}^{n} (1/2)^i]$ for some $n \geq 2$. It follows that ℓ_q can be isometrically embedded into $L_q[0, 1]$. Hence, by Theorem 5.8, every graph is $(L_p[0, 1], \ell_q)$-embeddable. The result now follows from Proposition 3.1 and Lemma 5.7. ☐

Note that the proof of Lemma 5.9 fails for $q > 2$, since ℓ_q^3 cannot be isometrically embedded in ℓ_1 when $q > 2$ (see [21]). It is unknown to the authors if every graph is (ℓ_p, ℓ_q)-embeddable when $q > 2$ and $p < q$.

Theorem 5.10. *Let X be a normed space and $1 \leq p \leq q \leq 2$. If G is (X, ℓ_p)-embeddable, then it is (X, ℓ_q)-embeddable. Hence, the set $\ell(G, X) \cap [1, 2]$ is either empty or a closed interval containing 2.*

Proof. This follows from Proposition 3.1 and Lemma 5.9. ☐

While Theorem 5.10 guarantees that, so long as $1 \leq p \leq q \leq 2$, (X, ℓ_p)-embeddability implies (X, ℓ_q)-embeddability, the converse is not true. For example, while K_4 is (ℓ_2^3, ℓ_2)-embeddable (Theorem 2.6), it is not (ℓ_2^3, ℓ_1)-embeddable. This latter statement will be proven later in Section 6 (see Theorem 6.7).

6 | Characterising Forbidden Minors for Embeddability

We shall now focus on the special case of X being a normed plane and Y being a normed space of strictly higher dimension. Our goal is to characterise (X, Y)-embeddability by identifying a finite list of forbidden minors, as suggested by the celebrated Robertson-Seymour theorem [6]. We do so with the following result.

Theorem 6.1. *Let X be a normed plane and Y an infinite-dimensional normed space. Then a graph G is (X, Y)-embeddable if and only if either:*

i. *X is not isometrically isomorphic to ℓ_∞^2 and G contains no K_4 minor.*
ii. *X is isometrically isomorphic to ℓ_∞^2 and G contains no W_4 or $K_4 +_e K_4$ minor (see Figure 2).*

The proof of Theorem 6.1 is split into two sub-cases; either X is not isometrically isomorphic to ℓ_∞^2 (Theorem 6.6), or it is isometrically isomorphic to ℓ_∞^2 (Theorem 6.11). In fact, the former proof gives a slightly stronger result, since it only requires that $\dim Y \geq 3$.

6.1 | Embedding the Complete Graph of Size 4

In Theorem 2.5, it was shown that a graph G is (ℓ_2^2, ℓ_2)-embeddable if and only if it contains no K_4 minor. In this subsection, we extend this result to (X, Y)-embeddable graphs, where X is a normed plane that is not isometrically isomorphic to ℓ_∞^2 and $\dim Y \geq 3$.

For the next result we define the following. A subset S of a normed space X is called an *equilateral set* if $\|x - y\| = 1$ for all $x, y \in S$.

Theorem 6.2. *Let X be a normed space.*

- i. *If $\dim X = 2$ then:*
  - a. *the maximal size of an equilateral set in X is 3 if and only if X is not isometrically isomorphic to ℓ_∞^2, and*
  - b. *the maximal size of an equilateral set in X is 4 if and only if X is isometrically isomorphic to ℓ_∞^2.*
- ii. *If $\dim X = 3$ then there exists an equilateral set in X of size 4.*

Proof.

- i. By ([22], Theorem 4), the maximal size of an equilateral set in a normed plane is either 3 or 4. Moreover, the maximal size 4 is achieved if and only if the 4 points in the equilateral set are the vertices of a closed ball in X. This latter property is only possible when X is isometrically isomorphic to ℓ_∞^2.
- ii. By ([22], Theorem 4), the maximal size of an equilateral set in a 3-dimensional normed space is at least 4.

