

The Vlasov Bivector: A Parameter-Free Approach to Vlasov Kinematics

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Abstract

Plasma kinematics is typically performed on either the mass shell or a lab bundle, 7-dimensional phase spaces equipped with a Vlasov vector field. This choice of phase space encodes the parametrisation of the differential equations governing particle dynamics. By replacing the Vlasov vector field and related quantities over a phase space with a bivector on an 8-dimensional conic sub-bundle of the tangent bundle, we construct a parameter-free version of Vlasov theory. The advantages of this formalism include compatibility with light-like and ultra-relativistic particles, non-metric connections, and metric-free or pre-metric theories. Additionally, this formalism applies to theories where no time-phase space can exist for topological reasons e.g. when we wish to consider all geodesics, including space-like geodesics. We also use this formalism to derive a simple formula which enables the transformation of a Vlasov field from one phase space to another in such a way that the trajectories of the particles in the base space are unchanged.

Our formalism also has implications for numerical simulations. By extending quantities such as the particle density form onto an 8-dimensional conic sub-bundle, we can extend existing numerical integrators, such as the Agile-Numerical-Integrator, to relativistic scenarios. The extension of the particle density form also allows for the generalisation of the current 3-form onto the conic bundle. We also discuss the complicated relationship between the stress-energy form and phase space, how our formalism illustrates this, and the corresponding challenges in developing a parameter-free Einstein-Vlasov system. The relationship between our formalism and sprays or semi-sprays is explored, and examples from Finsler geometry are given.

1 Introduction

In Vlasov and Boltzmann theories of kinematics, one is interested in the dynamics of a scalar field over a time-phase space which represents a particle density. The fields are typically functions of time, 3 positional coordinates, and 3 velocity or momentum coordinates. The dynamic equations are written in terms of a first order operator on this scalar field. In the case of the Vlasov equation, the action of the operator on the scalar field is zero, whereas for the Boltzmann equation, the right-hand side is non-zero to account for collisions. This 7-dimensional time-phase space, which we call the *kinematic domain*, is a subspace of the 8-dimensional tangent bundle. It corresponds to the chosen parameterisation one uses for the underlying solutions of the chosen force equation. Sometimes there is a natural choice for this 7-dimensional space, for example the mass shell in the case of an Einstein-Vlasov system (see [1]). In other cases, it has to be chosen, e.g. for null geodesics. Furthermore, there are cases where it is not even possible to construct a kinematic domain; for example, in the case where we consider all geodesics, including space-like geodesics.

In this article, we investigate an alternative approach that does not require a choice of kinematic domain. We work on an 8-dimensional conic bundle which is a subset of the tangent bundle, and we replace the Vlasov vector field,

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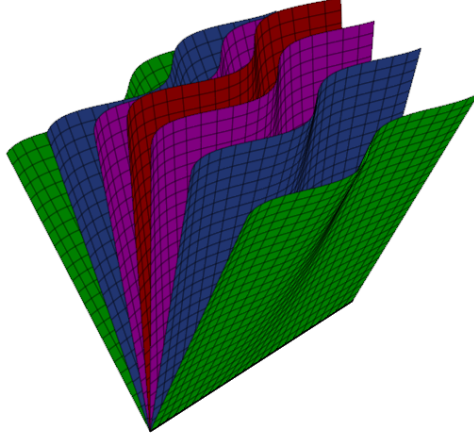


Figure 1: The Vlasov Bivector integrates to form leaves, like the pages of a book. The density of the leaves also corresponds to the particle distribution.

with the Vlasov bivector. This corresponds to not choosing a parameterisation for the underlying ordinary differential equations (ODEs) of the system. The integral 2-dimensional surfaces of this bivector are depicted in figure 1. There are many advantages to this formalism which we discuss in the following subsection. Although we concentrate on the geodesic or Lorentz force equations and their corresponding Vlasov fields, this approach can be applied to the kinematics models for any 2nd order ODEs, on any arbitrary dimensional base manifold.

In this article, we first summarise the standard kinematic domain approach plus its relationship with the conic sub-bundle, and make the link with sprays and semi-sprays. This includes the means by which we transform a Vlasov field between kinematic domains. We then define the Vlasov bivector and particle density 6-form, and we give the equations of motion for the latter. We show how to go between the approaches, when the kinematic domain approach exists. We also show how to calculate the current 3-form, the analogue to the stress-energy tensor, needed for the Einstein-Vlasov system. Next, we discuss the implications of our formalism for numerical methods, and apply our formalism to extend the agile numerical integrator (ANI) to relativistic contexts.

In this article, we use sprays and Finsler geometry as examples to put this work in context. However, for the reader unfamiliar with these concepts, all statements about sprays, semi-sprays and Finsler geometry may be safely ignored. The details linking our work and sprays are given in section A.1.

1.1 The Standard Vlasov Approach

When performing Vlasov kinematics, a 7-dimensional time-phase space E is chosen upon which to construct the particle density scalar field f_E . We will refer to E as the kinematic domain, and it is a bundle over the spacetime manifold M . The first order partial differential equation governing the dynamics of f_E can be represented by the Vlasov vector field $W_E \in \Gamma TE$,

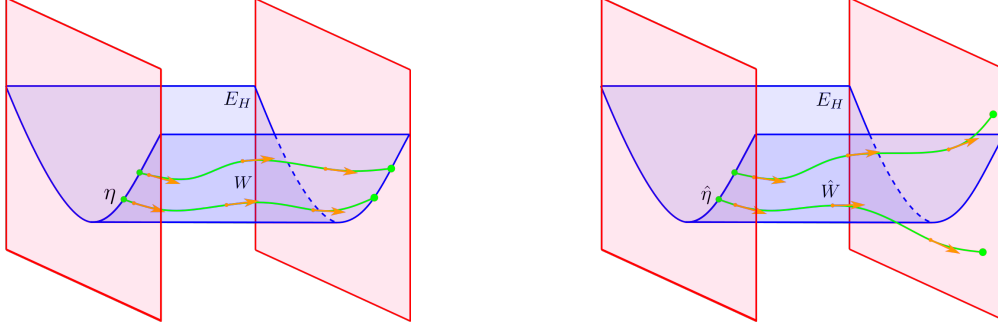
$$W_E \langle f_E \rangle = 0, \quad (1)$$

where the angled brackets $\langle \bullet \rangle$ indicate the action of a vector field on a scalar field. Note that it is also referred to as the Liouville vector field in some literature, e.g. [2].

Throughout this article, we use the language of differential geometry on the tangent bundle to describe Vlasov kinematic models. For an overview of this technique, see [3]. The solutions to the underlying second-order ordinary differential equations (ODEs) are called trajectories and correspond to the worldlines of the particles. The tangent vectors to the trajectories, that is, the velocities of particles, are curves in the kinematic domain, called the prolongation of the trajectories. These prolongations are the integral curves of the Vlasov vector field.

The kinematic domain $E \subset TM$ is a sub-bundle of the tangent bundle over the base manifold M and there are many choices for this sub-bundle. For example, if M has a spacetime metric, g (with signature $(-, +, +, +)$), then one choice is to use the unit mass shell E_H , also known as the upper unit hyperboloid:

$$E_H = \{\underline{v} \in TM : g(\underline{v}, \underline{v}) = -1, \underline{v} \text{ is future pointing}\}. \quad (2)$$



(a) Prolongations for a metric compatible connection, which remain on the mass shell.

(b) Prolongations for a non-metric compatible connection, which do not remain on the mass shell.

Figure 2: Illustration of integral curves, i.e the prolongations, in the mass shell for the case of a Vlasov field built from a force equation with a metric compatible connection (fig. 2a) and a non metric compatible connection (fig. 2b). The mass shell is represented by the blue sheet, integral curves η , $\hat{\eta}$ of the Vlasov fields W , \hat{W} are given by the green lines. The Vlasov fields themselves are depicted by the orange arrows.

This corresponds to proper-time parameterisation and is the natural choice for the relativistic Vlasov-Maxwell [4] system and the Einstein Vlasov system [1]. When performing plasma kinematics with the Vlasov-Maxwell system, we typically formulate the Lorentz force equation in terms of a metric compatible connection. One disadvantage of the upper unit hyperboloid is that prolongations of trajectories for a non-metric compatible connection will not, in general, remain on the mass shell (see lemma A.6 for an example). This is visualised in fig. 2.

Another choice of kinematic domain is the lab-time bundle, E_t where $t \in \Gamma\Lambda^0 M$ is a lab-time. This t has the property that if \underline{v} is future pointing then $\underline{v}\langle t \rangle > 0$. We also refer to t as a time-slicing scalar, because it slices spacetime into hypersurfaces. The events on each hypersurface are designated to be simultaneous. The term lab-time refers to any scalar field t , not necessarily the time coordinate of an inertial frame or associated with a lab. Thus it can also refer to the rest frame of an accelerating particle. The lab-time bundle, E_t , consists of vectors \underline{v} such that $\underline{v}\langle t \rangle = 1$ subject to some additional restrictions, such as being time-like. It is defined in eq. (21) below, once we have defined the conic bundle, U .

Working on E_t adds an additional term to the Vlasov equation relative to the Vlasov field on E_H . Unlike the unit hyperboloid, it does not require a metric compatible connection. It is also useful when other quantities are defined with reference to a global time (see, e.g. [5]). The disadvantage of this approach is that one has to choose a lab-time coordinate and there may be complicated transformations from one lab-time coordinate to another. An illustration of kinematic domains as ‘‘slices’’ of a conic bundle is given in fig. 3.

Given two kinematic domains, say E and \hat{E} , it is possible to transform the Vlasov field $W_E \in \Gamma TE$ to another Vlasov field $W_{\hat{E}} \in \Gamma T\hat{E}$ in such a way that the trajectories in the base space are unaffected. This transformation corresponds to a reparameterisation of the trajectories associated with the Vlasov field. For example, the geodesic equation on the unit hyperboloid E_H becomes the pre-geodesic equation on the lab-time bundle E_t , and by analogy the Lorentz force equation becomes the pre-Lorentz force equation. This extra force term, which is always proportional to the velocity, gives rise to an extra term in the Vlasov field.

1.2 Motivation for the Parameter Free Approach

The primary goal of this work is to present the formalism of the Vlasov field and the corresponding Vlasov equation in a way which does not require choosing a kinematic domain, or equivalently, a parameterisation. It is a considerable abstraction to pass from the 7-dimensional kinematic formalism to the 8-dimensional formalism, and several new concepts must be defined, such as the Vlasov bivector and the particle density form. We identify here a number of reasons why this new formalism is justified.

- It removes the arbitrariness of the kinematic domain and the need for additional terms. Thus this approach is fundamentally free of the choice of parameterisation.
- It works when modelling light-like particles where the kinematic domain is 6-dimensional, and hence one cannot

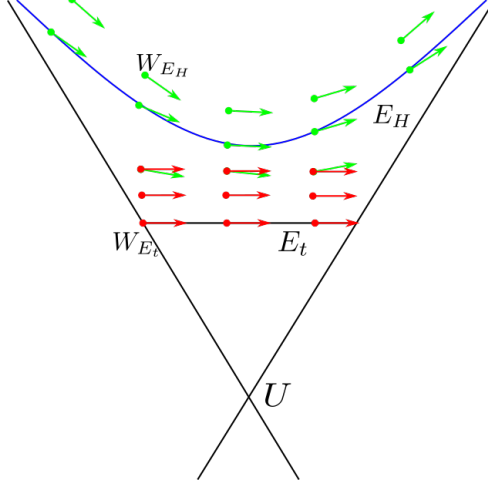


Figure 3: The 7–dimensional kinematic domains E_H and E_t are given by slices of the 8–dimensional conic bundle $U \subset TM$. The unit hyperboloid E_H is given by the dark blue hyperbola and a lab-time bundle E_t by the horizontal black line. The Vlasov fields W_{E_H} and W_{E_t} (green and red arrows respectively) are tangent to their respective kinematic domains.

use the hyperboloid bundle, E_H . One could use a lab-time bundle $E_t = \{\underline{v} \in TM : \underline{v}\langle t \rangle = 1 \text{ and } g(\underline{v}, \underline{v}) = 0\}$. In the case of Minkowski spacetime, with t given by the usual Minkowski time coordinate, the prolongations of light-like geodesics remain on E_t . However, in general where there is gravity or where we use an arbitrary time scalar field, this is not the case, and the prolongations of light-like geodesics will not remain on E_t . See appendix lemma A.7 for an example of this. Thus one has two choices, either to use our approach given here or to use the pre-geodesic equation.

- In particle accelerators, charged particles travel at speeds very close to the speed of light, and one can use an ultra-relativistic approximation. Although massless particles are not affected by an electromagnetic fields, ultra-relativistic particles do respond in the limit where the electromagnetic field becomes infinite. In [6], the authors consider the ultra-relativistic approximation for a charged fluid. The approach given here will enable the ultra-relativistic approximation for a kinetic description of charged particles.
- Sometimes it is necessary to work in multiple kinematic domains for practical reasons and hence it is necessary to transform the Vlasov equation between them. One can construct this transformation by considering the underlying ODEs, rescaling them, and then reconstructing the new corresponding Vlasov equation. The advantage of starting with the parameterisation free approach, is that it gives the formula for this transformation directly. This can be visualised in fig. 3. This is analogous to the advantages of working in coordinate-free notation. If one is subsequently given a coordinate system, one can easily calculate the corresponding coordinate quantities from the coordinate-free quantities. Also, given two coordinate systems, the transformation of these coordinate-quantities is also derived from the coordinate free definitions. Likewise the formula for passing from one kinematic domain to another falls out of our kinematic domain free definition of the Vlasov field. Additionally, working with coordinate free tools often provides geometric insight (see [7], for example).
- The numerical simulation of plasmas, both terrestrial and astrophysical are of intensive interest. For these one may attempt to solve the Maxwell-Vlasov system directly [8] (especially in 1 or 2 dimensions) or use a particle-in-cell (PIC) code. There are many codes in current use [9]. These generally use a lab-frame [10] which updates all the fields and particles from one timesteps to the next. These timesteps define the time-slicing scalar field, that is a field on spacetime the contours which define all simultaneous events. We know from relativity that such a notion of simultaneity is not universal and is a choice. When one can ignore the effect of gravity, it is usual to use an inertial frame.

In section section 5.2, we leverage our formalism to generalise the agile numerical integrator (ANI), found in [11]. We generate the equations for a Lorentz boost of the 1–dimensional Vlasov-Poisson equation. The current use

of the ANI is to systems where the velocities are either non-relativistic or only mildly relativistic. This avoids velocities bunching just below $v = c$. This prevents the ANI being used to model a bunch of particles being accelerated to ultra-relativistic velocities, such as in a particle accelerator. Using a series of Lorentz boosts, one can model such bunches. Here, we constantly work in a frame where the velocities have $v \ll c$, even though the actual velocities, relative to the lab, are ultra-relativistic.

One could also use an arbitrary non-inertial frame, for example one adapted to a prescribed accelerating orbit. This is a natural thing to do when considering moments moments of a distribution, for example, [5]. Our technology will trivially give you the Vlasov system in this case. Starting with the adapted time-slicing scalar field, it is trivial to use eq. (54) to deduce the Vlasov field in this adapted coordinate system. An alternative would require calculating all the pseudo-forces in the equations of motion and constructing the Vlasov field from these. The same approach can be used in general relativity with the contours of the time-slicing scalar field being the geodesic hypersurfaces which are orthogonal to a prescribed worldline, [12].

- In this formalism, we work on the conic bundle $U \subset \check{T}M$ (where $\check{T}M = TM \setminus \{0\}$, the subset of TM which excludes the zero vectors), which is a bundle over M and has the same dimension as TM . The formal definition of the conic bundle is given in definition 1.5. There are cases where there cannot exist a kinematic domain for topological reasons. For example, suppose we wish to consider all solutions to the autoparallel equation. In spacetime this corresponds to all time-like, light-like and space-like geodesics simultaneously. In this case $U = \check{T}M$. Thus we would have to pick an initial velocity for each direction in order to construct the phase space E upon which we define our Vlasov field. Each fibre E_p would then be topologically equivalent to the quotient set T_pM / \sim where $\underline{v} \sim \lambda \underline{v}$ for $\lambda \neq 0$. This set is the real projective $(n-1)$ -space $\mathbb{R}P^{n-1}$ in the case where M is a n -dimensional spacetime manifold. However, if n is odd, $n \geq 3$, $\mathbb{R}P^{n-1}$ cannot be embedded into T_pM , and hence no kinematic domain E exists. By contrast, our formalism remains valid in this case. For proof that $\mathbb{R}P^{2n}$ cannot be embedded into \mathbb{R}^{2n+1} see cor. 8.9 [13].
- As seen in fig. 2, the prolongations of solutions to the Lorentz force equation do not, in general, remain on E_H if the connection is not metric compatible. An example is given in the lemma A.6. By contrast, our approach works with force equations constructed from any connection, and more general models of acceleration [14].
Furthermore, our approach works even when the manifold does not possess a metric. For example, using the autoparallel equations, one can construct the Vlasov equations with just a connection and no metric. Thus it is compatible with pre-metric formalisms of dynamics (see [15–17]). For electrodynamics, without a metric, one would need to consider a force tensor, analogous to the electromagnetic field, but which maps vectors to vectors.
- This formalism can be generalised to Finsler spacetimes [18]. Furthermore, our formalism is not intrinsically dependant on any geometric objects beyond a base manifold. By casting objects from Finsler geometry based kinematic theories (e.g. [19]), the dependence of these objects upon the Finsler metric can be better highlighted in a way analogous to pre-metric electromagnetism.
- The clock hypothesis is defined using the metric. Particles which decay must have a notions of time. However for stable particles, one can argue that the description of the particles motion should not be defined in terms of proper time which does not effect the particles. Our approach may be useful for treatments which do not impose the clock hypothesis, e.g. Mashoon electrodynamics [20–23].
- Finally, there is a philosophical argument. There is vague distinction between the kinematics and the dynamics of a system. In the case of particle dynamics, the kinematics simply state that we are interested in curves that satisfy some unspecified ODE, whereas the dynamics prescribe the ODE. It is the dynamics that require the connection, and maybe a metric. By contrast, for the standard Vlasov field on E_H , the kinematics are defined on a kinematic domain which requires a metric. Thus, the metric is introduced at the kinematic stage rather than the dynamics stage.

1.3 The Vlasov Bivector and the Transport equations

The primary object of interest in this paper is the Vlasov bivector, Ψ . The Vlasov bivector is constructed on the conic bundle U as opposed to a given kinematic domain. Informally we may consider these to be the generalisation of a Vlasov field W_E on E to a geometric object on U . Although we use the language of bivectors here, one can use

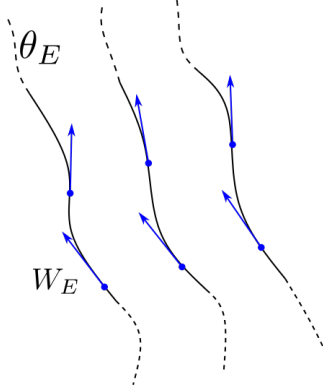


Figure 4: Diagram of the transport equations on a $(2n - 1)$ -dimensional kinetic domain E using form submanifolds. The details of submanifolds are described in [24]. The form manifolds of $\theta_E \in \Gamma\Lambda^{2n-2}E$ (black lines) do not terminate ($d\theta_E = 0$) and are tangent to the vector field $W_E \in \Gamma TE$, represented by blue arrows ($i_{W_E}\theta_E = 0$).

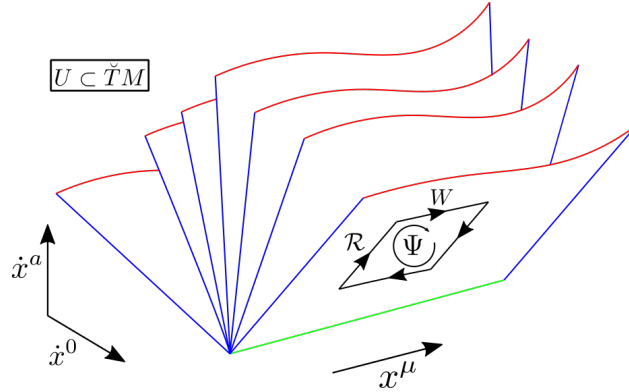


Figure 5: Sketch of an integrable Vlasov bivector Ψ . Here, a possible form for Ψ is $\Psi = \mathcal{R} \wedge W$. Notice that since Ψ is integrable, the bivectors ‘knit-together’ to form the leaves of a foliation, also depicted in fig. 1. The ambient space is $U \subset \check{T}M$. The green line is to indicate the absence of 0-vectors in our space. The density of the leaves corresponds to the velocity density of our one particle distributions function with higher density towards the middle and lower density towards the sides. Viewing this diagram as the particle density form θ , then the observation that the leaves are tangent to Ψ and that they have no boundary, is equivalent to the transport equations given in eq. (86).

the equivalent language of foliations to describe a Vlasov bivector. Each Ψ is constructed in such a way that the 2-dimensional leaves, see fig. 1 and fig. 5, that compose their foliations intersect any given kinematic domain E to produce 1-dimensional curves, visualised in fig. 6. These curves are exactly the integral curves of a Vlasov field associated with Ψ which is tangent to E .

Vlasov bivectors also have the added benefit of accounting for all projectively related Vlasov fields. Consequently, the Vlasov bivector accounts for all parametrisations in tandem without explicit reference to any of them. Furthermore, the equation for transforming Vlasov fields between kinematic domains can be easily derived from the Vlasov bivector, hence the choice to describe our formalism in terms of bivectors instead of foliations. Vlasov bivectors are the main topic of discussion in section 3.

In general, it is not informative to attempt to extend the particle density function f_E to U . Instead, we first replace f_E with the particle density 6-form θ_E . This is depicted in fig. 4. This 6-form has two conditions, equivalent to eq. (1), which together we call the *transport equations*. The first states that θ_E is closed, $d\theta_E = 0$ which corresponds to the form manifolds not terminating (i.e. no particles being created or lost). The second states that the Vlasov field is tangent to θ_E i.e. $i_{W_E}\theta_E = 0$. Furthermore, the transport equations apply to a more general set of theories than simply the Vlasov equation. The transport equations are discussed in section 2.5.

The geometric notion of the transport equations can be more easily translated into our formalism. In our case,

the particle density form on E , θ_E , is replaced with the particle density form $\theta \in \Gamma\Lambda^{2n-2}U$. This $(2n-2)$ -form on a $2n$ -dimensional manifold is also depicted in figure fig. 1, using the idea of form-submanifolds described in [24]. The closure of θ corresponds to the fact that the form-submanifolds do not have boundaries in U , they are also tangent to all projectively related Vlasov fields W . In section 4.1 the transport equations are generalised to U in terms of the Vlasov bivector and the particle density form on U .

Having established the Vlasov bivector, W , and the particle density form θ_E , we show how to relate them to the corresponding Vlasov vector W_E and density form θ_E . Of course, this is only possible if we are dealing with a time-orientable system. We say that the conic bundle is time-orientable if we can write U in terms of a disjoint union of future pointing and past pointing vectors, discussed in section section 1.6. When U is time-orientable, we show how to translate between W and W_E , and between θ and θ_E .

1.4 Signposting

The remainder of section 1 is devoted to the notation and conventions used throughout this paper. In section 2 we discuss kinematic domains and the various objects built on them. We begin by introducing the kinematic domain and its properties in section 2.1: a generalisation of the mass shell and lab-time bundles upon which we can perform kinematics. In this subsection we also introduce the Vlasov field W on the conic bundle U and explore its relationship with the Vlasov field W_E on kinematic domains. A useful tool for describing a kinematic domain is a kinematic indicator, these are introduced and discussed in section 2.2. Integral curves of Vlasov fields are discussed in section 2.3. We also discuss the conditions for which two different Vlasov fields correspond to the same trajectories. In section 2.4 we discuss how to transform Vlasov fields between kinematic domains in such a way that the trajectories of particles are preserved. In section 2.5 we introduce the transport equations, a geometric method of interpreting the Vlasov equation on E . We also discuss some advantages of this approach. Next, section 2.6 gives an example of how we may transform Vlasov fields between kinematic domains.

In section 3 we introduce the Vlasov bivector Ψ . First, the necessary properties to define Vlasov bivectors are discussed in section 3.1 and section 3.2. Vlasov bivectors themselves, their benefits, and geometric interpretation are given in section 3.3. We also illustrate the correspondence between bivectors and foliations here.

Section 4 deals with the particle density form on U and its applications. The transport equations are generalised to the conic bundle using the Vlasov bivector in section 4.1. We also discuss the geometric interpretation of these new transport equations here. In section 4.2, we explore the conditions necessary to define a particle density form on U given a particle density form on a kinematic domain E and vice versa. We then apply the particle density on U to define a current $(n-1)$ -form on M in section 4.3. We also show that this current form is the same as the current form typically defined on E . In section section 4.4 we define the stress-energy 3-forms. We see that unlike the case of the current, these depend on the choice of kinematic domain.

