A relation of families of two and three-dimensional magnetic Weyl-Dirac operators

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Abstract

Based on ideas mainly developed in Erdős and Solovej, 2001 and Aharonov and Casher, 1979, we consider a certain class of magnetic potentials, whose corresponding magnetic field is parallel to a particular class of Conformal Killing Fields, and the related magnetic Dirac operators. We deduce an equation interlacing these Dirac operators on $\mathbb{R}^3$ with a class of Dirac operators on $\mathbb{R}^2$. In the process, we introduce a one-parameter family of typically non-Hausdorff coordinate spaces, on members of which we map $\mathbb{R}^3$ using a Riemann-type submersion and show that standard Differential Geometry results hold for such spaces too.

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N. Alexandrakis

Κατὰ τὸν Δαίμονα Ἐαυτοῦ

Declaration

I declare that the work presented in this thesis is, to the best of my knowledge and belief, original and my own work, unless otherwise stated, cited, or commonly known. The material has not been submitted, either in whole or in part, for a degree at this, or any other University. This thesis does not exceed the maximum permitted word length of 80,000 words, including appendices and footnotes, but excluding the bibliography. An estimate of the word count is: 22970

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Contents

Chapter 1: Introduction

Motivated by quantum mechanics, the magnetic Weyl-Dirac equation is a modification to the stationary version of the massless Dirac equation (Weyl equation), which describes the spin of massless fermions named Weyl fermions. Usually, these equations can be viewed as eigenvalue problems of the associated operators (Weyl-Dirac and Dirac operator) and the non-trivial and bounded in L^2 eigenstates that correspond to the 0 eigenvalue are called zero-modes. Generally speaking, these operators are "square roots" of Schrödinger operators, in the sense that when the operator is applied to itself, we obtain a Schrödinger operator. The peculiarity of the Dirac equation quickly sparked the development of the related theory, such as *Spin Geometry*, while also settling the ground for applications of *Clifford Algebra*. Early particular advances include the celebrated *Atiyah-Singer index theorem* and the *Lichnerowicz formula*, see mainly Berline, Getzler, and Vergne, 2003 or Friedrich, 2000 and Cnops, 2012 for more details. Some related open questions are: For which potentials A can the Weyl-Dirac operator have a zero mode? If it has, in which space do they lie? What happens asymptotically as the magnetic potential becomes stronger ($A \mapsto tA$, as t grows to infinity)? What generic properties can we obtain about the Weyl-Dirac operator when the potential possesses certain singularities? Such questions have partially been answered in pieces of literature such as Elton, 2016,Elton, 2000,Elton, 2018, Elton, 2002, Erdős and Solovej, 2001 and Arai, 1993. A particular physical motivation for the study of these operators stems from the study of properties of graphene, see Eshghi and Mehraban, 2017 and the references there (mainly in the introduction) for more details.

1.1 Preliminaries

1.1.1 Basic nomenclature

Unless otherwise stated, the following rules on notation are used throughout the text:

Einstein's summation convention may be used (if there is no risk of confusion): when an index repeats as subscript and as superscript, we denote that the summation is taken over the set of indices. For example, consider the standard inner product of vectors (denoted in **bold**) in $\mathbb{R}^n$: $\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{R}^n} := a_i b^i = \sum_{i=1}^n a_i b_i$ (or just $\sum_i a_i b_i$) for $\mathbf{a}$,$\mathbf{b}$ $(= (b_1, ..., b_n)$, resp. for $\mathbf{a}) \in \mathbb{R}^n$. Similar to more abstract vector spaces. Also, the inner product is a special case of the "dot" product $\mathbf{a}.\mathbf{b}$ ("term by term"-product).

We will denote the imaginary unit by "i". The symbols "i, j, k" will usually be used as indices, mainly as subscripts, unless another convention is being followed.

The sets of positive and non-negative natural numbers will be denoted as $\mathbb{N}$ and $\mathbb{N}_0$, respectively. The sets of complex, real, rational, irrational numbers and integers are denoted as usual (with subscripts $\pm, *$ used to denote the respective positive/negative, zero-excluding subsets where applicable).

The standard *Pauli matrices* (which act on $\mathbb{C}^2-$vectors) are denoted as:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We set $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ the "vector" in $SU(2)$ with entries the standard *Pauli matrices* (see at the end of this subsection), which will often be treated as a typical $\mathbb{R}^3$ vector, by which the product will be denoted as $\boldsymbol{\sigma}.\mathbf{c} = \mathbf{c}.\boldsymbol{\sigma} = c_i \sigma^i$ for any $\mathbf{c} \in \mathbb{R}^3$.

The gradient (grad) of a scalar function and curl and divergence (div) of a vector are denoted as $\boldsymbol{\nabla}$,$\boldsymbol{\nabla}\times$ and $\boldsymbol{\nabla}.$ $(= \langle \boldsymbol{\nabla}, \cdot \rangle)$ respectively. These symbols may be used interchangeably with grad, curl, div since the former enables us to focus on the operation on $\mathbb{R}^3$ whilst the latter can be used more generally, such as in *curvilinear coordinates.*

Vector fields are denoted in big capitals (as *magnetic potential* and *fields*) with the exception of vector fields that correspond to standard orthonormal basis-vectors, which are denoted as $\mathbf{e}_i$, and whose respective dual *1-forms* are denoted as $\mathbf{e}^i$. The set of smooth vector fields mapping a subset $U \subseteq \mathbb{R}^n$ (for $n \in \mathbb{N}$) to $\mathbb{R}^k$ will be denoted as $\mathfrak{X}(U;\mathbb{R}^k)$. When it is obvious what the co-domain is, it will be denoted simply as $\mathfrak{X}(U)$.

Magnetic potentials and *fields* are denoted respectively as $\boldsymbol{A}(\mathbf{x}) \equiv \boldsymbol{A}$, $\mathbf{B}(\mathbf{x}) \equiv \mathbf{B} \in \mathbb{R}^3$ pointwise (here "$\equiv$" means set/denote), $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. We set the *standard magnetic Weyl-Dirac operator*, acting on $\mathbb{C}^2-$valued functions on $\mathbb{R}^3$, with magnetic potential $\mathbf{A}$, as:

$$\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} := \boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}} - \mathbf{A}) = \sum_{i=1}^{3} \sigma_i(-\mathrm{i}\nabla_{x_i} - A_i(\mathbf{x})) \tag{1.1.1}$$

where $\boldsymbol{\mathcal{D}} := (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3) = -\mathrm{i}\boldsymbol{\nabla}$ ($-$i times the gradient in 3 dimensions). Analytically: $\boldsymbol{\mathcal{D}} := -\mathrm{i}(\nabla_{x_1}, \nabla_{x_2}, \nabla_{x_3})$ with $\nabla_{x_i} = \frac{\partial}{\partial x_i}$, the partial derivative with respect to x_i (when on $\mathbb{R}^3$ or $\mathbb{R}^2$). When there is no risk of confusion, we may denote the partial derivatives simply as ∇_i or ∂_{x_i}.

In the case where dimensions are two, i.e. $\tilde{\mathbf{x}} = (x, y) \in \mathbb{R}^2$ and the magnetic potential is in $\mathbb{R}^2$ (pointwise), the latter will be denoted as $\tilde{\boldsymbol{A}}$ and the respective version of (1.1.1) will be referred to as (massless) *Dirac operator* in two dimensions. The tilde, "$\tilde{}$", will be used on two-dimensional objects and/or objects defined on two (real) dimensions.

When an operator is decomposed into other operators, as in Lemma 2.3.4, the scalars in the decomposition stand for "scalar times the identity matrix I_2". The dot "$\cdot$" when used on the right or inside of an operator, and isn't being followed by another object, denotes an arbitrary element of the space of the domain of definition of that operator.

The bracket $[\cdot,\cdot]$ denotes the *commutator* of two operators and the bracket $\{\cdot,\cdot\}$ the *anticommutator* (not to be confused with the *Poisson bracket* in the *Dynamical System's* literature). Unless there are no operators involved, $\{,\}$ denotes a set with 2 elements (this type of bracket is used to denote arbitrary, finite of infinite, distinct sets too).

The set of k-continuously differentiable functions from $N \subseteq \mathbb{R}^n$ to $M \subseteq \mathbb{R}^m$ are denoted as $C^k(N, M)$ or $C^k(N)$ if the target set is obvious (such as $\mathbb{R}^l$ for some $l \in \mathbb{N}$). Here $k \in \mathbb{N}_0 \cup \{\infty\}$, $n, m \in \mathbb{N}$ and in the case where $k = \infty$ the corresponding function will be called *smooth*. If moreover, the set consists only of compactly supported functions on N, then it will be denoted as $C_c^k(N, M)$ (resp. $C_c^k(N)$). Similarly, the respective spaces of p-integrable functions and the respective *Sobolev spaces* are denoted as usual (L^p and $W^{l,p}$ resp.); if the subscript "*loc*" is used, it will indicate their respective locally integrable sub-spaces. The spaces of *holomorphic* and *real-analytic* functions on some $U \subseteq \mathbb{C}$ and $I \subseteq \mathbb{R}$ respectively will be denoted as $\mathcal{A}(U)$ and $C^\omega(I)$. For a set $K \neq \emptyset$, the space of K-valued n-forms on a (smooth) manifold M will be denoted as $\Omega^n(M, K)$.

Let $n \in \mathbb{N}$, the unit sphere in $\mathbb{R}^{n+1}$ will be denoted as $\mathbb{S}^n$, whilst the open unit disk on $\mathbb{R}^2$ (equivalently for $\mathbb{C}$) will be denoted as $\mathbb{D}$. Moreover, we will often work on $\tilde{\mathbb{D}} := \mathbb{D} \times \{0\} \subseteq \mathbb{R}^3$.

Definition 1.1.1. Consider an open $U \subseteq \mathbb{R}^n$, for $n = 2$ or 3, and a magnetic Weyl-Dirac Operator $\boldsymbol{\mathcal{D}_A}$. A *zero-mode* on U is a continuously differentiable, square integrable, $\mathbb{C}^2$-valued function ψ that satisfies $\boldsymbol{\mathcal{D}_A}\psi = 0$.

In other words, a zero-mode is a[1] $C^1(U, \mathbb{C}^2)$, square-integrable eigenfunction of the Weyl-Dirac operator, with corresponding eigenvalue 0 (if it exists).

These $\mathbb{C}^2-$valued functions are usually elements of an associated Hilbert space, Ψ, with an inner product defined via the standard inner product in complex spaces:

$$\langle \psi_1, \psi_2 \rangle_\Psi := \int_{U'} \langle \psi_1, \psi_2 \rangle_{\mathbb{C}^2} d\mathbf{x}$$

for $\psi_{1,2} = \begin{pmatrix} \psi_{1,2}^{up} \\ \psi_{1,2}^{down} \end{pmatrix} \in \Psi$ ($\in \mathbb{C}^2$ pointwise), $U' \subseteq \mathbb{R}^n$ for $n \in \mathbb{N}$, and

$$\langle \psi_1, \psi_2 \rangle_{\mathbb{C}^2} := \psi_1^{up} \overline{\psi_2^{up}} + \psi_1^{down} \overline{\psi_2^{down}}.$$

More generally, $\mathbb{C}^2-$valued functions on $U \subseteq \mathbb{R}^n$ will be called **Spinors**.

[1] In cases where the potential is smooth, standard theory on *Elliptic regularity* can guarantee that a zero-mode is actually smooth, i.e. it lies in $C^\infty(U, \mathbb{C}^2)$, see Beck, 2016. It can also be easily shown, that since it belongs to $L^2(U, \mathbb{C}^2)$, then by the equation $\boldsymbol{\mathcal{D}_A}\psi = 0$ we get that $\psi \in W^{1,2}(U, \mathbb{C}^2)$

1.1.2 Structure of the Thesis

In this thesis, we produce a generalisation of the results in Erdős and Solovej, 2001 first, and then Aharonov and Casher, 1979. In particular:

In this Chapter, we start by briefly mentioning a few earlier results in the area from previous decades. Then we introduce some elementary quantities we'll be working with throughout this thesis. This can serve as an index, as it contains (generally non-trivial) formulas of quantities that are used in the following chapters.

In *Chapter 2*, we first briefly present and comment on some key results that have been produced since the 1970s, and introduce some of the basic tools we use throughout this text, giving a brief description. In the end, we also introduce a result, complementary to Elton, 2018 and define an operator, named Q, whose spectral properties are key to studying the Weyl-Dirac operator.

In *Chapter 3*, we introduce (without proofs) non-standard tools and results and definitions that will be used in this Thesis. as well as special cases of (magnetic) Dirac operators in two dimensions. In particular, we define the base space that we're using, *sections line bundles* on that base, $Spin^c$ structure and $Spin^c$*-bundles.* These are generalized versions of the respective ones in Erdős and Solovej, 2001. Having defined these, we introduce connections, Clifford multiplication, and Weyl-Dirac operators and a submersion map $F : \mathbb{R}^3 \to \mathbb{C} \equiv \mathbb{R}^2$ of Riemann type.

In *Chapter 4*, we prove properties of the aforementioned objects and results in Chapter 2 and provide justification on the well-posedness of the definitions we use.

In *Chapter 5*, the results mentioned above are used to conclude a relation between the standard Weyl-Dirac operators in three dimensions and Weyl-Dirac operators in two dimensions. This is prescribed by the aforementioned submersion, particularly by studying how 1-forms, Clifford multiplication, $spin^c$ connections change when moving from 3 dimensions to 2 through the aforementioned submersion.

Lastly, in *Chapter 6* we have the Appendix, which consists of a Glossary, where all the useful terms and formulae can be found quickly, as well as certain proofs of particular properties and results regarding quantities used in this thesis.

1.1.3 Brief review of early and recent literature

After more than 90 years since its derivation, the Dirac equation is still of particular interest to Physicists and Mathematicians. The latter are interested in generalizing explicit results obtained in low dimensions such as in $\mathbb{R}^3$ to higher dimensions on arbitrary manifolds (Cnops, 2012,Friedrich, 2000), while the former are interested in applications, for example, Eshghi and Mehraban, 2017 where the authors analytically solve particular classes of the Weyl-Dirac equation in order to obtain information for bound states of graphene.

Through the years, and especially in the second half of the 20th century, several important ideas for studying these equations were developed. Some examples include the techniques applied in Aharonov and Casher, 1979 as well as in Loss and Yau, 1986, where the authors construct explicit examples of magnetic fields and associated zero-modes. See the introduction in Tri, 2009 and the references therein for a more thorough presentation of these and related results.

In the late 90's, Elton, 2000 constructed some new examples of magnetic potentials that produce zero modes. Also, Erdős and Solovej, 2001, produced some results essentially generalising Loss and Yau, 1986 and Aharonov and Casher, 1979 to certain, low-dimensional manifolds, which we will be focusing on presenting this work in Chapter 3. Interestingly, the study in Erdős and Solovej, 2001 involves considering Weyl-Dirac operators on the 2-sphere, which is a compact manifold and therefore the Weyl-Dirac operator has a discrete spectrum on it (more information on that is found here Abrikosov Jr, 2002) (Reed, Simon, et al., 1972 also provides some useful information). Moreover, the 2-sphere and the plane are conformally equivalent metric spaces, which means that the induced (from the plane) Weyl-Dirac on the 2-sphere is weighted. However, when dealing with zero-modes (i.e. eigenfunctions that correspond to the zero eigenvalues), the weight at the front of the operator does not matter since the zero-mode will annihilate the operator regardless of the weight the operator it has at the front.

1.2 The Geometric framework: Magnetic potentials, fields and related differential forms

The Dirac operator carries a vast range of possibilities for developements in a number of areas in Geometry. To see that it suffices to consider the results of Loss and Yau, 1986 and Aharonov and Casher, 1979, intertwining the number of zero-modes for certain Dirac operator and the *flux* of the operator's **magnetic field**. The said magnetic field can naturally identified as 2-form given by the exterior derivative of the 1-form that corresponds the magnetic potential that produces it. Analytically, for the magnetic potential $\mathbf{A}(\mathbf{x}) = (A_1(\mathbf{x}), A_2(\mathbf{x}), A_3(\mathbf{x}))$ and the corresponding magnetic field $\mathbf{B}(\mathbf{x}) = (B_1(\mathbf{x}), B_2(\mathbf{x}), B_3(\mathbf{x})) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x})$ we have the following identifications

$$\alpha = A_1(\mathbf{x})dx_1 + A_2(\mathbf{x})dx_2 + A_3(\mathbf{x})dx_3 \tag{1.2.1}$$

$$\beta = d\alpha = B_1(\mathbf{x})dx_2 \wedge dx_3 + B_2(\mathbf{x})dx_1 \wedge dx_3 + B_3(\mathbf{x})dx_1 \wedge dx_2 \tag{1.2.2}$$

where

$$B_1(\mathbf{x}) = \partial_{x_2} A_3(\mathbf{x}) - \partial_{x_3} A_2(\mathbf{x}),$$

$$B_2(\mathbf{x}) = -\partial_{x_1} A_3(\mathbf{x}) + \partial_{x_3} A_1(\mathbf{x}),$$

$$\& \quad B_3(\mathbf{x}) = \partial_{x_1} A_2(\mathbf{x}) - \partial_{x_2} A_1(\mathbf{x}).$$

Another nice geometric property of Dirac operators, is that they transform neatly under conformal changes of metric/transformation. In particular, when moving from $\mathbb{R}^3$ with the standard Euclidean metric to a space with a conformally equivalent metric with weight $\tilde{\Omega}$, we have the following transformation rule:

$$\mathcal{D}_{\mathbf{A}} \mapsto \mathcal{D}_{\mathbf{A}}^{\tilde{\Omega}} := \tilde{\Omega}^{-2} \mathcal{D}_{\mathbf{A}} \tilde{\Omega} \tag{1.2.3}$$

as proved in Erdős and Solovej, 2001 when moving from $\mathbb{R}^3$ to $\mathbb{S}^3$ using the standard stereographic projection. This identity is particularly useful as it allows as to study zero-modes when working on either manifold; this is because the existence of zero modes is not affected by $\tilde{\Omega}$ as they are eigenvectors corresponding to the 0-eigenvalue.

We'll be focusing on Weyl-Dirac operators, $\mathcal{D}_{\mathbf{A}}$, such that the magnetic field $\mathbf{B} := \boldsymbol{\nabla} \times \mathbf{A}$ is parallel to a particular class of *conformal Killing fields* (on and in $\mathbb{R}^3$ with the usual metric). The latter are vector fields in $\mathbb{R}^3$ that satisfy:

$$\partial_i X_j + \partial_j X_i = \frac{2}{3} \boldsymbol{\nabla} \cdot \mathbf{X}. \tag{1.2.4}$$

Remark 1.2.1. The notion of conformal Killing fields extends on a generic manifold of dimension $n \in \mathbb{N}$ with pseudo-Riemannian metric $g = g_{ij}$, for $i, j \in \{1, 2, ..., n\}$. In this case, the aforementioned condition becomes:

$$\nabla_{\mathbf{e}_i} X_j + \nabla_{\mathbf{e}_j} X_i = \frac{2}{n} g_{ij} \sum_{k=1}^{n} \nabla_{\mathbf{e}_k} X_k \tag{1.2.5}$$

where $g_{ij} = g(\mathbf{e}_i, \mathbf{e}_j)$ for a (local) basis e_i, $i = 1, 2, .., n$ on the (tangent bundle of the) manifold. Here, ∇ denotes the *Levi-Civita connection.*

In particular, we'll be working with Dirac operators with corresponding magnetic fields parallel to the following (family of)*conformal Killing field(s)*, $\mathbf{X} : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\mathbf{X}(\mathbf{x}) := \kappa_1 \mathbf{W}_1(\mathbf{x}) + \kappa_2 \mathbf{W}_2(\mathbf{x}) \tag{1.2.6}$$

where $\kappa_{1,2} > 0$,

$$\mathbf{W}_1(\mathbf{x}) = (-2x_2, 2x_1, 0), \tag{1.2.7}$$

and

$$\mathbf{W}_2(\mathbf{x}) = (2x_1x_3, 2x_2x_3, 1 - x_1^2 - x_2^2 + x_3^2) = (2x_1x_3, 2x_2x_3, 1 - |\mathbf{x}|^2 + 2x_3^2). \tag{1.2.8}$$

The vector fields $\mathbf{W}_{1,2}$ generate rotations on planes parallel to the x_1x_2-plane and on half-planes with the x_3-plane as its boundary (being perpendicular to the x_1x_2-plane). Their integral curves are circles (of period π/κ_1, π/κ_2) on such planes. These integral curves also lie on tori centred at the origin.

We also set the (positive) parameter:

$$\kappa := \frac{\kappa_1}{\kappa_2}. \tag{1.2.9}$$

The limiting cases where $\kappa_1 = 0$ or $\kappa_2 = 0$ have been covered in Elton, 2018, where the author proves that the (standard) Weyl-Dirac operator attains no zero-modes.

The case where $\kappa_1 = \kappa_2$ (> 0) has been covered in Erdős and Solovej, 2001, where the authors generalize the Aharonov and Casher, 1979 zero-mode construction to generic 2 and 3-dimensional Riemannian manifolds.

We'll be focusing on the generic case when $\kappa_{1,2} > 0$ and $\kappa_1 \neq \kappa_2$, and we'll map the $\mathbb{R}^3$ to $\mathbb{R}^2$ by mapping integral curves of $\mathbf{X}(\mathbf{x})$ to discrete sets of points on $\mathbb{R}^2$ via a map F.

We'll also be reducing the standard, weighted 3 dimensional Dirac operator $\mathcal{D}_{\mathbf{A}}^{\Omega}$ to two 2-dimensional ones, $\tilde{\mathcal{D}}_{\tilde{A}}^{\omega_{1,2}}$, and by studying the latter (based ideas from Aharonov and Casher, 1979), we'll be able to convert results on the two-dimensional Dirac operators to the three-dimensional by "inverting" the reduction process.
The weights arise naturally from the fact that the map F is non-linear.

1.3 Spin^c-bundles and Clifford multiplication

In this section, we'll briefly introduce some more objects we'll be working with and some further, non-standard notation. Let M be an orientable Riemannian manifold of dimension m. The *tangent space* of M at a point p will be denoted as T_pM, its elements will be viewed as vector fields or just sections as usual if we're talking about the respective *tangent bundle*. The *cotangent space* of M at p will be denoted as T_p^*M and its elements are differential 1-forms (dual to vector fields in T_pM). The cotangent space can naturally be equipped with the metric that defines the inner product, as in the case of the tangent space. Recall that the space of real and complex k-forms (where $k \in \{0, 1, 2, ..., m\}$, where $m = \dim(M)$) over M will be denoted as $\Omega^k_{\mathbb{R}}(M)$ or $\Omega^k_{\mathbb{C}}(M)$ respectively (when there no risk of confusion, we may drop the subscripts).

The space of Spinors is a (complex) vector space Ψ (with the inner product, $\langle , \rangle_\Psi$ - identical to the one defined by (2.3.1), see the end of Section 1.1.1 too) which attains an isometry σ (usually in $\in SU(2)$) such that $\sigma = \sigma^*$ and $\mathrm{Tr}(\sigma) = 0$. The space of such isometries, Ψ^2, is a vector space that is produced by the standard Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ with the inner product:

$$\langle \sigma_\alpha, \sigma_\beta \rangle_{\Psi^2} := \frac{1}{2}\{\sigma_\alpha, \sigma_\beta\} = \frac{1}{2}(\sigma_\alpha\sigma_\beta + \sigma_\beta\sigma_\alpha) = \mathrm{Tr}(\sigma_\alpha\sigma_\beta)I_2 = \langle \alpha, \beta \rangle_{T^*M} I_2. \qquad (1.3.1)$$

The sigma-matrices are elements of the space of *Endomorphisms of Spinors*, which is $SU(2)$. In this setting, they are defined via a linear map (σ) from the space $\Omega^1(T^*M)$ ($T^*M := \bigcup_{p \in M} T_p^*M$) to $SU(2)$, such that $\sigma(e^i) = \sigma_i$ for $i = 1, 2, 3$ (the standard Pauli matrices) with $\{e^i\}_{i=1}^3$ the basis of T^*M (with the inner product $\langle , \rangle_{T^*M}$ defined by the metric).

At this stage, we are able to define a simplified, but convenient version of the **Clifford multiplication**: Let M a 3-dimensional, orientable Riemannian manifold. The Clifford multiplication, $\cdot_{cl}$, of a 1-form $\alpha \in T^*M$ (or a vector $\mathbf{a} \in TM$) with a spinor ψ is defined as $\alpha \cdot_{cl} \psi := \sigma(\alpha)\psi$ (resp. $\mathbf{a} \cdot_{cl} \psi := \sigma(\mathbf{a}^*)\psi$ more generally. Clifford multiplication can be extended to 0-forms via $\sigma(f(x)) := f(x)I_2$ (for any scalar function f) and to 2-forms

$e^i \wedge e^j$ via the identification $e^i \wedge e^j \mapsto -\mathrm{i}\epsilon_{ijk}e^k$ (for distinct $i, j, k \in \{1, 2, 3\}$), where ϵ_{ijk} is the *Levi-Civita symbol* in 3 dimensions.

This is done by the linearity of the map σ with the relations $\sigma(e^i \wedge e^j) := \sigma(-\mathrm{i}\epsilon_{ijk}e^k) = -\mathrm{i}\epsilon_{ijk}\sigma(e^k) = -\mathrm{i}\epsilon_{ijk}\sigma(e^i)\sigma(e^j)$ ("distributive" property of Clifford multiplication on 2-forms). Similarly, we can inductively extend it to m-forms, generalising it naturally. For a comprehensive introduction, with application to Physics, we refer the reader to Doran et al., 2003.

Two-dimensional complex vector bundles with inner product and isometry as above are called **Spin$^{\mathbf{c}}$ spinor bundles**, and their *sections* are called **Spinor fields**.

A *Spin^c spinor bundle* that admits an anti-linear bundle isometry, $\mathcal{C}$, sending each section to its perpendicular (satisfying $\langle \psi, \mathcal{C}\psi \rangle_{\mathbb{C}^2} = 0$) if the matrix is real values, or/and a reflection of it (or a combination of those), satisfying $\mathcal{C}^2\psi = -\psi$, is simply called a **Spin spinor bundle**. In practice, the existence of such isometry means that if we find just one nowhere vanishing *Spinor field*, ψ_1, then by applying $\mathcal{C}$ to it, we get an orthogonal basis of spinors made by $\psi_1, \psi_2(= \mathcal{C}\psi_1)$.

Remark 1.3.1. Interestingly, having a local basis $\{\xi_-, \xi_+\}$ of orthonormal Spinor fields, we can create a (local) orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ (and $\mathbf{e}_3$ additionally) for TN, where N is a 2-dimensional (resp. 3-)orientable Riemannian manifold, by imposing the condition $\alpha(\mathbf{e}_1) + \mathrm{i}\alpha(\mathbf{e}_2) = \langle \xi_-, \sigma(\alpha)\xi_+ \rangle_\Psi$ (resp. $\alpha(e_3) = \langle \xi_+, \sigma(\alpha)\xi_+ \rangle_\Psi$ additionally) for all (differential) 1-forms α, see Erdős and Solovej, 2001.

Further information on Spin/Spin^c structures and Clifford Algebras can be found in Friedrich, 2000, which includes the standard, more abstract definition of the Clifford multiplication and defines the Dirac operator accordingly.

We finish this subsection with Remark 1.3.2 below and by noting that the standard Weyl-Dirac operator[2] $\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} := \boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}} - \boldsymbol{A}) = -\mathrm{i}\boldsymbol{\sigma}.(\boldsymbol{\nabla} - \mathrm{i}\boldsymbol{A})$ is essentially given by the Spin^c connection[3] (see (2.1.1) for a full definition) $\boldsymbol{\nabla} - \mathrm{i}\boldsymbol{A}$, which transforms under conformal changes of metric as illustrated in Erdős and Solovej, 2001 (transformations under linear changes of variables are trivial). Recall, the magnetic field $\boldsymbol{A} = (A_1, A_2, A_3)$ can be treated as the real 1-form $\alpha = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 \in \Omega^1(\mathbb{R}^3) = T^*\mathbb{R}^3$, since $\alpha(\mathbf{e}_i) = A_i$, $i = 1, 2, 3$.

Remark 1.3.2. In order to study the Weyl-Dirac operator on a (Riemannian) manifold, it has to admit a Spin^c structure. That is, it should enable us to define Spinor bundles/fields, and endomorphisms on the respective defined spaces. Such manifolds are called **Spin^c Manifolds**. Orientable Riemannian manifolds are Spin or Spin^c (TM can be properly defined locally or globally), or manifolds that admit a Laplacian; however, there are non-orientable, pseudo-Riemannian manifolds that are Spin as well, not necessarily non-orientable hyperbolic manifolds though. A manifold's capacity to admit a Spin^c structure has to do with its algebro-topological properties, in particular, the Stiefel-Whitney class. Moreover, in the late 1950s, it was proved that the only spheres that are parallelizable (and hence attain Spin or Spin^c structure) are $\mathbb{S}^n$ for n=1,3,7; see Atiyah and Hirzebruch, 1961 for more information and the references there (mainly in the introduction).

[2]In other pieces of literature, Weyl-Dirac operator is defined as $\mathrm{i}\mathcal{D}_{\mathrm{A}}$, however, in this case the operator is not self-adjoint (with respect to the inner product defined by (2.3.1)) in any space of interest, not even formally.

[3]In the Quantum mechanics literature, this connection is usually called the *Berry connection* and the induced "Riemannian" curvature is called *Berry curvature*.

Chapter 2: Definitions and derivations of basic tools and semi-standard results

2.1 A few analytic proofs on basic results

In standard Differential Geometry, the Levi-Civita connection is often viewed as a "natural" generalization of the directional derivative to Riemannian manifolds. To generalize this notion to Spin^c spinor bundles we need to define the Spin^c connections accordingly, as well as check how they behave under conformal changes of metrics.

2.1.1 Connections on Spin^c-bundles and their conformal transformations

Following Erdős and Solovej, 2001, we define Spin^c-connections by imposing the conditions below:

Definition 2.1.1. A *Spin^c-connection* on a *Spinor bundle*, Ψ over a Riemannian manifold as base space, is a connection ∇ on Ψ that satisfies:

$$\mathbf{X}\langle \xi, \eta \rangle_{\Psi'} = \langle \nabla_{\mathbf{X}} \xi, \eta \rangle_{\Psi'} + \langle \xi, \nabla_{\mathbf{X}} \eta \rangle_{\Psi'}, \tag{2.1.1}$$

$$[\nabla_{\mathbf{X}}, \sigma(\alpha)] = \sigma(\nabla^{LC}_{\mathbf{X}} \alpha). \tag{2.1.2}$$

Here Ψ' (typically $\mathbb{C}^2$ in this context) is the Hilbert space on which the fibers of Ψ live, and ∇^{LC} is the *Levi-Civita* connection, acting on 1-forms α.

In this Thesis, we'll be working with (complex) vector bundles over certain Riemann surfaces as base spaces. For this reason, the aforementioned definition is satisfactory for our purposes. However, projecting these Riemann surfaces onto $\mathbb{R}^2 \equiv \mathbb{C}$ produce more generic geometric objects such as *orbifolds* (if $\kappa \in \mathbb{Q}_{>0}$)and non-hausdord spaces (if $\kappa \in \mathbb{R}_{>0} \setminus \mathbb{Q}$). On these peculiar objects, we can't define operations such as differentiation per se; however, these spaces are locally homeomorphic to the Euclidean plane (identified as $\mathbb{C}$, as every Riemann surface is) and therefore differentiation and every other useful operation can be well-defined and the aforementioned definition of a *spin*^{c} connection is satisfactory. More information on connections on 1-dimensional complex vector bundles are found on *Chapter 1* in Kobayashi, 2014.

Condition (2.1.1) represents the *compatibility with the metric* (induced by the underlying inner product) and (2.1.2) represents the *Leibniz rule* since $\nabla_{\mathbf{X}}(\sigma(\alpha)\xi) = \sigma(\alpha)\nabla_{\mathbf{X}}\xi + [\nabla_{\mathbf{X}}, \sigma(\alpha)]\xi$ for each *spinor field* ξ.

We'll be working with weighted versions of the metric spaces $\mathbb{R}^3$ and $\mathbb{R}^2$ - equipped with weights denoted as Ω and ω respectively. Specifically, given a metric g on $\mathbb{R}^n$, we define the weighted version of that metric, with corresponding weight Ω as $g_\Omega := \Omega^2 g$ (resp. ω if $n = 2$). In particular, the **Levi-Civita connection**[4], ∇^{LC}, changes under a conformal transformation: The Levi-Civita connections ∇^{LC},$\nabla^{LC,\Omega}$, corresponding to the standard Euclidean metric and g_Ω respectively satisfy:

$$\nabla^{LC,\Omega}_{\mathbf{X}}\mathbf{Y} = \nabla^{LC}_{\mathbf{X}}\mathbf{Y} + \Omega^{-1}\mathbf{X}(\Omega)\mathbf{Y} + \Omega^{-1}\mathbf{Y}(\Omega)\mathbf{X} - \Omega^{-1}(\mathbf{X}.\mathbf{Y})d\Omega^* \tag{2.1.3}$$

where $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(\mathbb{R}^n)$ are arbitrary vector fields in $\mathbb{R}^n$.

To prove this, we just need to consider the *Koszul formula* and the fact that the Levi-Civita connection is uniquely characterised by being *Torsion-free*:

$$\nabla^{LC}_{\mathbf{X}}\mathbf{Y} - \nabla^{LC}_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}]$$

[4]We use the same symbol as the gradient (although we slightly abuse notation by not writing it in **bold**) because this connection is a convenient generalization of the directional derivative (from Euclidean spaces to Riemannian manifolds).

and *preserves the metric*:

$$(\nabla^{LC}_{\mathbf{X}} g)(\mathbf{Y}, \mathbf{Z}) = 0$$

for all vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{X}(\mathbb{R}^n)$ (respectively for the conformal metric).

Moreover, the Levi-Civita connection corresponding to the conformal metric acts on 1-forms (dual of vector fields) according to:

$$\nabla^{LC,\Omega}_{\mathbf{X}} \alpha = \nabla_{\mathbf{X}} \alpha - \Omega^{-1} \mathbf{X}(\Omega) \alpha + \Omega^{-1}(\alpha, d\Omega) \boldsymbol{X}^{*} - \Omega^{-1} \alpha(\mathbf{X}) d\Omega \tag{2.1.4}$$

by the identity $\Omega^{-2}(\nabla^{LC,\Omega}_{\mathbf{X}} \alpha)^* = \nabla^{LC,\Omega}_{\mathbf{X}}(\Omega^{-2}\alpha^*)$ where α^*(resp. $\boldsymbol{X}^*$) denotes the dual vector field (resp. one-form) corresponding to α (resp. $\boldsymbol{X}^*$).

In (2.1.3)-(2.1.4) we just saw how the Levi-Civita connection changes under a conformal change of variables. Whereas, for Spin^c connections, we have:

Proposition 2.1.2. Let ∇ be a spin^c-connection on a spinor bundle endowed with a metric g and Clifford multiplication σ. Then following the conformal change of metrics $g \to g_\Omega := \Omega^2 g$, for some weight Ω on M, the Clifford multiplication transforms to $\sigma_\Omega = \Omega^{-1}\sigma$ and the spin^c-connection becomes:

$$\nabla^{\Omega}_{\mathbf{X}} = \nabla_{\mathbf{X}} + \frac{1}{4}\Omega^{-1}[\sigma(X^*), \sigma(d\Omega)]. \tag{2.1.5}$$

Proof. Regarding the transformation $\sigma \to \Omega^{-1}\sigma = \sigma_\Omega$, this follows from the requirement that σ_Ω needs to respect (1.3.1) when $\langle \alpha, \beta \rangle_{T^*M} \to \Omega^{-2}\langle \alpha, \beta \rangle_{T^*M}$ for all 1-forms $\alpha, \beta \in T^*M$.

Now, we check condition (2.1.2): Let α an arbitrary (smooth) 1-form (in $\mathbb{R}^2$ or $\mathbb{R}^3$), $\mathbf{X}$,$\boldsymbol{X}^*$ an arbitrary (smooth) vector field and its dual 1-form respectively and set $\sigma_\Omega(\alpha) = \Omega^{-1}\sigma(\alpha)$. By (2.1.4) we obtain:

$$\sigma_\Omega\big(\nabla^{LC,\Omega}_{\mathbf{X}} \alpha\big) = \Omega^{-1}\sigma\big(\nabla^{LC}_{\mathbf{X}} \alpha - \Omega^{-1} d\Omega(\mathbf{X})\alpha + \Omega^{-1}(d\Omega, \alpha) X^* - \Omega^{-1}\alpha(\mathbf{X}) d\Omega\big)$$

$$= \Omega^{-1}\sigma(\nabla^{LC}_{\mathbf{X}} \alpha) - \Omega^{-2} d\Omega(\mathbf{X})\sigma(\alpha) + \Omega^{-2}(d\Omega, \alpha)\sigma(\mathbf{X}^*) - \Omega^{-2}\alpha(\mathbf{X})\sigma(d\Omega)$$

$$
\begin{aligned}
&= \Omega^{-1}\sigma(\nabla^{LC}_{\mathbf{X}}\alpha) - \Omega^{-2}d\Omega(\mathbf{X})\sigma(\alpha) + \Omega^{-2}\langle d\Omega, \alpha\rangle I_2\sigma(\boldsymbol{X}^*) - \Omega^{-2}\langle \mathbf{X}^*, \alpha\rangle I_2\sigma(d\Omega) \\
&= [\nabla_{\mathbf{X}}, \sigma_\Omega(\alpha)] + \frac{\Omega^{-2}}{2}\{\sigma(d\Omega), \sigma(\alpha)\}\sigma(\boldsymbol{X}^*) - \frac{\Omega^{-2}}{2}\{\sigma(\boldsymbol{X}^*), \sigma(\alpha)\}\sigma(d\Omega) \\
&= [\nabla_{\mathbf{X}}, \sigma_\Omega(\alpha)] + \frac{1}{4\Omega^2}\{\sigma(\boldsymbol{X}^*), \{\sigma(d\Omega), \sigma(\alpha)\}\} - \frac{1}{4\Omega^2}\{\{\sigma(\boldsymbol{X}^*), \sigma(\alpha)\}, \sigma(d\Omega)\} \\
&= [\nabla_{\mathbf{X}}, \sigma_\Omega(\alpha)] + \frac{1}{4\Omega}[[\sigma(\boldsymbol{X}^*), \sigma(d\Omega)], \sigma_\Omega(\alpha)].
\end{aligned}
\tag{2.1.6}
$$

(One power of Ω went inside the anticommutator to give us σ_Ω) where we've used $\alpha(\mathbf{X}) = \langle \boldsymbol{X}^*, \alpha\rangle$ from standard Differential Geometry, and the inner product of 1-forms on the RHS is a natural extension of the standard inner product to 1-forms. The first term of the last equality in (2.1.6) comes from the fact that:

$$[\nabla^{LC}_{\mathbf{X}}, \sigma_\Omega(\alpha)] = [\nabla_{\mathbf{X}}, \Omega^{-1}\sigma(\alpha)] = \Omega^{-1}\sigma(\nabla^{LC}_{\mathbf{X}}\alpha) + d\Omega^{-1}(\mathbf{X})\sigma(\alpha).$$

We've also used the identity $[[A, B], C] = \{\{A, B\}, C\} - \{\{A, C\}, B\}$ and:

$$\{\sigma(\boldsymbol{X}^*), \sigma(\alpha)\}\sigma(d\Omega) = \frac{1}{2}\big(\{\sigma(\boldsymbol{X}^*), \sigma(\alpha)\}\sigma(d\Omega) + \sigma(d\Omega)\{\sigma(\boldsymbol{X}^*), \sigma(\alpha)\}\big) =$$

$$\frac{1}{2}\{\sigma(d\Omega), \{\sigma(\boldsymbol{X}^*), \sigma(\alpha)\}\} \quad \Big(= \sigma(d\Omega)\{\sigma(\boldsymbol{X}^*), \sigma(\alpha)\}\Big)$$

since $\{\sigma(\boldsymbol{X}^*), \sigma(\alpha)\}$ is scalar times the identity matrix I_2. Similarly:

$$\{\sigma(\boldsymbol{X}^*), \sigma(\alpha)\}\sigma(d\Omega) = \frac{1}{2}\{\sigma(d\Omega), \{\sigma(\boldsymbol{X}^*), \sigma(\alpha)\}\} = \sigma(d\Omega)\{\sigma(\boldsymbol{X}^*), \sigma(\alpha)\}.$$

Lastly, imposing (2.1.2) gives $[\nabla^{\Omega}_{\mathbf{X}}, \sigma_\Omega(\alpha)] = \sigma_\Omega(\nabla^{LC,\Omega}_{\mathbf{X}}\alpha)$ and by (2.1.6) we get (2.1.5). □

2.1.2 Rotations in $\mathbb{R}^3$ and quaternions

In this subsection, we will investigate the correspondence between Special Unitary matrices ($SU(2)$) and rotations in $\mathbb{R}^3$ ($SO(3)$). There is a double covering (2:1 surjective homomorphism) $SU(2) \mapsto SO(3)$, with *antipodal* elements $\pm a \in SU(2)$ being mapped to the same image in $SO(3)$, which we are going to take advantage of. Convex (resp. linear over $\mathbb{C}$) combinations of the Pauli matrices together with the identity produce $SU(2)$ (resp. $M_2(\mathbb{C})$) exhaustively. In essence, quaternions can be expressed in terms of *Pauli matrices*, with their algebra capturing the inner and cross product of vectors in $\mathbb{R}^3$.

First, consider (generally complex) scalars a_0, b_0 and vectors $\mathbf{a}, \mathbf{b}$ and the matrices: $S_a = a_0 + \mathrm{i}\mathbf{a}.\boldsymbol{\sigma}$, and $S_b = b_0 + \mathrm{i}\mathbf{b}.\boldsymbol{\sigma} \in SU(2)$, then

$$\begin{aligned} S_a \cdot S_b = (a_0 I_2 + \mathrm{i}\mathbf{a}.\boldsymbol{\sigma})(b_0 I_2 + \mathrm{i}\mathbf{b}.\boldsymbol{\sigma}) &= a_0 \mathbf{a}_0 I_2 + \mathrm{i}(a_0\mathbf{b} + b_0\mathbf{a}).\boldsymbol{\sigma} - a_i b^j(\delta_{ij} I_2 + \mathrm{i}\epsilon_{ijk}\sigma_k) \\ &= (a_0 b_0 - \mathbf{a}.\mathbf{b})I_2 + \mathrm{i}(a_0\mathbf{b} + b_0\mathbf{a} - \mathbf{a}\times\mathbf{b}).\boldsymbol{\sigma} \end{aligned} \tag{2.1.7}$$

Also, $\det(S_a) = a_0^2 + a_1^2 + a_2^2 + a_3^2$, and for $S_a^* = \overline{a}_0 I_2 - \mathrm{i}\overline{\mathbf{a}}.\boldsymbol{\sigma}$ we have that if $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ then $S_a \cdot S_a^* = |(a_0, \mathbf{a})|^2 I_2 = (a_0^2 + a_1^2 + a_2^2 + a_3^2)I_2 = \det(S_a)I_2 \implies S_a \in SU(2) \iff \det(S_a) = 1$, as it can be verified by simple calculations, in fact let $a_i = x_i + \mathrm{i}y_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, we have:

$$\begin{aligned} S_a \cdot S_a^* &= (a_0\overline{a}_0 + \mathbf{a}\overline{\mathbf{a}})I_2 + \mathrm{i}(\overline{a}_0\mathbf{a} - a_0\overline{\mathbf{a}} + \mathbf{a}\times\overline{\mathbf{a}}).\boldsymbol{\sigma} \\ &= \Big(\sum_{i=0}^{3} x_i^2 + y_i^2\Big) I_2 + (-x_0\mathbf{y} + y_0\mathbf{x} + 2\mathbf{x}\times\mathbf{y}).\boldsymbol{\sigma} \end{aligned} \tag{2.1.8}$$

Clearly $S_a \in SU(2)$ iff $\sum_{i=0}^{3}(x_i^2 + y_i^2) = 1$ and $-x_0\mathbf{y} + y_0\mathbf{x} + 2\mathbf{x}\times\mathbf{y} = 0$. We consider the inner product of the LHS of the latter equation with $\mathbf{x}\times\mathbf{y}$ (by which we immediately get $|\mathbf{x}\times\mathbf{y}| = 0 \implies \mathbf{x} \parallel \mathbf{y}$ since $\mathbf{x}\times\mathbf{y} \perp \mathbf{x}, \mathbf{y}$) and we get $y_0\mathbf{x} = x_0\mathbf{y}$.

Also: $\det(S_a) = a_0^2 + a_1^2 + a_2^2 + a_3^2 = (|(x_0, \mathbf{x})|^2 - |(y_0, \mathbf{y})|^2) + 2\mathrm{i}(x_0 y_0 + \mathbf{x}.\mathbf{y})$
$= (x_0^2 + x_1^2 + x_2^2 + x_3^2 - y_0^2 - y_1^2 - y_2^2 - y_3^2) + 2\mathrm{i}(x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3)$

and so for S_a to be in $SU(2)$ we need $|(x_0, \mathbf{x})|^2 = |(y_0, \mathbf{y})|^2 + 1$ and $x_0 y_0 = -\mathbf{x}.\mathbf{y}$, by the latter we get $x_0 y_0 \mathbf{y} = -(\mathbf{x}.\mathbf{y})\mathbf{y}$ and $x_0 y_0 \mathbf{x} = -(\mathbf{x}.\mathbf{y})\mathbf{x}$. Multiplying by $\mathbf{y}$ and $\mathbf{x}$ respectively and by $x_0\mathbf{y} = y_0\mathbf{x}$ we get: $|\mathbf{x}|^2(|\mathbf{y}|^2 + y_0^2) = 0 = |\mathbf{y}|^2(|\mathbf{x}|^2 + x_0^2)$.

Therefore, either $(x_0, \mathbf{x})$ or $(y_0, \mathbf{y})$ is zero, the first case cannot be true since then $\det(S_a) < -1$ whilst we want it to be 1, therefore we're left with the case $(\mathbf{y}, y_0) = \mathbf{0}$, i.e. $(a_0, \mathbf{a}) = (a_0, a_1, a_2, a_3) \in \mathbb{S}^3$ since it is a real vector with $|(a_0, \mathbf{a})|^2 = 1$ (for $S_a \in SU(2)$).

<u>Rotations in $\mathbb{R}^3$</u>: Now, we'll investigate the relation between quaternions/Pauli matrices and rotation of vectors in $\mathbb{R}^3$. In particular, if we consider a unit vector $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ and a (unit quaternion $q_n = \cos(\theta/2) + (\mathrm{i}n_1 + \mathrm{j}n_2 + \mathrm{k}n_3)\sin(\theta/2)$, then for a(n) (imaginary) quaternion $q = \mathrm{i}q_1 + \mathrm{j}q_2 + \mathrm{k}q_3$ the map $q \mapsto q' = q_n q q_n^{-1}$ returns a(n) (imaginary) quaternion $q' = \mathrm{i}q'_1 + \mathrm{j}q'_2 + \mathrm{k}q'_3$ such that the vector $(q'_1, q'_2, q'_3) \in \mathbb{R}^3$ is the anti-clockwise rotation of the vector (q_1, q_2, q_3) by a angle θ around the vector $\mathbf{n}$. We show this in the rest of this sub-section.

Consider a unit vector $\mathbf{n} \in \mathbb{R}^3$ and it's anti-clockwise rotation by θ around an axis in a direction $\mathbf{a} \in \mathbb{R}^3$, let R_a be *the matrix of the rotation* and $\mathbf{n}'$ the resulting vector, i.e. $\mathbf{n}' = R_a\mathbf{n}$. We decompose $\mathbf{n}$ to $\mathbf{n}_\parallel = (\mathbf{a}.\mathbf{n})\mathbf{a}$ and $\mathbf{n}_\perp = \mathbf{n} - \mathbf{n}_\parallel$ the projection of $\mathbf{n}$ to the "$\mathbf{a}$-axis" (respectively the parallel and vertical to $\mathbf{a}$ components of $\mathbf{n}$). Similarly, we have $\mathbf{n}'_\parallel$ and $\mathbf{n}'_\perp$ for $\mathbf{n}'$. Clearly, $\mathbf{n}'_\parallel = \mathbf{n}_\parallel$ and standard Geometry gives us $\mathbf{n}'_\perp = \cos(\theta)\mathbf{n}_\perp + \sin(\theta)\mathbf{a} \times \mathbf{n}_\perp = \cos(\theta)(\mathbf{n} - \mathbf{n}_\parallel) + \sin(\theta)\mathbf{a} \times (\mathbf{n} - \mathbf{n}_\parallel)$

$$\begin{aligned}\implies R\mathbf{n} = \mathbf{n}' = \mathbf{n}'_\perp + \mathbf{n}'_\parallel &= \cos(\theta)(\mathbf{n} - \mathbf{n}_\parallel) + \sin(\theta)\mathbf{a} \times (\mathbf{n} - \mathbf{n}_\parallel) + \mathbf{n}'_\parallel \\ &= \cos(\theta)(\mathbf{n} - (\mathbf{a}.\mathbf{n}_\parallel)\mathbf{a}) + \sin(\theta)\mathbf{a} \times (\mathbf{n} - (\mathbf{a}.\mathbf{n}_\parallel)\mathbf{a}) + (\mathbf{a}.\mathbf{n}_\parallel)\mathbf{a} \\ &= (1 - \cos(\theta))(\mathbf{a}.\mathbf{n})\mathbf{a} + \cos(\theta)\mathbf{n} + \sin(\theta)(\mathbf{a} \times \mathbf{n}).\end{aligned}$$

It is clear that an anti-clockwise rotation by an angle θ of a vector/point around a vector $\mathbf{a}$, has the same result as a rotation by an angle $2\pi - \theta$ (or just by $-\theta$ clockwise) around the vector -$\mathbf{a}$.

By the discussion before, we get that an S_a can be written as:

$$S_a = \begin{pmatrix} a_0 + \mathrm{i}a_3 & a_2 + \mathrm{i}a_1 \\ -a_2 + \mathrm{i}a_1 & a_0 - \mathrm{i}a_3 \end{pmatrix}. \tag{2.1.9}$$

We set $\tilde{\mathbf{a}} = \mathrm{i}a_3 + \mathrm{j}a_2 + \mathrm{k}a_1$ the "complex" part of the *quaternion*: A generalisation of complex number from 2 to 4 real dimensions, with 1 real part; quaternions were introduces by R. W. Hamilton in 1863 to help model rotations in mechanics.

We set:

$$\underset{\sim}{\mathbf{a}} = \frac{\tilde{\mathbf{a}}}{|\tilde{\mathbf{a}}|} = \frac{\tilde{\mathbf{a}}}{\sqrt{1 - |a_0|^2}} \tag{2.1.10}$$

the associated unitary quaternion (clearly $\underset{\sim}{\mathbf{a}} = \tilde{\mathbf{a}}$ if $|a_0| = 0$). We have: $a_0 = \sqrt{\frac{1}{2}(\cos(\theta) + 1)} \geq 0$ and if > 0 set $\hat{\mathbf{a}} = \dfrac{\sin(\theta)}{2a_0}\tilde{\mathbf{a}}$. It can easily be seen that $a_0^2 + |\hat{\mathbf{a}}|^2 = 1$ as well as $\cos(\theta) = 2a_0^2 - 1 = a_0^2 - |\hat{\mathbf{a}}|^2$, $\sin(\theta)\tilde{\mathbf{a}} = 2a_0\hat{\mathbf{a}}$ and lastly,

$1-\cos(\theta) = 2(1-a_0^2) = 2|\hat{\mathbf{a}}|^2$, so:

$$R_a\mathbf{n} = (a_0^2 - |\hat{\mathbf{a}}|^2)\mathbf{n} + 2(\hat{\mathbf{a}}.\mathbf{n})\hat{\mathbf{a}} + 2a_0\hat{\mathbf{a}} \times \mathbf{n}. \tag{2.1.11}$$

Since the map R_a remains invariant with respect to the reversal $(a_0, \hat{\mathbf{a}}) \mapsto (-a_0, -\hat{\mathbf{a}})$ we get that the respective map $S_a \mapsto R_a$ is a 2:1 covering. The existence of such R_a's is clear since $SU(2) \lhd U(2) \lhd GL(2)$ and therefore by conjugating an element $\mathbf{n}.\boldsymbol{\sigma}$ ($|\mathbf{n}| = 1$) of $SU(2)$ in $U(2)$ or $GL(n)$ we get an alternative element $\mathbf{n}'.\boldsymbol{\sigma}$ of $SU(2)$ which can be simpler to work with when transforming Spinors. Clearly there are two rotation matrices $R_\pm$ such that $\mathbf{n}' = R_\pm\mathbf{n}$. Analytically, we have the following stronger lemma:

Lemma 2.1.3. Conjugation of *Pauli matrices* in $SU(2)$ gives us alternative *Pauli matrices*, in terms of rotations in $\mathbb{R}^3$. Specifically, consider S_a as in (2.1.9) above, and R_a as in (2.1.11), these two quantities are related via:

$$(R_a\mathbf{n}).\boldsymbol{\sigma} = S_a^*(\mathbf{n}.\boldsymbol{\sigma})S_a. \tag{2.1.12}$$

Proof. Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ be the standard Pauli matrices, $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ and $S_a = a_0 I_2 + \mathrm{i}\hat{\mathbf{a}}.\boldsymbol{\sigma} \in SU(2)(a_0 \in \mathbb{R},\ \hat{\mathbf{a}} = (a_1, a_2, a_3) \in \mathbb{R}^3 : a_0^2 + |\hat{\mathbf{a}}|^2 = 1)$, then:

$$\begin{gathered} S_a^*(\mathbf{n}.\boldsymbol{\sigma})S_a = \\ (a_0 I_2 - \mathrm{i}\hat{\mathbf{a}}.\boldsymbol{\sigma})\mathbf{n}.\boldsymbol{\sigma}(a_0 I_2 + \mathrm{i}\hat{\mathbf{a}}.\boldsymbol{\sigma}) = a_0^2\mathbf{n}.\boldsymbol{\sigma} - \mathrm{i}(\hat{\mathbf{a}}.\boldsymbol{\sigma})(\mathbf{n}.\boldsymbol{\sigma}) + \mathrm{i}(\mathbf{n}.\boldsymbol{\sigma})(\hat{\mathbf{a}}.\boldsymbol{\sigma}) + (\hat{\mathbf{a}}.\boldsymbol{\sigma})\mathbf{n}.\boldsymbol{\sigma}(\hat{\mathbf{a}}.\boldsymbol{\sigma}) \\ = a_0^2\mathbf{n}.\boldsymbol{\sigma} + \mathrm{i}n_i a_j(\sigma^i\sigma^j - \sigma^j\sigma^i) + (\hat{\mathbf{a}}.\boldsymbol{\sigma})n_i a_j \sigma^i\sigma^j \\ = a_0^2\mathbf{n}.\boldsymbol{\sigma} - 2\epsilon_{ijk}n^i a^j\sigma^k + (\hat{\mathbf{a}}.\boldsymbol{\sigma})n^i a^j(\delta_{ij}I_2 + \mathrm{i}\epsilon_{ijk}\sigma_k) \\ = a_0^2\mathbf{n}.\boldsymbol{\sigma} - 2(\mathbf{n}\times\hat{\mathbf{a}}).\boldsymbol{\sigma} + a_i n_k a_j\sigma^i\sigma^k\sigma^j = a_0^2\mathbf{n}.\boldsymbol{\sigma} + 2(\hat{\mathbf{a}}\times\mathbf{n}).\boldsymbol{\sigma} + a_i n_k a_j(\delta_{ik}I_2 + \mathrm{i}\epsilon_{ikl}\sigma_l)\sigma_j \end{gathered} \tag{2.1.13}$$

and proceed by writing $\mathrm{i}\epsilon_{ikl}\sigma_l\sigma_j = \mathrm{i}\epsilon_{ikl}(\delta_{lj}I_2 + \mathrm{i}\epsilon_{ljm}\sigma_m) = \mathrm{i}\epsilon_{ikj}I_2 - \epsilon_{ikl}\epsilon_{ljm}\sigma_m$ and notice the following property of the Levi-Civita symbol: $\epsilon_{ikl}\epsilon_{ljm} = \delta_{ij}\delta_{km} - \delta_{im}\delta_{jk}$ and re-write the "RHS" of the very last part of (2.1.13) as:

$$a_0^2\mathbf{n}.\boldsymbol{\sigma} + 2(\hat{\mathbf{a}}\times\mathbf{n}).\boldsymbol{\sigma} + a_i n_k a_j\delta_{ik}\sigma_j + \mathrm{i}a_i n_k a_j\epsilon_{ikj}I_2 - a_i n_k a_j(\delta_{ij}\delta_{km} - \delta_{im}\delta_{jk})\sigma_m =$$

$$a_0^2\mathbf{n}.\boldsymbol{\sigma} + 2(\hat{\mathbf{a}}\times\mathbf{n}).\boldsymbol{\sigma} + (\hat{\mathbf{a}}.\mathbf{n})(\hat{\mathbf{a}}.\boldsymbol{\sigma}) + \mathrm{i}((\hat{\mathbf{a}}\times\mathbf{n}).\hat{\mathbf{a}})I_2 - |\hat{\mathbf{a}}|^2(\mathbf{n}.\boldsymbol{\sigma}) + (\hat{\mathbf{a}}.\mathbf{n})(\hat{\mathbf{a}}.\boldsymbol{\sigma}) =$$

$$(a_0^2 - |\hat{\mathbf{a}}|^2)\mathbf{n}.\boldsymbol{\sigma} + 2(\hat{\mathbf{a}} \times \mathbf{n}).\boldsymbol{\sigma} + 2(\hat{\mathbf{a}}.\mathbf{n})(\hat{\mathbf{a}}.\boldsymbol{\sigma}) = (R_a\mathbf{n}).\boldsymbol{\sigma}$$

by (2.1.11) we conclude $(R_a\mathbf{n}).\boldsymbol{\sigma} = S_a^*(\mathbf{n}.\boldsymbol{\sigma})S_a$. $\square$

Remark 2.1.4. The group of unitary *quaternions* is isomorphic to the group of special unitary matrices via the map $S : Q_8 \mapsto SU(2)$, $S(a) := S_a$ as in (2.1.9) (equivalently for certain obvious choices of a_0, a_1, a_2, a_3 in the RHS of (2.1.9)), $a = (a_0, \mathbf{a}) \in \mathbb{S}^3$ (with $a_0 \in \mathbb{R}$, $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$) viewed as a quaternion.

More information can be found in Stillwell, 2008. Also, quaternions are also regarded as a Clifford algebra; information on this and Clifford algebras "in higher dimensions" can be found in Atiyah, Bott, and Shapiro, 1964.

2.2 Further preliminaries and special cases

2.2.1 Basic facts on the Weyl-Dirac operator

In this section, we present some special cases of potentials $\boldsymbol{A}$ that make up the Weyl-Dirac operator $\boldsymbol{\mathcal{D}_A}$, for which we are able to obtain explicit solutions to $\boldsymbol{\mathcal{D}_A}\psi = 0$, even zero-modes. First, we notice that the Weyl-Dirac operator is *gauge invariant*, that is, certain qualitative information is preserved under unitary transformations $\psi \mapsto e^{-\mathrm{i}\phi}\psi$ (for some scalar function ϕ). Specifically:

$$\boldsymbol{\mathcal{D}_A}(e^{-\mathrm{i}\phi}\psi) = e^{-\mathrm{i}\phi}\boldsymbol{\mathcal{D}_A}\psi - e^{-\mathrm{i}\phi}(\boldsymbol{\sigma}.\boldsymbol{\nabla}\phi)\psi = e^{-\mathrm{i}\phi}\boldsymbol{\mathcal{D}_{A'}}\psi \quad \text{for } \boldsymbol{A'} = \boldsymbol{A} + \boldsymbol{\nabla}\phi \qquad (2.2.1)$$

This means that if $e^{-\mathrm{i}\phi}\psi$ is an eigenvector of $\boldsymbol{\mathcal{D}_A}$ with eigenvalue λ, then ψ is an eigenvector of $\boldsymbol{\mathcal{D}_{A'}}$ for $\boldsymbol{A'}$ easily derived from $\boldsymbol{A}$ and ϕ by (2.2.1), corresponding to the same eigenvalue λ, and *vice-versa*.

Therefore, in order to study the existence of zero-modes, it is enough to focus on classes of potentials identified by the magnetic field they produce (since $\boldsymbol{\nabla}\times\boldsymbol{A'} = \boldsymbol{\nabla}\times\boldsymbol{A}$ for $\boldsymbol{A}, \boldsymbol{A'}$ as above). In fact, by standard differential geometry, we know that if $\boldsymbol{A}, \boldsymbol{A'}$ satisfy $\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{\nabla}\times\boldsymbol{A'}$ then there exists a scalar ϕ such that $\boldsymbol{A'} = \boldsymbol{A} + \boldsymbol{\nabla}\phi$. Since qualitative information such as the existence of zero modes is not affected under such transformations $\boldsymbol{A} \mapsto \boldsymbol{A'}$, in applications, we're free to choose "simple" potentials, even divergence free if convenient. Moreover, for each potential $\boldsymbol{A}$ there exists a ϕ such that $\boldsymbol{A'} := \boldsymbol{A} + \boldsymbol{\nabla}\phi$ is divergeless (see Bourne, 2018 for more information):

$$0 = \boldsymbol{\nabla}.\boldsymbol{A'} = \boldsymbol{\nabla}.\boldsymbol{A} + \boldsymbol{\nabla}.\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}.\mathbf{A} + \Delta\phi \iff -\Delta\phi = \boldsymbol{\nabla}.\mathbf{A} \qquad (2.2.2)$$

which is a (free) Poisson equation and always has a solution, which can even be found explicitly with the help of the associated Green's functions[5]:

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y})\boldsymbol{\nabla}.\boldsymbol{A}(\mathbf{y})d\mathbf{y}, \quad \mathbf{x} = (x_1, .., x_n), \quad \mathbf{y} = (y_1, .., y_n)$$

[5]These are fundamental solutions of the Laplace equation in particular, and are found by looking for radially symmetric solutions, a recipe prescribed by the orthogonal ("rotational") symmetry of the Laplace operator.

where $\Phi(\mathbf{x}) = -\frac{1}{2\pi}\ln(|\mathbf{x}|)$ if $n = 2$ and $\frac{1}{n(n-2)|\mathrm{Vol}(\mathbb{B}^n)||\mathbf{x}|^{n-2}}$ for $n \geq 3$, where $\mathrm{Vol}(\mathbb{B}^n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, the volume of the unit-ball in $\mathbb{R}^n$.

Specifically for uni-directional fields, we have:

Lemma 2.2.1. Let $\mathbf{A}(\mathbf{x}) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ a smooth vector field on $\mathbb{R}^3$ that satisfies: $\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x}) = (0, 0, f(x_1, x_2))$ for some continuous $f : \mathbb{R}^2 \mapsto \mathbb{R}$. Then we can decompose: $\boldsymbol{A}_g(\mathbf{x}) = (A_{1,g}(x_1, x_2), A_{2,g}(x_1, x_2), 0) + \boldsymbol{\nabla} g(\mathbf{x})$ for a scalar function $g(\mathbf{x})$.

Proof. Let $\boldsymbol{A}(\mathbf{x}) = (A_1(\mathbf{x}), A_2(\mathbf{x}), A_3(\mathbf{x}))$ and $f(x_1, x_2)$ such that:

$$\nabla_1 A_2 - \nabla_2 A_1 = f(x_1, x_2)$$

where $\nabla_i = \nabla_{x_i}$, for $i = 1, 2$.

We pick (wlog) $\tilde{A}_1(x_1, x_2)$, $\tilde{A}_2(x_1, x_2)$ such that $\nabla_1 \tilde{A}_2 - \nabla_2 \tilde{A}_1 = f(x_1, x_2)$ (standard theory of PDEs guarantees the existence of such $\mathbf{A}(\mathbf{x})$, $\tilde{\boldsymbol{A}}_{1,2}(x_1, x_2)$ for any given piecewise continuous $f(x_1, x_2)$).