$\square$

Lemma 6.3. *Let X, Y be normed spaces where $\dim X \geq 2$ and let G be a (X, Y)-embeddable graph. Suppose G' is formed from G by adding a vertex v_0 and edges v_0v_1, v_0v_2, where $v_1, v_2 \in V$ are adjacent vertices in G. Then G' is (X, Y)-embeddable also.*

Proof. Choose any $q' \in Y^{V \cup \{v_0\}}$. By Proposition 3.3 we may assume that $\dim Y \geq 2$. Define $q \in Y^V$ to be the point with $q_v := q'_v$ for all $v \in V$. As G is (X, Y)-embeddable then there exists $p \in X^V$ where $f_{G,X}(p) = f_{G,Y}(q)$. Define $d_i := \|q'_{v_i} - q'_{v_0}\|_Y$ for each $i \in \{1, 2\}$. As $\|q'_{v_1} - q'_{v_2}\|_Y = \|p_{v_1} - p_{v_2}\|_X$ then,

$$d_1 + d_2 \geq \|p_{v_1} - p_{v_2}\|_X, d_1 + \|p_{v_1} - p_{v_2}\|_X \geq d_2, d_2 + \|p_{v_1} - p_{v_2}\|_X \geq d_1. \tag{2}$$

Define for each $i \in \{1, 2\}$ the sets

$$D_i := \{x \in X : \|x - p_{v_i}\|_X \leq d_i\} \quad \text{and} \quad S_i := \{x \in X : \|x - p_{v_i}\|_X = d_i\}.$$

As Equation (2) holds, the set $D_1 \cap D_2$ is non-empty but does not contain D_1 or D_2. It follows that the set $S_1 \cap S_2$, is non-empty; this can be seen by traversing the boundary of S_1 from the point $p_{v_1} + \frac{d_1}{\| p_{v_1} - p_{v_2} \|}(p_{v_1} - p_{v_2})$ (which is not contained in D_2) to the point $p_{v_1} + \frac{d_1}{\| p_{v_1} - p_{v_2} \|}(p_{v_2} - p_{v_1})$ (which is contained in D_2) and noting that the set S_2 must be intersected at some point during the path. Hence there exists $x \in X$ where $\|p_{v_i} - x\|_X = d_i$ for each $i \in \{1, 2\}$. If we set $p' \in X^{V \cup \{v_0\}}$ to be the realisation where $p'_v = p_v$ for all $v \in V$ and $p'_{v_0} = x$, then $f_{G',X}(p') = f_{G',Y}(q')$ as required. $\square$

Proposition 6.4 ([23], Proposition 7.3.1). *For any graph $G = (V, E)$, the following are equivalent:*

- i. *G does not contain K_4 as a minor, and $G + vw$ does contain K_4 as a minor for every distinct pair $v, w \in V$ where $vw \notin E$.*
- ii. *G can be formed from K_2 by a sequence moves where we add a vertex connected to pairs of adjacent vertices.*

Lemma 6.5. *Let X, Y be normed spaces where $\dim X \geq 2$. If G contains no K_4 minor then G is (X, Y)-embeddable.*

Proof. Since (X, Y)-embeddability is a minor-closed property, we may assume that G is maximal in the set of K_4 minor-free graphs, i.e., for any $e \notin E$, the graph $G + e$ contains a copy of K_4 as a minor. By Proposition 6.4 and Lemma 6.3, G is (X, Y)-embeddable, as K_2 is clearly (X, Y)-embeddable. $\square$

With this, we can now prove our first key result of the section.

Theorem 6.6. *Let X be a normed plane not isometrically isomorphic to ℓ_∞^2, and let Y be a normed space with $\dim Y \geq 3$. Then G is (X, Y)-embeddable if and only if G contains no K_4 minor.*

Proof. Suppose G is (X, Y)-embeddable and contains K_4 as a minor. Since (X, Y)-embeddability is a minor-closed property, it suffices to assume that $G = K_4$. By Theorem 6.2, there exists $q \in Y^V$ where $f_{G,Y}(q) = (1, 1, 1, 1)$. As G is (X, Y)-embeddable then there exists $x_1, x_2, x_3, x_4 \in X$ so that if we set $p_{v_i} = x_i$ for each $i \in \{1, 2, 3, 4\}$ then $f_{G,X}(p) = (1, 1, 1, 1)$. However this contradicts Theorem 6.2.

The converse follows from Lemma 6.5. ☐

For next result we first recall that a normed space X is *strictly convex* if for all points $x, y \in X$ with $\|x\| = \|y\| = 1$, we have $\|x + y\| = 2$ if and only if $x = y$.