In section 5, numerical applications for the framework are discussed. We begin in section 5.1 with a discussion of the application of lab time in simulations, particularly in relativistic cases. We then discuss a particular integration scheme, the ANI and its generalisation in section 5.2. This includes the introduction of an integration scheme which only uses one interpolation, and an application of our formalism: we perform a Lorentz boost on a Vlasov-Poisson system to to write the equation for integrating the Vlasov field in a new frame. We conclude in section 6.

In the appendices, we discuss the relationship between our work and sprays in section A.1, some useful auxiliary lemmas are contained in section A.2, and proofs which are not discussed in the body of the work are given in section A.3.

1.5 Notation and Conventions for general manifolds

Throughout this paper Greek indices will run from 0 to $n-1$ ($\mu, \nu = 0, 1, 2, \dots, n-1$) and Latin indices will run from 1 to $n-1$ ($a, b = 1, 2, \dots, n-1$), unless specified otherwise. We will use M to denote a connected n -dimensional manifold and TM to denote the tangent bundle over M (all manifolds are assumed to be smooth). Points in M will generally be denote $p, q \in M$. Vectors on M which are points in TM (and consequently the conic bundle U) will be denoted $\underline{u} \in U$. The space of vector fields over an arbitrary smooth manifold N of dimension ℓ is denoted ΓTN and its elements are denoted $X, Y \in \Gamma TN$. Maps between abstract manifolds are denoted $\Theta : N \rightarrow P$. The space of scalar fields is denoted $\Gamma\Lambda^0 N$ and we denote its elements $f, g, h, F, G, H \in \Gamma\Lambda^0 N$ depending on context. The space of r -forms is given by $\Gamma\Lambda^r N$ and we denote its elements by $\alpha, \beta \in \Gamma\Lambda^r N$. Although we consider both vectors at points and vector fields, we only consider r -form fields. Hence when referring to r -forms, the word field is implicit. A field

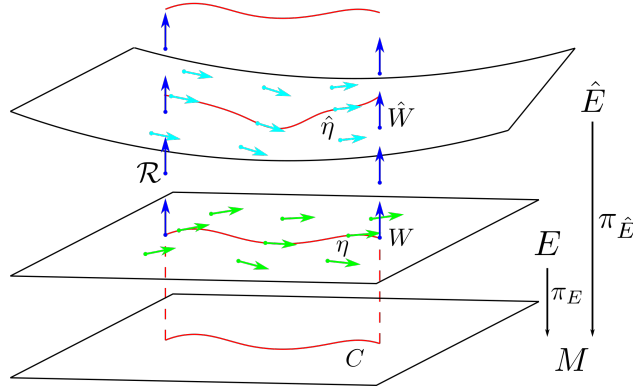


Figure 6: Given a two dimensional manifold M and two different kinematic domains E and \hat{E} , we can observe the relationship between two Vlasov fields W and \hat{W} related by theorem 2.18. The Vlasov field W (resp. \hat{W}), depicted by green (light blue) arrows, are tangent to E (\hat{E}) and generate integral curves η ($\hat{\eta}$), depicted by red lines. Both integral curves project down into the same curve on M , denoted C , also a red line. The radial vector field \mathcal{R} is denoted by dark blue arrows. This diagram can be identified with fig. 3.

Object Type	Set of Objects	Generic	Specific
Manifolds	N.A.	N, P, K	M, E, U
Scalar Fields	$\Gamma\Lambda^0 N$	h, k	f, F, g
Vector Fields	ΓTN	X, Y, Z, V	W
p -forms	$\Gamma\Lambda^p N$	$\alpha, \beta, \gamma, \omega$	$\theta, \theta_E, \chi, \mathcal{J}, \mathcal{J}_E$
Bivectors	$\Gamma\mathcal{B}^2(N)$	Φ	Ψ

Table 1: Table of symbols use throughout. The objects in the column marked specific are reserved for specific objects e.g. $W \in \Gamma TU$ is reserved for the Vlasov field, θ is used for the particle density form etc.

evaluated at a point will be denoted by $X|_p$ and the action of a vector field on a scalar field $X\langle h \rangle$. A table of generic and specific symbols is given in table 1.

The internal contraction of an r -form α by a vector field X is given by $i_X \alpha$ (or $\alpha : X$ for the special case when α is a 1-form), and the Lie derivative of α along X is denoted by $L_X \alpha$. Contraction by coordinate partial derivatives are denoted $i_{\partial_\mu}^{(x)} = i_{\partial_\mu^{(x)}}$ etc. A similar convention is used for Lie derivatives along a coordinate vector field where we denote $L_{\partial_\mu^{(x)}} = L_\mu^{(x)}$ etc.

Definition 1.1 (Pullback and Pushforward). Given a map between manifolds $\Theta : N \rightarrow P$, then the *pushforward* of a vector at a point $\underline{v} \in T_p N$ is given by

$$(\Theta_* \underline{v}) \langle h \rangle = \underline{v} \langle h \circ \Theta \rangle. \quad (3)$$

The *pullback* of a scalar field $h \in \Gamma\Lambda^0 P$ is given by

$$\Theta^* h = (h \circ \Theta). \quad (4)$$

We may also pullback forms of arbitrary degree r according to the following rules:

$$\Theta^*(dh) = d(\Theta^* h) \quad \text{and} \quad \Theta^*(\alpha \wedge \beta) = \Theta^* \alpha \wedge \Theta^* \beta, \quad (5)$$

for $h \in \Gamma\Lambda^0 N$, $\alpha \in \Gamma\Lambda^1 N$, $\beta \in \Gamma\Lambda^{r-1} N$. The pullback and pushforward satisfy the compatibility property:

$$\alpha : \Theta_* \underline{v} = \Theta^* \alpha : \underline{v}, \quad (6)$$

for any 1-form $\alpha \in \Gamma\Lambda^1 N$ and point vector $\underline{v} \in T_p N$. Observe that we pushforward vectors at points, while we pull back r -forms.

Definition 1.2 (Tangential Vector Fields). Given an embedding $\Theta : K \hookrightarrow N$, a vector field $X \in \Gamma TN$ is *tangent* to the submanifold K if, there exists a vector field $Y \in \Gamma TK$ such that

$$\Theta_*(Y|_p) = X|_{\Theta(p)}, \quad \forall p \in K. \quad (7)$$

Note that if Y exists then it is unique. In this case, we say that $Y \in \Gamma TK$ is induced by X . Given an r -form $\alpha \in \Gamma \Lambda^r N$ then

$$\Theta^*(i_X \alpha) = i_Y(\Theta^* \alpha). \quad (8)$$

Definition 1.3 (Scalar Lift). Given a scalar field $h \in \Gamma \Lambda^0 N$, the scalar lift defines a *scalar field* $\hat{h} \in \Gamma \Lambda^0 TN$ given by

$$\hat{h}|_{\underline{u}} = \underline{u}\langle h \rangle. \quad (9)$$

Definition 1.4 (Induced Coordinates). Given a coordinate system (x^0, \dots, x^{n-1}) for the patch $K \subset N$, this induces a coordinate system $(\bar{x}^0, \dots, \bar{x}^{n-1}, \dot{x}^0, \dots, \dot{x}^{n-1})$ for $\pi^{-1}(K) \subset TN$, where $\bar{x}^\mu = \pi^* x^\mu$ and \dot{x}^μ is the scalar lift. In this coordinate system an arbitrary vector field $X \in \Gamma TN$ can be written

$$X = \bar{X}^\mu \frac{\partial}{\partial \bar{x}^\mu} + \hat{X}^\mu \frac{\partial}{\partial \dot{x}^\mu} \quad \text{where} \quad \bar{X}^\mu = X\langle \bar{x}^\mu \rangle \text{ and } \hat{X}^\mu = X\langle \dot{x}^\mu \rangle. \quad (10)$$

Every vector field over TN can be defined using eq. (10) and consequently can be defined entirely by its action on scalar fields of type $\pi^* h$ and \hat{h} for all $h \in \Gamma \Lambda^0 N$.

Throughout, we will use the shorthand

$$\partial_\mu^{(x)} = \frac{\partial}{\partial x^\mu}, \quad \partial_\mu^{(\bar{x})} = \frac{\partial}{\partial \bar{x}^\mu}, \quad \partial_\mu^{(\dot{x})} = \frac{\partial}{\partial \dot{x}^\mu}. \quad (11)$$

Note that $\partial_\mu^{(x)} \in \Gamma TM$ while $\partial_\mu^{(\bar{x})} \in \Gamma TTM$ and $\partial_\mu^{(\dot{x})} \in \Gamma TTM$. In induced coordinates, the scalar lift of a scalar field $h \in \Gamma \Lambda^0 M$ is given by

$$\hat{h} = \dot{x}^\mu \partial_\mu^{(x)} h. \quad (12)$$

In general, we will not distinguish between \bar{x} and x unless necessary.

When convenient, the exterior product of coordinate differential forms will be written

$$dx^0 \wedge \dots \wedge dx^{n-1} = dx^{0, \dots, n-1}. \quad (13)$$

1.6 Notation and Conventions specific for this article

Let M be a spacetime manifold and $\pi : TM \rightarrow M$, the tangent bundle over M . The slit tangent bundle $\check{T}M = TM \setminus \{0\}$ is the subset of TM which excludes the zero vectors.

Definition 1.5 (Conic Sub-Bundle). A *conic sub-bundle* is a subset of the slit tangent bundle $U \subset \check{T}M$, which satisfies the following properties:

1. $\pi(U) = M$
2. If $\underline{u} \in U$ then for any $\lambda \neq 0$ we have $\lambda \underline{u} \in U$.

We also use π to also denote the projection $\pi : U \rightarrow M$. A formal discussion of conic bundles can be found in [18]. We additionally assume the properties necessary for U to have a smooth sub-bundle structure.

Definition 1.6 (Causal Indicator). We say U is *time-orientable* if $U = U^+ \cup U^-$ where $U^+ \cap U^- = \emptyset$ and $\underline{u} \in U^+$ if and only if $-\underline{u} \in U^-$. We call $\underline{u} \in U^+$ *future pointing*. The *causal indicator* is the scalar field $\sigma \in \Gamma \Lambda^0 U$ given by

$$\sigma|_{\underline{u}} = \begin{cases} 1, & \text{if } \underline{u} \in U^+ \\ -1, & \text{if } \underline{u} \in U^- \end{cases} \quad (14)$$

Examples of structures described by conic bundles include the collection of time-like vectors over a spacetime manifold, the light cone, and the causal region of a time-orientable manifold. In the case where we consider only the time-like vectors, the corresponding conic bundle has an open sub-bundle structure. In the case where we consider the causally connected regions of spacetime (the former case plus the light cone), the corresponding conic bundle is a sub-bundle with a boundary. However, it is not a closed set since it excludes the zero vectors.

An important example of a conic bundle which is not time-orientable is $U = \check{T}M$, for any manifold of dimension greater than 1. This is because each fibre, $\check{T}_p M$, is connected.

As discussed in the introduction, when dealing with kinematic domains we restrict our attention to time-orientable conic bundles. By contrast, when dealing with the Vlasov bivector approach, we do not have to make this restriction. Thus, the bivector approach can be applied to the case when $U = \check{T}M$.

Definition 1.7 (Radial Vector Field). The *radial vector field*, denoted by $\mathcal{R} \in \Gamma TU$, is the unique vector field that satisfies the following condition: for any $h \in \Gamma \Lambda^0 M$ and $\underline{u} \in U$ we have

$$\mathcal{R}\langle \pi^* h \rangle = 0, \quad \text{and} \quad \mathcal{R}\langle \dot{h} \rangle|_{\underline{u}} = \dot{h}|_{\underline{u}} = \underline{u}\langle h \rangle. \quad (15)$$

In local induced coordinates \mathcal{R} is given by

$$\mathcal{R} = \dot{x}^\mu \partial_\mu^{(\dot{x})}. \quad (16)$$

This can be seen by acting \mathcal{R} on the induced coordinates: $\mathcal{R}\langle \bar{x}^\mu \rangle = 0$, $\mathcal{R}\langle \dot{x}^\mu \rangle = \dot{x}^\mu$. Within the literature, the radial vector field is known by many names, perhaps the most common of which is the vertical vector field. We have opted to refer to it as the radial vector field as it exhibits many useful radial properties.

Definition 1.8 (Homogeneity). A scalar field $G \in \Gamma \Lambda^0 U$ is said to be *homogeneous of degree k* (also known as radially homogeneous or fibre-wise homogeneous) if

$$G|_{\lambda \underline{u}} = \lambda^k G|_{\underline{u}} \quad \text{for all} \quad \lambda \in \mathbb{R} \setminus \{0\}. \quad (17)$$

Note that by Euler's theorem of homogeneous functions, a function $G \in \Gamma \Lambda^0 U$ is (fibre-wise) k -homogeneous if and only if it satisfies

$$\mathcal{R}\langle G \rangle = kG. \quad (18)$$

2 Vlasov Systems and Kinematic Domains

2.1 Kinematic Domains and the Vlasov Picture

In order to discuss the Vlasov field we must first establish the space over which it is defined. We call such submanifolds kinematic domains. For this section we assume that U is time-orientable in the sense of definition 1.6, and that $E \subset U^+$, that is all points in E are future pointing. As stated in the introduction, this constraint is not needed for the generalisation to Vlasov bivectors, discussed below. We assume that each spacetime manifold M (and hence TM) is connected.

Definition 2.1 (Kinematic Domain). A *Kinematic domain* is a $(2n-1)$ -dimensional submanifold $E \subset U^+$ which satisfies the following properties:

1. $\pi_E : E \rightarrow M$ is surjective,
2. E is connected,
3. For any $\underline{u} \in U$ there exists a unique $\underline{v} \in E$ and $\lambda \neq 0$ such that $\underline{u} = \lambda \underline{v}$.

The inclusion map for E into U is given by $\Sigma_E : E \hookrightarrow U$.

For example, the unit mass shell, or as we refer to it throughout the remainder of this paper, the upper unit hyperboloid, given in eq. (2), is a kinematic domain. We can write this as

$$E_H = \{ \underline{u} \in U : F_H = 1 \text{ and } \sigma|_{\underline{u}} = 1 \}, \quad (19)$$

where

$$F_H : U \rightarrow \mathbb{R}; \quad \underline{u} \mapsto -g_{\pi(\underline{u})}(\underline{u}, \underline{u}). \quad (20)$$

Another example of a kinematic domain is the lab-time bundle, with respect to a time-slicing scalar field t ,

$$E_t = \{\underline{v} \in U : \underline{v}\langle t \rangle = 1\}. \quad (21)$$

Recall that a time-slicing scalar $t \in \Gamma\Lambda^0 M$, requires that if $\sigma(\underline{v}) = 1$ then $\underline{v}\langle t \rangle > 0$.

An example from Finsler geometry¹ is defined in [25], where the Finsler spacetime (M, L, F) admits an observer bundle

$$\mathcal{O} = \bigcup_{p \in M} S_p, \quad (22)$$

where

$$\mathcal{O}_p = \left\{ \underline{u} \in T_p M : L|_{\underline{u}} = \pm 1, g_{\mu\nu}^{(L)}|_{\underline{u}} \text{ has signature } (L, -L, -L, -L) \right\}, \quad (23)$$

and S_p is a non-empty closed connected component of \mathcal{O}_p . Here $g_{\mu\nu}^{(L)}$ is the metric induced by the fundamental function L .

We refer to two types of Vlasov fields throughout this paper: Vlasov fields on kinematic domains (denoted $W_E \in \Gamma TE$ for $E \subset U$), and Vlasov fields on U (denoted $W \in \Gamma TU$).

Definition 2.2 (Vlasov field on E). Given a kinematic domain E , a *Vlasov field on E* , $W_E \in \Gamma TE$, is a vector field that satisfies the horizontal condition

$$\pi_{E*}(W_E|_{\underline{v}}) = \underline{v}, \text{ or equivalently, } W_E \langle \pi_E^* h \rangle = \dot{h}, \quad (24)$$

for any $\underline{v} \in E$ and $h \in \Gamma\Lambda^0 E$. This equivalence is due to the fact $W_E|_{\underline{v}} \langle \pi_E^* h \rangle = \pi_{E*}(W_E|_{\underline{v}}) \langle h \rangle = \underline{v} \langle h \rangle = \dot{h}|_{\underline{v}}$, for all $\underline{v} \in E$ and $h \in \Gamma\Lambda^0 M$.

Given a particle density function $f_E \in \Gamma\Lambda^0 E$, the Vlasov field $W_E \in \Gamma TE$ satisfies eq. (1). The flow of the Vlasov field defines a set of integral curves which trace out the paths of particles through phase space (see section 2.3). The projections of these integral curves into the base space M are the trajectories of the particles described by f_E .

An example of the Vlasov field on E_H is the Lorentz force Vlasov equation. In local coordinates (x^μ, v^a) on E_H , with the embedding given by $\dot{x}^a = v^a$. Define v^0 by solving the constraint $g_{\mu\nu} v^\mu v^\nu = -1$ and choosing the root, such that $v^\mu \partial_\mu^{(x)} \in U^+$ (note that here the x^μ in $\partial_\mu^{(x)}$ is a coordinate on M). In this coordinate system, the Lorentz force Vlasov field is given by

$$W_{E_H} = v^\mu \partial_\mu^{(x)} - \left(\Gamma_{\nu\rho}^a v^\nu v^\rho - \frac{q}{m} g^{a\nu} \mathcal{F}_{\nu\rho} v^\rho \right) \partial_a^{(v)}. \quad (25)$$

Here $\mathcal{F} \in \Gamma\Lambda^2 M$ is the Faraday 2-form which satisfies the Maxwell equations. Dividing by v^0 gives the usual Vlasov field on E_H .

Definition 2.3 (Vlasov Field on U). Denoted $W \in \Gamma TU$, a *Vlasov field on U* is a vector field with the following defining properties:

1. W is horizontal,

$$\pi_*(W|_{\underline{u}}) = \underline{u}, \text{ or equivalently, } W \langle \pi^* h \rangle = \dot{h}, \quad (26)$$

for any point vector $\underline{u} \in U$ and $h \in \Gamma\Lambda^0 M$;

2. W is radially quadratic,

$$W \langle \dot{h} \rangle|_{\lambda \underline{u}} = \lambda^2 W \langle \dot{h} \rangle|_{\underline{u}}, \quad (27)$$

for any $h \in \Gamma\Lambda^0 M$, $\underline{u} \in U$, and $\lambda \in \mathbb{R}^+$.

From eq. (27) and eq. (18) the radially quadratic property can equivalently be stated as

$$\mathcal{R} \langle W \langle \dot{h} \rangle \rangle = 2W \langle \dot{h} \rangle. \quad (28)$$

¹Recall, examples from Finsler geometry may be ignored for readers unfamiliar with this geometry. For this reason we do not give definitions of standard Finsler objects.

Lemma 2.4. *Let $W \in \Gamma TU$ be horizontal. Then W is radially quadratic and hence a Vlasov field if and only if*

$$[\mathcal{R}, W] = W. \quad (29)$$

Proof. Let W be radially quadratic then for any $f \in \Gamma \Lambda^0 M$, $W\langle \dot{f} \rangle$ is a 2-homogeneous scalar field. It follows that $\mathcal{R}\langle W\langle \dot{f} \rangle \rangle = 2W\langle \dot{f} \rangle$ by eq. (28). Hence $[\mathcal{R}, W]\langle \dot{f} \rangle = \mathcal{R}\langle W\langle \dot{f} \rangle \rangle - W\langle \mathcal{R}\langle \dot{f} \rangle \rangle = W\langle \dot{f} \rangle$. From eq. (26), we also have

$$[\mathcal{R}, W]\langle \pi^* f \rangle = \mathcal{R}\langle \dot{f} \rangle = \dot{f} = W\langle \pi^* f \rangle,$$

and hence $[\mathcal{R}, W] = W$.

Suppose that $[\mathcal{R}, W] = W$, then for any $f \in \Gamma \Lambda^0 M$ we have

$$W\langle \dot{f} \rangle = [\mathcal{R}, W]\langle \dot{f} \rangle = \mathcal{R}\langle W\langle \dot{f} \rangle \rangle - W\langle \dot{f} \rangle. \quad (30)$$

Rearranging the above gives $\mathcal{R}\langle W\langle \dot{f} \rangle \rangle = 2W\langle \dot{f} \rangle$. Hence W is radially quadratic. \square

In fig. 3 we see two Vlasov fields on U that represent the same trajectories on M , just with different parametrisations. One is tangent to the unit hyperboloid E_H while the other is tangent to a lab-time bundle E_t . In section 2.4 we see the formula for transforming from one Vlasov field to another in such a way that the particle trajectories are unaffected.

The Vlasov field on U can be written in local coordinates as

$$W = \dot{x}^\mu \partial_\mu^{(x)} + \varphi^\mu \partial_\mu^{(x)}, \quad (31)$$

where φ^μ are 2-homogeneous scalar fields, $\varphi^\mu|_{\lambda \underline{u}} = \lambda^2 \varphi^\mu|_{\underline{u}}$. The Vlasov field W on U can be reduced to a Vlasov field on E provided it is tangent to it.

Definition 2.5 (Extension of a Vlasov Field on E). Given a kinematic domain E with inclusion map $\Sigma_E : E \hookrightarrow U$ and a Vlasov field $W_E \in \Gamma TE$, we call $W \in \Gamma TU$ the extension of W_E if W is tangent to E and W_E is induced by W , as in eq. (7),

$$\Sigma_{E*} \left(W_E|_{\underline{v}} \right) = W|_{\Sigma_E(\underline{v})}, \quad (32)$$

for all $\underline{v} \in E$.

Lemma 2.6. *Let W be a Vlasov field and G be any k -homogeneous scalar field. Then for any $\lambda \neq 0$ and $\underline{u} \in U$,*

$$W|_{\lambda \underline{u}} \langle G \rangle = \lambda^{k+1} W|_{\underline{u}} \langle G \rangle.$$

Proof. Let G be a k -homogeneous function. First note that

$$\partial_\mu^{(x)} \left(G|_{\lambda \underline{u}} \right) = \lambda \left(\partial_\mu^{(x)} G \right)|_{\lambda \underline{u}},$$

by the chain rule. Hence we have

$$\left(\partial_\mu^{(x)} G \right)|_{\lambda \underline{u}} = \lambda^{-1} \partial_\mu^{(x)} \left(G|_{\lambda \underline{u}} \right) = \lambda^{-1} \partial_\mu^{(x)} \left(\lambda^k G|_{\underline{u}} \right) = \lambda^{k-1} \left(\partial_\mu^{(x)} G \right)|_{\underline{u}}.$$

We also have

$$\left(\partial_\mu^{(x)} G \right)|_{\lambda \underline{u}} = \partial_\mu^{(x)} \left(\lambda^k G|_{\underline{u}} \right) = \lambda^k \left(\partial_\mu^{(x)} G \right)|_{\underline{u}}.$$

By expanding the Vlasov field in coordinates we get

$$\begin{aligned} W|_{\lambda \underline{u}} \langle G \rangle &= \left(\dot{x}^\mu \partial_\mu^{(x)} G + \varphi^\mu \partial_\mu^{(x)} G \right)|_{\lambda \underline{u}} = \lambda \underline{u} \lambda^k \left(\partial_\mu^{(x)} G \right)|_{\underline{u}} + \lambda^2 \varphi^\mu|_{\underline{u}} \lambda^{k-1} \left(\partial_\mu^{(x)} G \right)|_{\underline{u}} \\ &= \lambda^{k+1} \left(\dot{x}^\mu \partial_\mu^{(x)} G + \varphi^\mu \partial_\mu^{(x)} G \right)|_{\underline{u}} = \lambda^{k+1} W|_{\underline{u}} \langle G \rangle, \end{aligned}$$

hence the result. \square

It is worth noting at this point that what we call Vlasov fields on U are referred to as sprays and W_E are referred to as semi-sprays in literature (see [26] or [27] for an overview of the theory of sprays and semi-sprays). The correspondence between these objects is explored in section A.1.