It is sufficient to show that for all such $A_i(\mathbf{x}), \tilde{A}_i(\mathbf{x})$ $(i = 1, 2)$ there exists a scalar $g(\mathbf{x})$ such that:

$$A_i(\mathbf{x}) = \tilde{A}_i(x_1, x_2) + \nabla_{x_i} g(\mathbf{x}) \text{ for } i = 1, 2 \text{ and } A_3(\mathbf{x}) = \nabla_{x_3} g(\mathbf{x})$$

By the former equation, we have $g(\mathbf{x}) = \int A_3(\mathbf{x}) dx_3 + h(x_1, x_2)$ for some $h(x_1, x_2)$ which has to satisfy the latter as well, equivalently:

$$\nabla_2 h = (A_2 - \tilde{A}_2) - \int \nabla_2 A_3 dx_3, \text{ and}$$
$$\nabla_1 h = A_1 - \tilde{A}_1 - \int \nabla_1 A_3 dx_3.$$

By the condition $\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x}) \parallel (0, 0, 1)$, and since $\tilde{A}_{1,2}$ are arbitrary functions of (x_1, x_2) we can simplify (after slightly abusing notation) the RHS's of the above equations to just $\nabla_2 h = -\tilde{A}_2$ and $\nabla_1 h = -\tilde{A}_1$. Finally, the existence as well as the regularity of such $h(x_1, x_2)$ is guaranteed by the standard theory of PDEs. □

2.2.2 Special cases of the Weyl-Dirac operator in two dimensions

If we're looking for planar zero-modes (x_1, x_2-dependent), by Lemma 1.3.1 above we can consider the magnetic field to be two-dimensional. Also, since the zero mode will not depend on x_3, the two-dimensional Dirac operator can be defined as:

$$\boldsymbol{\mathcal{D}}_{\tilde{\boldsymbol{A}}} := \sigma_1(-\mathrm{i}\nabla_1 - \tilde{A}_1) + \sigma_2(-\mathrm{i}\nabla_2 - \tilde{A}_2) \tag{2.2.3}$$

More explicitly:

$$\boldsymbol{\mathcal{D}}_{\tilde{\boldsymbol{A}}} = \begin{pmatrix} 0 & -\mathrm{i}(\nabla_1 - \mathrm{i}\nabla_2) - (\tilde{A}_1 - \mathrm{i}\tilde{A}_2) \\ -\mathrm{i}(\nabla_1 + \mathrm{i}\nabla_2) - (\tilde{A}_1 + \mathrm{i}\tilde{A}_2) & 0 \end{pmatrix} \tag{2.2.4}$$

notice that $\nabla_1 - \mathrm{i}\nabla_2 = 2\partial_z$ (for $z = x + \mathrm{i}y$) and $\nabla_1 + \mathrm{i}\nabla_2 = 2\partial_{\bar{z}}$ and we can turn the respective PDEs into two ODEs with respect to z and $\bar{z}$, as in Aharonov and Casher, 1979.

In certain cases where the potential $\mathbf{A}(\mathbf{x})$ is x_1, x_2 - dependent, we can easily obtain non-trivial solutions explicitly (even zero modes) for the Weyl-Dirac operator, especially when the corresponding magnetic field is uni-directional, for example:

$$\boldsymbol{A}(\mathbf{x}) = \big(-x_2, x_1, 0\big) \implies \boldsymbol{B}(\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x}) = 2(0, 0, 1) \tag{2.2.5}$$

We want to construct a specific magnetic potential $\boldsymbol{A}_c$ such that the corresponding magnetic field $\boldsymbol{\nabla} \times \boldsymbol{A}_c(\mathbf{x})$ is compactly supported on a set of co-dimension 2, such the x_3-axis in $\mathbb{R}^3$. To construct it, we proceed heuristically: Assume that it is of the form $\boldsymbol{A}_c(\mathbf{x}) = c(\mathbf{x})\boldsymbol{A}(\mathbf{x})$ for some scalar function $c(\mathbf{x})$, then its curl is:

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{A}_c(\mathbf{x}) &= \boldsymbol{\nabla} c(\mathbf{x}) \times \mathbf{A}(\mathbf{x}) + c(\mathbf{x})(\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x})) = \boldsymbol{\nabla} c(\mathbf{x}) \times \mathbf{A}(\mathbf{x}) + (0, 0, c(\mathbf{x})) \\ &= \big(x_1\partial_{x_3} c(\mathbf{x}), x_2\partial_{x_3} c(\mathbf{x}), x_1\partial_{x_1} c(\mathbf{x}) + x_2\partial_{x_2} c(\mathbf{x}) + 2c(\mathbf{x})\big). \end{aligned} \tag{2.2.6}$$

We want the last quantity to be zero almost everywhere, and so we set $\partial_{x_3} c = 0$, i.e. $c = c(x_1, x_2)$. The differential operator $x_1\partial_{x_1} + x_2\partial_{x_2}$ is the differential operator $r\partial_r$

(with $r = \sqrt{x_1^2 + x_2^2}$), expressed in Cartesian coordinates. Indeed, let $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \arctan(x_2/x_1)$, then the differential form dr evaluates:

$$dr = (\partial_{x_1} r) dx_1 + (\partial_{x_2} r) dx_2 = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} dx_1 + \frac{x_2}{\sqrt{x_1^2 + x_2^2}} dx_2$$

by which we immediately see (using the anticommutativity property of differential forms) that its dual, the vector field ∂_r satisfies:

$$\partial r = (\partial_{x_1} r)\partial_{x_1} + (\partial_{x_2} r)\partial_{x_2} = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}\partial_{x_1} + \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\partial_{x_2} \implies r\partial_r = x_1\partial_{x_1} + x_2\partial_{x_2}$$

We'll look for spherically symmetric solutions of: $x_1\partial_{x_1} c + x_2\partial_{x_2} c + 2c = 0,\ r \neq 0$, that is $c = c(r)$ for $r = \sqrt{x_1^2 + x_2^2}$, which turns the equation for c into the ODE:

$$c'(r) + \frac{2}{r}c(r) = 0, \quad r \neq 0 \quad \implies c(r) = c_0 r^{-2} \text{ for some } c_0 \in \mathbb{R} \tag{2.2.7}$$

We just saw that for all non-zero $c_0 \in \mathbb{R}$ the magnetic potential:

$$\mathbf{A}_{c_0}(\mathbf{x}) = \Big(-\frac{c_0 x_2}{x_1^2 + x_2^2}, \frac{c_0 x_1}{x_1^2 + x_2^2}, 0 \Big) \tag{2.2.8}$$

generates a magnetic field $\mathbf{B}_{c_0}(\mathbf{x}) := \boldsymbol{\nabla} \times \mathbf{A}_{c_0}(\mathbf{x})$ that is singular on the x_3-axis.

Writing it as a differential 1-form helps us work on different manifolds easily since the exterior derivative (which can be identified as the curl in $\mathbb{R}^3$) can easily be transformed when changing variables (coordinates). We have:

$$\alpha_{c_0} = -\frac{c_0 x_2}{x_1^2 + x_2^2} dx_1 + \frac{c_0 x_1}{x_1^2 + x_2^2} dx_2. \tag{2.2.9}$$

It is worth mentioning that this is a standard example of a differential 1-form that is closed but not exact when (x_1, x_2) belongs in a region punctured at the origin (such a region is not contractible and therefore *Poincaré's lemma* does not apply). In the case of such magnetic potentials the corresponding Weyl-Dirac operator $\mathcal{D}_{A_{c_0}}$ has two linearly independent and real zero modes, namely $\psi_{c_0}^+(r) = (r^{-c_0/2}, 0)^T$ and $\psi_{c_0}^-(r) = (0, r^{c_0/2})^T$ in Polar coordinates, respectively in Cartesian:

$$\psi_{c_0}^+(x_1, x_2) = \big((x_1^2 + x_2^2)^{-c_0/4}, 0\big)^T, \quad \psi_{c_0}^-(x_1, x_2) = \big(0, (x_1^2 + x_2^2)^{c_0/4}\big)^T. \tag{2.2.10}$$

However, there is no value of c_0 such that $\psi^{\pm}_{c_0}$ is a zero mode on $\mathbb{R}^2$, because by integration we'll have a power of r that will tend to infinity either at 0 or at ∞. This is consistent with the *Aharonov-Casher Theorem*, since the corresponding *magnetic 2-form* satisfies $\beta_{c_0} = d\alpha_{c_0} = 0$, i.e. the total *flux* satisfies:

$$\Phi(\alpha_{c_0}) = \frac{1}{2\pi}\lim_{R\to\infty}\int_{D(0,R)} d\alpha_{c_0} = \frac{1}{2\pi}\lim_{R\to\infty}\int_{D(0,R)} \beta_{c_0} = 0.$$

However, this is not the case for $\tilde{\alpha}_c = -cx_2dx_1 + cx_1dx_2$ (as in (2.2.5), for $c \neq 0$), which satisfies:

$$\Phi(\tilde{\alpha}_c) := \frac{1}{2\pi}\lim_{R\to\infty}\int_{D(0,R)} d\tilde{\alpha}_c = \frac{c}{2\pi}\lim_{R\to\infty}\int_{D(0,R)} 2dx_1dx_2 = sgn(c)\infty$$

The corresponding $\mathcal{D}_{\tilde{\boldsymbol{A}}_{c_0}}$ has solutions

$$\tilde{\psi}^+_c(x_1,x_2) = \big(e^{-\frac{c(x_1^2+x_2^2)}{2}},0\big)^T, \text{ and } \tilde{\psi}^-_c(x_1,x_2) = \big(0, e^{\frac{c(x_1^2+x_2^2)}{2}}\big)^T$$

which are easily found by heuristically searching for real-valued spinors. Complex valued spinors can be obtained by multiplying $\tilde{\psi}^{\pm}_c$ with an (arbitrary in this case) entire function $f(x_1 \pm \mathrm{i}x_2)$. Clearly, only $\tilde{\psi}^{sgn(c)}_c \in L^2(\mathbb{R}^2)$, hence it is a zero-mode. In search of real zero-modes, we can re-write the two-dimensional Weyl-Dirac Operator as:

$$\boldsymbol{\mathcal{D}}_{\tilde{\boldsymbol{A}}} = \begin{pmatrix} 0 & -\nabla_2 - \tilde{A}_1 - \mathrm{i}(\nabla_1 - \tilde{A}_2) \\ \nabla_2 - \tilde{A}_1 - \mathrm{i}(\nabla_1 + \tilde{A}_2) & 0 \end{pmatrix} \tag{2.2.11}$$

essentially separating the real and imaginary parts of the operator.

This makes things a lot easier since if we're looking for a "real" zero-mode $\psi_{re} = (\psi^+_{re}, \psi^-_{re})^T$ ($\psi^{\pm}_{re}$ are the "up" and "down" components of the Spinor respectively), for each component we have:

$$(\pm\nabla_2 - \tilde{A}_1)\psi^{\pm}_{re} = 0 \quad \& \quad (\nabla_1 \pm \tilde{A}_2)\psi^{\pm}_{re} = 0.$$

These equations can easily be solved by standard techniques and obtain unique (up to scalar multiplication) respective solutions:

$$\psi^{\pm}_{re} = e^{\pm(\int \tilde{A}_1dx_2 - \int \tilde{A}_2dx_1)}. \tag{2.2.12}$$

As we can see, the signs on the exponents of the last two equations are opposite, which means that if one solution decays at infinity (at any rate), then the other will explode. Consequently, we can have at most one solution in L^2 (zero-mode), either "up" or "down". In the case of the following class of potentials (with $c, c_0 > 0$), we have:

$$\tilde{\boldsymbol{A}}_{c,c_0}(x_1, x_2) = \frac{c_0}{c + x_1^2 + x_2^2}(-x_2, x_1). \tag{2.2.13}$$

which satisfies

$$\Phi(\tilde{\boldsymbol{A}}_{c,c_0}) := \frac{1}{2\pi} \lim_{R\to\infty} \int_{D(0,R)} d\gamma_{c,c_0} = 2c_0.$$

The respective solutions $\psi_{re}^{\pm}$ that emerge are (corresponding to the up/down components):

$$\psi_{re}^{\pm}(x_1, x_2) = \left(c + x_1^2 + x_2^2\right)^{\mp c_0/2} \tag{2.2.14}$$

and we have a zero-mode iff $|c_0| > 4$, which can only be $\psi_{re}^{-sgn(c_0)}$.

Clearly, we have no zero-modes in the case where $c_0 = 0$ (and the case $c = 0$ corresponds to (2.2.8)). In the case where $c < 0$, the same problem can be studied similarly in the open disk $\{x_1^2 + x_2^2 < -c\}$ or its complement.

These particular examples are a simplified illustration of the Aharonov-Casher ideas (Aharonov and Casher, 1979), although we mostly looked for real solutions instead of complex ones. The results are perfectly in line with these ideas and the last example will be used in the next Chapter where we construct zero-modes for the three-dimensional Weyl-Dirac operator based on that.

2.3 Operators commuting with the Weyl-Dirac operator

In the last few decades, certain methods have been developed in order to investigate the existence and certain properties of zero modes of *Dirac* or *Weyl-Dirac* operator $\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} := \boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}} - \boldsymbol{A})$. A simple, analytic approach is presented and explained in Schluter, Wietschorke, and Greiner, 1983.

Regarding the question of existence of zero modes of the *Weyl-Dirac* operator, it turns out that in some cases the problem is equivalent to investigating the existence of zero modes of the *Pauli Operator* $\boldsymbol{\mathcal{P}}_{\boldsymbol{A}} := \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}^2$. This equivalency lies on the fact that $\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}$, if defined on certain domains, like $\mathcal{H} := C_c^\infty(\mathbb{R}^3, \mathbb{C}^2)$ or $W^{1,2}(\mathbb{R}^3, \mathbb{C}^2)$ the operator is self-adjoint (see Elton, 2018) when associated with the inner product

$$\langle z, w\rangle_{\mathcal{H}} := \int_{\mathbb{R}^3} \langle z, w\rangle_{\mathbb{C}^2} d\mathbf{x}, \quad d\mathbf{x} = dx_1 dx_2 dx_3 \text{ with } \langle z, w\rangle_{\mathbb{C}^2} = z_1\overline{w_1} + z_2\overline{w_2}. \quad (2.3.1)$$

In fact, $\phi \in \mathcal{H}$ is a zero mode of $\mathcal{D}_A$ iff it is a zero mode of $\mathcal{D}_A^2$ since:

$$0 = ||\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}\phi||^2 = \langle \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}\phi, \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}\phi\rangle_{\mathcal{H}} = \langle \phi, \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}^2\phi\rangle_{\mathcal{H}}. \quad (2.3.2)$$

The second order operator $\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}^2$ is sometimes easier to work with as we have the *Lichnerowicz formula* (see Erdős and Solovej, 2001 and/or Berline, Getzler, and Vergne, 2003Friedrich, 2000 the graduate books in *references*). The Lichnerowicz formula involves the Laplacian as well as curvature terms. In matrix form, when working on $\mathbb{R}^3$ (where we have 0 scalar curvature) it can be written as:

$$\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}^2 = \boldsymbol{P}^2 \otimes I_2 - \boldsymbol{\sigma}.\boldsymbol{B} = \begin{pmatrix} P^2 - B_3 & B_1 - \mathrm{i}B_2 \\ B_1 + \mathrm{i}B_2 & P^2 + B_3 \end{pmatrix} = \sum_{i=1}^{3}(-\mathrm{i}\nabla_i - A_i)^2 I_2 - \boldsymbol{\sigma}.\boldsymbol{B} \quad (2.3.3)$$

with $\mathbf{P}^2 = (-\mathrm{i}\boldsymbol{\nabla} - \boldsymbol{A})^2$.

In some cases, the RHS in equation (2.3.3) can be diagonalised, and under some conditions on $\mathbf{A}$, the problem boils down to finding solutions to Poisson-type equations, on which there is an abundance of helpful results, Aharonov and Casher, 1979.

2.3.1 Commuting second order operators with Spin fields

Standard operator theory gives us that when two (bounded) operators commute, they have related eigenstates (not necessarily the same eigenvalues, unless perhaps in non-degenerate cases). Under certain conditions, certain such relations still hold when one of the operators in unbounded. We will start the study of zero modes of the Weyl-Dirac equation by studying whether a weighted version of the aforementioned Pauli operator commutes with a certain spin field - whose eigenfunctions are simpler to study.

Generality is not affected by the fact that these operators are weighted, because we're looking for zero-modes. Hence the respective eigenvalue problem is equivalent to the one related to the non-weighted version of the operator.

Consider a magnetic potential $\mathbf{A}(\mathbf{x})$ (simply noted as $\mathbf{A}$), a 3-dimensional vector field $\mathbf{X}(\mathbf{x})$ on $\mathbb{R}^3$, and $\Omega : \mathbb{R}^3 \mapsto \mathbb{R}_{\geq 0}$ a weight on $\mathbb{R}^3$ (positive in an open region of interest). Then we have the following:

Proposition 2.3.1. Consider the operator $(\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\cdot))^2 := \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\cdot))$ and the spin field $\tilde{\boldsymbol{S}} = \boldsymbol{\sigma}.\boldsymbol{X}$. We have that $\tilde{\boldsymbol{S}}$ and $(\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\cdot))^2$ commute iff $\Omega\mathbf{X}$ is a conformal Killing field, parallel to $\mathbf{B} := \boldsymbol{\nabla} \times \boldsymbol{A}$, $\Omega^{-1}\mathbf{X}$ is irrotational and $\boldsymbol{\nabla}.(\Omega^{-2}\mathbf{X}) = 0$.

Remark 2.3.2. Regarding the conformal Killing field $\Omega\mathbf{X}$ we have: $\Omega\mathbf{X}.\boldsymbol{\nabla} \times (\Omega\mathbf{X}) = \Omega\mathbf{X}.\boldsymbol{\nabla} \times (\Omega^2(\Omega^{-1}\mathbf{X})) = \Omega\mathbf{X}.((\boldsymbol{\nabla}\Omega^2) \times (\Omega^{-1}\mathbf{X}) + \Omega^2\boldsymbol{\nabla} \times (\Omega^{-1}\mathrm{X})) = 2\Omega\mathbf{X}.(\boldsymbol{\nabla}\Omega \times \mathbf{X}) = 0$, since $\Omega^{-1}\mathbf{X}$ is irrotational. The magnetic field is parallel to $\mathbf{X}$ (and so to $\Omega\mathbf{X}$ too) Therefore the conditions imposed in *Proposition 2.3.1* include the required conditions for the main result (*Theorem 1.1*) in Elton, 2018, i.e. the Weyl-Dirac operator with a magnetic potential that produces such a field possesses no zero modes.

Corollary 2.3.3. Under *Proposition 2.3.1* we have that for such potentials $\mathbf{A}$, the corresponding Weyl-Dirac operator has no zero modes. Moreover, $|\mathbf{X}|$ is constant.

Proof. $\boldsymbol{\nabla} \times \Omega^{-1}\mathbf{X} = 0 \implies \Omega\mathbf{X}.\boldsymbol{\nabla} \times (\Omega\mathbf{X}) = \Omega^3\mathbf{X}.(\boldsymbol{\nabla} \times (\Omega^{-1}\mathbf{X})) + \mathbf{X}.(\boldsymbol{\nabla}\Omega^2 \times \mathbf{X}) = 0$, and since it is parallel to the magnetic field $\mathbf{B}$, we have the conditions of the main result in Elton, 2018 (*Theorem 1.1*), i.e. no zero-modes. The fact that $|\mathbf{X}|$ is constant, follows by the conformal Killing field condition, after multiplying by $2X_j$ and summing up for all j to obtain: $\Omega\nabla_i|\mathbf{X}|^2 = 0$. □

In order to prove the aforementioned this *Proposition 1.3.1*, we'll use the following auxiliary results::

Lemma 2.3.4. For all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $k \in \{1,2,3\}$ and for $\mathbf{A}$, $\mathbf{B}$, $\tilde{\boldsymbol{S}}$ and $\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\cdot)$ as previously defined, the following identities hold:

$$\text{i)} \quad [\boldsymbol{\sigma}.\mathbf{a}, \boldsymbol{\sigma}.\mathbf{b}] = \mathrm{i}\boldsymbol{\sigma}.(\mathbf{a} \times \mathbf{b}), \tag{2.3.4}$$

$$\text{ii)} \quad [\sigma_k, \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}^2] = -2\mathrm{i}(\boldsymbol{\sigma} \times \mathbf{B})_k, \tag{2.3.5}$$

$$\text{iii)} \quad [\tilde{\boldsymbol{S}}, \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}] = 2\boldsymbol{\sigma}.(\mathbf{X} \times \boldsymbol{\nabla}) - 2\mathrm{i}\boldsymbol{\sigma}.(\mathbf{X} \times \mathbf{A}) + \mathrm{i}\boldsymbol{\nabla}.\mathbf{X} - \boldsymbol{\sigma}.\boldsymbol{\nabla}\times\mathbf{X}, \tag{2.3.6}$$

$$\text{iv)} \quad (\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\cdot))^2 = \Omega^2\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}^2 + \Omega(\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}(\Omega))\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} + 2\Omega(\boldsymbol{\mathcal{D}}\Omega.\boldsymbol{\mathcal{D}}) + \Omega(\Delta\Omega) - \sum_{i=1}^{3}(\nabla_i\Omega)^2. \tag{2.3.7}$$

<u>*Proof of Lemma 2.3.4*</u>. Now, we prove the equations (2.3.4) to (2.3.7):

i) Let $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^3$, we have:

$$[\boldsymbol{\sigma}.\mathbf{a}, \boldsymbol{\sigma}.\mathbf{b}] = [\sigma_i a^i, \sigma_j b^j] = \sigma_i\sigma_j a^i b^j - \sigma_j\sigma_i a^i b^j = (\sigma_i\sigma_j - \sigma_j\sigma_i)a^i b^j = 2\mathrm{i}\epsilon_{ijk}\sigma_k a^i b^j$$

where ϵ_{ijk} is the *Levi-Civita* symbol, and so we obtain: $[\boldsymbol{\sigma}.\mathbf{a}, \boldsymbol{\sigma}.\mathbf{b}] = 2\mathrm{i}\boldsymbol{\sigma}.(\mathbf{a} \times \mathbf{b})$.

ii) Regarding (2.3.5) we have that since $\boldsymbol{\mathcal{D}}_{\mathbf{A}}^2$ can be decomposed to a multiple (that commutes with σ_i) of the identity matrix plus $\boldsymbol{\sigma}.\mathbf{B}$ we obtain:

$$[\sigma_k, \boldsymbol{\mathcal{D}}_{\mathbf{A}}^2] = [\sigma_k, \boldsymbol{\sigma}.\mathbf{B}] = [\sigma_k, \sigma_i B^i] = \sigma_k\sigma_i B^i - \sigma_i\sigma_k B^i = (\sigma_k\sigma_i - \sigma_i\sigma_k)B^i = 2\mathrm{i}\epsilon_{kij}\sigma_j B^i \implies$$

$$[\sigma_k, \boldsymbol{\mathcal{D}}_{\mathbf{A}}^2] = 2\mathrm{i}(\boldsymbol{\sigma} \times \mathbf{B}).$$

iii) For (2.3.6), we have:

$$[\tilde{S}, \boldsymbol{\mathcal{D}}_{\mathbf{A}}] = [\sigma_i X^i, \sigma_j \mathcal{D}^j - \sigma_j A^j] = [\sigma_i X^i, \sigma_j \mathcal{D}^j] - [\sigma_i X^i, \sigma_j A^j] = [\sigma_i X^i, \sigma_j \mathcal{D}^j] - 2\mathrm{i}\boldsymbol{\sigma}.(\mathbf{X}\times\mathbf{A})$$

by eq. (1.2.4) holds and by the linearity of the commutator. The other term satisfies:

$$[\sigma_i X^i, \sigma_j \mathcal{D}^j] = \sigma_i X^i \sigma_j \mathcal{D}^j - \sigma_j \mathcal{D}^j(\sigma_i X^i\cdot) = \sigma_i X^i \sigma_j \mathcal{D}^j - \sigma_j X^i \sigma_i \mathcal{D}^j - \sigma_j\sigma_i \mathcal{D}^j(X^i) \implies$$

$$[\sigma_i X^i, \sigma_j \mathcal{D}^j] = \mathrm{i}\epsilon_{ijk}\sigma_k X^i \mathcal{D}^j + \mathrm{i}\boldsymbol{\nabla}.\mathbf{X} - \epsilon_{ijk}\sigma_k \nabla_i X^j = \boldsymbol{\sigma}.(\mathbf{X}\times\boldsymbol{\nabla}) + \mathrm{i}\boldsymbol{\nabla}.\mathbf{X} - \boldsymbol{\sigma}.\boldsymbol{\nabla}\times\mathbf{X}.$$

and (2.3.6) follows. Lastly, we have:

iv) $(\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\cdot))^2 := \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\cdot)) = \boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}}-\mathbf{A})\Omega(\boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}}-\mathbf{A})\Omega\cdot)$.

where: $\boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}}-\mathbf{A})(\Omega\cdot) = \Omega(\boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}}-\mathbf{A})) + \boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}\Omega = \Omega\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} + \boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}\Omega$ and so:

$(\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\cdot))^2 = \boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}}-\mathbf{A})\Omega(\Omega\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} + \boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}\Omega) = \boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}}-\mathbf{A})(\Omega^2\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} + \Omega(\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}\Omega)) =$

$\Omega^2\boldsymbol{\mathcal{D}}^2_{\mathbf{A}} + \Omega(\sigma_i\sigma_j(\mathcal{D}^j\Omega))(\mathcal{D}^i - A^i) + 2\sigma_j(\Omega_i\mathcal{D}^i(\Omega))(\mathcal{D}_j - A_j) + (\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}(\Omega))^2 + \Omega\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}(\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}(\Omega))$

$$\implies (\boldsymbol{\mathcal{D}}_{\mathbf{A}}(\Omega\cdot))^2 = \Omega^2\boldsymbol{\mathcal{D}}^2_{\mathbf{A}} + \Omega(\boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}}\Omega))\boldsymbol{\mathcal{D}}_{\mathbf{A}} - 2((\nabla_i\Omega)\nabla^i) + \sum_{i=1}^{3}(\mathcal{D}_i(\Omega))^2 + \Omega\Delta\Omega.$$

□

Corollary 2.3.5. The quantity $[\tilde{\boldsymbol{S}}, (\boldsymbol{\mathcal{D}}_{\mathbf{A}}(\Omega\cdot))^2]$ satisfies:

$$\Omega^{-1}[\tilde{\boldsymbol{S}}, (\boldsymbol{\mathcal{D}}_{\mathbf{A}}(\Omega\cdot))^2] = \Omega[\tilde{\boldsymbol{S}}, \boldsymbol{\mathcal{D}}^2_{\mathbf{A}}] + [\tilde{\boldsymbol{S}}, \boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}(\Omega)]\boldsymbol{\mathcal{D}}_{\mathbf{A}} + (\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}(\Omega))[\tilde{\boldsymbol{S}}, \boldsymbol{\mathcal{D}}_{\mathbf{A}}] + 2\sigma_i(\nabla_j\Omega)\nabla^j X^i. \tag{2.3.8}$$

Proof. By (2.3.7) and the identity $[U, VW] = [U, V]W + V[U, W]$. □

Proof of Prop. 2.3.1: We will start by analysing each term on the RHS of (2.3.6).

i) $[\tilde{\boldsymbol{S}}, \boldsymbol{\mathcal{D}}^2_{\mathbf{A}}] = [\tilde{\boldsymbol{S}}, \sum_i(-\mathrm{i}\nabla_i - A_i)^2 I_2 - \boldsymbol{\sigma}.\mathbf{B}] = [\sigma_i X^i, \sum_i(-\mathrm{i}\nabla_i - A^i)^2 I_2 - \sigma_i B^i] \implies$

$[\tilde{\boldsymbol{S}}, \boldsymbol{\mathcal{D}}^2_{\mathbf{A}}] = [\sigma_i X^i, \sum_i(-\mathrm{i}\nabla_i - A_i)^2 I_2] - [\sigma_i X^i, \sigma_i B^i]$.

Regarding the first term of the RHS of the last equation, we have:

$$[\sigma_i X^i, \textstyle\sum_i(-\mathrm{i}\nabla_i - A_i)^2 I_2] = [\sigma_i X^i, -\Delta + \sum_i(2A_i\nabla_i + A_i^2) + \mathrm{i}\boldsymbol{\nabla}.\mathbf{A}].$$

Since σ_i commutes with the identity matrix we have: $[\sigma_i, \boldsymbol{\nabla}.\mathbf{A}] = [\sigma_i, A_j^2] = 0$, and by linearity of the Poisson bracket we get:

$$[\tilde{\boldsymbol{S}}, \boldsymbol{\mathcal{D}}^2_{\mathbf{A}}] = -[\sigma_i X^i, \Delta] + 2A_j[\sigma_i X^i, \nabla^j] - [\sigma_i X^i, \sigma_j B^j] \tag{2.3.9}$$

where $[\sigma_i X^i, -\Delta] = -\sigma_i X^i \Delta + \Delta(\sigma_i X^i \cdot) = 2(\nabla_i X^j)\sigma_j \nabla^i + \sigma_j \Delta X^j$
Regarding the next term in the RHS of (2.3.8) we have:

$$\begin{gathered}
[\tilde{\boldsymbol{S}}, \boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}(\Omega)]\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} = [\boldsymbol{\sigma}.\boldsymbol{X}, \boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}(\Omega)].(\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}\cdot) - [\boldsymbol{\sigma}.\mathbf{X}, \boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}(\Omega)].(\boldsymbol{\sigma}.\mathbf{A}) \\
= 2\mathrm{i}\big(\boldsymbol{\sigma}.(\mathbf{X} \times \boldsymbol{\mathcal{D}}(\Omega))\big).\big((\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}) - (\boldsymbol{\sigma}.\mathbf{A})\big) = 2\big(\boldsymbol{\sigma}.(\mathbf{X} \times \boldsymbol{\nabla}\Omega)\big).\big((\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}) - (\boldsymbol{\sigma}.\mathbf{A})\big) \\
= -2\mathrm{i}\big(\boldsymbol{\sigma}.(\mathbf{X} \times \boldsymbol{\nabla}\Omega)\big).\big(\boldsymbol{\sigma}.\boldsymbol{\nabla}\big) - 2\big(\boldsymbol{\sigma}.(\mathbf{X} \times \boldsymbol{\nabla}\Omega)\big).\big(\boldsymbol{\sigma}.\mathbf{A}\big) \\
= 2\boldsymbol{\sigma}.\big((\mathbf{X} \times \boldsymbol{\nabla}\Omega) \times \boldsymbol{\nabla}\big) - 2\mathrm{i}(\mathbf{X} \times \boldsymbol{\nabla}\Omega).\boldsymbol{\nabla} - 2\mathrm{i}\boldsymbol{\sigma}.\big((\mathbf{X} \times \boldsymbol{\nabla}\Omega) \times \mathbf{A}\big) - 2(\mathbf{X} \times \boldsymbol{\nabla}\Omega).\mathbf{A}.
\end{gathered} \tag{2.3.10}$$

Lastly, considering the last term on the RHS of (2.3.8) we get (from (2.3.6)):

$$(\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}(\Omega))[\tilde{\boldsymbol{S}}, \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}] = -\mathrm{i}\big(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega\big).\big(2\boldsymbol{\sigma}.(\mathbf{X} \times \boldsymbol{\nabla}) - 2\mathrm{i}\boldsymbol{\sigma}.(\mathbf{X} \times \mathbf{A}) + \mathrm{i}\boldsymbol{\nabla}.\mathbf{X} - \boldsymbol{\sigma}.\boldsymbol{\nabla}\times\mathbf{X}\big).$$

We will first deal with thfirst-orderer (with respect to differentiation) term:

$$-\mathrm{i}\big(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega\big).\big(2\boldsymbol{\sigma}.(\mathbf{X} \times \boldsymbol{\nabla})\big) = 2\boldsymbol{\sigma}.(\boldsymbol{\nabla}\Omega \times (\mathbf{X} \times \boldsymbol{\nabla})) - 2\mathrm{i}\boldsymbol{\nabla}\Omega.(\mathbf{X} \times \boldsymbol{\nabla}) \tag{2.3.11}$$

where from standard theory we have $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}).\mathbf{c} = -(\mathbf{b} \times \mathbf{a}).\mathbf{c}$ which means that: $-2\mathrm{i}\boldsymbol{\nabla}\Omega.(\mathbf{X} \times \boldsymbol{\nabla}) = 2\mathrm{i}(\mathbf{X} \times \boldsymbol{\nabla}\Omega).\boldsymbol{\nabla}$ and thus (2.3.8) we get:

$$[\tilde{\boldsymbol{S}}, (\boldsymbol{\mathcal{D}}_{\mathbf{A}}(\Omega\cdot))^2] = 2\Omega^2(\nabla_i X^j)\sigma_j \nabla^i + 2\Omega\boldsymbol{\sigma}.\big((\mathbf{X} \times \boldsymbol{\nabla}\Omega) \times \boldsymbol{\nabla}\big) + 2\Omega\boldsymbol{\sigma}.(\boldsymbol{\nabla}\Omega \times (\mathbf{X} \times \boldsymbol{\nabla})) + 0s \tag{2.3.12}$$

where by $0s$ we denote the "zero-order terms" (with respect to differentiation).

Considering the triple cross product formula: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -(\mathbf{c}.\mathbf{b})\mathbf{a} + (\mathbf{c}.\mathbf{a})\mathbf{b}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}.\mathbf{a})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$ and the anti-commutative property of the exterior product we get (after setting $\mathbf{a} = \mathbf{X}$, $\mathbf{b} = \boldsymbol{\nabla}\Omega$, $\mathbf{c} = \boldsymbol{\nabla}$):

$$[\tilde{\boldsymbol{S}}, (\boldsymbol{\mathcal{D}}_{\mathbf{A}}(\Omega\cdot))^2] = 2\Omega^2(\nabla_i X_j)\sigma^j \nabla^i \cdot + 2\Omega\boldsymbol{\sigma}.\big((\mathbf{X}.\boldsymbol{\nabla}\cdot)\boldsymbol{\nabla}\Omega\big) - 2\Omega\boldsymbol{\sigma}.\big(\mathbf{X}.\boldsymbol{\nabla}\Omega\big)\boldsymbol{\nabla} \cdot + 0s \tag{2.3.13}$$

where:

$$2\Omega\boldsymbol{\sigma}.\big((\mathbf{X}.\boldsymbol{\nabla})\boldsymbol{\nabla}\Omega\big) = 2\Omega\boldsymbol{\sigma}.\big(\boldsymbol{\nabla}\Omega(\mathbf{X}.\boldsymbol{\nabla})\big) = 2\Omega\boldsymbol{\sigma}.\big((\boldsymbol{\nabla}\Omega)X_i\nabla^i\big) = 2\Omega X_i(\boldsymbol{\nabla}\Omega)_j\sigma^j\nabla^i \tag{2.3.14}$$

and

$$-2\Omega\boldsymbol{\sigma}.\big(\mathbf{X}.\boldsymbol{\nabla}\Omega\big)\boldsymbol{\nabla} = -2\Omega\big(X_j\nabla^j\Omega)\big)\sigma_i\nabla^i \tag{2.3.15}$$

so

$$[\tilde{\boldsymbol{S}}, (\boldsymbol{\mathcal{D}}_{\mathbf{A}}(\Omega\cdot))^2] = 2\Omega\Big(\big(\Omega\nabla_i X_j + X_i(\boldsymbol{\nabla}\Omega)_j\big)\sigma^j\nabla^i - (X_k\nabla^k\Omega)\sigma_i\nabla^i + 0s\Big), \tag{2.3.16}$$

In order for $\tilde{\boldsymbol{S}}$ and $(\boldsymbol{\mathcal{D}}_{\mathbf{A}}(\Omega\cdot))^2$ to commute we require:

$$\Omega\nabla_i X_j + X_i\nabla_j\Omega - (X_k\nabla^k\Omega)\delta_{ij} = 0. \tag{2.3.17}$$

In particular $\Omega\nabla_i X_j + X_i\nabla_j\Omega = 0$ for all $i \neq j$.

However, $\mathbf{X}.\boldsymbol{\nabla}\Omega = X_k\nabla^k\Omega$ and so equation (2.3.17) for $i = j$ simplifies to

$$\big(\nabla_i(\Omega\mathbf{X})_i\big) = \mathbf{X}.\boldsymbol{\nabla}\Omega. \tag{2.3.18}$$

and summing it all over i we obtain $\boldsymbol{\nabla}.(\Omega\mathbf{X}) = 3\mathbf{X}.\boldsymbol{\nabla}\Omega$ or equivalently:
$\Omega\boldsymbol{\nabla}.\mathbf{X} - 2\mathbf{X}.\boldsymbol{\nabla}\Omega = 0 \implies \Omega^{-2}\boldsymbol{\nabla}.\mathbf{X} - 2\Omega^{-3}\mathbf{X}.\boldsymbol{\nabla}\Omega = 0$ or simply

$$\boldsymbol{\nabla}.(\Omega^{-2}\mathbf{X}) = 0. \tag{2.3.19}$$

Returning to equation (2.3.17) for $i \neq j$ and setting a the RHS of it and b the RHS of the "same" equation but with i and j reversed we get, from standard linear algebra that (2.3.17) holds iff $a - b = a + b = 0$, i.e. $a = b = 0$, in particular (the case of subtraction):

$$\Omega\nabla_i X_j + X_i\nabla_j\Omega - \Omega\nabla_j X_i - X_j\nabla_i\Omega = 0.$$

Multiplying the last equation by Ω^{-2} and grouping together the terms that include the same partial derivatives, we get: $\nabla_i(\Omega^{-1}X_j) - \nabla_j(\Omega^{-1}X_i) = 0$ or equivalently

$$\boldsymbol{\nabla} \times \big(\Omega^{-1}\mathbf{X}\big) = 0. \tag{2.3.20}$$

Now, in the case of addition, we get:

$$\Omega\nabla_i X_j + X_i\nabla_j\Omega + \Omega\nabla_j X_i + X_j\nabla_i\Omega = 2\delta_{ij}(\mathbf{X}.\boldsymbol{\nabla}\Omega)$$

or equivalently (since the RHS of this equation equation $(2/3)\delta_{\mathrm{ij}}(\boldsymbol{\nabla}.\mathbf{X})$):

$$\nabla_{\mathrm{i}}(\Omega \mathrm{X}_{\mathrm{j}}) + \nabla_{\mathrm{j}}(\Omega \mathrm{X}_{\mathrm{i}}) = \frac{2}{3}\delta_{\mathrm{ij}}\boldsymbol{\nabla}.(\Omega\mathbf{X}). \tag{2.3.21}$$

This equation is precisely the condition that defines a conformal Killing field $(\Omega\mathbf{X})$.

To prove that the magnetic field $\mathbf{B}$ should be parallel to $\mathbf{X}$ recall:

$$\begin{aligned}
[\tilde{\boldsymbol{S}}, \boldsymbol{\mathcal{D}}^2_{\mathbf{A}}] &= [\tilde{\boldsymbol{S}}, \textstyle\sum_{i=1}^3(-\mathrm{i}\nabla_i - A_i)^2 - \boldsymbol{\sigma}.\mathbf{B}] = [\sigma_i X^i, \textstyle\sum_{j=1}^3(-\mathrm{i}\nabla_j - A_j)^2] - [\sigma_i X^i, \sigma_j B^j] \\
&= [\sigma_i X^i, -\Delta + 2\mathrm{i}A_j\nabla^j + \textstyle\sum_{j=1}^3 A_j^2 + \mathrm{i}\boldsymbol{\nabla}.\mathbf{A}] - \mathrm{i}\boldsymbol{\sigma}.(\mathbf{X}\times\mathbf{B}) \\
&= -[\sigma_i X^i, \Delta] + 2\mathrm{i}[\sigma_i X^i, A_j\nabla^j] + [\sigma_i X^i, \textstyle\sum_{j=1}^3 A_j^2] + \mathrm{i}[\sigma_i X^i, \boldsymbol{\nabla}.\mathbf{A}] - \mathrm{i}\boldsymbol{\sigma}.(\mathbf{X}\times\mathbf{B}) \\
&= -[\sigma_i X^i, \Delta] + 2\mathrm{i}[\sigma_i X^i, A_j\nabla^j] - \mathrm{i}\boldsymbol{\sigma}.(\mathbf{X}\times\mathbf{B}) \\
&= 2\sigma_i\nabla_j X^i\nabla^j + \sigma_i\Delta X^i + 2\mathrm{i}\sigma_i A_j\nabla^j X^i - \mathrm{i}\boldsymbol{\sigma}.(\mathbf{X}\times\mathbf{B})
\end{aligned}$$

$$\implies \Omega^2[\tilde{\boldsymbol{S}}, \boldsymbol{\mathcal{D}}^2_{\mathbf{A}}] = 2\Omega^2\mathrm{i}\sigma_i A_j\nabla^j X^i - \Omega^2\mathrm{i}(\boldsymbol{\sigma}.(\mathbf{X}\times\mathbf{B})) + \Omega^2\sigma_i\Delta X^i + 1s. \tag{2.3.22}$$

here by $1s$ we denote the first-order (regarding differentiation) terms. We proceed:

$$\begin{aligned}
\Omega[\tilde{\boldsymbol{S}}, \boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}\Omega]\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} &= -\mathrm{i}\Omega[\sigma_i X^i, \sigma_j\nabla^j\Omega](\boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}} - \mathbf{A})) = 2\Omega(\boldsymbol{\sigma}.(\mathbf{X}\times\boldsymbol{\nabla}\Omega))(\boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}} - \boldsymbol{A})) \\
&= -2\Omega(\boldsymbol{\sigma}.(\mathbf{X}\times\boldsymbol{\nabla}\Omega))(\boldsymbol{\sigma}.\mathbf{A}) + 2\Omega(\boldsymbol{\sigma}.(\mathbf{X}\times\boldsymbol{\nabla}\Omega))\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}, \text{ so:}
\end{aligned}$$

$$\Omega[\tilde{\boldsymbol{S}}, \boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}\Omega]\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} = -2\Omega\mathrm{i}\boldsymbol{\sigma}.((\mathbf{X}\times\boldsymbol{\nabla}\Omega)\times\mathbf{A}) - 2\Omega(\mathbf{X}\times\boldsymbol{\nabla}\Omega).\mathbf{A} + 1s. \tag{2.3.23}$$

$$\begin{aligned}
\Omega(\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}\Omega)[\tilde{\boldsymbol{S}}, \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}] &= -\Omega\mathrm{i}(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega)(2\boldsymbol{\sigma}.(\mathbf{X}\times\boldsymbol{\nabla}) - 2\mathrm{i}\boldsymbol{\sigma}.(\mathbf{X}\times\mathbf{A}) + \mathrm{i}\boldsymbol{\nabla}.\mathbf{X} - \boldsymbol{\sigma}.(\boldsymbol{\nabla}\times\mathbf{X}) \\
&= -\Omega\mathrm{i}(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega)(-2\mathrm{i}\boldsymbol{\sigma}.(\mathbf{X}\times\mathbf{A}) + \mathrm{i}\boldsymbol{\nabla}.\mathbf{X} - \boldsymbol{\sigma}.(\boldsymbol{\nabla}\times\mathbf{X}) + 1s \\
&= -2\Omega(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega)(\boldsymbol{\sigma}.(\mathbf{X}\times\mathbf{A})) + \Omega(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega)\boldsymbol{\nabla}.\mathbf{X} + \Omega\mathrm{i}(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega)(\boldsymbol{\sigma}.(\boldsymbol{\nabla}\times\mathbf{X})) + 1s
\end{aligned}$$

$$\begin{aligned}
\implies \Omega(\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}\Omega)[\tilde{\boldsymbol{S}}, \boldsymbol{\mathcal{D}}_{\mathbf{A}}] &= -2\Omega\boldsymbol{\sigma}.(\boldsymbol{\nabla}\Omega\times(\mathbf{X}\times\mathbf{A})) - 2\Omega(\boldsymbol{\nabla}\Omega.(\mathbf{X}\times\mathbf{A})) + \\
&\Omega(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega)\boldsymbol{\nabla}.\mathbf{X} - \Omega\sigma.(\boldsymbol{\nabla}\Omega\times(\boldsymbol{\nabla}\times\mathbf{X})) + \Omega\mathrm{i}(\boldsymbol{\nabla}\Omega.(\boldsymbol{\nabla}\times\mathbf{X})) + 1\mathrm{s}
\end{aligned} \tag{2.3.24}$$

and lastly:

$$-2\Omega[\tilde{\boldsymbol{S}}, \boldsymbol{\nabla}\Omega.\boldsymbol{\nabla}] = 2\Omega\sigma_i(\nabla_j\Omega)\nabla^j X^i \tag{2.3.25}$$

by standard differential geometry, the terms $-2\Omega(\mathbf{X}\times\boldsymbol{\nabla}\Omega).\mathbf{A}$ and $-2\Omega(\boldsymbol{\nabla}\Omega).(\mathbf{X}\times\mathbf{A})$ cancel each other out. We also notice that

$$\boldsymbol{\nabla}\Omega.(\boldsymbol{\nabla}\times\mathbf{X}) = \boldsymbol{\nabla}\Omega.(\boldsymbol{\nabla}\times\big(\Omega(\Omega^{-1}X)\big)) = \boldsymbol{\nabla}\Omega.(\Omega\boldsymbol{\nabla}\times\big(\Omega^{-1}\mathbf{X}\big)) + \boldsymbol{\nabla}\Omega.(\boldsymbol{\nabla}\Omega\times(\Omega^{-1}\mathbf{X}))$$

and since $\Omega^{-1}\mathbf{X}$ is irrotational and $\boldsymbol{\nabla}\Omega\times(\Omega^{-1}\mathbf{X})\perp\boldsymbol{\nabla}\Omega$ we get $\boldsymbol{\nabla}\Omega.(\boldsymbol{\nabla}\times X)=0$.

We proceed by grouping together the remaining terms that do not include products with the components of $\mathbf{A}$ or $\mathbf{B}$. For the component of σ_i we see that the following is true:

$$\Omega(\Omega\Delta \mathrm{X_i} + (\nabla_\mathrm{i}\Omega)\boldsymbol{\nabla}.\mathbf{X} - (\boldsymbol{\nabla}\Omega\times(\boldsymbol{\nabla}\times\mathbf{X}))_\mathrm{i} + 2(\nabla_\mathrm{j}\Omega)\nabla^\mathrm{j}\mathrm{X_i})\sigma_\mathrm{i} = 0. \tag{2.3.26}$$

In fact, we have:

$$\begin{gathered}(\boldsymbol{\nabla}\Omega\times(\boldsymbol{\nabla}\times\mathbf{X}))_\mathrm{i} = (-1)^\mathrm{i}\big((\nabla_\mathrm{j}\Omega)(\boldsymbol{\nabla}\times\mathbf{X})_\mathrm{k} - (\nabla_\mathrm{k}\Omega)(\boldsymbol{\nabla}\times\mathbf{X})_\mathrm{j}\big) = \\ (-1)^i\big((\nabla_j\Omega)(-1)^k(\nabla_i X_j - \nabla_j X_i) - (\nabla_k\Omega)(-1)^j(\nabla_i X_k - \nabla_k X_i)\big) \\ = -(-1)^{i+k}\nabla_j\Omega\nabla_j X_i + (-1)^{j+k}\nabla_k\Omega\nabla_k X_i + (-1)^{i+k}\nabla_j\Omega\nabla_i X_j - (-1)^{j+k}\nabla_k\Omega\nabla_i X_k \\ = -\nabla_j\Omega\nabla^j X_i + \nabla_j\Omega\nabla_i X^j.\end{gathered}$$

The latest equality came from the fact that i, j, k fill the set $\{1,2,3\}$ while varying in a circular manner, so $\mathrm{k} = \mathrm{i} + 2\mathrm{mod}3 = \mathrm{j} + 1\mathrm{mod}3$ and we also added and subtracted $\nabla_i\Omega\nabla_j X_i$. Therefore, equation (2.3.26) just above becomes:

$$\Omega\Delta\mathrm{X_i} + (\nabla_\mathrm{i}\Omega)\boldsymbol{\nabla}.\mathbf{X} - \nabla_\mathrm{j}\Omega\nabla_\mathrm{i}\mathrm{X^j} + 3(\nabla_\mathrm{j}\Omega)\nabla^\mathrm{j}\mathrm{X_i} = 0.$$

We proceed by noticing that taking (2.3.17), multiplying it with $\nabla_i\Omega$ and summing all over i we get $\nabla_i\Omega\nabla^i X_j + X_i\nabla^i\Omega\nabla_j\Omega - X_k\nabla_i\Omega\nabla^k\Omega\delta^i_j = 0$ which implies $\nabla_i\Omega\nabla^i X_j = 0$ for each $j = 1,2,3$. That leaves us with

$$\Omega\Delta\mathrm{X_i} + (\nabla_\mathrm{i}\Omega)\boldsymbol{\nabla}.\mathbf{X} - \nabla_\mathrm{j}\Omega\nabla_\mathrm{i}\mathrm{X^j} = 0$$

which is produced by taking the i-th derivative of (2.3.17) and summing all over i and switching i and j, i.e. this terms evaluates at 0 as well.

After switching $\mathbf{A}$ with $\boldsymbol{\nabla}$ we notice the terms regarding the components A should satisfy the same equations as the first order (regarding differentiation) terms; therefore, it has already been proven that they equal 0.

The last remaining term is $-\Omega^2 \mathrm{i}\boldsymbol{\sigma}.(\mathbf{X} \times \mathbf{B})$, which is 0 iff $\mathbf{X} \times \mathbf{B} = 0$, i.e. iff $\mathbf{X}$ is parallel to $\mathbf{B}$. This completes the proof. □

2.3.2 Commuting diagonal first-order operators with Dirac operators

As we just saw, the set of zero-order hermitian operators that commute with the Weyl-Dirac operator (hence potentially shedding some light on the eigenstates of the latter) is very limited. Even when they do, we don't get zero modes for the Weyl-Dirac operator. So, we're going to start looking at first-order operators. Now, since we're interested in square-integrable eigenstates, we'll investigate whether "simple" first-order operators (with diagonal first-order terms), like the operator(s) Q in Erdős and Solovej, 2001 and Elton, 2018, commute with a suitably weighted version of the Weyl-Dirac operator, $\boldsymbol{\mathcal{D}}_{\mathbf{A}}^{\Omega} := \Omega^{-2}\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\cdot)$. In this case, the set of resulting commuting operators is still relatively small, but we get some fruitful results.

Standard operator theory yields that when two normal operators commute, they share a common basis for their eigenvectors. The operators $\boldsymbol{\mathcal{D}}_{\mathbf{A}}^{\Omega}$ and $Q_{\mathbf{X}}$ (see (2.3.32) below), are unbounded, but their respective resolvents $(\lambda_1 I_2 - \boldsymbol{\mathcal{D}}_{\mathbf{A}}^{\Omega})^{-1}$ and $(\lambda_2 I_2 - Q_{\mathbf{X}})^{-1}$, $\lambda_{1,2} \in \mathbb{C}\backslash\mathbb{R}$ are bounded (and normal since $\boldsymbol{\mathcal{D}}_{\mathbf{A}}^{\Omega}$ and $Q_{\mathbf{X}}$ are self-adjoint in $L^2_{\Omega}(\mathbb{R}^3, \mathbb{C}^2)$) (see (3.5.4)), and they commute with one another. Hence these operators share a common spectral resolution. Moreover, since we're working on a particular weighted version of $\mathbb{R}^3$ the Dirac operator has compact resolvent and so they have a common basis of eigenvectors, see Reed and Simon, 1981.

Also, given that $(\lambda_1 I_2 - \boldsymbol{\mathcal{D}}^{\Omega})^{-1}$ is compact, the vector field $\Omega^{-1}\mathbf{A}$ is bounded (see

(5.2.16) and the discussion right below) and $(\lambda_1 I_2 - \boldsymbol{\mathcal{D}}^{\Omega}_{\mathbf{A}})^{-1} = (\lambda_1 I_2 - \boldsymbol{\mathcal{D}}^{\Omega} + \Omega^{-1}\boldsymbol{\sigma}.\mathbf{A})^{-1}$ then the spectrum $\sigma(\lambda_1 I_2 - \boldsymbol{\mathcal{D}}^{\Omega} + \Omega^{-1}\boldsymbol{\sigma}.\mathbf{A})^{-1} \setminus \{0\}$ is equal to the respective point spectrum, $\sigma_p(\lambda_1 I_2 - \boldsymbol{\mathcal{D}}^{\Omega} + \Omega^{-1}\boldsymbol{\sigma}.\mathbf{A})^{-1} \setminus \{0\}$.

Complementary to prop. 5.2 i) in Elton, 2018, we have the following result:

Proposition 2.3.6. Consider non-negative functions Ω_1, Ω_2 on $\mathbb{R}^3$, positive in a region of interest. Let $\mathbf{X}$ a smooth vector field on $\mathbb{R}^3$, and a first-order operator $Q_{\mathbf{X}} = -\mathrm{i}(\mathbf{X}.\boldsymbol{\nabla}) + Q_0$, where Q_0 is some zero-order operator. Then for $Q_{\mathbf{X}}$ to commute with $\Omega_1 \boldsymbol{\mathcal{D}}_A(\Omega_2 \cdot)$, $\mathbf{X}$ must be a *conformal Killing field.* Moreover, this implies that $(\Omega_1\Omega_2)(\mathbf{x}) = c|\mathbf{X}(\mathbf{x})|$, for some positive constant c.

Proof. We have: $\Omega_1 \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega_2\psi) = -\mathrm{i}\Omega_1(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega_2)\psi + \Omega_1\Omega_2\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}\psi$ where ψ is a spinor on $\mathbb{R}^3$. This expression can be rewritten as:

$$-\mathrm{i}\Omega_1(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega_2) + \Omega_1\Omega_2\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} = -\mathrm{i}\Omega_1(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega_2) + \Omega_1\Omega_2\boldsymbol{\sigma}.(-\mathrm{i}\boldsymbol{\nabla} - \mathbf{A}).$$

Due to it being a zero-order operator acting on Weyl-Dirac spinors, Q_0 has the form $a_0 I_2 + Y_1\sigma_1 + Y_2\sigma_2 + Y_3\sigma_3 = a_0 I_2 + \boldsymbol{\sigma}.\mathbf{Y}$ (for $\mathbf{Y} = (Y_1, Y_2, Y_3) \in \mathbb{R}$ pointwise). Where a_0 and Y_i, $i \in \{1, 2, 3\}$ are (not necessarily constant) complex scalars.

We start by separating the brackets and identifying 2nd-order terms. By linearity of the commutator, we have:

$$[\Omega_1\boldsymbol{\mathcal{D}}_A(\Omega_2\cdot), Q_{\mathbf{X}}] = [-\mathrm{i}\Omega_1(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega_2) + \Omega_1\Omega_2\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}, Q_{\mathbf{X}}] =$$

$$[-\mathrm{i}\Omega_1(\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega_2) + \Omega_1\Omega_2\boldsymbol{\sigma}.(-\mathrm{i}\boldsymbol{\nabla} - \mathbf{A}), -\mathrm{i}(\mathbf{X}.\boldsymbol{\nabla}) + Q_0]$$

$$= [-\mathrm{i}\Omega_1\Omega_2\boldsymbol{\sigma}.\boldsymbol{\nabla}, -\mathrm{i}\mathbf{X}.\boldsymbol{\nabla}] + [-\mathrm{i}\Omega_1\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega_2, -\mathrm{i}\mathbf{X}.\boldsymbol{\nabla}] + [-\mathrm{i}\Omega_1\Omega_2\boldsymbol{\sigma}.\mathbf{A}, -\mathrm{i}\mathbf{X}.\boldsymbol{\nabla}]$$

$$+\Omega_1\Omega_2[\boldsymbol{\sigma}.(-\mathrm{i}\boldsymbol{\nabla} - \mathbf{A}), Q_0] + \Omega_1[-\mathrm{i}\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega_2, Q_0]$$

$$= -\Omega_1\Omega_2\sigma_i(\nabla^i X_j)\nabla^j + \Omega_1(\mathbf{X}.\boldsymbol{\nabla}\Omega_2)\boldsymbol{\sigma}.\boldsymbol{\nabla} + \Omega_1[-\mathrm{i}\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega_2, -\mathrm{i}\mathbf{X}.\boldsymbol{\nabla}] +$$

$$[-\mathrm{i}\Omega_1\Omega_2\boldsymbol{\sigma}.\mathbf{A}, -\mathrm{i}\mathbf{X}.\boldsymbol{\nabla}] + \Omega_1\Omega_2[-\mathrm{i}\boldsymbol{\sigma}.\boldsymbol{\nabla}, Q_0] + \Omega_1\Omega_2[-\boldsymbol{\sigma}.\mathbf{A}, Q_0] + \Omega_1[-\mathrm{i}\boldsymbol{\sigma}.\boldsymbol{\nabla}\Omega_2, Q_0]$$

where it is easy to see how the 2nd-order terms (who are only present in the first from

the right bracket in the last equation) cancel each other out.
Regarding the first-order terms, we have that the coefficient of ∇_j is

$$\sigma_i\Omega_1\Omega_2\nabla_i X_j - (X_i\nabla^i(\Omega_1+\Omega_2))\sigma_j - 2\epsilon_{ijk}\sigma_i\Omega_1\Omega_2 Y_k,$$

recall ϵ_{ijk} is the Levi-Civita symbol. Equivalently, we have the following form for the aforementioned first-order coefficient:

$$\Omega_1\Omega_2(\sigma_i\nabla_i X_j - (X_i(\Omega_1\Omega_2)^{-1}\nabla^i(\Omega_1\Omega_2))\sigma_j - 2\epsilon_{ijk}\sigma_i Y_k) =$$

$$\Omega_1\Omega_2(\sigma_i\nabla_i X_j - (X_i\nabla^i(\ln(\Omega_1\Omega_2)))\sigma_j - 2\epsilon_{ijk}\sigma_i Y_k).$$

Since $\Omega_1\Omega_2$ is positive on a region of interest, the latter quantity is zero iff:

$$\sigma_i\nabla_i X_j - (\mathbf{X}.\boldsymbol{\nabla}(\ln(\Omega_1\Omega_2)))\sigma_j - 2\epsilon_{ijk}\sigma_i Y_k = 0. \tag{2.3.27}$$

Multiplying this by σ_i and summing for all $i \in \{1,2,3\}$ we have:

$$\nabla_i X_j - 2\epsilon_{ijk}Y_k = 0,$$

if $i \neq j$. We can multiply by ϵ_{ijk} and add for all distinct i, j and get $\epsilon_{ijk}\nabla_i X_j - 4Y_k = 0$ for all k other than i, j. This is equivalent to: $\mathbf{Y}(\mathbf{x}) = \dfrac{1}{4}\boldsymbol{\nabla}\times\mathbf{X}(\mathbf{x})$.

For $i = j$, we get:

$$\nabla_i X_i - (\mathbf{X}.\boldsymbol{\nabla}(\ln(\Omega_1\Omega_2))) = 0,$$

where by summing for all $i \in \{1,2,3\}$ we get

$$\boldsymbol{\nabla}\cdot\mathbf{X} = 3(\mathbf{X}.\boldsymbol{\nabla}(\ln(\Omega_1\Omega_2))) \implies \mathbf{X}.\boldsymbol{\nabla}\ln(\Omega_1\Omega_2) = \frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{X}, \tag{2.3.28}$$

However, we notice that the solution to the differential equation (2.3.28) above (with unknown function $(\Omega_1\Omega_2)(\mathbf{x})$) is $|\mathbf{X}(\mathbf{x})|$ (up to a multiplicative constant). So, without loss of generality we can set $(\Omega_1\Omega_2)(\mathbf{x}) = c|\mathbf{X}(\mathbf{x})|$, for some $c \in \mathbb{R}_{>0}$ and (2.3.27) becomes:

$$\sigma_i\nabla_i X_j - (1/3)(\boldsymbol{\nabla}\cdot\mathbf{X})\sigma_j - 2\epsilon_{ijk}\sigma_i Y_k = 0. \tag{2.3.29}$$

By $\mathbf{Y}(\mathbf{x}) = (1/4)\boldsymbol{\nabla} \times \mathbf{X}(\mathbf{x})$ we get that for all distinct $i, j, k \in \{1, 2, 3\}$ we have:

$$Y_k = (1/4)\epsilon_{ijk}(\nabla_i X_j - \nabla_j X_i) \implies$$

$$-2\epsilon_{ijk} Y_k = -(1/2)(\nabla_i X_j - \nabla_j X_i).$$

So (2.3.27) becomes

$$\frac{1}{2}\sigma_i(\nabla_i X_j + \nabla_j X_i) = \frac{1}{3}\sigma_j(\boldsymbol{\nabla} \cdot \mathbf{X}).$$

Given that $i \neq j$, the linear independence of Pauli matrices gives us that

$$\nabla_i X_j + \nabla_j X_i = 0.$$

If, however, $i = j$ we have $\nabla_i X_i = \frac{1}{3}\boldsymbol{\nabla} \cdot \mathbf{X}$. Combining the last two relations we get:

$$\nabla_i X_j + \nabla_j X_i = \frac{2}{3}(\boldsymbol{\nabla} \cdot \mathbf{X})\delta_{ij}$$

which is precisely the condition that defines *Conformal Killing Fields* on $\mathbb{R}^3$ with the standard Euclidean metric. □

Following Elton, 2018 and the calculation we've made so far, we understand that a neat and useful choice for Q_0 is $\frac{1}{4}\boldsymbol{\sigma}.(\boldsymbol{\nabla} \times \mathbf{A}) - \frac{2\mathrm{i}}{3}\boldsymbol{\nabla} \cdot \mathbf{X} - \mathbf{X}.\mathbf{A}$. Moreover, one can make the ansatz $Q_{\mathbf{X}} = \mathbf{X}.\hat{\boldsymbol{P}}_{\boldsymbol{A}} + \tilde{Q}_0$, where $\hat{\boldsymbol{P}}_{\boldsymbol{A}} := -\mathrm{i}\boldsymbol{\nabla} - \mathbf{A}$ is the magnetic momentum operator and for $\tilde{Q}_0 = \tilde{a}_0(\mathbf{x})I_2 + \boldsymbol{\sigma}.\mathbf{Y}(\mathbf{x})$ is just some suitable first order operator to be determined (in our case it's $(1/4)\boldsymbol{\nabla} \times \mathbf{X}(\mathbf{x})$). This operator is of great importance in mathematical physics as when squared it produces a *magnetic Schrödinger Operator*. The form of the condition (2.3.27) will remain intact. Lastly, in order for $Q_{\mathbf{X}}$ and $\Omega_1\boldsymbol{\mathcal{D}}_{\mathbf{A}}^{\Omega_2}$ to commute, we see by doing the respective calculation that the rest of the zero order terms have to satisfy:

$$\mathbf{X} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \mathrm{Re}(\tilde{a}_0) = 0, \tag{2.3.30}$$

and

$$\boldsymbol{\nabla}(\mathbf{X}.\boldsymbol{\nabla} \ln{(\Omega_2)} + \mathrm{Im}(\tilde{a}_0)) + \frac{1}{4}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{X}) = 0. \tag{2.3.31}$$

Clearly, the *magnetic field*, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ need not necessarily be parallel to $\mathbf{X}$. However, for the sake of producing a commuting operator, $Q_{\mathbf{X}}$, as simple as possible, as well as generalizing the results in Erdős and Solovej, 2001, we can pick $\tilde{a}_0$ to be imaginary (i.e. $\mathrm{Re}(\tilde{a}_0) = 0$). In this case, its spectrum and eigenstates of $Q_{\mathbf{X}}$, (hence the eigenstates of $\Omega_1 \boldsymbol{\mathcal{D}}_{\mathbf{A}}^{\Omega_2}$ as well, if $(\Omega_1\Omega_2)(\mathbf{x}) = const.|\mathbf{X}(\mathbf{x})|$) are easier to find. So, for a *conformal Killing field* $\mathbf{X}$ and a *magnetic potential* $\mathbf{A}$ that produces a *magnetic field* $\mathbf{B}$, parallel to $\mathbf{X}$, we define the operator:

$$Q_{\mathbf{X}} := \mathbf{X}.(-\mathrm{i}\boldsymbol{\nabla} - \mathbf{A}) + \frac{1}{4}\boldsymbol{\sigma}.(\boldsymbol{\nabla} \times \boldsymbol{X}) - \frac{2\mathrm{i}}{3}\boldsymbol{\nabla} \cdot \mathbf{X}. \tag{2.3.32}$$

This operator acts on *Weyl-Dirac* (or just $\mathbb{C}^2$-)spinors. In Erdős and Solovej, 2001 we have that $\Omega_2(\mathbf{x}) = |\mathbf{X}|^{-1}$ and $\Omega_1(\mathbf{x}) = (\Omega_2(\mathbf{x}))^{-2} = |\mathbf{X}|^2$.

Moreover, for $\Omega := |\mathbf{X}|^{-1}$, the operator $Q_{\mathbf{X}}$ is self-adjoint on the *Sobolev space* $W_{\Omega}^{1,2}(\mathbb{R}^3)$, and to see that, it suffices to study the first-order term $-\mathrm{i}\nabla$. Considering, $\phi \in C_c(\mathbb{R}^3)$, and vol^3 (often denoted as $d^3\mathrm{vol}$) the standard form on $\mathbb{R}^3$, we have:

$$\int_{\mathbb{R}^3} \overline{-\mathrm{i}\mathbf{X}.\boldsymbol{\nabla}\psi}\phi\Omega^3\mathrm{vol}^3 = \int_{\mathbb{R}^3} \psi\Omega^3(-\mathrm{i}X_j\boldsymbol{\nabla}^j(\Omega^{-3}\phi))\Omega^3\mathrm{vol}^3.$$

And from the details of the previous proof, we get that the RHS is

$$\int_{\mathbb{R}^3} \big(\overline{\psi}(\mathbf{X}.(-\mathrm{i}\boldsymbol{\nabla}\phi)) + \overline{\psi}\phi(-\mathrm{i}\boldsymbol{\nabla}.\mathbf{X} - 3\mathrm{i}\Omega^{-1}\mathbf{X}.\boldsymbol{\nabla}\Omega)\big)\Omega^3\mathrm{vol}^3 =$$

$$\int_{\mathbb{R}^3} \big(\overline{\psi}(\mathbf{X}.(-\mathrm{i}\boldsymbol{\nabla}\phi)) - \mathrm{i}\overline{\psi}\phi\Omega^{-1}(\Omega\boldsymbol{\nabla}.\mathbf{X} + 3\mathbf{X}.\boldsymbol{\nabla}\Omega)\big)\Omega^3\mathrm{vol}^3 = \int_{\mathbb{R}^3} \overline{\psi}(\mathbf{X}.(-\mathrm{i}\boldsymbol{\nabla}\phi))\Omega^3\mathrm{vol}^3.$$

Regarding the divergence term in (2.3.32), we have:

Proposition 2.3.7. The term $-\frac{2\mathrm{i}}{3}\boldsymbol{\nabla} \cdot \mathbf{X}(\mathbf{x})$ does not contribute to the spectrum of the operator $Q_{\mathbf{X}}$ (they have equal spectrum). It only transforms its eigenvectors by a scalar multiplication by $|\mathbf{X}(\mathbf{x})|^2$.