Theorem 6.7. *Let X be a strictly convex normed space with* $\dim X \geq 2$ *and let Y be a normed space that is not strictly convex. Then a graph G is (X, Y)-embeddable if and only if it contains no K_4 minor.*

Proof. By Lemma 6.5, if G contains no K_4 minor then it is (X, Y)-embeddable. Suppose G contains K_4 as a minor. Since (X, Y)-embeddability is a minor-closed property, it suffices to assume that $G = K_4$. As Y is not strictly convex, there exists $x, y \in Y$ where $\|x\| = \|y\| = 1, x \neq y$ and $\|x + y\| = 2$. Let q be the realisation of K_4 in Y with

$$q_{v_1} = 0, q_{v_2} = x, q_{v_3} = y, q_{v_4} = x + y.$$

As X is strictly convex, no such $p \in X^V$ exists with $f_{G,X}(p) = f_{G,Y}(q)$. Hence G is not (X, Y)-embeddable, as required. ☐

Example 6.8. By Theorem 6.7, the (ℓ_2^2, ℓ_∞)-embeddable graphs are exactly those that contain no K_4 minor. Thus, by Theorem 2.5, (ℓ_2^2, ℓ_2)-embeddability is equivalent to (ℓ_2^2, ℓ_∞)-embeddability. Moreover, if $\dim Y = \infty$ then, by Theorem 5.1, (ℓ_2^2, Y)-embeddability is equivalent to both (ℓ_2^2, ℓ_2)- and (ℓ_2^2, ℓ_∞)-embeddability.

Note that (ℓ_2^3, ℓ_2)-embeddability is not equivalent to (ℓ_2^3, ℓ_∞)-embeddability (compare Theorem 2.6 and Theorem 6.7). If $\dim Y = \infty$ then, by Theorem 5.1(*i*), the forbidden minors for (ℓ_2^3, Y)-embeddability are also forbidden minors for (ℓ_2^3, ℓ_∞)-embeddability, and hence contain a K_4 minor. By Theorem 5.1(*ii*), the forbidden minors for (ℓ_2^3, Y)-embeddability include the forbidden minors for (ℓ_2^3, ℓ_2)-embeddability. In particular, K_5 and $K_{2,2,2}$ are forbidden minors for (ℓ_2^3, Y)-embeddability.

6.2 | Embedding into ℓ_∞^2

Let M be a real $n \times n$ symmetric matrix. We say that M is a *Euclidean distance matrix* if there exists a map $p : \{1, ..., n\} \to \ell_2$ where $\|p_i - p_j\|_2^2 = M_{i,j}$ for all $i, j \in \{1, ..., n\}$.

Theorem 6.9 ([24], Theorem 1). *Let M be a real $n \times n$ symmetric matrix with $M_{ii} = 0$ for all $i \in \{1, ..., n\}$. Define A to be the $(n-1) \times (n-1)$ symmetric matrix with $A_{ij} = M_{in} + M_{jn} - M_{ij}$ for all $i, j \in \{1, ..., n-1\}$. Then M is a Euclidean distance matrix if and only if A is positive semidefinite.*

Lemma 6.10 ([11], Section 5). *There exist no realisations of W_4 or $K_4 +_e K_4$ in ℓ_∞^2 such that the edge lengths of the embedded graph satisfy the given edge weights in Figure 3.*

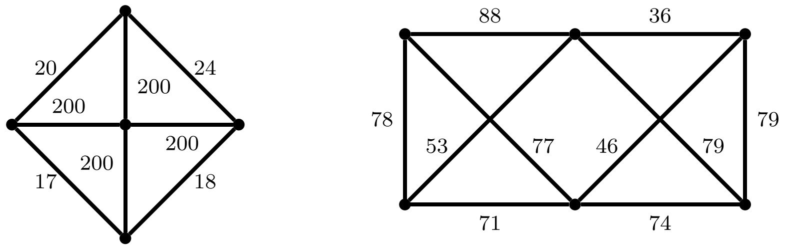

FIGURE 3 | The edge lengths assigned to W_4 (left) and $K_4 +_e K_4$ (right).

For the next result we recall that a *(symmetric) partial matrix* is a real-valued indexed tuple $\widetilde{M} = (m_{ij})_{(i,j)\in E}$ for some indexing set $E \subset \{(i,j) : 1 \leq i \leq j \leq n\}$. A *completion* of a partial matrix $\widetilde{M}$ is a symmetric matrix M where $M_{ij} = m_{ij}$ for each $(i,j) \in E$.