We now give the example of the Lorentz force equation and the corresponding Vlasov fields adapted to the unit hyperboloid E_H and the lab bundle E_t . Here U is the conic bundle of time-like vectors. The Vlasov field on U adapted to E_H is given by

$$W = \dot{x}^\mu \partial_\mu^{(\dot{x})} + \left(\frac{q}{m} \sigma \sqrt{F_H} g^{\mu\nu} F_{\nu\rho} \dot{x}^\rho - \Gamma_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho \right) \partial_\mu^{(\dot{x})}, \quad (33)$$

where F_H is as defined in eq. (20). The inclusion of the factor $\sigma \sqrt{F_H}$ is to ensure the $\partial_\mu^{(\dot{x})}$ term is 2- and future pointing. The parameterisation associated with the trajectories of this Vlasov field is proper time τ . The Vlasov field on U describing particles subject to the Lorentz force in the lab-time bundle W_{E_t} with associated lab-time t is given by

$$\begin{aligned} \hat{W} &= \dot{x}^\mu \partial_\mu^{(\dot{x})} + \left(\frac{q}{m} \sigma \sqrt{F_H} g^{\mu\nu} F_{\nu\rho} \dot{x}^\rho - \Gamma_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho \right) \partial_\mu^{(\dot{x})} \\ &\quad - \left(\frac{q}{m} \sigma \sqrt{F_H} g^{\lambda\nu} F_{\nu\rho} \dot{x}^\rho \frac{\partial t}{\partial x^\lambda} - \Gamma_{\nu\rho}^\lambda \dot{x}^\nu \dot{x}^\rho \frac{\partial t}{\partial x^\lambda} + \dot{x}^\nu \dot{x}^\rho \frac{\partial t}{\partial x^\nu \partial x^\rho} \right) \frac{\dot{x}^\mu}{i} \partial_\mu^{(\dot{x})}. \end{aligned} \quad (34)$$

When using a lab-time bundle it is often convenient to choose a coordinate system (t, x^1, x^2, x^3) adapted to the lab-time t . In this coordinate system eq. (34) becomes

$$\hat{W} = \dot{x}^\mu \partial_\mu^{(\dot{x})} + \left(\frac{q}{m} \sigma \sqrt{F_H} g^{\mu\nu} F_{\nu\rho} \dot{x}^\rho - \Gamma_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho \right) \partial_\mu^{(\dot{x})} - \left(\frac{q}{m} \sigma \sqrt{F_H} g^{0\nu} F_{\nu\rho} \dot{x}^\rho - \Gamma_{\nu\rho}^0 \dot{x}^\nu \dot{x}^\rho \right) \frac{\dot{x}^\mu}{i} \partial_\mu^{(\dot{x})}, \quad (35)$$

where $x^0 = t$. We see below in subsection 2.6, that we can demonstrate the transformation between eq. (33) and eq. (34) using the underlying 2nd order ODEs as done in lemma 2.17. However, it is much easier to calculate these transformations after we have defined the kinematic indicator in the following subsection. This leads to the main result of this section in theorem 2.18.

2.2 Kinematic Indicators

A kinematic domain can be defined in terms of a scalar field on U , we call this field a kinematic indicator.

Definition 2.7 (Kinematic Indicator). A *kinematic indicator* for E , is a non-vanishing scalar field $F \in \Gamma\Lambda^0 U$ with non-zero integer degree of homogeneity k , $F|_{\lambda\underline{v}} = \lambda^k F|_{\underline{v}}$, such that F is positively valued on U^+ , and E is given by

$$E = \{ \underline{u} \in U : F|_{\underline{u}} = a \text{ and } \sigma|_{\underline{u}} = 1 \}, \quad (36)$$

for some positive number a . Note that if k is odd the inclusion of the $\sigma|_{\underline{u}} = 1$ condition is unnecessary.

For example F_H , given in eq. (20) is a kinematic indicator for E_H , the unit hyperboloid. A second example $F = i$, is a kinematic indicator for E_t , the lab-time bundle.

Lemma 2.8. Given an arbitrary kinematic domain E , there exists a unique 1-homogeneous kinematic indicator $F \in \Gamma\Lambda^0 U$ such that $E = \{ \underline{u} \in U : F|_{\underline{u}} = 1 \}$. For each $\underline{u} \in U$ this is given by

$$F|_{\underline{u}} = \lambda \quad \text{where } \underline{u} = \lambda \underline{v} \text{ for a unique } \underline{v} \in E. \quad (37)$$

Proof. For each $\underline{u} \in U$, there exists a unique $\lambda \in \mathbb{R} \setminus \{0\}$ and $\underline{v} \in E$ such that $\underline{u} = \lambda \underline{v}$ by the conic properties of U (definition 1.5). We define then $F|_{\underline{u}} = \lambda$. Since each component of E is connected F is smooth and hence $F \in \Gamma\Lambda^0 U$. To see that F is unique, suppose there is another scalar F' which is 1-homogeneous and is such that such that $E = \{ \underline{u} \in U : F'|_{\underline{u}} = 1 \}$ and pick any $\underline{u} \in U$. Then $F'|_{\underline{u}} = F'|_{\lambda\underline{v}} = \lambda = F|_{\underline{u}}$. \square

Given a kinematic indicator F for a kinematic domain E as defined in lemma 2.8, we may define another scalar for some $k \in \mathbb{Z}$, $k \neq 0$ and $a \in \mathbb{R}^+$ by

$$\hat{F} = a F^k. \quad (38)$$

Not only is \hat{F} a k -homogeneous kinematic indicator for E , it is also unique for the chosen k and a .

Lemma 2.9. Let F be a kinematic indicator for E as given by lemma 2.8 and let \hat{F} be given by eq. (38). \hat{F} is a kinematic indicator for E .

Proof. Let \hat{E} be a contour of \hat{F} as given by eq. (36). For and $\underline{v} \in E$ notice that $F^k|_{\underline{v}} = 1$ so that $\hat{F}|_{\underline{v}} = aF^k|_{\underline{v}} = a$. Hence $\underline{v} \in E$ implies $\underline{v} \in \hat{E}$. The converse can be proved similarly to show $\underline{v} \in \hat{E}$ implies $\underline{v} \in E$. Hence \hat{F} is a kinematic indicator for E . \square

Lemma 2.10. Let $a \in \mathbb{R}^+$ and $k \in \mathbb{N}$. If \hat{F} is a k -homogeneous kinematic indicator for E such that eq. (36) is satisfied, then \hat{F} is uniquely given by eq. (38).

Proof. \hat{F} satisfies the following properties:

$$\hat{F}|_{\underline{v}} = a \quad \text{and} \quad \hat{F}|_{\lambda \underline{u}} = \lambda^k \hat{F}|_{\underline{u}},$$

for all $\underline{v} \in E, \underline{u} \in U$ and $\lambda \neq 0$. These two functions can be related by

$$\hat{F}|_{\underline{v}} = a = aF|_{\underline{v}}, \quad \text{and} \quad \frac{(\sigma \hat{F})|_{\lambda \underline{u}}}{\hat{F}|_{\lambda \underline{u}}} = \sigma|_{\lambda \underline{u}} \lambda^k = \sigma|_{\lambda \underline{u}} \left(\frac{F|_{\lambda \underline{u}}}{F|_{\underline{u}}} \right)^k, \quad (39)$$

for all $\underline{v} \in E, \underline{u} \in U$ and $\lambda \neq 0$.

By the conic property, for any $\underline{u} \in U$ there exists a unique $\underline{v} \in E$ and $\lambda \neq 0$ such that $\underline{u} = \lambda \underline{v}$. By plugging such a \underline{v} into eq. (39) we get

$$\frac{\hat{F}|_{\lambda \underline{v}}}{a} = \left(\frac{F|_{\lambda \underline{v}}}{1} \right)^k \implies \hat{F}|_{\underline{u}} = aF^k|_{\underline{u}}.$$

Hence $\hat{F} = aF^k$. \square

In the case of the upper unit hyperboloid F_H (eq. (20)), the kinematic indicator is 2-homogeneous. The equivalent 1-homogeneous kinematic indicator is given by

$$\hat{F}_H|_{\underline{u}} = \sigma|_{\underline{u}} \sqrt{F_H|_{\underline{u}}}. \quad (40)$$

where σ is given by definition 1.6.

At this point, we may observe some similarities with foliations of the tangent bundle based on Finsler metrics [28]. The function F which defines E can be a Finsler metric; however, it is not a necessary condition. The only requirement for F is that it be homogeneous of some degree.

Definition 2.11 (Compatible Vlasov Field). Given a kinematic indicator F with associated kinematic domain E , a Vlasov field $W \in \Gamma TU$ is said to be compatible with F if it satisfies $W \langle F \rangle = 0$. The restriction of W to a compatible E is then denoted $W_E \in \Gamma TE$ as given by eq. (32).

Note that by lemma A.5, W is tangent to a kinematic domain E with kinematic indicator F if and only if W is compatible with F .

Lemma 2.12. Given a kinematic domain E and a Vlasov field $W_E \in \Gamma TE$, there exists a unique Vlasov field $W \in \Gamma TU$ which is an extension of W_E , or equivalently W_E is induced by W .

Proof. First observe that for any $f \in \Gamma \Lambda^0 M$ and $\underline{v} \in E$,

$$(\dot{f} \circ \Sigma_E)|_{\underline{v}} = \dot{f}|_{\Sigma_E(\underline{v})} = \dot{f}|_{\underline{v}}.$$

It follows that $\dot{f} \circ \Sigma_E = \dot{f}$.

For all $\underline{u} \in U$ and $f \in \Gamma \Lambda^0 M$, define $W \in \Gamma TU$

$$\pi_*(W|_{\underline{u}}) = \underline{u} \quad \text{and} \quad W|_{\underline{u}} \langle \dot{f} \rangle = \lambda^2 W_E|_{\underline{v}} \langle \dot{f} \rangle,$$

where $\underline{u} = \lambda \underline{v}$ by the conic property for $\underline{v} \in E$ and $\lambda \neq 0$. By construction W is radially quadratic and horizontal so it remains to show that it is unique.

Let both W and \hat{W} induce W_E as defined above and set $X = \hat{W} - W$. Observe that $\pi_* X = 0$ and

$$X|_{\underline{u}} \langle \dot{f} \rangle = \lambda^2 (W_E - W_E)|_{\underline{v}} \langle \dot{f} \rangle = 0.$$

Hence it follows that $X = 0$ and W is unique.

To see that W is compatible with F (and hence tangent to E by lemma A.5) consider the following. Without loss of generality, we may assume F is 1-homogeneous. By lemma 2.6 we have $W|_{\underline{u}} \langle F \rangle = \lambda^2 W|_{\underline{v}} \langle F \rangle$. It follows that

$$W|_{\underline{u}} \langle F \rangle = \lambda^2 W|_{\underline{v}} \langle F \rangle = \lambda^2 (\Sigma_{E^*} W_E|_{\underline{v}}) \langle F \rangle = \lambda^2 W_E|_{\underline{v}} \langle F \circ \Sigma_E \rangle = \lambda^2 W_E|_{\underline{v}} \langle 1 \rangle = 0.$$

□

2.3 Trajectories, Prolongations, and the Horizontal Condition on U

The notion of integral curves for Vlasov fields on kinematic domains can be extended to integral curves on U . The Vlasov field W on U generates a set of integral curves on U , denoted η . The projections of these curves onto M are exactly the trajectories of the particles and are denoted C . Since we restrict our attention here to integral curves of horizontal vector fields, the terms prolongation and integral curve can be used interchangeably.

Trajectories can be considered in terms of maps from intervals of the real line $\mathcal{I} \subset \mathbb{R}$ into the spacetime manifold M :

$$C : \mathcal{I} \hookrightarrow M. \quad (41)$$

These trajectories can be parameterised by choosing a parameter, $s \in \Gamma \Lambda^0 \mathcal{I}$ with $ds \neq 0$. We then write the parameterised trajectories as $C(s)$.

Definition 2.13 (Prolongations). For some $s \in \Gamma \Lambda^0 \mathcal{I}$ and $\partial_s \in \Gamma T \mathcal{I}$, the *prolongation* of a curve $C : \mathcal{I} \hookrightarrow M$ is given by

$$\eta : \mathcal{I} \hookrightarrow U, \quad \eta(s_0) = C_*(\partial_s|_{s_0}), \quad \forall s_0 \in \mathcal{I}. \quad (42)$$

Lemma 2.14. $\eta : \mathcal{I} \hookrightarrow U$ is the prolongation of some trajectory C if and only if

$$(\pi \circ \eta)_*(\partial_s) = \eta. \quad (43)$$

Proof. First, suppose η is the prolongation of C . Then $C_*(\partial_s|_{s_0}) \in T_{C(s_0)} M$. Hence $\pi(\eta(s_0)) = \pi(C_*(\partial_s|_{s_0})) = C(s)$, i.e. $\pi \circ \eta = C$ and hence $(\pi \circ \eta)_*(\partial_s) = C_*(\partial_s) = \eta$.

Conversely, assuming eq. (43), let $C = \pi \circ \eta$, then $C_*(\partial_s) = (\pi \circ \eta)_*(\partial_s) = \eta$. □

Let $\hat{\mathcal{I}} \subset \mathbb{R}$ be coordinated by \hat{s} and let $\hat{C} : \hat{\mathcal{I}} \hookrightarrow M$ be an alternative parameterisation of C . I.e. there exists a diffeomorphism $\hat{s} = \hat{s}(s)$ such that

$$\hat{C}(\hat{s}(s)) = C(s). \quad (44)$$

Note that although the two parameterisations define the same curve, the prolongations η and $\hat{\eta}$ do not coincide. Consequently, the tangent vectors along the prolongations (i.e. acceleration) belong to different spaces: $\eta_*(\partial_s|_{s_0}) \in T_{\eta(s_0)} U$ and $\hat{\eta}_*(\partial_{\hat{s}}|_{\hat{s}_0}) \in T_{\hat{\eta}(\hat{s}_0)} U$ even when $C(s_0) = \hat{C}(\hat{s}_1)$.

Definition 2.15. An integral curve η of W is said to lie along E if

$$\eta(s) \in E, \quad \forall s \in \mathcal{I}. \quad (45)$$

This is a necessary condition for η to be an integral curve of W_E .

If W_E is induced by W , η is an integral curve of W , and η lies along E , then η is also an integral curve of W_E .

Lemma 2.16. All integral curves η of a vector field $X \in \Gamma T U$ are prolongations if and only if X is horizontal.

Proof. Let $\underline{u} \in U$ and let η be an integral curve of X i.e. $X|_{\eta(s_1)} = \eta_*(\partial_s|_{s_1})$ for all $s_1 \in \mathcal{I}$, such that $\eta(s_0) = \underline{u}$ for some $s_0 \in \mathcal{I}$.

Suppose first that η is a prolongation. Then

$$\pi_* X|_{\underline{u}} = \pi_* X|_{\eta(s_0)} = \pi_* \eta_*(\partial_s|_{s_0}) = (\pi \circ \eta)_*(\partial_s|_{s_0}) = \eta(s_0) = \underline{u}. \quad (46)$$

Hence X is horizontal.

Suppose now that X is horizontal. We have $\pi_* X|_{\underline{u}} = \underline{u} = \eta(s_0)$. Then,

$$\eta(s_0) = \pi_* X|_{\eta(s_0)} = \pi_* \eta_*(\partial_s|_{s_0}) = (\pi \circ \eta)_*(\partial_s|_{s_0}). \quad (47)$$

Hence η is a prolongation. □

The integral curves η of Vlasov fields $W \in \Gamma TU$ correspond to trajectories C in the base space M through the following relation:

$$C = \pi \circ \eta. \quad (48)$$

In local coordinates (x^μ, \dot{x}^μ) let $C^\mu(s_0) = x^\mu|_{C(s_0)}$ then

$$\dot{C}^\mu(s_0) = \left. \frac{dC^\mu}{ds} \right|_{s_0} \quad \text{and} \quad \ddot{C}^\mu(s_0) = \left. \frac{d^2C^\mu}{ds^2} \right|_{s_0}. \quad (49)$$

These trajectories can be expressed in terms of a parameterised system of second-order differential equations in terms of the coefficients of W :

$$\ddot{C}^\mu|_{s_0} = \varphi^\mu|_{\dot{C}(s_0)}, \quad (50)$$

where s is a parameter corresponding to W and φ^μ are the coefficients of W as given in eq. (31). Due to the projection in eq. (48), there is a class of integral curves that produce the same trajectories. The vector fields which produce these curves differ only by a term $k\mathcal{R}$ where $k \in \Gamma\Lambda^0 U$ is a 1-homogeneous function. The converse is also true, and a full proof of this statement can be found in [26]. These vector fields are said to be projectively related in the literature (see definition A.2 for a definition). We present here a proof of only the former statement to illustrate this result in our mathematical notation.

Lemma 2.17. *Let $W, \hat{W} \in \Gamma TU$ be Vlasov fields and let $k \in \Gamma\Lambda^0 U$ be a 1-homogeneous scalar field. If W and \hat{W} are related by*

$$\hat{W} = W + k\mathcal{R} \quad (51)$$

then they have the same trajectories up to a parameterisation. That is, if the trajectories C of W are parameterised by s then \hat{W} has the same trajectories C only parameterised by \hat{s} where

$$\left. \frac{d^2\hat{s}}{ds^2} \right|_{s_0} + k|_{\underline{u}} \left. \frac{d\hat{s}}{ds} \right|_{s_0} = 0, \quad \frac{d\hat{s}}{ds} > 0, \quad (52)$$

and $\underline{u} = C_*(\partial_s|_{s_0}) \in U$.

Proof. Notice that if W and \hat{W} are related by eq. (51) then in local coordinates we have

$$\hat{\varphi}^\mu = \varphi^\mu + k\dot{x}^\mu,$$

where φ^μ (resp. $\hat{\varphi}^\mu$) are the $\partial_\mu^{(\dot{x})}$ coefficients of W (\hat{W}), see eq. (31). Let the trajectories of W be denoted by $C(s)$. These trajectories satisfy eq. (50). Define a new parameter $\hat{s} = \hat{s}(s)$ by eq. (52) and set $\hat{s}_0 = \hat{s}(s_0)$.

The trajectory $C(\hat{s}(s))$ can be shown to satisfy

$$\left. \frac{d^2C^\mu}{ds^2} \right|_{s_0} = \left. \frac{d^2\hat{s}}{ds^2} \right|_{s_0} \left. \frac{dC^\mu}{d\hat{s}} \right|_{\hat{s}_0} + \left(\left. \frac{d\hat{s}}{ds} \right|_{s_0} \right)^2 \left. \frac{d^2C^\mu}{d\hat{s}^2} \right|_{\hat{s}_0}.$$

We therefore have

$$\begin{aligned} \frac{d^2 C^\mu}{d\hat{s}^2} \Big|_{\hat{s}_0} &= \left(\frac{d\hat{s}}{ds} \Big|_{s_0} \right)^{-2} \left(\frac{d^2 C^\mu}{ds^2} \Big|_{s_0} - \frac{d^2 \hat{s}}{ds^2} \Big|_{s_0} \frac{dC^\mu}{d\hat{s}} \Big|_{\hat{s}_0} \right) = \left(\frac{d\hat{s}}{ds} \Big|_{s_0} \right)^{-2} \left(\varphi^\mu|_{C_*(\partial_s|_{s_0})} + k|_{C_*(\partial_s|_{s_0})} \frac{d\hat{s}}{ds} \Big|_{s_0} \frac{dC^\mu}{d\hat{s}} \Big|_{\hat{s}_0} \right) \\ &= \varphi^\mu|_{\hat{u}} + k|_{\hat{u}} \frac{dC^\mu}{d\hat{s}} \Big|_{\hat{s}_0} = (\varphi^\mu + k\dot{x}^\mu)|_{\hat{u}} = \hat{\varphi}^\mu|_{\hat{u}}, \end{aligned}$$

where $\hat{u} = C_*(\partial_s|_{\hat{s}_0})$. The third line is due to the 2-homogeneity of φ^μ and 1-homogeneity of k . Hence $C(\hat{s})$ are the trajectories associated with \hat{W} . Hence, both W and \hat{W} have the same trajectories up to a reparameterisation. \square

In the case of special relativity, with local coordinates (t, x^a) , where t is a lab-time coordinate and $s = t$, the Lorentz force equation may be written in velocity coordinates as

$$v^0 = 1, \quad \frac{d\vec{v}}{dt} = \frac{1}{\gamma} \frac{q}{m} (\vec{E} + \vec{v} \times \vec{B}) - \frac{1}{\gamma^3} \frac{q}{m} (\vec{v} \cdot \vec{E}) \vec{v}. \quad (53)$$

where \vec{v} is the 3-velocity of the particle and γ is the Lorentz factor. We see that in this case the new (second) term in eq. (51) correspond to the second term in eq. (53), which arises from $\frac{d\gamma}{dt}$.

2.4 Transforming Between Kinematic Domains

Suppose we are given a Vlasov field W_E on a kinematic domain E with kinematic indicator F . Using this data we can construct a new Vlasov field \hat{W} which is compatible with a new kinematic indicator \hat{F} with associated kinematic domain \hat{E} . This defines a new Vlasov field $\hat{W}_{\hat{E}}$ on \hat{E} which produces the same trajectories in the base space M as the initial Vlasov field W_E .

Once we have promoted W_E into a Vlasov field W on U using lemma 2.12, we can use it to construct a new Vlasov field \hat{W} which is tangent to another kinematic domain \hat{E} . A visualisation of the following lemma can be found in fig. 6.

Theorem 2.18. *Let E and \hat{E} be kinematic domains with kinematic indicator F and \hat{F} respectively. Given a Vlasov field $W \in \Gamma TU$ which is compatible with F we may construct a new Vlasov field $\hat{W} \in \Gamma TU$ given by*

$$\hat{W} = W - \frac{W\langle\hat{F}\rangle}{\mathcal{R}\langle\hat{F}\rangle} \mathcal{R} \quad (54)$$

which is compatible with \hat{F} and corresponds to the same trajectories as W up to parameterisation.

Proof. It is clear that $\hat{W}\langle\hat{F}\rangle = 0$. To see that the resultant vector field is horizontal observe that

$$\pi_*(\hat{W}|_{\underline{u}}) = \pi_*(W|_{\underline{u}}) - \frac{W\langle\hat{F}\rangle}{\mathcal{R}\langle\hat{F}\rangle} \Big|_{\underline{u}} \pi_*(\mathcal{R}|_{\underline{u}}) = \pi_*(W|_{\underline{u}}) = \underline{u}$$

To see that \hat{W} is radially quadratic observe that

$$\begin{aligned} \hat{W}|_{\lambda\underline{v}}\langle\hat{h}\rangle &= W|_{\lambda\underline{v}}\langle\hat{h}\rangle - \frac{W\langle\hat{F}\rangle}{\mathcal{R}\langle\hat{F}\rangle} \Big|_{\lambda\underline{v}} \mathcal{R}|_{\lambda\underline{v}}\langle\hat{h}\rangle = \lambda^2 W|_{\underline{v}}\langle\hat{h}\rangle - \frac{\lambda^{k+1} W|_{\underline{v}}\langle\hat{F}\rangle}{\lambda^k \mathcal{R}|_{\underline{v}}\langle\hat{F}\rangle} (\lambda\underline{v}\langle\hat{h}\rangle) \\ &= \lambda^2 \left(W|_{\underline{v}}\langle\hat{h}\rangle - \frac{W\langle\hat{F}\rangle}{\mathcal{R}\langle\hat{F}\rangle} \Big|_{\underline{v}} \mathcal{R}|_{\underline{v}}\langle\hat{h}\rangle \right) = \lambda^2 \hat{W}|_{\underline{v}}\langle\hat{h}\rangle, \end{aligned}$$

for any $h \in \Gamma^0 M$, $\underline{v} \in U$, $\lambda \neq 0$. Hence \hat{W} is a valid Vlasov field. To see that \hat{W} corresponds to the same set of trajectories as W , notice that $W\langle\hat{F}\rangle/\mathcal{R}\langle\hat{F}\rangle$ in eq. (54) is a 1-homogeneous function. Hence, by lemma 2.17, W and \hat{W} have the same trajectories up to parameterisation. \square

2.5 The Transport Equations on Kinematic Domains

A method of interpreting the Vlasov equation in a kinematic domain E can be given in terms of the transport equations [29]. The transport equations are written in terms of a particle density form which codifies the phase space trajectories corresponding to a given Vlasov field. With some additional structure (a choice of non-vanishing top form, otherwise known as a measure) we can recover the usual Vlasov equation (i.e. a particle density function $f_E \in \Gamma\Lambda^0 E$ such that $W_E \langle f_E \rangle = 0$) from the transport equations on our preferred choice of kinetic domain E .

Definition 2.19 (Transport Equations on E). Consider a kinematic domain $E \subset U$, a $(2n-2)$ -form $\theta_E \in \Gamma\Lambda^{2n-2} E$, and a Vlasov field $W_E \in \Gamma TE$. The *transport equations on E* are written

$$d\theta_E = 0, \quad i_{W_E} \theta_E = 0. \quad (55)$$

A visualisation of the transport equations is given in fig. 4. These geometric statements are equivalent to the Vlasov equation under appropriate conditions. We call θ_E a particle density form on E .