Proof. The vector field $\mathbf{X} = (X_1, X_2, X_3)$ is a *Killing vector field*, i.e. it satisfies

$$\nabla_i X_j + \nabla_j X_i = \frac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\mathbf{X}$$

and we notice: $\mathbf{X}.\boldsymbol{\nabla}|\mathbf{X}|^2 = \mathbf{X}.\boldsymbol{\nabla}(X_1^2 + X_2^2 + X_3^2) =$

$$2(X_1, X_2, X_3).\begin{pmatrix} X_1\nabla_1 X_1 + X_2\nabla_1 X_2 + X_3\nabla_1 X_3 \\ X_1\nabla_2 X_1 + X_2\nabla_2 X_2 + X_3\nabla_2 X_3 \\ X_1\nabla_3 X_1 + X_2\nabla_3 X_2 + X_3\nabla_3 X_3 \end{pmatrix} = 2\sum_{i=1}^{3} X_i^2\nabla_i X_i +$$

$$2(X_1X_2\nabla_1 X_2 + X_1X_3\nabla_3 X_3 + X_1X_2\nabla_2 X_1 + X_2X_3\nabla_2 X_3 + X_3X_1\nabla_3 X_1 + X_3X_2\nabla_3 X_2)$$

However, the fact that $\mathbf{X}$ is conformal implies:

$$X_iX_j\nabla_i X_J + X_jX_i\nabla_j X_i = X_iX_j(\nabla_i X_J + X\nabla_j X_i) = 0,\ \ \forall\ \ i \neq j \in (1, 2, 3).$$

and

$$\nabla_i X_i = \frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{X}.$$

Hence

$$\mathbf{X}.\boldsymbol{\nabla}|\mathbf{X}|^2 = \frac{2}{3}|\mathbf{X}|^2\boldsymbol{\nabla}\cdot\mathbf{X}$$

and

$$-\frac{2\mathrm{i}}{3}\boldsymbol{\nabla}\cdot\mathbf{X} = -\mathrm{i}\frac{1}{|\mathbf{X}|^2}(\mathbf{X}.\boldsymbol{\nabla}|\mathbf{X}|^2) = -\mathrm{i}\mathbf{X}.\boldsymbol{\nabla}\ln\big(|\mathbf{X}|^2\big) = -\mathrm{i}\nabla_{\mathbf{X}}\ln(|\mathbf{X}|)$$

so if we make the transformation $u(\mathbf{x}) \mapsto |\mathbf{X}(\mathbf{x})|^2 u(\mathbf{x})$ $(= e^{\ln|\mathbf{X}|^2}u(\mathbf{x}))$ we get:

$$-\mathrm{i}\mathbf{X}.\boldsymbol{\nabla}\big(|\mathbf{X}|^2 u(\mathbf{x})\big) = -\mathrm{i}|\mathbf{X}|^2\mathbf{X}.\boldsymbol{\nabla}u(\mathbf{x}) - \mathrm{i}(\mathbf{X}.\boldsymbol{\nabla}|\mathbf{X}|^2)\boldsymbol{\nabla}u(\mathbf{x})$$

$$= -\mathrm{i}|\mathbf{X}|^2\mathbf{X}.\boldsymbol{\nabla}u(\mathbf{x}) - \frac{2\mathrm{i}}{3}(|\mathbf{X}|^2\boldsymbol{\nabla}\cdot\mathbf{X})\boldsymbol{\nabla}u(\mathbf{x}) = |\mathbf{X}|^2(-\mathrm{i}\mathbf{X}.\boldsymbol{\nabla} - \frac{2\mathrm{i}}{3}(\boldsymbol{\nabla}\cdot\mathbf{X})u(\mathbf{x}))$$

which equals $\lambda|\mathbf{X}|^2u(\mathbf{x})$ if λ and $u(\mathbf{x})$ (resp. $|\mathbf{X}|^2u(\mathbf{x})$) are a pair of eigenvalue-eigenvector $(\lambda, u(\mathbf{x}))$ (resp. $(\lambda, |\mathbf{X}|^2u(\mathbf{x}))$).

By considering the zero-order terms $-\mathbf{X}.\mathbf{A}$ and $\frac{1}{4}\boldsymbol{\sigma}.(\boldsymbol{\nabla}\times\boldsymbol{X})$ we get that the eigenvalues for the operator $Q_{\mathbf{X}}$ remain intact when we remove the term $\frac{2\mathrm{i}}{3}\boldsymbol{\nabla}\cdot\mathbf{X}$. □

Corollary 2.3.8. To describe the spectrum of the operator $Q_{\mathbf{X}}$ qualitatively, it suffices to study the spectrum of the operator

$$\tilde{Q}_{\mathbf{X}} := -\mathrm{i}\mathbf{X}.\boldsymbol{\nabla} + \frac{1}{4}\boldsymbol{\sigma}.(\boldsymbol{\nabla} \times \boldsymbol{X}). \tag{2.3.33}$$

The proof of this corollary follows from the previous two lemmas. In particular, the term $-\mathbf{X}.\mathbf{A}$ only adds a constant to the eigenvalues and transforms the eigenvectors by scalar multiplication.

Chapter 3: Summary of results

In this Chapter, we present the main results of the Thesis (without the proofs), starting with outlning the process of submersing $\mathbb{R}^3$ into $\mathbb{R}^2$ and finishing with the statement of the main Theorem, 5.2.5.

3.1 Outline of the reduction process

In this paragraph, we'll briefly describe the aforemention process of reducing the 3-dimensional space to 2 dimensions. In particular, we'll be working with weighted spaces of $\mathbb{R}^3$ and $\mathbb{R}^2$. The main reason is that the process we apply to submerse $\mathbb{R}^3$ to $\mathbb{R}^2$ does not preserve lengths between different the charts of $\mathbb{R}^3$ we'll be working with, so we'll need to introduce weights on $\mathbb{R}^2$, as well as on $\mathbb{R}^3$ that's not only "compatible" (length-preserving) with the aforementioned weights on $\mathbb{R}^2$, but also a suitable choice so that the respective weighted version of the Weyl-Dirac operator commutes with particular operators (see operator (2.3.32) and (2.3.6)) whose eigenstates are easier to study.

We construct the submersion map, $F : \mathbb{R}^3 \mapsto \mathbb{R}^2 \equiv \mathbb{C}$, is done by utilizing the symmetries of the magnetic field $\mathbf{A}(\mathbf{x})$, in particular, the magnetic potential, $\boldsymbol{\nabla} \times \mathbf{A}$ it produces. Since the latter is parallel to the *conformal Killing field* $\mathbf{X}(\mathbf{x})$, we'll use it's symmetries, such as the *integral curves* of $\mathbf{X}(\mathbf{x})$ and map every such curve to a (set of) point(s) on the plane.

This is done by considering two charts of $\mathbb{R}^3$: $\mathbb{R}^3_1 := \mathbb{R}^3 \setminus \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\}$ and $\mathbb{R}^3_2 := \mathbb{R}^3 \setminus \{(0, 0, x_3) \in \mathbb{R}^3\}$; which is $\mathbb{R}^3$ excluding the sets containing the singularities the components $\mathbf{W}_2(\mathbf{x})$ and $\mathbf{W}_1(\mathbf{x})$ of $\mathbf{X}(\mathbf{x})$ respectively.

Essentially, the map F is a collection of two maps $\mathbb{R}^3_{1,2} \mapsto \mathbb{R}^2$.

The way a point $\mathbf{x}$ in $\mathbb{R}^3_{1,2}$ are mapped to points on $\mathbb{C}$ is by first following the (flow of the) integral curve (denoted as $\gamma(t)$; $\gamma(0) = \mathbf{x}$ and is unique given $\mathbf{x} \in \mathbb{R}^3$), $\mathbf{x}$ belongs to until

a) $\boldsymbol{\gamma}(t) \in \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1;\}$ if we're working on $\mathbb{R}^3_1$, ($\mathbf{x} \in \mathbb{R}^3_1$) and then mapped to $\mathbb{C}$ using a radial function (denoted as f_1) while keeping the argument (phase) the same, and if otherwise

b) $\boldsymbol{\gamma}(t) \in \{(x_1, 0, x_3) : x_1 > 0\}$; then map the respective point on the half-plane $\{(x_1, 0, x_3) : x_1 > 0\}$, to the unit disk $\{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1;\}$ using a Cayley transform and lastly use a radially symmetric function f_2 to map it to $\mathbb{C}$ similar to the previous case.

The (generally multi-valued) maps that map the aforementioned curves onto $\mathbb{R}^2$ are: $\chi_{\kappa,1}, \chi_{\kappa^{-1},2} : \mathbb{R}^3_{1,2} \to \mathbb{C}$ (or simply denoted as $\chi_{1,2}$ if there is no need emphasizing the parameter κ) as:

$$\chi_{\kappa,1}(\mathbf{x}) = \frac{|\tilde{z}_1|}{1 + |\mathbf{x}|^2 + |\tilde{z}_2|}\tilde{e}(\tilde{z}_1)(\tilde{e}(\tilde{z}_2))^{-\kappa} \tag{3.1.1}$$

and

$$\chi_{\kappa^{-1},2}(\mathbf{x}) = \frac{1 + |\mathbf{x}|^2 + |\tilde{z}_2| - |\tilde{z}_1|}{1 + |\mathbf{x}|^2 + |\tilde{z}_2| + |\tilde{z}_1|}(\tilde{e}(\tilde{z}_1))^{-\frac{1}{\kappa}}\tilde{e}(\tilde{z}_2). \tag{3.1.2}$$

Using these maps, we can introduce functions (single-valued maps) $\chi^{\epsilon}_{\kappa,1} : \mathbb{R}^3_1 \to \mathbb{C}$ and $\chi^{\epsilon}_{\kappa^{-1},2} : \mathbb{R}^3_2 \to \mathbb{C}$ by considering particular branches, $\epsilon_{\kappa,1}$, $\epsilon_{\kappa^{-1},2}$, of the exponentials $\tilde{e}(\tilde{z}_1)(\tilde{e}(\tilde{z}_2))^{-\kappa}$ and $(\tilde{e}(\tilde{z}_1))^{-\frac{1}{\kappa}}\tilde{e}(\tilde{z}_2)$ respectively.

Then, these maps $\chi_{\kappa^{\pm 1}}$ are composed with functions $\tilde{F}_{\kappa^{\pm 1}} : \mathbb{D} \to \mathbb{C}$ respectively, of the form

$$\tilde{F}_{\kappa^{\pm 1}}(z_0) = z_0 f_{\kappa^{\pm 1}}(|z_0|^2), \quad z_0 \in \mathbb{D} \tag{3.1.3}$$

where $f_{\kappa^{\pm 1}} : [0, 1) \to \mathbb{R}_+$ satisfy:

$$\frac{f'_{\kappa^{\pm 1}}(s)}{f_{\kappa^{\pm 1}}(s)} = \frac{1}{2s}\Big(-1 + \sqrt{1 + \frac{(\kappa^{\pm 1})^2 s}{(1-s)^2}}\Big) \tag{3.1.4}$$

and we finally get the map(s) $F_{1,2} : \mathbb{R}^3_{1,2} \mapsto \mathbb{C}$

$$F_j(\mathbf{x}) = \chi_j(\mathbf{x}) f_{\kappa^{3-2j}}(\chi_j(\mathbf{x})) \tag{3.1.5}$$

where $\chi_1(\mathbf{x})$ is $\chi_{\kappa,1}(\mathbf{x})$ and $j = 1, 2$.

The aforementioned two copies of $\mathbb{R}^2$ (equiv. $\mathbb{C}$) are equipped with weights $\omega_{1,2}$ will be denoted as $(\mathbb{R}^2, \omega_{1,2})$ (equiv. $(\mathbb{C}, \omega_{1,2})$) respectively. Points on these two copies are identified by the (generally multivalued) map τ, which is formally defined taking values $\mathbb{C}^*$ to itself as:

$$\tau(z) = \kappa(1+\kappa)^{-1-1/\kappa} z^{-1/\kappa}, \quad \kappa > 0 \tag{3.1.6}$$

The union of these (weighted) copies will be noted as $\mathbb{O}_{\boldsymbol{\kappa}}$ (see (3.3.1) for full definition - and the discussion below it for a well-posed definition of the map τ).

The main work of this thesis consists firstly of defining Dirac operators on the aforementioned *orbit space* $\mathbb{O}_{\boldsymbol{\kappa}}$ (i.e. the space of orbits/integral curves of $\mathbf{X}$) and lifting them to respective Weyl-Dirac operators on $\mathbb{R}^3$ across a Riemannian-type submersion (i.e. submersion from one Riemannian manifold to another that respects the metrics - Do Carmo and Flaherty Francis, 1992) map F that maps points (in particular, curves) on $\mathbb{R}^3$ to (sets of) points on $\mathbb{C}$ (identified as $\mathbb{R}^2$). The derivation of the map F yields it to be multivalued; however, choosing different branches of a suitable Riemann surface, we are able to obtain well-defined, single-valued maps. This process includes defining/constructing forms, line and Spin^c bundles on $\mathbb{O}_\kappa$ and connections and Spin^c-connections on these bundles respectively; as well as *Clifford multiplication* that acts on the Spin^c bundles. Then we study how these building blocks are lifted to $\mathbb{R}^3$ across the map(s) F, (see also (4.1.60), (4.1.52)).

This work leads to an interesting relation between between (families of) Dirac operators on the orbit space $\mathbb{O}_\kappa$ and (families of) Weyl-Dirac operators on $\mathbb{R}^3$ (see (5.2.5)). This relation sheds light on the number of zero-modes in strong fields.

3.2 List of useful quantities and definitions

This section can be treated as collection of every non-standard tool used in this Thesis. We provide the definitions and formulas we've came up with in order to pursue the study of the aforementioned peculiar class of Dirac operators.

3.2.1 New variables and useful scalar functions

For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ we set the complex variables:

$$\tilde{z}_1(\mathbf{x}) \equiv \tilde{z}_1 := 2x_1 + 2x_2\mathrm{i} \tag{3.2.1}$$

and

$$\tilde{z}_2(\mathbf{x}) \equiv \tilde{z}_2 := 1 - |\mathbf{x}|^2 + 2x_3\mathrm{i}. \tag{3.2.2}$$

For $z \in \mathbb{C}^*$, we can define the function $\tilde{e} : \mathbb{C}^* \to C(0,1)$ (the unit circle on the complex plane) by:

$$\tilde{e}(z) := \frac{z}{|z|} = e^{\mathrm{i}\theta}, \quad \theta = Arg(z). \tag{3.2.3}$$

Recall that we'll be working with the vector field(s)

$$\mathbf{X}(\mathbf{x}) := \kappa_1 \mathbf{W}_1(\mathbf{x}) + \kappa_2 \mathbf{W}_2(\mathbf{x})$$

given $\kappa_{1,2} \in \mathbb{R}_{>0}$ which define the constant (parameter) κ by $\kappa := \kappa_1/\kappa_2$ and

$$\mathbf{X}^{ES}(\mathbf{x}) = \mathbf{W}_1(\mathbf{x}) + \mathbf{W}_2(\mathbf{x}). \tag{3.2.4}$$

The latter is the vector field considered in Erdős and Solovej, 2001.

Also, recall that the components of these vectors fields are given by

$$\mathbf{W}_1(\mathbf{x}) = (-2x_2, 2x_1, 0), \quad \mathbf{W}_2(\mathbf{x}) = (2x_1x_3, 2x_2x_3, 1 - |\mathbf{x}|^2 + 2x_3^2) \tag{3.2.5}$$

and they have fixed points on the x_3-axis and the unit circle on the x_1x_2-plane respectively (the variables $\tilde{z}_{1,2}$ are zero there respectively).

We have the weight $\Omega : \mathbb{R}^3 \to \mathbb{R}_+$

$$\Omega(\mathbf{x}) = \frac{1}{|\mathbf{X}(\mathbf{x})|} = \frac{1}{(\kappa_1^2|\mathbf{W}_1(\mathbf{x})|^2 + \kappa_2^2|\mathbf{W}_2(\mathbf{x})|^2)^{1/2}} \tag{3.2.6}$$

These allow us to rewrite:

$$\Omega(\mathbf{x}) = \frac{1}{(4(\kappa_1^2 - \kappa_2^2)(x_1^2 + x_2^2) + \kappa_2^2(1 + |\mathbf{x}|^2)^2)^{1/2}} = \frac{1}{\sqrt{\kappa_1^2|\tilde{z}_1|^2 + \kappa_2^2|\tilde{z}_2|^2}} \tag{3.2.7}$$

as since simple calculations show that $|\tilde{z}_1|^2 = |\mathbf{W}_1(\mathbf{x})|^2$ and $|\tilde{z}_2|^2 = |\mathbf{W}_2(\mathbf{x})|^2$.

We proceed by presenting some information on the submersion.

3.2.2 On the formula for the submersion

Recall that we map subsets $\mathbb{R}^3$ to $\mathbb{C} \equiv \mathbb{R}^2$ by submersing two charts [6] to $\mathbb{R}^2$:

$$\mathbb{R}_1^3 := \mathbb{R}^3 \setminus \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\} \tag{3.2.8}$$

using which we can define single-valued maps from $\mathbb{R}_{1,2}^3 \to \mathbb{C}$ and

$$\mathbb{R}_2^3 := \mathbb{R}^3 \setminus \{(0, 0, x_3) \in \mathbb{R}^3 : x_3 \in \mathbb{R}\}. \tag{3.2.9}$$

This submersion we'll be working with is denoted as: $F = \operatorname{Re} F + \mathrm{i} \operatorname{Im} F : \mathbb{R}^3 \to \mathbb{C} \equiv \mathbb{R}^2$, or more analytically as $F \equiv F_{1,2} : \mathbb{R}_{1,2}^3 \to \mathbb{C} \equiv \mathbb{R}^2$. In particular we have the following

Theorem 3.2.1. There exist surjective maps $F_{1,2} : \mathbb{R}_{1,2}^3 \to \mathbb{C}$, such that $\mathbf{X}.\boldsymbol{\nabla} F_{1,2} = 0$ and $|\boldsymbol{\nabla} \operatorname{Im} F_{1,2}| = |\boldsymbol{\nabla} \operatorname{Re} F_{1,2}|$.

Corollary 3.2.2. Up to a choice of a suitable branch of a Riemann surface due to the emerging multivaluedness, $F_{1,2}$ defines a Riemannian submersion from $(\mathbb{R}_{1,2}^3, \Omega)$ to $(\mathbb{C}, \omega_{1,2})$.

[6] The sets that have been excluded from the unit circle on the x_1x_2-plane and the x_3-axis are two integral curves for $\mathbf{X}$, each one even corresponding to the vector field $\mathbf{W}_1$ and $\mathbf{W}_2$ respectively.

Note that a Riemannian submersion, f, from a (Riemannian) manifold M to N (all smooth) is an isomorphism between the orthogonal complement of $\ker(df)$ and the tangent space(s) of N that is an isometry. See Do Carmo and Flaherty Francis, 1992 for more information. See the proof on 4.1.10 for more details, which also includes the following identities:

$$\boldsymbol{\nabla} F_{1,2}(\mathbf{x}) = F_{1,2}(\mathbf{x})(\mathbf{P}_{1,2}(\mathbf{x}) + \mathrm{i}\mathbf{Q}_{1,2}(\mathbf{x})) \tag{3.2.10}$$

where

$$\mathbf{Q}_1(\mathbf{x}) = \frac{2}{|\tilde{z}_1|^2}\mathbf{W}_1(\mathbf{x}) - \frac{2\kappa}{|\tilde{z}_2|^2}\mathbf{W}_2(\mathbf{x}) = \frac{2}{|\mathbf{W}_1(\mathbf{x})|^2}\mathbf{W}_1(\mathbf{x}) - \frac{2\kappa}{|\mathbf{W}_2(\mathbf{x})|^2}\mathbf{W}_2(\mathbf{x}) \tag{3.2.11}$$

and

$$\mathbf{Q}_2(\mathbf{x}) = -\frac{2}{\kappa|\tilde{z}_1|^2}\mathbf{W}_1(\mathbf{x}) + \frac{2}{|\tilde{z}_2|^2}\mathbf{W}_2(\mathbf{x}) = -\frac{1}{\kappa}\mathbf{Q}_1(\mathbf{x}), \tag{3.2.12}$$

while

$$\mathbf{P}_1(\mathbf{x}) = \frac{2}{|\tilde{z}_2|^2|\tilde{z}_1|^2}\sqrt{|\tilde{z}_2|^2 + \kappa^2|\tilde{z}_1|^2}\mathbf{W}_1(\mathbf{x})\times\mathbf{W}_2(\mathbf{x}) = \frac{2\kappa_2^{-1}|\mathbf{X}(\mathbf{x})|}{|\mathbf{W}_1(\mathbf{x})|^2|\mathbf{W}_2(\mathbf{x})|^2}\mathbf{W}_1(\mathbf{x})\times\mathbf{W}_2(\mathbf{x}). \tag{3.2.13}$$

Similarly, we get:

$$\mathbf{P}_2(\mathbf{x}) = -\frac{1}{\kappa}\mathbf{P}_1(\mathbf{x}) \tag{3.2.14}$$

where $\epsilon^b_{b'}(z)$ ($z \in \mathbb{C}^*$, $b \in \mathbb{R}$) denotes a branch of the exponential $\tilde{e}^b(z) := (z/|z|)^b$, indexed by b'. These functions are found by mapping each $\mathbf{x} \in \mathbb{R}^3_1$ (resp. $\mathbb{R}^3_2$), to the points where the integral curve that $\mathbf{x}$ belongs to and the unit (resp. the $\{(x_1, 0, x_3), x_1 > 0\}$) intersect (with the respective points on said half-plane then mapped to the unit disk via a Cayley transform).

3.2.3 Weights on the plane and further terms

In applications, we'll have $s \equiv s_{1,2}$ depending on whether we're working on $\mathbb{R}^3_{1,2}$, where:

$$s_{1,2} = |\chi_{1,2}(\mathbf{x})|^2. \tag{3.2.15}$$

We have the functions $h_{1,2} : [0, 1) \mapsto [0, \infty)$ defined as

$$h_{1,2}(s) := s f_{1,2}^2(s) \tag{3.2.16}$$

where $f_{1,2}$ satisfy 3.1.4, with $f_{1,2}(s) \to 1$ as $s \to 0$(for neatness) and are increasing (so invertible).

We also set:

$$\xi_1 = \frac{2\sqrt{s_1}}{1 - s_1}, \quad \xi_2 = \frac{2\sqrt{s_2}}{1 - s_2}. \tag{3.2.17}$$

Equivalently

$$\xi_j = \Big(\frac{|\tilde{z}_1|}{|\tilde{z}_2|}\Big)^{3-2j}, \quad j = 1, 2. \tag{3.2.18}$$

The following "mass" terms will also be involved in our calculations:

$$m_{1,2}(w) := \frac{1}{\kappa_{2,1}} \frac{1 + h_{1,2}^{-1}(w)}{((1 - h_{1,2}^{-1}(w))^2 + 4\kappa^{\pm 2} h_{1,2}^{-1}(w)^2)^{1/2}} = \frac{1}{\kappa_{2,1}} \frac{1 + s_{1,2}}{\sqrt{(1 - s_{1,2})^2 + 4\kappa^{\pm 2} s_{1,2}}}. \tag{3.2.19}$$

where $w \geq 0$, and

$$M(\mathbf{x}) := \frac{|\mathbf{X}^{ES}(\mathbf{x})|}{|\mathbf{X}(\mathbf{x})|} = \frac{(|\mathbf{W}_1(\mathbf{x})|^2 + |\mathbf{W}_2(\mathbf{x})|^2)^{1/2}}{(\kappa_1^2 |\mathbf{W}_1(\mathbf{x})|^2 + \kappa_2^2 |\mathbf{W}_2(\mathbf{x})|^2)^{1/2}} = \frac{1 + |\mathbf{x}|^2}{(\kappa_1^2 |\tilde{z}_1|^2 + \kappa_2^2 |\tilde{z}_2|^2)^{1/2}}. \tag{3.2.20}$$

The latter can equivalently be written as:

$$M(\mathbf{x}) = \frac{(|\tilde{z}_1|^2 + |\tilde{z}_2|^2)^{1/2}}{(\kappa_1^2 |\tilde{z}_1|^2 + \kappa_2^2 |\tilde{z}_2|^2)^{1/2}} = \tag{3.2.21}$$

or equivalently,

$$M(\mathbf{x}) = \frac{(\xi_1^2 + 1)^{1/2}}{\kappa_2(\kappa^2 \xi_1^2 + 1)^{1/2}} = \frac{(1 + \xi_2^2)^{1/2}}{\kappa_1(1 + \kappa^{-2}\xi_2^2)^{1/2}}. \tag{3.2.22}$$

The last two equations satisfy:

$$m_{1,2}(F_{1,2}(\mathbf{x})) = M(\mathbf{x}) \tag{3.2.23}$$

This follows by substituting equations (3.2.15) to (3.2.22) and doing simple calculations.

The weights on $\mathbb{R}^2$ satisfying:

$$\omega_j(w) = \frac{1}{\kappa_{j'}} \frac{\sqrt{h_j^{-1}(w)}(1 - h_j^{-1}(w))}{\sqrt{w}((1 - h_j^{-1}(w))^2 + 4\kappa^{2(3-2j)} h_j^{-1}(w))}, \quad w \geq 0 \tag{3.2.24}$$

Here, h_j^{-1} $(j = 1, 2)$ is the inverse of $h_j : [0, 1) \mapsto \mathbb{R}_{\geq 0}$ defined as $h_{1,2}(s) = s f_{1,2}^2(s)$ which is equal to $|\tilde{F}_{\kappa^{3-2j}}(\chi_j(\mathbf{x}))|^2$ if $s = |\chi_j(\mathbf{x})|^2$.

The weight(s) $\omega_{1,2}$ above may also be written in complex coordinates as

$$\omega_{1,2}^c(z, \overline{z}) := \omega_{1,2}(|z|^2). \tag{3.2.25}$$

Slightly abusing notation, (3.2.24) (and (3.2.25) resp.), may be denoted simply as $\omega_{1,2}^c(z)$ (resp. $\omega_{1,2}(x, y)$ for $x, y \in \mathbb{R}^2$ corresponding to $z = x + \mathrm{i}y$) depending on whether we're working on $\mathbb{C}$ or $\mathbb{R}^2$.

Regarding the aforementioned weights we have:

Proposition 3.2.3. The weights $\omega_{1,2}$ and Ω on $\mathbb{R}^2$ and $\mathbb{R}^3$ and the maps $F_{1,2}$ satisfy:

$$\omega_1(|F_1(\mathbf{x})|^2)|\boldsymbol{\nabla} \operatorname{Re} F_1(\mathbf{x})| = \omega_2(|F_2(\mathbf{x})|^2)|\boldsymbol{\nabla} \operatorname{Re} F_2(\mathbf{x})| = \Omega(\mathbf{x}). \tag{3.2.26}$$

Also, the additional equation are satisfied to ensure that the map(s) F define a partial isometry/Riemannian submersion:

$$|\boldsymbol{\nabla} \operatorname{Re} F_{1,2}(\mathbf{x})| = |\boldsymbol{\nabla} \operatorname{Im} F_{1,2}(\mathbf{x})|. \tag{3.2.27}$$

In particular, the weights $\omega_{1,2}$ satisfy:

$$\omega_1(|z|^2) = \frac{\omega_2(|\tau(z)|^2)}{|z|^{1+\frac{1}{\kappa}}}, \quad z \neq 0. \tag{3.2.28}$$

The latter follows from the fact that $F_2 = \tau(F_1)$, which implies

$$\boldsymbol{\nabla} F_2(\mathbf{x}) = \tau'(F_1(\mathbf{x}))\boldsymbol{\nabla} F_1(\mathbf{x}) = C_\kappa F_1^{-1-\frac{1}{\kappa}}(\mathbf{x})\boldsymbol{\nabla} F_1(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}_1^3 \cap \mathbb{R}_2^3 \tag{3.2.29}$$

where $C_\kappa = \kappa(1+\kappa)^{-1-\frac{1}{\kappa}}$.

3.3 The building blocks of the Weyl-Dirac operator on orbifold-like spaces

3.3.1 Orbit spaces

In this paragraph, we look at the aforementioned quantities in more detail and introduce some more. The Weyl-Dirac operator is typically defined on Riemannian manifolds, where the underlying geometry is well-understood. Here, we define on more generalized spaces like *pq-football* type of orbifolds and certain *non-Hausdorff* spaces. However, we first need to define the objects that make up the operator: the base space (also referred to as the *orbit space*), forms on that space, line bundles and connections thereon, and *Spin*^{c} bundles and corresponding *Spin*^{c} connections and *Clifford multiplication* on the aforementioned bundles.

Definition 3.3.1. Let $\kappa := \kappa_1/\kappa_2 \in \mathbb{R}_{>0}$ and $j \in \{1, 2\}$. We *define* the spaces $\mathbb{C}_{\kappa^{3-2j}}$ as the complex plane equipped with the weighted Euclidean metrics, with corresponding weight(s) ω_j, see (4.2.15), (4.2.16). respectively

We'll also be working with certain topological spaces, including what we'll refer to as the *orbit space* (which is going to be the space the integral curves/orbits of $\mathbf{X}$ "live") we have:

Definition 3.3.2. The complex plane(s) $\mathbb{C}$ quotient-ed by $e^{2\pi i\kappa^{3-2j}}$ will be denotes as $\mathbb{O}_{\kappa^{3-2j}}$. The pair (collection) of these two spaces, will be denoted $\mathbb{O}_\kappa$ and will be referred to as the *orbit space.*

The maps $F_{1,2}$ (see the beginning of subsection 3.2.2) map points (curves of $\mathbf{X}(\mathbf{x})$) in $\mathbb{R}^3_{1,2}$ to (sets of) points z on $\mathbb{C}$, related via certain rotations. Moreover, given a point $\mathbf{x} \in \mathbb{R}^3_1 \cap \mathbb{R}^3_2$, the maps $F_{1,2}$ map it to (generally distinct) points $z_{1,2}$ respectively; we'll identify these points by the following map $\tau : \mathbb{C}_\kappa \to \mathbb{C}_{1/\kappa}$ (given a parameter $\kappa > 0$):

$$\tau(z) = \kappa(1+\kappa)^{-1-1/\kappa} z^{-1/\kappa}. \tag{3.3.1}$$

This map is multivalued for $\kappa \in \mathbb{Q}_{>0} \setminus \{1\}$, because if we write the complex number z

in its polar form $z = re^{i\theta}$:

$$z^{-1/\kappa} = r^{-1/\kappa}(e^{\mathrm{i}\theta})^{-1/\kappa} = \{r^{-1/\kappa}e^{-\mathrm{i}\frac{\theta}{\kappa}-\frac{2n\pi\mathrm{i}}{\kappa}} : n \in \mathbb{N}\} \tag{3.3.2}$$

i.e. the map $z \to z^{-\frac{1}{\kappa}}$ is a set-valued map; which has finitely many elements if $\kappa \in \mathbb{Q}_{>0}$ and infinitely many if $\mathbb{R}_{>0} \setminus \mathbb{Q}$. In other words, the logarithmic Riemann surface covers the complex plane finitely or infinitely many times (if $\kappa \in \mathbb{Q}_{>0}$ or $\mathbb{R}_{>0} \setminus \mathbb{Q}$). This means we have a make a choice for the value we're working with. We see that the aforementioned spaces $\mathbb{C}_{\kappa^{\pm 1}}$ (and their respective weighted versions) can be though as a branch (after being quotient-ed by the respective group actions) of the logarithmic function with arguments $e^{2\pi\kappa^{\pm 1}\mathrm{i}}$. In particular, the expression $z^{-\frac{1}{\kappa}}$ can be written as

$$z^{-\frac{1}{\kappa}} = e^{-\frac{1}{\kappa}\log(z)}$$

which allows us to make the natural choice of value to work with, by considering the principal branch of the logarithmic function. This removes the ambiguity of multivalued-ness and makes the map τ a well-defined, holomorphic map between (logarithmic) Riemann surfaces. See Hille, 2012 and Forster, 2012.

More generally, having made a particular choice of value (by choosing a particular branch of the corresponding Riemann surface - the one of the logarithm). In particular, given such a choice for τ, we have (at least formally):

$$F_2(\mathbf{x}) = \tau(F_1(\mathbf{x})), \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3_1 \cap \mathbb{R}^3_2 \tag{3.3.3}$$

Remark 3.3.3. If $\kappa \in \mathbb{Q}$, each space $\mathbb{O}_{\kappa^{\pm 1}}$ is an *orbifold*. In particular, $\mathbb{O}_{\boldsymbol{\kappa}} \equiv \mathbb{S}^2$ iff $\kappa = 1$ (see Erdős and Solovej, 2001). If $\kappa \in \mathbb{R}_{>0} \setminus \mathbb{Q}$, $\mathbb{O}_{\kappa^{\pm 1}}$ is not even a Hausdorff space. However, we'll still be able to do calculus on it using standard complex analysis. In any case, each such space is equivalent to the unit sphere (excluding a pole) in $\mathbb{R}^3$, quotiented by a group action generating a rotation.

3.3.2 Line bundles on $\mathbb{O}_{\boldsymbol{\kappa}}$

We'll work with *sections of line bundles*, indexed by $k_{1,2} \in \mathbb{Z}$, on our base space, which will then be used to produce the related *spin*^{c} bundles and their corresponding sections. In essence, the latter are the "direct" sum of two line bundles. Such sections are defined as follows:

Definition 3.3.4. Given $k_1, k_2 \in \mathbb{Z}$, we define a *section of the line bundle* $\tilde{L}_{2k_2\kappa_2+2k_1\kappa_1}$, as the collection of the maps $u_{2,1}$ satisfying:

$$\tilde{u}_1(e^{2\kappa\pi \mathrm{i}}z) = e^{2k_1\kappa\pi \mathrm{i}}\tilde{u}_1(z), \tag{3.3.4}$$

$$\tilde{u}_2(e^{\frac{2\pi \mathrm{i}}{\kappa}}z) = e^{\frac{2k_2}{\kappa}\pi \mathrm{i}}\tilde{u}_2(z), \tag{3.3.5}$$

$$\tilde{e}^{-k_2}(z_2)\tilde{u}_2(z_2) = \tilde{e}^{k_1}(z_1)\tilde{u}_1(z_1), \tag{3.3.6}$$

where $z_{1,2}$ satisfy:

$$z_2 = \tau(z_1) \tag{3.3.7}$$

Remark 3.3.5. Sections of this line bundle is the only object associated with the bundle that is useful to us, therefore we'll limit ourselves to just defining this; skipping the intricate definition of the line bundle on $\mathbb{O}_\kappa$ (which is a generalization of the respective line bundles on Erdős and Solovej, 2001). These sections (and their respective space) will be denoted $\tilde{u} \equiv (u_1, u_2) \in \Gamma_{\tilde{L}}(\mathbb{O}_{\boldsymbol{\kappa}})$, where the latter symbol denotes the space of such sections of the line bundle on $\mathbb{O}_\kappa$.

Given $z_{1,2} \in \mathbb{C}^*$ such that $z_2 = \tau(z_1)$, we have: $\tilde{u}_2(\tau(e^{2m\pi \mathrm{i}}z_1)) = \tilde{u}_2(e^{-\frac{2m\pi \mathrm{i}}{\kappa}}\tau(z_1)) = \tilde{u}_2(R_{-\frac{m}{\kappa}}\tau(z_1)) = \tilde{e}^{-2m\frac{k_2\pi}{\kappa}\mathrm{i}}\tilde{u}_2(\tau(z_1))$ which makes these rotations perfectly consistent with the definition of our orbit space.

The first two conditions are in order for the bundle to be *well-defined*, whereas the last condition captures the compatibility of the bundle(s) (by mapping fibres from one bundle to another) and so will be referred to as the *compatibility condition*. However, the two charts also need to be compatible with each other, and for that, they need to satisfy the last condition (3.3.6) respectively for z_1, z_2 belonging in each respective chart, satisfying $z_2 = \tau(z_1)$.

The fact $\mathbb{C} \setminus \{0\}$ is not simply connected (instead, it is homomorphic to a circle) indicates that transition maps from one circle that is quotient-ed by the group action $e^{2\kappa\pi \mathrm{i}}$ to another one quotient-ed by $e^{2\pi\mathrm{i}/\kappa}$ have to satisfy an equation of that form (3.3.6). This equation captures how many times the fibers $\tilde{u}_2$ of the (line) bundle on $\mathbb{O}_{\kappa^{-1}}$ rotate depending on the degree of the map τ and how many times rotation of the fibers $\tilde{u}_1$ of the (line) bundle on $\mathbb{O}_\kappa$ rotate as z_1 makes a full rotation around the origin.

Recall that this map τ is multivalued, unless $\kappa \neq 1$. Moreover, given any choice of value for $\tau(z_1)$ we make, will be consistent with the compatibility condition of the bundle, (3.3.6), given the well-defined condition, (3.3.5), as shown in (6.1.3).

In the case where we have the trivial bundle ($k_1 = k_2 = 0$), the aforementioned *sections* are simple, complex valued functions on $\mathbb{C}_{\kappa^{\pm 1}}$. The said sections of line bundles, produce the eigenvectors of the operator $-\mathrm{i}\mathbf{X}.\boldsymbol{\nabla}$, which is the truncation of order 1 of the operator $Q_{\mathbf{X}}$, (2.3.32). It is reasonable to start by studying these objects, in order to obtain information about the eigenvectors of our Dirac operators. Orthonormal sections of the *spin*^{c} bundles, which are produced by section of the aforementioned line bundles, act as basis for the solution space of the Dirac operator.

To see the correspondence: *sections of* $\tilde{L}_{2k_1\kappa_1+2k_2\kappa_2}$ $\leftrightarrow$ *eigenvectors of* $-\mathrm{i}\mathbf{X}.\boldsymbol{\nabla}$, we have the following:

Proposition 3.3.6. Let $k_{1,2} \in \mathbb{Z}$ and the associated (complex) line bundle $\tilde{L}_{2k_1\kappa_1+2k_2\kappa_2}$. Consider a section u of $\tilde{L}_{2k_1\kappa_1+2k_2\kappa_2}$. Then, letting $\mu = (2k_1+1)\kappa_1 + (2k_2+1)\kappa_2$, u can be lifted to a single function $P^{1,2}_{\mu-}u_{1,2} : \mathbb{R}^3_{1,2} \to \mathbb{R}$ that satisfies:

$$-\mathrm{i}\mathbf{X}.\boldsymbol{\nabla}\big(P^{1,2}_{\mu-}u_{1,2}\big)(\mathbf{x}) = \lambda(P^{1,2}_{\mu-}u_{1,2})(\mathbf{x}) \iff \lambda = 2k_1\kappa_1 + 2k_2\kappa_2 \tag{3.3.8}$$

Moreover, the respective lifting operator is given by

$$(P^j_{\mu-}u_j)(\mathbf{x}) = \tilde{e}^{k_{j'}+k_j\kappa^{3-2j}}(z_{j'})(F^*_j u_j)(\mathbf{x}), \quad j, j' \in \{1,2\}, \quad j \neq j'. \tag{3.3.9}$$

The proof diverges from the purpose of this thesis so it's not included.

This operator takes section of the line bundle $\tilde{L}_{2k_1\kappa_1+2k_2\kappa_2}$ and maps them to typical (smooth) $\mathbb{C}$-valued functions on $\mathbb{R}^3$.

Remark 3.3.7. The subscript "$\mu-$" of P^j, on the right-hand side of (3.3.9) simply denote that $\tilde{u}$ is a section of the line bundle $\tilde{L}_{\mu-(\kappa_1+\kappa_2)}$ ($\mu \equiv (2k_1+1)\kappa_1 + (2k_2+1)\kappa_2$). Respectively if $k_{1,2} \mapsto k_{1,2}+1$ we'll use the "$\mu+$" subscript ($\mu\pm = \mu \pm (\kappa_1+\kappa_2)$).

3.3.3 Forms and connections on line bundles

Definition 3.3.8. Let $n = 0, 1, 2$. A *global* n-form, $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2)$ on the base space, orbit-space ($\mathbb{O}_{\boldsymbol{\kappa}}$) is a form that is invariant under certain rotations and preserved under the transition map τ:

$$R^*_{\boldsymbol{\kappa}^{3-2j}}\tilde{\alpha}_j = \tilde{\alpha}_j, \quad j = 1, 2 \tag{3.3.10}$$

$$\tau^*\tilde{\alpha}_2 = \tilde{\alpha}_1 \tag{3.3.11}$$

Remark 3.3.9. A 0-form on the orbit space $\mathbb{O}_{\boldsymbol{\kappa}}$ is a collection of two complex-valued functions, defined on $\mathbb{C}$ each.

Example 3.3.10. The "mass" terms $m_{1,2}$, (3.2.19), on the plane and $M(\mathbf{x})$, 3.2.22 can be viewed as examples of 0-forms on $\mathbb{C}$ and $\mathbb{R}^3$ respectively. Moreover, they satisfy:

$$(F^*_{1,2}m_{1,2})(\mathbf{x}) := m_{1,2}(F_{1,2}(\mathbf{x})) = M(\mathbf{x}) \tag{3.3.12}$$

As well as the well-defined condition

$$(\tau^*m_2)(z_1) = m_2(\tau(z_1)) = m_2(z_2) = m_1(z_1)$$

where $z_{1,2} = F_{1,2}(\mathbf{x})$ when working on $\mathbb{R}^3$, which for $\mathbf{x} \in \mathbb{R}^3_1 \cap \mathbb{R}^3_2$ implies

$$(F^*_2m_2)(\mathbf{x}) = m_2(F_2(\mathbf{x})) = m_1(F_1(\mathbf{x})) = (F^*_1m_1)(\mathbf{x}) \tag{3.3.13}$$

Remark 3.3.11. The aforementioned forms, constitute a *global* forms, i.e. a form that can be defined throughout the *orbit space*. However, we need to define *connections* acting on the line bundles $\tilde{L}_{2k_1\kappa_1+2k_2\kappa_2}$ ($k_{1,2} \in \mathbb{Z}$), and consequently *spin*^{c} for this, we need *local* 1-forms (that act as the zero-order coefficients when composed with vector fields). That is, 1-forms that are defined on $\mathbb{C}$ and satisfy (3.3.10) but not necessarily (3.3.11). In particular, the local 1-forms on $\mathbb{O}_\kappa$, $\tilde{c} \equiv (\tilde{c}^1, \tilde{c}^2)$, that are of interest satisfy an equation of the form:

$$\tau^* \tilde{c}^2_z = -\frac{\lambda}{2\kappa_1}\tilde{\zeta}_z + \tilde{c}^1_z \tag{3.3.14}$$

where

$$\tilde{\zeta}_z = \frac{1}{2\mathrm{i}|z|^2}(\overline{z}dz - zd\overline{z}) = \frac{1}{2\mathrm{i}}\Big(\frac{1}{z}dz - \frac{1}{\overline{z}}d\overline{z}\Big), \tag{3.3.15}$$

This is the formula, 1-forms need to satisfy in order for us to construct well-defined connections on $\tilde{L}_\lambda$ that act as building blocks to *spin*^{c} connections. Moreover, we'll be working on weighted versions of $\mathbb{R}^2$, with weights $\omega_{1,2}$ as discussed before. Also, in the previous Chapter (in particular (2.1.5)), which introduce extra terms to the connection (see (3.4.15)). Therefore, by slightly *abusing notation*, the condition that 1-forms (who define the magnetic potential) needs to satisfy:

$$\tau^* \tilde{\alpha}^2_z = -\frac{\mu}{2\kappa_1}\tilde{\zeta}_z + \tilde{\alpha}^1_z, \quad \mu = (2k_1+1)\kappa_1 + (2k_2+1)\kappa_2. \tag{3.3.16}$$

Remark 3.3.12. ***Lifts of 1-forms via the map*** P
In the end of the previous sub-section, we introduced the maps P_λ ($k_{1,2} \in \mathbb{Z}$) act and lift sections of the bundle L_λ. It is apparent to see how these it behaves on objects like $(\tilde{c}\tilde{u})(\tilde{\mathbf{x}}) := (\tilde{c}_{\tilde{\mathbf{x}}}(\partial_x + \partial_y))\tilde{u}(x,y) \in \Omega^1(\tilde{U}) \otimes \Gamma(\tilde{U} \times \mathbb{C})$ where $\tilde{U}$ is a connected component of $\mathbb{C}$ and $\tilde{\mathbf{x}} = (x, y)$. Given such 1-form $\tilde{c}$ we have:

$$(P_\lambda(\tilde{c}\tilde{u}))(\mathbf{x}) = ((F^*\tilde{c}_{\tilde{\mathbf{x}}})(\partial_x + \partial_y))((P_\lambda \tilde{u})(\mathbf{x}))$$

where $P \equiv P_{2k_2\kappa_2 + 2k_1\kappa_1}$ and $\tilde{u} \in \tilde{L}_{2k_2\kappa_2+2k_1\kappa_1}$, and $\tilde{\alpha}$ satisfies (4.2.3) (with $\lambda = 2k_2\kappa_2 + 2k_1\kappa_1$). Where we've chosen a branch of $\tilde{e}(\tilde{z}_1)e^{-\kappa}(\tilde{z}_2)$, $e^{-1/\kappa}(\tilde{z}_1)\tilde{e}(\tilde{z}_2)$ on the map F, and the form $\tilde{\alpha}_{\mathbf{x}} \equiv \tilde{\alpha}_z$ is the (pair of) 1-form(s) on $\mathbb{C}$ satisfying (4.2.3).

Now, we'll look at *connections* on relevant *line bundles*. Given an open, connected $\tilde{U} \subseteq \mathbb{C}$ and $\tilde{\alpha} \in \Omega^1(\mathbb{C})$. A mapping $\tilde{\nabla}^{\tilde{\alpha}} : \mathfrak{X}(\tilde{U}) \times \Gamma(\tilde{U} \times \mathbb{C}) \mapsto \Gamma(\tilde{U} \times \mathbb{C})$ given by:

$$\tilde{\nabla}^{\tilde{\alpha}} = d - \mathrm{i}\tilde{\alpha} \tag{3.3.17}$$

defines a *connection* acting on $\Gamma(\tilde{U} \times \mathbb{C})$.

Moreover, a connection on a line bundle $\tilde{L}_\lambda$ can be defined as a map

$$\tilde{\nabla}^{\tilde{\alpha}} : \Gamma(\tilde{L}_\mu) \mapsto \Omega^1(\mathbb{O}_\kappa, \tilde{L}_\mu)$$

mapping sections of the line bundle $\tilde{L}_\mu$ to $\tilde{L}_\mu$-valued 1-forms on the base space $\mathbb{O}_\kappa$.

The "d" in equation (3.3.17) above denotes the differential (first-order coefficient) part of the connection. In particular, given an $\tilde{\boldsymbol{X}} = \tilde{X}_z\partial_z + \tilde{X}_{\bar{z}}\partial_{\bar{z}} \in \mathfrak{X}(U)$ and $u \in \Gamma(U \times \mathbb{C})$, we have the formula

$$\tilde{\nabla}^{\tilde{\alpha}}_{\tilde{\boldsymbol{X}}} u = \tilde{\boldsymbol{X}} u - \mathrm{i}\tilde{\alpha}(\tilde{\boldsymbol{X}})u. \tag{3.3.18}$$

Remark 3.3.13. Is it straightforward to see that for any $\tilde{\boldsymbol{X}} \in \mathfrak{X}(\mathbb{C})$, $\nabla^{\tilde{\alpha}}_{\tilde{\boldsymbol{X}}}$ satisfies,

$$\tilde{\boldsymbol{X}}\langle u_1, u_2\rangle_{\mathbb{C}} = \langle \nabla^{\tilde{\alpha}}_{\tilde{\boldsymbol{X}}} u_1, u_2\rangle_{\mathbb{C}} + \langle u_1, \nabla^{\tilde{\alpha}}_{\tilde{\boldsymbol{X}}} u_2\rangle_{\mathbb{C}}. \tag{3.3.19}$$

This equation confirms that $\nabla^{\tilde{\alpha}}_{\tilde{\boldsymbol{X}}}$ is, indeed, a connection. Recall $\langle u_1, u_2\rangle_{\mathbb{C}} = u_1\overline{u}_2$.

As a direct consequence of the definition of $P_\lambda \equiv P^j_\lambda$, $j \in \{1, 2\}$, we have:

Corollary 3.3.14. Let $\tilde{\alpha}_j \in \Omega^1(\mathbb{C})$ and $\alpha_j := F^*_j\tilde{a}^j \in \Omega^1(\mathbb{R}^3_j)$, $j = 1, 2$, we have:

$$P_\lambda(d - \mathrm{i}\tilde{a}_j)u = (d - \mathrm{i}\alpha_j)P_\lambda u. \tag{3.3.20}$$

Remark 3.3.15. Although the aforementioned corollary holds for any $\tilde{\alpha}_j \in \Omega^1(\mathbb{C})$ and related $\alpha_j := F^*_j\tilde{a}^j \in \Omega^1(\mathbb{R}^3_j)$, we'll use this result for local 1-forms $\tilde{\alpha}^j$ on $\Omega^1(\mathbb{O}_{\boldsymbol{\kappa}})$ that satisfy (3.3.10) and (3.3.14). See the proof of (5.1.11) for more details.

3.4 Spin^c bundles, spin^c connections and Clifford multiplication

Now that we've defined *line bundles* on the orbit space and connections acting on sections of them, we re ready to define Spin^c bundles (bundles whose sections that are $\mathbb{C}^2$-valued) and connections acting on sections of those, as well as $SU(2)$-automorphisms (Clifford multiplication). These will help us to define *Weyl-Dirac operators*.

Definition 3.4.1. Given, $\mu = (2k_1+1)\kappa_1 + (2k_2+1)\kappa_2$ $(k_{1,2} \in \mathbb{Z})$, the corresponding *Spin*^{c} bundles on $\mathbb{O}_{\boldsymbol{\kappa}}$ are given by the direct sum:

$$S_\mu = \tilde{L}_{\mu-\tilde{\kappa}} \oplus \tilde{L}_{\mu+\tilde{\kappa}}, \quad \tilde{\kappa} = \kappa_1 + \kappa_2. \tag{3.4.1}$$

The "sections" of the aforementioned bundle, are denoted as $\begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix}$.

The respective compatibility conditions (3.3.6) can be summarized as

$$\begin{pmatrix} \tilde{u}_1^+(z) \\ \tilde{u}_1^-(z) \end{pmatrix} = \begin{pmatrix} e^{k_1}(z)e^{k_2}(\tau(z)) & 0 \\ 0 & e^{k_1+1}(z)e^{k_2+1}(\tau(z)) \end{pmatrix} \begin{pmatrix} \tilde{u}_2^+(\tau(z)) \\ \tilde{u}_2^-(\tau(z)) \end{pmatrix}, \quad z \neq 0 \tag{3.4.2}$$

These sections can then be lifted to sections of the (trivial) *spin*^{c} bundle on $\mathbb{R}^3$ by considering the respective lifts $P_{\mu\pm}$ on each respective bundle $\tilde{L}_{\mu\pm} \equiv \tilde{L}_{\mu\pm\tilde{\kappa}}$ and the considering the "direct sum" of those.

Specifically, we define the map(s) $\mathcal{P}_\mu : \Gamma(L_\mu) \to \Gamma(\mathbb{R}^3 \times \mathbb{C}^2)$:

$$(\mathcal{P}_\mu \tilde{u})(\mathbf{x}) = U(\mathbf{x}) \begin{pmatrix} (P_{\mu-}\tilde{u}^+)(\mathbf{x}) \\ (P_{\mu+}\tilde{u}^-)(\mathbf{x}), \end{pmatrix} \tag{3.4.3}$$

where the matrix $U(\mathbf{x}) \in SU(2)$ (pointwise) is introduced in order to revert the resulting Dirac operator back into it's standard form (with the standard Pauli matrices). In particular, the aforementioned $SU(2)-$matrix $U(\mathbf{x})$ is given by:

$$U(\mathbf{x}) = U_0(\mathbf{x})U_1(\mathbf{x}) \tag{3.4.4}$$

with

$$U_0(\mathbf{x}) = \frac{1}{(8|\mathbf{X}(\mathbf{x})|v(\mathbf{x}))^{1/2}} \begin{pmatrix} \overline{w} & -\overline{\tilde{z}_1 w} \\ \tilde{z}_1 w & w \end{pmatrix}, \quad U_1(\mathbf{x}) = \begin{pmatrix} \mathrm{i}\gamma^e(\mathbf{x}) & 0 \\ 0 & -\mathrm{i}\overline{\gamma}^e(\mathbf{x}) \end{pmatrix} \tag{3.4.5}$$

where:

$$w = \kappa_2 x_3 + \kappa_1 \mathrm{i} \tag{3.4.6}$$

$$\gamma^e(\mathbf{x}) = \tilde{e}(|\mathbf{X}(\mathbf{x})| + \kappa_1|\mathbf{X}^{ES}(\mathbf{x})| + (\kappa_1 - \kappa_2)\tilde{z}_2) \tag{3.4.7}$$

and

$$v(\mathbf{x}) = \frac{\Omega^{-1}(\mathbf{x}) - \kappa_2(1 - |\mathbf{x}|^2 + 2x_3^2)}{4(x_1^2 + x_2^2)} = \frac{\Omega^{-1}(\mathbf{x}) - X_3(\mathbf{x})}{|\tilde{z}_1|^2} = \frac{|w|^2}{\Omega^{-1}(\mathbf{x}) + X_3(\mathbf{x})}. \tag{3.4.8}$$

More details can be found in the first section of *Chapter 5*.

The process of defining *Clifford multiplication* on $\mathbb{O}_{\boldsymbol{\kappa}}$ is done by first defining it as usual on the two charts, $\mathbb{O}_{\kappa^{3-2j}}$, of $\mathbb{O}_{\boldsymbol{\kappa}}$. Recall these charts are identified as the respective weighted versions of $\mathbb{C}$, quotient-ed by the corresponding group action. The definition of Clifford multiplication and then imposing the proper compatibility conditions.

Definition 3.4.2. Given a 1-form $\tilde{\alpha} \equiv \tilde{\alpha}^j_{\tilde{\boldsymbol{x}}} := \tilde{\alpha}^j_{\tilde{x}} d\tilde{x} + \tilde{\alpha}^j_{\tilde{y}} d\tilde{y} \in \Omega^1(\mathbb{O}_{\kappa^{3-2j}})$, $j = 1, 2$, we define the (weighted) *Clifford multiplication*, $\sigma_{\omega_j} : T^*\mathbb{O}_{\kappa^{3-2j}} \to End(\mathbb{S}_\mu)|_{\mathbb{O}_{\kappa^{3-2j}}}$ (the space of Endomorphisms of the *spin*^{c}-bundle S_μ, restricted on $\mathbb{O}_{\kappa^{3-2j}}$) as the traceless, self-adjoint isometry:

$$\sigma_{\omega_j}(\tilde{\alpha}^j_{\tilde{\boldsymbol{x}}}) = \omega_j^{-1} \begin{pmatrix} 0 & \tilde{\alpha}^j_{\tilde{x}} - \mathrm{i}\tilde{\alpha}^j_{\tilde{y}} \\ \tilde{\alpha}^j_{\tilde{x}} + \mathrm{i}\tilde{\alpha}^j_{\tilde{y}} & 0 \end{pmatrix}, \quad j = 1, 2 \tag{3.4.9}$$

where $\omega_j = \omega_j(\tilde{x}^2 + \tilde{y}^2)$ is introduced in (3.2.24). If $\tilde{\alpha}$ is a *global 1-form*, then the pair of these *sigma matrices*, as described above, will constitute what we define as the *Clifford multiplication* on S_μ, $\sigma_\omega : T^*\mathbb{O}_\kappa \to End(\mathbb{S}_\mu)|_{\mathbb{O}_{\boldsymbol{\kappa}}}$

This definition follows from choosing the standard Pauli matrices being chosen corresponding to the standard basis $\sigma_1 = \sigma(d\tilde{x})$ and $\sigma_2 = \sigma(d\tilde{y})$ and the condition that the map σ is linear. Equation (3.4.9) can be re-written as

$$\sigma_{\omega_j}(\tilde{\alpha}^j) = \omega_j^{-1}(\tilde{\alpha}^j_{\tilde{x}}\sigma_1 + \tilde{\alpha}^j_{\tilde{y}}\sigma_2),$$

with the (inverted) weights ω_j^{-1} being introduced so that the map(s) $\sigma_{\omega_j} : T^*\mathbb{O}_{\kappa^{3-2j}} \to End(\mathbb{S}_\mu)|_{\mathbb{O}_{\kappa^{3-2j}}}$ will be an isometry with respect to the (weighted) metric $\langle\cdot,\cdot\rangle_{\Omega^1(\mathbb{O}_{\kappa^{3-2j}})} := \omega_j^2\langle\cdot,\cdot\rangle_{\Omega^1(\mathbb{C})}$ on the co-tangent bundle. Given the definition of forms on $\mathbb{O}_{\boldsymbol{\kappa}}$ and the properties they satisfy, and considering real-valued 1-forms on $\mathbb{O}_{\boldsymbol{\kappa}}$, $\tilde{\alpha}^j := \tilde{\alpha}^j_- dz + \tilde{\alpha}^j_+ d\bar{z}$ (satisfying $\tilde{\alpha}^j_- = \overline{\tilde{\alpha}^j_+}$ so that the form is real-valued):

$$R_{\kappa^{3-2j}}\tilde{\alpha}^j_\pm = e^{\pm i2\pi\kappa^{3-2j}}\tilde{\alpha}^j_\pm, \quad j = 1, 2. \tag{3.4.10}$$

By imposing the conditions interlacing the basis of spinor bundles with the basis of the tangent bundle on the manifold, see Erdős and Solovej, 2001 (*Proposition 7*) we get that the index $(2k_1\kappa_1 + 2k_2\kappa_2)$ of the line bundles that compose the spin^c bundle must differ by $2\kappa_1 + 2\kappa_2$.

We proceed with elaborating on the structure of the *spin*^{c} bundles on $\mathbb{O}_{\boldsymbol{\kappa}}$ and set the matrix in (3.4.2):

$$\mathcal{E}_{k_1,k_2}(z) = \begin{pmatrix} \tilde{e}^{k_1}(z)\tilde{e}^{k_2}(\tau(z)) & 0 \\ 0 & \tilde{e}^{k_1+1}(z)\tilde{e}^{k_2+1}(\tau(z)) \end{pmatrix} \tag{3.4.11}$$

which maps sections of the *spin*^{c} bundle on $\mathbb{O}_\kappa$ to the respective sections on $\mathbb{O}_{\kappa^{-1}}$. We've defined it like this because if $z = z_1$ (which is what we consider when working towards the main result of this work), we have $Arg(z_1) = 2\kappa_1\pi \pmod{2\pi}$and $Arg(z_2) = Arg(\tau(z_1)) = 2\kappa_2\pi$, making the exponents $e^{k_j}(z_j)$ $(j = 1, 2)$ equal to $2k_j\kappa_j$ (and $e^{k_j+1}(z_j)$ equal to $(2k_j + 2)\kappa_j$) acting on sections of line bundle indexed by $2k_j$ (resp. $2k_j + 2$). This is part of a generalisation of the respective construction in the case of Erdős and Solovej, 2001, being consistent with/satisfying all the respective properties related to the Clifford multiplication.

We need to check whether *Clifford multiplication* defined on $\mathbb{O}_{\boldsymbol{\kappa}}$ is compatible with the (3.4.2), while respecting the (conformal) metrics (with weights $\omega_{1,2}$ respectively). We want:

$$\sigma_{\omega_1}(\tilde{\alpha}^1_z)\begin{pmatrix}\tilde{u}^+_1(z) \\ \tilde{u}^-_1(z)\end{pmatrix} = \mathcal{E}_{k_1,k_2}(z)\sigma_{\omega_2}(\tau^*\tilde{\alpha}^2_z)\begin{pmatrix}\tilde{u}^+_2(\tau(z)) \\ \tilde{u}^-_2(\tau(z))\end{pmatrix}$$

i.e.

$$\omega_1^{-1}(|z|^2)\sigma(\tilde{\alpha}_z^1)\begin{pmatrix}\tilde{u}_1^+(z)\\ \tilde{u}_1^-(z)\end{pmatrix} = \omega_2^{-1}(|\tau(z)|^2)\mathcal{E}_{k_1,k_2}(z)\sigma(\tau^*\tilde{\alpha}_z^2)\begin{pmatrix}\tilde{u}_2^+(\tau(z))\\ \tilde{u}_2^-(\tau(z))\end{pmatrix}$$

which given (3.4.2) is true iff

$$\mathcal{E}_{k_1,k_2}(z) = |\tau(z)|^{-1-\frac{1}{\kappa}}\sigma^{-1}(\tilde{\alpha}_z^1)\mathcal{E}_{k_1,k_2}(z)\sigma(\tilde{\alpha}_{\tau(z)}^2)$$

which can easily be verified by calculations considering the definition/behaviour of σ on 1-forms $\tilde{\alpha}^{1,2}$. Thus, we've showed that Clifford multiplication on S_μ is well-defined. Recall here that we have $\tau^*\tilde{\alpha}_z^2 = \tilde{\alpha}_z^1$. It is also easy to see that this σ_j respects (1.3.1) while still being an endomorphism

Note that on $\mathbb{C}^*$, for distinct $j, j' \in \{1,2\}$, the aforementioned matrix $\mathcal{E}_{k_1,k_2}$ satisfies:

$$\mathcal{E}_{k_1,k_2}(z) = \begin{pmatrix}\tilde{e}^{k_j}(\tilde{z}_j)\tilde{e}^{k_{j'}}(\tilde{z}_{j'}) & 0\\ 0 & \tilde{e}^{(k_j+1)}(\tilde{z}_j)\tilde{e}^{(k_{j'}+1)}(\tilde{z}_{j'})\end{pmatrix}, \quad \text{for } \ z = z_1 \tag{3.4.12}$$

on $\mathbb{R}_1^3 \cap \mathbb{R}_2^3$. This matrix can further be factorized as $\mathcal{E}_{k_1,k_2} = \mathcal{E}_{k_1}\mathcal{E}_{k_2}$ where

$$\mathcal{E}_k(z) = \begin{pmatrix}\tilde{e}^k(z) & 0\\ 0 & \tilde{e}^{k+1}(z)\end{pmatrix}. \tag{3.4.13}$$

Remark 3.4.3. It can be shown that the aforementioned *Spin*^{c} bundles, are (up to diffeomorphism) the only *Spin*^{c} bundles on $\mathbb{O}_{\boldsymbol{\kappa}}$. The proof of this is very closely related to the definition of *Clifford multiplication* below, which given the definition of forms, is the only natural definition of Clifford multiplication. A sketch of a proof is the following:

The dimension of the base space (and of the related tangent bundle, which is 2), combined with the fact that the $\mathbb{O}_\kappa \cap \mathbb{O}_{\kappa^{-1}}$ (the intersection of each chart of the base space) is topologicaly equivalent with the unit circle and with stardard decomposition techniques of unitary matrices allow us to classify (up to diffeomorphism) the bundles on $\mathbb{O}_{\boldsymbol{\kappa}}$, subject to the compatibility condition (3.4.2) written (roughly speaking) as "$U(1) \times$ (*diagonal* $U(2)$)". The "$U(1)$" factor is responsible for capturing the degree of the transition map from unit circle to itself, and the "diagonal U(2)" ensures that all the other relevant conditions are satisfied (in particular the correspondence of the basis of the bundle with a local basis of the base space, and the compatibility condition). These ideas follow from the respective proof in Erdős and Solovej, 2001.

Considering how 1-forms lift from the plane to $\mathbb{R}^3$ via F, we get that the aforementioned *Clifford multiplication* can be lifted as well. In particular, we have (See *Chapter 5.1* for the proof):

Proposition 3.4.4. ***Lifts of Pauli matrices via $\mathcal{P}$.***
Given $k_{2,1} \in \mathbb{Z}$, and the weights ω, Ω on $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, (5.2.3), the map $\mathcal{P}_\mu$ satisfies:

$$\mathcal{P}_\mu(\sigma_\omega(\tilde{\alpha})\tilde{\psi}) = \sigma_\Omega(F^*\tilde{\alpha})(\mathcal{P}_\mu\tilde{\psi}) \tag{3.4.14}$$

Lastly, by introducing the forms:

$$\tilde{w}_z^{1,2} := \tilde{\omega}_{1,2}^c(z)\tilde{\zeta}_z, \tag{3.4.15}$$

where

$$\tilde{\omega}_{1,2}^c(z) := -\frac{1}{\kappa_1+\kappa_2}\frac{\omega_{1,2}'(|z|^2)}{\omega_{1,2}(|z|^2)} = -\frac{1}{\tilde{\kappa}}\frac{\omega_{1,2}'(|z|^2)}{\omega_{1,2}(|z|^2)} \tag{3.4.16}$$

and

$$\tilde{\zeta}_z = \frac{1}{2\mathrm{i}|z|^2}(\overline{z}dz - zd\overline{z}) = \frac{1}{2\mathrm{i}}\Big(\frac{1}{z}dz - \frac{1}{\overline{z}}d\overline{z}\Big),$$

to account for the conformal change of metric, we can now define *Spin*^{c} connections on the *spin*^{c}-bundle, S_μ (recall $\mu = (2k_1+1)\kappa_1 + (2k_2+1)\kappa_2)$).