Theorem 6.11. *Let X be a normed plane isometrically isomorphic to ℓ_∞^2, and let Y be an infinite-dimensional normed space. Then a graph is (X, Y)-embeddable if and only if it contains no W_4 or $K_4 +_e K_4$ minor (see Figure 2).*

Proof. By Proposition 3.2, we may assume $X = \ell_\infty^2$. By Theorem 5.1, for the forward implication it suffices to consider the case where $Y = \ell_2$.

The matrix $M(W_4)$ below is a completion of the partial matrix formed from the squares of the edge weights of W_4 given in Figure 3, where bold numbers represent entries added to complete the partial matrix:

$$M(W_4) := \begin{bmatrix} 0 & 324 & \mathbf{245} & 576 & 40000 \\ 324 & 0 & 289 & \mathbf{294} & 40000 \\ \mathbf{245} & 289 & 0 & 400 & 40000 \\ 576 & \mathbf{294} & 400 & 0 & 40000 \\ 40000 & 40000 & 40000 & 40000 & 0 \end{bmatrix}.$$

Similarly, the matrix $M(K_4 +_e K_4)$ is a completion of the partial matrix formed from the squares of the edge weights of $K_4 +_e K_4$ given in Figure 3, where bold numbers represent the added values:

$$M(K_4 +_e K_4) := \begin{bmatrix} 0 & \mathbf{3003} & 5041 & 5929 & 5476 & 2116 \\ \mathbf{3003} & 0 & 2809 & 7744 & 6241 & 1296 \\ 5041 & 2809 & 0 & 6084 & \mathbf{4765} & \mathbf{2595} \\ 5929 & 7744 & 6084 & 0 & \mathbf{6545} & \mathbf{4655} \\ 5476 & 6241 & \mathbf{4765} & \mathbf{6545} & 0 & 6241 \\ 2116 & 1296 & \mathbf{2595} & \mathbf{4655} & 6241 & 0 \end{bmatrix}.$$

By applying Theorem 6.9 combined with a simple computational eigenvalue check, we see that both $M(W_4)$ and $M(K_4 +_e K_4)$ are Euclidean distance matrices. Hence there exist realisations p and q of W_4 and $K_4 +_e K_4$ respectively in ℓ_2 that realise the edge weights shown in Figure 3. By Lemma 6.10, there exist no realisations of either graph in ℓ_∞^2 that satisfy the given edge weights in Figure 3. Hence W_4 and $K_4 +_e K_4$ are not (ℓ_∞^2, ℓ_2)-embeddable, and so any (ℓ_∞^2, ℓ_2)-embeddable graph must contain no W_4 or $K_4 +_e K_4$ minor.

For the converse, any graph that contains no W_4 or $K_4 +_e K_4$ minor is $(\ell_\infty^2, \ell_\infty)$-embeddable by Theorem 2.7. Hence any graph that contains no W_4 or $K_4 +_e K_4$ minor is (ℓ_∞^2, Y)-embeddable by Theorem 5.1. □

Remark 6.12. The matrices $M(W_4)$ and $M(K_4 +_e K_4)$ in the proof of Theorem 6.11 were obtained as follows: We first applied a method similar to what is sketched out in [25] using a semidefinite program solver in Julia [26] to obtain completed matrices. After this, we then rounded the entries of each matrix to obtain $M(W_4)$ and $M(K_4 +_e K_4)$. We next verified that both $M(W_4)$ and $M(K_4 +_e K_4)$ are Euclidean distance matrices by computing the unique A matrix in each case (as described in Theorem 6.9) and checking that it is positive semidefinite. This last step was performed by computing the exact eigenvalues of each A matrix.

6.3 | Embedding into ℓ_2^2 From Any Normed Space

Proposition 3.3 and Theorem 6.6 characterise exactly which graphs are (ℓ_2^2, Y)-embeddable when $\dim Y \neq 2$; i.e., all graphs if $\dim Y = 1$, K_4 minor-free graphs if $\dim Y \geq 3$. We also know from Theorem 6.7 that, when Y is a normed plane that is not strictly convex, a graph is (ℓ_2^2, Y)-embeddable if and only if it contains no K_4 minor. We now improve this latter result by dropping the requirement that Y is not strictly convex, and hence obtain a full characterisation for the (ℓ_2^2, Y)-embeddable graphs for any choice of Y. We begin with the following result of Alonso and Benítez.