Let E be a kinematic domain, $W_E \in \Gamma TE$ a Vlasov field on E , and let $\Omega_E \in \Gamma\Lambda^{2n-1} E$ be a measure on E such that

$$L_{W_E} \Omega_E = 0. \quad (56)$$

We can relate a particle density $(2n-2)$ -form $\theta_E \in \Gamma\Lambda^{2n-2} E$ and a particle density function (PDF) $f_E \in \Gamma\Lambda^0 E$ via

$$\theta_E = f_E i_{W_E} \Omega_E. \quad (57)$$

Lemma 2.20. Given $\theta_E \in \Gamma\Lambda^{2n-2} E$ such that $i_{W_E} \theta_E = 0$ then $f_E \in \Gamma\Lambda^0 E$, given by eq. (57), exists and is unique. Furthermore, θ_E satisfies the transport equations eq. (55) if and only if f_E satisfies eq. (1), that is $W_E \langle f_E \rangle = 0$.

Proof. Let (x^0, \dots, x^{2n-2}) define a coordinate system on E such that $\partial_0^{(x)} = W_E$. Then,

$$\Omega_E = \Omega_0 dx^0 \wedge \dots \wedge dx^{2n-2}, \quad \text{and } \theta_E = \theta_E^\mu i_\mu^{(x)} \Omega_E,$$

where $\mu = 0, \dots, 2n-2$. We then have

$$0 = i_{W_E} \theta_E = i_0^{(x)} \theta_E = \theta_E^0 i_0^{(x)} i_0^{(x)} \Omega_E + \theta_E^a i_0^{(x)} i_a^{(x)} \Omega_E = \theta_E^a i_0^{(x)} i_a^{(x)} \Omega_E,$$

where $a = 1, \dots, 2n-2$. Hence $\theta_E^a = 0$. It follows that $\theta_E = \theta_E^0 i_0^{(x)} \Omega$ i.e. $\theta_E = f_E i_{W_E} \Omega_E$.

From eq. (57), we have

$$d\theta_E = df_E \wedge i_{W_E} \Omega_E = i_{W_E} df \wedge \Omega_E = W_E \langle f_E \rangle \Omega_E,$$

hence the transport equations for θ_E are equivalent to eq. (1). \square

As a corollary of lemma 2.20, the transport equations and the Vlasov equation, eq. (1), can equivalently be cast as

$$L_{W_E} (f_E \Omega_E) = 0. \quad (58)$$

An example of the transport equations can be seen in [25]. However, these equations are subtly different in that a form $\omega_E = i_{W_E} \Omega_E$ with $L_{W_E} \Omega_E = 0$ is defined and the transport equations are cast as $d\omega_E = 0$ and $i_{W_E} \omega_E = 0$. Note that the particle density function f_E is absent from this definition.

Given a volume form $\Omega \in \Gamma\Lambda^{2n} U$ on U , such that $L_W \Omega = 0$, and kinematic domain $E \subset U$ with inclusion map $\Sigma_E : E \hookrightarrow U$, the typical choice of top form on E is given by

$$\Omega_E = \Sigma_E^* (i_{\mathcal{R}} \Omega). \quad (59)$$

Lemma 2.21. Let $\Omega \in \Gamma\Lambda^{2n} U$ such that $L_W \Omega = 0$. Let $\Omega_E \in \Gamma\Lambda^{2n-1} E$ be defined by eq. (59), then eq. (56) holds.

Proof. Observe that

$$L_{W_E} \Omega_E = L_{W_E} (\Sigma_E^* i_{\mathcal{R}} \Omega) = \Sigma_E^* (L_W i_{\mathcal{R}} \Omega) = \Sigma_E^* (i_{[W, \mathcal{R}]} \Omega + i_{\mathcal{R}} L_W \Omega) = -\Sigma_E^* i_W \Omega = -i_{W_E} \Sigma_E^* \Omega = 0,$$

since the degree of Ω is greater than the dimension of E . \square

In the instance that the chosen kinematic domain is the upper unit hyperboloid E_H as given in eq. (19), then the volume form typically chosen is

$$\Omega_{E_H} = -\frac{\det(g)}{\dot{x}_0} dx^0 \wedge \dots \wedge dx^{n-1} \wedge d\dot{x}^1 \wedge \dots \wedge d\dot{x}^{n-1}. \quad (60)$$

Furthermore, if W_{E_H} is the Vlasov field corresponding to the Lorentz force it can be shown that $L_{W_{E_H}} \Omega_{E_H} = 0$ and hence the transport equations equipped with Ω_{E_H} are equivalent to the standard Vlasov equation. The choice of volume form in (60) is given by (59) where

$$\Omega = -\det(g) dx^0 \wedge \dots \wedge dx^{n-1} \wedge d\dot{x}^1 \wedge \dots \wedge d\dot{x}^n. \quad (61)$$

It is a straightforward exercise to show that $L_W \Omega = 0$ for the above Ω and W given by eq. (33).

It is often the case that we know the measure only on the time-slices of E , as is the case as seen in section 5.2. In the following lemma, we show that this is sufficient information to specify Ω uniquely. In this lemma we consider the scalar field s on E to be the time slicing, and α to be the measure on each time-slice. For example if the particle current density at time s is of the form $\int \dot{x}^\mu f_E \omega_0 d\dot{x}^1 \dots d\dot{x}^{n-1}$, where we integrate over the velocities, then we set $\alpha = \omega_0 d\dot{x}^1 \dots d\dot{x}^{n-1}$. The other data required is the constant $\lambda \in \mathbb{R}$ which enables us to extend the uniqueness to the entirety of U . Of course, Ω will depend on the choice of α and λ .

Lemma 2.22. *Given an $\alpha \in \Gamma \Lambda^{2n-2} E$ and $\lambda \in \mathbb{R}$. Let $s \in \Gamma \Lambda^0 E$, such that $ds \neq 0$ on E , let $S_{s_0} = \{\underline{v} \in E, s|_{\underline{v}} = s_0\} \subset E$ and define $\Xi_{s_0} : S_{s_0} \rightarrow E$. If there exists an $\Omega \in \Gamma \Lambda^{2n} U$ satisfies the following $L_W \Omega = 0$, $L_{\mathcal{R}} \Omega = \lambda \Omega$ and $\Xi_{s_0}^* i_{W_E} \Sigma_E^* i_{\mathcal{R}} \Omega = \Xi_{s_0}^* \alpha$ for each s_0 then Ω is unique.*

Proof. Suppose there exists $\Omega, \hat{\Omega} \in \Gamma \Lambda^{2n} U$ satisfying the relevant conditions and define $\omega = \Omega - \hat{\Omega}$. Define a coordinate system on U such that $\mathcal{R} = \partial_0^{(x)}$, $W = \partial_1^{(x)}$. Let $\omega = \omega'(x^0, \dots, x^{2n-1}) dx^0 \wedge \dots \wedge dx^{2n-1}$. Then $L_W \Omega = 0$ and $L_W \hat{\Omega} = 0$ implies $L_W \omega = 0$. Furthermore,

$$\lambda \omega = L_{\mathcal{R}} \omega = (\partial_0^{(x)} \omega') dx^0 \wedge \dots \wedge dx^{2n-1} + \omega' L_0^{(x)} dx^0 \wedge \dots \wedge dx^{2n-1} = (\partial_0^{(x)} \omega') dx^0 \wedge \dots \wedge dx^{2n-1},$$

hence,

$$\omega = \omega'(x^2, \dots, x^{n-1}) e^{\lambda x^0} dx^0 \wedge \dots \wedge dx^{2n-1}.$$

Also, since $\Xi_{s_0}^* i_{W_E} \Sigma_E^* i_{\mathcal{R}} \Omega = \alpha$ (similarly for $\hat{\Omega}$) for each s_0 , we have $\Xi_{s_0}^* i_{W_E} \Sigma_E^* i_{\mathcal{R}} \omega = 0$, thus $\Omega'_0(x^2|_{s_0}, \dots, x^{n-1}|_{s_0}) = 0$ for all s_0 so $\omega' = 0$ and hence $\omega = 0$. \square

We observe that Ω given by (61) satisfies $L_{\mathcal{R}} \Omega = n\Omega$. Therefore by setting $s = x^0$ and $\alpha = (-\dot{x}^0 \det g / \dot{x}_0) dx^1 \dots dx^{n-1} \wedge d\dot{x}^1 \dots d\dot{x}^{n-1}$, lemma 2.22 tells us that eq. (61) is unique.

There are instances however when it is advantageous to cast the Vlasov equation in terms of the transport equations even when the flow of the measure is not preserved by the Vlasov field (i.e. $L_{W_E} \Omega_E \neq 0$). In [30] an attempt is made to account for the radiation reaction within the Vlasov field over a 4-dimensional spacetime manifold M . To accomplish this, the Vlasov equation is cast as

$$L_{W_Q} (f_Q \omega_Q) = 0. \quad (62)$$

Here the 10-dimensional manifold Q is a subset of the double copy of the tangent bundle $Q \subset TM \oplus TM$ where the first copy of the tangent bundle contains normalised time-like velocity vectors (as on the upper unit hyperboloid), and the second copy contains acceleration vectors which are orthogonal to these velocities. The measure $\omega_Q \in \Gamma \Lambda^{10} Q$ is formed on Q by pulling back a specific 10 form related to the measure on $TM \oplus TM$. In this case, the flow of ω_Q along W_Q is not conserved:

$$L_{W_Q} \omega_Q = \frac{3}{\tau} \omega_Q, \text{ or equivalently } W_Q \langle f_Q \rangle + \frac{3}{\tau} f_Q = 0, \quad (63)$$

where $\tau = q^2 / 6\pi m$. The latter equation is similar to eq. (1), but the additional term on the LHS is to account for losses due to radiation. Note that, although the Vlasov equation is altered, the transport equations remain the same:

$$d\theta_Q = 0, \quad i_{W_Q} \theta_Q = 0, \quad (64)$$

where

$$\theta_Q = i_{W_Q} (f_Q \omega_Q). \quad (65)$$

2.6 Example: The unit hyperboloid and lab-time Vlasov vector fields for the Lorentz force equation.

In eq. (33), eq. (34) and eq. (35), we gave the Vlasov vector field adapted to the upper unit hyperboloid, an arbitrary time slicing and a time slicing with respect to coordinate time. We stated that one method to find the transformation is to calculate 2nd order ODEs, perform the reparameterisation, and then recalculate the corresponding Vlasov vector field, which we do here.

Lemma 2.23. *Equation eq. (33) can be transformed into eq. (34), by reparameterising the Lorentz force ODE.*

Proof. Let t be the parameterisation of the trajectories C of the Vlasov field W_{E_H} (i.e. eq. (33)). The Lorentz force ODE is given in coordinates by

$$\frac{d^2 C^\mu}{d\tau^2} = \frac{q}{m} \sigma \sqrt{F_H} g^{\mu\nu} F_{\nu\rho} \frac{dC^\rho}{d\tau} - \Gamma_{\nu\rho}^\mu \frac{dC^\nu}{d\tau} \frac{dC^\rho}{d\tau}.$$

Let W_{E_t} , given by eq. (34), be parameterised by t . Assuming the existence of a relationship $t = t(\tau)$ the above ODE can be reparameterised as

$$\frac{d^2 C^\mu}{dt^2} = \frac{q}{m} \sigma \sqrt{F_H} g^{\mu\nu} F_{\nu\rho} \frac{dC^\rho}{dt} - \Gamma_{\nu\rho}^\mu \frac{dC^\nu}{dt} \frac{dC^\rho}{dt} - \left(\frac{dt}{d\tau}\right)^{-3} \frac{d^2 t}{d\tau^2} \frac{dC^\mu}{dt}. \quad (66)$$

Let the integral curves of W_{E_H} and W_{E_t} be given by η_{E_H} and η_{E_t} respectively. Let $t_0 = t(\tau_0)$ then we have the following relations:

$$\left. \frac{dt}{d\tau} \right|_{\tau_0} = i|_{\eta_{E_H}(\tau_0)},$$

$$\begin{aligned} \left. \frac{d^2 t}{d\tau^2} \right|_{\tau_0} &= \eta_{E_H^*}(\partial_\tau|_{\tau_0})(i) = W_{E_H}|_{\eta_{E_H}(\tau_0)}(i) = W_{E_H}|_{\eta_{E_H}(\tau_0)} \left\langle \dot{x}^\nu \frac{\partial t}{\partial x^\nu} \right\rangle \\ &= \left(\frac{q}{m} \sigma \sqrt{F_H} g^{\lambda\nu} F_{\nu\rho} \dot{x}^\rho \frac{\partial t}{\partial x^\lambda} - \Gamma_{\nu\rho}^\lambda \dot{x}^\nu \dot{x}^\rho \frac{\partial t}{\partial x^\lambda} + \dot{x}^\nu \dot{x}^\rho \frac{\partial t}{\partial x^\nu \partial x^\rho} \right) \Big|_{\eta_{E_H}(\tau_0)} \\ &= \frac{q}{m} \left(\sigma \sqrt{F_H} g^{\lambda\nu} F_{\nu\rho} \right) \Big|_{\eta_{E_H}(\tau_0)} \frac{dC^\rho}{d\tau} \Big|_{\tau_0} \frac{\partial t}{\partial x^\lambda} \Big|_{\tau_0} - \Gamma_{\nu\rho}^\lambda \Big|_{\eta_{E_H}(\tau_0)} \frac{dC^\nu}{d\tau} \Big|_{\tau_0} \frac{dC^\rho}{d\tau} \Big|_{\tau_0} \frac{\partial t}{\partial x^\lambda} \Big|_{\tau_0} + \frac{dC^\nu}{d\tau} \Big|_{\tau_0} \frac{dC^\rho}{d\tau} \Big|_{\tau_0} \frac{\partial t}{\partial x^\nu \partial x^\rho} \Big|_{\tau_0} \\ &= \left(\frac{dt}{d\tau} \Big|_{\tau_0} \right)^2 \left(\frac{q}{m} \left(\sigma \sqrt{F_H} g^{\lambda\nu} F_{\nu\rho} \right) \Big|_{\eta_{E_t}(t_0)} \frac{dC^\rho}{dt} \Big|_{t_0} \frac{\partial t}{\partial x^\lambda} \Big|_{t_0} - \Gamma_{\nu\rho}^\lambda \Big|_{\eta_{E_t}(t_0)} \frac{dC^\nu}{dt} \Big|_{t_0} \frac{dC^\rho}{dt} \Big|_{t_0} \frac{\partial t}{\partial x^\lambda} \Big|_{t_0} \right. \\ &\quad \left. + \frac{dC^\nu}{dt} \Big|_{t_0} \frac{dC^\rho}{dt} \Big|_{t_0} \frac{\partial t}{\partial x^\nu \partial x^\rho} \Big|_{t_0} \right) \end{aligned}$$

Inserting the above relationships into eq. (66) we get

$$\begin{aligned} \frac{d^2 C^\mu}{dt^2} &= \frac{q}{m} \sigma \sqrt{F_H} g^{\mu\nu} F_{\nu\rho} \frac{dC^\rho}{dt} - \Gamma_{\nu\rho}^\mu \frac{dC^\nu}{dt} \frac{dC^\rho}{dt} \\ &\quad - (i)^{-1} \left(\frac{q}{m} \sigma \sqrt{F_H} g^{\lambda\nu} F_{\nu\rho} \frac{dC^\rho}{dt} \frac{\partial t}{\partial x^\lambda} - \Gamma_{\nu\rho}^\lambda \frac{dC^\nu}{dt} \frac{dC^\rho}{dt} \frac{\partial t}{\partial x^\lambda} + \frac{dC^\nu}{dt} \frac{dC^\rho}{dt} \frac{\partial t}{\partial x^\nu \partial x^\rho} \right) \frac{dC^\mu}{dt}. \end{aligned} \quad (67)$$

We may identify this with the $\partial_\mu^{(\dot{x})}$ term in eq. (34) using eq. (50). Hence W_{E_t} corresponds to the reparameterised Lorentz force equation above. If we were to choose an adapted coordinate system ($x^0 = t, x^a, \dot{x}^\mu$) then we would have

$$\frac{\partial t}{\partial x^\mu} = \delta_\mu^0, \quad \frac{\partial t}{\partial x^\mu \partial x^\nu} = 0. \quad (68)$$

Plugging these into eq. (67) gives the $\partial_\mu^{(\dot{x})}$ coefficient of eq. (35). \square

Alternatively, we can use the transformation formula eq. (54).

Lemma 2.24. Equation eq. (33) can be transformed into eq. (34) and hence eq. (35), using the transformation formula eq. (54).

Proof. Follows from

$$\frac{W_{E_H} \langle i \rangle}{\mathcal{R} \langle i \rangle} \mathcal{R}^\mu = \frac{q}{m} \sqrt{F_H g^{\lambda\nu}} F_{\nu\rho} \dot{x}^\rho \frac{\partial t}{\partial x^\lambda} \frac{\dot{x}^\mu}{i} - \Gamma_{\nu\rho}^\lambda \dot{x}^\nu \dot{x}^\rho \frac{\partial t}{\partial x^\lambda} \frac{\dot{x}^\mu}{i} + \dot{x}^\nu \dot{x}^\rho \frac{\partial t}{\partial x^\nu \partial x^\rho} \frac{\dot{x}^\mu}{i}. \quad (69)$$

□

Thus we see that the use of the transformation formula greatly simplifies the task of re-adapting the Vlasov vector field from one kinematic domain to another. Furthermore, we will see below that eq. (54) can be trivially derived from the Vlasov bivector. This is shown in lemma 3.21 below.

3 The Vlasov Bivectors

Here we present the Vlasov Bivector, the key goal of this article. The advantages of this approach have been listed in section section 1.2. The Vlasov bivector Ψ can always be expressed as

$$\Psi = \mathcal{R} \wedge W, \quad (70)$$

for any appropriate $W \in \Gamma TU$. It is trivial to see that any Vlasov fields related by eq. (51) and eq. (54) produce the same Vlasov bivector, and so represents an entire class of equivalent Vlasov fields. Throughout this section, Ψ will be reserved for Vlasov bivectors while an arbitrary bivector will be denoted Φ .

3.1 Bivectors

First, we write the definition of bivectors, and explore some of their important properties.

Definition 3.1 (Bivector). A *bivector* over a manifold N is an exterior product of vector fields $X_\mu, Y_\mu \in \Gamma TN$ over N , denoted by $\sum_\mu X_\mu \wedge Y_\mu$, for an arbitrary summation index μ . These follow the standard rules for exterior products, namely ‘f’-linearity and antisymmetry.

Definition 3.2 (Simple Bivectors). A bivector over N , Φ is said to be *simple* if there exists $X, Y \in \Gamma TN$ such that $\Phi = X \wedge Y$. The space of all simple bivectors (fields) over N is denoted by $\Gamma \mathcal{B}^2(N)$.

We require that a Vlasov bivector be simple and take the form eq. (70). In order to formally write the definition of Ψ we first observe the following property of simple bivectors.

Definition 3.3 (Bivector Pairing). A bivector $\sum_\mu X_\mu \wedge Y_\mu$ acts on a pair of scalar fields $F, G \in \Gamma \Lambda^0 U$ according to

$$\sum_\mu (X_\mu \wedge Y_\mu) \langle F, G \rangle = \sum_\mu (X_\mu \langle F \rangle Y_\mu \langle G \rangle - X_\mu \langle G \rangle Y_\mu \langle F \rangle). \quad (71)$$

Similarly we may define a vector field using

$$\sum_\mu (X_\mu \wedge Y_\mu) \langle F, \bullet \rangle = \sum_\mu (X_\mu \langle F \rangle Y_\mu - Y_\mu \langle F \rangle X_\mu). \quad (72)$$

In the case that we have a simple bivector acting on a scalar field we write

$$(X \wedge Y) \langle F, \bullet \rangle = X \langle F \rangle Y - Y \langle F \rangle X. \quad (73)$$

Lemma 3.4. Given a bivector Φ and a non-zero vector field $X \in \Gamma TN$, $X \wedge \Phi = 0$ if and only if there exists some $Y \in \Gamma TN$ such that $\Phi = X \wedge Y$.

Proof. Suppose that $X \wedge \Phi = 0$ and let N have dimension ℓ . Define a local coordinate system such that $X = \partial_0$ when we may write $\Phi = \Phi^{0a} \partial_0 \wedge \partial_a + \frac{1}{2} \Phi^{bc} \partial_b \wedge \partial_c$, where $a, b, c = 1, \dots, \ell - 1$. Since $\partial_0 \wedge \Phi = \frac{1}{2} \Phi^{bc} \partial_0 \wedge \partial_b \wedge \partial_c = 0$, we must have that each $\Phi^{bc} = 0$. Hence $\Phi = \partial_0 \wedge (\Phi^{0a} \partial_a) = X \wedge Y$. The converse holds by the properties of the exterior product ($X \wedge X = 0$). □

Definition 3.5 (Specially Related Pairs of Vector Fields). Two pairs of vector fields $X_1, X_2 \in \Gamma TN$ and $Y_1, Y_2 \in \Gamma TN$ are said to be *specially related* if there exists $\alpha, \beta, \gamma, \delta \in \Gamma \Lambda^0 N$ satisfying

$$(\alpha\delta - \beta\gamma)|_p = 1 \quad \forall p \in N,$$

such that

$$Y_1 = \alpha X_1 + \beta X_2, \text{ and } Y_2 = \gamma X_1 + \delta X_2.$$

Lemma 3.6. *Two pairs of vector fields, $X_1, X_2 \in \Gamma TN$ and $Y_1, Y_2 \in \Gamma TN$, are specially related if and only if $X_1 \wedge X_2 = Y_1 \wedge Y_2$.*

Proof. See section A.2. □

For a Vlasov bivector to contain all the necessary information to perform plasma kinematics, it must satisfy three conditions: the radial condition, the horizontal condition, and a third condition. There are three equivalent ways to express the third condition: integrability, being radially cubic, and being expressible as in eq. (70).

We begin by discussing integrability. A geometric interpretation of a bivector is a network of infinitesimal rectangles whose sides are defined by a pair of vectors. When these bivectors ‘knit together’ to form smooth surfaces, the vector-distribution spanned by the components of the bivector is integrable. These surfaces are depicted in figure fig. 5. More formally, we may consider integrability using the Frobenius theorem and the language of vector distributions (a method of smoothly assigning vector subspaces of $T_x M$ to each point $x \in M$).

Definition 3.7 (Tangent Bivector). A bivector Φ is tangent to a surface $K \subset N$ if, given a representation for the bivector $\Phi = X \wedge Y$, $X, Y \in \Gamma TN$, both X and Y are tangent to K in the sense described in definition 1.2.

From lemma 3.6 it is clear that definition 3.7 is independent of the representation of the bivector: if $\Phi = X \wedge Y$ and $\Phi = Z \wedge V$ and X and Y are tangent to K , then by lemma 3.6 Z and V are linear combinations of X and Y and hence tangent to K .

Definition 3.8 (Integrability). A simple bivector $\Phi \in \Gamma \mathcal{B}^2(N)$ is said to be *integrable* if there exists $X, Y \in \Gamma TN$ and $\alpha, \beta \in \Gamma \Lambda^0 N$ such that if $\Phi = X \wedge Y$ then $[X, Y] = \alpha X + \beta Y$. By lemma 3.6, if one representation of Φ satisfies the integrability condition, then so do all other representations of Φ .

Lemma 3.9. *If a bivector $\Phi \in \Gamma \mathcal{B}^2(N)$ is integrable then for any representation $\Phi = X \wedge Y$, $X, Y \in \Gamma TN$, we have*

$$[X, Y] = \alpha X + \beta Y, \tag{74}$$

for some $\alpha, \beta \in \Gamma \Lambda^0 N$.

Proof. See section A.3. □

It follows from lemma 3.6 and lemma 3.9 that integrability is a well defined property. That is, given a simple bivector, if one representation satisfies the integrability condition, then all representations do.

The Frobenius theorem states that a vector distribution over a manifold M is integrable if and only if the Lie bracket of any two vectors within the distribution also lies within the distribution. An integrable vector distribution then admits a collection of maximal connected integral manifolds which form a foliation of M . Given an integrable bivector $\Phi = X \wedge Y$ we can form a vector distribution over U which is spanned by X and Y . This generates a 2-dimensional foliation of U such that X and Y are tangent to the leaves of our foliation at each point. We may identify the leaves of this foliation with the form manifolds of the particle density form.

We also introduce the null condition here. This is a property an arbitrary bivector may have which is necessary for defining the transport equations on U .

Definition 3.10 (Null condition). Given a $(2n-2)$ -form α , and a bivector $\Phi \in \Gamma \mathcal{B}^2(N)$, then *the null condition* is given by

$$\text{Null}(\Phi, \alpha) \text{ is true} \iff i_X \alpha = 0 \text{ and } i_Y \alpha = 0, \text{ for any } X, Y \in \Gamma TN \text{ such that } \Phi = X \wedge Y. \tag{75}$$

To see that the null condition is well defined notice that if $X \wedge Y = Z \wedge V$ then $\text{Null}(X \wedge Y, \alpha)$ holds if and only if $\text{Null}(Z \wedge V, \alpha)$ holds. This is due to lemma 3.6 and the linearity of the contraction mapping.