Definition 3.4.5. Given a 1-form α, a *Spin*^{c} connection that acts on the *spin*^{c} bundle(s) S_μ on $\mathbb{O}_\kappa$, is defined as:

$$\tilde{\nabla}^{\omega,\tilde{\alpha}} := \begin{pmatrix} d - \mathrm{i}\tilde{\alpha} + \mathrm{i}\tilde{\kappa}\tilde{w}^c & 0 \\ 0 & d - \mathrm{i}\tilde{\alpha} - \mathrm{i}\tilde{\kappa}\tilde{w}^c \end{pmatrix} \tag{3.4.17}$$

see (3.4.15) and (3.4.16) for the component $\tilde{w}^c$ and its relation to ω, here $\tilde{\kappa} = \kappa_1 + \kappa_2$.

The formula right above can be re-written as:

$$\tilde{\nabla}^{\omega,\tilde{\alpha}} := \begin{pmatrix} d - \mathrm{i}\tilde{\alpha} - \mathrm{i}\frac{\omega'}{\omega}\tilde{\zeta} & 0 \\ 0 & d - \mathrm{i}\tilde{\alpha} + \mathrm{i}\frac{\omega'}{\omega}\tilde{\zeta} \end{pmatrix} \tag{3.4.18}$$

and simply denoted as $\tilde{\nabla}^{\omega,\tilde{\alpha}}$ or in tensor-notation:

$$\tilde{\nabla}^{\tilde{w}^c,\tilde{\alpha}} := (d - \mathrm{i}\tilde{\alpha}) \otimes I_2 + \mathrm{i}\tilde{\kappa} w^c \sigma_3 \tag{3.4.19}$$

where $\lambda = 2k_1\kappa_1 + 2k_2\kappa_2,\ k_{1,2} \in \mathbb{Z}$. Unlike $\tilde{\alpha}_{1,2},\ \tilde{\alpha}_z^{1,2}$ are *local* 1-forms (except when $k_1 = k_2 = 0$ when both are global). The form $\tilde{w}^c$ has corresponding "λ" equal to 1.

Given the duality *vector fields*$\leftrightarrow$*1-forms* we get that vector fields on $\mathbb{O}_{\boldsymbol{\kappa}}$ are defined subject to satisfying the conditions:

$$\tau_* \tilde{X}_1 = \tilde{X}_2, \tag{3.4.20}$$

$$(R_{\kappa^{3-2j}})_* \tilde{X}^j = (R_{\kappa^{3-2j}})_* \tilde{X}_j^z \partial_z + (R_{\kappa^{3-2j}})_* \tilde{X}_j^{\bar{z}} \partial_{\bar{z}} = \tilde{X}_j^z \partial_z + \tilde{X}_j^{\bar{z}} \partial_{\bar{z}}, \quad j = 1,2 \tag{3.4.21}$$

Recall that R_a is a rotation by $2\pi a$ (quantized as $e^{2\pi a \mathrm{i}}$ here) and the subscript $*$ denotes the pushforward. As a consequence of (3.4.20) and (3.4.21) we have respectively:

$$\tilde{X}_1^z(\tau(z), \overline{\tau(z)}) = C_\kappa z^{-1-\frac{1}{\kappa}} \tilde{X}_2^z(z, \bar{z}), \quad \tilde{X}_1^{\bar{z}}(\tau(z), \overline{\tau(z)}) = C_\kappa \bar{z}^{-1-\frac{1}{\kappa}} \tilde{X}_2^{\bar{z}}(z, \bar{z}), \quad z \in \mathbb{C}^* \tag{3.4.22}$$

$$(R_{\kappa^{3-2j}})_* \tilde{X}_j^z = e^{2\mathrm{i}\pi\kappa^{3-2j}} \tilde{X}_j^z, \quad (R_{\kappa^{3-2j}})_* \tilde{X}_j^{\bar{z}} = e^{-2\mathrm{i}\pi\kappa^{3-2j}} \tilde{X}_j^{\bar{z}}, \quad j = 1,2 \tag{3.4.23}$$

Before proceeding to state the highlights of this work, we present how *spin*^{c} connections are lifted across F. Combing the results on lifts of connections on line bundles at the end of previous sections, alongside with a number of lemmas in the beginning of *Chapter 5*, we've proved:

Proposition 3.4.6. ***Lifts of spin^c connections via $\mathcal{P}$.***
Given $k_{2,1} \in \mathbb{Z}$ and $\mu = (2k_1+1)\kappa_1 + (2k_2+1)\kappa_2$. The map $\mathcal{P}_\mu$ as defined in (5.1.16), when acting on a spinor after a *spin*^{c} connection $\tilde{\nabla}^\omega$ has been applied to, satisfies:

$$\mathcal{P}_\mu \tilde{\nabla}^\omega \tilde{\psi} = \Big(\nabla^\Omega - \mathrm{i}\kappa_1\kappa_2 \Omega(\mathbf{x}) M(\mathbf{x}) \big(\sigma(\cdot) - 2(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})\mathbf{X}_{\mathbf{x}}^*\big)\Big)(\mathcal{P}_\mu \tilde{\psi}) \tag{3.4.24}$$

where $\sigma(\cdot) = (dx_1, dx_2, dx_3).\boldsymbol{\sigma}$ $(= (dx_1, dx_2, dx_3).(\sigma_1, \sigma_2, \sigma_3))$, and $\tilde{\nabla}^\omega$ is defined in (3.4.17) after setting $\tilde{\alpha} = 0$.

The proof can be found at the end of *Chapter 5.2*.

3.5 The main results of the thesis

In this section we define *Weyl-Dirac* operators on the orbit space $\mathbb{O}_{\boldsymbol{\kappa}}$ and describe how the said operators on $\mathbb{O}_{\boldsymbol{\kappa}}$ are related to certain Weyl-Dirac operators on $\mathbb{R}^3$.

3.5.1 Dirac operators on the orbit space

Recall that the orbit space $\mathbb{O}_{\boldsymbol{\kappa}}$ is a collection of two planes, equipped with weights $\omega_{1,2}$, see (3.2.24) and the discussion right below), satisfying certain "compatibility" conditions (3.2.28). A *Dirac operator* on $\mathbb{R}^2 \equiv \mathbb{C}$ with metric weight omega is defined as follows:

Definition 3.5.1. Consider $\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2$, a local basis of coordinate vector fields, and $\tilde{\nabla}^{\tilde{\omega},\tilde{\alpha}}$ a Spin^c-connection as defined in (3.4.17) with weight ω and respective magnetic potential defined by the form $\tilde{\alpha}$. The Weyl-Dirac operator on $\mathbb{R}^2$, with corresponding magnetic potential given by $\tilde{\boldsymbol{A}}(\tilde{x},\tilde{y}) := (\tilde{\alpha}(\tilde{\boldsymbol{e}}_1), \tilde{\alpha}(\tilde{\boldsymbol{e}}_2))|_{(\tilde{x},\tilde{y})}$, and weighted metric ω is defined as

$$\mathcal{D}^{\omega}_{\tilde{A}} := -\mathrm{i}\sigma(\tilde{e}_1)\tilde{\nabla}^{\omega,\tilde{\alpha}}_{\tilde{\boldsymbol{e}}_1} - \mathrm{i}\sigma(\tilde{e}_2)\tilde{\nabla}^{\omega,\tilde{\alpha}}_{\tilde{\boldsymbol{e}}_2} \tag{3.5.1}$$

where $\tilde{e}_{1,2}$ are the 1-forms dual to the vector fields $\tilde{\boldsymbol{e}}_{1,2}$.

These operators acts on sections of the *Spin*^{c}-bundle(s) $\Gamma(\mathbb{C},\mathbb{C}^2)$.

Now, in order to define a *Dirac* operator on $\mathbb{O}_{\boldsymbol{\kappa}}$ we just need to consider two suitable copies of the Dirac operator defined right above. In particular we need the forms and weights in the corresponding copies of the operator to satisfy the the respective compatibility conditions ((3.2.28) for the weights and (3.3.16) for the forms).

Definition 3.5.2. The Dirac operator on $\mathbb{O}_{\boldsymbol{\kappa}}$ is defined as the collection of two copies (indelxed by $j = 1, 2$) of Dirac operators, as defined in (3.5.1), with the respective coordinate vector fields $\tilde{\boldsymbol{e}}_{1,j}, \tilde{\boldsymbol{e}}_{2,j}$, 1-forms $\tilde{\alpha}^j$ satisfying (3.4.20), (3.4.21), and weights ω_j satisfying $\omega_1 = |z|^{-1-\frac{1}{\kappa}}\omega_2 \circ \tau$. This operator will be denoted as $\tilde{\mathcal{D}}^{\omega}_{\tilde{A}}$.

These operators acts on sections of the *Spin*^{c}-bundle(s) S_μ.

3.5.2 Interlacing 2 and 3-dimensional Dirac operators

The main aim of this study is to investigate the existence of zero-modes for a certain family of 3-dimensional Weyl-Dirac operators. We have a submersion that maps 2 and 3 dimensions, and since the 2-dimensional Dirac operators are generally easier to study, we want find a relation between the 2 and 3-dimensional Dirac operators according the said submersion.

Dirac operators are automorphisms of sections of a *spin*^{c} bundle. So, we want to find an operator, indexed by the same index as the respective spin^c bundle on $\mathbb{O}_{\boldsymbol{\kappa}}$ that "connects" these two. In other words, given an L_μ-section $\tilde{\psi}$, we're looking for an operator $\mathcal{P}_\mu$ that, roughly speaking satisfies a relation such as

$$\mathcal{D}^{\Omega}_{\mathbf{A}^\nu}(\mathcal{P}_\mu\tilde{\psi}) = \mathcal{P}_\mu\big(\tilde{\mathcal{D}}^{\omega}_{\tilde{A}}\tilde{\psi}\big) + \text{(0-order terms)}.$$

where $\tilde{A}$, A are the magnetic potentials in 2 and 3 dimensions respectively, $\tilde{\mathcal{D}}^{\omega}_{\tilde{A}}$ is the 2-dimensional Dirac operator and $\mathcal{D}^{\Omega}_{\mathbf{A}^\nu}$ is the 3-dimensional Dirac operator given by:

$$\mathcal{D}^{\Omega}_{\mathbf{A}^\nu} := \sum_{j=1}^{3}\sigma(e^j)\nabla^{\Omega,\alpha}_{\mathbf{e}_j} = \sum_{j=1}^{3}\sigma(e^j)\big(\nabla_{\mathbf{e}_j} - \alpha^\nu(\mathbf{e}_j) + \frac{1}{4\Omega}[\sigma(e^j),\sigma(d\Omega)]\big) \tag{3.5.2}$$

where

$$\alpha^\nu_{\mathbf{x}} = \nu\Omega^2\mathbf{X}^*|_{\mathbf{x}} + F^*_j(\tilde{\alpha}'_j)|_{\mathbf{x}}, \tag{3.5.3}$$

for $\nu\in\mathbb{R}$ (which gives us the *flux*) and some $\tilde{\alpha}_j\in\Omega^1(\mathbb{O}_\kappa)$, is the respective 1-form (evaluated at $\mathbf{x}$) producing the magnetic potential by $\mathbf{A}^\nu(\mathbf{x}) = (\alpha^\nu_{\mathbf{x}}(\mathbf{e}_1), \alpha^\nu_{\mathbf{x}}(\mathbf{e}_2), \alpha^\nu_{\mathbf{x}}(\mathbf{e}_3))$ (recall that $\Omega(\mathbf{x}) = |\mathbf{X}(\mathbf{x})|^{-1}$).

Moreover, it is shown that any such magnetic 1-form of interest, i.e. 1-form that produces magnetic potentials whose corresponding magnetic field is parallel to $\mathbf{X}(\mathbf{x})$ is the form (3.5.3).

Finally, we get:

Theorem 3.5.3. Let $\mu = (2k_1+1)\kappa_1 + (2k_2+1)\kappa_2$, $k_{1,2} \in \mathbb{Z}$, $\nu \in \mathbb{R}$ and the Dirac operators $\mathcal{D}^{\Omega}_{\mathbf{A}^{\nu}}$ and $\mathcal{D}^{\omega}_{\tilde{\boldsymbol{A}}}$ as defined in (3.5.2) and (3.5.2) respectively. Then for any section $\tilde{\psi}$ of the *spin*^{c} bundle, (3.4.1), on $\mathbb{O}_{\boldsymbol{\kappa}}$, we have:

$$\mathcal{D}^{\Omega}_{\mathbf{A}^{\nu}}(\mathcal{P}_{\mu}\tilde{\psi}) = \mathcal{P}_{\mu}\big((\tilde{\mathcal{D}}^{\omega}_{\tilde{A}} + \kappa_1\kappa_2 m I_2 + (\mu-\nu)\sigma_3)\tilde{\psi}\big) \tag{3.5.4}$$

where $\mathcal{P}_{\mu}$ is defined in (5.1.16) and m is defined in (3.2.19).

Remark 3.5.4. The Dirac operators $\mathcal{D}^{\Omega}_{\mathbf{A}^{\nu}}$ and $\tilde{\mathcal{D}}^{\omega}_{\tilde{A}}$, (3.5.2) and (3.5.1), also act on elements $\psi \in W^{1,2}_{\tilde{\Omega}}(\mathbb{R}^3,\mathbb{C}^2)$, $\tilde{\psi} \in W^{1,2}_{\tilde{\omega}_j}(\mathbb{R}^2,\mathbb{C}^2)$ (resp. $L^2_{\tilde{\Omega}}(\mathbb{R}^3,\mathbb{C}^2)$ and $L^2_{\tilde{\omega}_j}(\mathbb{R}^2,\mathbb{C}^2)$) satisfying $\mathcal{D}^{\Omega}_{A}\psi \in L^2_{\Omega}(\mathbb{R}^3,\mathbb{C}^2)$ and $\tilde{\mathcal{D}}^{\omega}_{\tilde{A}}\tilde{\psi} \in L^2_{\Omega}(\mathbb{R}^2,\mathbb{C}^2)$ (respectively $\in \mathcal{D}'(\mathbb{R}^3)$ and $\mathcal{D}'(\mathbb{R}^2)$ - which are corresponding spaces of distributions). The respective "weighted" spaces L^2_{Ω}, $L^2_{\omega_j}$ and $W^{1,2}_{\Omega}$, $W^{1,2}_{\omega_j}$ are defined as the spaces of $\mathbb{C}^2$-valued functions on $\mathbb{R}^3$, $\mathbb{R}^2$ respectively, such that they (and their first derivatives - in the sense of distributions - for the cases of the Sobolev spaces W) are bounded with respect to the norms:

$$||\psi||_{L^2_{\tilde{\Omega}}(\mathbb{R}^3,\mathbb{C}^2)} := \Big(\int_{\mathbb{R}^3} \Omega^3(\mathbf{x})\langle\psi,\psi\rangle_{\mathbb{C}^2} d\mathbf{x}\Big)^{1/2} \tag{3.5.5}$$

and

$$||\psi||_{W^{1,2}_{\tilde{\Omega}}(\mathbb{R}^3,\mathbb{C}^2)} := \Big(\int_{\mathbb{R}^3} \Omega^3(\mathbf{x})\big(\langle\psi,\psi\rangle_{\mathbb{C}^2} + \sum_{i=1}^{3}\langle\partial_{x_i}\psi,\partial_{x_i}\psi\rangle_{\mathbb{C}^2}\big)d\mathbf{x}\Big)^{1/2} \tag{3.5.6}$$

for $\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$ and $d\mathbf{x} = dx_1dx_2dx_3$. Respectively, for the case when the weight is ω_j, the formulas still hold by replacing $\Omega \mapsto \omega$ and $3 \mapsto 2$ (as we are in 2 dimensions).

Chapter 4: Line and Spin^c Bundles on a quotient space and a Riemann-type submersion

In a paper by Erdős and Solovej, 2001, the authors demonstrate the existence of zero-modes for magnetic fields that are parallel to:

$$\mathbf{X}^{ES}(\mathbf{x}) := \mathbf{W}_1(\mathbf{x}) + \mathbf{W}_2(\mathbf{x}) \tag{4.0.7}$$

where:

$$\mathbf{W}_1(\mathbf{x}) := (-2x_2, 2x_1, 0) \quad \& \quad \mathbf{W}_2(\mathbf{x}) := (2x_1x_2, 2x_2x_3, 1 - x_1^2 - x_2^2 + x_3^2).$$

The two components of the sum right above are the fields denoted by $\mathbf{W}_1(\mathbf{x})$ and $\mathbf{W}_2(\mathbf{x})$ respectively. In Elton, 2018, it is proved that the Weyl-Dirac operator possesses no zero-modes when the magnetic field is parallel to $\mathbf{W}_1(\mathbf{x})$, $\mathbf{W}_2(\mathbf{x})$ or $(0, 0, 1)$. Here, we study the existence of zero-modes in cases where the magnetic field is parallel to a linear combination $\kappa_1\mathbf{W}_1(\mathbf{x}) + \kappa_2\mathbf{W}_2(\mathbf{x})$ for $\kappa_{1,2} \in \mathbb{R}_{>0}$, $\kappa_1 \neq \kappa_2$. For such $\kappa_1, \kappa_2 > 0$, recall we've defined

$$\mathbf{X}(\mathbf{x}) := \kappa_1\mathbf{W}_1(\mathbf{x}) + \kappa_2\mathbf{W}_2(\mathbf{x}). \tag{4.0.8}$$

The Dirac operator is rather complicated, mainly because it strongly couples the two components of a Weyl spinor it acts on. The contributions of Elton, 2018,Elton, 2016 and Erdős and Solovej, 2001 respectively make use of the first-order operator $Q_{\mathbf{X}}$, (for $\mathbf{X}$ defined as $\mathbf{W}_1, \mathbf{W}_2$ or $\mathbf{X}^{ES}$ respectively) , which has diagonal first-order terms and commutes with the Weyl-Dirac Operator. The latter fact is particularly useful as it implies that these two operators have the related eigenstates, and that enables us to get information on zero-modes of $\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}$. In this chapter, we'll investigate the spectrum of this respective operator, as well as construct a suitable submersion from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Unlike the *Erdős-Solovej* case, the respective geometric objects we'll be working with here, are not Riemannian manifolds. Instead, we get a collection of two *orbifolds* when $\kappa \in \mathbb{Q}_{>0}$, and a more general object otherwise. However, we can treat both cases, concerning the calculations performed, as if they were Riemannian manifolds.

In the first section of this chapter, we define the basic objects we'll be working with throughout this thesis.

4.1 A family of multivalued Riemannian submersions

We want to derive with a map F, that maps points in $\mathbb{R}^3$ to points on $\mathbb{R}^2$ (equivalently $\mathbb{C}$), such that it is constant along the vector field $\mathbf{X}$ and respects orthonormality of the normal and binormal directions of $\mathbf{X}$. In particular, we want F to satisfy:

$$\mathbf{X}.\boldsymbol{\nabla} F = 0. \tag{4.1.1}$$

This is a necessary condition. Moreoever, we'll find such a map to submerse charts of $(\mathbb{R}^3, \Omega)$ into copies of $(\mathbb{R}^2, \omega)$, $(\mathbb{R}^2, \omega_{1,2})$, where $\Omega(\mathbf{x}) = |\mathbf{X}(\mathbf{x})|^{-1}$ and $\omega_{1,2}$ are given by (4.2.15) and (4.2.16). Particular versions of Dirac operators on such weighted space can commute with the operator(s) $Q_{\mathbf{X}}$

The construction of the submersion F will be broken into two steps:

First step consists of mapping $\mathbb{R}^3$ onto the open unit disk $\mathbb{D}$ on $\mathbb{C}$. This is done using the following maps $\chi_{\kappa^{-1},2}$, $\chi_{\kappa,1}$ defined on charts $\mathbb{R}^3_2 := \mathbb{R}^3 \setminus \{(0,0,x_3) \in \mathbb{R}^3 | x_3 \in \mathbb{R}\}$ and $\mathbb{R}^3_1 := \mathbb{R}^3 \setminus \{(x_1,x_2,0) \in \mathbb{R}^3 | x_1^2 + x_2^2 = 1\}$ respectively:

$$\chi_{\kappa,1}(\mathbf{x}) = \frac{|\tilde{z}_1|}{1 + |\mathbf{x}|^2 + |\tilde{z}_2|} \tilde{e}(\tilde{z}_1)(\tilde{e}(\tilde{z}_2))^{-\kappa} \tag{4.1.2}$$

and

$$\chi_{\kappa^{-1},2}(\mathbf{x}) = \frac{1 + |\mathbf{x}|^2 + |\tilde{z}_2| - |\tilde{z}_1|}{1 + |\mathbf{x}|^2 + |\tilde{z}_2| + |\tilde{z}_1|} (\tilde{e}(\tilde{z}_1))^{-\frac{1}{\kappa}} \tilde{e}(\tilde{z}_2) \tag{4.1.3}$$

where $\tilde{z}_1 = x_1 + \mathrm{i}x_2$, $\tilde{z}_2 = 1 - |\mathbf{x}|^2 + 2\mathrm{i}x_3$ and $\tilde{e}(z) = z/|z|$ for $z \in \mathbb{C}^*$.

Remark 4.1.1. The subscripts κ, κ^{-1} may be dropped from the χ's, F's and $\tilde{F}$'s ((3.1.1),(3.1.2) and (4.1.6),(4.1.7)) for the sake of simplicity and neatness if there is no need emphasizing the value of the parameter κ.

Now, we can define a map υ such that $\upsilon(\chi_{\kappa,1}(\mathbf{x})) = \chi_{\kappa^{-1},2}(\mathbf{x})$. To find such a function explicitly, we just need to pay attention to the radial and angular factors in the polar form of $\chi_{\kappa^{\mp 1}}(\mathbf{x})$. By the equation in the top half of this page, we have (where $\chi_{\kappa^{\mp 1}} = \chi_{\kappa^{\mp 1}}(\mathbf{x})$):

$$\frac{1 - |\chi_{\kappa,1}(\mathbf{x})|}{1 + |\chi_{\kappa,1}(\mathbf{x})|} = |\chi_{\kappa^{-1},2}(\mathbf{x})|,$$

$$\arg(\chi_{\kappa^{-1},2}) = \tilde{e}^{-\frac{1}{\kappa}}(\tilde{z}_1)\tilde{e}(\tilde{z}_2) = \left(\tilde{e}(\tilde{z}_1)\tilde{e}(\tilde{z}_2)^{-\kappa}\right)^{-\frac{1}{\kappa}} = \arg(\chi_{\kappa,1})^{-\frac{1}{\kappa}}.$$

Therefore, slightly abusing notation we define the map $\upsilon : \mathbb{D} \to \mathbb{D}$ and its inverse respectively:

$$\upsilon(\chi_{\kappa,1}) := \frac{1 - |\chi_{\kappa,1}|}{1 + |\chi_{\kappa,1}|} \arg(\chi_{\kappa,1})^{-\frac{1}{\kappa}} \tag{4.1.4}$$

$$\upsilon^{-1}(\chi_{\kappa^{-1},2}) = \frac{1 - |\chi_{\kappa^{-1},2}|}{1 + |\chi_{\kappa^{-1},2}|} \arg(\chi_{\kappa^{-1},2})^{-\kappa} \equiv \chi_{\kappa,1}(\mathbf{x}). \tag{4.1.5}$$

The maps χ and υ are obviously multivalued, but we can work with single-valued versions of them by considering branches of the respective exponentials

Remark 4.1.2. These maps $\chi_{\kappa^{\mp 1}}$ can be derived if we consider a point $\mathbf{x} \in \mathbb{R}^3_1$ (resp. $\mathbb{R}^3_2$) and we write it in the parametric form $\mathbf{x} = \gamma(t)$ where $\gamma(\cdot)$ is the integral curve of $\mathbf{X}(\mathbf{x})$ passing through $\mathbf{x}$ and follow the integral curve (forward of backwards) until it reaches the unit disk $\mathbb{D}$ on the x_1x_2-plane (resp. the right x_1x_3-plane and then get it mapped onto $\mathbb{D}$ via a Cayley transform).

The *second step* involves finding maps $\tilde{F}_\kappa, \tilde{F}_{\kappa^{-1}}$ from the unit disk on the complex plane (naturally identified with $\mathbb{D}$) to $\mathbb{C}$ such that the compositions with $\chi_{1,2} : \mathbb{R}^3_{1,2} \mapsto \mathbb{D}$ produce a Riemann-type submersion. In particular, we want them to preserve the orthonormality of the two normal and binormal vectors of $\mathbf{X}$.

$$F_{\kappa,1} := \tilde{F}_\kappa \circ \chi_{\kappa,1} \tag{4.1.6}$$

and

$$F_{\kappa^{-1},2} := \tilde{F}_{\kappa^{-1}} \circ \chi_{\kappa^{-1},2} \tag{4.1.7}$$

defined on $\mathbb{R}^3_1$ and $\mathbb{R}^3_2$ respectively, produce a (generally multivalued) submersion of Riemann-type. In other words, we want them to preserve the orthonormality of the normal and bi-normal vectors (of the integral curve $\gamma(\cdot;\rho,t_0)$ of the vector field $\mathbf{X}$).

This search of maps, $\tilde{F}_{\kappa^{\pm1}}$ consists of looking for functions with a "radially symmetric" component $f_{\kappa^{\pm1}}$, in the form:

$$\tilde{F}_{\kappa^{\pm1}}(\tilde{z},\overline{\tilde{z}}) = \tilde{z} f_{\kappa^{\pm1}}(\tilde{z}\overline{\tilde{z}}) = \tilde{z} f_{\kappa^{\pm1}}(|\tilde{z}|^2)$$

which may be loosely be denoted as $\tilde{F}_{\kappa^{\pm1}}(\tilde{z})$ (where $\tilde{z} = \chi_{\kappa^{\mp1}}$ respectively). We're considering $f_{\kappa^{\pm1}}$ (and consequently $\tilde{F}_{\kappa^{\pm1}}$) to be a smooth function of $\tilde{z}^2$ so that it will be smooth rear $|\tilde{z}| = 0$.

The derivation of these maps partially relies on information given by the integral curves of the vector field $\mathbf{X}_\kappa$ (or equivalently $\mathbf{X}_\kappa(\mathbf{x}) := \kappa_2^{-1}\mathbf{X}(\mathbf{x})$ if $\kappa_2 \neq 0$ since the shape and geometric properties of these curves remain intact). This information is given in the next subsection.

4.1.1 Integral curves of the vector field X

Following the respective results in Elton, 2018, by simply inspecting the vector field $\mathbf{X}(\mathbf{x})$, the integral curves of $\mathbf{X}$ are given by:

$$\boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)), \tag{4.1.8}$$

where

$$\gamma_1(t) = \frac{(1-\rho^2)\cos(2\kappa_1 t + t_0 - t_0')}{1+\rho^2+2\rho\cos(2\kappa_2 t + t_0 + t_0')}, \tag{4.1.9}$$

$$\gamma_2(t) = \frac{(1-\rho^2)\sin(2\kappa_1 t + t_0 - t_0')}{1+\rho^2+2\rho\cos(2\kappa_2 t + t_0 + t_0')}, \tag{4.1.10}$$

$$\gamma_3(t) = \frac{2\rho\sin(2\kappa_2 t + t_0 + t_0')}{1+\rho^2+2\rho\cos(2\kappa_2 t + t_0 + t_0')}, \tag{4.1.11}$$

for[7] $\rho \in [0,1)$ and $\boldsymbol{\gamma}(t) = \big(0, 0, \tan(2\kappa_2 t + t_0 + t_0')\big)$ for $\rho = 1$ ($t \in \mathbb{R}$, t_0, $t_0' \in [0, 2\pi)$).

Remark 4.1.3. The parameters t_0, $t_0' \in [0, 2\pi)$ and $\rho \in [0,1]$ are used to describe the initial condition $\boldsymbol{\gamma}(0) = \mathbf{x}_0 \in \mathbb{R}^3$. In particular, the integral curves of $\mathbf{X}$ lie on a torus, parametrised by ρ, t_0', with ρ describing the "thickness" of the torus.

The fact that these curves are integral curves of $\mathbf{X}(\mathbf{x})$ can be seen easily by inspecting the integral curves of each of its components ($\kappa_1 \mathbf{W}_1(\mathbf{x})$ and $\kappa_2 \mathbf{W}_2(\mathbf{x})$), which are respectively:

$$\boldsymbol{\gamma}^1(t) = c_0(\cos(2\kappa_1 t + t_0 - t_0'), \sin(2\kappa_1 t + t_0 - t_0'), c) \tag{4.1.12}$$

(for $\kappa_1 \mathbf{W}_1(\mathbf{x})$) where $c_0, c \in \mathbb{R}$, and for $\rho \in (0,1)$, c_1, $c_2 \in \mathbb{R}$ such that $c_1^2 + c_2^2 = 1 - \rho^2$, the case for $\kappa_2 \mathbf{W}_2(\mathbf{x})$:

$$\boldsymbol{\gamma}^2(t) = \frac{1}{1 + \rho^2 + 2\rho\cos(2\kappa_2 t + t_0 + t_0')}\Big(c_1, c_2, 2\rho\sin(2\kappa_2 t + t_0 + t_0')\Big). \tag{4.1.13}$$

This comes essentially from a direct inspection of the cases where $\kappa_2 = 0$ and $\kappa_1 = 0$ respectively, see Elton, 2018.

4.1.2 Derivation of maps from $\mathbb{R}^3$ to the unit disk

Given that we have the formulae for $\boldsymbol{\gamma}(t)$, we can find the points on them that are on the unit disk (and the plane $\{(x_1, 0, x_3) : x_1 > 0\}$) explicitly, given[8] $t \in \mathbb{R}$. In particular, we have:

Proposition 4.1.4. Let $\mathbf{x} \in \mathbb{R}^3_{1,2}$. The integral curve(s) of $\mathbf{X}$ intersects the open unit disk $\tilde{\mathbb{D}} := \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1\}$ at points given by:

$$\chi_{\kappa,1}(\mathbf{x}) = \frac{2\sqrt{x_1^2 + x_2^2}}{1 + |\mathbf{x}|^2 + \sqrt{(1 - |\mathbf{x}|^2)^2 + 4x_3^2}}\tilde{e}(x_1 + \mathrm{i}x_2)\tilde{e}^{-\kappa}(1 - |\mathbf{x}|^2 + 2\mathrm{i}x_3) \tag{4.1.14}$$

[7] In the case where $\kappa := \kappa_1/\kappa_2 = 1$, ρ corresponds to distinct Hopf fibers. In particular $(1-\rho^2)/(1+\rho^2) = \sin(\eta)$ (and $2\rho/(1+\rho^2) = \cos(\eta)$) where $\eta \in [0, \pi/2]$ is one of the Hopf coordinates by the standard Hopf parametrization. Note that the variable ρ corresponds to the rational parametrization of the circle.

[8] If $\kappa = p/q$ for $p, q \in \mathbb{Z}$, $q \neq 0$ with $gcd(p,q) = 1$, then $\boldsymbol{\gamma}$ is periodic with period $T = q\pi$.

and

$$\chi_{\frac{1}{\kappa},2}(\mathbf{x}) = \frac{1+|\mathbf{x}|^2+|\tilde{z}_2|-|\tilde{z}_1|}{1+|\mathbf{x}|^2+|\tilde{z}_2|+|\tilde{z}_1|}\tilde{e}(x_1+\mathrm{i}x_2)^{-\frac{1}{\kappa}}\tilde{e}(1-|\mathbf{x}|^2+2\mathrm{i}x_3). \tag{4.1.15}$$

Proof. This is done by solving the equation $\gamma_3(t+t_1;\rho,t_0)=0$ (which collapses the integral curve down to $x_3=0$ for $t_1\in\mathbb{R}$, i.e. $\sin(2\kappa_2(t+t_1)+2t_0)=0$ which implies $2\kappa_2(t+t_1)+t_0+t_0'=k\pi$ and $\cos(2\kappa_2(t+t_1)+t_0+t_0')=(-1)^k$, however, we want the points which are in the unit disk, so by the previous formulae we get that we need $\cos(2\kappa_2(t+t_1)+t_0+t_0')=1$.

This means k must be even and so $2\kappa_2(t+t_1)+t_0+t_0'=2k_1\pi$ for some $k_1\in\mathbb{Z}$ means that $t_1=k_1\pi-t_0-t$. In this case, we notice that $(1-\rho^2)/(1+\rho)^2=(1-\rho)/(1+\rho)$ we can define the map $\chi_{\kappa,1}:\mathbb{R}^3_1\mapsto\mathbb{D}$ using the formula:

$$\frac{1-\rho}{1+\rho}e^{\mathrm{i}((k_1+1)\kappa\pi-(1+\kappa)t_0)} = \frac{1-\rho}{1+\rho}e^{\mathrm{i}2k_1\kappa\pi}e^{-\mathrm{i}2(1+\kappa)t_0} \tag{4.1.16}$$

where we've identified $\mathbf{x}=(x_1,x_2,x_3)\in\mathbb{R}^3_1$ naturally with $\boldsymbol{\gamma}(t;\rho,t_0)$ for $(t;\rho,t_0)$ such that $\mathbf{x}=\boldsymbol{\gamma}(t;\rho,t_0)$. Note that we can re-write the previous formula by writing ρ and t_0 in terms of x_1,x_2,x_3. To proceed, we set

$$\tilde{z}_1 := x_1+\mathrm{i}x_2 = \frac{(1-\rho^2)\cos(2\kappa_1 t+t_0-t_0'))}{1+\rho^2+2\rho\cos(2\kappa_2 t+t_0+t_0')}+\mathrm{i}\frac{(1-\rho^2)\sin(2\kappa_1 t+t_0-t_0')}{1+\rho^2+2\rho\cos(2\kappa_2 t+t_0+t_0')} \tag{4.1.17}$$

and

$$\tilde{z}_2 = 1-|\mathbf{x}|^2+2\mathrm{i}x_3. \tag{4.1.18}$$

In terms of (t,ρ,t_0) we get:

$$\begin{aligned}1-|\mathbf{x}|^2 &= 1-\frac{(1-\rho^2)^2}{(1+\rho^2+2\rho\cos(2\kappa_2 t+t_0+t_0'))^2}-\frac{(2\rho)^2\sin^2(2\kappa_2 t+t_0+t_0'))}{(1+\rho^2+2\rho\cos(2t+t_0+t_0'))^2}\\ &= 1-\frac{\sin^2(\eta)}{(1+\cos(\eta)\cos(2\kappa_2 t+t_0+t_0'))^2}-\frac{\cos^2(\eta)(1+\cos^2(2\kappa_2 t+t_0-t_0'))}{(1+\cos(\eta)\cos(2\kappa_2 t+t_0+t_0'))^2}\\ &= \frac{(1+\cos(\eta)\cos(2\kappa_2 t+t_0+t_0')^2-\sin^2(\eta)-\cos^2(\eta)(1+\cos^2(2\kappa_2 t+t_0+t_0'))}{(1+\cos(\eta)\cos(2\kappa_2 t+t_0+t_0'))^2}\\ &= \frac{-2\cos(\eta)\cos(2\kappa_2 t+t_0+t_0')+\cos^2(\eta)\cos^2(2\kappa_2 t+t_0+t_0)+\cos^2(\eta)\cos^2(2\kappa_2 t+t_0+t_0')}{(1+\cos(\eta)\cos(2\kappa_2 t+t_0+t_0'))^2}\end{aligned}$$

where we've substituted $(1-\rho^2)/(1+\rho^2) = \sin(\eta)$ $(2\rho(1+\rho^2)^{-1} = \cos(\eta))$ and

$$|\tilde{z}_2|^2 = (1-|\mathbf{x}|^2)^2 + 4x_3^2.$$

Also, $\tilde{e}(\tilde{z}_1) = e^{\mathrm{i}2\kappa(t-t_0)}$ and $\tilde{e}(\tilde{z}_2) = e^{\mathrm{i}2(t+t_0)}$ which turn (3.1.14) (for $k_1 = 0$) to

$$\frac{1-\rho}{1+\rho} e^{-\mathrm{i}2(1+\kappa)t_0} = \frac{1-\rho}{1+\rho}\tilde{e}(\tilde{z}_1)\tilde{e}^{-\kappa}(\tilde{z}_2) =: \chi_{\kappa,1}(\mathbf{x}). \tag{4.1.19}$$

Which is a map from $\mathbb{R}^3$ to the unit disk in $\mathbb{R}^2$ (recall, $\tilde{z}_{1,2}$ and ρ are functions of $\mathbf{x}$). In the $\mathbf{x} = (x_1, x_2, x_3)$ variables:

$$\chi_{\kappa,1}(\mathbf{x}) = \frac{2\sqrt{x_1^2 + x_2^2}}{1 + |\mathbf{x}|^2 + \sqrt{(1-|\mathbf{x}|^2)^2 + 4x_3^2}}\tilde{e}(x_1 + \mathrm{i}x_2)\tilde{e}^{-\kappa}(1 - |\mathbf{x}|^2 + 2\mathrm{i}x_3) \tag{4.1.20}$$

where we easily see that $\chi_{\kappa,1}(x_1, x_2, 0) = x_1 + \mathrm{i}x_2$ is multi-valued otherwise.

- First, we map points $\mathbf{x}$ of $\boldsymbol{\gamma}(t)$ to the first point on the half-plane $\{(x_1, 0, x_3) : x_1 > 0\}$ that $\mathbf{x}$ is mapped to when moving forward (or backwards) in time t along $\boldsymbol{\gamma}(t)$.

- Secondly, we map the half-plane $\{(x_1, 0, x_3) : x_1 > 0\}$ onto the open unit disk $\tilde{\mathbb{D}} := \{(x_1, x_2, 0) : x_1^2 + x_2^2 < 1\}$. This is done using a Cayley transform.

Lastly, we map the planar unit open disk $\tilde{\mathbb{D}}$ to the whole plane, as we did before. This time, however, to ease the calculations and make the results neater, we "normalise" the vector field $\mathbf{X}(\mathbf{x}) \to \mathbf{X}_\kappa(\mathbf{x}) = \frac{1}{\kappa_2}\mathbf{X}(\mathbf{x})$; this only affects the parametrization (turning t to $\kappa_2^{-1}t$), the extrinsic geometric properties remain intact.

Step 1:

Given a $(t, t_0, \rho) \in [0, \pi/\kappa) \times [0, \pi/\kappa) \times [0, 1)$ $(= \mathbb{R}/(\pi\kappa^{-1}) \times \mathbb{R}/(\pi\kappa^{-1}) \times [0,1))$ we just need to find a $\min\{t_1 > 0\}$ such that $\boldsymbol{\gamma}_\kappa(t + t_1, t_0, \rho) \in \{(x_1, 0, x_3) : x_1 > 0\}$, i.e. $\gamma_{\kappa,1}(t + t_1, t_0, \rho) > 0$ that $\gamma_{\kappa,2}(t + t_1, t_0, \rho) = 0 \implies \sin(2\kappa(t + t_1) - 2t_0) = 0$ i.e. $2\kappa(t + t_1) - t_0 = n\pi$ for some $n \in \mathbb{Z}$, i.e. $t_1 = \dfrac{n\pi + 2t_0}{2\kappa} - t$.

However, we also want $\cos(2\kappa(t + t_1) + 2t_0) > 0$ which means that we want n to be an even integer, i.e $n = 2k$ for some $k \in \mathbb{Z}$ (clearly $\cos(2\kappa(t + t_1) - 2t_0) = 1$).

We can pick κ to be 1 without loss of generality, or 0 if we are going backwards in time, let it be 0.

Now, the half-plane $\{(x_1, 0, x_3) : x_1 > 0\}$ can naturally be identified with the complex half-plane with complex variable $x_1 + \mathrm{i}x_3$ (where now $x_{1,3}$ are functions of (t, t_0, ρ), effectively of (t_0, ρ)). We have:

$$\tilde{x}_1 = \frac{1-\rho^2}{1+\rho^2+2\rho\cos\big(2(t+\frac{t_0}{\kappa}-t)+2t_0\big)} = \frac{1-\rho^2}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}. \tag{4.1.21}$$

Recall $\rho \in [0, 1)$ so the above is clearly positive, and after the re-scaling (resp. re-parametrisation), we have $\mathbf{X} \to \kappa_2^{-1}\mathbf{X}$ (resp. $t \to \kappa_2^{-1}t$) we have respectively:

$$\tilde{x}_3 = \frac{2\rho\sin\big(2(1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}. \tag{4.1.22}$$

Step 2: We want to map $\tilde{x}_1 + \mathrm{i}\tilde{x}_3$ onto the unit disk $\tilde{\mathbb{D}}$. We will do that with the help of a *Cayley transform*: $z \mapsto e^{\mathrm{i}\theta}(az+b)(z+c)^{-1}$ for suitable choices of $a, b, c \in \mathbb{C}$ and $\theta \in [0, 2\pi)$. A relatively simple and neat choice of such constants will be to pick $a = 1$, $b = -1$, so that the point $1 + \mathrm{i}0 \equiv (1, 0)$ will be mapped to the origin, and now by picking $c = 1$ we make sure that the modulus will be less than 1. Then we have:

$$\begin{aligned}\tilde{z} - 1 &= \Big(\frac{1-\rho^2}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)} - 1\Big) + \frac{2\rho\sin\big(2(1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}\mathrm{i} \\ &= \frac{-2\rho(\rho+\rho\cos\big(2(1+\frac{1}{\kappa})t_0)\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)} + \frac{2\rho\sin\big(2(1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}\mathrm{i}\end{aligned}$$

and

$$\begin{aligned}\tilde{z} + 1 &= \Big(\frac{1-\rho^2}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)} + 1\Big) + \frac{2\rho\sin\big(2(1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}\mathrm{i} \\ &= \frac{2(1+\rho\cos\big((1+\frac{1}{\kappa})t_0)\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)} + \frac{2\rho\sin\big(2(1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}\mathrm{i}.\end{aligned}$$

Therefore

$$(\tilde{z}+1)(\overline{\tilde{z}}+1) = |\tilde{z}+1|^2 = |\tilde{z}|^2 + \tilde{z} + \overline{\tilde{z}} + 1 = |\tilde{z}|^2 + 2\,\mathrm{Re}(\tilde{z}) + 1 =$$

$$\frac{(1-\rho^2)^2+4\rho^2\sin^2(2(1+\frac{1}{\kappa})t_0))}{\big(1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)\big)^2}+\frac{2(1-\rho^2)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}+1$$

$$=\frac{(1-\rho^2)^2+4\rho^2-4\rho^2\cos^2(2(1+\frac{1}{\kappa})t_0))}{\big(1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)\big)^2}+\frac{2(1-\rho^2)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}+1$$

$$=\frac{(1+\rho^2)^2-4\rho^2\cos^2(2(1+\frac{1}{\kappa})t_0))}{\big(1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)\big)^2}+\frac{2(1-\rho^2)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}+1$$

$$=\frac{1+\rho^2-2\rho\cos\big((1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}+\frac{2(1-\rho^2)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}+1$$

$$=\frac{3-\rho^2-2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}+\frac{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}$$

$$\implies |\tilde{z}+1|^2=\frac{4}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}.$$

and $(\tilde{z}-1)(\overline{\tilde{z}}+1)=|\tilde{z}|^2-\overline{\tilde{z}}+\tilde{z}-1=|\tilde{z}|^2+2\mathrm{i}\,\mathrm{Im}(\tilde{z})-1=$

$$\frac{(1-\rho^2)^2+4\rho^2\sin^2(2(1+\frac{1}{\kappa})t_0)}{\big(1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)\big)^2}+\frac{4\rho\sin\big(2(1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}\mathrm{i}-1$$

$$=\frac{(1+\rho^2)^2-4\rho^2\cos^2(2(1+\frac{1}{\kappa})t_0)}{\big(1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)\big)^2}+\frac{4\rho\sin\big(2(1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}\mathrm{i}-1$$

$$=\frac{1+\rho^2-2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}+\frac{4\rho\sin\big(2(1+\frac{1}{\kappa})t_0\big)}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}\mathrm{i}-1$$

$$=\frac{-4\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)+4\rho\sin\big(2(1+\frac{1}{\kappa})t_0\big)\mathrm{i}}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}=-\frac{4\rho}{1+\rho^2+2\rho\cos\big(2(1+\frac{1}{\kappa})t_0\big)}e^{-\mathrm{i}2(1+\frac{1}{\kappa})t_0}$$

therefore: $\dfrac{\tilde{z}-1}{\tilde{z}+1}=-\rho e^{-\mathrm{i}2(1+\frac{1}{\kappa})t_0}$ and so for $\theta=\pi$ we have: $e^{\mathrm{i}\pi}\dfrac{\tilde{z}-1}{\tilde{z}+1}=\rho e^{-\mathrm{i}2(1+\frac{1}{\kappa})t_0}$.

Now, by conjugating the latter we obtain a neat map:

$$\chi_{\frac{1}{\kappa},2}(\mathbf{x}(t,t_0,\rho)):=\rho e^{\mathrm{i}2(1+\frac{1}{\kappa})t_0} \tag{4.1.23}$$

in "local" coordinates (t,ρ,t_0). In order to revert it into Euclidean coordinates, we notice: $\dfrac{|\tilde{z}_1|}{1+|\mathbf{x}|^2+|\tilde{z}_2|}=\dfrac{1-\rho}{1+\rho}$

and setting the LHS as A we have solve for ρ and we have: $\rho = \dfrac{1-A}{1+A}$, analytically:

$$\rho = \frac{1+|\mathbf{x}|^2 + \sqrt{(1-|\mathbf{x}|^2)^2 + 4x_3^2} - 2\sqrt{x_1^2+x_2^2}}{1+|\mathbf{x}|^2 + \sqrt{(1-|\mathbf{x}|^2)^2 + 4x_3^2} + 2\sqrt{x_1^2+x_2^2}} = \frac{1+|\mathbf{x}|^2+|\tilde{z}_2|-|\tilde{z}_1|}{1+|\mathbf{x}|^2+|\tilde{z}_2|+|\tilde{z}_1|}.$$

Recall $\tilde{z}_1 = x_1 + \mathrm{i}x_2$ and $\tilde{z}_2 = 1 - |\mathbf{x}|^2 + 2\mathrm{i}x_3$ and since we have: $\tilde{e}(\tilde{z}_1) = e^{2\kappa(t-t_0)\mathrm{i}}$ and $\tilde{e}(\tilde{z}_2) = e^{2(t+t_0)\mathrm{i}}$ we get that $\tilde{e}(\chi_{\frac{1}{\kappa},2}(\mathbf{x})) = \tilde{e}(\tilde{z}_1)^{-\frac{1}{\kappa}}\tilde{e}(\tilde{z}_2)$ and so

$$\chi_{\frac{1}{\kappa},2}(\mathbf{x}) = \frac{1+|\mathbf{x}|^2+|\tilde{z}_2|-|\tilde{z}_1|}{1+|\mathbf{x}|^2+|\tilde{z}_2|+|\tilde{z}_1|}\tilde{e}(\tilde{z}_1)^{-\frac{1}{\kappa}}\tilde{e}(\tilde{z}_2). \tag{4.1.24}$$

When there is no serious risk of confusion, we'll abuse notation and denote $\chi_{\kappa,1}$ and $\chi_{\frac{1}{\kappa},2}$ simply as $\chi_{\kappa^{\pm 1}}$ respectively. □

4.1.3 Mapping the unit disk to the complex plane

In order to deduce a (suitable) Riemannian-type submersion from (charts of) $\mathbb{R}^3$ to copies of $\mathbb{R}^2$, we'll look for functions $\tilde{F} : \mathbb{D} \mapsto \mathbb{C}$ such that when composed with $\chi_{\kappa,1}(\mathbf{x})$ we'll get a function $F \equiv F_{1,2}$ that's a Riemannian submersion from $\mathbb{R}^3$ to a Riemann surface. To find such $F_{1,2}$, we make the following ansatz:

$$\tilde{F}_{\kappa^{\pm 1}}(\tilde{z}) = \tilde{z} f_{\kappa^{\pm 1}}(|\tilde{z}|^2) \tag{4.1.25}$$

where $\tilde{r} = \sqrt{\tilde{z}\bar{\tilde{z}}} = |\tilde{z}|$ (we've put a square in $f_{\kappa^{\pm 1}}$ in order to ensure smoothness at the origin), and compose it with $\chi_{\kappa^{\pm 1}}$. Regarding the case of $\tilde{F}_1$ composed with χ_1, by slightly abusing of notation we have:

$$(\tilde{F}_\kappa \circ \chi_\kappa)(\mathbf{x}) = \tilde{F}_\kappa(\chi_\kappa(\mathbf{x})) = |\chi_\kappa(\mathbf{x})| f_\kappa(|\chi_\kappa(\mathbf{x})|^2)\tilde{e}(\tilde{z}_1)\tilde{e}^{-\kappa}(\tilde{z}_2) \tag{4.1.26}$$

This is going to be a radial scaling of χ_κ. To have a Riemannian submersion, we want: $d_t(\tilde{F}_\kappa \circ \chi_\kappa(\boldsymbol{\gamma}(t))) = 0 \implies$

$$\mathbf{X}(\mathbf{x}).\boldsymbol{\nabla}\big(\tilde{F}_\kappa \circ \chi_\kappa(\mathbf{x})\big) = 0, \text{ (where } \mathbf{x} = \gamma(t) \text{ for an integral curve } \gamma \text{ of } \mathbf{X} \text{ and } t \in \mathbb{R}).$$

We want to calculate: $\boldsymbol{\nabla}|\chi_\kappa(\mathbf{x})| = \boldsymbol{\nabla}\Big(\dfrac{2\sqrt{x_1^2+x_2^2}}{1+|\mathbf{x}|^2+\sqrt{(1-|\mathbf{x}|^2)^2+4x_3^2}}\Big)$, and to do that we re-write: $\dfrac{2\sqrt{x_1^2+x_2^2}}{1+|\mathbf{x}|^2+\sqrt{(1-|\mathbf{x}|^2)^2+4x_3^2}} = \dfrac{|\tilde{z}_1|}{1+|\mathbf{x}|^2+|\tilde{z}_2|} = \dfrac{1+|\mathbf{x}|^2-|\tilde{z}_2|}{|\tilde{z}_1|}$.

We proceed with these calculations in steps. First, we calculate the gradients:

$$\boldsymbol{\nabla}|\tilde{z}_2| = \boldsymbol{\nabla}\sqrt{|\tilde{z}_2|^2} = \boldsymbol{\nabla}\sqrt{\tilde{z}_2\overline{\tilde{z}_2}} = \frac{1}{2|\tilde{z}_2|}(\overline{\tilde{z}_2}\boldsymbol{\nabla}\tilde{z}_2 + \tilde{z}_2\boldsymbol{\nabla}\overline{\tilde{z}_2}), \tag{4.1.27}$$

$$\boldsymbol{\nabla}|\tilde{z}_1| = \boldsymbol{\nabla}\sqrt{|\tilde{z}_1|^2} = \boldsymbol{\nabla}\sqrt{\tilde{z}_1\overline{\tilde{z}_1}} = \frac{1}{2|\tilde{z}_1|}(\overline{\tilde{z}_1}\boldsymbol{\nabla}\tilde{z}_1 + \tilde{z}_1\boldsymbol{\nabla}\overline{\tilde{z}_1}). \tag{4.1.28}$$

Remark 4.1.5. Note that when defining $\tilde{F}_\kappa(z)$ we've naturally identified the usual plane $\mathbb{R}^2$ as the complex plane ($\mathbb{C}$). Assuming that $\tilde{F}_\kappa(z)$ can produce a Riemannian submersion from $\mathbb{R}^3_1$ to $\mathbb{R}^2$ (when composed with χ_κ), i.e. it (locally) preserves lengths and orthogonality of the basis of some 2-dimensional subspace of $\mathbb{R}^3$, without loss of generality, we assume that $\boldsymbol{\nabla}\operatorname{Re}(\tilde{F}_\kappa)$, $\boldsymbol{\nabla}\operatorname{Im}(\tilde{F}_\kappa)$ (or equivalently $\operatorname{Re}\boldsymbol{\nabla}\tilde{F}_\kappa$ and $\operatorname{Im}\boldsymbol{\nabla}\tilde{F}_\kappa$) correspond to the vectors whose lengths and orthogonality is preserved (i.e. they have the same length and they are orthogonal too). In fact, since we just need to show that some two orthogonal linear combinations of vectors $d_{t_0}(\tilde{F}_\kappa\circ\chi_\kappa(\mathbf{x}(t,t_0,\rho)))$ and $d_\rho(\tilde{F}_\kappa\circ\chi_\kappa(\mathbf{x}(t,t_0,\rho)))$ (both of these vectors are perpendicular to $d_t(\tilde{F}_\kappa\circ\chi_\kappa(\mathbf{x}(t,t_0,\rho))$, producing a plane) remain orthogonal (and have the same length) under $d\tilde{F}_\kappa$: $span\{d_{t_0}(\tilde{F}_\kappa\circ\chi_\kappa(\mathbf{x}(t,t_0,\rho)))\}^\perp \mapsto \mathbb{C}\equiv\mathbb{R}^2$ and both are linear combinations of $\operatorname{Re}\boldsymbol{\nabla}\big(\tilde{F}_\kappa\circ\chi_\kappa\big)$ and $\operatorname{Im}\boldsymbol{\nabla}\Big(\tilde{F}_\kappa\circ\chi_\kappa\Big)$. Then, since for any $\tilde{F}_{\kappa^{\pm1}}$ we can choose a linear transformation that satisfies this, we can pick $\tilde{F}_\kappa$ (which is yet unknown) to satisfy this in the first place.

Consider $\tilde{F}_\kappa$in the form (4.1.25) is a fruitful idea because it preserves angles and re-scales the radial part of the complex number $r:[0,1)\mapsto[0,\infty)$. Given the fact that $\boldsymbol{\nabla}\tilde{F}_{\kappa,1}$ is a complex vector, the ideas above can be translated to the equations $(\boldsymbol{\nabla}\tilde{F}_{\kappa,1})^2 = 0 = (\overline{\boldsymbol{\nabla}\tilde{F}_{\kappa,1}})^2$, equivalently:

$$(\operatorname{Re}\boldsymbol{\nabla}\tilde{F}_\kappa)^2 - (\operatorname{Im}\boldsymbol{\nabla}\tilde{F}_\kappa)^2) \pm 2\mathrm{i}(\operatorname{Re}\boldsymbol{\nabla}\tilde{F}_\kappa).(\operatorname{Im}\boldsymbol{\nabla}\tilde{F}_\kappa) = 0, \tag{4.1.29}$$

i.e. $(\operatorname{Re}\boldsymbol{\nabla}\tilde{F}_\kappa)^2 = (\operatorname{Im}\boldsymbol{\nabla}\tilde{F}_\kappa)^2 \implies |\operatorname{Re}\boldsymbol{\nabla}\tilde{F}_\kappa| = |\operatorname{Im}\boldsymbol{\nabla}\tilde{F}_\kappa|$ and

$(\mathrm{Re}\,\boldsymbol{\nabla}\tilde{F}_\kappa).(\mathrm{Im}\,\boldsymbol{\nabla}\tilde{F}_\kappa) = 0$ (persistence of lengths and orthogonality respectively).

We have:

Lemma 4.1.6. The gradient of $\chi_\kappa(\mathbf{x})$ satisfies:

$$\boldsymbol{\nabla}|\chi_\kappa| = \frac{|\chi_\kappa|}{2|\tilde{z}_2|(x_1^2+x_2^2)}\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x}). \tag{4.1.30}$$

Proof. We calculate:

$$\begin{aligned}\boldsymbol{\nabla}\Big(\frac{|\tilde{z}_1|}{1+|\mathbf{x}|^2+|\tilde{z}_2|}\Big) &= \frac{1}{1+|\mathbf{x}|^2+|\tilde{z}_2|}\boldsymbol{\nabla}|\tilde{z}_1| + |\tilde{z}_1|\boldsymbol{\nabla}\big(1+|\mathbf{x}|^2+|\tilde{z}_2|\big)^{-1} \\ &= \frac{1}{1+|\mathbf{x}|^2+|\tilde{z}_2|}\cdot\frac{1}{\sqrt{x_1^2+x_2^2}}\begin{pmatrix}x_1\\x_2\\0\end{pmatrix} - \frac{|\tilde{z}_1|}{(1+|\mathbf{x}|^2+|\tilde{z}_2|)^2}\left(\begin{pmatrix}2x_1\\2x_2\\2x_3\end{pmatrix} + \boldsymbol{\nabla}|\tilde{z}_2|\right) \\ &= \frac{(x_1^2+x_2^2)^{-1/2}}{1+|\mathbf{x}|^2+|\tilde{z}_2|}\begin{pmatrix}x_1\\x_2\\0\end{pmatrix} - \frac{4\sqrt{x_1^2+x_2^2}}{(1+|\mathbf{x}|^2+|\tilde{z}_2|)^2}\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} - |\tilde{z}_2|^{-1}\begin{pmatrix}x_1(1-|\mathbf{x}|^2)\\x_2(1-|\mathbf{x}|^2)\\-x_3(1+|\mathbf{x}|^2)\end{pmatrix}\right),\end{aligned} \tag{4.1.31}$$

which simplifies to

$$\begin{aligned}&\boldsymbol{\nabla}\Big(\frac{|\tilde{z}_1|}{1+|\mathbf{x}|^2+|\tilde{z}_2|}\Big) \\ &= |\chi_\kappa|\left(\frac{1}{x_1^2+x_2^2}\begin{pmatrix}x_1\\x_2\\0\end{pmatrix} - \frac{2}{1+|\mathbf{x}|^2+|\tilde{z}_2|}\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} - |\tilde{z}_2|^{-1}\begin{pmatrix}x_1(1-|\mathbf{x}|^2)\\x_2(1-|\mathbf{x}|^2)\\-x_3(1+|\mathbf{x}|^2)\end{pmatrix}\right)\right)\end{aligned} \tag{4.1.32}$$

$\implies$

$$\begin{aligned}&\boldsymbol{\nabla}\Big(\frac{|\tilde{z}_1|}{1+|\mathbf{x}|^2+|\tilde{z}_2|}\Big) = \\ &|\chi_{\kappa,1}|\Big(\frac{1}{x_1^2+x_2^2} - \frac{2}{1+|\mathbf{x}|^2+|\tilde{z}_2|}\Big(1-\frac{1-|\mathbf{x}|^2}{|\tilde{z}_2|}\Big)\Big)\begin{pmatrix}x_1\\x_2\\0\end{pmatrix} - \frac{|\chi_{\kappa,1}|}{|\tilde{z}_2|}\begin{pmatrix}0\\0\\x_3\end{pmatrix}.\end{aligned} \tag{4.1.33}$$

The coefficient of the first term in the RHS is

$$|\chi_{\kappa,1}|\left(\frac{(1+|\mathbf{x}|^2+|\tilde{z}_2|)|\tilde{z}_2|-2(x_1^2+x_2^2)(|\tilde{z}_2|+|\mathbf{x}|^2-1)}{(1+|\mathbf{x}|^2+|\tilde{z}_2|)|\tilde{z}_2|(x_1^2+x_2^2)}\right).$$

By observing that the numerator inside the parentheses is

$$(1+|\mathbf{x}|^2-2(x_1^2+x_2^2))|\tilde{z}_2|+|\tilde{z}_2|^2-2(x_1^2+x_2^2)|\mathbf{x}|^2+2(x_1^2+x_2^2)=$$

$$\begin{aligned}
&(1-x_1^2-x_2^2+x_3^2)|\tilde{z}_2|+(1-|\mathbf{x}|^2)^2+4x_3^2-2(x_1^2+x_2^2)|\mathbf{x}|^2+2(x_1^2+x_2^2)\\
&=(1-x_1^2-x_2^2+x_3^2)|\tilde{z}_2|+(1-|\mathbf{x}|^2)^2+4(x_1^2+x_2^2+x_3^2)-2(x_1^2+x_2^2)|\mathbf{x}|^2-2(x_1^2+x_2^2)\\
&=(1-x_1^2-x_2^2+x_3^2)|\tilde{z}_2|+(1-|\mathbf{x}|^2)^2+4|\mathbf{x}|^2-2(x_1^2+x_2^2)|\mathbf{x}|^2-2(x_1^2+x_2^2)\\
&=(1-x_1^2-x_2^2+x_3^2)|\tilde{z}_2|+(1+|\mathbf{x}|^2)^2-2(x_1^2+x_2^2)(1+|\mathbf{x}|^2)\\
&=(1-x_1^2-x_2^2+x_3^2)|\tilde{z}_2|+(1+|\mathbf{x}|^2)(1+|\mathbf{x}|^2-2(x_1^2+x_2^2))\\
&=(1-x_1^2-x_2^2+x_3^2)|\tilde{z}_2|+(1+|\mathbf{x}|^2)(1-x_1^2-x_2^2+x_3^2)\\
&=(1-x_1^2-x_2^2+x_3^2)(1+|\mathbf{x}|^2+|\tilde{z}_2|).
\end{aligned}$$

Therefore, we have:

$$\boldsymbol{\nabla}|\chi_{\kappa,1}|=\frac{|\chi_{\kappa,1}|}{|\tilde{z}_2|(x_1^2+x_2^2)}\left((1-x_1^2-x_2^2+x_3^2)\begin{pmatrix}x_1\\x_2\\0\end{pmatrix}-(x_1^2+x_2^2)\begin{pmatrix}0\\0\\x_3\end{pmatrix}\right)\implies$$

$$\boldsymbol{\nabla}|\chi_{\kappa,1}|=\frac{|\chi_{\kappa,1}|}{2|\tilde{z}_2|(x_1^2+x_2^2)}\mathbf{W}_1(\mathbf{x})\times\mathbf{W}_2(\mathbf{x}) \tag{4.1.34}$$

where $\mathbf{W}_1(\mathbf{x})=(-2x_2,2x_1,0)$ and $\mathbf{W}_2(\mathbf{x})=(2x_1x_3,2x_2x_3,1-|\mathbf{x}|^2+2x_3^2)$. □

This is an important step towards deriving the equations since:

$$\begin{aligned}
\boldsymbol{\nabla}\big(\tilde{F}_\kappa\circ\chi_\kappa(\mathbf{x})\big)&=\boldsymbol{\nabla}\tilde{F}_\kappa(\chi_\kappa(\mathbf{x}))=(\boldsymbol{\nabla}\chi_\kappa)\tilde{f}_\kappa(|\chi_\kappa|^2)+\chi_\kappa\tilde{f}'_\kappa(|\chi_{\kappa,1}|^2)\boldsymbol{\nabla}|\chi_{\kappa,1}|^2\\
&=r\tilde{f}'_{\kappa,1}(r^2)(\boldsymbol{\nabla}r)\tilde{e}(\chi_\kappa)+\tilde{f}_\kappa(r^2)(\boldsymbol{\nabla}r^2)\tilde{e}(\chi_\kappa)+r\tilde{f}_\kappa(r^2)\boldsymbol{\nabla}\tilde{e}(\chi_\kappa)
\end{aligned}$$

where we calculate the gradient of $\tilde{e}(\chi_{\kappa,1})$ by calculating each one of its components $(\tilde{e}(\tilde{z}_{2,1}))$ individually, and we do that by noticing:

$$\boldsymbol{\nabla}\tilde{e}(\tilde{z}_{2,1}) = \boldsymbol{\nabla}\left(\frac{\tilde{z}_{2,1}}{|\tilde{z}_{2,1}|}\right) = \frac{\mathrm{i}\tilde{z}_{2,1}}{|\tilde{z}_{2,1}|}\boldsymbol{\nabla}\arg(\tilde{z}_{2,1}). \tag{4.1.35}$$

For future reference, we have:

Lemma 4.1.7. The auxiliary variables $\tilde{z}_{1,2}$ satisfy:

$$\boldsymbol{\nabla}\arg(\tilde{z}_1) = \frac{1}{2(x_1^2+x_2^2)}\mathbf{W}_1(\mathbf{x}) \tag{4.1.36}$$

and

$$\boldsymbol{\nabla}\arg(\tilde{z}_2) = -\frac{2}{(1-|\mathbf{x}|^2)^2+4x_3^2}\mathbf{W}_2(\mathbf{x}). \tag{4.1.37}$$

Proof. We have:

$$\boldsymbol{\nabla}\arg(\tilde{z}_{2,1}) = -\frac{\mathrm{i}\overline{\tilde{z}}_{2,1}}{|\tilde{z}_{2,1}|}\boldsymbol{\nabla}\left(\frac{\tilde{z}_{2,1}}{|\tilde{z}_{2,1}|}\right) = -\frac{\mathrm{i}\overline{\tilde{z}}_{2,1}}{|\overline{\tilde{z}}_{2,1}|}\left(\frac{1}{|\tilde{z}_{2,1}|}\boldsymbol{\nabla}\tilde{z}_{2,1} + \tilde{z}_{2,1}\boldsymbol{\nabla}|\tilde{z}_{2,1}|^{-1}\right)$$

$$= -\frac{\mathrm{i}\overline{\tilde{z}}_{2,1}}{|\tilde{z}_{2,1}|}\left(\frac{1}{|\tilde{z}_{2,1}|}\boldsymbol{\nabla}\tilde{z}_{2,1} - \frac{\tilde{z}_{2,1}}{|\tilde{z}_{2,1}|^2}\boldsymbol{\nabla}|\tilde{z}_{2,1}|\right) = -\frac{\mathrm{i}\overline{\tilde{z}}_{2,1}}{|\tilde{z}_{2,1}|^2}\left(\boldsymbol{\nabla}\tilde{z}_{2,1} - \frac{\tilde{z}_{2,1}}{2|\tilde{z}_{2,1}|^2}\boldsymbol{\nabla}\left(\tilde{z}_{1,2}\overline{\tilde{z}}_{1,2}\right)\right)$$

$$= -\frac{\mathrm{i}\overline{\tilde{z}}_{2,1}}{|\tilde{z}_{2,1}|^2}\left(\boldsymbol{\nabla}\tilde{z}_{2,1} - \frac{\tilde{z}_{2,1}}{2|\tilde{z}_{2,1}|^2}\left(\overline{\tilde{z}}_{2,1}\boldsymbol{\nabla}\tilde{z}_{2,1} + \tilde{z}_{2,1}\boldsymbol{\nabla}\overline{\tilde{z}}_{2,1}\right)\right).$$