Theorem 6.13 ([27], Corollary, pg. 323). *Let Y be a normed plane which is not isometrically isomorphic to ℓ_2^2 and let $\epsilon \in (0,2)$. If ϵ is not an element of the set*

$$S := \{2\cos(k\pi/2n) : n, k \in \mathbb{N}, 1 \leq k \leq n-1\},$$

then there exist unit vectors $a, b \in Y$ such that $\|a-b\|_Y = 1$ and $\|a+b\|_Y \neq \sqrt{4-\epsilon^2}$.

Theorem 6.14. *Let Y be any normed space.*

i. *If $\dim Y = 1$ or Y is a normed plane that is isometrically isomorphic to ℓ_2^2, then every graph is (ℓ_2^2, Y)-embeddable.*
ii. *If $\dim Y \geq 3$ or Y is a normed plane that is not isometrically isomorphic to ℓ_2^2, then a graph is (ℓ_2^2, Y)-embeddable if and only if it contains no K_4 minor.*

Proof. As noted above, the remaining case to check is exactly when Y is a normed plane that is not isometrically isomorphic to ℓ_2^2. By Lemma 6.5, if G contains no K_4 minor then it is (ℓ_2^2, Y)-embeddable. Suppose G contains K_4 as a minor. Since (ℓ_2^2, Y)-embeddability is a minor-closed property, it suffices to assume that $G \cong K_4$. Label the vertices of G by v_1, v_2, v_3, v_4.

Choose $\varepsilon \in (0,2)\backslash S$, with S being the set defined in Theorem 6.13. It is easy to see that, up to isometry, there exist exactly two realisations of K_4 in ℓ_2^2 such that the edges $v_1v_2, v_1v_3, v_2v_4, v_3v_4$ have length 1 and the edge v_2v_3 has length ε: the realisation p where $\|p_{v_1} - p_{v_4}\|_Y = \sqrt{4-\varepsilon^2}$, and the realisation p' where $\|p'_{v_1} - p'_{v_4}\|_Y = 0$. By Theorem 6.13, there exists $a, b \in Y$ such that $\|a\|_Y = \|b\|_Y = 1$, $\|a-b\|_Y = \varepsilon$ and $\|a+b\|_Y \neq \sqrt{4-\varepsilon^2}$. Since $\|a-b\|_Y < 2$, we also have that $\|a+b\|_Y \neq 0$. Define q to be the realisation of K_4 in Y with $q_{v_1} = 0$, $q_{v_2} = a$, $q_{v_3} = b$ and $q_{v_4} = a+b$. Since

$$\|q_{v_1} - q_{v_2}\|_Y = \|q_{v_1} - q_{v_3}\|_Y = \|q_{v_2} - q_{v_4}\|_Y = \|q_{v_3} - q_{v_4}\|_Y = 1$$

and $\|q_{v_2} - q_{v_3}\|_Y = \|a-b\|_Y = \varepsilon$, but $\|q_{v_1} - q_{v_4}\|_Y = \|a+b\|_Y \notin \{0, \sqrt{4-\varepsilon^2}\}$, we have that $f_{G,Y}(q) \notin f_{G,\ell_2^2}((\ell_2^2)^V)$, that is, G is not (ℓ_2^2, Y)-embeddable. ☐

7 | Embeddability for Countably Infinite Graphs

For this final section we shall now allow a graph $G = (V, E)$ to have countably infinite vertex and edge sets. Our definitions of embeddability extend immediately to countably infinite graphs.

Definition 7.1. A *tower* in G is a sequence of finite subgraphs $(G_n)_{n\in\mathbb{N}}$ with $G_n = (V_n, E_n)$, where $V_n \subset V_{n+1}$ and $E_n \subset E_{n+1}$ for each $n \in \mathbb{N}$. A tower is *complete* if $\bigcup_{n\in\mathbb{N}} V_n = V$ and $\bigcup_{n\in\mathbb{N}} E_n = E$.

The following lemma is a well-known application of Tychonoff's theorem. We provide the proof for completeness.