3.2 Horizontal Bivectors

The radial and the horizontal conditions are dealt with in tandem. We now exclusively work on the submanifold $U \subset \check{T}M$.

Definition 3.11 (Radial Bivectors). A simple bivector $\Phi \in \Gamma\mathcal{B}^2(U)$ is called *radial* if

$$\mathcal{R} \wedge \Phi = 0, \quad (76)$$

where $\mathcal{R} \in \Gamma TU$ is the radial vector field.

Definition 3.12 (Horizontal Bivectors). A simple radial bivector $\Phi \in \Gamma\mathcal{B}^2(U)$ is called *horizontal* if for any $f, h \in \Gamma\Lambda^0 M$ it satisfies

$$\Phi\langle\pi^* f, \dot{h}\rangle = -\dot{f}\dot{h}. \quad (77)$$

The space of horizontal bivectors is denoted $\Gamma\mathcal{B}_H^2(U)$.

Note that in order for a bivector to be horizontal it must also be simple and radial. Hence, from this point onwards, when we refer to a horizontal bivector, we also assume it to be simple and radial.

Lemma 3.13. A bivector $\Phi = \mathcal{R} \wedge X$ is horizontal if and only if $X \in \Gamma TU$ is horizontal.

Proof. Suppose first that $\Phi = \mathcal{R} \wedge X$ for some horizontal $X \in \Gamma TU$. We then have $\mathcal{R} \wedge \Psi = 0$ automatically, hence Φ is radial. To see that Φ is horizontal observe that

$$\Phi\langle\pi^* f, \dot{h}\rangle = \mathcal{R}\langle\pi^* f\rangle X\langle\dot{h}\rangle - \mathcal{R}\langle\dot{h}\rangle X\langle\pi^* f\rangle = -\dot{f}\dot{h},$$

for any $f, h \in \Gamma\Lambda^0 M$.

Suppose now that Φ is horizontal. By lemma 3.4 (since Φ is radial and simple) there exists some $X \in \Gamma TU$ such that

$$\Phi = \mathcal{R} \wedge X.$$

By the horizontal condition

$$-\dot{f}\dot{h} = \Phi\langle\pi^* f, \dot{h}\rangle = \mathcal{R}\langle\pi^* f\rangle X\langle\dot{h}\rangle - \mathcal{R}\langle\dot{h}\rangle X\langle\pi^* f\rangle = -X\langle\pi^* f\rangle\dot{h},$$

for any $f, h \in \Gamma\Lambda^0 M$. Hence $X\langle\pi^* f\rangle = \dot{f}$, that is X is horizontal. \square

Lemma 3.14. Let $X \in \Gamma TU$ and $\Phi \in \Gamma\mathcal{B}^2(U)$ such that $\Phi = \mathcal{R} \wedge X$. If X is horizontal and Φ is integrable then there exists $\alpha \in \Gamma\Lambda^0 U$ such that

$$[\mathcal{R}, X] = X + \alpha\mathcal{R}. \quad (78)$$

Proof. Since X is horizontal, observe that for any $f \in \Gamma\Lambda^0 M$ we have

$$[\mathcal{R}, X]\langle\pi^* f\rangle = \mathcal{R}\langle X\langle\pi^* f\rangle\rangle - X\langle\mathcal{R}\langle\pi^* f\rangle\rangle = \mathcal{R}\langle\dot{f}\rangle = \dot{f} = X\langle\pi^* f\rangle.$$

By the integrability of Φ we also have $[\mathcal{R}, X] = \alpha\mathcal{R} + \beta X$ so that we also have

$$[\mathcal{R}, X]\langle\pi^* f\rangle = \alpha\mathcal{R}\langle\pi^* f\rangle + \beta X\langle\pi^* f\rangle = \beta X\langle\pi^* f\rangle.$$

Combining the last two equations gives us that $\beta = 1$. Hence the result holds. \square

3.3 Vlasov Bivectors

With this we have all the necessary ingredients to define a Vlasov bivector. There are several equivalent properties which ensure that a bivector contains sufficient structure. Given a horizontal bivector $\Psi \in \Gamma\mathcal{B}_H^2(U)$, these three properties are, the existence of a Vlasov field $W \in \Gamma TU$ such that $\Psi = \mathcal{R} \wedge W$, integrability, and another property we call the radially cubic property. Representations and integrability are discussed in section 3.1, and the radially cubic property is discussed below.

Definition 3.15 (Radially Cubic). A horizontal bivector $\Psi \in \Gamma\mathcal{B}_H^2(U)$ is *radially cubic* if for any $f, h \in \Gamma\Lambda^0 M$, $\underline{u} \in U$, and $\lambda \neq 0$ we have

$$\Psi\langle \dot{f}, \dot{h} \rangle|_{\lambda \underline{u}} = \lambda^3 \Psi\langle \dot{f}, \dot{h} \rangle|_{\underline{u}}. \quad (79)$$

We see below in theorem 3.18 that we can generate a Vlasov field W , using a Vlasov bivector and a kinematic indicator. A natural choice to use is a "lab-time" kinematic indicator i . However in general there is no $t \in \Gamma\Lambda^0 M$ such that $\dot{i} \neq 0$ on all of U . As an intermediate step, we can always define the following kinematic indicator using the coordinate system (x^0, \dots, x^{n-1}) on M ,

$$F_{\text{crd}} = \sum_{\mu=0}^{n-1} (\dot{x}^\mu)^2. \quad (80)$$

It is clear this is not a physical kinematic indicator since it depends on the coordinate system. It is homogeneous degree 2. Since this is even, one would need the causality indicator to define a kinematic domain, eq. (36).

Definition 3.16 (Coordinate based Vlasov field). Let (x^0, \dots, x^{n-1}) be a local coordinate system on M and let $\Psi \in \Gamma\mathcal{B}_H^2(U)$, we define the *coordinate based Vlasov field* by

$$W_{\text{crd}} = \frac{\Psi\langle F_{\text{crd}}, \bullet \rangle}{2F_{\text{crd}}}. \quad (81)$$

Lemma 3.17. Let $\Psi \in \Gamma\mathcal{B}_H^2(U)$ be radially cubic and then W_{crd} as given by eq. (81) is a Vlasov field such that

$$\Psi = \mathcal{R} \wedge W_{\text{crd}}. \quad (82)$$

Furthermore, given another representation for Ψ , say $\Psi = \mathcal{R} \wedge X$ for some $X \in \Gamma TU$, then

$$W_{\text{crd}} = X - \frac{X\langle F_{\text{crd}} \rangle}{2F_{\text{crd}}} \mathcal{R}. \quad (83)$$

Proof. See section A.3 □

Theorem 3.18. Let $\Psi \in \Gamma\mathcal{B}_H^2(U)$ be a horizontal bivector. The following properties are equivalent:

- (i) There exists a Vlasov field $W \in \Gamma TU$ such that $\Psi = \mathcal{R} \wedge W$.
- (ii) Ψ is integrable (definition 3.8).
- (iii) Ψ is radially cubic (definition 3.15).

Proof. Suppose first that there exists a Vlasov field W such that $\Psi = \mathcal{R} \wedge W$. To see that Ψ is radially cubic observe that, for some $f, h \in \Gamma\Lambda^0 M$, any $\underline{u} \in U$, and $\lambda \neq 0$, it follows that

$$\Psi\langle \dot{f}, \dot{h} \rangle|_{\lambda \underline{u}} = \mathcal{R}\langle \dot{f} \rangle W\langle \dot{h} \rangle|_{\lambda \underline{u}} - \mathcal{R}\langle \dot{h} \rangle W\langle \dot{f} \rangle|_{\lambda \underline{u}} = \lambda \mathcal{R}\langle \dot{f} \rangle \lambda^2 W\langle \dot{h} \rangle|_{\underline{u}} - \lambda \mathcal{R}\langle \dot{h} \rangle \lambda^2 W\langle \dot{f} \rangle|_{\underline{u}} = \lambda^3 \Psi\langle \dot{f}, \dot{h} \rangle|_{\underline{u}}.$$

Also, since W is radially quadratic we have $[\mathcal{R}, W] = W$ by lemma 2.4. Hence Ψ is integrable. That is, property (i) implies properties (ii) and (iii).

By lemma 3.17, if Ψ is radially cubic then W_{crd} as defined by eq. (81) is a Vlasov field and $\Psi = \mathcal{R} \wedge W_{\text{crd}}$. Hence property (iii) implies (i).

Suppose now that Ψ is integrable. Let W_{crd} be given by eq. (81). By lemma 3.17, $\Psi = \mathcal{R} \wedge W_{\text{crd}}$ and W_{crd} is horizontal. It remains to show that W_{crd} is radially quadratic. By lemma 3.13 there exists a horizontal vector field

$X \in \Gamma TU$ such that $\Psi = \mathcal{R} \wedge X$ and by lemma 3.17, W_{crd} and X are related by eq. (83). The integrability of Ψ tells us $[\mathcal{R}, X] = \alpha\mathcal{R} + \beta X$ for some $\alpha, \beta \in \Gamma\Lambda^0 U$. It follows that $\beta = 1$ from lemma 3.14. We then have that

$$\begin{aligned} [\mathcal{R}, W_{\text{crd}}] &= [\mathcal{R}, X] - \left[\mathcal{R}, \frac{X\langle \dot{f} \rangle}{\dot{f}} \mathcal{R} \right] = [\mathcal{R}, X] - \mathcal{R} \left\langle \frac{X\langle \dot{f} \rangle}{\dot{f}} \right\rangle \mathcal{R} = X + \alpha\mathcal{R} - \left(\frac{\mathcal{R}\langle X\langle \dot{f} \rangle \rangle}{\dot{f}} - \frac{X\langle \dot{f} \rangle}{\dot{f}} \right) \mathcal{R} \\ &= X + \alpha\mathcal{R} - \frac{1}{\dot{f}} [\mathcal{R}, X] \langle \dot{f} \rangle \mathcal{R} = X + \alpha\mathcal{R} - \frac{1}{\dot{f}} (X\langle \dot{f} \rangle + \alpha\dot{f}) \mathcal{R} = X - \frac{X\langle \dot{f} \rangle}{\dot{f}} \mathcal{R} = W_{\text{crd}}. \end{aligned}$$

Hence W_{crd} is radially quadratic by lemma 2.4. It follows that W_{crd} is a Vlasov field and property (ii) implies property (i). \square

Definition 3.19 (Vlasov Bivectors). A horizontal bivector $\Psi \in \Gamma\mathcal{B}_H^2(U)$ defines a *Vlasov bivector* if it satisfies one of the following equivalent properties:

1. There exists a Vlasov field $W \in \Gamma TU$ such that $\Psi = \mathcal{R} \wedge W$.
2. Ψ is integrable (definition 3.8).
3. Ψ is radially cubic (definition 3.15).

The space of Vlasov vectors over U is denoted by $\Psi \in \Gamma\mathcal{B}_V^2(U)$.

Lemma 3.20. Let $\Psi \in \Gamma\mathcal{B}_V^2(U)$, let E have a kinematic indicator F of order k , and let

$$W_F = \frac{\Psi\langle F, \bullet \rangle}{kF}, \quad (84)$$

Then W_F is a Vlasov field which is compatible with F and

$$\Psi = \mathcal{R} \wedge W_F. \quad (85)$$

Proof. $\Psi = \mathcal{R} \wedge W_F$ and W_F is horizontal by the same logic as lemma 3.17. Furthermore, by lemma 3.17, $\Psi = \mathcal{R} \wedge W_{\text{crd}}$ where W_{crd} is given by eq. (81). To see that W_F is radially quadratic observe that

$$\begin{aligned} W_F|_{\lambda\underline{u}}\langle \dot{f} \rangle &= \left(\frac{\Psi\langle F, \dot{f} \rangle}{kF} \right) \Big|_{\lambda\underline{u}} = \left(\frac{\mathcal{R}\langle F \rangle W_{\text{crd}}\langle \dot{f} \rangle - \mathcal{R}\langle \dot{f} \rangle W_{\text{crd}}}{kF} \right) \Big|_{\lambda\underline{u}} = \left(W_{\text{crd}}\langle \dot{f} \rangle - \frac{\dot{f}}{kF} W_{\text{crd}}\langle F \rangle \right) \Big|_{\lambda\underline{u}} \\ &= \left(\lambda^2 W_{\text{crd}}\langle \dot{f} \rangle - \frac{\lambda\dot{f}}{\lambda^k kF} \lambda^{k+1} W_{\text{crd}}\langle F \rangle \right) \Big|_{\underline{u}} = \lambda^2 \left(W_{\text{crd}}\langle \dot{f} \rangle - \frac{W_{\text{crd}}\langle F \rangle}{kF} \mathcal{R}\langle \dot{f} \rangle \right) \Big|_{\underline{u}} \\ &= \lambda^2 W_F|_{\underline{u}}\langle \dot{f} \rangle. \end{aligned}$$

The Vlasov field W_F is compatible with F , i.e. $W_F\langle F \rangle = 0$, since $\Psi\langle F, F \rangle = 0$. It follows that

$$\frac{\Psi\langle F, \bullet \rangle}{kF} = \frac{\Psi\langle F, \bullet \rangle}{\mathcal{R}\langle F \rangle} = \frac{\mathcal{R}\langle F \rangle W_F - W_F\langle F \rangle \mathcal{R}}{\mathcal{R}\langle F \rangle} = W_F.$$

\square

With lemma 3.6 and lemma 3.9 combined we may note that Vlasov bivectors constructed from projectively related Vlasov fields represents the same object; that is, if there exists a 1-homogeneous function $k \in \Gamma\Lambda^0 U$ such that $\hat{W} = W + k\mathcal{R}$, then $\Psi = \mathcal{R} \wedge W = \mathcal{R} \wedge \hat{W}$. In figure fig. 5, the foliations generated by \mathcal{R}, W are the same of those generated by \mathcal{R}, \hat{W} . We can use the Vlasov bivector to derive the transformation formula eq. (54).

Lemma 3.21. If $\Psi = \mathcal{R} \wedge W = \mathcal{R} \wedge \hat{W}$ and \hat{W} is compatible with kinematic indicators \hat{F} then eq. (54) holds.

Proof. Consider the action of Ψ on the scalar field \hat{F} :

$$\Psi\langle \hat{F}, \bullet \rangle = \mathcal{R}\langle \hat{F} \rangle W - W\langle \hat{F} \rangle \mathcal{R} = \mathcal{R}\langle \hat{F} \rangle \hat{W} - \hat{W}\langle \hat{F} \rangle \mathcal{R} = \mathcal{R}\langle \hat{F} \rangle \hat{W}.$$

Hence eq. (54) \square

Here we show that the Vlasov Bivector Ψ , knit together as leaves, as depicted in fig. 5. Since the leaves are tangent to \mathcal{R} , then they must open out like a book.

Theorem 3.22. *Given a Vlasov bivector Ψ , then for each $\underline{u} \in U$ there exists a 2-dimensions surface $K \subset U$ such that $\underline{u} \in K$ and Ψ is a tangent to K .*

Proof. From theorem 3.18 we can write $\Psi = \mathcal{R} \wedge W$ such that \mathcal{R} and W are in involution, eq. (29). The result now follows directly from Frobenius Theorem, which can be found in [31]. \square

4 Particle Density $(2n-2)$ -forms

In this section we introduce the particle density $(2n-2)$ -form on U , $\theta \in \Gamma\Lambda^{2n-2}U$. This may be depicted pictorially in fig. 1 and fig. 5. It is the generalisation of the particle density form $\theta_E \in \Gamma\Lambda^{2n-2}E$, and it is subject to the transport equations on U , defined below. This reformulation of the transport equations has the same advantages as mentioned in the introduction.

4.1 Transport Equations on U

Definition 4.1 (Transport Equations on U). Given a particle density form $\theta \in \Gamma\Lambda^{2n-2}U$ and a Vlasov bivector $\Psi \in \Gamma\mathcal{B}_V^2(U)$, the *transport equations on U* are given by

$$\text{Null}(\Psi, \theta) \text{ holds and } d\theta = 0, \quad (86)$$

where the null condition is given by definition 3.10.

Notice that if $\text{Null}(\Psi, \theta)$ holds and $\Psi = \mathcal{R} \wedge W$ then both W and \mathcal{R} are tangent to the form manifolds of θ . That is, $i_W\theta = 0$ and $i_{\mathcal{R}}\theta = 0$. This is consistent with the visualisation in fig. 5. We can identify the form manifolds of θ with the leaves of a foliation generated by a Vlasov bivector. Since θ is a closed form, the form manifolds associated with it are smooth surfaces. Furthermore, the velocity density profile of the particle distribution is reflected in these form manifolds: the closer together the surfaces, the greater the local velocity density of the particles (and vice versa).

Definition 4.2 (Populated Systems on U). Given a Vlasov bivector Ψ and a particle density $(2n-2)$ -form $\theta \in \Gamma\Lambda^{2n-2}U$ which satisfies the transport equations on U eq. (86), we define the pair (Ψ, θ) to be a populated system on U .

As stated in the introduction, defining a populated system does not require a time orientation. Since for any given radial $\{\lambda\underline{u} \in U, \lambda > 0\}$ is disconnected from $\{\lambda\underline{u} \in U, \lambda < 0\}$ there is no relationship between $\theta|_{\underline{u}}$ and $\theta|_{-\underline{u}}$. For example, in eq. (70) the particle density form can be considered nonzero on U^+ and zero on U^- .

4.2 Relating the particle density on U with the particle density on E

For this subsection we assume that U is time-orientable and that there is a kinematic domain E . We relate the particle densities $\theta \in \Gamma\Lambda^{2n-2}U$ and $\theta_E \in \Gamma\Lambda^{2n-2}E$. Furthermore we assume that θ only contains particles which lie on U^+ .

Definition 4.3 (Future Pointing Particle Density). A particle density form $\theta \in \Gamma\Lambda^{(2n-2)}U$ is called future pointing if U is time-orientable and

$$\theta|_{U^-} = 0. \quad (87)$$

For future pointing particle densities we let the restriction $\theta^+ \in \Gamma\Lambda^{2n-2}U^+$ be

$$\theta^+ = \theta|_{U^+}. \quad (88)$$

A populated system (Ψ, θ^+) formed from a future pointing particle density is called a future pointing populated system.

Lemma 4.4. *Let (Ψ, θ^+) define a future pointing populated system on U , satisfying the transport equations. Given a kinematic domain $E \subset U^+$ with kinematic indicator F , let W_E be given by eq. (32) where W is given by eq. (84), and let $\theta_E \in \Gamma\Lambda^{2n-2}E$ be given by*

$$\theta_E = \Sigma_E^* \theta^+. \quad (89)$$

Then W_E and θ_E satisfy the transport equations on E .

Proof. Observe that

$$d\theta_E = d\Sigma_E^* \theta = \Sigma_E^* d\theta = 0, \text{ and } i_{W_E} \theta_E = i_{W_E} \Sigma_E^* \theta = \Sigma_E^* (i_W \theta) = 0.$$

Hence θ_E with W_E satisfy definition 2.19. \square

Lemma 4.5. *Given a kinematic domain E with $\theta_E \in \Gamma\Lambda^{2n-2}E$ and $W_E \in \Gamma TE$ such that θ_E satisfies the transport equations on E , define the map,*

$$\Pi_E : U^+ \rightarrow E; \quad \Pi(\lambda \underline{v}) = \underline{v} \text{ where } \underline{v} \in E \text{ and } \lambda > 0. \quad (90)$$

Let

$$\theta^+ = \Pi_E^* \theta_E, \quad (91)$$

let θ be future pointing, given by eq. (87) and eq. (88), and let $W \in \Gamma TU$ be given by eq. (32). The system $(\Psi = \mathcal{R} \wedge W, \theta)$ satisfies the transport equations on U .

Proof. See section A.3. \square

Lemma 4.6. *Given a kinematic domain E and particle densities θ_E and θ where θ is future pointing satisfying the relevant transport equations, then eq. (89) holds if and only if eq. (91) holds.*

Proof. See section A.3. \square

A direct application of lemma 4.4 and lemma 4.5 to numerical simulations, is given in section 5.2, where they are used to construct a particle density function in a new kinematic as a way to deduce quantities in a boosted frame.

4.3 The current associated with the particle density form

For a populated system on E , (W_E, θ_E) can construct the current \mathcal{J}_E associated with θ_E by integrating over each fibre. In differential geometry, this integration can be expressed very naturally using the language of de Rham pushforwards. The conservation of charge $d\mathcal{J}_E = 0$ follows from the transport equation $d\theta_E = 0$ and the fact the exterior derivative commutes with the de Rham pushforward.

To write the current density for a populated system on U , it is necessary to introduce a support 1-form χ . The goal of this subsection is to first define the current form \mathcal{J} on U , then show that the result is independent of the choice of this support 1-form, and to show charge is conserved.

Definition 4.7 (Integration along a fibre). Let $\pi : K \rightarrow M$ where K is an oriented k -dimensional vector bundle over M . A form $\alpha \in \Gamma\Lambda^q K$ (for $n \leq k \leq 2n$) is said to have vertical compact support if for each $p \in M$ the restriction $\alpha|_{\pi^{-1}(p)}$ has compact support. Given a form with vertical compact support $\alpha \in \Gamma\Lambda^q K$ the *integral along the fibre* (otherwise known as the de Rham pushforward) $\pi_{\zeta} \alpha \in \Gamma\Lambda^{n-r} M$ is defined by

$$\int_K \pi^* \beta \wedge \alpha = \int_M \beta \wedge \pi_{\zeta} \alpha, \quad (92)$$

for all forms $\beta \in \Gamma\Lambda^r M$ with compact support such that $q + r = \dim(K)$. For a comprehensive overview see [32].

Definition 4.8 (Current forms from E). Let U be time-orientable, let $E \subset U^+$ be a kinematic domain and let $\theta_E \in \Gamma\Lambda^{2n-2}E$ define a particle density $2n-2$ -form satisfying the transport equations on E . We define the current $(n-1)$ -form $\mathcal{J}_E \in \Gamma\Lambda^{n-1}M$ by

$$\mathcal{J}_E = \pi_{E\zeta}(\theta_E). \quad (93)$$

Here, $\pi_E : E \rightarrow M$ is the projection from the bundle E to M .

Since the above definition relies on a choice of kinematic domain E , we propose the following generalisation. For this, we need to define a support form, which is a 1-form on U .

Definition 4.9 (Support Form). Given any $\underline{u} \in U$ let $\mathfrak{R}_{\underline{u}} = \{\lambda \underline{u} : \lambda > 0\}$ and $\hat{\mathfrak{R}}_{\underline{u}} : \mathbb{R}^+ \hookrightarrow U$. A *support form* $\chi \in \Gamma\Lambda^1 U$ is a 1-form such that for all $\underline{u} \in U$, $\hat{\mathfrak{R}}_{\underline{u}}^* \chi$ has compact support on $\mathfrak{R}_{\underline{u}}$ and satisfies

$$\int_{\mathbb{R}^+} \hat{\mathfrak{R}}_{\underline{u}}^* \chi = 1, \quad (94)$$

for each $\underline{u} \in U$.

Observe that replacing \underline{u} with $\lambda \underline{u}$ for some $\lambda > 0$ does not change the integration in eq. (94). Recall that $\theta|_{\underline{u}}$ may be unrelated to $\theta|_{-\underline{u}}$ and in general both can be non-zero. Thus we need a support form to have support on both $\mathfrak{R}_{\underline{u}}$ and $\mathfrak{R}_{-\underline{u}}$ so that both sides contribute.

Definition 4.10 (Current forms from U). Let $\theta \in \Gamma\Lambda^{2n-2} U$ be a particle density form satisfying the transport equations on U (definition 4.1). The *current form on U* , $\mathcal{J} \in \Gamma\Lambda^{n-1} M$ is given by

$$\mathcal{J} = \pi_{\zeta}(\chi \wedge \theta), \quad (95)$$

for any support form $\chi \in \Gamma\Lambda^1 U$.

Although we need to impose additional structure through the inclusion of the support form, we may show that the current form from U is independent of our choice of support form.

Lemma 4.11. Let N be an ℓ dimensional manifold, $\alpha \in \Gamma\Lambda^{\ell-1} N$, $\beta \in \Gamma\Lambda^1 N$, $t \in \Gamma\Lambda^0 N$ and $X \in \Gamma TN$ such that

$$L_X \alpha = 0, \quad i_X \alpha = 0, \quad \text{and} \quad X \langle t \rangle = 1. \quad (96)$$

Let $K_{t_0} = \{p \in N : t|_p = t_0\}$ for some value t_0 and define the embedding $\Sigma_{t_0} : K_{t_0} \hookrightarrow N$. Lastly, let $\eta_p(t)$ denote the integral curve of X passing through p . Then

$$\int_N \alpha \wedge \beta = \int_{p \in K_{t_0}} \Sigma_{t_0}^* \alpha \left(\int_{\mathbb{R}} \eta_p^* \beta \right), \quad (97)$$

provided $\eta_p^* \beta$ has compact support on the domain of each η_p .