We recall: $\boldsymbol{\nabla}|\tilde{z}_1| = \boldsymbol{\nabla}\sqrt{x_1^2+x_2^2} = (x_1^2+x_2^2)^{-1/2}(x_1,x_2,0)^T$ or equivalently:

$$\boldsymbol{\nabla}|\tilde{z}_1| = \boldsymbol{\nabla}\sqrt{\tilde{z}_1\overline{\tilde{z}}_1} = \frac{1}{2|\tilde{z}_1|}\boldsymbol{\nabla}\left(\tilde{z}_1\overline{\tilde{z}}_1\right) = \frac{1}{2|\tilde{z}_1|}\boldsymbol{\nabla}\left(\tilde{z}_1\overline{\tilde{z}}_1\right) = \frac{1}{2|\tilde{z}_1|}\left(\overline{\tilde{z}}_1\boldsymbol{\nabla}\tilde{z}_1 + \tilde{z}_1\boldsymbol{\nabla}\overline{\tilde{z}}_1\right). \tag{4.1.38}$$

Analytically: $\boldsymbol{\nabla}\tilde{z}_1 = \boldsymbol{\nabla}(x_1+\mathrm{i}x_2) = (1,\mathrm{i},0)^{\mathrm{T}}$, $\boldsymbol{\nabla}\overline{\tilde{z}}_1 = \boldsymbol{\nabla}(x_1-\mathrm{i}x_2) = (1,-\mathrm{i},0)^{\mathrm{T}}$ and $\boldsymbol{\nabla}\tilde{z}_2 = \boldsymbol{\nabla}\left(1-|\mathbf{x}|^2+2\mathrm{i}x_3\right) = \boldsymbol{\nabla}\left(1-x_1^2-x_2^2-x_3^2+2\mathrm{i}x_3\right) = (-2x_1,-2x_2,-2x_3+2\mathrm{i})$ and

$$\boldsymbol{\nabla}\overline{z}_2 = \boldsymbol{\nabla}\left(1-|\mathbf{x}|^2-2\mathrm{i}x_3\right) = \boldsymbol{\nabla}\left(1-x_1^2-x_2^2-x_3^2-2\mathrm{i}x_3\right) = (-2x_1,-2x_2,-2x_3-2\mathrm{i}),$$

and we calculate $\overline{\tilde{z}}_2\boldsymbol{\nabla}\tilde{z}_1 = (x_1-\mathrm{i}x_2)\boldsymbol{\nabla}(x_1+\mathrm{i}x_2) = (x_1-\mathrm{i}x_2)(1,\mathrm{i},0)^{\mathrm{T}}$ and so

$$\tilde{z}_1 \boldsymbol{\nabla} \overline{\tilde{z}}_1 = (x_1 + \mathrm{i}x_2) \boldsymbol{\nabla}(x_1 - \mathrm{i}x_2) = (x_1 + \mathrm{i}x_2)(1, -\mathrm{i}, 0)^{\mathrm{T}}$$

$$\implies \boldsymbol{\nabla} \tilde{z}_1 - \frac{\tilde{z}_1}{2|\tilde{z}_1|^2}(\tilde{z}_1 \boldsymbol{\nabla} \overline{\tilde{z}}_1 + \overline{\tilde{z}}_1 \boldsymbol{\nabla} \tilde{z}_1) = \begin{pmatrix} 1 \\ \mathrm{i} \\ 0 \end{pmatrix} - \frac{x_1 + \mathrm{i}x_2}{x_1^2 + x_2^2} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

$$\implies \frac{\overline{\tilde{z}}_1}{|\tilde{z}_1|^2} \boldsymbol{\nabla} \tilde{z}_1 - \frac{|\tilde{z}_1|^2}{2|\tilde{z}_1|^4}(\tilde{z}_1 \boldsymbol{\nabla} \overline{\tilde{z}}_1 + \overline{\tilde{z}}_1 \boldsymbol{\nabla} \tilde{z}_1) = \frac{x_1 - \mathrm{i}x_2}{x_1^2 + x_2^2} \begin{pmatrix} 1 \\ \mathrm{i} \\ 0 \end{pmatrix} - \frac{1}{x_1^2 + x_2^2} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}.$$

That is,

$$\boldsymbol{\nabla} \arg(\tilde{z}_1) = -\frac{\mathrm{i}}{x_1^2 + x_2^2} \begin{pmatrix} -\mathrm{i}x_2 \\ \mathrm{i}x_1 \\ 0 \end{pmatrix} = \frac{1}{2(x_1^2 + x_2^2)} \mathbf{W}_1(\mathbf{x}). \tag{4.1.39}$$

Similarly, by (4.1.27), we compute

$$\boldsymbol{\nabla}|\tilde{z}_2| = \frac{1}{\sqrt{(1 - |\mathbf{x}|^2)^2 + 4x_3^2}} \left((1 - |\mathbf{x}|^2 - 2\mathrm{i}x_3) \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 + \mathrm{i} \end{pmatrix} + (1 - |\mathbf{x}|^2 + \mathrm{i}x_3) \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 - \mathrm{i} \end{pmatrix} \right)$$

$$= \frac{1}{\sqrt{(1 - |\mathbf{x}|^2)^2 + 4x_3^2}} \begin{pmatrix} -2x_1(1 - |\mathbf{x}|^2) \\ -2x_2(1 - |\mathbf{x}|^2) \\ 2x_3(1 + |\mathbf{x}|^2) \end{pmatrix}.$$

We could bring it all together to compute

$$\boldsymbol{\nabla} \tilde{e}^{-\kappa}(\tilde{z}_2) =$$

$$-\kappa \tilde{e}^{-\kappa-1}(\tilde{z}_2) \boldsymbol{\nabla} \tilde{e}(\tilde{z}_2) = -\kappa \mathrm{i} \tilde{e}^{-\kappa-1}(\tilde{z}_2) \tilde{e}(\tilde{z}_2) \boldsymbol{\nabla} \arg(\tilde{z}_2) = -\kappa \tilde{e}^{-\kappa}(\tilde{z}_2) \frac{|\tilde{z}_2|}{\tilde{z}_2} \boldsymbol{\nabla}\Big(\frac{\tilde{z}_2}{|\tilde{z}_2|}\Big)$$

$$= -\kappa \tilde{e}^{-\kappa}(\tilde{z}_2) \frac{|\tilde{z}_2|}{\tilde{z}_2} \big(|\tilde{z}_2|^{-1} \boldsymbol{\nabla} \tilde{z}_2 + \tilde{z}_2 \boldsymbol{\nabla} |\tilde{z}_2|^{-1}\big) = -\kappa \tilde{e}^{-\kappa}(\tilde{z}_2) \Big(\frac{1}{\tilde{z}_2} \boldsymbol{\nabla} \tilde{z}_2 - \frac{1}{|\tilde{z}_2|} \boldsymbol{\nabla} |\tilde{z}_2|\Big)$$

$$= -\kappa \tilde{e}^{-\kappa}(\tilde{z}_2) \Big(\frac{1}{\tilde{z}_2} \boldsymbol{\nabla} \tilde{z}_2 - \frac{1}{|\tilde{z}_2|} \boldsymbol{\nabla} |\tilde{z}_2|\Big) = -\kappa \tilde{e}^{-\kappa}(\tilde{z}_2) \Big(\frac{1}{\tilde{z}_2} \boldsymbol{\nabla} \tilde{z}_2 - \frac{1}{|\tilde{z}_2|} \boldsymbol{\nabla} \sqrt{\tilde{z}_2 \overline{\tilde{z}}_2}\Big)$$

$$= \quad -\kappa\tilde{e}^{-\kappa}(\tilde{z}_2)\left(\frac{1}{1-|\mathbf{x}|^2+2\mathrm{i}x_3}\begin{pmatrix}-2x_1\\-2x_2\\-2x_3+2\mathrm{i}\end{pmatrix}-\frac{1}{(1-|\mathbf{x}|^2)^2+4x_3^2}\begin{pmatrix}-2x_1(1-|\mathbf{x}|^2)\\-2x_2(1-|\mathbf{x}|^2)\\2x_3(1+|\mathbf{x}|^2)\end{pmatrix}\right)$$

$$= -\kappa\tilde{e}^{-\kappa}(\tilde{z}_2)\left(\frac{1-|\mathbf{x}|^2-2\mathrm{i}x_3}{(1-|\mathbf{x}|^2)^2+4x_3^2}\begin{pmatrix}-2x_1\\-2x_2\\-2x_3+2\mathrm{i}\end{pmatrix}-\frac{1}{(1-|\mathbf{x}|^2)^2+4x_3^2}\begin{pmatrix}-2x_1(1-|\mathbf{x}|^2)\\-2x_2(1-|\mathbf{x}|^2)\\2x_3(1+|\mathbf{x}|^2)\end{pmatrix}\right)$$

$$= -\frac{2\kappa(\tilde{e}(\tilde{z}_2))^{-\kappa}}{(1-|\mathbf{x}|^2)^2+4x_3^2}\begin{pmatrix}2\mathrm{i}x_1x_3\\2\mathrm{i}x_2x_3\\\mathrm{i}(1-|\mathbf{x}|^2+2x_3^2)\end{pmatrix} = -\frac{2\mathrm{i}\kappa(\tilde{e}(\tilde{z}_2))^{-\kappa}}{(1-|\mathbf{x}|^2)^2+4x_3^2}\mathbf{W}_2(\mathbf{x})$$

$$\Longrightarrow \boldsymbol{\nabla}\arg(\tilde{z}_2) = -\frac{2}{(1-|\mathbf{x}|^2)^2+4x_3^2}\mathbf{W}_2(\mathbf{x}). \tag{4.1.40}$$

□

4.1.4 Properties of weights on $\mathbb{R}^2$ and $\mathbb{R}^3$

Proposition 4.1.8. The weights, $\omega_{1,2}$, Ω on $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, satisfy:

$$\omega_{1,2}(|F_{1,2}(\mathbf{x})|^2)|\mathbf{P}_{1,2}(\mathbf{x})| = \omega_{1,2}(|F_{1,2}(\mathbf{x})|^2)|\mathbf{Q}_{1,2}(\mathbf{x})| = \Omega(\mathbf{x}) \tag{4.1.41}$$

for $\mathbf{x} \in \mathbb{R}^3_{1,2}$.

Proof. From the last part of the proof of the previous proposition, in particular, (4.2.30), and for distinct $j, j' = 1, 2$ we have:

$$\omega_j(|F_j(\mathbf{x}))|^2) = \frac{1}{\kappa_{j'}} \frac{\sqrt{h_j^{-1}(|F_j(\mathbf{x})|^2)}(1 - h_j^{-1}(|F_j(\mathbf{x})|^2))}{|F_j(\mathbf{x})|((1 - h_j^{-1}(|F_j(\mathbf{x})|^2))^2 + 4\kappa^{3-2j} h_j^{-1}(|F_j(\mathbf{x})|^2))}$$

$$= \frac{\sqrt{s_j}(1 - s_j)}{\kappa_{j'}\sqrt{h_1(s_j)}((1 - s_j)^2 + 4\kappa^{2(3-2j)} s_j)}.$$

Now, for Ω we have:

$$\Omega(\mathbf{x}) = \frac{1}{|\mathbf{X}(\mathbf{x})|} = \frac{1}{(\kappa_1^2|\tilde{z}_1|^2 + \kappa_2^2|\tilde{z}_2|^2)^{1/2}} = \frac{1}{\kappa_2|\tilde{z}_2|\sqrt{1 + \kappa^2\xi_1^2}} = \frac{1}{\kappa_1|\tilde{z}_1|\sqrt{1 + \kappa^{-2}\xi_2^2}},$$

recall $\xi_1 = |\tilde{z}_1||\tilde{z}_2|^{-1}$ and $\xi_2 = \xi_1^{-1}$. Now, from (4.1.48) and having set $s_1 = |\chi_{\kappa,1}|^2 \equiv |\chi_1|^2$, by (3.2.17), we get: $\xi_1^2 = \dfrac{4s_1}{(1 - s_1)^2} \implies$

$$\left(\sqrt{1 + \kappa^2\xi_1^2}\right)^{-1} = \left(\sqrt{1 + \frac{4\kappa^2 s_1}{(1 - s_1)^2}}\right)^{-1} = \left((1 - s_1)^{-1}\sqrt{(1 - s_1)^2 + 4\kappa^2 s_1}\right)^{-1}$$

$$= \frac{1 - s_1}{\sqrt{(1 - s_1)^2 + 4\kappa^2 s_1}}.$$

Similarly,

$$\left(\sqrt{1 + \kappa^{-2}\frac{\xi_2^2}{4}}\right)^{-1} = \left(\sqrt{1 + \frac{4^2\kappa^{-2} s_2}{4(1 - s_2)^2}}\right)^{-1} = \frac{1 - s_2}{\sqrt{(1 - s_2)^2 + 4\kappa^2 s_2}}$$

$$\implies \Omega(\mathbf{x}) = \frac{1-s_1}{\kappa_2|\tilde{z}_2|\sqrt{(1-s_1)^2+4\kappa^2 s_1}} = \frac{1-s_2}{\kappa_1|\tilde{z}_1|\sqrt{(1-s_2)^2+4\kappa^{-2} s_2}}.$$

So,

$$\omega_j(F_j(\mathbf{x}))|\mathbf{Q}_j(\mathbf{x})| = \frac{\sqrt{s_j}(1-s_j)}{\kappa_{j'}((1-s_j)^2+4\kappa^{2(3-2j)}s_j)}|\mathbf{Q}_j(\mathbf{x})|.$$

Now,

$$|\mathbf{Q}_1(\mathbf{x})| = \sqrt{\frac{1}{x_1^2+x_2^2}+\frac{4\kappa^2}{(1-|\mathbf{x}|^2)^2+4x_3^2}} = \sqrt{\frac{1}{x_1^2+x_2^2}+\frac{4\kappa^2}{(1-|\mathbf{x}|^2)^2+4x_3^2}}$$

$$= \frac{1}{\sqrt{x_1^2+x_2^2}}\sqrt{1+\frac{4\kappa^2(x_1^2+x_2^2)}{(1-|\mathbf{x}|^2)^2+4x_3^2}} = \frac{1}{|\tilde{z}_1|}\sqrt{1+\kappa^2\xi_1^2} = \frac{1}{|\tilde{z}_1|}\sqrt{1+\frac{4\kappa^2 s_1}{(1-s_1)^2}}$$

$$= \frac{1}{|\tilde{z}_1|}\frac{\sqrt{(1-s_1)^2+4\kappa^2 s_1}}{(1-s_1)}$$

$$\implies \omega_1(F_1(\mathbf{x}))|\mathbf{Q}_1(\mathbf{x})| = \frac{\sqrt{s_1}}{\kappa_2|\tilde{z}_1|\sqrt{((1-s_1)^2+4\kappa^2 s_1)}}.$$

However, $\dfrac{\sqrt{s_1}}{|\tilde{z}_1|} = \dfrac{\sqrt{s_1}}{\xi_2|\tilde{z}_2|}$ and now because $\xi_1 = \dfrac{\sqrt{s_1}}{1-s_1}$ we have:

$$\omega_1(|F_1(\mathbf{x})|^2)|\mathbf{Q}_1(\mathbf{x})| = \frac{(1-s_1)}{\kappa_2|\tilde{z}_2|\sqrt{(1-s_1)^2+4\kappa^2 s_1}} = \Omega(\mathbf{x}).$$

Similarly for $\mathbf{Q}_2(\mathbf{x})$. From the definition of $\mathbf{Q}(\mathbf{x})$ and $\mathbf{P}(\mathbf{x})$. Simple calculations show that $|\mathbf{P}_{1,2}(\mathbf{x})| = |\mathbf{Q}_{1,2}(\mathbf{x})|$. □

As a result, the map $F \equiv F_{1,2} : \mathbb{R}^3_{1,2} \to \mathbb{C} \equiv \mathbb{R}^2$ defines a Riemannian submersion on the respective spaces with weights Ω and ω. This is true because, for $F := F_{1,2} \equiv (\operatorname{Re} F_{1,2}, \operatorname{Im} F_{1,2}) : \mathbb{R}^3_{1,2} \to \mathbb{R}^2$ to act as a submersion from 3 (Euclidean) dimensions to 2, we need to introduce (smooth) weights, $\omega_{1,2} : \mathbb{R}^2 \to \mathbb{R}_{>0}$ in order to make this map a partial isometry. where $\mathbb{C}_{1,2} \ni z_{1,2} = x_{1,2} + \mathrm{i}y_{1,2} \equiv (x_{1,2}, y_{1,2}) \in \mathbb{R}^2$. The equation corresponding to the imaginary parts is the same since $|\boldsymbol{\nabla} \operatorname{Re} F_{\kappa,1}(\mathbf{x})| = |\boldsymbol{\nabla} \operatorname{Im} F_{\kappa,1}(\mathbf{x}))|$. Simple calculation. This condition is satisfied for any choice of weights satisfying (4.2.7) introduced in (4.2.15), (4.2.16).

4.1.5 The complete formula for the submersion

We want to obtain a Riemann-type submersion, F, from the $\mathbb{R}^3$ (or at least, charts of it) to $\mathbb{R}^2$ (or copies of it) such that $\mathbf{X}.\boldsymbol{\nabla} F(\mathbf{x}) = 0$. In this particular, we derive such maps $F_{1,2}$ from $\mathbb{R}^3_{1,2}$ to $\mathbb{C} \equiv \mathbb{R}^2$ explicitly. First, slightly abusing notation, we write:

$$\boldsymbol{\nabla} F_{1,2}(\mathbf{x}) = \boldsymbol{\nabla} \operatorname{Re} F_{1,2}(\mathbf{x}) + \mathrm{i} \boldsymbol{\nabla} \operatorname{Im} F_{1,2}(\mathbf{x})$$

and we notice that if we make the ansatz $F_j(\mathbf{x}) = \tilde{F}_j(\chi_j(\mathbf{x})) = \chi_j(\mathbf{x}) f_{\kappa^{3-2j}}(|\chi_j(\mathbf{x})|^2)$ (where the κ-dependence has been dropped from χ's for simplicity, $j = 1, 2$) we have:

$$\boldsymbol{\nabla} F_{1,2} = f_{\kappa^{\pm 1}}(|\chi_{1,2}|^2) \boldsymbol{\nabla} \chi_{1,2} + \chi_{1,2} \boldsymbol{\nabla} |\chi_{1,2}|^2 f'_{\kappa^{\pm 1}}(|\chi_{1,2}|^2)$$

for some $f : [0, 1) \mapsto [0, \infty)$. To find a formula for $F_{1,2}$ we'll use the following:

Lemma 4.1.9. The maps $\chi_{\kappa,1} : \mathbb{R}^3_1 \to \mathbb{D}$, $\chi_{\frac{1}{\kappa},2} : \mathbb{R}^3_2 \to \mathbb{D}$ satisfy:

$$\boldsymbol{\nabla} \chi_{\kappa,1} = \chi_{\kappa,1} \left(\frac{1}{2|\tilde{z}_2|(x_1^2 + x_2^2)} \mathbf{W}_1 \times \mathbf{W}_2 + \frac{\mathrm{i}}{2(x_1^2 + x_2^2)} \mathbf{W}_1 - \frac{2\kappa \mathrm{i}}{(1 - |\mathbf{x}|^2)^2 + 4x_3^2} \mathbf{W}_2 \right), \tag{4.1.42}$$

$$\boldsymbol{\nabla} \chi_{\frac{1}{\kappa},2} = \chi_{\kappa^{-1},2} \left(- \frac{4|\chi_{\kappa,1}|}{(1 - |\chi_{\kappa,1}|^2)|\tilde{z}_2||\tilde{z}_1|^2} \mathbf{W}_1 \times \mathbf{W}_2 - \frac{\mathrm{i}}{2\kappa(x_1^2 + x_2^2)} \mathbf{W}_1 + \frac{2\mathrm{i}}{(1 - |\mathbf{x}|^2)^2 + 4x_3^2} \mathbf{W}_2 \right). \tag{4.1.43}$$

Proof. We have:

$$\boldsymbol{\nabla} \chi_{\kappa,1} = \big(\boldsymbol{\nabla} |\chi_{\kappa,1}|\big) \tilde{e}(\tilde{z}_1) \tilde{e}^{-\kappa}(\tilde{z}_2) + |\chi_{\kappa,1}| \big(\big(\boldsymbol{\nabla} \tilde{e}(\tilde{z}_1\big) \tilde{e}^{-\kappa}(\tilde{z}_2))\big)$$

$$= \big(\boldsymbol{\nabla} |\chi_{\kappa,1}|\big) \tilde{e}(\tilde{z}_1) \tilde{e}^{-\kappa}(\tilde{z}_2) + |\chi_{\kappa,1}| \big(\boldsymbol{\nabla} \tilde{e}(\tilde{z}_1)\big) \tilde{e}^{-\kappa}(\tilde{z}_2) + |\chi_{\kappa,1}| \tilde{e}(\tilde{z}_1) \boldsymbol{\nabla} \tilde{e}^{-\kappa}(\tilde{z}_2)$$

$$= \frac{|\chi_{\kappa,1}|}{2} \tilde{e}(\tilde{z}_1) \tilde{e}^{-\kappa}(\tilde{z}_2) \left(\frac{1}{|\tilde{z}_2|(x_1^2 + x_2^2)} \mathbf{W}_1 \times \mathbf{W}_2 + \frac{\mathrm{i}}{(x_1^2 + x_2^2)} \mathbf{W}_1 - \frac{4\kappa \mathrm{i}}{(1 - |\mathbf{x}|^2)^2 + 4x_3^2} \mathbf{W}_2 \right)$$

$$\Longrightarrow$$

$$\boldsymbol{\nabla} \chi_{\kappa,1} = \chi_{\kappa,1} \left(\frac{1}{2|\tilde{z}_2|(x_1^2 + x_2^2)} \mathbf{W}_1 \times \mathbf{W}_2 + \frac{\mathrm{i}}{2(x_1^2 + x_2^2)} \mathbf{W}_1 - \frac{2\kappa \mathrm{i}}{(1 - |\mathbf{x}|^2)^2 + 4x_3^2} \mathbf{W}_2 \right). \tag{4.1.44}$$

Regarding the other map, we have:

$$\boldsymbol{\nabla}\chi_{\frac{1}{\kappa},2} = \big(\boldsymbol{\nabla}|\chi_{\kappa^{-1},2}|\big)e^{-\frac{1}{\kappa}}(\tilde{z}_1)\tilde{e}(\tilde{z}_2) + |\chi_{\kappa^{-1},2}|\big((\boldsymbol{\nabla}e^{-\frac{1}{\kappa}}(\tilde{z}_1)\tilde{e}(\tilde{z}_2))\big)$$

$$= \big(\boldsymbol{\nabla}|\chi_{\kappa^{-1},2}|\big)e^{-\frac{1}{\kappa}}(\tilde{z}_1)\tilde{e}(\tilde{z}_2) + |\chi_{\kappa,1}|\big(\boldsymbol{\nabla}e^{-\frac{1}{\kappa}}(\tilde{z}_1)\big)\tilde{e}(\tilde{z}_2) + |\chi_{\kappa,1}|e^{-\frac{1}{\kappa}}(\tilde{z}_1)\boldsymbol{\nabla}\tilde{e}(\tilde{z}_2)$$

$$= \chi_{\kappa^{-1},2}\bigg(-\frac{4|\chi_{\kappa,1}|}{(1-|\chi_{\kappa,1}|^2)|\tilde{z}_2||\tilde{z}_1|^2}\mathbf{W}_1\times\mathbf{W}_2 + \mathrm{i}\Big(-\frac{1}{2\kappa(x_1^2+x_2^2)}\mathbf{W}_1 + \frac{2}{(1-|\mathbf{x}|^2)^2+4x_3^2}\mathbf{W}_2\Big)\bigg).$$

□

Now we're ready for:

Theorem 4.1.10. There exist surjective maps $F_{1,2} : \mathbb{R}^3_{1,2} \to \mathbb{C}$, such that $\mathbf{X}.\boldsymbol{\nabla}F_{1,2} = 0$ and $|\boldsymbol{\nabla}\operatorname{Im}F_{1,2}| = |\boldsymbol{\nabla}\operatorname{Re}F_{1,2}|$. Consequently, $F_{1,2}$ defines a Riemannian-type submersion from $\mathbb{R}^3_{1,2}$ to $\mathbb{C}$ up to a choice of a suitable branch of a Riemann surface due to the emerging multivaluedness. These maps satisfy:

$$F_1(\mathbf{x}) = (\tilde{F}_\kappa \circ \chi_{\kappa,1})(\mathbf{x}) = C_1 e^{\int \frac{\sqrt{1+\kappa^2\xi_1^2}}{\xi_1\sqrt{1+\xi_1^2}}d\xi_1}\tilde{e}(\tilde{z}_1)\epsilon_a^{-\kappa}(\tilde{z}_2), \quad \xi_1 = |\tilde{z}_1|/|\tilde{z}_2| \tag{4.1.45}$$

and

$$F_2(\mathbf{x}) = f_{\kappa^{-1},2}(|\chi_{\kappa^{-1},2}|^2) = C_2 e^{\int \frac{\sqrt{1+\frac{\xi_2^2}{\kappa^2}}}{\xi_2\sqrt{1+\xi_2^2}}d\xi_2}\epsilon_{a'}^{-1/\kappa}(\tilde{z}_1)\tilde{e}(\tilde{z}_2), \quad \xi_2 = |\tilde{z}_2|/|\tilde{z}_1| \tag{4.1.46}$$

where $\epsilon^b_{b'}(z)$ ($z \in \mathbb{C}^*$, $b \in \mathbb{R}$) denotes a branch of the exponential $\tilde{e}^b(z) := (z/|z|)^b$, indexed by b' in the sense $\epsilon^b_{b'}(z) = e^{b(\mathrm{i}Arg(z)+\mathrm{i}2\pi b')}$.

Proof. We have,

$$\boldsymbol{\nabla}F_{\kappa,1} = f_{\kappa,1}(|\chi_{\kappa,1}|^2)\boldsymbol{\nabla}\chi_{\kappa,1} + \chi_{\kappa,1}(\boldsymbol{\nabla}|\chi_{\kappa,1}|^2)f'_{\kappa,1}(|\chi_{\kappa,1}|^2)$$

$$= f_{\kappa,1}(|\chi_{\kappa,1}|^2)\boldsymbol{\nabla}\chi_{\kappa,1} + \chi_{\kappa,1}(\boldsymbol{\nabla}\big(\chi_{\kappa,1}\overline{\chi}_{\kappa,1}\big))f'_{\kappa,1}(|\chi_{\kappa,1}|^2)$$

$$= f_{\kappa,1}(|\chi_{\kappa,1}|^2)\boldsymbol{\nabla}\chi_{\kappa,1} + \chi_{\kappa,1}\big((\boldsymbol{\nabla}\chi_{\kappa,1})\overline{\chi}_{\kappa,1} + \chi_{\kappa,1}\boldsymbol{\nabla}\overline{\chi}_{\kappa,1})\big)f'_{\kappa,1}(|\chi_{\kappa,1}|^2)$$

$$= f_{\kappa,1}(|\chi_{\kappa,1}|^2)\boldsymbol{\nabla}\chi_{\kappa,1} + 2\chi_{\kappa,1}\operatorname{Re}(\overline{\chi}_{\kappa,1}\boldsymbol{\nabla}\chi_{\kappa,1})f'_{\kappa,1}(|\chi_{\kappa,1}|^2).$$

where

$$\overline{\chi}_{\kappa,1}\boldsymbol{\nabla}\chi_{\kappa,1} =$$

$$\overline{\chi}_{\kappa,1}\chi_{\kappa,1}\bigg(\frac{1}{2|\tilde{z}_2|(x_1^2+x_2^2)}\mathbf{W}_1\times\mathbf{W}_2 + \mathrm{i}\Big(-\frac{1}{2(x_1^2+x_2^2)}\mathbf{W}_1 - \frac{2\kappa}{(1-|\mathbf{x}|^2)^2+4x_3^2}\mathbf{W}_2\Big)\bigg)$$

$\Longrightarrow$

$$2\,\mathrm{Re}(\overline{\chi}_{\kappa,1}\boldsymbol{\nabla}\chi_{\kappa,1}) = \frac{|\chi_{\kappa,1}|^2}{|\tilde{z}_2|(x_1^2+x_2^2)}\mathbf{W}_1\times\mathbf{W}_2 = \frac{4|\chi_{\kappa,1}|^2}{|\tilde{z}_2||\tilde{z}_1|^2}\mathbf{W}_1\times\mathbf{W}_2$$

$\Longrightarrow$

$$(F_1)^{-1}\boldsymbol{\nabla}F_{\kappa,1} = (F_1)^{-1}\big(f_\kappa(|\chi_{\kappa,1}|^2)\boldsymbol{\nabla}\chi_{\kappa,1} + \chi_{\kappa,1}\frac{|\chi_{\kappa,1}|^2}{|\tilde{z}_1|^2|\tilde{z}_2|}\big(\mathbf{W}_1\times\mathbf{W}_2\big)f'_\kappa(|\chi_{\kappa,1}|^2)\big)$$

$$= \frac{4}{|\tilde{z}_1|^2|\tilde{z}_2|}\Big(\frac{1}{2}+\frac{|\chi_{\kappa,1}|^2 f'_{\kappa,1}(|\chi_{\kappa,1}|^2)}{f_{\kappa,1}(|\chi_{\kappa,1}|^2)}\Big)\mathbf{W}_1\times\mathbf{W}_2 - \mathrm{i}\Big(-\frac{1}{2(x_1^2+x_2^2)}\mathbf{W}_1 + \frac{2\kappa}{(1-|\mathbf{x}|^2)^2+4x_3^2}\mathbf{W}_2\Big).$$

Notice that we can safely divide by F_1 because we work away from singularities on this particular chart, i.e. $F_1 \neq 0$.

Also, recall that: $|\tilde{z}_1|^2 = |2x_1+2\mathrm{i}x_2|^2 = 4x_1^2+4x_2^2 = |\mathbf{W}_1|^2$ and $|\tilde{z}_2|^2 = |1-|\mathbf{x}|^2+2x_3\mathrm{i}|^2 = (1-|\mathbf{x}|^2)^2+4x_3^2 = |\mathbf{W}_2|^2$.

We notice that the real and imaginary parts (corresponding to the two directions defining the plane(s) that $\mathbb{R}^3_{1,2}$ is submersed to) are perpendicular. So, we have:

$$(F_1)^{-2}(\boldsymbol{\nabla}F_1)^2 =$$

$$\frac{16}{|\tilde{z}_1|^4|\tilde{z}_2|^2}\Big(\frac{1}{2}+\frac{|\chi_{\kappa,1}|^2 f'_\kappa(|\chi_{\kappa,1}|^2)}{f_\kappa(|\chi_{\kappa,1}|^2)}\Big)^2|\mathbf{W}_1\times\mathbf{W}_2|^2 - \frac{|\mathbf{W}_1|^2}{4(x_1^2+x_2^2)^2} - \frac{4\kappa^2|\mathbf{W}_2|^2}{((1-|\mathbf{x}|^2)^2+4x_3^2)^2}$$

$$= (\mathrm{Re}(\boldsymbol{\nabla}F_1))^2 - (\mathrm{Im}(\boldsymbol{\nabla}F_1))^2.$$

However, given that we wish F is a Riemannian submersion, we should have that the lengths of the $\mathrm{Re}(\boldsymbol{\nabla}F_1), \mathrm{Im}(\boldsymbol{\nabla}F_1)$ should be equal. I.e. $(\boldsymbol{\nabla}F_1)^2 = 0$ or simply $|\mathrm{Re}(\boldsymbol{\nabla}F_1)| = |\mathrm{Im}(\boldsymbol{\nabla}F_1)|$. In other words:

$$\frac{16}{|\tilde{z}_1|^4|\tilde{z}_2|^2}\left(\frac{1}{2}+\frac{|\chi_{\kappa,1}|^2 f'_\kappa(|\chi_{\kappa,1}|^2)}{f_\kappa(|\chi_{\kappa,1}|^2)}\right)^2|\mathbf{W}_1\times\mathbf{W}_2|^2=\frac{|\mathbf{W}_1|^2}{4(x_1^2+x_2^2)^2}+\frac{4\kappa^2|\mathbf{W}_2|^2}{((1-|\mathbf{x}|^2)^2+4x_3^2)^2}.$$

Given that $\mathbf{W}_1$,$\mathbf{W}_2$ are perpendicular, we notice that $|\mathbf{W}_1\times\mathbf{W}_2|=|\mathbf{W}_1||\mathbf{W}_2|$ and the equation above simplifies to:

$$\frac{16}{|\tilde{z}_1|^2}\left(\frac{1}{2}+\frac{|\chi_{\kappa,1}|^2 f'_\kappa(|\chi_{\kappa,1}|^2)}{f_\kappa(|\chi_{\kappa,1}|^2)}\right)^2=\frac{|\mathbf{W}_1|^2}{4(x_1^2+x_2^2)^2}+\frac{4\kappa^2|\mathbf{W}_2|^2}{((1-|\mathbf{x}|^2)^2+4x_3^2)^2}.$$

And with the RHS simplified, we get the following:

$$\frac{1}{|\tilde{z}_1|^2}\left(\frac{1}{2}+\frac{|\chi_{\kappa,1}|^2 f'_\kappa(|\chi_{\kappa,1}|^2)}{f_\kappa(|\chi_{\kappa,1}|^2)}\right)^2=\frac{1}{16(x_1^2+x_2^2)}+\frac{\kappa^2}{4((1-|\mathbf{x}|^2)^2+4x_3^2)}=\frac{1}{4|\tilde{z}_1|^2}+\frac{\kappa^2}{4|\tilde{z}_2|^2}$$

$$\implies \left(\frac{1}{2}+\frac{|\chi_{\kappa,1}|^2 f'_\kappa(|\chi_{\kappa,1}|^2)}{f_\kappa(|\chi_{\kappa,1}|^2)}\right)^2=\frac{1}{2}+\frac{\kappa^2}{2}\left(\frac{|\tilde{z}_1|}{|\tilde{z}_2|}\right)^2$$

$$\implies$$

$$\frac{|\chi_{\kappa,1}|^2 f'_\kappa(|\chi_{\kappa,1}|^2)}{f_\kappa(|\chi_{\kappa,1}|^2)}=-\frac{1}{2}+\sqrt{1+\kappa^2\left(\frac{|\tilde{z}_1|}{|\tilde{z}_2|}\right)^2}=-\frac{1}{2}+\frac{1}{2}\sqrt{1+\kappa^2\left(\frac{|\tilde{z}_1|}{|\tilde{z}_2|}\right)^2}. \qquad (4.1.47)$$

We chose the "+" sign when applying the square root so that f_κ is increasing and $f_\kappa(|\chi_{\kappa,1}|^2)\to\infty$ as $|\chi_{\kappa,1}|\to 1$ (i.e. as $x_1^2+x_2^2\to 1$ and $x_3\to 0$).

Now, we've set $\xi_1=\dfrac{|\tilde{z}_1|}{|\tilde{z}_2|}$ the new "independent" variable and we have:

$$|\chi_{\kappa,1}|^2=\frac{|\tilde{z}_1|^2}{(1+|\mathbf{x}|^2+|\tilde{z}_2|)^2}=\frac{|\tilde{z}_1|^2}{(\sqrt{|\tilde{z}_1|^2+|\tilde{z}_2|^2}+|\tilde{z}_2|)^2}=\frac{\left(\frac{|\tilde{z}_1|}{|\tilde{z}_2|}\right)^2}{\left(1+\sqrt{1+\left(\frac{|\tilde{z}_1|}{|\tilde{z}_2|}\right)^2}\right)^2}$$

$$\implies |\chi_{\kappa,1}|^2=\frac{\xi_1^2}{(1+\sqrt{1+\xi_1^2})^2}.$$

Setting $z = \sqrt{1+\xi_1^2}$ (≥ 1) as a new dummy variable, we get the quadratic for z: $|\chi_{\kappa,1}|^2(1+z)^2 = |\chi_{\kappa,1}|^2(1+2z+z^2) = z^2-1$, which gives:

$$(1-|\chi_{\kappa,1}|^2)z^2 - 2z - (1+|\chi_{\kappa,1}|^2) = 0$$

Solving this quadratic, we get $z = \dfrac{1+|\chi_{\kappa,1}|^2}{1-|\chi_{\kappa,1}|^2}$ and then solving for $\xi_1 = \sqrt{z^2-1}$ keeping in mind that $\xi_1 > 0$ we get:

$$\xi_1 = \frac{2|\chi_{\kappa,1}|}{1-|\chi_{\kappa,1}|^2} \tag{4.1.48}$$

So we set

$$g_\kappa(\xi_1) := f_\kappa(|\chi_{\kappa,1}|^2) = f_\kappa\Big(\frac{\xi_1^2}{(1+\sqrt{1+\xi_1^2})^2}\Big) > 0,$$

and we have:

$$g'_\kappa(\xi_1) = \Big(\frac{d}{d\xi_1}\Big(\frac{\xi_1^2}{(1+\sqrt{1+\xi_1^2})^2}\Big)\Big) f'_\kappa\Big(\frac{\xi_1^2}{(1+\sqrt{1+\xi_1^2})^2}\Big)$$

$$= \frac{2\xi_1}{(1+\sqrt{1+\xi_1^2})^2\sqrt{1+\xi_1^2}} f'_\kappa(|\chi_{\kappa,1}|^2)$$

$$\implies |\chi_{\kappa,1}|^2 \frac{f'_\kappa(|\chi_{\kappa,1}|^2)}{f_\kappa(|\chi_{\kappa,1}|^2)} = |\chi_{\kappa,1}|^2 \frac{(1+\sqrt{1+\xi_1^2})^2\sqrt{1+\xi_1^2}}{2\xi_1}\frac{g'_\kappa(\xi_1)}{g_\kappa(\xi_1)} = \frac{\xi_1\sqrt{1+\xi_1^2}}{2}\frac{g'_\kappa(\xi_1)}{g_\kappa(\xi_1)}$$

So the equation becomes

$$\frac{\xi_1\sqrt{1+\xi_1^2}}{2}\frac{g'_\kappa(\xi_1)}{g_\kappa(\xi_1)} = \frac{1}{2}\Big(-1+\sqrt{1+\kappa^2\xi_1^2}\Big). \tag{4.1.49}$$

From these equations, we see that $g_{\kappa,1}(\xi_1)$ is increasing. This is essentially a first-order linear equation and can be solved explicitly via standard methods. We have:

$$\frac{g'_\kappa(\xi_1)}{g_\kappa(\xi_1)} = -\frac{1}{\xi_1\sqrt{1+\xi_1^2}} + \frac{\sqrt{1+\kappa^2\xi_1^2}}{\xi_1\sqrt{1+\xi_1^2}}$$

and using standard trigonometric substitutions, such as setting $\theta_1 = \tan(\xi_1)$, we eventually get (also see Edition, 2007 for a table of indefinite integrals):

$$\ln(g_{\kappa,1}(\mathbf{x})) = C_1 + \Big(\frac{1}{2}\ln\left(\frac{\sqrt{1+\xi_1^2}+1}{\sqrt{1+\xi_1^2}-1}\right) + \int \frac{\sqrt{1+\kappa^2\xi_1^2}}{\xi_1\sqrt{1+\xi_1^2}}d\xi_1\Big)$$

$\implies$

$$\ln(g_{\kappa}(\xi_1)) = C_1 + \Big(\ln\left(\sqrt{\frac{\sqrt{1+\xi_1^2}+1}{\sqrt{1+\xi_1^2}-1}}\right) + \int \frac{\sqrt{1+\kappa^2\xi_1^2}}{\xi_1\sqrt{1+\xi_1^2}}d\xi_1\Big)$$

$\implies$

$$g_{\kappa}(\xi_1) = C_1 + \left(\sqrt{\frac{\sqrt{1+\xi_1^2}+1}{\sqrt{1+\xi_1^2}-1}}\right)e^{\int \frac{\sqrt{1+\kappa^2\xi_1^2}}{\xi_1\sqrt{1+\xi_1^2}}d\xi_1} \tag{4.1.50}$$

Therefore

$$f_{\kappa}(|\chi_{\kappa,1}|^2) := C_1\left(\frac{\sqrt{1+\xi_1^2}+1}{\sqrt{1+\xi_1^2}-1}\right)^{1/2} e^{\int \frac{\sqrt{1+\kappa^2\xi_1^2}}{\xi_1\sqrt{1+\xi_1^2}}d\xi_1} \tag{4.1.51}$$

where $\xi_1 = \dfrac{2|\chi_{\kappa,1}|}{1-|\chi_{\kappa,1}|^2}$, and $C_1 > 0$ is a constant. To complete the picture, the maps $F_1(\mathbf{x})$, $\tilde{F}_{\kappa,1}$ are given by:

$$F_1(\mathbf{x}) = (\tilde{F}_{\kappa} \circ \chi_{\kappa,1})(\mathbf{x}) = C_1 e^{\int \frac{\sqrt{1+\kappa^2\xi_1^2}}{\xi_1\sqrt{1+\xi_1^2}}d\xi_1}\tilde{e}(\tilde{z}_1)\epsilon_a^{-\kappa}(\tilde{z}_2) \tag{4.1.52}$$

where recall $\epsilon_a^{-\kappa}(\tilde{z}_2)$ is a branch of the $(\tilde{e}(\tilde{z}_2))^{-\kappa}$ (recall $\tilde{z}_1 = 2x_1 + 2\mathrm{i}x_2$, $\tilde{z}_2 = 1 - |\mathbf{x}|^2 + 2\mathrm{i}x_3$).

By MacLaurin asymptotic expansion, we see: $g_{\kappa,1}(\xi_1) \to C_1$ $(= O(1))$ as $\xi_1 \to 0$ and ∞ as $\xi_1 \to \infty$ (it is also increasing). Using this fact, we get that $\xi_1 \to 0 \implies \tilde{F}_{\kappa}(\chi_{\kappa,1}) \to 0$, the integral may be considered definite with endpoints from 0(limiting) to ξ_1. Similarly for the "second" chart, we have:

$$\boldsymbol{\nabla}\chi_{\kappa^{-1},2} = \big(\boldsymbol{\nabla}|\chi_{\kappa^{-1},2}|\big)e^{-\frac{1}{\kappa}}(\tilde{z}_1)\tilde{e}(\tilde{z}_2) + |\chi_{\kappa^{-1},2}|\big((\boldsymbol{\nabla}e^{-\frac{1}{\kappa}}(\tilde{z}_1)\tilde{e}(\tilde{z}_2))\big)$$

$$= \big(\boldsymbol{\nabla}|\chi_{\kappa^{-1},2}|\big)e^{-\frac{1}{\kappa}}(\tilde{z}_1)\tilde{e}(\tilde{z}_2) + |\chi_{\kappa^{-1},2}|\big(\boldsymbol{\nabla}e^{-\frac{1}{\kappa}}(\tilde{z}_1)\big)\tilde{e}(\tilde{z}_2) + |\chi_{\kappa^{-1},2}|e^{-\frac{1}{\kappa}}(\tilde{z}_1)\boldsymbol{\nabla}\tilde{e}(\tilde{z}_2)$$

$$= \chi_{\kappa^{-1},2}\bigg(-\frac{4|\chi_{\kappa,1}|}{(1-|\chi_{\kappa,1}|^2)|\tilde{z}_2|}\mathbf{W}_1\times\mathbf{W}_2 + \mathrm{i}\Big(-\frac{1}{2\kappa(x_1^2+x_2^2)}\mathbf{W}_1 + \frac{2}{(1-|\mathbf{x}|^2)^2+4x_3^2}\mathbf{W}_2\Big)\bigg).$$

$$\boldsymbol{\nabla}F_2 = f_2(|\chi_{\kappa^{-1},2}|^2)\boldsymbol{\nabla}\chi_{\kappa^{-1},2} + \chi_{\kappa^{-1},2}(\boldsymbol{\nabla}|\chi_{\kappa^{-1},2}|^2)f'_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)$$

$$= f_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)\boldsymbol{\nabla}\chi_{\kappa^{-1},2} + \chi_{\kappa^{-1},2}(\boldsymbol{\nabla}\big(\chi_{\kappa^{-1},2}\overline{\chi}_{\kappa^{-1},2})\big)f'_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)$$

$$= f_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)\boldsymbol{\nabla}\chi_{\kappa^{-1},2} + \chi_{\kappa^{-1},2}\big((\boldsymbol{\nabla}\chi_{\kappa^{-1},2})\overline{\chi}_{\kappa^{-1},2} + \chi_{\kappa^{-1},2}\boldsymbol{\nabla}\overline{\chi}_{\kappa^{-1},2})\big)f'_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)$$

$$\Longrightarrow$$

$$\boldsymbol{\nabla}F_{\kappa^{-1}} = f_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)\boldsymbol{\nabla}\chi_{\kappa^{-1},2} + 2\chi_{\kappa^{-1},2}\operatorname{Re}(\overline{\chi}_{\kappa^{-1},2}\boldsymbol{\nabla}\chi_{\kappa^{-1},2})f'_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)$$

By (4.1.43) we have:

$$\boldsymbol{\nabla}\chi_{\kappa^{-1},2} = \big(\boldsymbol{\nabla}|\chi_{\kappa^{-1},2}|\big)e^{-\frac{1}{\kappa}}(\tilde{z}_1)\tilde{e}(\tilde{z}_2) + |\chi_{\kappa^{-1},2}|\big((\boldsymbol{\nabla}e^{-\frac{1}{\kappa}}(\tilde{z}_1)\tilde{e}(\tilde{z}_2))\big)$$

$$= \chi_{\kappa^{-1},2}\bigg(-\frac{4|\chi_{\kappa,1}|}{(1-|\chi_{\kappa,1}|^2)|\tilde{z}_2||\tilde{z}_1|^2}\mathbf{W}_1\times\mathbf{W}_2 + \mathrm{i}\Big(-\frac{1}{2\kappa(x_1^2+x_2^2)}\mathbf{W}_1 + \frac{2}{(1-|\mathbf{x}|^2)^2+4x_3^2}\mathbf{W}_2\Big)\bigg)$$

$$\Longrightarrow$$

$$2\operatorname{Re}(\overline{\chi}_{\kappa^{-1},2}\boldsymbol{\nabla}\chi_{\kappa^{-1},2}) = -\frac{8|\chi_{\kappa^{-1},2}|^2|\chi_{\kappa,1}|}{(1-|\chi_{\kappa,1}|^2)|\tilde{z}_2||\tilde{z}_1|^2}\mathbf{W}_1\times\mathbf{W}_2.$$

Given that $F_{\kappa^{-1},2} \neq 0$ in an area of interest, we can have:

$$(\boldsymbol{\nabla}F_2)^2 = 0 \iff (F_2)^{-2}(\boldsymbol{\nabla}F_2)^2 = 0$$

$$\Longrightarrow \frac{|\chi_{\kappa,1}|^2}{(1-|\chi_{\kappa,1}|^2)^2(x_1^2+x_2^2)}\left(1+2|\chi_{\kappa^{-1},2}|^2\frac{f'_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)}{f_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)}\right)^2 = \frac{1}{\kappa^2(x_1^2+x_2^2)} + \frac{4}{(1-|\mathbf{x}|^2)^2+4x_3^2}$$

$$\Longrightarrow$$

$$\left(1+2|\chi_{\kappa^{-1},2}|^2\frac{f'_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)}{f_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)}\right)^2 = \left(\frac{1-|\chi_{\kappa,1}|^2}{2|\chi_{\kappa,1}|}\right)^2\left(\frac{1}{\kappa^2} + \frac{4(x_1^2+x_2^2)}{(1-|\mathbf{x}|^2)^2+4x_3^2}\right). \quad (4.1.53)$$

The last factor in the RHS can be written as $\dfrac{1}{\kappa^2} + \dfrac{|\tilde{z}_1|^2}{|\tilde{z}_2|^2}$.

We set a new variable $\xi_2 = 1/\xi_1 = \dfrac{|\tilde{z}_2|}{|\tilde{z}_1|} = \dfrac{1-|\chi_{\kappa,1}|^2}{2|\chi_{\kappa,1}|}$ (we want $\xi_2 \to \infty$ when $\xi_1 \to 0$ and vice versa). By (4.1.48), we have:

$$|\chi_{\kappa,1}|^2 = \frac{\xi_1^2}{(1+\sqrt{1+\xi_1^2})^2} = \frac{\xi_1^2}{\left(1+\xi_1\sqrt{1+\left(\frac{1}{\xi_1}\right)^2}\right)^2} = \frac{1}{\left(\frac{1}{\xi_1}+\sqrt{1+\xi_2^2}\right)^2}$$

$$\Longrightarrow\ |\chi_{\kappa,1}| = \frac{1}{\xi_2+\sqrt{1+\xi_2^2}}.$$

We also have:

$$|\chi_{\kappa^{-1},2}|^2 = \left(\frac{1-|\chi_{\kappa,1}|}{1+|\chi_{\kappa,1}|}\right)^2 = \frac{\left(1-\frac{1}{\xi_2+\sqrt{1+\xi_2^2}}\right)^2}{\left(1+\frac{1}{\xi_2+\sqrt{1+\xi_2^2}}\right)^2} = \left(\frac{\xi_2+\sqrt{1+\xi_2^2}-1}{\xi_2+\sqrt{1+\xi_2^2}+1}\right)^2.$$

As in the case of "first" chart, we set: $g_{\kappa^{-1}}(\xi_2) = f_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)$ and we have:

$$\frac{d}{d\xi_2} g_{\kappa^{-1}}(\xi_2) = f'_{\kappa^{-1}}(|\chi_{\kappa^{-1}}|^2)\frac{d}{d\xi_2}(|\chi_{\kappa^{-1},2}|^2),$$

where

$$\frac{d}{d\xi_2}(|\chi_{\kappa^{-1}}|^2) = \frac{d}{d\xi_2}\left(\frac{\xi_2+\sqrt{1+\xi_2^2}-1}{\xi_2+\sqrt{1+\xi_2^2}+1}\right)^2$$

$$= 4\frac{(\xi_2+\sqrt{1+\xi_2^2}-1)(\xi_2+\sqrt{1+\xi_2^2})}{(\xi_2+\sqrt{1+\xi_2^2}+1)^3\sqrt{1+\xi_2^2}}.$$

Thus

$$2|\chi_{\kappa^{-1},2}|^2\frac{f'_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)}{f_{\kappa^{-1}}(|\chi_{\kappa^{-1},2}|^2)} =$$

$$\frac{1}{2}\left(\frac{\xi_2+\sqrt{1+\xi_2^2}-1}{\xi_2+\sqrt{1+\xi_2^2}+1}\right)^2\frac{(\xi_2+\sqrt{1+\xi_2^2}+1)^3\sqrt{1+\xi_2^2}}{(\xi_2+\sqrt{1+\xi_2^2}-1)(\xi_2+\sqrt{1+\xi_2^2})}\frac{g'_{\kappa^{-1}}(\xi_2)}{g_{\kappa^{-1}}(\xi_2)}$$

$$= \frac{(\xi_2+\sqrt{1+\xi_2^2}+1)(\xi_2+\sqrt{1+\xi_2^2}-1)(\sqrt{1+\xi_2^2})}{2(\xi_2+\sqrt{1+\xi_2^2})}\frac{g'_{\kappa^{-1}}(\xi_2)}{g_{\kappa^{-1}}(\xi_2)}$$

$$= \frac{\big((\xi_2 + \sqrt{1+\xi_2^2})^2 - 1\big)\sqrt{1+\xi_2^2}}{\xi_2 + \sqrt{1+\xi_2^2}} \frac{g'_{\kappa^{-1}}(\xi_2)}{g_{\kappa^{-1}}(\xi_2)}$$

$$= \frac{(\xi_2^2 + 1 + \xi_2^2 + 2\xi_2\sqrt{1+\xi_2^2} - 1)\sqrt{1+\xi_2^2}}{\xi_2 + \sqrt{1+\xi_2^2}} \frac{g'_{\kappa^{-1}}(\xi_2)}{g_{\kappa^{-1}}(\xi_2)}$$

$$= \frac{\big(\xi_2^2 + \xi_2\sqrt{1+\xi_2^2}\big)\sqrt{1+\xi_2^2}}{\xi_2 + \sqrt{1+\xi_2^2}} \frac{g'_{\kappa^{-1}}(\xi_2)}{g_{\kappa^{-1}}(\xi_2)} = \xi_2\sqrt{1+\xi_2^2}\frac{g'_{\kappa^{-1}}(\xi_2)}{g_{\kappa^{-1}}(\xi_2)}.$$

So (4.1.53) becomes:

$$1 + \xi_2\sqrt{1+\xi_2^2}\frac{g'_{\kappa^{-1}}(\xi_2)}{g_{\kappa^{-1}}(\xi_2)} = \xi_2\sqrt{\frac{1}{\kappa^2} + \frac{1}{\xi_2^2}} = \sqrt{1 + \frac{\xi_2^2}{\kappa^2}}. \tag{4.1.54}$$

$$\frac{g'_{\kappa^{-1}}(\xi_2)}{g_{\kappa^{-1}}(\xi_2)} = -\frac{1}{\xi_2\sqrt{1+\xi_2^2}} + \frac{1}{\xi_2}\sqrt{\frac{1+\frac{\xi_2^2}{\kappa^2}}{1+\xi_2^2}} \tag{4.1.55}$$

$$\Longrightarrow \frac{g'_{\kappa^{-1}}(\xi_2)}{g_{\kappa^{-1}}(\xi_2)}d\xi_2 = \Big(-\frac{1}{\xi_2\sqrt{1+\xi_2^2}} + \frac{1}{\xi_2}\sqrt{\frac{1+\frac{\xi_2^2}{\kappa^2}}{1+\xi_2^2}}\Big)d\xi_2.$$

Again, we've picked the "+" sign when taking the square root in (4.1.53) so that we can have $g_{\kappa^{-1}}(\xi_2) \to 0$ (resp. ∞) as $\xi_2 \to 0$ (resp. ∞) respectively, and is increasing.

So similarly as in the "first" case (for xi_1), we get:

$$\ln(g_{\kappa^{-1},2}(\xi_2)) = Const. + \frac{1}{2}\ln\Big(\frac{\sqrt{1+\xi_2^2}+1}{\sqrt{1+\xi_2^2}-1}\Big) + \int \frac{\sqrt{1+\frac{\xi_2'^2}{\kappa^2}}}{\xi_2'\sqrt{1+\xi_2'^2}}d\xi_2'$$

and by exponentiating, we get:

$$g_{\kappa^{-1}}(\xi_2) = C_2\Big(\frac{\sqrt{\xi_2^2+1}+1}{\sqrt{\xi_2^2+1}-1}\Big)^{1/2} e^{\int \frac{\sqrt{1+\frac{\xi_2'^2}{\kappa^2}}}{\xi_2'\sqrt{1+\xi_2'^2}}d\xi_2'} \tag{4.1.56}$$

for some constant $C_2 > 0$, where again, using asymptotics (as in the case for g_κ), we get that $g_{\kappa^{-1}} \to C_{\kappa^{-1}} = O(1)$ (and non-zero). Equation (4.1.46) follows. $\square$

Remark 4.1.11. The changes of variables $\chi_{1,2} \to \xi_{1,2}$ also satisfy the following identities, confirming the previous calculation for ξ_1:

$$|\chi_{\kappa,1}|\xi_1 = 1 - |\chi_{\kappa,1}|\sqrt{1+\xi_1^2} \implies |\chi_{\kappa,1}|\sqrt{1+(\xi_1)^2} = 1 - \xi_1|\chi_{\kappa,1}|$$

$$\implies |\chi_{\kappa,1}|^2(1+\xi_1^2) = (1-\xi_1|\chi_{\kappa,1}|)^2$$

$$\implies |\chi_{\kappa,1}|^2 = 1 - 2\xi_1|\chi_{\kappa,1}| \implies \xi_1 = \frac{1-|\chi_{\kappa,1}|^2}{2|\chi_{\kappa,1}|} \quad \left(= \frac{2|\chi_{\kappa^{-1}}|}{1-|\chi_{\kappa^{-1}}|^2}\right).$$

Moreover, we have that the exponent is given via the following formula (for $\kappa \neq 1$) and satisfies:

$$\int^{\xi_1} \frac{\sqrt{1+\kappa^2\xi_1'^2}}{\xi_1'\sqrt{1+\xi_1'^2}} d\xi_1' =$$

$$\frac{1}{2}\ln\left(\frac{|\sqrt{1+\kappa^2\xi_1^2} - \sqrt{1+\xi_1^2}|}{\sqrt{1+\xi_1^2} + \sqrt{1+\kappa^2\xi_1^2}}\right) + \kappa\ln\left(\frac{|\kappa\sqrt{1+\xi_1^2} - \sqrt{1+\kappa^2\xi_1^2}|}{\sqrt{|\kappa^2-1|}}\right). \tag{4.1.57}$$

which yields:

$$|F_1(\mathbf{x})| = Const.\left(\frac{|\sqrt{1+\kappa^2\xi_1^2} - \sqrt{1+\xi_1^2}|}{\sqrt{1+\kappa^2\xi_1^2} + \sqrt{1+\kappa^2\xi_1^2}}\right)^{1/2}\left(\frac{\sqrt{1+\xi_1^2} + \kappa\sqrt{1+\kappa^2\xi_1^2}}{\sqrt{|\kappa^2-1|}}\right)^{\kappa} \tag{4.1.58}$$

On the other hand, the indefinite integral on the exponent satisfies:

$$\int^{\xi_2} \frac{\sqrt{1+\frac{\xi_2'^2}{\kappa^2}}}{\xi_2'\sqrt{1+\xi_2'^2}} d\xi_2' =$$

$$\frac{1}{2}\ln\left(\frac{|\sqrt{1+(\xi_2/\kappa)^2} - \sqrt{1+\xi_2^2}|}{\sqrt{1+(\xi_2/\kappa)^2} + \sqrt{1+\xi_2^2}}\right) + \frac{1}{\kappa}\ln\left(\frac{\sqrt{1+\xi_2^2} - \kappa\sqrt{1+(\xi_2/\kappa)^2}}{\sqrt{|\kappa^2-1|}}\right). \tag{4.1.59}$$

so for $F_2(\mathbf{x})$ we have that this is equal to:

$$(\tilde{F}_{\kappa^{-1}} \circ \chi_{\kappa^{-1}})(\mathbf{x}) = \chi_{\kappa^{-1}} f_{\kappa^{-1}}(|\chi_{\kappa^{-1}}(\mathbf{x})|^2) = \chi_{\kappa^{-1}} g_{\kappa^{-1}}(\xi_2) = |\chi_{\kappa^{-1}}| g_{\kappa^{-1}}(\xi_2) e^{-\frac{1}{\kappa}}(\tilde{z}_1)\tilde{e}(\tilde{z}_2)$$

$\implies$

$$|\chi_{\kappa^{-1}}|^2 = \left(\frac{\xi_2 + \sqrt{1+\xi_2^2} - 1}{\xi_2 + \sqrt{1+\xi_2^2} + 1}\right)^2$$

$$= \frac{\xi_2^2 + 1 + 2\xi_2\sqrt{1+\xi_2^2} - 2\xi_2 - 2\sqrt{1+\xi_2^2} + \xi_2^2 + 1}{(\xi_2^2 + 1 + 2\xi_2\sqrt{1+\xi_2^2} + 2\xi_2 + 2\sqrt{1+\xi_2^2} + \xi_2^2 + 1)}$$

$$= \frac{\xi_2^2 + 1 + \xi_2\sqrt{1+\xi_2^2} - \xi_2 - \sqrt{1+\xi_2^2}}{\xi_2^2 + 1 + \xi_2\sqrt{1+\xi_2^2} + \xi_2 + \sqrt{1+\xi_2^2}} = \frac{(\sqrt{1+\xi_2^2} + \xi_2)(\sqrt{1+\xi_2^2} - 1)}{(\sqrt{1+\xi_2^2} + \xi_2)(\sqrt{1+\xi_2^2})}$$

$$= \frac{\sqrt{1+\xi_2^2}(\sqrt{1+\xi_2^2} - 1) + \xi_2(\sqrt{1+\xi_2^2} - 1)}{\sqrt{1+\xi_2^2}(\sqrt{1+\xi_2^2} + 1) + \xi_2(\sqrt{1+\xi_2^2} + 1)} = \frac{\sqrt{1+\xi_2^2} - 1}{\sqrt{1+\xi_2^2} + 1}.$$

$$|\chi_{\kappa^{-1}}|\left(\frac{\sqrt{\xi_2^2+1}+1}{\sqrt{\xi_2^2+1}-1}\right)^{1/2} = \left(|\chi_{\kappa^{-1}}|^2\frac{\sqrt{\xi_2^2+1}+1}{\sqrt{\xi_2^2+1}-1}\right)^{1/2} = 1$$

$$\implies |\chi_{\kappa^{-1},2}|g_{\kappa^{-1},2}(\xi_2) = C_2 e^{\int^{\xi_2} \frac{\sqrt{1+\frac{\xi_2'^2}{\kappa^2}}}{\xi_2'\sqrt{1+\xi_2'^2}}d\xi_2'}$$

$\implies$

$$|F_2(\mathbf{x})| = Const.\left(\frac{|\sqrt{1+(\xi_2/\kappa)^2} - \sqrt{1+\xi_2^2}|}{\sqrt{1+(\xi_2/\kappa)^2} + \sqrt{1+\xi_2^2}}\right)^{1/2}\left(\frac{\sqrt{1+\xi_2^2} + \kappa\sqrt{1+(\xi_2/\kappa)^2}}{\sqrt{|\kappa^2-1|}}\right)^{1/\kappa} \tag{4.1.60}$$

Recall $\xi_2 = \xi_2(\mathbf{x})$. Now, keeping in mind that $F_{\kappa^{-1},2} = \tau(F_{\kappa,1})$ (recall $\tau(z) = C_\kappa z^{-1/\kappa}$) as well as

$$|\boldsymbol{\nabla}\operatorname{Re}F_{\kappa,1}(\mathbf{x})| = |\boldsymbol{\nabla}\operatorname{Im}F_1(\mathbf{x})| = |F_1(\mathbf{x})|\sqrt{\frac{1}{x_1^2+x_2^2} + \frac{4\kappa^2}{(1-|\mathbf{x}|^2)^2 + 4x_3^2}}$$

$$= \kappa|F_1(\mathbf{x})|\sqrt{\frac{1}{\kappa^2(x_1^2+x_2^2)} + \frac{4}{(1-|\mathbf{x}|^2)^2 + 4x_3^2}} = \kappa|C_\kappa^\kappa||F_2(\mathbf{x})|^{-\kappa-1}|\operatorname{Im}\boldsymbol{\nabla}F_2(\mathbf{x})|.$$

The last equation was derived using $F_2 = \tau(F_1)$ and

$$|\boldsymbol{\nabla}\operatorname{Im}F_2(\mathbf{x})| = |\boldsymbol{\nabla}\operatorname{Re}F_2(\mathbf{x})| = |F_2(\mathbf{x})|\sqrt{\frac{1}{\kappa^2(x_1^2+x_2^2)} + \frac{4}{(1-|\mathbf{x}|^2)^2 + 4x_3^2}}.$$

So, in order for F to define a submersion of (weighted) spaces $\mathbb{R}^3$ to $(\mathbb{R}^2, \omega)$, we therefore need the weights, $\omega_{1,2}$ (corresponding to the copies $\mathbb{R}^2_{1,2}$ of plane, respectively) to satisfy:

$$\kappa\omega_1|C_\kappa^\kappa||F_2|^{-\kappa-1}|\operatorname{Im}\boldsymbol{\nabla}F_2| = \omega_2|\boldsymbol{\nabla}\operatorname{Im}F_2| \tag{4.1.61}$$

which is equivalent to

$$\omega_1(z) = \kappa|z|^{1+1/\kappa} C_\kappa^{-1} \omega_2(\tau(z)), \quad z \in \mathbb{C}_{\neq 0} \tag{4.1.62}$$

Recall that, when these weights are used in relation to the process of submersing $\mathbb{R}^3$ to $\mathbb{R}^2$, the argument $z = x + y\mathrm{i}$ is naturally identified with the point (x, y) on the plane. In particular, when this point (x, y) is the image of the map F, the variable z effectively becomes a function of $\mathbf{x} = (x_1, x_2, x_3)$.

Now, if we equip $\mathbb{R}^3$ with a weight Ω then a sufficient and necessary condition for F to define a submersion is:

$$\omega_1(|F_1(\mathbf{x})|^2)|\boldsymbol{\nabla} \operatorname{Re} F_1(\mathbf{x})| = \omega_2(|F_2(\mathbf{x})|^2)|\boldsymbol{\nabla} \operatorname{Re} F_2(\mathbf{x})| = \Omega(\mathbf{x}) \tag{4.1.63}$$

and similarly for $\operatorname{Im} F_1$, $\operatorname{Im} F_2$.