Lemma 7.2. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of non-empty compact Hausdorff spaces where for each $n \leq m$ there exists a continuous map $\pi_{n,m} : A_m \to A_n$. Suppose that $\pi_{n,n}$ is the identity map and $\pi_{n,\ell} = \pi_{n,m} \circ \pi_{m,\ell}$ for all $n \leq m \leq \ell$. Then the inverse limit*

$$A := \left\{ (a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n : \pi_{n,m}(a_m) = a_n \quad \text{for all} \quad n \leq m \right\}$$

is a non-empty compact subset of $\prod_{n\in\mathbb{N}} A_n$.

Proof. By Tychonoff's theorem, $\prod_{n\in\mathbb{N}} A_n$ is compact; further, since the product of Hausdorff spaces is Hausdorff, $\prod_{n\in\mathbb{N}} A_n$ is Hausdorff also. Since A is a closed subset of $\prod_{n\in\mathbb{N}} A_n$, it is a compact subset.

For any $n \leq m$ we note that $\pi_{1,n}(A_n) \supset \pi_{1,m}(A_m)$. As each $\pi_{1,n}(A_n)$ is a non-empty compact subset of a Hausdorff space, by Cantor's intersection theorem there exists $x \in \cap_{n\in\mathbb{N}} \pi_{1,n}(A_n)$. Define for each $k \in \mathbb{N}$ the set

$$B_k := \left\{ (a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n : a_1 = x, \pi_{n,k}(a_k) = a_n \quad \text{for all} \quad n \leq k \right\}.$$

By our choice of $x \in A_1$, each B_k is a non-empty compact subset of $\prod_{n \in \mathbb{N}} A_n$, and we note that $B_k \supset B_\ell$ for all $k \leq \ell$. By Cantor's intersection theorem, $\bigcap_{n \in \mathbb{N}} B_n$ is a non-empty set which is contained in A. ☐

Theorem 7.3. *Let $G = (V, E)$ be a connected graph with countable vertex set. Then the following are equivalent for any normed spaces X, Y where X is finite-dimensional:*

1. *G is (X, Y)-embeddable.*
2. *Every subgraph of G is (X, Y)-embeddable.*
3. *G contains a complete tower of connected (X, Y)-embeddable subgraphs.*

Proof. It is immediate that (i) ⇒ (ii) ⇒ (iii). Suppose (iii) holds, i.e. there exists a complete tower $(G_n)_{n \in \mathbb{N}}$ where each G_n is connected and (X, Y)-embeddable. Choose any $q \in Y^V$ and $v_0 \in V_1$. For each $n \in \mathbb{N}$, let

$$A_n := \{p \in X^{V_n} : p_{v_0} = 0, f_{G_n,X}(p) = f_{G_n,Y}(q|_{V_n})\}.$$

As each G_n is connected, X is finite-dimensional and each $f_{G_n,X}$ is continuous, each set A_n is a non-empty compact Hausdorff space. For every $n \leq m$, define the continuous map

$$\pi_{n,m} : A_m \to A_n, (p_v)_{v \in V_m} \mapsto (p_v)_{v \in V_n}.$$

It is immediate that $\pi_{n,n}$ is the identity map and $\pi_{n,\ell} = \pi_{n,m} \circ \pi_{m,\ell}$ for all $n \leq m \leq \ell$. Then by Lemma 7.2 there exists $(p_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} A_n$ where $\pi_{n,m}(p_m) = p_n$ for all $n \leq m$. If we define $p \in X^V$ to be the unique point where $p_v := (p_n)_v$ for $v \in V_n$ then $f_{G,X}(p) = f_{G,Y}(q)$. Hence G is (X, Y)-embeddable as required. ☐

Remark 7.4. It is worth noting that Theorem 7.3 requires that X is finite-dimensional. To see why this is required, take G to be the complete graph with a countably infinite set of vertices, $X = \ell_p$ for some $1 \leq p < 2$, and $Y = \ell_2$. By Theorem 5.1, every finite subgraph is (ℓ_p, ℓ_2)-embeddable. Suppose for contradiction the graph G is also (ℓ_p, ℓ_2)-embeddable. Choose a realisation q of G such that the set $D := \{q_v : v \in V\}$ is a dense subset of ℓ_2. By our assumption, there exists an equivalent realisation $\tilde{q}$ in ℓ_p. Hence there exists an isometry $f : D \to \ell_p$. As ℓ_p is complete, we can extend this map to an isometry $h : \ell_2 \to \ell_p$. However, no such isometry can exist (see e.g. [28], Section 4), which gives the desired contradiction.

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Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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