Proof. See section A.3. □

Lemma 4.12. Let the current form $\mathcal{J} \in \Gamma\Lambda^{n-1} M$ as given by definition 4.10. \mathcal{J} is independent of the choice of support form $\chi \in \Gamma\Lambda^1 U$.

Proof. Observe that since θ satisfies the transport equations (definition 4.1) we have $i_{\mathcal{R}} \theta = 0$ and $d\theta = 0$ so $L_{\mathcal{R}} \theta = 0$. It follows that

$$L_{\mathcal{R}}(\pi^* \phi \wedge \theta) = \pi^*(L_{\pi_* \mathcal{R}} \phi) \wedge \theta - \pi^* \phi \wedge L_{\mathcal{R}} \theta = 0,$$

since $\pi_* \mathcal{R} = 0$, for any test form ϕ . We also have $i_{\mathcal{R}}(\pi^* \phi \wedge \theta) = 0$ so we may apply lemma 4.11 in the following way. Let $r \in \Gamma\Lambda^0 U$ such that $dr \neq 0$ and $\mathcal{R} \langle r \rangle = 1$, then let $\eta_{\underline{u}}$ be an integral curve of \mathcal{R} passing through $\underline{u} \in U$ and let $K_{r_0} = \{\underline{u} \in U : r|_{\underline{u}} = r_0\}$ with $\Sigma_{r_0} : K_{r_0} \hookrightarrow U$ for some value r_0 . We then have

$$\int_U \pi^* \phi \wedge \theta \wedge \chi = \int_{\underline{u} \in K_{r_0}} \Sigma_{r_0}^* (\pi^* \phi \wedge \theta) \left(\int_{\mathbb{R}} \eta_{\underline{u}}^* \chi \right) = \int_{K_{r_0}} \Sigma_{r_0}^* (\pi^* \phi \wedge \theta).$$

Hence \mathcal{J} is independent of the choice of support form χ . □

Lemma 4.13. The current $(n-1)$ -form on U , $\mathcal{J} = \pi_{\zeta}(\chi \wedge \theta)$, satisfies the continuity equation

$$d\mathcal{J} = 0. \quad (98)$$

Proof. The exterior derivative commutes with the de Rham pushforward and $d\theta = 0$ so we have

$$d\mathcal{J} = d\pi_\zeta(\chi \wedge \theta) = \pi_\zeta(d\chi \wedge \theta).$$

Since \mathcal{J} is independent of our choice of χ by lemma 4.12 it suffices to pick χ such that $d\chi = 0$. By picking $r \in \Gamma\Lambda^0 U$ such that $dr \neq 0$ and $\mathcal{R}\langle r \rangle = 1$ we may define a coordinate system (x^μ, r, y^a) . By choosing

$$\chi = \chi_r(r)dr,$$

where $\chi_r(r)$ is a function in r with compact support, to comply with eq. (94), we have $d\chi = 0$. Hence $d\mathcal{J} = 0$, and charge is conserved. \square

Lemma 4.14. *Let U be time-orientable, E be a kinematic domain and θ is future time pointing. Let θ and θ_E be related by eq. (89) and eq. (91). The current form on U , $\mathcal{J} \in \Gamma\Lambda^{n-1}M$ (definition 4.10) and the current form on E , $\mathcal{J}_E \in \Gamma\Lambda^{n-1}M$, definition 4.8, are identical.*

Proof. Let F be the 1-homogeneous kinematic indicator associated with E , define

$$r = \log F,$$

$K_{r_0} = \{\underline{u} \in U : r|_{\underline{u}} = r_0\}$, and $\Sigma_{r_0} : K_{r_0} \hookrightarrow U$. First observe that $E = K_0$ and $\mathcal{R}\langle r \rangle = 1$. By application of lemma 4.11 we have for any test form $\phi \in \Gamma_0\Lambda^1 M$,

$$\int_U \pi^* \phi \wedge \theta \wedge \chi = \int_{p \in K_0} \Sigma_0^*(\pi^* \phi \wedge \theta) \left(\int_{\mathbb{R}} \eta_p^* \chi \right) = \int_{K_0} \Sigma_0^*(\pi^* \phi \wedge \theta) = \int_E \Sigma_E^*(\pi^* \phi \wedge \theta) = \int_E \pi_E^* \phi \wedge \theta_E.$$

Hence for any E , $\mathcal{J} = \mathcal{J}_E$. \square

Using the coordinate expression for the fibre integral in [33], the coefficients of the current vector on the unit hyperboloid E_H (i.e. the 4 current when $\dim M = 4$) are given by

$$\mathcal{J}_{E_H}^\mu = \int_{E_H} \dot{x}^\mu f_{E_H} \frac{\sqrt{-\det g}}{\dot{x}_0} d\dot{x}^{1, \dots, n-1}, \quad (99)$$

where

$$\mathcal{J}_{E_H}^\mu = \star(dx^\mu \wedge \mathcal{J}_{E_H}). \quad (100)$$

4.4 Discussion about the Stress-Energy $n-1$ -form

Given a kinematic domain E and $\alpha \in \Gamma\Lambda^1 M$, the stress-energy $(n-1)$ -form can be expressed

$$\tau_\alpha^E = \pi_{E\zeta}(\hat{\alpha}\theta_E), \quad (101)$$

where $\hat{\alpha} \in \Gamma\Lambda^0 U$ is given by $\hat{\alpha}|_{\underline{u}} = \alpha : \underline{u}$ for $\underline{u} \in U$. The stress-energy $n-1$ -form for the unit hyperboloid E_H can be converted into the usual stress-energy tensor

$$T_{E_H}^{\mu\nu} = \int_{E_H} \dot{x}^\mu \dot{x}^\nu f_{E_H} \frac{\sqrt{-\det g}}{\dot{x}_0} d\dot{x}^{1, \dots, n-1}, \quad (102)$$

using the relationship

$$T_{E_H}^{\mu\nu} = \star(dx^\mu \wedge \tau_{dx^\nu}^{E_H}), \quad (103)$$

where \star is the Hodge dual. See lemma A.8 for proof of this statement. This can be substituted into the right hand side of Einstein's equations to complete the Einstein-Vlasov system on E_H .

Due to the similarities between the definitions of the current $(n-1)$ -form, eq. (93), and the stress-energy $(n-1)$ -form, eq. (101)), it may be tempting to try defining the stress-energy $(n-1)$ -form on U by

$$\tau = \pi_\zeta(\chi \wedge \hat{\alpha}\theta). \quad (104)$$

Unfortunately, unlike for \mathcal{J} , the stress-energy $(n-1)$ -form depends on the choice of χ . This is because $d(\hat{\alpha}\theta) \neq 0$ in general, so that lemma 4.14 does not apply.

For an explicit example of τ_α^E depending on E , consider the charged free dust in Minkowski space time so that in Cartesian coordinates $W = \dot{x}^\mu \partial_\mu^{(x)}$. In this case, $\tau_\alpha^{E_H}$ and $\tau_{E_t \alpha}$ are given by

$$\tau_\alpha^{E_H} = i_\mu^{(x)} dx^{0, \dots, n-1} \int_{E_H} f_{E_H} \alpha_a \frac{\dot{x}^a}{\dot{x}^0} \dot{x}^\mu d\dot{x}^1, \dots, \dot{x}^{n-1}, \quad (105)$$

$$\tau_\alpha^{E_t} = i_\mu^{(x)} dx^{0, \dots, n-1} \int_{E_t} f_{E_t} \alpha_a \dot{x}^a \dot{x}^\mu d\dot{x}^1, \dots, \dot{x}^{n-1}. \quad (106)$$

in their respective adapted coordinate systems.

Furthermore, in general, the stress-energy form is not necessarily divergenceless (in the sense put forth in [34]) for an arbitrary lab frame with no electromagnetic interaction. Using the external covariant derivative D ,

$$D\tau_\alpha^E = d\tau_\alpha^E - dx^\mu \wedge \tau_{\nabla_\mu \alpha}^E, \quad (107)$$

the divergenceless condition can then be written $D\tau_\alpha^E = 0$. In [35] it's shown that $D\tau_\alpha^{E_H} = 0$. However, for an arbitrary lab bundle E_t , it can be shown that

$$D\tau_{dx^\mu}^{E_t} = 0, \quad D\tau_{dx^0}^{E_t} = dx^{0, \dots, n-1} \int_{\pi^{-1}(\bullet)} \Gamma_{\mu\nu}^0 \dot{x}^\nu \dot{x}^\mu d\dot{x}^1, \dots, \dot{x}^{n-1}. \quad (108)$$

in an adapted coordinate system (x^μ, \dot{x}^a) .

The usual stress-energy form is $\tau_\alpha^{E_H}$, which is needed for the Einstein-Vlasov system. This implies there is a preferred kinematic domain, E_H . One may ask if there are still advantages for using the formalism in this article. First, it makes it clear how the Vlasov equations i.e. the Vlasov bivector and the transport equations depend on the metric. This may be less explicit in the usual treatment. This is especially relevant if one needs to vary the metric, as is done in [36], for instance. Second, it reveals the relationship between the stress-energy tensor and the kinematic domain. Since we do not have a formulation of the stress-energy form/tensor which does not depend on the kinematic domain, we cannot assert that Einstein-Vlasov systems are parameter free under our formalism.

Another use of our approach, is when considering non-metric compatible connections. As stated in the introduction, the trajectories no longer remain on E_H , fig. 2. However, using the Vlasov bivector and the particle density form θ , this is no longer a problem. After calculating θ , one can choose E_H to calculate $\tau_\alpha^{E_H}$.

It is an open question to see if there is a relationship between the stress-energy forms for different E 's or χ 's or if there exists an object similar to the stress-energy form can be defined which does not depend on the choice of E or χ .

5 Application to Numerical Simulation of Plasmas

This work can be applied to numerical simulations of plasmas. Such simulations typically involve solving the Maxwell-Vlasov system, either directly (particularly in one or two dimensions), or through PIC methods.

The usual method for describing the relativistic Maxwell-Vlasov system is in terms of proper time. However, for numerical simulations it is much easier to use the lab time. This is explained in section section 5.1.

A particular numerical simulation procedure is the agile numerical integrator (ANI). To date this method is mainly used where velocities are non-relativistic. If velocities start approaching the speed of light, simulation becomes increasingly challenging as it would require many grid points near $v = c$. We show how one can extend the applicability of the ANI, using a series of Lorentz transformations, to scenarios where the velocities of particles become large, but the relative velocities between the particles remains small. This is ideal for cases of particle acceleration. This is detailed in section section 5.2.

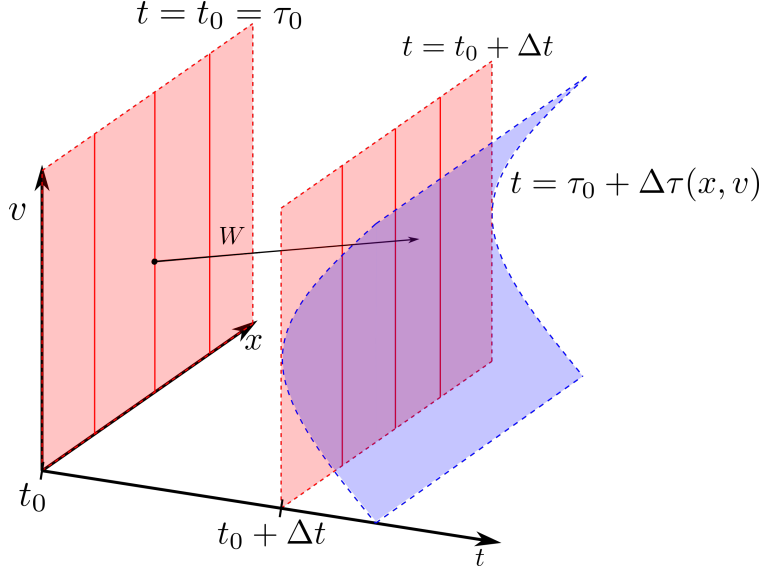


Figure 7: The advantages of using a lab-frame when doing numerical simulations. The red planes represent time slices. We integrate over v to evaluate the charge and current. These are the fibres depicted in dark red. By contrast, using proper-time to step forward from $\tau_0 = t_0$ to $\tau_0 + \Delta\tau_0$ results in the blue surface which cannot be directly integrated.

5.1 The Use of Lab Time

These simulations invariably use a lab-frame. Recall that the term lab-frame in this article refers to any kinematic domain E_t , which is determined by a time-slicing scalar field $t \in \Gamma\Lambda^0 M$. This may be the frame of an actual lab, but can include the rest frame of a particle.

It is challenging to use the proper-time Lorentz force equations to update the particle trajectory. This is because, assuming $x^0(\tau)$ all coincided, the updated $x^0(\tau + \Delta\tau)$ would not coincide. This would make it difficult to sum over the particles in the same cell to obtain the charge or current. The same problem arises when attempting to solve the Maxwell-Vlasov system with the proper time version of Vlasov's equation, since the integration over velocities cannot be performed directly. This is depicted in figure fig. 7.

Assuming a numerical simulation scheme is using a lab-frame E_t , one needs to choose time-slicing scalar field t . This decision is made alongside many others, such as coordinates, grid strategy, integrator, etc. The usual choice for Minkowski spacetime is an inertial frame. However, there may be advantages with using a non-inertial frame. For example, in a particle accelerator, one could use the frame adapted to the design orbit. For uniform acceleration, this would correspond to Rindler coordinates.

In general relativity there are generally no preferred coordinate systems, and the choice of time-slicing may be arbitrary. In this case, it may be useful to be able to run the simulations in different coordinate systems and compare the result.

Let t be one time-slicing and $\hat{t} = \hat{t}(t, x^a)$ be a new time-slicing. Then from eq. (12)

$$\hat{t} = \frac{\partial \hat{t}}{\partial t} t + \frac{\partial \hat{t}}{\partial x^a} x^a, \quad (109)$$

since \hat{t} is a kinematic indicator it can be directly placed into eq. (54) to give the transformed Vlasov field.

5.2 The Agile Numerical Integrator

Our formalism enables a generalisation of work such as [11] wherein a novel integration scheme, the agile numerical (ANI), is derived from the treatment of the Vlasov field as the divergence of a flux density \mathcal{J} over phase space. As stated in the introduction, the ANI is mainly used to model particles with non-relativistic velocities. Here we show how to extend, using a Lorentz transformation, the ANI to higher velocities, giving it application for particle accelerators, or even the electrons, inside the bubble of a LWFA.

In the aforementioned paper, the linearised Vlasov-Poisson system is described by the flux density

$$\mathcal{J} = f\hat{t} + v_x f \hat{x} - \frac{q}{m} \frac{\partial \phi}{\partial x} f \hat{v}_x, \quad (110)$$

where \hat{t} (resp. \hat{x} , \hat{v}_x) are unit vectors in the t (resp. x , v_x) direction, and ϕ is the electric potential satisfying the Poisson equation. This enables the Vlasov equation to be cast as

$$\nabla \cdot \mathcal{J} = 0, \quad (111)$$

or an equivalent flux integral.

We can identify the coefficients of the phase-space flux density eq. (110) with the coefficients of the particle density form definition 2.19. In particular, assuming the Vlasov-Poisson system to be written on a lab-time bundle E_t , we may make the following identifications: $v_x = \dot{x}$, $\hat{t} = \partial_t$, $\hat{x} = \partial_x$, $\hat{v}_x = \partial_{\dot{x}}$, and $\Omega_{E_t} = dt \wedge dx \wedge d\dot{x}$ which satisfies eq. (56). This allows us to recognise the notion of a velocity field on a phase space \underline{u} from [11] with our notion of a Vlasov field W_{E_t} on a kinematic domain E_t . From here we can make the identification,

$$\mathcal{J} = f_{E_t} W_{E_t}. \quad (112)$$

Consequently eq. (111) is equivalent to the transport equations eq. (58), and the Vlasov equation can be written as

$$L_{W_{E_t}} \left(f_{E_t} \Omega_{E_t} \right) = 0. \quad (113)$$

Consequently, our formalism allows for the generalisation of the ANI. We illustrate this through an application by performing a Lorentz boost to the Vlasov-Poisson system.

The main benefit of the ANI is that by characterising the Vlasov equation as a vector field, the particle density function (PDF) can be evolved independently along each dynamical axis. This allows us to write the equation for the evolution of the PDF with fewer interpolations than a method invoking Strang splitting. For the relativistic Vlasov-Poisson system, the discretised Vlasov equation yields

$$f(t + \Delta t, x, \dot{x}) = f(t, x - \dot{x}\Delta t, \dot{x}) + f\left(t, x, \dot{x} - \frac{q}{m} \mathcal{E} F_H^{3/2} \Delta t\right) - f(t, x, \dot{x}), \quad (114)$$

where F_H is given by eq. (20), which is equal to $F_H = 1 - \dot{x}^2$ here, and \mathcal{E} is the electric field.

Note that this scheme requires 2 interpolations, one in x and one in \dot{x} . By gridding phase space in intervals width $(\Delta t, \Delta x, \Delta \dot{x})$ and indexing the nodes on the grid such that $f_{j,k}^n = f(n\Delta t, j\Delta x, k\Delta \dot{x})$, the above equation can be written

$$f_{j,k}^{n+1} = \sum_{j'} \mu_{j,j'}^0(x - \dot{x}\Delta t) f_{j',k}^n + \sum_{k'} \mu_{k,k'}^1\left(\dot{x} - \frac{q}{m} \mathcal{E} F_H^{3/2} \Delta t\right) f_{j,k'}^n - f_{j,k}^n \quad (115)$$

where the $\mu_{j,j'}^0(x)$ and $\mu_{k,k'}^1(\dot{x})$ are the interpolation parameters e.g. [37].

The number of interpolations can be halved by defining the gridpoints as $x = j\Delta x$ and $\dot{x} = k\Delta x/\Delta t$. Substituting into (114) gives

$$f_{j,k}^{n+1} = f_{j-k,k}^n + \sum_{k'} \mu_{k,k'}^1\left(\dot{x} - \frac{q}{m} \mathcal{E} F_H^{3/2} \Delta t\right) f_{j,k'}^n - f_{j,k}^n. \quad (116)$$

We can still obey the Courant stability conditions, by choosing Δt such that $\Delta t \leq \Delta x/V_0$ and $\Delta t \leq (\Delta x / (\frac{q}{m} \mathcal{E}_0 F_H^{3/2}))^{1/2}$, where V_0 is the maximum velocity of the system, and \mathcal{E}_0 is the maximum electric field amplitude. When the above grid spacing is chosen, we call this the matched grid case.

As stated in the introduction, if the particles become ultra-relativistic so that $\dot{x} \rightarrow c$, the use of the ANI becomes challenging as, to resolve the velocities, we would need many velocity grid points near $\dot{x} = c$. Using eq. (115), this is possible, but not ideal, whereas, using eq. (116) this is not possible as the grid size $\Delta \dot{x}$ is fixed. Assuming the energy spread of the particles remains small, the solution is to perform a Lorentz boost, and continue the simulation in a grid adapted to this new rest frame. We show here how our technology can be used to transform the Vlasov field W_{E_t} and the PDF f_{E_t} to the new frame.

Suppose we have a Vlasov-Poisson system on the lab bundle E_t with PDF f_{E_t} , and corresponding Vlasov field W_{E_t} . To find the equivalent of the PDF evolution equation (eq. (114)) in the boosted frame $E_{t'}$, we first need to deduce $W_{E_{t'}}$

using theorem 2.18. Next, we find an appropriate choice of Ω to calculate θ_{E_t} using eq. (57) and use this to deduce $\theta_{E_{t'}}$ by promoting θ_{E_t} to θ^+ using lemma 4.5. Then we use θ^+ to calculate $\theta_{E_{t'}}$ through lemma 4.6, which gives us a relationship between f_{E_t} and $f_{E_{t'}}$ using eq. (57).

Using eq. (34) we can write the 1D1V Vlasov-Poisson field on U (the light cone), in a coordinate system adapted to E_t , (t, x, i, \dot{x}) , using the Lorentz metric $g_{\mu\nu} = \text{diag}(-1, 1)$ as

$$W = i\partial_t + \dot{x}\partial_x - \frac{q}{m}\mathcal{E}\frac{F_H^{3/2}}{i}\partial_{\dot{x}}, \quad (117)$$

where $\mathcal{E} = \mathcal{E}(t, x)$ is the electric field, and F_H is eq. (20) (here, $F_H = t^2 - \dot{x}^2$). Note that W is horizontal and vertically quadratic. Since $W\langle i \rangle = 0$, W is tangent to E_t by lemma A.5, we can write

$$W_{E_t} = \partial_t + \dot{x}\partial_x - \frac{q}{m}\mathcal{E}(F_H \circ \Sigma_{E_t})^{3/2}\partial_{\dot{x}}, \quad (118)$$

since $\Sigma_{E_t^*}(W_{E_t}) = W|_{E_t}$. That is, W is the extension of W_{E_t} . This is consistent with the 1D1V Vlasov field used to derive Landau oscillations (before linearisation) when $F_H = 1$ in the non-relativistic case. Consider now a Lorentz boost to a frame moving with constant velocity V , $(t, x) \mapsto (t', x') = (\gamma t - V\gamma x, \gamma x - V\gamma t)$, where $\gamma = (1 - V^2)^{-1/2}$. It is straightforward to show that on U we have

$$i' = \gamma i - V\gamma \dot{x}, \quad \dot{x}' = \gamma \dot{x} - V\gamma i, \quad i = \gamma i' + \gamma V \dot{x}', \quad \dot{x} = \gamma \dot{x}' + \gamma V i', \quad (119)$$

$$\partial_t = \gamma \partial_{t'} - \gamma V \partial_{x'}, \quad \partial_x = \gamma \partial_{x'} - \gamma V \partial_{t'}, \quad \partial_i = \gamma \partial_{i'} - \gamma V \partial_{\dot{x}'}, \quad \partial_{\dot{x}} = \gamma \partial_{\dot{x}'} - \gamma V \partial_{i'}. \quad (120)$$

The lab bundle associated with this boost is given by $E_{t'}$. Using eq. (54) and the above, we get

$$W' = W - \frac{W\langle i' \rangle}{\mathcal{R}\langle i' \rangle} \mathcal{R} = i\partial_t + \dot{x}\partial_x - \frac{q}{m}\mathcal{E}\frac{F_H^{3/2}}{i}\partial_{\dot{x}} - \frac{q}{m}\mathcal{E}\frac{V\gamma F_H^{3/2}}{i'i'} \mathcal{R} = i'\partial_{t'} + \dot{x}'\partial_{x'} - \frac{q}{m}\mathcal{E}\frac{F_H^{3/2}}{i'}\partial_{\dot{x}'}. \quad (121)$$

From the same reasoning as for eq. (118) it follows that

$$W_{E_{t'}} = \partial_{t'} + \dot{x}'\partial_{x'} - \frac{q}{m}\mathcal{E}(F_H \circ \Sigma_{E_{t'}})^{3/2}\partial_{\dot{x}'}, \quad (122)$$

where (t', x', i', \dot{x}') is the coordinate system adapted to $E_{t'}$.

To get the particle density function in the new frame we first need the correct volume element $\Omega \in \Gamma\Lambda^4 U$,

$$\Omega = \frac{-\det g}{F_H^{3/2}} dt \wedge dx \wedge di \wedge d\dot{x}. \quad (123)$$

To see that $L_W\Omega = 0$ observe that (noting $\det g = -1$)

$$\begin{aligned} L_W\Omega &= di_W\Omega = d\left(\frac{1}{F_H^{3/2}}\left(idx \wedge di \wedge d\dot{x} - \dot{x}dt \wedge di \wedge d\dot{x} + \frac{q}{m}\mathcal{E}\frac{F_H^{3/2}}{i}dt \wedge dx \wedge di\right)\right) \\ &= \left(\partial_t i + \partial_x \dot{x} - \frac{q}{m}\mathcal{E}\partial_{\dot{x}}\frac{1}{i}\right) dt \wedge dx \wedge di \wedge d\dot{x} = 0. \end{aligned} \quad (124)$$

A similar calculation shows $L_{W'}\Omega = 0$. From lemma 2.22, by setting $s = t$ and $\alpha = (F_H \circ \Sigma_{E_t})^{-3/2} dx \wedge d\dot{x}$, and noting that $L_{\mathcal{R}}\Omega = -\Omega$ we see that Ω is unique.