Remark 4.1.12. The value $C_\kappa > 0$ in

$$C_\kappa = \kappa(1+\kappa)^{-1-1/\kappa}, \quad \kappa > 0 \tag{4.1.64}$$

we notice that $C_{1/\kappa} = C_\kappa^\kappa$. This is probably the most neat choice of C_κ (considering it a function of κ) if we think $C_{1/\kappa}$ as the constant such that $\tau^{-1}(z_2) = C_{1/\kappa} z_2^{-\kappa}$ for $z_2 = F_{\kappa^{-1},2} \in \mathbb{C}_2$ (pointwise). The results are not particularly affected by the choice of value, so we'll pick that for the sake of neatness.

Picking weight-functions $\omega_{1,2} : \mathbb{R}^2 \to \mathbb{R}_{>0}$ and $\Omega : \mathbb{R}^3 \to \mathbb{R}_{>0}$ on (the two respective copies of) $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively we have the following.

Corollary 4.1.13. The maps $F_{1,2} : \mathbb{R}^3_{1,2} \to \mathbb{C}$, define a Riemannian submersion from the weighted space(s) $(\mathbb{R}^3_{1,2}, \Omega)$ to $(\mathbb{R}^2, \omega_{1,2})$.

Remark 4.1.14. The aforementioned weights ω, are defined on $\mathbb{R}^2$, which is naturally identified by $\mathbb{C}$. However, the variables $z = x + y\mathrm{i} \in \mathbb{C}$ and $(x, y) \in \mathbb{R}^2$ may be used interchangeably as arguments, slightly abusing notation, because working on $\mathbb{C}$ is easier than on $\mathbb{R}^2$.

4.2 On the construction Weyl-Dirac operators on $\mathbb{O}_\kappa$

In this section, we'll construct the building blocks of *Weyl-Dirac* operators that act on sections of a suitable *Spin*^{c} *bundle*. This *Spin*^{c} *bundle* is made up of two line bundles on $\mathbb{O}_\kappa$ (by taking their direct, tensor product). In particular, we define *1-forms* on $\mathbb{O}_\kappa$, *connections* and *Clifford multiplication* acting on sections of the aforementioned line bundles, *Spin*^{c} connections acting on sections of the respective *Spin*^{c} bundle. Recall $\mathbb{O}_\kappa$ is a pair of complex planes, with non-zero points on each plane identified via the map $\tau(z) = \kappa(1+\kappa)^{-1-1/\kappa} z^{-1/\kappa}$.

In the previous section, we saw that the submersion which works for us is given in terms of a (countable) multivalued map (finite or infinite), at least formally. In other words, each point $\mathbf{x} \in \mathbb{R}^3$ is mapped to a countable set of points on $\mathbb{R}^2$ (identified with $\mathbb{C}$). Hence, constructing line or *Spin*^{c} bundles on that space may only be possible when suitable equivalence relations are met.

Recall definition (3.4.1) of *Spin*^{c}-bundles and note that sections of these bundles are naturally identified with $\mathbb{C}^2$-valued maps $\begin{pmatrix} u^+ \\ u^- \end{pmatrix}$, with ("up" and "down") components $u^\pm$ from respective sections from $\tilde{L}^\mp$. Each such component satisfies (3.3.4), (3.3.5) (for each respective $k_{1,2} \in \mathbb{Z}$) on each respective chart of the orbit space $\mathbb{O}_\kappa$ and (3.3.6) on the overlap of these charts.

Remark 4.2.1. The charts for the base space for the complex line and *Spin*^{c} bundles can naturally be identified with the quotient space(s): $\mathbb{S}^2_{N/S}/\sim_{\kappa^{\mp 1}}$. This space is the unit sphere $\mathbb{S}^2$ on $\mathbb{R}^3$ excluding the North/South pole, respectively when considering points $z_{2,1} \in \mathbb{C}$ ($\equiv \mathbb{R}^2$, naturally identified as the punctured sphere using stereographic projections) on each such chart, identified if they are related via an expression of the form $z_1 = e^{2k\kappa\pi \mathrm{i}} z_2$ or $z_1 = e^{2k\frac{\pi}{\kappa}\mathrm{i}} z_2$ (for some $k \in \mathbb{Z}$) respectively. That is identification via a rotation by $2\pi\kappa$ or $2\pi\kappa^{-1}$ around the "z"-axis. The identification of $\mathbb{C}$ (equivalently $\mathbb{R}^2$) with the punctured unit sphere $\mathbb{S}^2_{N/S}$ on $\mathbb{R}^3$ is done via stereographic projection from the North/South pole respectively. This identification does not affect any qualitative properties of the space, as stereographic projections are conformal (they preserve angles). Essentially, the stereographic projection sets up a conformal isomorphism between $\mathbb{S}^2 \setminus \{(0,0,\pm 1)\}$ and $\mathbb{C}^*$. There is a group action of $\mathbb{Z}^2$ on $\mathbb{C}^*$ via $(m,n) : z \mapsto e^{2n\pi\kappa \mathrm{i} + 2m\pi\frac{1}{\kappa}\mathrm{i}}$.

Remark 4.2.2. In order to identify suitable classes of 1-forms and vector fields on $\mathbb{O}_{\boldsymbol{\kappa}}$, we'll work on 2 charts of $\mathbb{O}_{\boldsymbol{\kappa}}$, namely two copies of $\mathbb{R}^2$. However, as it's been evident from the previous section, it is often more convenient to work on $\mathbb{C}$ rather than $\mathbb{R}^2$. So, a 1-form on $\mathbb{R}^2$ given by $\tilde{\alpha}_x(x,y)dx + \tilde{\alpha}_y(x,y)dy$ on $\mathbb{R}^2$ may be identified as a 1-form on $\mathbb{C}$ given by $\tilde{\alpha}_+(z,\overline{z})dz + \tilde{\alpha}_-(z,\overline{z})d\overline{z}$. These two forms may be used interchangeably.

We proceed by defining the basic objects we'll be working with on $\mathbb{O}_{\tilde{\kappa}}$.

4.2.1 Forms and U(1)-connections on line bundles on $\mathbb{O}_{\boldsymbol{\kappa}}$

Before we define *Weyl-Dirac operators* on the aforementioned *Spin*^{c}-bundle, we first need to define[9] a $U(1)$-*connection*, $\tilde{\nabla} = d - \mathrm{i}\tilde{\alpha}^j$, acting on each of the components of the fibre, where $\tilde{\alpha}^j$ is a real 1-form on the $\mathbb{O}_{\kappa^{3-2j}}$. Then, we want these connections to respect (3.3.6)-(3.3.5) as well as note how they behave under a change of variables.

Now, let $z = x + \mathrm{i}y$, we have: $dz = dx + \mathrm{i}dy$, $d\overline{z} = dx - \mathrm{i}dy$ and $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$.

Recall the transition map τ from (3.1.6). Let's first see how a $U(1)$-connection changes under change of variables $(z,\overline{z}) \mapsto (\tau(z), \overline{\tau(z)})$:

Let $\tilde{\mathbf{X}} \equiv \tilde{\mathbf{X}}(z,\overline{z}) = \tilde{X}_+(z,\overline{z})\partial_z + \tilde{X}_-(z,\overline{z})\partial_{\overline{z}}$, a vector field on $\mathbb{C}$, and $f(z,\overline{z})$ a function on $\mathbb{C}$. The push-forward of the vector-field $\tilde{\mathbf{X}}(z,\overline{z})$ is[10]:

$$\tau_*\tilde{\mathbf{X}} = \tilde{X}_+(z,\overline{z})\tau_*\partial_z + \tilde{X}_-(z,\overline{z})\tau_*\partial_{\overline{z}}.$$

We have:

$$\partial_z = (\partial_z\tau(z))\partial_{\tau(z)} + (\partial_z\overline{\tau(z)})\partial_{\overline{\tau(z)}} = \tau'(z)\partial_{\tau(z)} = -\frac{C_\kappa}{\kappa}z^{-\frac{1}{\kappa}-1}\partial_{\tau(z)} = -\frac{C_\kappa}{\kappa z^{\frac{1}{\kappa}+1}}\partial_{\tau(z)}.$$

[9] Recall this object acts like functions $f = f(\mathbf{x})$ as $\tilde{\nabla}_{\mathbf{X}} f := (df)_{\mathbf{X}} - \mathrm{i}\tilde{\alpha}(\mathbf{X})f = \mathbf{X}(f) - \mathrm{i}\tilde{\alpha}(\mathbf{X})f$ for all vector fields $\mathbf{X} = \mathbf{X}(\mathbf{x})$.

[10] We abuse notation and by τ we mean the map(ing)/change of variables $(z,\overline{z}) \mapsto (\tau(z), \overline{\tau(z)})$

By recalling (3.1.6), we have: $\partial_{\overline{z}}\tau(z) = \partial_z\overline{\tau(z)} = \partial_z\tau(\overline{z}) = 0)$ and

$$\partial_{\overline{z}} = (\partial_{\overline{z}}\tau(z))\partial_{\tau(z)} + (\partial_{\overline{z}}\overline{\tau(z)})\partial_{\overline{\tau(z)}} = \overline{\tau'(z)}\partial_{\overline{\tau(z)}} = -\frac{\overline{C_\kappa}}{\kappa}\overline{\tau(z)}^{-\frac{1}{\kappa}-1}\partial_{\overline{\tau(z)}} = -\frac{\overline{C_\tau}}{\kappa\overline{z}^{\frac{1}{\kappa}+1}}\partial_{\overline{\tau(z)}}$$

i.e. $\tau_*\partial_z = -\kappa^{-1}C_\kappa z^{-1-\frac{1}{\kappa}}\partial_{\tau(z)}$ and $\tau_*\partial_{\overline{z}} = -\kappa^{-1}\overline{C_\kappa}\overline{z}^{-1-\frac{1}{\kappa}}\partial_{\overline{\tau(z)}}$,

$$\partial_{\tau(z)} = -\frac{\kappa z^{\frac{1}{\kappa}+1}}{C_\kappa}\partial_z \quad \text{and} \quad \partial_{\overline{\tau(z)}} = -\frac{\kappa\overline{z}^{\frac{1}{\kappa}+1}}{\overline{C_\kappa}}\partial_{\overline{z}}$$

Also, for $\tilde{\alpha}_z = \tilde{\alpha}_+(z,\overline{z})dz + \tilde{\alpha}_-(z,\overline{z})d\overline{z}$, we have:

$$(\tau^{-1})^*\tilde{\alpha} = (\tau^{-1})^*(\tilde{\alpha}_+(z,\overline{z})dz + \tilde{\alpha}_-(z,\overline{z})d\overline{z}) =$$

$$\tilde{\alpha}_+(\tau^{-1}(z),\overline{\tau^{-1}(z)})d\tau^{-1}(z) + \tilde{\alpha}_-(\tau^{-1}(z),\overline{\tau^{-1}(z)})d\overline{\tau^{-1}(z)}$$

Recall, we have: $\tau^{-1}(z) = C_\kappa^\kappa z^{-\kappa}$ and so

$$d\tau^{-1}(z) = C_\kappa^\kappa dz^{-\kappa} = -\kappa C_\kappa^\kappa z^{-\kappa-1}dz = -\frac{\kappa C_\kappa^\kappa}{z^{\kappa+1}}dz = -\frac{\kappa(\tau^{-1}(z))^{1+\frac{1}{\kappa}}}{C_\kappa}dz$$

and

$$d\overline{\tau^{-1}(z)} = \overline{C_\kappa^\kappa}d\overline{z}^{-\kappa} = -\kappa\overline{C_\kappa^\kappa}\overline{z}^{-\kappa-1}d\overline{z} = -\frac{\kappa\overline{C_\kappa^\kappa}}{\overline{z}^{\kappa+1}}d\overline{z} = -\frac{\kappa(\overline{\tau^{-1}(z)})^{1+\frac{1}{\kappa}}}{\overline{C_\kappa}}d\overline{z}$$

therefore

$$((\tau^{-1})^*)\tilde{\alpha}_{\tau(z)} = -\kappa\tilde{\alpha}_+(z,\overline{z})\frac{z^{1+\frac{1}{\kappa}}}{C_\kappa}dz - \kappa\tilde{\alpha}_-(z,\overline{z})\frac{\overline{z}^{1+\frac{1}{\kappa}}}{\overline{C_\kappa}}d\overline{z}$$

and since $dz(\partial_{\overline{z}}) = 0$ (resp. $d\overline{z}(\partial_z) = 0$), we get the expected identity:

$$(\tau^{-1})^*\tilde{\alpha}_{\tau(z)}(\tau_*\tilde{\mathbf{X}}) = \tilde{\alpha}_+(z,\overline{z}) + \tilde{\alpha}_-(z,\overline{z}) = \tilde{\alpha}_z(\tilde{\mathbf{X}}) = \tilde{\alpha}_{\tau^{-1}(\tau(z))}(\tau_*^{-1}(\tau_*\tilde{\mathbf{X}}))$$

This is true for any $\tilde{\alpha} \in \Omega^1(\mathbb{C})$. However, our main goal, for now, is to identify connections that respect the conditions (3.3.4) and (3.3.5), i.e. given a rotation operator $R_\rho(\cdot) = e^{2\pi\rho\mathrm{i}}(\cdot)$ we have:

$$\tilde{\nabla}_{R_{\rho*}\tilde{\mathbf{X}}}u(R_\rho z) = R_\rho^k\tilde{\nabla}_{\tilde{\mathbf{X}}}u(z) \tag{4.2.1}$$

for $\tilde{\mathbf{X}}(z,\overline{z}) = \tilde{X}_+(z,\overline{z})dz + \tilde{X}_-(z,\overline{z})d\overline{z}$ where $u(z)$ is either of $u_{2,1}(z_{2,1})$ and k is either of $\dfrac{\lambda}{\kappa_{2,1}}$ respectively.

Proposition 4.2.3. Equation (4.2.1) is true iff the zero-order component, $\tilde{\alpha}$ of the connection $\tilde{\nabla}$ satisfies $R_\rho^*\tilde{\alpha} = \tilde{\alpha}$.

Proof. We calculate:

$$(R_\rho)_*\tilde{\mathbf{X}} = \tilde{X}_+(z,\overline{z})(R_\rho)_*\partial_z + \tilde{X}_-(z,\overline{z})(R_\rho)_*\partial_{\overline{z}}$$

$$= e^{2\pi \mathrm{i}\rho}\tilde{X}_+(z,\overline{z})\partial_{R_\rho z} + e^{-2\pi\mathrm{i}\rho}\tilde{X}_-(z,\overline{z})\partial_{\overline{R_\rho z}} = R_\rho\tilde{X}_+(z,\overline{z})\partial_{R_\rho z} + \overline{R_\rho}\tilde{X}_-(z,\overline{z})\partial_{\overline{R_\rho z}}$$

These equations are easy to derive since $R_\rho(z) = e^{2\pi\mathrm{i}\rho}z$ ($\implies \overline{R_\rho(z)} = e^{-2\pi\mathrm{i}\rho}\overline{z}$), set:

$$e^{2\pi\mathrm{i}\rho}\partial_{R_\rho z}u(R_\rho z, \overline{R_\rho z}) = e^{2\pi\mathrm{i}\rho}\partial_z u|_{R_\rho z} = \partial_z(u\circ R_\rho)|_z = \partial_z(R_\rho^k u)|_z = R_\rho^k\partial_z u|_z$$

and $e^{-2\pi\mathrm{i}\rho}\partial_{\overline{R_\rho z}}u(R_\rho z, \overline{R_\rho z}) = e^{-2\pi\mathrm{i}\rho}\partial_{\overline{z}}u|_{\overline{R_\rho z}} = \partial_{\overline{z}}(u\circ R_\rho)|_z = \partial_{\overline{z}}R_\rho^k u|_z = R_\rho^k\partial_{\overline{z}}u|_z$

$$\implies R_{\rho^*}\tilde{\mathbf{X}} = e^{2\pi\mathrm{i}\rho}\tilde{X}_+(z,\overline{z})\partial_{R_\rho z}u(R_\rho z,\overline{R_\rho z}) + e^{-2\pi\mathrm{i}\rho}\tilde{X}_-(z,\overline{z})\partial_{\overline{R_\rho z}}u(R_\rho z,\overline{R_\rho z}) = R_\rho^k\tilde{\mathbf{X}}u.$$

We have $(R_{\rho*}\tilde{\mathbf{X}})u|_z (= R_\rho^k d_{\tilde{\mathbf{X}}}u|_z)$ for all differentiable u defined (at least locally) at z and $R_\rho z$. Therefore:

$$\tilde{\nabla}_{R_{\rho*}\tilde{\mathbf{X}}}u|_{R_\rho z} \equiv \tilde{\nabla}_{R_{\rho*}\tilde{\mathbf{X}}}u(R_\rho z) = d_{R_{\rho*}\tilde{\mathbf{X}}}u|_{R_\rho z} - \mathrm{i}(\tilde{\alpha}_{R_\rho z}(R_{\rho*}\tilde{\mathbf{X}}))u(R_\rho z)$$

$$= d_{R_{\rho*}\tilde{\mathbf{X}}^z}u(R_\rho z) - \mathrm{i}(R_\rho^*\tilde{\alpha})(\tilde{\mathbf{X}}^z)u(R_\rho z) = R_\rho^k d_{\tilde{\mathbf{X}}}u|_z - \mathrm{i}R_\rho^k\tilde{\alpha}_z(\tilde{\mathbf{X}})u(z) = R_\rho^k\tilde{\nabla}_{\tilde{\mathbf{X}}}u(z)$$

$$\text{iff } R_\rho^*\tilde{\alpha} = \tilde{\alpha}.$$

□

This condition ensures that the connection(s) is(are) well-defined on the (line) bundle where u lives in.

Remark 4.2.4. Obviously, if $\tilde{\mathbf{X}}$ is an infinitesimal generator of rotation by ρ, i.e. if $(R_\rho)_*\tilde{\mathbf{X}} = \tilde{\mathbf{X}}$, that would be enough, if imposed. However, the set of right invariant (with respect to rotations R_ρ) vector fields is relatively narrow, as is the set of 1-forms subject to the condition $R_\rho^*\tilde{a} = \tilde{a}$, which is related to the magnetic potential and we'll see that for certain choices of $\tilde{\mathbf{X}}$ there are fruitful relations between these two.

However, given that we have two spaces, $\mathbb{O}_{\kappa^{\pm 1}}$, and the compatibility condition (3.3.6) between the two charts, we should have connections on these, $\tilde{\nabla}_{\tilde{\mathbf{X}}}$, that respect this compatibility condition on these bundles, i.e. for $z_{2,1} \in \mathbb{C}_{\neq 0}$ s.t. $z_2 = \tau(z_1)$ and $\tilde{\mathbf{X}}^{2,1} : \tilde{\mathbf{X}}^2 = \tau_* \tilde{\mathbf{X}}^1$ we should have:

$$e^{2\pi i k_2} \tilde{\nabla}^{\tilde{\alpha}^2}_{\tilde{\mathbf{X}}^2} u_2(z_2) = e^{2\pi i k_1} \tilde{\nabla}^{\tilde{\alpha}^1}_{\tilde{\mathbf{X}}^1} u_1(z_1). \tag{4.2.2}$$

Proposition 4.2.5. Let $\lambda = 2k_1\kappa_1 + 2k_2\kappa_2$ for some $k_{1,2}\mathbb{Z}$ and $u_{1,2} \in L_\lambda$. In order for the condition (4.2.2) to be true we must have:

$$\tau^* \tilde{\alpha}^2_z = -\frac{\lambda}{2\kappa_1} \tilde{\zeta}_z + \tilde{\alpha}^1_z \tag{4.2.3}$$

where $\tilde{\alpha}^{1,2}_z$ are the respective connection 1-forms (producing the 0-order coefficients) for $\tilde{\nabla}^{\tilde{\alpha}^{1,2}}$.

Proof. We set $\tilde{\nabla}^{\tilde{\alpha}^{2,1}} = d - i\tilde{\alpha}^{2,1}$, recall $z_2 = \tau(z_1)$ and re-write (3.3.6) as:

$$u_1(z_1) = e^{-k_1}(z_1) e^{-k_2}(\tau(z_1)) u_2(\tau(z_1))$$

$$\implies \tilde{\nabla}^1_{\tilde{\mathbf{X}}^1} u_1(z_1) = \tilde{\nabla}^1_{\tilde{\mathbf{X}}^1}(e^{-k_1}(z_1) e^{-k_2}(\tau(z_1)) u_2(\tau(z_1)))$$

$$= e^{-k_1}(z_1) e^{-k_2}(\tau(z_1)) (\tilde{\nabla}^2_{\tau_* \tilde{\mathbf{X}}^1} u_2)(z_1) = e^{-k_1}(z_1) e^{-k_2}(\tau(z_1)) \tilde{\nabla}^2_{\tilde{\mathbf{X}}^2} u_2(z_2)$$

the top-left hand side equals $\tilde{\mathbf{X}}^1 u(z_1) - i a^1_{z_1}(\tilde{\mathbf{X}}^1) u(z_1)$ whilst the terms in the bottom-right equations equal

$$e^{-k_1}(z_1) e^{-k_2}(\tau(z_1)) \big((\tau_* \tilde{\mathbf{X}}^1) u(\tau(z_1)) - i a^2_{z_2}(\tau_* \tilde{\mathbf{X}}^1) u(\tau(z_1)) \big).$$

We have:

$$\tilde{\mathbf{X}}(z, \overline{z}) = \tilde{X}_+(z, \overline{z}) \partial_z + \tilde{X}_-(z, \overline{z}) \partial_{\overline{z}} \implies \tau_* \tilde{\mathbf{X}} = \tilde{X}_+(z, \overline{z}) \tau_* \partial_z + \tilde{X}_-(z, \overline{z}) \tau_* \partial_{\overline{z}}$$

$$= \tau'(z) \tilde{X}_+(z, \overline{z}) \partial_{\tau(z)} + \overline{\tau'(z)} \tilde{X}_-(z, \overline{z}) \partial_{\tau(\overline{z})} = -\frac{C_\kappa}{\kappa z^{1+\frac{1}{\kappa}}} \tilde{X}_+(z, \overline{z}) \partial_{\tau(z)} - \frac{\overline{C_\kappa}}{\kappa \overline{z}^{1+\frac{1}{\kappa}}} \tilde{X}_-(z, \overline{z}) \partial_{\overline{\tau(z)}}$$

Recall the equality right above is true since $\tau(z) = C_\kappa z^{-\frac{1}{\kappa}}$ (and $\overline{\tau(z)} = \overline{C_\kappa} \overline{z}^{-\frac{1}{\kappa}}$):

$$\tau_*\partial_z = (\partial_z\tau(z))\partial_{\tau(z)} + (\partial_z\overline{\tau(z)})\partial_{\overline{\tau(z)}} = \tau'(z)\partial_z + 0 = -\frac{C_\tau}{\kappa z^{1+\frac{1}{\kappa}}}\partial_{\tau(z)}, \text{ similarly}$$

$$\tau_*\partial_{\overline{z}} = (\partial_{\overline{z}}\tau(z))\partial_{\tau(z)} + (\partial_{\overline{z}}\overline{\tau(z)})\partial_{\overline{\tau(z)}} = 0 + (\partial_{\overline{z}}\overline{\tau(z)})\partial_{\overline{z}} = -\frac{\overline{C_\tau}}{\kappa \overline{z}^{1+\frac{1}{\kappa}}}\partial_{\overline{\tau(z)}}.$$

We evaluate $\tilde{\nabla}^1_{\tilde{\mathbf{X}}^1}(e^{-k_1}(z_1)e^{-k_2}(\tau(z_1))u_2(\tau(z_1)))$:

Regarding the differential part of the connection, we have: $d_{\tilde{\mathbf{X}}^1} = \tilde{\mathbf{X}}^1$
$\Longrightarrow$

$$\begin{aligned} d_{\tilde{\mathbf{X}}^1}(e^{-k_1}(z_1)e^{-k_2}(\tau(z_1))u_2(\tau(z_1))) &= \tilde{\mathbf{X}}^1(e^{-k_1}(z_1)e^{-k_2}(\tau(z_1))u_2(\tau(z_1))) \\ &= (\tilde{\mathbf{X}}^1(e^{-k_1}(z_1)))e^{-k_2}(\tau(z_1))u_2(\tau(z_1)) + e^{-k_1}(z_1)\tilde{\mathbf{X}}^1(e^{-k_2}(\tau(z_1)))u_2(\tau(z_1)) + \\ &\quad e^{-k_1}(z_1))e^{-k_2}(\tau(z_1))\tilde{\mathbf{X}}^1(u_2(\tau(z_1)). \end{aligned}$$

We set $z_1 = z$ for simplicity, as before, and calculate[11]:

$$\begin{aligned} \tilde{\mathbf{X}}^z_1 e^{-k_1}(z) = \tilde{\mathbf{X}}^1\Big(\frac{z}{|z|}\Big)^{-k_1} &= \tilde{X}^1_+(z,\overline{z})\partial_z\Big(\frac{z}{|z|}\Big)^{-k_1} + \tilde{X}^1_-(z,\overline{z})\partial_{\overline{z}}\Big(\frac{z}{|z|}\Big)^{-k_1} \\ &= \tilde{X}^1_+(z,\overline{z})\partial_z\Big(\frac{z}{\overline{z}}\Big)^{-k_1/2} + \tilde{X}^1_-(z,\overline{z})\partial_{\overline{z}}\Big(\frac{z}{\overline{z}}\Big)^{-k_1/2} \\ &= -\frac{k_1}{2}\tilde{X}^1_+(z,\overline{z})\Big(\frac{z}{\overline{z}}\Big)^{-k_1/2-1}\frac{1}{\overline{z}} + \frac{k_1}{2}\tilde{X}^1_-(z,\overline{z})\Big(\frac{z}{\overline{z}}\Big)^{-k_1/2-1}\frac{z}{\overline{z}^2} \\ &= -\frac{k_1}{2z}\tilde{X}^1_+(z,\overline{z})\Big(\frac{z}{\overline{z}}\Big)^{-k_1/2} + \frac{k_1}{2\overline{z}}\tilde{X}^1_-(z,\overline{z})\Big(\frac{z}{\overline{z}}\Big)^{-k_1/2} \end{aligned}$$

$$= -\frac{k_1}{2|z|^2}\big(\overline{z}\tilde{X}^1_+(z,\overline{z}) - z\tilde{X}^1_-(z,\overline{z})\big)\Big(\frac{z}{\overline{z}}\Big)^{-k_1/2} = -\frac{k_1}{2}e^{-k_1}(z)(\frac{\overline{z}}{|z|^2}dz - \frac{z}{|z|^2}d\overline{z})(\tilde{\mathbf{X}}^1(z,\overline{z}))$$

$$= -k_1e^{-k_1}(z)\frac{\mathrm{i}}{2\mathrm{i}}(\frac{\overline{z}}{|z|^2}dz - \frac{z}{|z|^2}d\overline{z})(\tilde{\mathbf{X}}^1(z,\overline{z})) = -\mathrm{i}k_1e^{-k_1}(z)\tilde{\zeta}(\tilde{\mathbf{X}}^1(z,\overline{z})).$$

We have $\tilde{\zeta}_z = \frac{\overline{z}}{|z|^2}dz - \frac{z}{|z|^2}d\overline{z} \in \Omega^1(\mathbb{C}_{\neq 0})$ and almost similarly we evaluate:

$$\tilde{\mathbf{X}}^1(e^{-k_2}(z_2)) = \tilde{\mathbf{X}}^1(e^{-k_2}(\tau(z))) = (\tau_*\tilde{\mathbf{X}}^1e^{-k_2})(z)$$

[11]In this setting, when we talk about the argument of a complex number (or function) we mean its principal value. Also, for all real (number or function) θ we set, $\sqrt{e^{\mathrm{i}\theta}} \equiv e^{\mathrm{i}\theta/2}$.

$$= \tau'(z)\tilde{X}^1_+(z,\overline{z})\partial_z\Big(\frac{z}{\overline{z}}\Big)^{-k_2} + \overline{\tau'(z)}\tilde{X}^1_-(z,\overline{z})\partial_{\overline{z}}\Big(\frac{z}{\overline{z}}\Big)^{-k_2}$$

$$= -\frac{\mathrm{i}k_2 e^{-k_2}(z)}{2\mathrm{i}}\Big(\frac{C_\kappa \overline{z}}{\kappa z^{3+\frac{1}{\kappa}}}\tilde{X}^1_+(z,\overline{z}) - \frac{\overline{C_\kappa} z}{\kappa \overline{z}^{3+\frac{1}{\kappa}}}\tilde{X}^1_-(z,\overline{z})\Big) = -\mathrm{i}k_2 e^{-k_2}(z)\tau^*\tilde{\zeta}_{(z,\overline{z})}(\tilde{\mathbf{X}}^z_1(z,\overline{z})).$$

It is clear that $(du_2(\tau(z_1)))(\tau_*\tilde{\mathbf{X}}^1) = (du_2(\tau(z_1))(\tau_*\tilde{\mathbf{X}}^1) = (du_2(z_2)(\tilde{\mathbf{X}}^1)$, so we just need to work with the 0-order terms, it suffices to have:

$$\tau^*\tilde{\alpha}^2_z(\tilde{\mathbf{X}}^2) = -\frac{k_1}{2}\zeta(\tilde{\mathbf{X}}^1) - \frac{k_2}{2}\tau^*\zeta(\tilde{\mathbf{X}}^1) + \tilde{\alpha}^1_z(\tilde{\mathbf{X}}^1). \tag{4.2.4}$$

However, we have: $2\mathrm{i}\tau^*\tilde{\zeta} = \tau^*(\frac{1}{z}dz - \frac{1}{\overline{z}}d\overline{z}) = \frac{1}{\tau(z)}(\tau^*dz) - \frac{1}{\overline{\tau(z)}}(\tau^*d\overline{z})$

$$= \frac{1}{\tau(z)}(\partial_z\tau(z) + \partial_z\overline{\tau(z)})dz - \frac{1}{\overline{\overline{\tau(z)}}}(\partial_{\overline{z}}\tau(z) + \partial_{\overline{z}}\overline{\tau(z)})d\overline{z}$$

$$= -\frac{1}{\tau(z)}(\partial_z\tau(z))dz + \frac{1}{\overline{\overline{\tau(z)}}}(\partial_{\overline{z}}\overline{\tau(z)})d\overline{z} = \frac{\tau(z)}{z\kappa\tau(z)}dz - \frac{\overline{\tau(z)}}{\overline{z}\kappa\overline{\tau(z)}}d\overline{z} = \frac{1}{\kappa}(\frac{1}{z}dz - \frac{1}{\overline{z}}d\overline{z})$$

$$\implies \tau^*\tilde{\zeta}_z = \frac{1}{\kappa}\tilde{\zeta}_z = \frac{\kappa_2}{\kappa_1}\tilde{\zeta}_z.$$

Therefore, the condition on the 1-form of the connection becomes (recalling $\lambda = 2k_1\kappa_1 + 2k_2\kappa_2$):

$$\tau^*\tilde{\alpha}^2_z = -\frac{\lambda}{2\kappa_1}\tilde{\zeta}_z + \tilde{\alpha}^1_z. \tag{4.2.5}$$

□

Now, we'd like to construct connections (acting) on the line bundles $\tilde{L}_\lambda$. Since sections of L_μ are naturally identified with $\mathbb{C}^2$-valued functions, we clearly need the coefficients of the connection to be 2×2 matrix-valued functions. We also need it to respect the conditions (2.1.1) and (2.1.2). It is clear, that for any $M_{2\times 2}(\mathbb{R}[x])-$connection one-form $\tilde{a}'$ and any vector field[12] $\tilde{\mathbf{X}}_{\tilde{\mathbf{x}}} \in \mathfrak{X}(\mathbb{R}^2)$, the connection $\nabla^{\tilde{a}} = d - \mathrm{i}\tilde{a}$ satisfies (2.1.1). However, before we identify such conditions on a', we first need to define *Clifford multiplication* on this bundle.

[12] We often work on $\mathbb{C}$ (identified as the complex plane) and vector fields $\tilde{\mathbf{X}}_z$, but we use the same letter, $\tilde{\mathbf{X}}$, to denote arbitrary vector fields on either. In equations, the subscript (z and $\tilde{x}$) apart from coordinates, automatically denotes whether this vector field belongs to $\mathfrak{X}(\mathbb{R}^2)$ or $\mathfrak{X}(\mathbb{C})$

4.2.2 Compatibility of forms and metrics

Let (x,y) the standard coordinates in $\mathbb{R}^2$:

$$\sigma(dx) = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\; \& \;\; \sigma(dy) = \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

$$\implies \sigma(dz) = \sigma(dx + \mathrm{i}dy) = \sigma_1 + \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\& \;\; \sigma(d\overline{z}) = \sigma(dx - \mathrm{i}dy) = \sigma_1 - \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

and so $\sigma_1 = \frac{1}{2}(\sigma(dz) + \sigma(d\overline{z}))$ and $\sigma_2 = \frac{1}{2\mathrm{i}}(\sigma(dz) - \sigma(d\overline{z})) = \frac{\mathrm{i}}{2}(-\sigma(dz) + \sigma(d\overline{z}))$

We need to have: $[\tilde{\nabla}_{\tilde{\mathbf{X}}}, \sigma(\tilde{\alpha})] = \sigma(\nabla^{LC}_{\tilde{\mathbf{X}}}\tilde{\alpha})$ for any form $\tilde{\alpha} \in \Omega^1(\mathbb{C})$, $\tilde{\mathbf{X}} \in \mathfrak{X}(\mathbb{C})$, where ∇^{LC} is the Levi-Civita connection acting on differential 1-forms. Let

$$\tilde{\alpha}_z = \tilde{\alpha}_+(z,\overline{z})dz + \tilde{\alpha}_-(z,\overline{z})d\overline{z} \in \Omega^1(\mathbb{C}), \text{ where } z = \tilde{x} + \mathrm{i}\tilde{y},$$

$$\tilde{\alpha}_\pm(z,\overline{z}) = \frac{1}{2}(\tilde{\alpha}_x(\tilde{\mathbf{x}}) \mp \mathrm{i}\tilde{\alpha}_y(\tilde{\mathbf{x}})), \quad \tilde{\mathbf{x}} \in \mathbb{R}^2, \text{and } \; \tilde{\alpha}_{1,2}(\tilde{\mathbf{x}}) \in \mathbb{C} \text{ (pointwise)}, \tilde{\boldsymbol{x}} = (\tilde{x},\tilde{y}) \in \mathbb{R}^2.$$

The map σ is linear over the space of complex functions, so to properly define a Clifford multiplication on the bundle L_μ (whose base-space can naturally be identified with $\mathbb{C}_{1,2}/R_{\kappa^\pm}$ where $R_{\kappa^{\pm 1}} = e^{2\pi\mathrm{i}\kappa^{\pm 1}}$ generated a group of rotations on the complex plane). The quotient denotes $z_1' \sim_{R_{\kappa^{\pm 1}}} z_2'$ iff $z_1' = e^{2k\pi\mathrm{i}\kappa^{\pm 1}} z_2'$ for some $k \in \mathbb{Z}$. The consideration of this equivalence class helps us tackle problems emerging from the multi-valuedness of $e^{2\pi\mathrm{i}\kappa^{\pm 1}}$. To properly define Clifford multiplication on these bundles, we need to have an inner product of differential 1-forms on each of these bundles, according to the compatibility condition:

Remark 4.2.6. In a normed vector space X, we can define a metric $|(\tilde{x},\tilde{y})|_X$ (with $\tilde{x},\tilde{y} \in X$) (such as a spin-c bundle, with X as the base manifold), via the norm $||\cdot||_X$ of this space X as $|(\tilde{x},\tilde{y})|_X := ||\tilde{x} - \tilde{y}||_X$. More generally, if X is an inner product space the norm then can be naturally induced by the inner product $\langle,\rangle_X$ of that space (for example the one in (1.3.1)) as $|(\tilde{x},\tilde{y})|_X := ||\tilde{x} - \tilde{y}||_X := \langle \tilde{x} - \tilde{y}, \tilde{x} - \tilde{y}\rangle_X$. However, in our case we have by definition a metric on a Riemannian manifold and in our case, we can consider conformal metrics $g^{1,2}_{\tilde{\mathbf{x}}} : T_{\tilde{\mathbf{x}}}\mathbb{S}^2_{S,N} \times T_{\tilde{\mathbf{x}}}\mathbb{S}^2_{S,N} \mapsto \mathbb{R}_{\geq 0}$ as sums of direct products of two differential one-forms (viewed as tensors of degree (0,1)), i.e. tensors of degree $(0,2)$: $g^{1,2}_{\tilde{\mathbf{x}}} = g^{1,2}_{ij}(\tilde{\mathbf{x}}) d\tilde{x}^i \otimes d\tilde{x}^j$ (recall this notation means the summation takes place over all possible duos of indices i and j) in local coordinates $\tilde{x}_i$, $i = 1, 2, ..., \dim X$. Farkas and Kra, 1980 discuss Riemannian metrics on Riemann surfaces such as the Riemann sphere.

Consider weights $\omega_{1,2} : \mathbb{C} \mapsto \mathbb{R}_{>0}$ and define the "weighted" norms $||\cdot||_{\mathbb{O}_{\kappa^{3-2j}}} : \mathbb{O}_{\kappa^{3-2j}} \mapsto \mathbb{R}_{>0}$ of the form:

$$||\cdot||_{\mathbb{O}_{\kappa^{3-2j}}} = \omega_j ||\cdot||_{\Omega^1(\mathbb{C})} \tag{4.2.6}$$

where the latter is the standard Euclidean norm in the space of forms, induced by the inner product $\langle , \rangle_{\Omega^1(\mathbb{C})}$ as

$$||\tilde{\alpha}^{1,2}||_{\Omega^1(\mathbb{C})} := \sqrt{\langle \tilde{\alpha}^{1,2}, \tilde{\alpha}^{1,2} \rangle_{\Omega^1(\mathbb{C})}} := \sqrt{\tilde{\alpha}^{1,2}(\tilde{\boldsymbol{A}}_{1,2})}$$

where $\tilde{\boldsymbol{A}}_{1,2}(\tilde{\mathbf{x}})$ are the vector fields dual to the 1-forms $\tilde{\alpha}^{1,2}$ respectively whose components are the respective components of the 1-forms $\tilde{\alpha}^{1,2}$ respectively. However, in order to define forms globally (all over) on $\mathbb{O}_{\boldsymbol{\kappa}}$ we need forms $\tilde{\alpha}^{1,2}$ defined on $\mathbb{C}_{1,2}$ respectively, which are compatible with each other with respect to the map τ, i.e. we want[13] $\tilde{\alpha}^1 = \tau^* \tilde{\alpha}^2$:

$$\tau^* \tilde{\alpha}^2_z = a^2_+(\tau(z), \overline{\tau(z)}) \frac{d\tau}{dz} dz + \tilde{\alpha}^2_-(\tau(z)), \overline{\tau(z)}) \frac{d\overline{\tau}}{d\overline{z}} d\overline{z}$$

$$= \tilde{\alpha}^2_+(\tau(z), \overline{\tau(z)}) \frac{d\tau}{dz} dz + \tilde{\alpha}^2_-(\tau(z)), \overline{\tau(z)}) \frac{d\overline{\tau}}{d\overline{z}} d\overline{z}$$

$$= -\frac{C_\kappa}{\kappa z^{1+\frac{1}{\kappa}}} \tilde{\alpha}^2_+(\tau(z), \overline{\tau(z)}) dz - \frac{\overline{C_\kappa}}{\kappa \overline{z}^{1+\frac{1}{\kappa}}} \tilde{\alpha}^2_-(\tau(z), \overline{\tau(z)}) d\overline{z} = \tilde{\alpha}^1_+(z, \overline{z}) dz + \tilde{\alpha}^1_-(z, \overline{z}) d\overline{z} =: \tilde{\alpha}^1_z$$

$$\text{if and only if } \tilde{\alpha}^1_-(z, \overline{z}) = -\frac{\overline{C_\kappa}}{\kappa \overline{z}^{1+\frac{1}{\kappa}}} \tilde{\alpha}^2_-(\tau(z), \overline{\tau(z)}) \ \& \ \tilde{\alpha}^1_+(z, \overline{z}) = -\frac{C_\kappa}{\kappa z^{1+\frac{1}{\kappa}}} \tilde{\alpha}^2_+(\tau(z), \overline{\tau(z)}).$$

[13]Notice that here we do not require compatibility with respect to a connection as in (2.1.2) (where we essentially consider two different 1-forms on the same bundle), but instead we require compatibility in terms of charts that give us a global connection

We also want the norms $||\cdot||_{\mathbb{O}_{\kappa^{3-2j}}}$ to be compatible with each other, for that we have:

Proposition 4.2.7. In order for (4.2.6) to hold, we need the weights $\omega_{1,2}$ to satisfy:

$$\omega_2(\tau(z)) = \frac{C_\kappa \omega_1(z)}{\kappa |z|^{1+\frac{1}{\kappa}}} \tag{4.2.7}$$

for some constant $C_\kappa \in \mathbb{R}_{>0}$ such that $\tau(z) = C_\kappa z^{-1/\kappa}$ (see remark 4.1.12).

Proof. Consider an arbitrary 1-form $\tilde{\alpha} \equiv (\tilde{\alpha}_1, \tilde{\alpha}_2) \in \Omega^1(\mathbb{O}_{\boldsymbol{\kappa}})$. We have that its norm satisfies:

$$||\tilde{\alpha}^1_z||_{\Omega^1(\mathbb{O}_\kappa)} = ||\tilde{\alpha}^2_{\tau(z)}||_{\Omega^1(\mathbb{O}_{\frac{1}{\kappa}})} \iff \omega_1(z)||\tilde{\alpha}^1_z||_{\Omega^1_{\mathbb{R}}(\mathbb{C})} = \omega_2(\tau(z))||\tilde{\alpha}^2_{\tau(z)}||_{\Omega^1_{\mathbb{R}}(\mathbb{C})}$$

$$\iff \omega_1(z)||\tilde{\alpha}^1_z||_{\Omega^1_{\mathbb{R}}(\mathbb{C})} = \omega_2(\tau(z))||\tilde{\alpha}^2_{\tau(z)}||_{\Omega^1_{\mathbb{R}}(\mathbb{C})}$$

$$\iff \frac{|C_\kappa|\omega_1(z)}{\kappa|z|^{1+\frac{1}{\kappa}}}\sqrt{\left(\tilde{\alpha}^2_-(\tau(z),\overline{\tau(z)})\right)^2 + \left(\tilde{\alpha}^2_+(\tau(z),\overline{\tau(z)})\right)^2} =$$

$$\iff \omega_2(\tau(z))\sqrt{\left(\tilde{\alpha}^2_-(\tau(z),\overline{\tau(z)})\right)^2 + \left(\tilde{\alpha}^2_+(\tau(z),\overline{\tau(z)})\right)^2}$$

$$\omega_2(\tau(z)) = \frac{C_\kappa \omega_1(z)}{\kappa |z|^{1+\frac{1}{\kappa}}}. \tag{4.2.8}$$

□

That's the necessary condition the weights $\omega_{1,2}$ need to satisfy in order for us to be able to define a weighted metric for the space made up by the two copies, $\mathbb{C}_{1,2}$ (resp. $\mathbb{R}^2_{1,2}$), of $\mathbb{C}$, whose points are identified by the map z. This is guaranteed by (4.1.62).

4.2.3 Clifford multiplication on the quotient space

Now that we have a comprehensive description of well-defined forms and related conformal metrics and norms on the *spin*^{c}-bundles S_μ, we can define *Clifford multiplication* on that bundle. By definition of the σ-matrices in 2 dimensions (with $\sigma(dx) = \sigma_1$, $\sigma(dy) = \sigma_2$) and the linear property of Clifford multiplication, we have:

$$\sigma(f(x,y)dx + g(x,y)dy) = f(x,y)\sigma(dx) + g(x,y)\sigma(dy) = f(x,y)\sigma_1 + g(x,y)\sigma_2.$$

We re-write the coordinates (x, y) as z and $\bar{z}$: $dz = dx + \mathrm{i}dy$, $d\bar{z} = dx - \mathrm{i}dy$, $dx = \frac{1}{2}(dz + d\bar{z})$ and $dx = \frac{1}{2\mathrm{i}}(dz - d\bar{z}) \implies \sigma_1 = \sigma(dx) = \frac{1}{2}\big(\sigma(dz) + \sigma(d\bar{z})\big)$ and $\sigma_2 = \sigma(dy) = \frac{1}{2\mathrm{i}}\big(\sigma(dz) - \sigma(d\bar{z})\big)$. Using these, by re-writing the (real) 1-form $\tilde{\alpha}_{\tilde{\boldsymbol{x}}} =$

In (3.4.9), we presented the natural generalization of "weighted" *Clifford multiplication*, corresponding to the conformal change of metric (with weight $\omega_{1,2}$)

$$g(\cdot,\cdot) \to g_{\omega_{1,2}}(\cdot,\cdot) := \omega_{1,2}^2 g(\cdot,\cdot).$$

Given the fact that we'll mostly be working on $\mathbb{C}$, we rewrite the *weighted Clifford multiplication*, $\sigma_{\omega_{1,2}}$ on the *Spin*^{c}-bundle over $\mathbb{C}_{1,2}$ $(\equiv \mathbb{R}^2_{1,2})$ as:

$$\sigma_{\omega_{1,2}}(\tilde{\alpha}_z^{1,2}) := \omega_{1,2}^{-1}(|z|^2)\sigma(\tilde{\alpha}_z^{1,2}) \tag{4.2.9}$$

where $\tilde{\alpha}_z^{1,2} = \tilde{\alpha}_+^{1,2}dz + \tilde{\alpha}_-^{1,2}d\bar{z} = \tilde{\alpha}_+^{1,2}(z,\bar{z})(dx + \mathrm{i}dy) + \tilde{\alpha}_-^{1,2}(z,\bar{z})(dx - \mathrm{i}dy)$ which equals $(\tilde{\alpha}_+^{1,2}(z,\bar{z}) + \tilde{\alpha}_-^{1,2}(z,\bar{z}))dx + \mathrm{i}(\tilde{\alpha}_+^{1,2}(z,\bar{z}) - \tilde{\alpha}_+^{1,2}(z,\bar{z}))dy \equiv 2\tilde{\alpha}_{\tilde{x}}^{1,2}(\tilde{\mathbf{x}})d\tilde{x} + 2\tilde{\alpha}_{\tilde{y}}^{1,2}(\tilde{\mathbf{x}})d\tilde{y}$ As a direct consequence of the definition of σ, given $\tilde{\alpha}_z^{1,2} = \tilde{\alpha}_+^{1,2}dz + \tilde{\alpha}_-^{1,2}d\bar{z}$ we have:

$$\sigma_{\omega_{1,2}}(\tilde{\alpha}_z^{1,2}) = \sigma_{\omega_{1,2}}(\tilde{\alpha}_+^{1,2}dz + \tilde{\alpha}_-^{1,2}d\bar{z})$$

$$= \omega_{1,2}^{-1}(|z|^2)\sigma(\tilde{\alpha}_+^{1,2}(z,\bar{z})dz + \tilde{\alpha}_-^{1,2}(z,\bar{z})d\bar{z})$$

$$= \omega_{1,2}^{-1}(|z|^2)\tilde{\alpha}_+^{1,2}(z,\bar{z})\sigma(dz) + \omega_{1,2}^{-1}(|z|^2)\tilde{\alpha}_-^{1,2}(z,\bar{z})\sigma(d\bar{z})$$

$$= 2\omega_{1,2}^{-1}(|z|^2)\begin{pmatrix} 0 & \tilde{\alpha}_{\tilde{x}}^{1,2}(\tilde{\mathbf{x}}) - \mathrm{i}\tilde{\alpha}_{\tilde{y}}^{1,2}(\tilde{\mathbf{x}}) \\ \tilde{\alpha}_{\tilde{x}}^{1,2}(\tilde{\mathbf{x}}) + \mathrm{i}\tilde{\alpha}_{\tilde{y}}^{1,2}(\tilde{\mathbf{x}}) & 0 \end{pmatrix}, \quad z = \tilde{x} + \mathrm{i}\tilde{y}, \;\; \tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y}).$$

As in Erdős and Solovej, 2001, simple calculation can easily show that for any sections of the *spin*^{c}-bundle S_μ. We've introduced these weights so that this definition respects/is consistent with the *inner product* for Pauli matrices as in (1.3.1), in fact, we first see that the norms are respected (here $\Psi^2(\mathbb{O}_{\kappa^{\pm 1}})$ is the space of endomorphism of Spinors/sections of S_μ):

$$||\sigma(\tilde{\alpha}_z^j)||_{\Psi^2(\mathbb{O}_{\kappa^{3-2j}})} := \langle \sigma(\tilde{\alpha}_z^j), \sigma(\tilde{\alpha}_z^j)\rangle_{\Psi^2(\mathbb{O}_{\kappa^{3-2j}})} = \frac{1}{2\omega_j^2(|z|^2)}\{\sigma(\tilde{\alpha}_z^j), \sigma(\tilde{\alpha}_z^j)\}$$

$$= \frac{1}{2\omega_j^2(|z|^2)} Tr(\sigma^2(\tilde{\alpha}_z^j)I_2) = \frac{1}{2} Tr(\sigma_{\omega_j}^2(\tilde{\alpha}_z^j)I_2).$$

However,

$$\frac{4}{\omega_j^2(|z|^2)} \tilde{\alpha}_-^j(z,\overline{z})\tilde{\alpha}_+^j(z,\overline{z})I_2 = \frac{4}{\omega_j^2(|z|^2)} \big(\tilde{\alpha}_x^j(\tilde{\mathbf{x}}) - \mathrm{i}\tilde{\alpha}_y^j(\tilde{\mathbf{x}})\big)\big(\tilde{\alpha}_x^j(\tilde{\mathbf{x}}) + \mathrm{i}\tilde{\alpha}_y^j(\tilde{\mathbf{x}})\big)I_2$$

$$= \frac{4}{\omega_j^2(|z|^2)} \big((\tilde{\alpha}_{\tilde{x}}^j(\tilde{\mathbf{x}}))^2 + (\tilde{\alpha}_{\tilde{y}}^j(\tilde{\mathbf{x}}))^2\big)I_2 = \frac{4}{\omega_j^2(|z|^2)} ||\tilde{\alpha}_{\tilde{\mathbf{x}}}^j||^2_{\Omega^1_{\mathbb{R}}(\mathbb{R}^2)} I_2 = \frac{4}{\omega_j^2(z)} \langle \tilde{\alpha}_{\tilde{\mathbf{x}}}^j, \tilde{\alpha}_{\tilde{\mathbf{x}}}^j \rangle_{\Omega^1(\mathbb{R}^2)} I_2$$

$$\implies \frac{2}{\omega_j^2(|z|^2)} \Big(Tr\big(\sigma^2(\tilde{\alpha}_{\tilde{\mathbf{x}}}^j)I_2\big)\Big) = \frac{1}{2}\Big(Tr\Big(\frac{4}{\omega_j^2(|z|^2)}\sigma^2(\tilde{\alpha}_{\tilde{\mathbf{x}}}^j)I_2\Big)\Big).$$

More generally, for $z = \tilde{x} + \mathrm{i}\tilde{y} \equiv \tilde{\mathbf{x}} = (\tilde{x}, \tilde{y})$ and $z' = \tilde{x}' + \mathrm{i}\tilde{y}' \equiv \tilde{\mathbf{x}}' = (\tilde{x}', \tilde{y}')$ we can easily see that the change from real to complex coordinates still respects (1.3.1):

$$\langle \tilde{\sigma}(\tilde{\alpha}_z^j), \tilde{\sigma}(\tilde{\alpha}_{z'}^j) \rangle_{\Psi^2(\mathbb{O}_{\kappa^{3-2j}})} =$$

$$\frac{1}{2\omega_j(|z|^2)\omega_j(|z'|^2)} \{\tilde{\sigma}(\tilde{\alpha}_z^j), \tilde{\sigma}(\tilde{\alpha}_{z'}^j)\} = \frac{1}{2\omega_j(|z|^2)\omega_j(|z'|^2)} \big(\tilde{\sigma}(\tilde{\alpha}_{z'}^j)\tilde{\sigma}(\tilde{\alpha}_{z'}^j) + \tilde{\sigma}(\tilde{\alpha}_z^j)\tilde{\sigma}(\tilde{\alpha}_{z'}^j)\big)$$

$$= \frac{2}{\omega_j(|z|^2)\omega_j(|z'|^2)} \big(\tilde{\alpha}_+^j(z,\overline{z})\tilde{\alpha}_-^j(z',\overline{z}') + \tilde{\alpha}_-^j(z,\overline{z})\tilde{\alpha}_+^j(z',\overline{z}')\big)I_2$$

$$= \frac{4}{\omega_j^2(z)} \big(\tilde{\alpha}_x^{1,2}(\tilde{x},\tilde{y})\tilde{\alpha}_x^j(\tilde{x}',\tilde{y}') + \tilde{\alpha}_y^j(\tilde{x},\tilde{y})\tilde{\alpha}_y^j(\tilde{x}',\tilde{y}')\big)I_2$$

$$= \frac{4}{\omega_j^2(z)} \langle \alpha_{\tilde{\mathbf{x}}}^j, \alpha_{\tilde{\mathbf{x}}'}^j \rangle_{\Omega^1(\mathbb{R}^2)} I_2 = \Big\langle \frac{2}{\omega_j(z)}\alpha_{\tilde{\mathbf{x}}}^j, \frac{2}{\omega_j(z)}\alpha_{\tilde{\mathbf{x}}'}^j \Big\rangle_{\Omega^1(\mathbb{R}^2)} I_2 =$$

$$= \frac{4}{\omega_j^2(z)} \Big(Tr\big(\sigma(\tilde{\alpha}_{\tilde{\mathbf{x}}}^j)\sigma(\tilde{\alpha}_{\tilde{\mathbf{x}}'}^j)\big)\Big)I_2 = \Big(Tr\Big(\Big(\frac{2}{\omega_j(z)}\sigma(\tilde{\alpha}_{\tilde{\mathbf{x}}}^j)\Big)\Big(\frac{2}{\omega_j(z)}\sigma(\tilde{\alpha}_{\tilde{\mathbf{x}}'}^j)\Big)\Big)\Big)I_2$$

$$= \langle \tilde{\sigma}(\tilde{\alpha}_{\tilde{\mathbf{x}}}^j), \tilde{\sigma}(\tilde{\alpha}_{\tilde{\mathbf{x}}'}^j) \rangle_{\Psi^2(\mathbb{O}_{\kappa^{3-2j}})}.$$

These automorphisms σ act on sections of S_μ, which can be thought of as spinors in the form $\begin{pmatrix} \tilde{u}_{up}^+ \\ \tilde{u}_{dn}^- \end{pmatrix}$, where the subscripts up,dn denote "up" and "down" respectively and the superscripts $\pm$ the denote the line bundle $\tilde{L}_{\mu\mp} := \tilde{L}_{\mu\mp(\kappa_1+\kappa_2)}$, where they belong to.

Remark 4.2.8. The aforementioned equations and definitions, most notably (4.2.7), hold for the case where the space(s) $\mathbb{C}_j$, $j = 1, 2$ (consequently $\mathbb{R}^2_j$, which are about to be equipped with weighted metric) are quotient by the from action $e^{2\pi\mathrm{i}\kappa^{3-2j}}$ respectively.

4.2.4 Spin^c connections and Weyl-Dirac Operators in local coordinates

We follow the reasoning for constructing a *Spin^c connection* in Erdős and Solovej, 2001 on different charts of our base ("orbit") space, $\mathbb{O}_\kappa$. 1-forms $\tilde{a}_z^{1,2}$ that satisfy (2.1.2) (so that they will be compatible with the connection). However, the connections $\tilde{\nabla}$ need to satisfy (4.2.2). Also, recall than under a conformal change of metric $g(\cdot,\cdot) \to \omega^2 g(\cdot,\cdot)$, a *spin^c* connection $\tilde{\nabla}^{\tilde{\alpha}}$ changes to

$$\tilde{\nabla}^{\tilde{a},\omega}_{\tilde{\mathbf{X}}} = \tilde{\nabla}^{\tilde{a}}_{\tilde{\mathbf{X}}} + \frac{1}{4\omega_{1,2}}[\sigma(\tilde{\mathbf{X}}^*), \sigma(d\omega_{1,2})]$$

for $\tilde{\mathbf{X}} \in \mathfrak{X}(\mathbb{R}^2)$ (or $\mathfrak{X}(\mathbb{C})$ depending on the coordinates we use) where ∇ is the respective standard (non-conformal) connection.

Analytically in two (Euclidean) dimensions, we let $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}(\tilde{x},\tilde{y}) \in \mathfrak{X}(\mathbb{R}^2)$, so:

$$\tilde{\mathbf{X}}_{(\tilde{x},\tilde{y})} = \tilde{X}_{\tilde{x}}(\tilde{x},\tilde{y})\partial_{\tilde{x}} + \tilde{X}_{\tilde{y}}(\tilde{x},\tilde{y})\partial_{\tilde{y}}, \text{ this implies}$$

$$\tilde{\mathbf{X}}^* = \tilde{X}_{\tilde{x}}(\tilde{x},\tilde{y})d\tilde{x} + \tilde{X}_{\tilde{y}}(\tilde{x},\tilde{y})d\tilde{y}$$

$$\implies \sigma(\tilde{\mathbf{X}}^*) = \tilde{X}_{\tilde{x}}(\tilde{x}_1,\tilde{x}_2)\sigma_1 + \tilde{X}_{\tilde{y}}(\tilde{x}_1,\tilde{x}_2)\sigma_2$$

(denoting $(\tilde{x}_1,\tilde{x}_2)$ the standard coordinates on the Euclidean plane) and

$$\sigma(d\omega_{1,2}) = \sigma(\partial_{\tilde{x}}\omega_{1,2}d\tilde{x} + \partial_{\tilde{y}}\omega_{1,2}d\tilde{y})$$
$$= \partial_{\tilde{x}}\omega_{1,2}\sigma(d\tilde{x}) + \partial_{\tilde{y}}\omega_{1,2}\sigma(d\tilde{y}) = (\partial_{\tilde{x}}\omega_{1,2})\sigma_1 + (\partial_{\tilde{y}}\omega_{1,2})\sigma_2$$

and so

$$[\sigma(\tilde{\mathbf{X}}^*), \sigma(d\omega_{1,2})] = [\tilde{X}_{\tilde{x}}(\tilde{x},\tilde{y})\sigma_1 + \tilde{X}_{\tilde{y}}(\tilde{x},\tilde{y})\sigma_2, (\partial_{\tilde{x}}\omega_{1,2})\sigma_1 + (\partial_{\tilde{y}}\omega_{1,2})\sigma_2]$$

$$= 2(\tilde{X}_{\tilde{x}}(\tilde{x},\tilde{y})\partial_{\tilde{y}}\omega_{1,2} - \tilde{X}_{\tilde{y}}(\tilde{x},\tilde{y})\partial_{\tilde{x}}\omega_{1,2})\sigma_3.$$

Without loss of generality, we assume $\omega_{1,2} \equiv \omega_{1,2}(|z|^2)$
$(= \omega_{1,2}(x_1^2 + x_2^2)$, with $\omega' = d\omega(\tilde{z})/d\tilde{z})$ so the quantity above can be written as:

$$[\sigma(\tilde{\mathbf{X}}^*), \sigma(d\omega_{1,2})] =$$

$$4\omega'_{1,2}(\tilde{y}\tilde{X}_{\tilde{x}}(\tilde{x},\tilde{y}) - \tilde{x}\tilde{X}_{\tilde{y}}(\tilde{x},\tilde{y}))\mathrm{i}\sigma_3 = 4\mathrm{i}\omega'_{1,2}(\tilde{y}d\tilde{x} - \tilde{x}d\tilde{y})(\tilde{\mathbf{X}})\sigma_3$$

$\Longrightarrow$

$$\nabla^{\omega_{1,2}}_{\tilde{\mathbf{X}}} = \nabla_{\tilde{\mathbf{X}}} + \frac{\omega'_{1,2}}{\omega_{1,2}}(\tilde{y}d\tilde{x} - \tilde{x}d\tilde{y})(\tilde{\mathbf{X}})\sigma_3. \tag{4.2.10}$$

Now, we can define the corresponding (weighted) *Weyl-Dirac* operator as:

$$\tilde{\mathcal{D}}^{\omega_{1,2}}_{\tilde{a}^{1,2}} := \frac{1}{\omega_{1,2}}\tilde{\boldsymbol{\sigma}}.(-\mathrm{i}\tilde{\boldsymbol{\nabla}}^{\omega_{1,2}} - \tilde{\boldsymbol{A}}^{1,2}(\tilde{\boldsymbol{x}})).$$

This definition is consistent with (2.1.5).

Here $\tilde{\boldsymbol{\sigma}} = (\sigma_1, \sigma_2)$, $\tilde{\boldsymbol{A}}^{1,2}(\tilde{\boldsymbol{x}}) = (\tilde{A}^{1,2}_{\tilde{x}}(\tilde{\boldsymbol{x}}), \tilde{A}^{1,2}_{\tilde{y}}(\tilde{\boldsymbol{x}}))$, $\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y})$ and [14]

$$\tilde{\boldsymbol{\nabla}}^{\omega_{1,2}} = (\tilde{\nabla}^{\omega_{1,2}}_{\tilde{x}}, \tilde{\nabla}^{\omega_{1,2}}_{\tilde{y}}) := (\nabla_{\tilde{x}} + \tilde{y}\frac{\omega'_{1,2}}{\omega_{1,2}}, \nabla_{\tilde{y}} - \tilde{x}\frac{\omega'_{1,2}}{\omega_{1,2}}),$$

the conformal "gradient" in two dimensions in standard coordinates defined as the "coordinates" of the conformal connection (4.2.10), $\tilde{\boldsymbol{\nabla}} = (\nabla_{\tilde{x}_1}, \nabla_{\tilde{x}_2})$ the usual gradient in two dimensions. Analytically, we can write:

$$\begin{aligned}
\tilde{\mathcal{D}}^{\omega_{1,2}}_{\tilde{a}^{1,2}} &= \frac{1}{\omega_{1,2}}\Big(- \mathrm{i}\sigma_1(\nabla^{\omega_{1,2}}_{\tilde{x}} - \tilde{A}^{1,2}_{\tilde{x}}(\tilde{\boldsymbol{x}})) - \sigma_2(-\mathrm{i}\nabla^{\omega_{1,2}}_{\tilde{y}} - \tilde{A}^{1,2}_{\tilde{y}}(\tilde{\boldsymbol{x}}))\Big) \\
&= \frac{1}{\omega_{1,2}}\Big(- \mathrm{i}\sigma_1\nabla^{\omega_{1,2}}_{\tilde{x}} - \sigma_1\tilde{A}^{1,2}_{\tilde{x}}(\tilde{\boldsymbol{x}}) - \mathrm{i}\sigma_2\nabla^{\omega_{1,2}}_{\tilde{y}} - \sigma_2\tilde{A}^{1,2}_{\tilde{y}}(\tilde{\boldsymbol{x}})\Big) \\
&= \frac{1}{\omega_{1,2}}\Big(\sigma_1\Big(- \mathrm{i}\tilde{\nabla}_{\tilde{x}} - \tilde{A}^{1,2}_{\tilde{x}}(\tilde{\boldsymbol{x}}) + \tilde{y}\frac{\omega'_{1,2}}{\omega_{1,2}}\Big) + \sigma_2\Big(- \mathrm{i}\tilde{\nabla}_{\tilde{y}} - \tilde{A}^{1,2}_{\tilde{y}}(\tilde{\boldsymbol{x}}) - \tilde{x}\frac{\omega'_{1,2}}{\omega_{1,2}}\Big)\Big).
\end{aligned}$$

These operators are clearly simpler to study than the typical *Weyl-Dirac* Operator in three dimensions. The good news is, that after studying such 2-dimensional operators, we can still translate the results into three dimensions via the lifting operator(s) $\mathcal{P}_\mu$, (5.1.16). However, we should now investigate how magnetic potentials change under such transformations, introduced by the map F.

[14] We use tilde when notating objects in two dimensions to distinguish between objects in three dimensions notated similarly. We use them now in order to avoid confusion with the coordinates in $\mathbb{R}^3$.

4.2.5 Useful differential forms

The magnetic potential(s) $\mathbf{A}(\mathbf{x})$, and respective fields $\mathbf{B}(\mathbf{x}) := \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x})$ on $\mathbb{R}^3$ are naturally associated with 1 and 2-forms $a_{\mathbf{x}}$ and $\beta_{\mathbf{x}} = d\alpha_{\mathbf{x}}$ respectively. In particular, the magnetic potentials we'll work with are of the form

$$\alpha_{\mathbf{x}} = \nu\alpha^b_{\mathbf{x}} + F^*\tilde{\alpha} \tag{4.2.11}$$

where

$$\alpha^b_{\mathbf{x}} := \frac{1}{|\mathbf{X}(\mathbf{x})|^2}(\mathbf{X}(\mathbf{x}).d\mathbf{x}) = \frac{1}{|\mathbf{X}(\mathbf{x})|^2}(X_1(\mathbf{x})dx_1 + X_2(\mathbf{x})dx_2 + X_3(\mathbf{x})dx_3) \tag{4.2.12}$$

and $\tilde{\alpha}$ is a 1-form on $\mathbb{O}_{\boldsymbol{\kappa}}$ and F^* denotes the *pull-back* to $\mathbb{R}^3$.