To write θ_{E_t} we first calculate $i_{W_{E_t}}\Sigma_{E_t}^*i_{\mathcal{R}}\Omega$:

$$\begin{aligned} i_{\mathcal{R}}\Omega &= \frac{i}{F_H^{3/2}}dt \wedge dx \wedge d\dot{x} - \frac{\dot{x}}{F_H^{3/2}}dt \wedge dx \wedge di, \\ \Sigma_{E_t}^*i_{\mathcal{R}}\Omega &= \frac{1}{(F_H \circ \Sigma_{E_t})^{3/2}}dt \wedge dx \wedge d\dot{x}, \\ i_{W_{E_t}}\Sigma_{E_t}^*i_{\mathcal{R}}\Omega &= \frac{1}{(F_H \circ \Sigma_{E_t})^{3/2}}dx \wedge d\dot{x} - \frac{\dot{x}}{(F_H \circ \Sigma_{E_t})^{3/2}}dt \wedge d\dot{x} - \frac{q}{m}\mathcal{E}dt \wedge dx. \end{aligned} \quad (125)$$

Hence we may write

$$\begin{aligned}\theta_{E_t} &= f_{E_t} i_{W_{E_t}} \Sigma_{E_t}^* i_R \Omega \\ &= \frac{f_{E_t}}{(F_H \circ \Sigma_{E_t})^{3/2}} \left(dx \wedge d\dot{x} - \dot{x} dt \wedge d\dot{x} - \frac{q}{m} \mathcal{E}(F_H \circ \Sigma_{E_t})^{3/2} dt \wedge dx \right).\end{aligned}\quad (126)$$

Recall the definition for $\Pi_{E_t} : U^+ \rightarrow E_t$ from eq. (90). Using this we can define $\theta^+ = \Pi_{E_t}^* \theta_{E_t}$. Define coordinates on E_t by $(t_{E_t}, x_{E_t}, \dot{x}_{E_t})$ (note that this is the same as (t, x, \dot{x}) in previous and subsequent equations due to abuse of notation). Using the fact that $(t, x, \dot{x}) \mapsto (t_{E_t}, x_{E_t}, \dot{x}_{E_t})$ where $\Sigma_{E_t}(t_{E_t}, x_{E_t}, \dot{x}_{E_t}) = (t, x, 1, \dot{x}/i)$,

$$\Pi_{E_t}^*(d\dot{x}) = d\left(\frac{\dot{x}}{i}\right) = \frac{1}{i} d\dot{x} - \frac{\dot{x}}{i^2} di \quad (127)$$

$$\begin{aligned}\Rightarrow \theta^+ &= \frac{f_{E_t} \circ \Pi_{E_t}}{(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t})^{3/2}} \left(dx \wedge \left(\frac{1}{i} d\dot{x} - \frac{\dot{x}}{i^2} di \right) - \frac{\dot{x}}{i} dt \wedge \left(\frac{1}{i} d\dot{x} - \frac{\dot{x}}{i^2} di \right) - \frac{q}{m} \mathcal{E}(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t})^{3/2} dt \wedge dx \right) \\ &= \frac{f_{E_t} \circ \Pi_{E_t}}{(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t})^{3/2}} \left(\frac{1}{i} dx \wedge d\dot{x} - \frac{\dot{x}}{i^2} dx \wedge di - \frac{\dot{x}}{i^2} dt \wedge d\dot{x} + \frac{\dot{x}^2}{i^3} dt \wedge di - \frac{q}{m} \mathcal{E}(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t})^{3/2} dt \wedge dx \right).\end{aligned}\quad (128)$$

$$\quad (129)$$

Now we perform the coordinate transformation corresponding to the Lorentz boost:

$$\begin{aligned}\frac{1}{i} dx \wedge d\dot{x} &= \frac{\gamma^2}{i} (dx' + V dt') \wedge (d\dot{x}' + V di') \\ &= \frac{\gamma^2}{i} (dx' \wedge d\dot{x}' + V dx' \wedge di' + V dt' \wedge d\dot{x}' + V^2 dt' \wedge di'), \\ -\frac{\dot{x}}{i^2} dx \wedge di &= -\frac{\dot{x}\gamma^2}{i^2} (dx' + V dt') \wedge (di' + V d\dot{x}') \\ &= -\frac{\dot{x}\gamma^2}{i^2} (dx' \wedge di' + V dx' \wedge d\dot{x}' + V dt' \wedge di' + V^2 dt' \wedge d\dot{x}'), \\ -\frac{\dot{x}}{i^2} dt \wedge d\dot{x} &= -\frac{\dot{x}\gamma^2}{i^2} (dt' + V dx') \wedge (d\dot{x}' + V di') \\ &= -\frac{\dot{x}\gamma^2}{i^2} (dt' \wedge d\dot{x}' + V dt' \wedge di' + V dx' \wedge d\dot{x}' + V^2 dx' \wedge di'), \\ \frac{\dot{x}^2}{i^3} dt \wedge di &= \frac{\dot{x}^2\gamma^2}{i^3} (dt' + V dx') \wedge (di' + V d\dot{x}') \\ &= \frac{\dot{x}^2\gamma^2}{i^3} (dt' \wedge di' + V dt' \wedge d\dot{x}' + V dx' \wedge di' + V^2 dx' \wedge d\dot{x}'), \\ -\frac{q}{m} \mathcal{E}(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t})^{3/2} dt \wedge dx &= -\frac{q}{m} \mathcal{E}(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t})^{3/2} \gamma^2 (dt' + V dx') \wedge (dx' + V dt') \\ &= -\frac{q}{m} \mathcal{E}(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t})^{3/2} dt' \wedge dx'.\end{aligned}\quad (130)$$

Gathering terms gives

$$\begin{aligned}
dx' \wedge d\dot{x}' &: \frac{\gamma^2}{i} \left(1 - 2\frac{\dot{x}V}{i} + \frac{V^2\dot{x}^2}{i^2} \right) = \frac{i'^2}{\gamma^3(i' + V\dot{x}')^3} \\
dx' \wedge di' &: \frac{\gamma^2}{i} \left(V - \frac{\dot{x}}{i}(1 + V^2) + \frac{V\dot{x}^2}{i^2} \right) = \frac{-\dot{x}'i'}{\gamma^3(i' + V\dot{x}')^3} \\
dt' \wedge d\dot{x}' &: \frac{\gamma^2}{i} \left(V - \frac{\dot{x}}{i}(1 + V^2) + \frac{\dot{x}^2V}{i^2} \right) = \frac{-\dot{x}'i'}{\gamma^3(i' + V\dot{x}')^3} \\
dt' \wedge di' &: \frac{\gamma^2}{i} \left(V^2 - 2\frac{\dot{x}V}{i} + \frac{\dot{x}^2}{i^2} \right) = \frac{\dot{x}'^2}{\gamma^3(i' + V\dot{x}')^3} \\
dt' \wedge dx' &: -\frac{q}{m} \mathcal{E}(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t})^{3/2}.
\end{aligned} \tag{131}$$

Therefore, θ^+ takes the form

$$\begin{aligned}
\theta^+ &= \frac{f_{E_t} \circ \Pi_{E_t}}{(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t})^{3/2}} \left(\frac{i'^2}{\gamma^3(i' + V\dot{x}')^3} dx' \wedge d\dot{x}' + \frac{-\dot{x}'i'}{\gamma^3(i' + V\dot{x}')^3} dx' \wedge di' + \frac{-\dot{x}'i'}{\gamma^3(i' + V\dot{x}')^3} dt' \wedge d\dot{x}' \right. \\
&\quad \left. + \frac{\dot{x}'^2}{\gamma^3(i' + V\dot{x}')^3} dt' \wedge di' - \frac{q}{m} \mathcal{E}(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t})^{3/2} dt' \wedge dx' \right).
\end{aligned} \tag{132}$$

Performing the pullback $\Sigma_{E_{t'}}^*$ on θ^+ gives

$$\begin{aligned}
\theta_{E_{t'}} &= \frac{f_{E_t} \circ \Pi_{E_t} \circ \Sigma_{E_{t'}}}{(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t} \circ \Sigma_{E_{t'}})^{3/2}} \left(\frac{1}{\gamma^3(1 + V\dot{x}')^3} dx' \wedge d\dot{x}' + \frac{-\dot{x}'}{\gamma^3(1 + V\dot{x}')^3} dt' \wedge d\dot{x}' \right. \\
&\quad \left. - \frac{q}{m} \mathcal{E}(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t} \circ \Sigma_{E_{t'}})^{3/2} dt' \wedge dx' \right).
\end{aligned} \tag{133}$$

To simplify observe that

$$\begin{aligned}
\Sigma_{E_t}^* F_H &= 1 - \dot{x}^2, \\
\Pi_{E_t}^* \Sigma_{E_t}^* F_H &= 1 - \frac{\dot{x}^2}{i^2} = \frac{-\dot{x}'^2 + i'^2}{\gamma^2(i' + V\dot{x}')^2}, \\
\Sigma_{E_{t'}}^* \Pi_{E_t}^* \Sigma_{E_t}^* F_H &= \frac{1 - \dot{x}'^2}{\gamma^2(1 + V\dot{x}')^2} = \frac{F_H \circ \Sigma_{E_{t'}}}{\gamma^2(1 + V\dot{x}')^2}.
\end{aligned} \tag{134}$$

Plugging into our expression for $\theta_{E_{t'}}$ yields

$$\theta_{E_{t'}} = \frac{f_{E_t} \circ \Pi_{E_t} \circ \Sigma_{E_{t'}}}{(F_H \circ \Sigma_{E_t} \circ \Pi_{E_t} \circ \Sigma_{E_{t'}})^{3/2}} \left(dx' \wedge d\dot{x}' - \dot{x}' dt' \wedge d\dot{x}' - \frac{q}{m} \mathcal{E}(F_H \circ \Sigma_{E_{t'}})^{3/2} dt' \wedge dx' \right). \tag{135}$$

Comparing this with the definition for $\theta_{E_{t'}} = f_{E_{t'}} i_{W_{E_{t'}}} \Sigma_{E_{t'}}^* i_{\mathcal{R}} \Omega$,

$$\theta_{E_{t'}} = \frac{f_{E_{t'}}}{(F_H \circ \Sigma_{E_{t'}})^{3/2}} \left(dx' \wedge d\dot{x}' - \dot{x}' dt' \wedge d\dot{x}' - \frac{q}{m} \mathcal{E}(F_H \circ \Sigma_{E_{t'}})^{3/2} dt' \wedge dx' \right), \tag{136}$$

allows us to make the identification

$$f_{E_{t'}} = f_{E_t} \circ \Pi_{E_t} \circ \Sigma_{E_{t'}}. \tag{137}$$

By observing that

$$\Sigma_{E_{t'}} : E_{t'} \hookrightarrow U; (t', x', \dot{x}') \mapsto (t', x', 1, \dot{x}') = (\gamma t - \gamma V x, \gamma x - \gamma V t, 1, \gamma \dot{x} - \gamma V i), \tag{138}$$

$$\Pi_{E_t} : U \rightarrow E_t; (\gamma t - \gamma V x, \gamma x - \gamma V t, 1, \gamma \dot{x} - \gamma V i) \mapsto (\gamma t - \gamma V x, \gamma x - \gamma V t, \gamma \dot{x} - \gamma V), \tag{139}$$

we may write

$$f_{E_{t'}}(t'(x, t), x'(x, t), \dot{x}'(\dot{x})) = f_{E_t}(\gamma t - \gamma V x, \gamma x - \gamma V t, \gamma \dot{x} - \gamma V). \quad (140)$$

It will now be necessary to create a new grid and redeposit the particles over this grid, using interpolation and eq. (140). To do this, we require a new choice of grid spacing, a reasonable option for the matched grid case would be

$$\Delta t' = \gamma \Delta t, \quad \Delta x' = \frac{\Delta x}{\gamma}, \quad \Delta x' = \frac{\Delta x'}{\Delta t'} = \frac{1}{\gamma^2} \frac{\Delta x}{\Delta t}. \quad (141)$$

On this new grid, we can recreate eq. (114) using eq. (140) and eq. (141) in order to execute the ANI in the new boosted bundle $E_{t'}$.

6 Conclusion

In this article, we have presented an alternative way of representing the Vlasov equation. The primary advantage of this representation is that it enables the user to write the equations of motion without reference to a kinematic domain. As a result, the transformation laws of the Vlasov field become apparent and transformations between frames can be easily calculated in our framework.

The generality of our formalism makes it applicable to a wide variety of situations where the usual tools of kinematics would otherwise fail. For example, these tools apply in situations where there cannot exist a time orientation, when one does not have a metric compatible connection, or when working in a pre-metric context. Additionally, one can investigate the Vlasov equation for light-like particles. This enables the extension of the ultra-relativistic approximation [6] from fluids to the Vlasov equation. This is particularly relevant when considering particles in particle accelerators as well as astrophysical plasmas near black holes and neutron stars.

The current 3-form and the transport equations are generalised to our formalism. However, the generalisation of stress-energy 3-form depends on the choice of kinematic domain or a support form. Furthermore, the divergence of the stress-energy 3-form, also depends on the choice of kinematic domain. As a result it may be divergence-free with respect to one E , but not another. Since the stress-energy tensor underlies the Einstein-Vlasov system, it will be important, in future work, to investigate the relationship between τ_α^E , and its divergence, for different kinematic domains.

As highlighted in section 5, our work has implications for numerical simulations, in particular when these simulations concern coordinate transformations, non inertial frames or general relativity. The formulation of the (relativistic) Vlasov equation [38] in terms of the particle density form, i.e. the transport equations, allows for the generalisation of some existing numerical integrators; for example, the agile numerical integrator [11]. Our formalism enables the extension of the ANI to include ultra-relativistic scenarios, that is, where the velocities approach the speed of light. For example, when electrons are accelerated up to ultra-relativistic velocities, but the spread in their energies remains small. In section 5.2, we show that this is achieved by performing a series of Lorentz boosts, so that the PDF is always in a frame where velocities are not close to the speed of light. These techniques may be generalised to other numerical integrators which see applications in accelerator physics. For future work, one can look at applying this technique to 2d Vlasov solvers. This will enable the modelling of the transfer for energy from high energy electrons to electromagnetic waves using a wiggler or undulator, generalising work such as [39].

There are many interesting plasma phenomena like Landau damping, two stream instability, and plasma echos which can be further investigated with these tools. The basic theory of these starts with a (relativistic) Poisson-Vlasov system, for example [38]. As such, they have a preferred inertial lab-frame and corresponding time-slicing. The power of the results in this article will become apparent if one wishes to reformulated the Poisson-Vlasov systems in a non-inertial lab-frame or with respect to a more complicated kinematic domain. For example, a scenario where there is a magnetic field in the inertial lab-frame, but no magnetic field in an adapted co-moving accelerating lab-frame.

There is clearly a deep relationship between the work presented here and the idea of trajectory and solutions of second-order ODEs which do not have a parameter prescribed. In a follow-up article, we will show how to define such trajectories and their connection to the leaves of the Vlasov bivector.

Other directions one may consider are to look at the relationship of this work with jet bundles and Finsler geometry. One can also look at how to generalise the Vlasov equation and the Boltzmann equation. In this latter case, we may have to replace the transport equation eq. (86), with $\text{Null}(\Psi, \theta)$ and $d\pi_\zeta \theta = 0$.

In summary we argue that the Vlasov bivector is the fundamental object to describe kinetic systems, since it is invariant under reparameterisation.

A Appendices

A.1 Sprays and Semi-Sprays

Vlasov fields can be formulated in terms of sprays and semi-sprays. For a detailed discussion of sprays the reader is directed towards [26] and [40]. For our purposes we define a spray as follows.

Definition A.1 (Spray). A *spray* on a smooth manifold N is a smooth vector field $X \in \Gamma T(\check{T}N)$ which is expressed in local adapted coordinates (x^μ, \dot{x}^μ) as

$$X = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + X^\mu \frac{\partial}{\partial \dot{x}^\mu}, \quad (142)$$

where $X^\mu = X^\mu(x, \dot{x})$ are local functions on $\check{T}N$ satisfying

$$X^\mu|_{\lambda \underline{u}} = \lambda^2 X^\mu|_{\underline{u}}, \quad \lambda > 0, \quad \underline{u} \in \check{T}N. \quad (143)$$

In the case where $\check{T}N$ is a conic bundle this can be shown to be equivalent to definition 2.3.

The trajectories of a spray are defined in the same way as described in section 2.3, satisfying the equation

$$\frac{d^2 C^\mu}{dt^2} = X^\mu \left(C^\mu(t), \frac{dC^\mu}{dt} \right). \quad (144)$$

Definition A.2 (Projectively related Sprays). Two sprays X and \hat{X} are *projectively related* if they have the same trajectories as point sets. That is, if $C(t)$ is a trajectory of X , then there exists a reparameterisation $t = t(s)$ such that $C(s) := C(t(s))$ is a geodesic of \hat{X} (and vice versa).

Lemma A.3. *Two sprays X and \hat{X} are projectively related if and only if there exists a 1-homogeneous scalar field $k \in \Gamma \Lambda^0(\check{T}N)$ such that*

$$\hat{X} = X + k\mathcal{R}, \quad (145)$$

where \mathcal{R} is the radial vector field on $\check{T}N$.

Proof. See Z. Shen, Differential Geometry of Spray and Finsler Spaces, pages 173-174 [26], and lemma 2.17. \square

In the literature a semi-spray is defined similarly to a spray on $\check{T}N$, only without the homogeneity property. For our purposes, it is productive to define sprays on some hypersurface $K \subset \check{T}N$. We restrict our attention to hypersurfaces defined similarly to lab-time bundles.

Definition A.4 (Semi-Spray). Given $s \in \Gamma \Lambda^0 N$ let $K = \{\underline{u} \in \check{T}N : s|_{\underline{u}} = 1\}$. A *semi-spray* $X_K \in \Gamma \check{T}K$ is a vector field given by

$$X_K = \frac{\partial}{\partial s} + \dot{x}^a \frac{\partial}{\partial x^a} + X_K^a(s, x^a, \dot{x}^a) \frac{\partial}{\partial \dot{x}^a}. \quad (146)$$

Unlike a spray, there are no homogeneity conditions on X_K^a . The semi-spray X_K corresponds to a set of ordinary differential equations locally expressed as

$$\frac{d^2 f^a}{ds^2} = X_K^a \left(s, f^a, \frac{df^a}{ds} \right). \quad (147)$$

Similarly to the case with a spray, $f(s)$ is a solution to eq. (147) if and only if its lift $\dot{f}(s) = (1, f^a(s), df^a(s)/ds)$ is an integral curve of X_K .

Observe that in the case where $K = E_s \subset U$, we may identify eq. (146) with eq. (35), a Vlasov field on a lab-time bundle eq. (21).

Given such a semi-spray on K we may construct a spray on $\check{T}N$ and vice versa. The full details of the lemma can be found in [26]. An example of this lemma in action is given below. If we are given a semi-spray determined by coefficients X_K^a over K (equipped with a choice of parameterisation s) then we may construct a spray over $\check{T}N$ with the following coefficients

$$\begin{cases} X^0(x, \dot{x}) = 0 \\ X^a(x, \dot{x}) = \dot{x}^0 \dot{x}^0 X_K^a(x^0, x^a, \dot{x}^a / \dot{x}^0), \end{cases} \quad (148)$$

where $s = x^0$. Note that under this construction, X_K is induced by X . This is an example of the quadratic extension described in lemma 2.12. The freedom to choose X^0 in eq. (148) roughly corresponds to the freedom to re-parameterise the spray according to eq. (145). In the instance where we restrict ourselves to a Vlasov field on a lab-time then section 2.2 can be identified with eq. (148).

A.2 Auxiliary Lemmas

Lemma A.5. *Let $X \in \Gamma TN$, $f \in \Gamma \Lambda^0 N$ be such that $df \neq 0$, and $f^{-1}\{0\} = K \subset N$ with $\Sigma_K : K \hookrightarrow N$. We have $X|_K \langle f \rangle = 0$ if and only if there exists a unique $Y \in \Gamma TK$ such that $X|_K = \Sigma_{K*} Y$ (i.e. X is tangent to K).*

Proof. Suppose X is tangent to K then we have

$$X|_K \langle f \rangle = \Sigma_{K*} Y \langle f \rangle = Y \langle f \circ \Sigma_K \rangle = Y \langle 0 \rangle = 0.$$

Suppose now that $X|_K \langle f \rangle = 0$. If a suitable vector field exists it is unique by the injectivity of Σ_K . Let ℓ be the dimension of N . Since $df \neq 0$ there exists a local coordinate system $\{x^0 = f, x^1, \dots, x^{\ell-1}\}$ where we may express X as $X = X^\mu \partial_\mu^{(x)}$, where $\mu = 0, \dots, \ell - 1$. Since $X \langle f \rangle = 0$, we have $X^0 = X \langle x^0 \rangle = 0$. In this coordinate system we also have $\Sigma_K : (x^0, \dots, x^{\ell-1}) \mapsto (0, x^1, \dots, x^{\ell-1})$. This allows us to define a vector field $Y \in \Gamma TK$ where locally $Y = Y^a \partial_a^{(x)}$, $Y^a = \Sigma_{K*} X^a$, for $a = 1, \dots, \ell - 1$. It follows that $X|_K = \Sigma_{K*} Y$ and hence X is tangent to K . \square

Lemma A.6. *Given a spacetime manifold M with a pseudo-Riemann metric g , there exists a Vlasov field $W \in \Gamma TU$ constructed from a force equation involving a non-metric compatible connection $\hat{\nabla}$ i.e. in local coordinates (x, \dot{x}) ,*

$$\hat{\nabla}_{\dot{C}} \dot{C} = i_{\dot{C}} \widetilde{\mathcal{F}} \quad \text{and} \quad W = \dot{x}^\mu \partial_\mu^{(x)} + \left(g^{\mu\nu} \mathcal{F}_{\nu\rho} \dot{x}^\rho - \hat{\Gamma}_{\rho\nu}^\mu \dot{x}^\nu \dot{x}^\rho \right) \partial_\mu^{(\dot{x})}, \quad (149)$$

where $\dot{C} = C_* (\partial_\tau)$ and τ is the chosen parameterisation of the trajectories, such that the integral curves of W will not lie on E_H .

Proof. Let ∇ denote the Levi-Civita connection built from g . Since $\hat{\nabla}$ is non-metric compatible, it has non-vanishing non-metricity:

$$\hat{Q} = \hat{\nabla} g \neq 0.$$

Letting F_H denote the kinematic indicator of E_H . We observe that W is not tangent to E_H since there exists \dot{C} such that

$$\begin{aligned} W|_{\dot{C}} \langle F_H \rangle &= \dot{C}_* (\partial_\tau) \langle F_H \rangle = \partial_\tau \langle F_H \circ \dot{C} \rangle = \partial_\tau C^* (g(\dot{C}, \dot{C})) = \dot{C} \langle g(\dot{C}, \dot{C}) \rangle = \hat{\nabla}_{\dot{C}} (g(\dot{C}, \dot{C})) = \hat{Q}(\dot{C}, \dot{C}, \dot{C}) + g(\hat{\nabla}_{\dot{C}} \dot{C}, \dot{C}) \\ &= \hat{Q}(\dot{C}, \dot{C}, \dot{C}) + g(i_{\dot{C}} \widetilde{\mathcal{F}}, \dot{C}) = \hat{Q}(\dot{C}, \dot{C}, \dot{C}) + i_{\dot{C}} i_{\dot{C}} \mathcal{F} = \hat{Q}(\dot{C}, \dot{C}, \dot{C}) \neq 0, \end{aligned}$$

where τ is a parameter and $\dot{C} = C_* (\partial_\tau)$. Hence W is not tangent to E_H by lemma A.5, and consequently, its integral curves will not remain on E_H . Conversely, if we replace $\hat{\nabla}$ with the Levi-Civita connection ∇ in eq. (149), then we may see that $W|_{\dot{C}} \langle F_H \rangle = 0$ since $\nabla g = 0$. Hence W is tangent to E_H and consequently its integral curves are confined to E_H . \square

Lemma A.7. *Let M be a Minkowsky spacetime manifold of dimension 2 and let $s \in \Gamma \Lambda^0 M$ define a lab-time function such that E_s defines a lab-time bundle (see eq. (21)). The null geodesics parameterised by the induced lab-time coordinate in general do not satisfy the geodesic equation. Consequently, the prolongations of lab-time parameterised curves must be expressed in terms of the pre-geodesic equation.*

Proof. Let $C : I \hookrightarrow M$ be a trajectory parameterised by $\tau \in \Gamma \Lambda^0 I$. Then the prolongation satisfies

$$\dot{C} = C_* (\partial_\tau), \quad \dot{C} \langle s \rangle = \frac{d(s \circ C)}{d\tau} = 1.$$

Let (t, x) define coordinates on M . Null trajectories in Minkowsky spacetime form straight lines, for example,

$$(t \circ C)(\tau) = (x \circ C)(\tau).$$

The trajectory above satisfies

$$\frac{\partial(s \circ C)}{\partial \tau} = \frac{\partial(x \circ C)}{\partial \tau} \frac{\partial s}{\partial x} + \frac{\partial(t \circ C)}{\partial \tau} \frac{\partial s}{\partial t} = \frac{\partial(x \circ C)}{\partial \tau} \left(\frac{\partial s}{\partial x} + \frac{\partial s}{\partial t} \right) = 1$$

By defining $X = x \circ C$ and $T = t \circ C$ we can write

$$\dot{X} = \left(\frac{\partial s}{\partial x} + \frac{\partial s}{\partial t} \right)^{-1} \frac{\partial}{\partial x}, \quad \dot{T} = \left(\frac{\partial s}{\partial x} + \frac{\partial s}{\partial t} \right)^{-1} \frac{\partial}{\partial t}.$$

It follows that

$$\nabla_{\dot{X}} \dot{X} \langle s \rangle = \frac{d}{d\tau} \left(\frac{\partial s}{\partial x} + \frac{\partial s}{\partial t} \right)^{-1} \frac{\partial s}{\partial x} \neq 0,$$

in general since there is no specific relationship between s , t and x . The same result follows for \dot{T} . Hence the null trajectories parameterised by the lab-time function cannot obey the geodesic equation, they must obey a pre-geodesic equation.