The respective magnetic potentials and fields on $\mathbb{R}^2$ will be denoted with the respective Greek characters with the tilde, $\tilde{\alpha}$ and $\tilde{\beta}$. The volume forms in the weighted versions of $\mathbb{R}^3$ and the two copies $\mathbb{R}^2_{1,2}$ of the plane, are denoted as (recall $\Omega(\mathbf{x}) = |\mathbf{X}(\mathbf{x})|^{-1}$)

$$\mathrm{vol}^3_{\Omega} = \Omega^3(\mathbf{x})dx_1 \wedge dx_2 \wedge dx_3 \tag{4.2.13}$$

and

$$\mathrm{vol}^2_{\omega_{1,2}} = \omega^2_{1,2}(|z|^2)\Big(-\frac{1}{2\mathrm{i}}\Big)dz \wedge d\bar{z} = \omega^2_{1,2}(|z|^2)d\tilde{x} \wedge d\tilde{y} \tag{4.2.14}$$

where $z = z_{1,2} = \tilde{x}_{1,2} + \mathrm{i}\tilde{y}_{1,2} \in \mathbb{C}_{1,2} \equiv \mathbb{R}^2_{1,2}$ (depending on the copy of the plane we're working on) and the weights $\omega_{1,2} : [0,\infty) \to [0,\infty)$ we'll be using on $\mathbb{R}^2$

$$\omega_1(w) = \frac{1}{\kappa_2}\frac{\sqrt{h_1^{-1}(w)}(1 - h_1^{-1}(w))}{w((1 - h_1^{-1}(w))^2 + 4\kappa^{-2}h_1^{-1}(w))} \tag{4.2.15}$$

and

$$\omega_2(w) = \frac{1}{\kappa_1}\frac{\sqrt{h_2^{-1}(w)}(1 - h_2^{-1}(w))}{w((1 - h_2^{-1}(w))^2 + 4\kappa^{-2}2h_2^{-1}(w))}. \tag{4.2.16}$$

In applications we have: $w = |z|^2 \equiv |z_j|^2 = |F_j(\mathbf{x})|^2 = |\tilde{F}_{\kappa^{3-2j}}(\chi_{\kappa^{3-2j}}(\mathbf{x}))|^2 = |\tilde{F}_{\kappa^{3-2j}}(\tilde{z}_j)|$. For the sake of neatness, we've defined

$$\tilde{\omega}^c_{1,2}(z,\bar{z}) := \omega_{1,2}(|z|^2). \tag{4.2.17}$$

The aforementioned formulas for the weights can be deduced by imposing the weight $\Omega(\mathbf{x})$ in $\mathbb{R}^3_{1,2}$ and looking for weights to impose on $\mathbb{R}^2$ such that the maps $F_{1,2} : \mathbb{R}^3_{1,2} \to \mathbb{R}^2$ can define a Riemannian submersion between the respective weighted spaces. See proposition 4.1.8 for details.

We'll also be working with the following 1-forms, $\tilde{w}^{1,2}$ corresponding to the respective conformal change of metric(s) on $\mathbb{R}^2_{1,2}$ (see subsection 4.1.2 for some of their essential propertial):

$$\tilde{w}^{1,2}_z := \tilde{\omega}^c_{1,2}(z)\tilde{\zeta}_z \tag{4.2.18}$$

where

$$\tilde{\omega}^c_{1,2}(z) := -\frac{1}{\kappa_1+\kappa_2}\frac{\omega'_{1,2}(|z|^2)}{\omega_{1,2}(|z|^2)} \tag{4.2.19}$$

and

$$\tilde{\zeta}_z = \frac{1}{2\mathrm{i}|z|^2}(\overline{z}dz - zd\overline{z}) = \frac{1}{2\mathrm{i}}\Big(\frac{1}{z}dz - \frac{1}{\overline{z}}d\overline{z}\Big). \tag{4.2.20}$$

We'll often work in real, regular planar coordinates $\tilde{\boldsymbol{x}} = (\tilde{x},\tilde{y})$ instead of the respective complex ones $z_{1,2} \equiv \tilde{x}_{1,2} + \mathrm{i}\tilde{y}_{1,2}$. In this case, we'll abuse notation again and denote z, when used as subscripts, as $\tilde{\boldsymbol{x}}$ in the aforementioned three formulas. In this case, the closed 1-form $\tilde{\zeta}$ will be written as:

$$\tilde{\zeta}_{\tilde{\boldsymbol{x}}} = \frac{1}{|\tilde{\boldsymbol{x}}|^2}(-\tilde{y}d\tilde{x} + \tilde{x}d\tilde{y}). \tag{4.2.21}$$

On $\mathbb{R}^3$, two useful 1-forms with similar notation are:

$$\zeta^1 = -x_2dx_1 + x_1dx_2 \tag{4.2.22}$$

and

$$\zeta^2 = x_1dx_1 + x_2dx_2. \tag{4.2.23}$$

Remark 4.2.9. When denoting forms on $\mathbb{R}^2$ or $\mathbb{C}$, we'll be slightly abusing notation and the only thing denoting whether we use the complex or real formulation (if the formula of the form is not written in full), as in (3.3.15) and (4.2.21) respectively, will be the subscript (resp. z or $\tilde{\boldsymbol{x}}$).

We further introduce the following scalar functions, $l_{1,2} : [0, \infty) \to [0, \infty)$:

$$l_1(w) = \frac{2}{\kappa_2} \frac{h_1^{-1}(w)}{(1 - h_1^{-1}(w))^2 + 4\kappa^2 h_1^{-1}(w)}, \tag{4.2.24}$$

$$l_2(w) = \frac{2}{\kappa_1} \frac{h_2^{-1}(w)}{(1 - h_2^{-1}(w))^2 + 4(1/\kappa)^2 h_2^{-1}(w)}. \tag{4.2.25}$$

We are now ready to define the 1-forms on $\mathbb{R}^2$ we'll mostly be working with:

$$\tilde{\gamma}_z^{1,2} = \frac{1}{2\mathrm{i}|z|^2} l_{1,2}(|z|^2)(\overline{z}dz - zd\overline{z}) = l_{1,2}(|z|^2)\tilde{\zeta}_z. \tag{4.2.26}$$

To define Weyl-Dirac Operators on $\mathbb{R}^2$ and $\mathbb{R}^3$ and how they are related via the Riemannian submersion defined by F, we first need to see how connections on $\tilde{\alpha} \in \mathbb{O}_\kappa$ behave when pulled-back via F. In particular, we'll need to define and study a specific class of such forms, corresponding to magnetic fields parallel to $\mathbf{X}(\mathbf{x})$. and we have:

Proposition 4.2.10. The 1-form $\tilde{\gamma}_z^{1,2}$ is smooth all over $\mathbb{C}$, is rotationally invariant and further satisfies:

$$\tau^* \tilde{\gamma}_z^2 = \tilde{\gamma}_z^1 - \frac{1}{2\kappa_1} \tilde{\zeta}_z \tag{4.2.27}$$

and $d\tilde{\gamma}_z^{1,2} = 4\kappa_1\kappa_2 m_{1,2}(z)\mathrm{vol}^2_{\omega_{1,2}} = 4\kappa_1\kappa_2 m_{1,2}(z)\tilde{\omega}^2_{1,2}(z) dz \wedge d\tilde{z}$.

Proof. The smoothness of $\tilde{\gamma}_z^{1,2}$ can be shown by noticing that:

$$l_{1,2}(|z|^2) = O(|\chi_{\kappa^{\pm 1}}|^2) \text{ as } |\chi_{\kappa^{\pm 1}}| \to 0,$$

since we have $|z|^2 = |\tilde{F}_{1,2}(\chi_{\kappa^{\pm 1}})|^2 = |\chi_{\kappa^{\pm 1}}|^2 f^2_{\kappa^{\pm 1}}(|\chi_{\kappa^{\pm 1}}|^2) = h(|\chi_{\kappa^{\pm 1}}|^2) = O(|\chi_{\kappa^{\pm 1}}|^2)$ (as $|\chi_{\kappa^{\pm 1}}| \to 0$) while its denominator is positive and bounded from below (for fixed $\kappa \in (0, 1)$). The latter can be seen by setting $s \equiv s_{1,2} = |\chi_{1,2}(\mathbf{x})|^2 \in [0, 1)$ and noticing:

$$(1 - s)^2 + 4\kappa^{\pm 2} s = 1 + s^2 + 2(2\kappa^{\pm 2} - 1)s =$$

$$1 - (2\kappa^{\pm 2} - 1)^2 + s^2 + 2(2\kappa^{\pm 2} - 1)s + (2\kappa^{\pm 2} - 1)^2 = 1 - (2\kappa^{\pm 2} - 1)^2 + (2\kappa^{\mp 2} - 1 + s)^2 \geq$$

$$1-(2\kappa^{\pm 2}-1)^2 = 4\kappa^{\pm 2}(1-\kappa^{\pm 2}) > 0$$

iff $\kappa^{\pm 2} \in (0,1)$, otherwise

$$(1-s)^2 + 4\kappa^{\pm 2}s = (1-s)^2 + 4s + 4(\kappa^{\pm 2}-1)s \geq (1-s)^2 + 4s = (1+s)^2 \geq 1.$$

So in each case:

$$(1-s_{1,2})^2 + 4\kappa^{\pm 2}s_{1,2} \geq \min\{4\kappa^{\pm 2}|1-\kappa^{\pm 2}|, 1\}. \tag{4.2.28}$$

It follows that $l_{1,2}(|z|^2)/|z|^2 = O(f^{-2}_{\kappa^{\pm 1}}(\chi_{\kappa^{\pm 1}}))$ as $\chi_{\kappa^{\pm 1}} \to 0$, which is $O(1)$ there (see (4.1.45) and the discussion below - in particular the limit is $\neq 0$). Moreover, the function $h_{1,2}(s)$ (mapping $[0,1)$ to $[0,\infty)$) is smooth & increasing, completing the argument for the smoothness of $\gamma_z^{1,2}$.

The fact that it is rotationally invariant follows from the fact that all terms appearing are radially symmetric.

Lastly, we have: $\tau^*\tilde{\gamma}_z^2 = \frac{1}{2\mathrm{i}}\tau^*\big((l_2(|z|^2)/|z|^2)\tilde{\zeta}_z\big) = \frac{1}{2\mathrm{i}}\tau^*(l_2(|z|^2)/|z|^2)\tau^*\tilde{\zeta}_z$

$$= \frac{1}{2\mathrm{i}|\tau(z)|^2}l_2(|\tau(z)|^2)\tau^*(\overline{z}dz - zd\tilde{z}) = -\frac{l_2(|\tau(z)|^2)}{\kappa}\tilde{\gamma}_z^1$$

where the last equality follows from the discussion before the derivation of (4.2.3).

Furthermore, we have:

$$l_2(|\tau(z)|^2) = \frac{2}{\kappa_1}\frac{\left(\frac{1-|\chi_{\kappa,1}|}{1+|\chi_{\kappa,1}|}\right)^2}{\left(1-\left(\frac{1-|\chi_{\kappa,1}|}{1+|\chi_{\kappa,1}|}\right)^2\right)^2 + \frac{4}{\kappa^2}\left(\frac{1-|\chi_{\kappa,1}|}{1+|\chi_{\kappa,1}|}\right)^2}$$

$$= \frac{2}{\kappa_1}\frac{\left(\frac{1-|\chi_{\kappa,1}|}{1+|\chi_{\kappa,1}|}\right)^2}{\left(\frac{(1+|\chi_{\kappa,1}|)^2-(1-|\chi_{\kappa,1}|)^2}{(1+|\chi_{\kappa,1}|)^2}\right)^2 + \frac{4}{\kappa^2}\left(\frac{1-|\chi_{\kappa,1}|}{1+|\chi_{\kappa,1}|}\right)^2}$$

$$= \frac{2}{\kappa_1}\frac{(1-|\chi_{\kappa,1}|)^2}{\frac{(2|\chi_{\kappa,1}|2)^2}{(1+|\chi_{\kappa,1}|)^2} + \frac{4}{\kappa^2}\big(1-|\chi_{\kappa,1}|\big)^2} = \frac{2}{\kappa_1}\frac{(1-|\chi_{\kappa,1}|^2)^2}{4|\chi_{\kappa,1}|^2 + \frac{4}{\kappa^2}\big(1-|\chi_{\kappa,1}|^2\big)^2}$$

$$= \frac{\kappa^2}{2\kappa_1}\frac{(1-|\chi_{\kappa,1}|^2)^2}{4\kappa^2|\chi_{\kappa,1}|^2+\left(1-|\chi_{\kappa,1}|^2\right)^2} = \frac{\kappa^2}{2\kappa_1}\left(1-\frac{4\kappa^2|\chi_{\kappa,1}|^2}{\left(\frac{1-|\chi_{\kappa,1}|}{1+|\chi_{\kappa,1}|}\right)^2+4\kappa^2|\chi_{\kappa,1}|^2}\right)$$

$$= \frac{\kappa^2}{2\kappa_1}(1-l_1(|z|^2)).$$

$$\implies -\frac{l_2(|\tau(z)|^2)}{\kappa}\zeta_z = \frac{\kappa}{2\kappa_1}(l_1(|z|^2)-1)\zeta_z = \frac{1}{2\kappa_2}l_1(|z|^2)\zeta_z - \frac{1}{2\kappa_2}\zeta_z = \tau^*\gamma_z^2.$$

Regarding the last fact, we have:

$$d\tilde{\gamma}_z^1 = (dl_1(z\bar{z}))\wedge\tilde{\zeta}_z + l_1(z\bar{z})\wedge d\tilde{\zeta}_z = dl_1(z\bar{z}))\wedge\tilde{\zeta}_z$$

where we used the fact that $d\tilde{\zeta}_z = 0$. We compute:

$$dl_1(z\bar{z}) = \partial_{\bar{z}}l_1(z\bar{z})d\bar{z} + \partial_z l_1(z\bar{z})dz = zl_1'(z\bar{z})d\bar{z} + \bar{z}l_1'(z\bar{z})dz$$

$$\implies dl_1(z\bar{z}))\wedge\tilde{\zeta}_z = (zl_1'(z\bar{z})d\bar{z}+\bar{z}l_1'(z\bar{z})dz)\wedge\tilde{\zeta}_z =$$

$$(zl_1'(z\bar{z})d\bar{z}+\bar{z}l_1'(z\bar{z})dz)\wedge\left(\frac{1}{z}dz-\frac{1}{\bar{z}}d\bar{z}\right) = -2l_1'(|z|^2)dzd\bar{z} = 4l_1'(|\tilde{\boldsymbol{x}}|^2)d\tilde{x}d\tilde{y}$$

where $\tilde{\boldsymbol{x}} = (\tilde{x},\tilde{y}) \in \mathbb{R}^2$.

We have:

$$-2l_1'(|z|^2) = -2\frac{(h_1^{-1})'(|z|^2)(1-h_1^{-1}(|z|^2))}{(1-h_1^{-1}(|z|^2))^2+\kappa^2 h_1^{-1}(|z|^2))^2}$$

and letting $z = \tilde{F}_1(\chi_1(\mathbf{x}))$ and noticing:

$$(h_1^{-1})'(w) = \frac{1}{h_1'(s)} = \frac{1}{f^2(s)+2sf(s)f'(s)} = \frac{1}{2sf^2(s)(\frac{1}{2s}+\frac{f'(s)}{f(s)})} = \frac{s}{h(s)\left(\sqrt{1+\frac{4\kappa^2 s}{(1-s^2)^2}}\right)}.$$

And noticing $h_1^{-1}(|z|^2) = s$ we get: $-2l_1(|z|^2) =$

$$-2F_1^*l_1 = \frac{-2s(1-s)(1-s^2)}{h_1(s)\sqrt{(1-s^2)^2+4\kappa^2 s}((1-s)^2+4\kappa^2 s)^2} = \frac{-2s(1-s)^2(1+s)}{h_1(s)((1-s)^2+4\kappa^2 s)^{5/2}} \tag{4.2.29}$$

where the last part came from (4.1.47) after setting $s = |\chi_1(\mathbf{x})|^2$, whilst by (4.2.15) and

(4.2.17) we have

$$F_1^*\tilde{\omega}_1(w) = \omega_1(|F_1(\mathbf{x})|^2) = \frac{1}{\kappa_2}\frac{\sqrt{h_1^{-1}(|F_1(\mathbf{x})|^2)}(1-h_1^{-1}(|F_1(\mathbf{x})|^2))}{|F_1(\mathbf{x})|((1-h_1^{-1}(|F_1(\mathbf{x})|^2))^2+4\kappa^{-2}h_1^{-1}(|F_1(\mathbf{x})|^2))} \tag{4.2.30}$$

and since $|F_1(\mathbf{x})|^2 = |\chi_1(\mathbf{x})|^2 f_1^2(|\chi_1(\mathbf{x})|^2)) = h_1(|\chi_1(\mathbf{x})|^2) = h_1(s)$, the RHS equates to:

$$\frac{1}{\kappa_2}\frac{\sqrt{s}(1-s)}{\sqrt{h_1(s)}((1-s)^2+4\kappa^{-2}s)} = \frac{\kappa_2\sqrt{s}(1-s)}{\sqrt{h_1(s)}(\kappa_2^2(1-s)^2+4\kappa_1^2 s)}.$$

We square this term, multiply it by $m_1(F_1(\mathbf{x}))$ and by (3.2.19) we get $-2F_1^*l_1 = 4\kappa_1\kappa_2 F_1^*(m_1(z))F_1^*(\tilde{\omega}_1^2(z))$, and so

$F_1^*(d\tilde{\gamma}_z^1) = 4\kappa_1\kappa_2 F_1^*(m_1(z)(\tilde{\omega}(z))^2 dz\wedge d\bar{z}) \implies d\tilde{\gamma}_z^1 = 4\kappa_1\kappa_2 m_1(z)\tilde{\omega}^2(z)dz\wedge d\bar{z}.$
Similarly for $F_2^*(d\tilde{\gamma}_z^2)$. □

Chapter 5: Interlacing Weyl-Dirac Operators on two and three dimensions

5.1 Lifts of spinors, connections on line and *Spin*^{c} bundles.

The primary aim of this thesis is to find a way to relate certain two-dimensional operators to standard, three-dimensional ones. Having seen how Weyl-Dirac operators behave under conformal change of variables, we want to transform the three-dimensional Weyl-Dirac operators according to the map F in particular. We'll construct lifts of Spinor bundles by lifting sections of the line bundles $\tilde{L}_{2k_1\kappa_1+2k_2\kappa_2}$ on $\mathbb{O}_\kappa$ to sections of line bundles on $\mathbb{R}^3$, for given $k_{1,2} \in \mathbb{Z}$, and considering the "direct sum" $\tilde{L}_{\mu-} \oplus \tilde{L}_{\mu+}$ (recall $\tilde{L}_{\mu\pm} = \tilde{L}_{\mu\pm(\kappa_1+\kappa_2)}$, $\mu = (2k_1+1)\kappa_1 + (2k_2+1)\kappa_2$). Since the *Weyl-Dirac* operator commutes with the operator $Q_{\mathbf{X}}$ and the eigenvectors of the latter have components of the form $\tilde{u}(z_0)e^{\mathrm{i}\lambda t}$, where $z_0 = F(\mathbf{x}(t)) \in \mathbb{C}$ (for some choice of value for $F(\mathbf{x})$) and t is the "time" variable in the integral curves γ of $\mathbf{X}(\mathbf{x})$ (see paragraph 3.1.1) that satisfies $\gamma(t) = \mathbf{x}$ if $\gamma(0) = x_0$.

In particular, we'll be lifting sections on $\tilde{L}_{\mu\pm}$ back to $\mathbb{R}^3$ via the following operator(s):

$$(P_{2k_1\kappa_1+2k_2\kappa_2}\tilde{u}_j)(\mathbf{x}) = \tilde{e}^{k_{j'}+k_j\kappa^{3-2j}}(z_{j'})F_j^*\tilde{u}_j, \quad \forall j \neq j' \in \{1,2\} \tag{5.1.1}$$

for $\mathbf{x} \in \mathbb{R}^3_j$, where the integers k_j corresponds to the choice of bundle and the phase factor $\tilde{e}^{k_{j'}+k_j\kappa^{3-2j}}(z_{j'}) = e^{\mathrm{i}\lambda t}$ for $\lambda = 2k_1\kappa_1 + 2k_2\kappa_2$ and t being dependent on $\mathbf{x}$ and the choice of $z_{j'}$.

5.1.1 Lifts of connections on the line bundles $\tilde{L}_\lambda$

In the previous chapter, we studied 1-forms, connections and Clifford multiplication defined on the *orbit space* $\mathbb{O}_{\boldsymbol{\kappa}}$ and acting on a section of the *Spin*^{c} bundles L_μ. In this subsection, we'll see how the aforementioned operator P_λ, (3.3.9), "commutes" with connections on the line bundle $\tilde{L}_\lambda$; we have:

Proposition 5.1.1. Consider $\tilde{\nabla}$, a connection on L_μ, and let $\tilde{\alpha} = \tilde{\alpha}' + \lambda\tilde{\gamma}$ for $\tilde{\alpha}' \in \Omega^1(\mathbb{O}_\kappa)$. Set $\alpha' = F^*\tilde{\alpha}' \in \Omega^1(\mathbb{R}^3)$. Then for a section $\psi \in \Omega^1(\mathbb{R}^3)$, we have

$$\big(d - \mathrm{i}(\alpha' + \lambda\alpha)\big)P_\lambda\psi = P_\lambda(\tilde{\nabla}\psi). \tag{5.1.2}$$

Proof. Letting $F \equiv F_j$, $j = 1, 2$, (depending on the chart we're on), and $\gamma \equiv \gamma^j$ respectively, we have

$$F^*\gamma = \frac{1}{2\mathrm{i}}F^*\Big(\frac{1}{z}dz - \frac{1}{\bar{z}}d\bar{z}\Big) = \frac{1}{2\mathrm{i}}\Big(\frac{1}{F}\boldsymbol{\nabla}F - \frac{1}{\overline{F}}\boldsymbol{\nabla}\overline{F}\Big) = \frac{1}{2\mathrm{i}}(\mathbf{P}_j + \mathrm{i}\mathbf{Q}_j) - (\mathbf{P} - \mathrm{i}\mathbf{Q}_j)) = \mathbf{Q}_j.$$

Also,

$$|\mathbf{X}|^2\mathbf{Q}_j^b = 2|\mathbf{X}|^2\Big(\frac{1}{|\mathbf{W}_j|^2}\mathbf{W}_j - \kappa^{3-2j}\frac{1}{|\mathbf{W}_{j'}|^2}\mathbf{W}_{j'}\Big)^b, \quad \textbf{for} \quad j' \neq j \in \{1, 2\}.$$

where the superscript b denotes the "flat" musical isomorphism on $(\mathbb{R}^3, \Omega)$.
Recalling $s_j = |\chi_j(\mathbf{x})|^2$, $j = 1, 2$ we have:

$$\frac{s_j}{(1-s_j)^2 + 4\kappa^{2(3-2j)}s_j} = \frac{s_j}{(1-s_j)^2}\frac{(1-s_j)^2}{(1-s_j)^2 + 4\kappa^{2(3-2j)}s_j} = \frac{|\tilde{z}_j|^2}{|\tilde{z}_{j'}|^2}\Big(1 + \kappa^{3-2j}\xi_j^2\Big)^{-1}$$

$$= \kappa_{j'}^2\frac{|\tilde{z}_j|^2}{\kappa_j^2|\tilde{z}_j|^2 + \kappa_{j'}^2|\tilde{z}_{j'}|^2} = \kappa_{j'}^2\frac{|\mathbf{W}_j|^2}{|\mathbf{X}|^2}.$$

Regarding γ^j respectively, we have:

$$F_j^*\xi_j = F_j^*(l_j\zeta) = (F_j^*l_j)F_j^*\tilde{\zeta}_z$$

$$= 2\frac{\kappa_j}{\kappa_{j'}^2}\frac{s_j}{(1-s_j)^2 + 4\kappa^{2(3-3j)}s_j}\mathbf{Q}_j$$

and

$$2\frac{\kappa_j}{\kappa_{j'}^2}\frac{s_j}{(1-s_j)^2+4\kappa^{2(3-2j)}s_j}|\mathbf{X}|^2\mathbf{Q}_j^b =$$

$$\kappa_j|\mathbf{W}_j|^2\Big(\frac{1}{|\mathbf{W}_j|^2}\mathbf{W}_j^b - \kappa^{3-2j}\frac{1}{|\mathbf{W}_{j'}|}\mathbf{W}_{j'}^b\Big) =$$

$$\kappa_j\mathbf{W}_j^b + \kappa_{j'}\mathbf{W}_{j'}^b - \frac{1}{\kappa_{j'}}\Big(\kappa_{1,2}^2\frac{|\mathbf{W}_j|^2}{|\mathbf{W}_j|^2} + \kappa_{j'}^2\Big)\mathbf{W}_{j'}^b = \alpha_b - \frac{1}{\kappa_{j'}}\frac{|\mathbf{X}|^2}{|\mathbf{W}_{j'}|^2}\mathbf{W}_{j'}^b.$$

So,

$$P_\lambda^j(\tilde{\nabla}\psi) = P_\lambda^j\big((d-\mathrm{i}(\tilde{\alpha}'+\lambda\xi))\big) = P_\lambda^j(d\psi) - \mathrm{i}(\alpha'+\lambda\alpha_b)P_\lambda^j\psi + \frac{\mathrm{i}\lambda}{\kappa_{j'}}\frac{1}{|\mathbf{W}_{j'}|^2}\mathbf{W}_{j'}^b P_\lambda^j\psi. \quad (5.1.3)$$

However,

$$\tilde{e}^{-k_j}(\tilde{z}_j)d\tilde{e}^{k_j}(\tilde{z}_j) = \mathrm{i}k_j|\mathbf{X}|^2\mathrm{Im}(\tilde{z}_2^{-1}\boldsymbol{\nabla}\tilde{z}_2)^b = 2\mathrm{i}k_j\frac{|\mathbf{X}|^2}{|\mathbf{W}_j|^2}\mathbf{W}_j^b$$

However,

$$\tilde{e}^{-k_j}(\tilde{z}_j)d\tilde{e}^{k_j}(\tilde{z}_j) = \mathrm{i}k_j|\mathbf{X}|^2\mathrm{Im}(\tilde{z}_2^{-1}\boldsymbol{\nabla}\tilde{z}_2)^b = 2\mathrm{i}k_j\frac{|\mathbf{X}|^2}{|\mathbf{W}_j|^2}\mathbf{W}_j^b$$

Also, simple calculations show: $\tilde{e}^{k_j}(z)d\tilde{e}^{-k_j}(z) = -\mathrm{i}k_j\tilde{\zeta}_z$, for all $k_j \in \mathbb{Z}$. So, by these and the definition of (3.3.9) we have (again, for distinct $j, j' = 1, 2$):

$$dP_\lambda^j\psi = d\big(\tilde{e}^{k_1}(\tilde{z}_1)\tilde{e}^{k_2}(\tilde{z}_2)F_j^*(e^{-k_j}\psi_j)\big) =$$

$$(d\tilde{e}^{k_1}(\tilde{z}_1))\tilde{e}^{k_2}(\tilde{z}_2)F_j^*(e^{-k_j}\psi_j) + \tilde{e}^{k_1}(\tilde{z}_1)(d\tilde{e}^{k_2}(\tilde{z}_2))F_j^*(e^{-k_j}\psi_j) +$$
$$\tilde{e}^{k_1}(\tilde{z}_1)\tilde{e}^{k_2}(\tilde{z}_2)d(F_j^*(\tilde{e}^{-k_j}(z)\psi_j(z))) = F_j^*d(\tilde{e}^{-k_j}(z)\psi_j(z)).$$

Thus $dP_\lambda\psi$ equals:

$$\Big(\frac{d\tilde{e}^{k_1}(\tilde{z}_1)}{\tilde{e}^{k_1}(\tilde{z}_1)} + \frac{d\tilde{e}^{k_2}(\tilde{z}_2)}{\tilde{e}^{k_2}(\tilde{z}_2)}\Big)\tilde{e}^{k_1}(\tilde{z}_1)\tilde{e}^{k_2}(\tilde{z}_2)F_j^*(\tilde{e}^{-k_j}(z)\psi_j(z))$$

$$\tilde{e}^{k_1}(\tilde{z}_1)\tilde{e}^{k_2}(\tilde{z}_2)F_j^*((d\tilde{e}^{-k_j}(z))\psi_j(z) + \tilde{e}^{-k_j}(z)d\psi_j(z)) =$$

$$2\mathrm{i}|\mathbf{X}|^2\Big(\frac{k_1}{|\mathbf{W}_1|^2}\mathbf{W}_1^b + \frac{k_2}{|\mathbf{W}_2|^2}\mathbf{W}_2^b - \frac{k_j}{|\mathbf{W}_j|^2}\mathbf{W}_j^b + \frac{k_j\kappa^{3-2j}}{|\mathbf{W}_{j'}|^2}\mathbf{W}_{j'}^b\Big)\tilde{e}^{k_1}(\tilde{z}_1)\tilde{e}^{k_2}(\tilde{z}_2)F_j^*(e^{-k_j}\psi_j) +$$
$$\tilde{e}^{k_1}(\tilde{z}_1)\tilde{e}^{k_2}(\tilde{z}_2)F_j^*(e^{-k_j}d\psi_j)$$

$$= 2\mathrm{i}(k_{j'} + \kappa^{3-2j}k_j)\frac{|\mathbf{X}|^2}{|\mathbf{W}_{j'}|^2}\mathbf{W}^b_{j'}P_\lambda\psi + P_\lambda(d\psi) = P_\lambda(\tilde{\nabla}\psi) + \mathrm{i}(\tilde{\alpha} + \lambda\alpha_b)P_\lambda\psi, \quad \text{by } (5.1.3).$$

□

5.1.2 Spin^c connections on sections on L_μ

In this subsection, we'll define *Spin*^{c} connections that act on the direct sum of the two line bundles $\tilde{L}_{\mu\pm}$, S_μ. These spin^c connections are going to be a tensor product of two connections on the line bundles $\tilde{L}_{\mu\pm}$ respectively. To describe these connections, we first establish some properties on the zero-order of these connections. In particular,

Lemma 5.1.2. Let $z \in \mathbb{C}$ and $s \in [0, 1)$ such that $h_j(s) = |z|^2$ (for j=1,2), then,

$$\tilde{w}^c_j(z) = \frac{1}{2\tilde{\kappa}}\Big(1 - \frac{1+s}{(1-s)^2 + \kappa^{2(3-2j)}s} + 8\kappa^{3-2j}\frac{s(1+s)}{(1-s)^2 + \kappa^{3-2j}s}\Big). \tag{5.1.4}$$

Proof. Letting our dummy variable $w_j = h_j(s_j)$, $(j = 1, 2)$ and recall that in applications, $s \equiv s_j = |\chi_j(\mathbf{x})|^2$ for $\mathbf{x} \in \mathbb{R}^3_j$. We have:

$$w_j\frac{ds_j}{dw_j} = \Big(\frac{d}{ds_j}\ln h_j(s_j)\Big)^{-1} = \frac{s_j(1-s_j)}{\sqrt{(1-s_j)^2 + \kappa^{3-2j}s_j}} \tag{5.1.5}$$

Then, for $q_j(s_j) := (1-s_j)^2 + \kappa^{3-2j}s_j$, we have:

$$\frac{w_j\omega_j'(w_j)}{\omega_j(w_j)} = \frac{w_j}{2}\frac{d}{dt}\ln\big(\omega_j^2(w_j)\big) = \frac{w_j}{2}\frac{d}{dt}\Big(\frac{1}{\kappa_{j'}}\frac{s_j(1-s_j)^2}{h_j(s_j)((1-s_j)^2 + \kappa^{3-2j}s_j)^2}\Big) =$$

$$\frac{w_j}{2}\frac{d}{dt}\Big(-\ln(w_j) + \ln\Big(\frac{s_j(1-s_j)^2}{((1-s_j)^2 + \kappa^{3-2j}s_j)^2}\Big)\Big) =$$

$$-\frac{1}{2} + \frac{w_j}{2}\frac{ds_j}{dt}\frac{d}{ds_j}\big(\ln(s_j) + 2\ln(1-s_j) - 2\ln\big((1-s_j)^2 + \kappa^{\pm}s_j\big)^2\big) =$$

$$-\frac{1}{2} + \frac{1}{2}\frac{s_j(1-s_j)}{(1-s_j)^2 + \kappa^{3-2j}s_j)^2}\Big(\frac{1}{s_j} + \frac{2}{1-s_j} - 2\frac{2}{1-s_j} - 2\frac{-2(1-s_j) + 4\kappa^{3-2j}}{(1-s_j)^2 + \kappa^{3-2j}s_j}\Big) =$$

$$-\frac{1}{2} + \frac{1+s_j}{2q_j(s_j)} - \frac{s_j(1-s_j)}{q_j^{1/2}(s_j)(1-s_j)q_j(s_j)}(2q_j(s_j) - 2(1-s_j)^2 + 4\kappa^{\pm 2}(1-s_j)) =$$

$$-\frac{1}{2}+\frac{1+s_j}{2\sqrt{q_j(s_j)}}-4\kappa^{2(3-2j)}\frac{s_j(1+s_j)}{(q_j(s_j))^{3/2}}.$$

□

Moreover, we have:

Lemma 5.1.3. The 1-forms $\tilde{w}_j^c$, with $j=1,2$ satisfy:

$$R_{\kappa^{3-2j}}\tilde{w}_j^c=\tilde{w}_j^c \quad \text{and} \quad \tau^*\tilde{w}_2^c=\tilde{w}_1^c-\frac{1}{2\kappa_1}\tilde{\zeta}. \tag{5.1.6}$$

Proof. The fact that $\tilde{w}_{1,2}^c$ are radially symmetric in z implies rotational invariance, consequently $R_\kappa\tilde{w}_1^c=\tilde{w}_1^c$ and $R_{\kappa^{-1}}\tilde{w}_2^c=\tilde{w}_2^c$. Also, we have:

$$\tau^*\tilde{w}^2=(\tau^*\tilde{w}_2^c)(\tau^*\tilde{\zeta})=-\tilde{\kappa}^{-1}(\tau^*\tilde{w}_2^c)\tilde{\zeta}.$$

Now for $w=|z|^2\in(0,\infty)$ (non-trivial z) and $s_{1,2}\in(0,1)$ satisfying $h_1(s_1)=w$ and $h_2(s_2)=C_\kappa^2w^{-1/\kappa}$, we have (for $q_{1,2}$ as defined in the previous lemma):

$$\frac{(1+s_2)^2}{q_2(s_2)}=\frac{(1-s_2)^2}{q_2(s_2)}+\frac{4s_2}{q_2(s_2)}=\kappa_1^2\frac{4s_1}{q_1(s_1)}+\kappa_1^2\frac{(1-s_1)^2}{q_1(s_1)}=\kappa_1^2\frac{(1+s_1)^2}{q_1(s_1)}$$

and considering the previous lemma we get:

$$\tilde{\kappa}^{-1}\tau^*\tilde{w}_2^c=-\frac{1}{\tilde{\kappa}^2}\Big(1-\frac{1+s_2}{\sqrt{q_2(s_2)}}+\frac{8}{\kappa^2}\frac{s_2(1+s_2)}{q_2^{3/2}(s_2)}\Big)$$

$$=-\frac{1}{\kappa\tilde{\kappa}}\Big(1-\kappa\frac{1+s_2}{\sqrt{q_2(s_2)}}+2\kappa\frac{(1-s_1)^2(1+s_1)}{q_1^{3/2}(s_1)}\Big)$$

$$=\frac{1}{2\tilde{\kappa}}\Big(1-\frac{1+s_1}{\sqrt{q_1(s_1)}}+8\kappa^2\frac{s_1(1+s_1)}{q_1^{3/2}(s_1)}\Big)-\frac{1+\kappa^{-1}}{2\tilde{\kappa}}=\tilde{w}_1^c-\frac{1}{2\kappa_1}\tilde{\zeta}.$$

□

As discussed in the previous chapter, on $\mathbb{C}\equiv\mathbb{R}^2$, the standard definition of the Clifford multiplication is given by

$$\tilde{\sigma}(\tilde{\alpha}) := \begin{pmatrix} 0 & \tilde{\alpha}_z \\ \tilde{\alpha}_{\overline{z}} & 0 \end{pmatrix} \tag{5.1.7}$$

for $\tilde{\alpha} = \tilde{\alpha}_z dz + \tilde{\alpha}_{\overline{z}} d\overline{z} \in \Omega^1(\mathbb{C})$. However, under a conformal change of metric, by a weight $\omega(|z|^2)$, a spin-c connection (given by $\nabla = (d - \mathrm{i}\tilde{\alpha}) \otimes I_2$) changes according to (2.1.5), or more analytically:

$$\tilde{\nabla}^{\omega} = \tilde{\nabla} + \frac{1}{4\omega}[\tilde{\sigma}(\cdot), \tilde{\sigma}(d\omega)] = (d - \mathrm{i}\tilde{\alpha}) \otimes I_2 + \frac{1}{4\omega}[\tilde{\sigma}(\cdot), \tilde{\sigma}(d\omega)] \tag{5.1.8}$$

where the $\cdot$ in $\tilde{\sigma}(\cdot)$ denotes some $SU(2)$-valued 1-form on $\sigma_1 d\tilde{x} + \sigma_2 d\tilde{y}$. Regarding the 0-order term that emerges from the conformal change of metric, we have the following lemma:

Lemma 5.1.4. Let a vector field $\tilde{\boldsymbol{X}} = \tilde{x}_1 \partial_{\tilde{x}_1} + \tilde{x}_2 \partial_{\tilde{x}_2} \in \mathfrak{X}(\mathbb{R}^2)$. The following identity is true:

$$\frac{1}{4\omega_{1,2}}[\tilde{\sigma}(\tilde{\boldsymbol{X}}^*), \tilde{\sigma}(d\omega_{1,2})] = \mathrm{i}\tilde{\kappa}\tilde{w}^c_{1,2}\sigma_3, \quad \tilde{\kappa} = \kappa_1 + \kappa_2. \tag{5.1.9}$$

Proof. Recall $\omega_{1,2} = \omega_{1,2}(|z|^2)$ which implies $d\tilde{\omega}_{1,2}(z) = \omega'(|z|^2)(\overline{z}dz + zd\overline{z})$. Now, given any vector field $\tilde{\boldsymbol{X}} = \tilde{x}_1 \dfrac{\partial}{\partial \tilde{x}_1} + \tilde{x}_2 \dfrac{\partial}{\partial \tilde{x}_2}$ on $\mathbb{C} \equiv \mathbb{R}^2$, it's dual, $\tilde{\boldsymbol{X}}^*$ satisfies:

$$\tilde{\boldsymbol{X}}^* = \tilde{x}_1 d\tilde{x}_1 + \tilde{x}_2 d\tilde{x}_2 = \frac{1}{2}(d\tilde{x}_1 - \mathrm{i}d\tilde{x}_2)(\tilde{X})(d\tilde{x}_1 + \mathrm{i}d\tilde{x}_2) + \frac{1}{2}(d\tilde{x}_1 + \mathrm{i}d\tilde{x}_2)(\tilde{X})(d\tilde{x}_1 - \mathrm{i}d\tilde{x}_2) =$$

$$\frac{1}{2}(d\overline{z}(\tilde{\boldsymbol{X}})dz + dz(\tilde{\boldsymbol{X}})d\overline{z}).$$

Therefore, $[\tilde{\sigma}(\tilde{\boldsymbol{X}}^*), \tilde{\sigma}(d\tilde{\omega}_{1,2})] =$

$$\left[\begin{pmatrix} 0 & d\overline{z}(\tilde{\boldsymbol{X}}) \\ dz(\tilde{\boldsymbol{X}}) & 0 \end{pmatrix}, 2\omega'_{1,2}(|z|^2) \begin{pmatrix} 0 & \overline{z} \\ z & 0 \end{pmatrix} \right] = 2(zd\overline{z}(\tilde{\boldsymbol{X}}) - \overline{z}dz(\tilde{\boldsymbol{X}}))\sigma_3$$

$$= -4\mathrm{i}|z|^2\omega'_{1,2}(|z|^2)\frac{1}{2\mathrm{i}}|z|^{-2}(\overline{z}dz - zd\overline{z})(\tilde{\boldsymbol{X}})\sigma_3.$$

The result follows from the definitions of $\tilde{\zeta}$ and $w^c_{1,2}$. □

Considering the lemma above and the formula for the change of connection under conformal change of metric, (5.1.8), we can define a *Spin*^{c} connection acting on sections of the $Spin^c$ bundle $S_\mu = \tilde{L}_{\mu-} \oplus \tilde{L}_{\mu+}$:

$$\tilde{\nabla}^\omega := \begin{pmatrix} d - \mathrm{i}\tilde{\alpha} + \mathrm{i}\tilde{\kappa}\tilde{w}^c & 0 \\ 0 & d - \mathrm{i}\tilde{\alpha} - \mathrm{i}\tilde{\kappa}\tilde{w}^c \end{pmatrix} \tag{5.1.10}$$

see (3.4.15) and (3.4.16) for the component $\tilde{w}^c$, here $\tilde{\kappa} = \kappa_1 + \kappa_2$. Equivalently, this connection can be written as:

$$\tilde{\nabla}^\omega := (d - \mathrm{i}\tilde{\alpha}) \otimes I_2 + \mathrm{i}\tilde{\kappa} w^c \sigma_3. \tag{5.1.11}$$

Remark 5.1.5. The aforementioned 1-form α on $\mathbb{O}_\kappa$, is actually a collection (pair) of forms $\tilde{\alpha}^j$ defined on $\mathbb{C}_j$ (for $j = 1, 2$ respectively), satisfying:

$$R_{\kappa^{\pm 1}} \tilde{\alpha}^{1,2} = \tilde{\alpha}^{1,2} \quad \text{and} \quad \tau^* \tilde{\alpha}^2 = \tilde{\alpha}^1 - \frac{\mu}{2\kappa_1}\tilde{\zeta} \tag{5.1.12}$$

while also having

$$R_{\kappa^{3-2j}}(\tilde{\alpha}^j \mp \tilde{\kappa}\tilde{w}_j) = \tilde{\alpha}^j \mp \tilde{\kappa}\tilde{w}_j \tag{5.1.13}$$

and

$$\tau^*(\tilde{\alpha}^2 \mp \tilde{\kappa}\tilde{w}^2) = \tilde{\alpha}^1 \mp \tilde{\kappa}\tilde{w}^1 - \frac{1}{2\kappa_1}(\mu \mp \tilde{\kappa})\tilde{\zeta} \tag{5.1.14}$$

Before we move on to lifts on Spinors (sections of S_μ) we introduce the following:

Lemma 5.1.6. Consider $\mu \in \Sigma := \{(2k_1+1)\kappa_1 + (2k_2+1)\kappa_2 : k_{1,2} \in \mathbb{Z}\}$ and $\tilde{\kappa} = \kappa_1 + \kappa_2$. Let $\tilde{\alpha}' \in \Omega^1(\mathbb{O}_\kappa)$ and set $\alpha' = F^*\tilde{\alpha}' \in \Omega^1(\mathbb{R}^3)$. Then,

$$(d - \mathrm{i}(\alpha' + \mu\alpha_b \mp \alpha_c))P_{\mu-\tilde{\kappa}}\psi_\pm = P_{\mu-\tilde{\kappa}}\big(d - \mathrm{i}(\tilde{\alpha}' + \mu\gamma \mp \tilde{\kappa}\tilde{w})\big) \tag{5.1.15}$$

for $\psi_\pm \in \Gamma(L_{\mu \mp \tilde{\kappa}})$.

Proof. Considering the notation introduced in the previous lemmas in this subsection, we have:

$$\frac{(1+s_j)^2}{q_j(s_j)} = \Big(1 + \frac{s_j}{(1-s_j)^2}\Big)\frac{(1-s_j)^2}{q_j(s_j)} = (1+\xi_j^2)(1+\kappa^{3-2j}\xi_j^2)^{-1}$$

$$= \kappa_{j'}^2 \frac{|\tilde{z}_1|^2 + |\tilde{z}_2|^2}{\kappa_1^2 |\tilde{z}_1|^2 + \kappa_2^2 |\tilde{z}_2|^2} = \kappa_{j'}^2 \frac{|\mathbf{X}^{ES}|^2}{|\mathbf{X}|^2}.$$

Following the calculations in the proof of *Proposition* 5.1.1 and (3.4.15), we get:

$$F_j^* \tilde{w}^j = F_j^* (\tilde{\omega}_j^c \tilde{\zeta}) = (F_j^* \tilde{\omega}_j^c)(F_j^* \tilde{\zeta})$$

$$= \frac{1}{\tilde{\kappa}} \Big(1 - \frac{1 + s_j}{\sqrt{q_j(s_j)}} + 8\kappa^{3-2j} \frac{s_j(1+s_j)}{(q_j(s_j))^{3/2}} \Big) |\mathbf{X}|^2 \mathbf{Q}_j^b$$

$$= \frac{|\mathbf{X}|^2}{\tilde{\kappa}} \Big(1 - \kappa_{j'} \frac{|\mathbf{X}^{ES}|}{|\mathbf{X}|} + 2\kappa_j^2 \kappa_{j'} \frac{|\mathbf{X}^{ES}|}{|\mathbf{X}|} \frac{|\mathbf{W}_j|^2}{|\mathbf{X}|^2} \Big) \Big(\frac{1}{|\mathbf{W}_j|^2} \mathbf{W}_j^b - \kappa^{3-2j} \frac{1}{|\mathbf{W}_{j'}|^2} \mathbf{W}_{j'}^b \Big)$$

$$= \frac{1}{\tilde{\kappa}} \Big(|\mathbf{X}|^2 - \kappa_{j'} |\mathbf{X}^{ES}| \Omega |\mathbf{X}|^2 + 2\kappa_{j'}^2 \kappa_{j'} |\mathbf{X}^{ES}| \Omega |\mathbf{W}_j|^2 \Big) \Big(\frac{1}{|\mathbf{W}_j|^2} \mathbf{W}_j^b - \kappa^{3-2j} \frac{1}{|\mathbf{W}_{j'}|^2} \mathbf{W}_{j'}^b \Big)$$

$$= \frac{1}{\tilde{\kappa}} \big(\kappa_j^2 (1 + \kappa_{j'} |\mathbf{X}^{ES}| \Omega) |\mathbf{W}_j|^2 + \kappa_{j'}^2 (1 - \kappa_{j'} |\mathbf{X}^{ES}| \Omega) |\mathbf{W}_{j'}|^2 \big) \frac{1}{|\mathbf{W}_{j'}|^2} \mathbf{W}_j^b -$$

$$\frac{1}{\tilde{\kappa}} \big(\kappa_j^2 (\kappa^{3-2j} + \kappa_{j'} |\mathbf{X}^{ES}| \Omega) |\mathbf{W}_j|^2 + \kappa_{j'}^2 (\kappa^{3-2j} - \kappa_{j'} |\mathbf{X}^{ES}| \Omega) |\mathbf{W}_{j'}|^2 \big) \frac{1}{|\mathbf{W}_{j'}|^2} \mathbf{W}_{j'}^b$$

$$= \frac{1}{\tilde{\kappa}} \Big(\kappa_j^2 (1 + \kappa_{j'} |\mathbf{X}^{ES}| \Omega) + \kappa_{j'}^2 (1 - \kappa_{j'} |\mathbf{X}| \Omega)) \frac{|\mathbf{W}_{j'}|^2}{|\mathbf{W}_j|^2} \Big) \mathbf{W}_j^b +$$

$$\frac{1}{\tilde{\kappa}} \Big(\kappa_j^2 (1 - \kappa_{j'} |\mathbf{X}^{ES}| \Omega) \frac{|\mathbf{W}_{j'}|^2}{|\mathbf{W}_j|^2} + \kappa_{j'}^2 (1 + \kappa_{j'} |\mathbf{X}| \Omega) \Big) \mathbf{W}_{j'}^b - \frac{1}{\kappa_{j'}} |\mathbf{X}|^2 \frac{1}{|\mathbf{W}_{j'}|^2} \mathbf{W}_j^b.$$

Keeping in mind that $-\frac{\kappa^{3-2j}}{\tilde{\kappa}} = \frac{1}{\tilde{\kappa}} - \frac{1}{\kappa_{j'}}$ the rest of the proof is almost identical to the one in lemma 5.1.1. □

5.1.3 Lifts of L_μ-sections, 1-forms and *Spin*^{c} connections

So now, we have all the tools to define a map $\mathcal{P}_\mu$, that lifts spinors from the bundle L_μ (2-dimensions) to three. Ideally, we would like to derive an equation of the form:

$$\boldsymbol{\mathcal{D}}_{\mathbf{A}}^{\Omega}((\mathcal{P}_\mu \tilde{u})(\mathbf{x})) = \mathcal{P}_\mu \tilde{\mathcal{D}}_{\tilde{A}}^{\omega} \tilde{u},$$

for some suitable weights Ω, $\omega \equiv \omega_{1,2}$, satisfying (3.2.26).

Define $\Sigma := \{(2k_1+1)\kappa_1 + (2k_2+1)\kappa_2 : (k_2,k_1) \in \mathbb{Z}^2\} \subseteq \mathbb{R}$. Given $\mu \in \Sigma$, we define $\mathcal{P}_\mu : \Gamma(L_\mu) \to \Gamma(\mathbb{R}^3 \times \mathbb{C}^2)$ (acting on $\tilde{u} = (\tilde{u}^+, \tilde{u}^-) \in S_\mu = \tilde{L}_{\mu-} \oplus \tilde{L}_{\mu+}$):

$$(\mathcal{P}_\mu \tilde{u})(\mathbf{x}) = U(\mathbf{x}) \begin{pmatrix} (P_{\mu-}\tilde{u}^+)(\mathbf{x}) \\ (P_{\mu+}\tilde{u}^-)(\mathbf{x}) \end{pmatrix}, \tag{5.1.16}$$

where $U(\mathbf{x})$ is a map introduced below to simplify the Weyl-Dirac operator. The superscripts $\pm$ on $\tilde{u}$ in the right-hand side of (5.1.16) simply denotes that $\tilde{u}^\pm$ is a section of the line bundle $\tilde{L}_{\mu\mp(\kappa_2+\kappa_1)}$.

We consider the following $SU(2)-$matrix:

$$U(\mathbf{x}) = U_0(\mathbf{x})U_1(\mathbf{x}), \tag{5.1.17}$$

where we've recalled:

$$U_0(\mathbf{x}) = \frac{1}{(2|\mathbf{X}(\mathbf{x})|v(\mathbf{x}))^{1/2}} \begin{pmatrix} \overline{w} & -\overline{\tilde{z}_1 w} \\ \tilde{z}_1 w & w \end{pmatrix}, \quad U_1(\mathbf{x}) = \begin{pmatrix} \mathrm{i}\gamma^e(\mathbf{x}) & 0 \\ 0 & -\mathrm{i}\overline{\gamma}^e(\mathbf{x}) \end{pmatrix}, \tag{5.1.18}$$

$$w = \kappa_2 x_3 + \kappa_1 \mathrm{i}, \tag{5.1.19}$$

$$\gamma^e(\mathbf{x}) = \tilde{e}(|\mathbf{X}(\mathbf{x})| + \kappa_1|\mathbf{X}^{ES}(\mathbf{x})| + (\kappa_1 - \kappa_2)\tilde{z}_2), \tag{5.1.20}$$

and

$$v(\mathbf{x}) = \frac{\Omega^{-1}(\mathbf{x}) - \kappa_2(1 - |\mathbf{x}|^2 + 2x_3^2)}{4(x_1^2 + x_2^2)} = \frac{\Omega^{-1}(\mathbf{x}) - X_3(\mathbf{x})}{|\tilde{z}_1|^2} = \frac{|w|^2}{\Omega^{-1}(\mathbf{x}) + X_3(\mathbf{x})}, \tag{5.1.21}$$

recall that for $z \in \mathbb{C} \setminus \{0\}$ we've set: $\tilde{e}(z) := z/|z| = e^{\mathrm{i}Arg(z)}$, and

$$\Omega^{-1}(\mathbf{x}) = |\mathbf{X}(\mathbf{x})| = \sqrt{\kappa_1^2|\mathbf{W}_1(\mathbf{x})|^2 + \kappa_2^2|\mathbf{W}_2(\mathbf{x})|^2} = \sqrt{\kappa_1^2|\tilde{z}_1|^2 + \kappa_2^2|\tilde{z}_2|^2}. \tag{5.1.22}$$

The matrix U will be particularly useful as it will help us to conjugate the non-standard sigma matrices that emerge through the process of submersing $\mathbb{R}^3$ in $\mathbb{R}^2$, with the standard ones. This is a consequence of the fact that $SU(2)$ is a universal double cover of $SO(3)$ rotating the coordinates back to the original one. In fact, we notice that by applying the Weyl-Dirac operator directly to $P_{\mu\mp(\kappa_1+\kappa_2)}\tilde{u}^{\pm}$, not only do we get extra zero-order terms, but also the Pauli matrices are non-constant. Some basic properties of the matrix $U(\mathbf{x})$ are summarized in the following two lemmas:

Lemma 5.1.7. The matrix $U(\mathbf{x})$ defined in (5.1.17) satisfies:

$$U(\mathbf{x})\sigma_3 U^*(\mathbf{x}) = \Omega(\mathbf{x})\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma} = |\mathbf{X}(\mathbf{x})|^{-1}\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma}. \tag{5.1.23}$$

Proof. We have:

$$U^*(\mathbf{x}) = (2\Omega^{-1}(\mathbf{x})v(\mathbf{x}))^{-1/2}\begin{pmatrix}\overline{\gamma}^e & 0\\ 0 & \gamma^e\end{pmatrix}\begin{pmatrix}w & \overline{\tilde{z}_1}v\\ -\tilde{z}_1 v & -w\end{pmatrix}.$$

One key thing to observe is:

$$\begin{pmatrix}\overline{\gamma}^e & 0\\ 0 & \gamma^e\end{pmatrix}\sigma_3\begin{pmatrix}\gamma^e & 0\\ 0 & \overline{\gamma}^e\end{pmatrix} = \begin{pmatrix}\overline{\gamma}^e & 0\\ 0 & \gamma^e\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\gamma^e & 0\\ 0 & \overline{\gamma}^e\end{pmatrix}$$

$$= \begin{pmatrix}\overline{\gamma}^e & 0\\ 0 & \gamma^e\end{pmatrix}\begin{pmatrix}\gamma^e & 0\\ 0 & -\overline{\gamma}^e\end{pmatrix} = \begin{pmatrix}|\gamma^e|^2 & 0\\ 0 & -|\overline{\gamma}^e|^2\end{pmatrix} = \sigma_3.$$

Therefore, we have:

$$U(\mathbf{x})\sigma_3 U^*(\mathbf{x}) = (2\Omega^{-1}(\mathbf{x})v(\mathbf{x}))^{-1}\begin{pmatrix}\overline{w} & -\overline{\tilde{z}_1}v\\ \tilde{z}_1 v & w\end{pmatrix}\sigma_3\begin{pmatrix}w & \overline{\tilde{z}_1}v\\ -\tilde{z}_1 v & \overline{w}\end{pmatrix}$$

$$= (2\Omega^{-1}(\mathbf{x})v(\mathbf{x}))^{-1}\begin{pmatrix}\overline{w} & -\overline{\tilde{z}_1}v\\ \tilde{z}_1 v & w\end{pmatrix}\begin{pmatrix}w & \overline{\tilde{z}_1}v\\ \tilde{z}_1 v & -\overline{w}\end{pmatrix}$$

$$= (2\Omega^{-1}(\mathbf{x})v(\mathbf{x}))^{-1}\begin{pmatrix}|w|^2 - |\tilde{z}_1 v|^2 & 2\overline{\tilde{z}_1}v\overline{w}\\ 2\tilde{z}_1 v w & -|w|^2 + |\tilde{z}_1 v|^2\end{pmatrix}.$$

We have, $|w|^2 = \kappa_1^2 + \kappa_2^2 x_3^2$ and

$$|w|^2 - |\tilde{z}_1 v|^2 = |w|^2 + |\tilde{z}_1 v|^2 - 2|\tilde{z}_1 v|^2$$

$$\text{while}\quad |w|^2 + |\tilde{z}_1 v|^2 = \kappa_1^2 + \kappa_2 x_3^2 + \frac{(\Omega^{-1}(\mathbf{x}) - \kappa_2(1 - |\mathbf{x}|^2 + 2x_3^2))^2}{4(x_1^2 + x_2^2)}$$

$$= \frac{(4\kappa_1^2 + 4\kappa_2^2 x_3^2)(x_1^2 + x_2^2) + (\Omega^{-1}(\mathbf{x}) - \kappa_2(1 - |\mathbf{x}|^2 + 2x_3^2))^2}{4(x_1^2 + x_2^2)}$$

$$= \frac{(4\kappa_1^2 + 4\kappa_2^2 x_3^2)(x_1^2 + x_2^2) + (\Omega^{-1}(\mathbf{x}))^2 + \kappa_2^2(1 - |\mathbf{x}|^2 + 2x_3^2)^2}{4(x_1^2 + x_2^2)}$$

$$-2\frac{\kappa_2 \Omega^{-1}(\mathbf{x})(1 - |\mathbf{x}|^2 + 2x_3^2)}{4(x_1^2 + x_2^2)}$$

$$= \frac{(\kappa_1^2 + \kappa_2^2 x_3^2)|\tilde{z}_1|^2 + \kappa_1^2|\tilde{z}_1|^2 + \kappa_2^2|\mathbf{W}_2|^2 + \kappa_2^2(1 - |\mathbf{x}|^2 + 2x_3^2)^2}{4(x_1^2 + x_2^2)}$$

$$-2\frac{\kappa_2 \Omega^{-1}(\mathbf{x})(1 - |\mathbf{x}|^2 + 2x_3^2)}{4(x_1^2 + x_2^2)}$$

$$= \frac{(\kappa_1^2 + \kappa_2^2 x_3^2)|\tilde{z}_1|^2 + 4\kappa_2^2 x_3^2(x_1^2 + x_2^2) + 2\kappa_2^2(1 - |\mathbf{x}|^2 + 2x_3^2)^2}{4(x_1^2 + x_2^2)}$$

$$-2\frac{\kappa_2 \Omega^{-1}(\mathbf{x})(1 - |\mathbf{x}|^2 + 2x_3^2)}{4(x_1^2 + x_2^2)}$$

$$\frac{2(\kappa_1^2 + \kappa_2^2 x_3^2)|\tilde{z}_1|^2 + 2\kappa_2^2(1 - |\mathbf{x}|^2 + 2x_3^2)^2 - 2\kappa_2 \Omega^{-1}(\mathbf{x})(1 - |\mathbf{x}|^2 + 2x_3^2)}{4(x_1^2 + x_2^2)}$$

$$= \frac{2\Omega^{-2}(\mathbf{x}) - 2\kappa_2 \Omega^{-1}(\mathbf{x})(1 - |\mathbf{x}|^2 + 2x_3^2)}{4(x_1^2 + x_2^2)}$$

$$= \frac{2\Omega^{-1}(\mathbf{x})(\Omega^{-1}(\mathbf{x}) - \kappa_2(1 - |\mathbf{x}|^2 + 2x_3^2))}{4(x_1^2 + x_2^2)} = 2\Omega^{-1} v(\mathbf{x}).$$

The second to last equality is true because $(\Omega(\mathbf{x}))^{-2} = |\mathbf{X}(\mathbf{x})|^2 =$

$$= \kappa_1^2|\mathbf{W}_1(\mathbf{x})|^2 + \kappa_2^2|\mathbf{W}_2(\mathbf{x})|^2 = \kappa_1^2|\tilde{z}_1|^2 + \kappa_2^2|\tilde{z}_2|^2 = (\kappa_1^2 + \kappa_2^2 x_3^2)|\tilde{z}_1|^2 + X_3^2(\mathbf{x}).$$

Therefore, we have:

$$|w|^2 - |\tilde{z}_1 v(\mathbf{x})|^2 = 2\Omega^{-1}(\mathbf{x})v(\mathbf{x}) - 2|\tilde{z}_1 v(\mathbf{x})|^2 = 2(\Omega^{-1}(\mathbf{x}) - |\tilde{z}_1|^2 v(\mathbf{x}))v(\mathbf{x})$$

$$= 2(\Omega^{-1}(\mathbf{x}) - (\Omega^{-1}(\mathbf{x}) - X_3(\mathbf{x})))v(\mathbf{x}) = 2X_3(\mathbf{x})v(\mathbf{x})$$

$$\implies U_0(\mathbf{x})\sigma_3 U_0^*(\mathbf{x}) = \frac{\Omega(\mathbf{x})}{2v(\mathbf{x})} \begin{pmatrix} 2X_3(\mathbf{x}) & 4(x_1 + \mathrm{i}x_2)(\mathrm{i}\kappa_1 + \kappa_2 x_3) \\ 4(x_1 - \mathrm{i}x_2)(\mathrm{i}\kappa_1 + \kappa_2 x_3) & -2X_3(\mathbf{x}) \end{pmatrix} v(\mathbf{x})$$

$$= \Omega(\mathbf{x}) \begin{pmatrix} X_3(\mathbf{x}) & X_1(\mathbf{x}) - \mathrm{i}X_2(\mathbf{x}) \\ X_1(\mathbf{x}) + \mathrm{i}X_2(\mathbf{x}) & -X_3(\mathbf{x}) \end{pmatrix} = \Omega(\mathbf{x})\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma} = \frac{1}{|\mathbf{X}(\mathbf{x})|}\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma} \implies$$

$$U_0^*(\mathbf{x})(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})U_0(\mathbf{x}) = |\mathbf{X}(\mathbf{x})|U_0^*(\mathbf{x})U_0(\mathbf{x})\sigma_3 U_0(\mathbf{x})U_0^*(\mathbf{x}) = |\mathbf{X}(\mathbf{x})|\sigma_3 \implies$$

$$U_1^*(\mathbf{x})U_0^*(\mathbf{x})(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})U_0(\mathbf{x})U_1(\mathbf{x}) = U^*(\mathbf{x})(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x}) = |\mathbf{X}(\mathbf{x})|U_1^*(\mathbf{x})\sigma_3 U_1(\mathbf{x}).$$

However, $U_1^*(\mathbf{x})\sigma_3 U_1^*(\mathbf{x}) = \begin{pmatrix} -\mathrm{i}\overline{\gamma^e} & 0 \\ 0 & \mathrm{i}\gamma^e \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{i}\gamma^e & 0 \\ 0 & -\mathrm{i}\overline{\gamma^e} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3.$

Therefore, we have:

$$U^*(\mathbf{x})(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x}) = |\mathbf{X}(\mathbf{x})|\sigma_3. \tag{5.1.24}$$

□

We also have the following two lemmas:

Lemma 5.1.8. The *phase factor* γ^e satisfies

$$-\overline{\gamma^e}^2 = \frac{-\mathrm{i}\kappa_1 X_3(\mathbf{x}) + \kappa_2 x_3 |X(\mathbf{x})|}{w\tilde{z}_2}. \tag{5.1.25}$$

Proof. We have: $\kappa_2(|\mathbf{X}^{ES}(\mathbf{x})| + (x_3 + \mathrm{i})\tilde{z}_2) =$

$$\kappa_2(x_3(1 + |\mathbf{x}|^2) + x_3(1 - |\mathbf{x}|^2) - 2x_3 + \mathrm{i}(1 - |\mathbf{x}|^2 + 2x_3^2)) = \mathrm{i}X_3(\mathbf{x})$$

$$\implies \mathrm{i}x_3|\mathbf{X}(\mathbf{x})| - \mathrm{i}\kappa X_3(\mathbf{x}) + w\tilde{z}_2 = \mathrm{i}x_3|\mathbf{X}(\mathbf{x})| - \mathrm{i}\kappa_1|\mathbf{X}^{ES}(\mathbf{x})| - \kappa_1(x_3 + \mathrm{i})\tilde{z}_2 + w\tilde{z}_2$$

$$= x_3(|\mathbf{X}(\mathbf{x})| - \kappa_1|\mathbf{X}^{ES}(\mathbf{x})| + (\kappa_2 - \kappa_1)\tilde{z}_2).$$

Similarly,

$$\kappa_2(|\mathbf{X}^{ES}(\mathbf{x})| + (x_3 + \mathrm{i})\overline{\tilde{z}_2}) = x_3(|\mathbf{X}(\mathbf{x})| - \kappa_1|\mathbf{X}^{ES}(\mathbf{x})| + (\kappa_2 + \kappa_1)\overline{\tilde{z}_2}).$$

Moreover, $(|\mathbf{X}(\mathbf{x})| - \kappa_1|\mathbf{X}^{ES}(\mathbf{x})|)(|\mathbf{X}(\mathbf{x})| + \kappa_1|\mathbf{X}^{ES}(\mathbf{x})|)$

$$= |\mathbf{X}(\mathbf{x})|^2 - \kappa_1^2|\mathbf{X}^{ES}(\mathbf{x})|^2 = 1(\kappa_2^2 - \kappa_1^2)|\mathbf{W}_2(\mathbf{x})|^2 = (\kappa_2^2 - \kappa_1^2)|\tilde{z}_2|^2.$$

Now, we have:

$$(|\mathbf{X}(\mathbf{x})| + \kappa_1|\mathbf{X}^{ES}(\mathbf{x})|)(x_3|\mathbf{X}(\mathbf{x})| - \mathrm{i}\kappa X_3(\mathbf{x}) + w\tilde{z}_1) =$$

$$x_3(|\mathbf{X}(\mathbf{x})| + \kappa_1|\mathbf{X}^{ES}(\mathbf{x})|)(|\mathbf{X}(\mathbf{x})| - \kappa_1|\mathbf{X}^{ES}(\mathbf{x})| + (\kappa_2 - \kappa_1)\tilde{z}_2) =$$

$$x_3\big((\kappa_2^2 - \kappa_1^2)|\tilde{z}_2|^2 + (\kappa_2 - \kappa_1)\tilde{z}_2(|\mathbf{X}(\mathbf{x})| + \kappa_1|\mathbf{X}^{ES}(\mathbf{x})|)\big) =$$

$$(\kappa_2 - \kappa_1)x_3(|\mathbf{X}(\mathbf{x})| + \kappa_1|\mathbf{X}^{ES}(\mathbf{x})| + (\kappa_1 + \kappa_2)\overline{\tilde{z}}_2) =$$

$$(\kappa_2 - \kappa_1)\tilde{z}_2(x_3|\mathbf{X}(\mathbf{x})| - \kappa\mathrm{i}X_3(\mathbf{x}) + w\overline{\tilde{z}}_2).$$

In other words, letting $L = |\mathbf{X}(\mathbf{x})| + \kappa_1|\mathbf{X}^{ES}(\mathbf{x})| + (\kappa_1 - \kappa_2)\tilde{z}_2$ we get

$$L(x_3(|\mathbf{X}(\mathbf{x})| - \kappa\mathrm{i}X_3(\mathbf{x}))) = -\overline{L}w\tilde{z}_2,$$

but $\gamma^e = \tilde{e}(L) \implies (\overline{L}/L) = -\overline{\gamma^e}^2$. □

Lemma 5.1.9. Consider $\Gamma := \tilde{e}(\tilde{z}_1)\tilde{e}(\tilde{z}_2)$ and $j \in \{1,2\}$. The matrix $U(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3_1 \cap \mathbb{R}^3_2$, satisfies:

$$U^*(\mathbf{x})(\mathbf{P}_j(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x}) = (-1)^j|\mathbf{Q}_j(\mathbf{x})|\begin{pmatrix} 0 & \overline{\Gamma} \\ \Gamma & 0 \end{pmatrix}, \tag{5.1.26}$$

$$U^*(\mathbf{x})(\mathbf{Q}_j(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x}) = (-1)^j|\mathbf{Q}_j(\mathbf{x})|\begin{pmatrix} 0 & -\mathrm{i}\overline{\Gamma} \\ -\mathrm{i}\Gamma & 0 \end{pmatrix}. \tag{5.1.27}$$

Proof. We have: $\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma} = \begin{pmatrix} X_3(\mathbf{x}) & \overline{w\tilde{z}_1} \\ w\tilde{z}_1 & X_3(\mathbf{x}) \end{pmatrix}$. Also, $|\overline{\tilde{z}}_1|^2 v(\mathbf{x}) = |\mathbf{X}(\mathbf{x})| - X_3(\mathbf{x})$ and $|w|^2 = v(\mathbf{x})(|\mathbf{X}(\mathbf{x})| + X_3(\mathbf{x})) \implies$

$$U_0^*(\mathbf{x})(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})U_0(\mathbf{x})$$

$$= \frac{1}{2|\mathbf{X}(\mathbf{x})|v(\mathbf{x})}\begin{pmatrix} w & \overline{\tilde{z}}_1 v(\mathbf{x}) \\ -\tilde{z}_1 v(\mathbf{x}) & \overline{w} \end{pmatrix}\begin{pmatrix} X_3(\mathbf{x}) & \overline{w\tilde{z}_1} \\ w\tilde{z}_1 & X_3(\mathbf{x}) \end{pmatrix}\begin{pmatrix} \overline{w} & -\overline{\tilde{z}}_1 v(\mathbf{x}) \\ \tilde{z}_1 v(\mathbf{x}) & w \end{pmatrix}$$

$$= \frac{1}{2v(\mathbf{x})}\begin{pmatrix} w & \overline{\tilde{z}}_1 v(\mathbf{x}) \\ -\tilde{z}_1 v(\mathbf{x}) & \overline{w} \end{pmatrix}\begin{pmatrix} \overline{w} & -\tilde{z}_1 w \\ \overline{\tilde{z}_1 w} & w \end{pmatrix} = |\mathbf{X}(\mathbf{x})|\sigma_3$$

$$\implies U^*(\mathbf{X}.\boldsymbol{\sigma})U = U_1^*U_0^*(\mathbf{X}.\boldsymbol{\sigma})U_0U_1 = U_0^*\sigma_3U_0 = |\mathbf{X}|U_1^*\sigma_3U_1 = |\mathbf{X}|\sigma_3.$$

By (6.1.13) (see the Appendix), we have:

$$\frac{\mathbf{P}_j(\mathbf{x})}{|\mathbf{Q}_j(\mathbf{x})|} = (-1)^j \frac{1}{|\mathbf{W}_1(\mathbf{x})||\mathbf{W}_2(\mathbf{x})|}\big(X_3^{ES}(\mathbf{x})(x_1,x_2,0) - |\tilde{z}_1|^2(0,0,1))\big)$$

$$= (-1)^j \frac{1}{|\mathbf{W}_1(\mathbf{x})||\mathbf{W}_2(\mathbf{x})|}\Big(\frac{1}{\kappa_2}X_3^{ES}(\mathbf{x})(x_1,x_2,0) - |\tilde{z}_1|^2(0,0,1)\Big),$$

and so:

$$-\kappa_2 x_3\overline{w} + X_3(\mathbf{x})v(\mathbf{x}) = (\mathrm{i}\kappa_1 - w)\overline{w} + |w|^2 + |\mathbf{X}(\mathbf{x})|v(\mathbf{x})$$

$$(\mathrm{i}\kappa_1 - w)\overline{w} + (|\mathbf{X}(\mathbf{x})| + X_3(\mathbf{x}))v(\mathbf{x}) - |\mathbf{X}(\mathbf{x})|v(\mathbf{x}) = -v(\mathbf{x})|\mathbf{X}(\mathbf{x})| + \mathrm{i}\kappa_1 X_3(\mathbf{x})$$

and

$$\overline{w}X_3(\mathbf{x}) + \kappa_2 x_3|\tilde{z}_1|^2 v(\mathbf{x}) =$$

$$(\kappa_2 x_3 - \kappa_1)X_3(\mathbf{x}) + \kappa_2 x_3(|\mathbf{X}(\mathbf{x})| - X_3(\mathbf{x})) = \kappa_2 x_3|\mathbf{X}(\mathbf{x})| - \kappa_1 X_3(\mathbf{x}).$$

So
$U_0^*(\mathbf{x})((2X_3(\mathbf{x})(x_1,x_2,0) - |\tilde{z}_1|^2\kappa_2 x_3(0,0,1)).\boldsymbol{\sigma})U_0(\mathbf{x}) =$

$$\frac{1}{2v(\mathbf{x})|\mathbf{X}(\mathbf{x})|}\begin{pmatrix} w & \overline{\tilde{z}_1 v(\mathbf{x})} \\ -\tilde{z}_1 v(\mathbf{x}) & \overline{w}\end{pmatrix}\begin{pmatrix} -\kappa_2 x_3|\tilde{z}_1|^2 & X_3(\mathbf{x})\overline{\tilde{z}}_S \\ X_3(\mathbf{x})\tilde{z}_1 & \kappa_2 x_3|\tilde{z}_1|^2\end{pmatrix}\begin{pmatrix} \overline{w} & \overline{\tilde{z}_1 v(\mathbf{x})} \\ -\tilde{z}_1 v(\mathbf{x}) & w\end{pmatrix},$$

the product of the last two matrices yields

$$\begin{pmatrix} |\tilde{z}_1|^2(-v(\mathbf{x})|\mathbf{X}(\mathbf{x})| + \mathrm{i}\overline{w}\kappa_1) & \overline{\tilde{z}_1}\kappa_1 X_3(\mathbf{x}) \\ \tilde{z}_1\kappa_1 X_3(\mathbf{x}) & |\tilde{z}_1|^2(-v(\mathbf{x})|\mathbf{X}(\mathbf{x})| - \mathrm{i}w\kappa_1)\end{pmatrix},$$

and by noticing:

$$-wv|\mathbf{X}| + \mathrm{i}\kappa_1|w|^2 + \mathrm{i}\kappa_2 x_3|\mathbf{X}(\mathbf{x})| - \kappa_1 v X_3 = -\mathrm{i}\kappa_1(|\mathbf{X}| + X_3) + \mathrm{i}\kappa_1|w|^2 = 0$$

and $|w|^2 = v(\mathbf{x})(|\mathbf{X}(\mathbf{x})| + X_3(\mathbf{x}))$, we get:

$$-|\tilde{z}_1|^2 v(\mathbf{x})(-v(\mathbf{x})\mathbf{X}(\mathbf{x})) + \mathrm{i}\kappa_1\overline{w}) + \overline{w}(\kappa_2 x_3|X(\mathbf{x})| - \mathrm{i}\kappa_1 X_3(\mathbf{x}))$$

$$= (|\mathbf{X}(\mathbf{x})| - X_3(\mathbf{x}))(v(\mathbf{x})|\mathbf{X}(\mathbf{x})| - \mathrm{i}\kappa_1\overline{w}) + \overline{w}(\kappa_2 x_3|\mathbf{X}(\mathbf{x})| - \mathrm{i}\kappa_1 X_3(\mathbf{x}))$$

$$= |\mathbf{X}(\mathbf{x})|(v(\mathbf{x})(|\mathbf{X}(\mathbf{x})| - X_3(\mathbf{x}) + |w|^2)$$

$$= \frac{v(\mathbf{x})|\mathbf{X}(\mathbf{x})|}{w}(w(|\mathbf{X}(\mathbf{x})| - X_3(\mathbf{x})) + \overline{w}(|\mathbf{X}(\mathbf{x})| + X_3(\mathbf{x})))$$

$$= \frac{2v(\mathbf{x})|\mathbf{X}(\mathbf{x})|}{w}(x_3\kappa_2|\mathbf{X}(\mathbf{x})| - \mathrm{i}\kappa_1 X_3(\mathbf{x})) = -2\kappa_2 v(\mathbf{x})|\mathbf{X}(\mathbf{x})|\tilde{z}_2\overline{\gamma^e}^2,$$

where we've used (5.1.25).