If, however, we choose to parameterise the null geodesics by the Minkowsky coordinate t then the above equation does satisfy $\nabla_{\dot{X}} \dot{X} = \nabla_{\dot{T}} \dot{T} = 0$. \square

Lemma A.8. *Let (M, g) be a spacetime manifold, let E be a kinematic domain with a local coordinate system (x^μ, \dot{x}^a) , let Ω_E be the volume form as given by eq. (60), and let θ_E be a particle density form built from Ω_E . The stress-energy 3-form τ_α^E eq. (101) is equivalent to the Einstein-Vlasov stress-energy tensor eq. (102).*

Proof. First observe that $\widehat{dx^\mu}|_{\underline{u}} = dx^\mu : \underline{u} = u^\mu = \dot{x}^\mu|_{\underline{u}}$. We may write θ_E as

$$\theta_E = f_E i_{W_E} \Omega_E = f_E \dot{x}^\mu i_\mu^{(x)} \Omega_E + f_E \varphi_E^a i_a^{(\dot{x})} \Omega_E,$$

where $\varphi_E^a = W_E \langle \dot{x}^a \rangle$. Here we use the coordinate-based definition for the de Rham pushforward as seen in [33]. Since only forms with maximal degree in the fibre coordinates contribute to the de Rham pushforward we have

$$\begin{aligned} (dx^\mu \wedge \tau_{dx^\nu}^E)|_p &= dx^\mu \wedge \pi_\zeta \left(\widehat{dx^\nu} \theta_E \right) \Big|_p = -(dx^\mu \wedge i_\rho^{(x)} d^4 x) \Big|_p \int_{E_p} \widehat{dx^\nu} \dot{x}^\rho f_E \frac{\det(g)}{\dot{x}_0} d^3 \dot{x} \\ &= -d^4 x|_p \int_{E_p} \dot{x}^\mu \dot{x}^\nu f_E \frac{\det(g)}{\dot{x}_0} d^3 \dot{x}. \end{aligned}$$

Since $\star 1 = \sqrt{\det(g)} d^4 x$ and $\star \star 1 = -1$ we have for each $p \in M$

$$\star (dx^\mu \wedge \tau_{dx^\nu}^E) \Big|_p = \int_{E_p} \dot{x}^\mu \dot{x}^\nu f_E \frac{\sqrt{\det(g)}}{\dot{x}_0} d^3 \dot{x},$$

so that eq. (102) holds. \square

A.3 Proofs

Proof of lemma 3.6. Let X_1, X_2 and Y_1, Y_2 be specially related. We then have

$$X_1 \wedge X_2 = (\alpha Y_1 + \beta Y_2) \wedge (\gamma Y_1 + \delta Y_2) = \alpha \delta Y_1 \wedge Y_2 + \beta \gamma Y_2 \wedge Y_1 = (\alpha \delta - \beta \gamma) Y_1 \wedge Y_2 = Y_1 \wedge Y_2.$$

Suppose $X_1 \wedge X_2 \neq 0$ then there exists a basis $\{X_1, \dots, X_n\}$ such that $X_i \wedge X_j \neq 0$ for $i \neq j$, $i, j = 1, \dots, n$. We may then express

$$Y_1 = \alpha^i X_i, \quad Y_2 = \beta^i X_i,$$

for scalar fields $\alpha^i, \beta^i \in \Gamma \Lambda^0 N$.

Hence, if we have $X_1 \wedge X_2 = Y_1 \wedge Y_2$, we observe that

$$X_1 \wedge X_2 = Y_1 \wedge Y_2 = \alpha^i X_i \wedge \beta^j X_j = (\alpha^1 \beta^2 - \alpha^2 \beta^1) X_1 \wedge X_2.$$

It follows that $\alpha^1 \beta^2 - \beta^1 \alpha^2 = 1$. It remains to be shown that the other coefficients $((\alpha^i, \beta^i)$ for $i > 2$) vanish. Since the coefficients of the terms $X_i \wedge X_j$ for i or $j > 2$ are vanishing, we have $\alpha^i \beta^j - \alpha^j \beta^i = 0$ for i or $j > 2$. We have the following set of equations:

$$\alpha^i \beta^j = \beta^i \alpha^j, \quad \text{for } i, j > 2, \quad (150)$$

$$\alpha^1 \beta^j = \alpha^j \beta^1, \quad \text{for } j > 2, \quad (151)$$

$$\alpha^2 \beta^j = \alpha^j \beta^2, \quad \text{for } j > 2. \quad (152)$$

Define column vectors

$$A = \begin{pmatrix} \alpha^3 \\ \vdots \\ \alpha^n \end{pmatrix}, \quad B = \begin{pmatrix} \beta^3 \\ \vdots \\ \beta^n \end{pmatrix}.$$

These column vectors satisfy the following relations

$$\alpha^1 B = \beta^1 A, \quad \text{by eq. (151),} \quad (153)$$

$$\alpha^2 B = \beta^2 A, \quad \text{by eq. (152).} \quad (154)$$

By multiplying eq. (153) by β^2 and then applying eq. (154) we observe that $\beta^2 \alpha^1 B = \beta^2 \beta^1 A = \beta^1 (\beta^2 A) = \beta^1 (\alpha^2 B)$ and hence $0 = (\beta^2 \alpha^1 - \beta^1 \alpha^2) B = B$. Since at least one of β^1, β^2 is non-zero we must also have $A = 0$ by eq. (153) or eq. (154). Hence X_1, X_2 and Y_1, Y_2 are specially related. \square

Proof of lemma 3.9. Since Φ is integrable there exists $V, Z \in \Gamma TN$ such that $\Phi = V \wedge Z$ and $[V, Z] = \gamma V + \delta Z$. Let Φ have another representation $\Phi = X \wedge Y$. By lemma 3.6 there exist scalar field $\lambda, \sigma, \rho, \kappa \in \Gamma \Lambda^0 N$ such that

$$X = \lambda V + \sigma Z, \quad \text{and } Y = \rho V + \kappa Z.$$

The Lie bracket of X and Y satisfies

$$[X, Y] = [\lambda V + \sigma Z, \rho V + \kappa Z] = [\lambda V, \rho V] + [\lambda V, \kappa Z] + [\sigma Z, \rho V] + [\sigma Z, \kappa Z].$$

Consider a term containing V and Z e.g.

$$[\lambda V, \kappa Z] = \lambda V \langle \kappa \rangle Z - \kappa Z \langle \lambda \rangle V + \lambda \kappa [V, Z] = (\lambda \kappa \gamma - \kappa Z \langle \lambda \rangle) V + (\lambda \kappa \delta + \lambda V \langle \kappa \rangle) Z.$$

Consider a term containing two of the same types e.g.

$$[\lambda V, \rho V] = \lambda V \langle \rho \rangle V - \rho V \langle \lambda \rangle V + \lambda \rho [V, V] = \lambda V \langle \rho \rangle V - \rho V \langle \lambda \rangle V.$$

Hence we may write

$$\begin{aligned} [X, Y] &= \lambda V \langle \rho \rangle V - \rho V \langle \lambda \rangle V + (\lambda \kappa \gamma - \kappa Z \langle \lambda \rangle) V + (\lambda \kappa \delta + \lambda V \langle \kappa \rangle) Z + (\sigma \rho \gamma - \rho V \langle \sigma \rangle) Z + (\sigma \rho \delta + \sigma Z \langle \rho \rangle) V \\ &\quad + \sigma Z \langle \kappa \rangle Z - \kappa Z \langle \sigma \rangle Z \\ &= (\lambda V \langle \rho \rangle - \rho V \langle \lambda \rangle + \lambda \kappa \gamma - \kappa Z \langle \lambda \rangle + \sigma \rho \delta + \sigma Z \langle \rho \rangle) V \\ &\quad + (\sigma Z \langle \kappa \rangle - \kappa Z \langle \sigma \rangle + \lambda \kappa \delta + \lambda V \langle \kappa \rangle + \sigma \rho \gamma - \rho V \langle \sigma \rangle) Z \\ &= \gamma' V + \delta' Z \end{aligned}$$

There exist $\lambda', \sigma', \rho', \kappa'$ such that $V = \lambda' X + \sigma' Y$ and $Z = \rho' X + \kappa' Y$. These are guaranteed to exist since $\lambda \kappa - \sigma \rho = 1$. It follows that

$$[X, Y] = (\gamma' \lambda' + \delta' \rho') X + (\gamma' \sigma' + \delta' \kappa') Y.$$

\square

Proof of lemma 3.17. By lemma 3.13 there exists a horizontal vector field $X \in \Gamma TU$ such that $\Psi = \mathcal{R} \wedge X$. Then W_{crd} and X are related by

$$W_{\text{crd}} = \frac{1}{2F_{\text{crd}}} (\mathcal{R}\langle F_{\text{crd}} \rangle X - X\langle F_{\text{crd}} \rangle \mathcal{R}) = X - \frac{X\langle F_{\text{crd}} \rangle}{2F_{\text{crd}}} \mathcal{R}.$$

To see that $\Psi = \mathcal{R} \wedge W_{\text{crd}}$ first suppose that $\Psi = \mathcal{R} \wedge X$ for some $X \in \Gamma TU$ then observe that

$$\mathcal{R} \wedge \frac{\Psi\langle F_{\text{crd}}, \bullet \rangle}{2F_{\text{crd}}} = \mathcal{R} \wedge \left(\frac{\mathcal{R}\langle F_{\text{crd}} \rangle}{2F_{\text{crd}}} X - \frac{X\langle F_{\text{crd}} \rangle}{2F_{\text{crd}}} \mathcal{R} \right) = \mathcal{R} \wedge X = \Psi.$$

Define local coordinates (x^μ, \dot{x}^μ) . To see that W_{crd} is horizontal observe that for any $f \in \Gamma \Lambda^0 M$ we have

$$W_{\text{crd}}\langle \pi^* f \rangle = \frac{\Psi\langle F_{\text{crd}}, \pi^* f \rangle}{2F_{\text{crd}}} = \sum_{\mu} \frac{\Psi\langle (\dot{x}^\mu)^2, \pi^* f \rangle}{2F_{\text{crd}}} = \sum_{\mu} \frac{\dot{x}^\mu \Psi\langle \dot{x}^\mu, \pi^* f \rangle}{F_{\text{crd}}} = \sum_{\mu} \frac{(\dot{x}^\mu)^2 f}{F_{\text{crd}}} = \dot{f}.$$

To see that W_{crd} is radially quadratic observe that for any $\underline{u} \in U$, $\lambda \in \mathbb{R}$, $f \in \Gamma \Lambda^0 M$,

$$\begin{aligned} W_{\text{crd}}|_{\lambda \underline{u}}\langle \dot{f} \rangle &= \frac{\Psi|_{\lambda \underline{u}}\langle F_{\text{crd}}, \dot{f} \rangle}{2F_{\text{crd}}|_{\lambda \underline{u}}} = \sum_{\mu} \left(\frac{\dot{x}^\mu}{F_{\text{crd}}} \right) \Big|_{\lambda \underline{u}} \Psi|_{\lambda \underline{u}}\langle \dot{x}^\mu, \dot{f} \rangle = \frac{\lambda}{\lambda^2} \sum_{\mu} \left(\frac{\dot{x}^\mu}{F_{\text{crd}}} \right) \Big|_{\underline{u}} \lambda^3 \Psi|_{\underline{u}}\langle \dot{x}^\mu, \dot{f} \rangle \\ &= \lambda^2 \frac{\Psi|_{\underline{u}}\langle (\dot{x}^\mu)^2, \dot{f} \rangle}{2F_{\text{crd}}|_{\underline{u}}} = \lambda^2 W_{\text{crd}}|_{\underline{u}}\langle \dot{f} \rangle. \end{aligned}$$

Hence W_{crd} is a Vlasov field. \square

Proof of lemma 4.5. Let F be the 1-homogeneous kinematic indicator for $E \subset U^+$. Define a foliation on U^+ of kinematic domains E_ℓ for $\ell \in \mathbb{R}^+$ where $E_1 = E$ and $F|_{E_\ell} = \ell$. We may define a coordinate system (x^μ, ℓ, ξ^a) for U^+ where ℓ is constant on each E_ℓ . We may choose ξ^a such that they are 0-homogeneous, and ℓ is 1-homogeneous by the 1-homogeneity of F . By lemma 2.6 we have for each $\lambda > 0$ and $\underline{u} \in U^+$,

$$W|_{\lambda \underline{u}}\langle x^\mu \rangle = \lambda W|_{\underline{u}}\langle x^\mu \rangle, \quad W|_{\lambda \underline{u}}\langle \xi^a \rangle = \lambda W|_{\underline{u}}\langle \xi^a \rangle. \quad (155)$$

The map

$$\Xi_\ell : E \rightarrow E_\ell ; \Xi_\ell(\underline{v}) = \ell \underline{v},$$

is well defined since $E_\ell = \{\ell \underline{v}, \underline{v} \in E\}$. Furthermore, it satisfies

$$\Pi_E \circ \Sigma_{E_\ell} \circ \Xi_\ell = \mathbb{1}_E \quad \text{i.e.} \quad \Xi_\ell = \left(\Pi_E \circ \Sigma_{E_\ell} \right)^{-1}.$$

Observe that we have

$$\Xi_\ell^* x^\mu = x^\mu, \quad \Xi_\ell^* \xi^a = \xi^a, \quad \Xi_\ell^* dx^\mu = dx^\mu, \quad \Xi_\ell^* d\xi^a = d\xi^a.$$

Lastly, define $\Omega \in \Gamma \Lambda^{2n-1} U$ by

$$\Omega = dx^0 \wedge \dots \wedge dx^{n-1} \wedge d\xi^1 \wedge \dots \wedge d\xi^{n-1}.$$

First observe that $d\theta^+ = 0$ since the exterior derivative commutes with the pullback. Notice also that $i_{\mathcal{R}}\theta^+ = 0$ since $\Pi_{E^*}\mathcal{R} = 0$ we have

$$i_{\mathcal{R}}\theta^+ = i_{\mathcal{R}}\Pi_E^*\theta_E = \Pi_E^*\left(i_{\Pi_{E^*}\mathcal{R}}\theta_E\right) = 0.$$

To see that $i_W\theta^+ = 0$ define coordinate functions y^0, \dots, y^{2n-2} as $y^k = x^k$ for $0 \leq k \leq n-1$, $y^k = v^{k-n+1}$ for $n \leq k \leq 2n-2$. Hence (y^0, \dots, y^{2n-2}) is a coordinate system for E while $(\ell, y^0, \dots, y^{2n-2})$ is a coordinate system for U^+ . With appropriate domains $y^k = \Pi_E^* y^k$ and $y^k = \Sigma_E^* y^k$. Hence for $\Sigma_E \circ \Pi_E : U^+ \rightarrow U^+$ we have $(\Sigma_E \circ \Pi_E)^*(y^k) = y^k$. Thus $\Omega = dy^0 \wedge \dots \wedge dy^{2n-2}$. Also,

$$(\Sigma_E \circ \Pi_E)^* f = f \quad \text{and} \quad (\Sigma_E \circ \Pi_E)^* i_k^{(y)} \Omega = i_k^{(y)} \Omega$$

where f is a scalar which such that $\mathcal{R}\langle f \rangle = 0$, so it is a function only of (y^0, \dots, y^{2n-2}) .

Since we have, for any $\underline{v} \in E$,

$$W|_{\lambda\underline{v}}\langle y^k \rangle = \lambda W|_{\underline{v}}\langle y^k \rangle$$

by eq. (155). Thus for $\underline{v} \in E$ (and $\lambda\underline{v} \in E_\lambda$)³⁴ we have

$$\left(\Xi_\lambda^* \left(i_{W_{E_\lambda}} dy^k \right) \right) \Big|_{\underline{v}} = \Xi_\lambda^* \left(\left(i_{W_{E_\lambda}} dy^k \right) \Big|_{\lambda\underline{v}} \right) = i_{W_{E_\lambda}} dy^k \Big|_{\lambda\underline{v}} = \lambda i_{W_E} dy^k \Big|_{\underline{v}} = \lambda i_{W_E} \Xi_\lambda^* \left(dy^k \Big|_{\lambda\underline{v}} \right) = \left(\lambda i_{W_E} \Xi_\lambda^* \left(dy^k \right) \right) \Big|_{\underline{v}}.$$

Since this is true for any $\underline{v} \in E$ we have (for $dy^k \in \Gamma\Lambda^1 E_\lambda$)

$$\Xi_\lambda^* \left(i_{W_{E_\lambda}} dy^k \right) = \lambda i_{W_E} \Xi_\lambda^* dy^k.$$

It follows that

$$\lambda i_{W_E} \Xi_\lambda^* \Sigma_{E_\lambda}^* \Omega = \Xi_\lambda^* \left(i_{W_{E_\lambda}} \Sigma_{E_\lambda}^* \Omega \right).$$

Since θ_E is an $(2n-2)$ -form on a $(2n-1)$ -dimensional manifold E , with coordinates (y^0, \dots, y^{2n-2}) we may write

$$\theta_E = \theta_E^k i_k^{(y)} \Sigma_E^* \Omega,$$

where $\theta_E^\mu = \theta_E^\mu(y^0, \dots, y^{2n-2})$. Thus we can extend θ_E^μ to a scalar field on U^+ so that $\mathcal{R}\langle \theta_E^k \rangle = 0$. Also $\theta_E^k = \Pi_E^* \theta_E^k = \Xi_\ell^* \theta_E^k$ for the appropriate domains. Therefore,

$$\theta^+ = \Pi_E^* \theta_E = \Pi_E^* \left(\theta_E^k i_k^{(y)} \Sigma_E^* \Omega \right) = \Pi_E^* \theta_E^k (\Sigma_E \circ \Pi_E)^* \left(i_k^{(y)} \Omega \right) = \theta_E^k i_k^{(y)} \Omega.$$

We then have for each λ

$$\begin{aligned} \Xi_\lambda^* \left(\Sigma_{E_\lambda}^* \left(i_W \theta^+ \right) \right) &= \Xi_\lambda^* \left(i_{W_{E_\lambda}} \Sigma_{E_\lambda}^* \Pi_E^* \theta_E \right) = \Xi_\lambda^* \left(i_{W_{E_\lambda}} \Sigma_{E_\lambda}^* \Pi_E^* \left(\theta_E^k i_k^{(y)} \Sigma_E^* \Omega \right) \right) = -\theta_E^k i_k^{(y)} \Xi_\lambda^* \left(i_{W_{E_\lambda}} \Sigma_{E_\lambda}^* \Pi_E^* \Sigma_E^* \Omega \right) \\ &= -\theta_E^k i_k^{(y)} \Xi_\lambda^* \left(i_{W_{E_\lambda}} \Sigma_{E_\lambda}^* \Omega \right) = -\lambda \theta_E^k i_k^{(y)} i_{W_E} \Xi_\lambda^* \left(\Sigma_{E_\lambda}^* \Omega \right) = \lambda i_{W_E} \Xi_\lambda^* \left(\theta_E^k i_k^{(y)} \Sigma_{E_\lambda}^* \Omega \right) \\ &= \lambda i_{W_E} \Xi_\lambda^* \left(\Sigma_{E_\lambda}^* \Pi_E^* \left(\theta_E^k i_k^{(y)} \Sigma_E^* \Omega \right) \right) = \lambda i_{W_E} \Xi_\lambda^* \left(\Sigma_{E_\lambda}^* \Pi_E^* \theta_E \right) = \lambda i_{W_E} \theta_E = 0 \end{aligned}$$

Since Ξ_λ is bijective we have $\Sigma_{E_\lambda}^* \left(i_W \theta^+ \right) = 0$. In order to see that this implies $i_W \theta^+ = 0$, observe the following result:

Given $\alpha \in \Gamma\Lambda^{2n-3} U^+$, if $i_{\mathcal{R}} \alpha = 0$ and $\Sigma_{E_\lambda}^* \alpha = 0$ for each $\lambda > 0$, then $\alpha = 0$.

This follows since $i_{\mathcal{R}} \alpha = 0$, we may write $\alpha = \alpha^{kj} i_k^{(y)} i_j^{(y)} \Omega$. For any $\lambda > 0$, $0 = \Sigma_{E_\lambda}^* \alpha = \Sigma_{E_\lambda}^* \left(\alpha^{kj} \right) i_k^{(y)} i_j^{(y)} \Sigma_{E_\lambda}^* \Omega$. Since the form on the RHS is non-zero we must have $\Sigma_{E_\lambda}^* \left(\alpha^{kj} \right) = 0$ for all $\lambda > 0$. Since U^+ is the union of E_λ then $\alpha^{jk} = 0$. It follows that (Ψ, θ^+) satisfy the transport equations on U^+ . \square

Proof of lemma 4.6. First suppose that $\theta^+ = \Pi_E^* \theta_E$. We have $\Sigma_E^* \theta^+ = \Sigma_E^* \Pi_E^* \theta_E = (\Pi \circ \Sigma_E)^* \theta_E = \theta_E$.

Suppose we now have $\theta_E = \Sigma_E^* \theta^+$. Define a coordinate system (ℓ, y^k) in the same way as in the proof for lemma 4.5, so that $(\Sigma_E \circ \Pi)^* y^k = y^k$. Since $d\theta^+ = 0$ and $i_{\mathcal{R}} \theta^+ = 0$ we have $\theta^+ = \theta^k i_k^{(y)} \Omega_X$, where θ^+ is a function of (y^0, \dots, y^{2n-2}) , and $\Omega_X = dy^0 \wedge \dots \wedge dy^{2n-2}$. We then have

$$\Pi_E^* \theta_E = \Pi_E^* \Sigma_E^* \theta^+ = \Pi_E^* \Sigma_E^* \left(\theta^k i_k^{(y)} \Omega_X \right) = \theta^k (\Sigma_E \circ \Pi_E)^* \left(i_k^{(y)} \Omega_X \right) = \theta^k i_k^{(y)} \Omega_X = \theta^+.$$

\square

Proof of lemma 4.11. Define a coordinate system on N by $(x^0, \dots, x^{\ell-2}, t)$ such that $X = \partial_t$. Since $i_X \alpha = 0$ it follows that $\alpha = \alpha_0 dx^0 \wedge \dots \wedge dx^{\ell-2}$ for some $\alpha_0 \in \Gamma \Lambda^0 N$. Let $\beta_0 = i_X \beta$ then we have

$$\begin{aligned}
\int_N \alpha \wedge \beta &= \int_N \alpha_0 dx^0 \wedge \dots \wedge dx^{\ell-2} \wedge (i_X \beta) dt \\
&= \int_{(x^0, \dots, x^{\ell-2}) \in K_{t_0}} \int_{t \in \mathbb{R}} \alpha_0(x^0, \dots, x^{\ell-2}) dx^0 \wedge \dots \wedge dx^{\ell-2} \wedge \beta_0(x^0, \dots, x^{\ell-2}, t) dt \\
&= \int_{(x^0, \dots, x^{\ell-2}) \in K_{t_0}} \alpha_0(x^0, \dots, x^{\ell-2}) dx^0 \wedge \dots \wedge dx^{\ell-2} \int_{t \in \mathbb{R}} \beta_0(x^0, \dots, x^{\ell-2}, t) dt \\
&= \int_{(x^0, \dots, x^{\ell-2}) \in K_{t_0}} \sum_{t_0}^* \alpha \int_{t \in \mathbb{R}} \beta|_{(x^0, \dots, x^{\ell-2}, t)} = \int_{(x^0, \dots, x^{\ell-2}) \in K_{t_0}} \sum_{t_0}^* \alpha \left(\int_{\mathbb{R}} \eta_{(x^0, \dots, x^{\ell-2}, t)}^* \beta \right).
\end{aligned}$$

□

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Data Statement

All data for this article is fully contained within this article.

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