So, since $\Gamma := \tilde{e}(\tilde{z}_1)\tilde{e}(\tilde{z}_2) = \dfrac{\tilde{z}_1\tilde{z}_2}{|\tilde{z}_1||\tilde{z}_2|}$, we have:

$$U_0^*(\mathbf{x})(2X_3(\mathbf{x})(x_1, x_2, 0) - |\tilde{z}_1|^2\kappa_2(x_3(0,0,1)).\boldsymbol{\sigma})U_0(\mathbf{x}) = \kappa_2\begin{pmatrix} 0 & -\overline{\tilde{z}_1\tilde{z}_2}(\gamma^e)^2 \\ -\tilde{z}_1\tilde{z}_2\overline{\gamma^e}^2 & 0 \end{pmatrix},$$

and so

$$\frac{1}{|\mathbf{Q}_j|}U^*(\mathbf{x})(\mathbf{P}_j.\boldsymbol{\sigma})U(\mathbf{x}) = (-1)^j\frac{1}{|\tilde{z}_1||\tilde{z}_2|}U_1^*(\mathbf{x})\begin{pmatrix} 0 & -\overline{\tilde{z}_1\tilde{z}_2}(\gamma^e)^2 \\ -\tilde{z}_1\tilde{z}_2\overline{\gamma^e}^2 & 0 \end{pmatrix}U_1(\mathbf{x})$$

$$= (-1)^j U_1^*(\mathbf{x})\begin{pmatrix} 0 & -\frac{\overline{\tilde{z}_1\tilde{z}_2}}{|\tilde{z}_1||\tilde{z}_2|}(\gamma^e)^2 \\ -\frac{\tilde{z}_1\tilde{z}_2}{|\tilde{z}_1||\tilde{z}_2|}\overline{\gamma^e}^2 & 0 \end{pmatrix}U_1(\mathbf{x})$$

$$= (-1)^j\begin{pmatrix} -\mathrm{i}\overline{\gamma}^e & 0 \\ 0 & \mathrm{i}\gamma^e \end{pmatrix}\begin{pmatrix} 0 & \frac{\overline{\tilde{z}_1\tilde{z}_2}}{|\tilde{z}_1||\tilde{z}_2|}(\gamma^e)^2 \\ \frac{\tilde{z}_1\tilde{z}_2}{|\tilde{z}_1||\tilde{z}_2|}\overline{\gamma^e}^2 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{i}\gamma^e & 0 \\ 0 & -\mathrm{i}\overline{\gamma}^e \end{pmatrix}$$

$$= (-1)^j\begin{pmatrix} 0 & \overline{\tilde{e}(\tilde{z}_1)\tilde{e}(\tilde{z}_2)} \\ \tilde{e}(\tilde{z}_1)\tilde{e}(\tilde{z}_2) & 0 \end{pmatrix} \implies U^*(\mathbf{x})(\mathbf{P}_j(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x}) = (-1)^j|\mathbf{Q}_j(\mathbf{x})|\begin{pmatrix} 0 & \overline{\Gamma} \\ \Gamma & 0 \end{pmatrix}.$$

Regarding $U^*(\mathbf{x})(\mathbf{Q}_j(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x})$, by using (6.1.12) and standard properties of *Pauli matrices* we have:

$$(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})(\mathbf{P}_j(\mathbf{x}).\boldsymbol{\sigma}) = (\mathbf{X}(\mathbf{x}).\mathbf{P}_j(\mathbf{x}))I_2 + \mathrm{i}(\mathbf{X}(\mathbf{x}) \times \mathbf{P}_j(\mathbf{x})).\boldsymbol{\sigma} = \mathrm{i}(\mathbf{X}(\mathbf{x}) \times \mathbf{P}_j(\mathbf{x})).\boldsymbol{\sigma}$$

$$\implies (\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})(\mathbf{P}_j(\mathbf{x}).\boldsymbol{\sigma}) = \mathrm{i}|\mathbf{X}(\mathbf{x})|\mathbf{Q}_j(\mathbf{x}).\boldsymbol{\sigma}$$

$$\implies \mathbf{Q}(\mathbf{x}).\boldsymbol{\sigma} = -\mathrm{i}|\mathbf{X}(\mathbf{x})|^{-1}(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})(\mathbf{P}_j(\mathbf{x}).\boldsymbol{\sigma})$$

$$\implies U^*(\mathbf{x})(\mathbf{Q}_j(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x}) = -\mathrm{i}|\mathbf{X}(\mathbf{x})|^{-1}U^*(\mathbf{x})(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})(\mathbf{P}_j(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x})$$

$$= -\mathrm{i}|\mathbf{X}(\mathbf{x})|^{-1}U^*(\mathbf{x})(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x})U^*(\mathbf{x})(\mathbf{P}_j(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x}).$$

Now, by lemma 5.1.7 we get $U^*(\mathbf{x})(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})(\mathbf{P}_j(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x}) = -\mathrm{i}|\mathbf{X}(\mathbf{x})|\sigma_3$, so

$$U^*(\mathbf{x})(\mathbf{Q}_j(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x}) = -\mathrm{i}\sigma_3 U^*(\mathbf{x})(\mathbf{P}(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x}) = (-1)^j\mathrm{i}|\mathbf{Q}_j(\mathbf{x})|\sigma_3 \begin{pmatrix} 0 & \overline{\Gamma} \\ \Gamma & 0 \end{pmatrix}$$

$$\implies (-1)^j\mathrm{i}|\mathbf{Q}(\mathbf{x})| \begin{pmatrix} 0 & \overline{\Gamma} \\ -\Gamma & 0 \end{pmatrix} \implies U^*(\mathbf{x})(\mathbf{Q}_j(\mathbf{x}).\boldsymbol{\sigma})U(\mathbf{x}) = (-1)^j\mathrm{i}|\mathbf{Q}_j(\mathbf{x})| \begin{pmatrix} 0 & \overline{\Gamma} \\ -\Gamma & 0 \end{pmatrix}.$$

□

Now we can check how it behaves under $SU(2)-$transforms. We have:

Proposition 5.1.10. ***Lifts of Pauli matrices via $\mathcal{P}$.***
Given $k_{2,1} \in \mathbb{Z}$, and the weights ω (5.2.3), Ω on $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, the map $\mathcal{P}_\mu$ satisfies:

$$\mathcal{P}_\mu(\sigma_\omega(\tilde{\alpha})\tilde{\psi}) = \sigma_\Omega(F^*\tilde{\alpha})(\mathcal{P}_\mu\tilde{\psi}). \tag{5.1.28}$$

Proof. Let $\tilde{U}_{2,1} \subseteq \mathbb{C}$ connected, on which we make a choice of a branch for $e^{-1/\kappa}(\tilde{z}_1)\tilde{e}(\tilde{z}_2)$, $\tilde{e}(\tilde{z}_1)e^{-\kappa}(\tilde{z}_2)$ respectively. We have:

$$\alpha_{\mathbf{x}} = F_j^*\tilde{\alpha}_z = (\tilde{\alpha}_{\tilde{x}} \circ F_j(\mathbf{x}))((\boldsymbol{\nabla}\operatorname{Re}F_j(\mathbf{x})).d\mathbf{x}) + (\tilde{\alpha}_{\tilde{y}} \circ F_j(\mathbf{x}))((\boldsymbol{\nabla}\operatorname{Im}F(\mathbf{x})).d\mathbf{x}).$$

Also, recall the complex formulation of 1-forms:

$\tilde{\alpha}_{\boldsymbol{x}} \equiv \tilde{\alpha}_z = \tilde{\alpha}_+ dz + \tilde{\alpha}_- d\tilde{z} = \tilde{\alpha}_+ dz + \overline{\tilde{\alpha}_+} d\overline{z}$ where $\tilde{\alpha}_+ = \tilde{\alpha}_+(z,\overline{z}) = \dfrac{1}{2}(\tilde{\alpha}_1(\tilde{\mathbf{x}}) - \tilde{\alpha}_2(\tilde{\mathbf{x}}))$, for $\tilde{\mathbf{x}} = (\tilde{x},\tilde{y})$, $z = \tilde{x} + \mathrm{i}\tilde{y}$.

So $F_{2,1}^*\tilde{\alpha}_z = F_{2,1}^*(\tilde{\alpha}_+(z,\overline{z})dz + \tilde{\alpha}_-(z,\overline{z})d\overline{z})$

$$= F_{2,1}\tilde{\alpha}_+(F_{2,1},\overline{F}_{2,1})(\boldsymbol{\nabla}F_{2,1}.d\mathbf{x}) + \overline{F}\tilde{\alpha}_-(F_{2,1},\overline{F}_{2,1})(\boldsymbol{\nabla}\overline{F}_{2,1}.d\mathbf{x})$$

$$= \tilde{\alpha}_+(F_{2,1},\overline{F}_{2,1})F_{2,1}(\mathbf{P}_{2,1} + \mathrm{i}\mathbf{Q}_{2,1}).d\mathbf{x} + \tilde{\alpha}_-(F_{2,1},\overline{F}_{2,1})\overline{F}_{2,1}(\mathbf{P}_{2,1} - \mathrm{i}\mathbf{Q}_{2,1}).d\mathbf{x}$$

$$\implies \sigma_\Omega(\alpha_{\mathbf{x}}) = \sigma_\Omega(F^*_{2,1}\tilde{\alpha}_z) = \frac{1}{\Omega}\sigma(F^*_{2,1}\tilde{\alpha}_z)$$

$$= \frac{1}{\Omega}\tilde{\alpha}_+(F_{2,1},\overline{F}_{2,1})F_{2,1}(\mathbf{P}_{2,1} + \mathrm{i}\mathbf{Q}_{2,1}).\boldsymbol{\sigma} + \frac{1}{\Omega}\tilde{\alpha}_-(F_{2,1},\overline{F}_{2,1})\overline{F}_{2,1}(\mathbf{P}_{2,1} - \mathrm{i}\mathbf{Q}_{2,1}).\boldsymbol{\sigma}$$

$$= \Omega^{-1}\big(F^*_{2,1}(z\tilde{\alpha}_+) + F^*_{2,1}(\bar{z}\tilde{\alpha}_-)\mathbf{P}_{2,1}.\boldsymbol{\sigma} + (F^*_{2,1}(z\tilde{\alpha}_+) - F^*_{2,1}(\bar{z}\tilde{\alpha}_-))\mathrm{i}\mathbf{Q}_{2,1}.\boldsymbol{\sigma}\big),$$

and using (5.1.26) and (5.1.27) we get (for $j, j' = 1, 2,\ j \neq j'$):

$$\sigma_\Omega(\alpha_{\mathbf{x}}) = (-1)^j \frac{|\mathbf{Q}_j(\mathbf{x})|}{\Omega(\mathbf{x})}(F^*_{j'}(z\tilde{\alpha}_+) + F^*_{j'}(\bar{z}\tilde{\alpha}_-))|U(\mathbf{x}) \begin{pmatrix} 0 & \overline{\Gamma} \\ \Gamma & 0 \end{pmatrix} U^*(\mathbf{x})$$

$$= (-1)^j \frac{|\mathbf{Q}_j(\mathbf{x})|}{\Omega(\mathbf{x})}(F^*_{j'}(z\tilde{\alpha}_+) - F^*_{j'}(\bar{z}\tilde{\alpha}_-))U(\mathbf{x}) \begin{pmatrix} 0 & \overline{\Gamma} \\ -\Gamma & 0 \end{pmatrix} U^*(\mathbf{x}) \tag{5.1.29}$$

$$= (-1)^j \frac{2|\mathbf{Q}_j(\mathbf{x})|}{\Omega(\mathbf{x})} U(\mathbf{x}) \begin{pmatrix} 0 & F^*_j(z\tilde{\alpha}_+)\overline{\Gamma} \\ F^*_j(\bar{z}\tilde{\alpha}_-)\Gamma & 0 \end{pmatrix} U^*(\mathbf{x}).$$

So recalling $(\mathcal{P}_\mu\tilde{u})(\mathbf{x})$

$$= U(\mathbf{x}) \begin{pmatrix} (P_{\mu-}\tilde{u}^+)(\mathbf{x}) \\ (P_{\mu+}\tilde{u}^-)(\mathbf{x}) \end{pmatrix} = U(\mathbf{x}) \begin{pmatrix} \tilde{e}^{k_j + k_{j'}\kappa^{2j-3}}(\tilde{z}_j) & 0 \\ 0 & \tilde{e}^{(k_j+1)+(k_{j'}+1)\kappa^{2j-3}}(\tilde{z}_j) \end{pmatrix} \begin{pmatrix} F^*\tilde{u}^+ \\ F^*\tilde{u}^- \end{pmatrix}$$

$\implies$

$$(\mathcal{P}_\mu\tilde{u})(\mathbf{x}) = U(\mathbf{x})\mathcal{E}_{k_1,k_2}(z_1) \begin{pmatrix} F^*\tilde{u}^+ \\ F^*\tilde{u}^- \end{pmatrix}$$

$$= U(\mathbf{x}) \begin{pmatrix} \tilde{e}^{k_j}(\tilde{z}_j)e^{k_{j'}}(\tilde{z}_{j'}) & 0 \\ 0 & \tilde{e}^{(k_j+1)}(\tilde{z}_j)e^{(k_{j'}+1)}(\tilde{z}_{j'}) \end{pmatrix} \begin{pmatrix} F^*\tilde{u}^+ \\ F^*\tilde{u}^- \end{pmatrix}. \tag{5.1.30}$$

Where we recall, that we've set

$$\mathcal{E}_{k_1,k_2}(z_1) = \begin{pmatrix} \tilde{e}^{k_j}(\tilde{z}_j)e^{k_{j'}}(\tilde{z}_{j'}) & 0 \\ 0 & \tilde{e}^{(k_j+1)}(\tilde{z}_j)e^{(k_{j'}+1)}(\tilde{z}_{j'}) \end{pmatrix} \tag{5.1.31}$$

and we have:

$$\sigma_\Omega(\alpha_{\mathbf{x}})(\mathcal{P}_\mu\tilde{u})(\mathbf{x}) = (-1)^{j+1}\frac{2|\mathbf{Q}_j(\mathbf{x})|}{\Omega(\mathbf{x})}U(\mathbf{x}) \begin{pmatrix} 0 & F^*_j(z\tilde{\alpha}_+)\overline{\Gamma} \\ F^*_j(z\tilde{\alpha}_-)\Gamma & 0 \end{pmatrix} \mathcal{E}_{k_1,k_2}(\tilde{z}_1) \begin{pmatrix} F^*\tilde{u}^- \\ F^*\tilde{u}^+ \end{pmatrix}.$$

So, in the case where the index is 1, we have:

$$\begin{pmatrix} 0 & F_1^*(z\tilde{\alpha}_+)\overline{\Gamma} \\ F_1^*(z\tilde{\alpha}_-)\Gamma & 0 \end{pmatrix} \mathcal{E}_{k_1,k_2}(\tilde{z}_1)$$

$$= \begin{pmatrix} 0 & F_1^*(z\tilde{\alpha}_+)\overline{\Gamma} \\ F_1^*(z\tilde{\alpha}_-)\Gamma & 0 \end{pmatrix} \begin{pmatrix} \tilde{e}^{k_1}(\tilde{z}_1)e^{k_2}(\tilde{z}_2) & 0 \\ 0 & \tilde{e}^{(k_1+1)}(\tilde{z}_1)e^{(k_2+1)}(\tilde{z}_2) \end{pmatrix}$$

$$= |F_1|\tilde{e}(\tilde{z}_1)\tilde{e}^{-\kappa}(\tilde{z}_2) \begin{pmatrix} 0 & (F_1^*\tilde{\alpha}_+)\overline{\Gamma} \\ (F_1^*\tilde{\alpha}_-)\Gamma & 0 \end{pmatrix} \begin{pmatrix} \tilde{e}^{k_1}(\tilde{z}_1)e^{k_2}(\tilde{z}_2) & 0 \\ 0 & \tilde{e}^{(k_1+1)}(\tilde{z}_1)e^{(k_2+1)}(\tilde{z}_2) \end{pmatrix}.$$

and we get:

$$\tilde{e}(\tilde{z}_1)\tilde{e}^{-\kappa}(\tilde{z}_2) \begin{pmatrix} 0 & (F_1^*\tilde{\alpha}_+)\overline{\Gamma} \\ (F_1^*\tilde{\alpha}_-)\Gamma & 0 \end{pmatrix} \begin{pmatrix} \tilde{e}^{k_1}(\tilde{z}_1)e^{k_2}(\tilde{z}_2) & 0 \\ 0 & \tilde{e}^{(k_{1,2}+1)}(\tilde{z}_{1,2})e^{(k_{2,1}+1)}(\tilde{z}_{2,1}) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (F_1^*\tilde{\alpha}_+)\overline{\Gamma}\tilde{e}(\tilde{z}_1)\tilde{e}^{-\kappa}(\tilde{z}_2) \\ (F_1^*\tilde{\alpha}_-)\Gamma\tilde{e}(\tilde{z}_1)\tilde{e}^{-\kappa}(\tilde{z}_2) & 0 \end{pmatrix} \begin{pmatrix} \tilde{e}^{k_1}(\tilde{z}_1)e^{k_2}(\tilde{z}_2) & 0 \\ 0 & \tilde{e}^{(k_1+1)}(\tilde{z}_1)e^{(k_2+1)}(\tilde{z}_2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (F_1^*\tilde{\alpha}_+)\tilde{e}^{(k_1+1)}(\tilde{z}_1)\tilde{e}^{(k_2-\kappa)}(\tilde{z}_2) \\ (F_1^*\tilde{\alpha}_-)\tilde{e}^{k_1}\tilde{e}^{(k_2+\kappa+1)}(\tilde{z}_2) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (F_1^*\tilde{\alpha}_+)\tilde{e}^{k_1}(\tilde{z}_1)\tilde{e}^{k_2}(\tilde{z}_2) \\ (F_1^*\tilde{\alpha}_-)\tilde{e}^{k_1+1}\tilde{e}^{(k_2+1)}(\tilde{z}_2) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{e}^{k_1}(\tilde{z}_1)e^{k_2}(\tilde{z}_2) & 0 \\ 0 & \tilde{e}^{(k_1+1)}(\tilde{z}_1)e^{(k_2+1)}(\tilde{z}_2) \end{pmatrix} \begin{pmatrix} 0 & (F_1^*\tilde{\alpha}_+) \\ (F_1^*\tilde{\alpha}_-) & 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \tilde{e}^{k_1}(\tilde{z}_1)e^{k_2}(\tilde{z}_2) & 0 \\ 0 & \tilde{e}^{(k_1+1)}(\tilde{z}_1)e^{(k_2+1)}(\tilde{z}_2) \end{pmatrix} \tilde{\sigma}(F^*\tilde{\alpha})$$

on $\mathbb{R}_1^3 \cap \mathbb{R}_2^3$, recall that $\tilde{\sigma}$ is the *Clifford multiplication* in two dimensions. Now, on $\mathbb{R}_1^3$ (resp. $\mathbb{R}_2^3$), and since $\tilde{\sigma}(F^*\tilde{\alpha}) = F^*(\tilde{\sigma}(\tilde{\alpha}))$, we have:

$$\begin{pmatrix} \tilde{e}^{k_1}(\tilde{z}_1)e^{k_2}(\tilde{z}_2) & 0 \\ 0 & \tilde{e}^{(k_1+1)}(\tilde{z}_1)e^{(k_2+1)}(\tilde{z}_2) \end{pmatrix} F^*(\tilde{\sigma}(\tilde{\alpha})) \begin{pmatrix} F^*\tilde{u}^+ \\ F^*\tilde{u}^- \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{e}^{k_{2,1}+k_{1,2}\kappa^{\pm 1}}(z_{2,1}) & 0 \\ 0 & \tilde{e}^{(k_{2,1}+1)+(k_{1,2}+1)\kappa^{\pm 1}}(z_{2,1}) \end{pmatrix} F^*(\tilde{\sigma}(\tilde{\alpha})) \begin{pmatrix} F^*\tilde{u}^+ \\ F^*\tilde{u}^- \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{e}^{k_{2,1}+k_{1,2}\kappa^{\pm 1}}(z_{2,1}) & 0 \\ 0 & \tilde{e}^{(k_{2,1}+1)+(k_{1,2}+1)\kappa^{\pm 1}}(z_{2,1}) \end{pmatrix} F^* \left(\tilde{\sigma}(\tilde{\alpha}) \begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix} \right)$$

$$= U^{-1}(\mathbf{x}) \mathcal{P}_\mu \tilde{\sigma}(\tilde{\alpha}) \begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix}.$$

Combining the above results we have:

$$\sigma_\Omega(\alpha_{\mathbf{x}})(\mathcal{P}_\mu \tilde{u})(\mathbf{x})$$

$$= \frac{|\mathbf{Q}_{1,2}(\mathbf{x})|}{\Omega(\mathbf{x})} |F_{1,2}(\mathbf{x})| \mathcal{P}_\mu \left(\tilde{\sigma}(\tilde{\alpha}) \begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix} \right) = \frac{|\boldsymbol{\nabla} \operatorname{Im} F_{1,2}(\mathbf{x})|}{\Omega(\mathbf{x})} \mathcal{P}_\mu \left(\tilde{\sigma}(\tilde{\alpha}) \begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix} \right)$$

$$= \frac{F_{1,2}^* \omega(|z|^2) |\boldsymbol{\nabla} \operatorname{Im} F_{1,2}(\mathbf{x})|}{\Omega(\mathbf{x})} \frac{1}{F^* \omega(|z|^2)} \mathcal{P}_\mu \left(\tilde{\sigma}(\tilde{\alpha}) \begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix} \right)$$

$$= \frac{F_{1,2}^* \omega(|z|^2) |\boldsymbol{\nabla} \operatorname{Im} F_{1,2}(\mathbf{x})|}{\Omega(\mathbf{x})} \mathcal{P}_\mu \left(\omega^{-1}(|z|^2) \tilde{\sigma}(\tilde{\alpha}) \begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix} \right)$$

$$= \frac{F_{1,2}^* \omega(|z|^2) |\boldsymbol{\nabla} \operatorname{Im} F_{1,2}(\mathbf{x})|}{\Omega(\mathbf{x})} \mathcal{P}_\mu \left(\tilde{\sigma}_\omega(\tilde{\alpha}) \begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix} \right) = \mathcal{P}_\mu \left(\tilde{\sigma}_\omega(\tilde{\alpha}) \begin{pmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix} \right).$$

□

Now that we have all the information about how *Clifford multiplication* "commutes" with the operator, we shall see how connections do so. We have the following:

Proposition 5.1.11. ***Lifts of Spin^c connections via $\mathcal{P}$.***
Given $k_{2,1} \in \mathbb{Z}$, the map $\mathcal{P}_\mu$, and $\mu = (2k_1+1)\kappa_1 + (2k_2+1)\kappa_2$ when acting on spinor under a spin^c connection has been applied to, satisfies:

$$\mathcal{P}_\mu \tilde{\nabla} \tilde{\psi} = \Big(\nabla^\Omega - \mathrm{i}\kappa_1\kappa_2 \Omega(\mathbf{x}) M(\mathbf{x}) \big(\sigma(\cdot) - 2(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})\mathbf{X}_{\mathbf{x}}^* \big) \Big) (\mathcal{P}_\mu \tilde{\psi}), \tag{5.1.32}$$

where $\nabla^\Omega = \nabla + \frac{1}{4\Omega}[\sigma(\cdot), \sigma(d\Omega)] = (d - \mathrm{i}\alpha) \otimes I_2 + \frac{1}{4\Omega}[\sigma(\cdot), \sigma(d\Omega)]$ for $\alpha = F^*\tilde{\alpha}' + \mu \mathbf{X}^*$ for $\tilde{\alpha}' \in \Omega^1(\mathbb{O}_\kappa)$.

To prove this proposition, we introduce the following quantities associated with it:

$$\Sigma^1 = \mathbf{W}_1(\mathbf{x}).\boldsymbol{\sigma} = \begin{pmatrix} 0 & -\mathrm{i}\overline{\tilde{z}_1} \\ \mathrm{i}\tilde{z}_1 & 0 \end{pmatrix} \tag{5.1.33}$$

and

$$\Sigma^2 = (2x_1, 2x_2, 0).\boldsymbol{\sigma} = \begin{pmatrix} 0 & \overline{\tilde{z}_1} \\ \tilde{z}_1 & 0 \end{pmatrix}. \tag{5.1.34}$$

Now, we have the following lemmas:

Lemma 5.1.12. The following equation holds:

$$\Omega(\mathbf{x})\overline{\Gamma(\mathbf{x})}d\Gamma(\mathbf{x}) = \mathrm{i}\frac{|\mathbf{X}(\mathbf{x})| - \kappa_1|\mathbf{X}^{ES}(\mathbf{x})|}{|\tilde{z}_2|^2}\mathbf{W}_2^b(\mathbf{x}) + \mathrm{i}\kappa_1\Omega^2\frac{1-\kappa_2 M(\mathbf{x})}{1-\Omega(\mathbf{x})X_3(\mathbf{x})}dx_3. \tag{5.1.35}$$

Where $\mathbf{W}_2^b(\mathbf{x})$ is the musical isomorphism of $\mathbf{W}_2(\mathbf{x})$ in $(\mathbb{R}^3, \Omega)$.

Proof. We have $|\gamma^e| = 1$, i.e. $|\gamma^e|^2 = \gamma^e\overline{\gamma}^e = 1 \implies 0 = (d\gamma^e)\bar{\gamma}^e + \gamma^e d\overline{\gamma}^e \implies$

$$\gamma^e d\overline{\gamma^e} = -\overline{\gamma}^e d\gamma^e = -\overline{\gamma^e d\overline{\gamma^e}} \implies \gamma^e d\overline{\gamma^e} \in \mathrm{i}\mathbb{R},$$

i.e. $\gamma^e d\overline{\gamma^e} = \mathrm{i}\,\mathrm{Im}(\gamma^e d\overline{\gamma}^e)$. Since, $|\gamma^e| = 1$, we also have:

$$d\gamma^e = d\frac{1}{\bar{\gamma}^e} = -\frac{1}{(\bar{\gamma}^e)^2}d\bar{\gamma}^e = -\frac{1}{\bar{\gamma}^e}\frac{d\bar{\gamma}^e}{\bar{\gamma}^e} = -\frac{1}{\bar{\gamma}^e}d\ln(\bar{\gamma}^e) = -\frac{1}{2\bar{\gamma}^e}d\big(\ln(\bar{\gamma}^e)^2\big)$$

$$= -\frac{1}{2\bar{\gamma}^e}\frac{d(\bar{\gamma}^e)^2}{(\bar{\gamma}^e)^2} = -\frac{1}{2\bar{\gamma}^e}\frac{d\Big(\frac{-\mathrm{i}\kappa_1 X_3(\mathbf{x})+\kappa_2 x_3|X(\mathbf{x})|}{w\tilde{z}_2}\Big)}{\frac{-\mathrm{i}\kappa_1 X_3(\mathbf{x})+\kappa_2 x_3|X(\mathbf{x})|}{w\tilde{z}_2}},$$

where we've used for the last equation (5.1.25). So, we have:

$$\bar{\gamma}^e d\gamma^e = -\frac{1}{2}\frac{d(-\mathrm{i}\kappa_1 X_3(\mathbf{x}) + \kappa_2 x_3|X(\mathbf{x})|)}{-\mathrm{i}\kappa_1 X_3(\mathbf{x}) + \kappa_2 x_3|X(\mathbf{x})|} + \frac{dw}{2w} + \frac{d\tilde{z}_2}{2\tilde{z}_2}.$$

However, $w = \kappa_2 x_3 + \kappa_1\mathrm{i}$, so $dw = \kappa_2 dx_3$ and

$$d\tilde{z}_2 = d(1 - |\mathbf{x}|^2 + 2x_3\mathrm{i}) = -2x_1dx_1 - 2x_2dx_2 + 2(-x_3+\mathrm{i})dx_3 = -2\zeta^2 + 2(-x_3+\mathrm{i})dx_3$$

$$\implies \quad \frac{d\tilde{z}_2}{\tilde{z}_2} = \frac{\bar{\tilde{z}}_2 d\tilde{z}_2}{|\tilde{z}_2|^2} = \frac{(1-|\mathbf{x}|^2-2x_3\mathrm{i})(-2\zeta^2+2(-x_3+\mathrm{i})dx_3)}{(1-|\mathbf{x}|^2)^2+4x_3^2}$$

$$= \frac{-2\bar{\tilde{z}}_2}{|\tilde{z}_2|^2}\zeta^2 + \frac{(-x_3(1-|\mathbf{x}|^2)+2x_3)+\mathrm{i}(1-|\mathbf{x}|^2+2x_3^2)}{|\tilde{z}_2|^2}$$

$$= \frac{-2\bar{\tilde{z}}_2}{|\tilde{z}_2|^2}\zeta^2 + \frac{x_3(1+|\mathbf{x}|^2)+\mathrm{i}(1-|\mathbf{x}|^2+2x_3^2)}{|\tilde{z}_2|^2}$$

$$\implies \frac{1}{w}dw + \frac{d\tilde{z}_2}{\tilde{z}_2} = \frac{\kappa_2}{w}dx_3 = \frac{-2\bar{\tilde{z}}_2}{|\tilde{z}_2|^2}\zeta^2 + \Big(\frac{\kappa_2\bar{w}}{|w|^2} + \frac{2}{|\tilde{z}_2|^2}(x_3|\mathbf{X}^{ES}(\mathbf{x})| + \frac{\mathrm{i}}{\kappa_2}\mathbf{X}_3(\mathbf{x}))\Big)dx_3$$

$$= -2\frac{(1-|\mathbf{x}|^2)}{|\bar{\tilde{z}}_2|^2}\zeta^2 + \Big(\frac{\kappa_2^2 x_3}{|w|^2} + 2\frac{x_3|\mathbf{X}^{ES}(\mathbf{x})|}{|\tilde{z}_2|^2}\Big)dx_3 + \mathrm{i}\Big(4\frac{x_3}{|\tilde{z}_2|^2}\zeta^2 + \big(-\frac{\kappa_1\kappa_2}{|w|^2} + \frac{1}{\kappa_2}X_3^{es}(\mathbf{x})\big)dx_3\Big)$$

$$\implies \mathrm{Im}\left(\frac{1}{w}dw + \frac{d\tilde{z}_2}{\tilde{z}_2}\right) = \frac{2}{|\tilde{z}_2|^2}\mathbf{W}_2(\mathbf{x}) - \frac{\kappa_1\kappa_2}{|w|^2}dx_3 = \frac{2\Omega^{-2}(\mathbf{x})}{|\tilde{z}_2|^2}\mathbf{W}_2^b(\mathbf{x}) - \frac{\kappa_1\kappa_2}{|w|^2}dx_3.$$

Moreover,

$$|x_3|\mathbf{X}(\mathbf{x})| - \mathrm{i}\kappa X_3(\mathbf{x})|^2 = x_3^2|\mathbf{X}(\mathbf{x})|^2 + \kappa^2 X_3^2(\mathbf{x}) = (x_3^2+\kappa^2)|\mathbf{X}(\mathbf{x})|^2 - \kappa^2(|\mathbf{X}(\mathbf{x})| - X_3^2(\mathbf{x}))$$

$$= (x_3^2+\kappa^2)|\mathbf{X}(\mathbf{x})|^2 - \kappa^2(X_1^2(\mathbf{x}) + X_2^2(\mathbf{x}))$$

$$= (x_3^2+\kappa^2)|\mathbf{X}(\mathbf{x})|^2 - \kappa_1^2(4\kappa_1^2x_2^2 + 4\kappa_2^2x_1^2x_3^2 + 4\kappa_1^2x_1^2 + 4\kappa_2^2x_2^2x_3^2)$$

$$= (x_3^2+\kappa^2)|\mathbf{X}(\mathbf{x})|^2 - \kappa^2(4\kappa_1^2(x_1^2+x_2^2) + 4x_3^2(x_1^2+x_2^2))$$

$$= (x_3^2+\kappa^2)|\mathbf{X}(\mathbf{x})|^2 - \kappa_1^2(\kappa^2+x_3^2)(4x_1^2+4x_2^2)) = \frac{|w|^2}{\kappa_2^2}(|\mathbf{X}(\mathbf{x})|^2 - 4\kappa^2|\tilde{z}_1|^2)$$

$$= \frac{|w|^2}{\kappa_2^2}(\kappa^2|\tilde{z}_1|^2 + \kappa_2|\tilde{z}_2|^2 - \kappa^2|\tilde{z}_1|^2) = |w|^2|\tilde{z}_2|^2$$

and

$$dX_3(\mathbf{x}) = \kappa_2(-2x_1dx_1 - 2x_2dx_2 - 2x_3dx_3 + 4x_3dx_3) = -2\kappa_2\zeta^2 + 2\kappa_2x_3dx_3$$

whereas $d|\mathbf{X}(\mathbf{x})| = \frac{1}{2|\mathbf{X}(\mathbf{x})|}d|\mathbf{X}(\mathbf{x})|^2 = \frac{\Omega(\mathbf{x})}{2}d|\mathbf{X}(\mathbf{x})|^2$ and:

$$d|\mathbf{X}(\mathbf{x})|^2 = \kappa_1^2 d|\tilde{z}_1|^2 + \kappa_2^2 d|\tilde{z}_2|^2$$

$$= 8\kappa_1^2(x_1dx_1 + x_2dx_2) + 2\kappa_2^2((1-|\mathbf{x}|^2)(-2x_1dx_1 - 2x_2dx_2 - 2x_3dx_3) + 2x_3)dx_3$$

$$= (8(\kappa_1^2-\kappa_2^2) + 4\kappa_2^2(1+|\mathbf{x}|^2))(x_1dx_1 + x_2dx_2) + 4\kappa_2^2x_3\big((|\mathbf{x}|^2-1)+2\big)dx_3$$

$$= (8(\kappa_1^2-\kappa_2^2) + 4\kappa_2^2(1+|\mathbf{x}|^2))\zeta^2 + 4\kappa_2^2x_3|\mathbf{X}^{ES}(\mathbf{x})|dx_3$$

$$= (8(\kappa_1^2-\kappa_2^2) + 4\kappa_2^2|\mathbf{X}^{ES}(\mathbf{x})|)\zeta^2 + 4\kappa_2^2x_3|\mathbf{X}^{ES}(\mathbf{x})|dx_3$$

$$\implies d|\mathbf{X}(\mathbf{x})| = \Omega(\mathbf{x})\big(4(\kappa_1^2-\kappa_2^2) + 2\kappa_2^2|\mathbf{X}^{ES}(\mathbf{x})|\big)\zeta^2 + 2\Omega(\mathbf{x})\kappa_2^2x_3|\mathbf{X}^{ES}(\mathbf{x})|dx_3.$$

So

$$\operatorname{Im}\big((x_3|\mathbf{X}(\mathbf{x})| + \mathrm{i}\kappa X_3(\mathbf{x}))d((x_3|\mathbf{X}(\mathbf{x})| - \mathrm{i}\kappa X_3(\mathbf{x}))\big) =$$

$$\operatorname{Im}\big((x_3|\mathbf{X}(\mathbf{x})| + \mathrm{i}\kappa X_3(\mathbf{x}))(x_3d|\mathbf{X}(\mathbf{x})| + |\mathbf{X}(\mathbf{x})|dx_3 - \mathrm{i}\kappa dX_3(\mathbf{x}))\big)$$

and the second factor, $x_3d|\mathbf{X}(\mathbf{x})| + |\mathbf{X}(\mathbf{x})|dx_3 - \mathrm{i}\kappa dX_3(\mathbf{x})$ is equal to:

$$x_3\Omega(4(\kappa_1^2-\kappa_2^2) + 2\kappa_2^2x_3^2\Omega|\mathbf{X}^{ES}|)\zeta^2 + 2\kappa_2^2|\mathbf{X}^{ES}|dx_3) + |\mathbf{X}|dx_3 - \mathrm{i}\kappa dX_3$$

$\implies$

$$\operatorname{Im}\big((x_3|\mathbf{X}(\mathbf{x})| + \mathrm{i}\kappa X_3(\mathbf{x}))d((x_3|\mathbf{X}(\mathbf{x})| - \mathrm{i}\kappa X_3(\mathbf{x}))\big) =$$

$$(4\kappa\Omega(\mathbf{x})x_3X_3(\mathbf{x})(\kappa_1^2-\kappa_2^2) + 2\kappa\kappa_2^2x_3\Omega(\mathbf{x})X_3(\mathbf{x})|\mathbf{X}^{ES}(\mathbf{x})|)\zeta^2 +$$

$$(2\kappa\kappa_2^2x_3^2\Omega(\mathbf{x})X_3(\mathbf{x})|\mathbf{X}^{ES}(\mathbf{x})| + 2\kappa X_3(\mathbf{x})|\mathbf{X}(\mathbf{x})|)dx_3 - \kappa x_3|\mathbf{X}(\mathbf{x})|dX_3(\mathbf{x})$$

$$= (4\kappa\Omega(\mathbf{x})x_3X_3(\mathbf{x})(\kappa_1^2-\kappa_2^2) + 2\kappa\kappa_2^2x_3X_3(\mathbf{x})\Omega(\mathbf{x})|\mathbf{X}^{ES}(\mathbf{x})| + 2\kappa\kappa_2x_3|\mathbf{X}(\mathbf{x})|)\zeta^2 +$$

$$(\kappa x_3X_3(\mathbf{x})|\mathbf{X}(\mathbf{x})| + 2\Omega(\mathbf{x})\kappa\kappa_2^2x_3^2X_3(\mathbf{x})|\mathbf{X}^{ES}(\mathbf{x})| - 2\kappa\kappa_2x_3^2|\mathbf{X}(\mathbf{x})|dx_3$$

where we've used the equation: $dX_3(\mathbf{x}) = -2\kappa_2\zeta^2 + 2\kappa_2x_3dx_3$. Now, we have:

$$(4\kappa\Omega(\mathbf{x})x_3X_3(\mathbf{x})(\kappa_1^2-\kappa_2^2) + 2\kappa\kappa_2^2x_3X_3(\mathbf{x})\Omega(\mathbf{x})|\mathbf{X}^{ES}(\mathbf{x})| + 2\kappa_1x_3|\mathbf{X}(\mathbf{x})|)\zeta^2$$

$$= 2x_3\Omega(\mathbf{x})(2\kappa X_3(\mathbf{x})(\kappa_1^2-\kappa_2^2) + \kappa\kappa_2^2X_3(\mathbf{x})(1+|\mathbf{x}|^2) + \kappa_2x_3X_3(\mathbf{x})|\mathbf{X}(\mathbf{x})|^2)\zeta^2.$$

Also,

$$\kappa\Omega(\mathbf{x})(X_3(\mathbf{x})|\mathbf{X}(\mathbf{x})| + 2\kappa_2^2x_3^2X_3(\mathbf{x})|\mathbf{X}^{ES}(\mathbf{x})| - 2\kappa_2x_3^2|\mathbf{X}(\mathbf{x})|^2)dx_3$$

$$= \kappa\Omega(\mathbf{x})(X_3(\mathbf{x})|\mathbf{X}(\mathbf{x})|^2 + 2\kappa_2^2x_3^2X_3(\mathbf{x})|\mathbf{X}^{ES}(\mathbf{x})| - 2\kappa_2x_3^2|\mathbf{X}(\mathbf{x})|)dx_3.$$

Re-writing the last two, we get respectively:

$$2x_3\Omega(\mathbf{x})(2\kappa X_3(\mathbf{x})(\kappa_1^2-\kappa_2^2)+\kappa\kappa_2^2X_3(\mathbf{x})(1+|\mathbf{x}|^2)+\kappa_1x_3X_3(\mathbf{x})|\mathbf{X}(\mathbf{x})|^2)\zeta^2$$

$$=2\kappa_1x_3\Omega(\mathbf{x})\big((1-|\mathbf{x}|^2+2x_3^2)(2\kappa_1^2-\kappa_2^2)+\kappa_2X_3(\mathbf{x})|\mathbf{x}|^2+|\mathbf{X}(\mathbf{x})|^2)\zeta^2$$

$$=2\kappa_1x_3\Omega(\mathbf{x})\big((1-|\mathbf{x}|^2+2x_3^2)(2\kappa_1^2-\kappa_2^2)+\kappa_2^2(1-|\mathbf{x}|^2+2x_3^2)|\mathbf{x}|^2+|\mathbf{X}(\mathbf{x})|^2)\zeta^2$$

$$=2\kappa_1x_3\Omega(\mathbf{x})\big((1-|\mathbf{x}|^2+2x_3^2)(2\kappa_1^2-\kappa_2^2+\kappa_2^2|\mathbf{x}|^2)+|\mathbf{X}(\mathbf{x})|^2\big)\zeta^2.$$

Expanding $|\mathbf{X}(\mathbf{x})|^2$, we have:

$$(1-|\mathbf{x}|^2+2x_3^2)(2\kappa_1^2-\kappa_2^2+\kappa_2^2|\mathbf{x}|^2)+|\mathbf{X}(\mathbf{x})|^2=$$

$$(1-|\mathbf{x}|^2+2x_3^2)(2\kappa_1^2-\kappa_2^2+\kappa_2^2|\mathbf{x}|^2)+4(\kappa_1^2+\kappa_2^2x_3^2)(x_1^2+x_2^2)+\kappa_2^2(1-|\mathbf{x}|^2+2x_3^2)^2$$

$$=(1-|\mathbf{x}|^2+2x_3^2)(2\kappa_1^2-\kappa_2^2+\kappa_2^2|\mathbf{x}|^2+\kappa_2^2(1-|\mathbf{x}|^2+2x_3^2))+4(\kappa_1^2+\kappa_2^2x_3^2)(x_1^2+x_2^2)$$

$$=(1-|\mathbf{x}|^2+2x_3^2)(2\kappa_1^2+2\kappa_2^2x_3^2)+(2\kappa_1^2+2\kappa_2^2x_3^2)(2x_1^2+2x_2^2)$$

$$=(1+|\mathbf{x}|^2)(2\kappa_1^2+2\kappa_2^2x_3^2)=2\kappa_1^2|\mathbf{X}^{ES}(\mathbf{x})|+2\kappa_2^2x_3^2|\mathbf{X}^{ES}(\mathbf{x})|.$$

Now, regarding the coefficient of dx_3 we have:

$$\kappa\Omega(\mathbf{x})(X_3(\mathbf{x})|\mathbf{X}(\mathbf{x})|^2+2\kappa_2^2x_3^2X_3(\mathbf{x})|\mathbf{X}^{ES}(\mathbf{x})|-2\kappa_2x_3^2|\mathbf{X}(\mathbf{x})|)$$

$$=\kappa\Omega(\mathbf{x})(\kappa_2(1-|\mathbf{x}|^2+2x_3^2)|\mathbf{X}(\mathbf{x})|^2+2\kappa_2^3x_3^2(1-|\mathbf{x}|^2+2x_3^2)|\mathbf{X}^{ES}(\mathbf{x})|-2\kappa_2x_3^2|\mathbf{X}(\mathbf{x})|^2)$$

$$=\kappa_1\Omega(\mathbf{x})((1-|\mathbf{x}|^2+2x_3^2)|\mathbf{X}(\mathbf{x})|^2+2\kappa_2^2x_3^2(1-|\mathbf{x}|^2+2x_3^2)|\mathbf{X}^{ES}(\mathbf{x})|-2x_3^2|\mathbf{X}(\mathbf{x})|^2)$$

$$=\kappa_1\Omega(\mathbf{x})((1-|\mathbf{x}|^2)|\mathbf{X}(\mathbf{x})|^2+2\kappa_2^2x_3^2(1-|\mathbf{x}|^2+2x_3^2)|\mathbf{X}^{ES}(\mathbf{x})|)$$

$$=\kappa_1\Omega\big(2(\kappa_1^2+\kappa_2^2x_3^2)|\mathbf{X}^{ES}|(1-|\mathbf{x}|^2+2x_3^2)+2|\mathbf{X}|^2-|\mathbf{X}^{ES}|((|\mathbf{X}|^2+2\kappa_1^2(1-|\mathbf{x}|^2+2x_3^2))\big)$$

$$=2\kappa_1|w|^2|\mathbf{X}^{ES}(\mathbf{x})|\Omega^{-1}(\mathbf{x})\mathbf{W}_2^b(\mathbf{x})+\kappa_1\Omega(\mathbf{x})(2(\kappa_2^2-\kappa_1^2)-\kappa_2(1+|\mathbf{x}|^2)|\tilde{z}_2|^2)dx_3.$$

Given that $|\mathbf{X}(\mathbf{x})|^2+2\kappa_1^2(1-|\mathbf{x}|^2+2x_3^2)=2\kappa_1^2|\mathbf{X}^{ES}(\mathbf{x})|^2+\kappa_2^2|\tilde{z}_2|^2$.

Finally,

$$\Omega\overline{\Gamma}d\Gamma = \frac{\mathrm{i}}{|\tilde{z}_2|^2}(\Omega^{-1} - \kappa_1|\mathbf{X}^{ES}|)\mathbf{W}_2^b - \frac{\mathrm{i}\kappa_1\Omega}{2|w|^2}(\kappa_2 + 2\Omega(\kappa_2^2 - \kappa_1^2) - \kappa_2^2\Omega|\mathbf{X}^{ES}|)dx_3$$

$$= \frac{\mathrm{i}}{|\tilde{z}_2|^2}(\Omega^{-1} - \kappa_1|\mathbf{X}^{ES}|)\mathbf{W}_2^b - \frac{\mathrm{i}\kappa_1\Omega^2}{2|w|^2}(\kappa_2|\mathbf{X}| + \kappa_2^2(1 - |\mathbf{x}|^2) - 2\kappa_1^2)dx_3$$

$$= \mathrm{i}\frac{|\mathbf{X}(\mathbf{x})| - \kappa_1|\mathbf{X}^{ES}(\mathbf{x})|}{|\tilde{z}_2|^2}\mathbf{W}_2^b - \frac{\mathrm{i}}{2}\kappa_1\Omega^2(\mathbf{x})\Big(\kappa_2\frac{|\mathbf{X}(\mathbf{x})| + X_3(\mathbf{x})}{|w|^2} - 2\Big)dx_3.$$

The result follows. □

Lemma 5.1.13. The matrix $U_0(\mathbf{x})$ satisfies:

$$dU_0U_0^* = \mathrm{i}\frac{\Omega(\mathbf{x})}{2v(\mathbf{x})}\kappa_1\kappa_2 dx_3\sigma_3 - 2\mathrm{i}\Omega(\mathbf{x})v(\mathbf{x})\zeta_1\sigma_3 + \mathrm{i}\frac{\Omega(\mathbf{x})}{2}(\kappa_2 dx_3\Sigma^1 - \kappa_2 x_3 d\Sigma^2 + \kappa_1 d\Sigma^2)$$
$$-\mathrm{i}\Omega(\mathbf{x})^2\frac{1 - \kappa_2 M(\mathbf{x})}{1 - \Omega(\mathbf{x})X_3(\mathbf{x})}(2v(\mathbf{x})\zeta_2 + \kappa_2 x_3 dx_3)(-\kappa_2 x_3\Sigma^1 + \kappa_1\Sigma^2). \tag{5.1.36}$$

Proof. We have the following identities that eventually lead to neat simplifications:

$$U_0U_0^* = I_2 \implies d(U_0U_0^*) = dI_2 = \mathbf{0}_{2x2}$$

$$\implies (dU_0)U_0^* + U_0dU_0^* = \mathbf{0}_{2x2} \implies (dU_0)U_0^* = -U_0dU_0^*$$

$$\implies (dU_0)U_0^* = \frac{1}{2}((dU_0)U_0^* - U_0dU_0^*)$$

$$= \frac{1}{4|\mathbf{X}|v}\left[\begin{pmatrix} d\overline{w} & -d(\overline{\tilde{z}_1}v) \\ d(\tilde{z}_1v) & dw \end{pmatrix}\begin{pmatrix} w & \overline{\tilde{z}_1}v \\ -\tilde{z}_1v & \overline{w} \end{pmatrix} - \begin{pmatrix} \overline{w} & -\overline{\tilde{z}_1}v \\ \tilde{z}_1v & w \end{pmatrix}\begin{pmatrix} dw & d(\overline{\tilde{z}_1}v) \\ -d(\tilde{z}_1v) & d\overline{w} \end{pmatrix}\right]$$
$$= V_1 + V_2 + V_3,$$

where

$$V_1(\mathbf{x}) = \mathrm{i}\frac{\Omega(\mathbf{x})}{2v(\mathbf{x})}\operatorname{Im}(wd\overline{w} + 4v^2(\mathbf{x})\tilde{z}_1 d\tilde{z}_1)\sigma_3, \tag{5.1.37}$$

$$V_2(\mathbf{x}) = \Omega(\mathbf{x})\begin{pmatrix} 0 & -\overline{w}d\overline{\tilde{z}_1} + \overline{\tilde{z}_1}dw \\ wd\tilde{z}_1 - \tilde{z}_1dw & 0 \end{pmatrix}, \tag{5.1.38}$$

and

$$V_3(\mathbf{x}) = \Omega(\mathbf{x})\frac{dv}{v}\begin{pmatrix} 0 & -\overline{w\tilde{z}_1} \\ w\tilde{z}_1 & 0 \end{pmatrix}. \tag{5.1.39}$$

We have: $w = \kappa_2 x_3 + \mathrm{i}\kappa_1 \implies dw = \kappa_2 dx_3 \implies \mathrm{Im}(wd\overline{w}) = \kappa_1\kappa_2 dx_3$, so

$$\mathrm{Im}(wd\overline{w}) = \kappa_1\kappa_2 dx_3 \ \& \ \mathrm{Im}(\tilde{z}_1 d\overline{\tilde{z}_1}) = \mathrm{Im}((x_1 + \mathrm{i}x_2)d(x_1 + \mathrm{i}x_2)) = -\zeta^1.$$

Hence, the V_i matrices above become,

$$V_1(\mathbf{x}) = \mathrm{i}\frac{\Omega(\mathbf{x})}{2v(\mathbf{x})}(\kappa_1\kappa_2 dx_3 - 4v^2(\mathbf{x})\zeta^1)\sigma_3, \tag{5.1.40}$$

$$V_2(\mathbf{x}) = \mathrm{i}\Omega(\mathbf{x})\left(\kappa_2\begin{pmatrix} 0 & -\mathrm{i}\overline{\tilde{z}_1} \\ \mathrm{i}\tilde{z}_1 & 0 \end{pmatrix}dx_3 - \kappa_2 x_3\begin{pmatrix} 0 & -\mathrm{i}d\overline{\tilde{z}_1} \\ \mathrm{i}d\tilde{z}_1 & 0 \end{pmatrix} + \kappa_1\begin{pmatrix} 0 & d\overline{\tilde{z}_1} \\ d\tilde{z}_1 & 0 \end{pmatrix}\right)$$

$$= \mathrm{i}\Omega(\mathbf{x})(\kappa_2 dx_3\Sigma^1 - \kappa_2 x_3 d\Sigma^1 + \kappa_1 d\Sigma^2) \tag{5.1.41}$$

and

$$\begin{aligned} V_3(\mathbf{x}) &= -\mathrm{i}\kappa_2 x_3\frac{\Omega(\mathbf{x})}{v(\mathbf{x})}\begin{pmatrix} 0 & -\mathrm{i}\overline{\tilde{z}_1} \\ \mathrm{i}\tilde{z}_1 & 0 \end{pmatrix}dv(\mathbf{x}) + \mathrm{i}\kappa_1\frac{\Omega(\mathbf{x})}{v(\mathbf{x})}\begin{pmatrix} 0 & \overline{\tilde{z}_1} \\ \tilde{z}_1 & 0 \end{pmatrix}dv(\mathbf{x}) \\ &= -\mathrm{i}\kappa_2 x_3\frac{\Omega(\mathbf{x})}{v(\mathbf{x})}\Sigma^1 dv(\mathbf{x}) + \mathrm{i}\kappa_1\frac{\Omega(\mathbf{x})}{v(\mathbf{x})}\Sigma^2 dv(\mathbf{x}). \end{aligned} \tag{5.1.42}$$

However,

$$2(\kappa_1^2 - \kappa_2^2) + \kappa_2^2|\mathbf{X}^{ES}(\mathbf{x})| = 2(\kappa_1^2 - \kappa_2^2) + \kappa_2^2(1 + |\mathbf{x}|^2) = 2\kappa_1^2 - \kappa_2^2 + \kappa_2^2|\mathbf{x}|^2$$

$$= 2\kappa_1^2 + \kappa_2^2(|\mathbf{x}|^2 - 1) = 2\kappa_1^2 + \kappa_2^2(2x_3^2 - (1 - |\mathbf{x}|^2 + 2x_3^2)$$

$$= 2(\kappa_1^2 + \kappa_2^2 x_3^2) - \kappa_2\kappa_2(1 - |\mathbf{x}|^2 + 2x_3^2) = 2(\kappa_1^2 + \kappa_2^2 x_3^2) - \kappa_2 X_3(\mathbf{x})$$

$$= 2|w|^2 - \kappa_2 X_3(\mathbf{x}) = 2\frac{|w|^2|\tilde{z}_1|^2}{|\tilde{z}_1|^2} - 8\kappa_2 X_3(\mathbf{x})\frac{(|w|^2|\tilde{z}_1|^2 + X_3^2(\mathbf{x})) - X_3^2(\mathbf{x})}{|\tilde{z}_1|^2} - \kappa_2 X_3(\mathbf{x})$$

$$= 8\frac{|\mathbf{X}(\mathbf{x})|^2 - X_3^2(\mathbf{x})}{|\tilde{z}_1|^2} - \kappa_2 X_3(\mathbf{x})$$

$$= 8\frac{|\mathbf{X}(\mathbf{x})| - X_3(\mathbf{x})}{|\tilde{z}_1|^2}(|\mathbf{X}(\mathbf{x})|+X_3(\mathbf{x}))-\kappa_2 X_3(\mathbf{x}) = 2v(\mathbf{x})\big((|\mathbf{X}(\mathbf{x})|+X_3(\mathbf{x}))-\frac{\kappa_2}{v(\mathbf{x})}X_3(\mathbf{x})\big).$$

Lastly,

$$dv = \frac{1}{|\tilde{z}_1|^2}\big(\frac{1}{2}\Omega d|\mathbf{X}|^2 - dX_3 - \frac{1}{4|\tilde{z}_1|^4}(|\mathbf{X}| - X_3)d|\tilde{z}_1|^2\big)$$

$$= \frac{1}{2|\tilde{z}_1|^2}\Big(\Omega(2(\kappa_1^2-\kappa_2^2)+\kappa_2^2|\mathbf{X}^{ES}|)\zeta^2+\kappa_2^2\Omega(\mathbf{x})x_3|\mathbf{X}^{ES}(\mathbf{x})|dx_3+\kappa_2 x_3\zeta^2-\kappa_2 x_3 dx_3-4v(\mathbf{x})\zeta^2\Big)$$

$$= \frac{\Omega}{2|\tilde{z}_1|^2}\Big((2(\kappa_1^2 - \kappa_2^2) + \kappa_2^2|\mathbf{X}^{ES}| + \kappa_2|\mathbf{X}| - 4v|\mathbf{X}|)\zeta^2 - \kappa_2 x_3(|\mathbf{X}| - \kappa_2|\mathbf{X}^{ES}|)dx_3\Big)$$

$$= \frac{\Omega}{2|\tilde{z}_1|^2}\Big(- 2v(1 - \frac{\kappa_2}{2v})(|\mathbf{X}| - X_3)\zeta^2 - \kappa_2 x_3(|\mathbf{X}| - X_3)\frac{|\mathbf{X}| - \kappa_2|\mathbf{X}^{ES}|}{|\mathbf{X}| - X_3}dx_3\Big)$$

$\Longrightarrow$

$$dv(\mathbf{x}) = -2\Omega(\mathbf{x})v(\mathbf{x})(1 - \frac{\kappa_2}{2v(\mathbf{x})} + \kappa_2 x_3)dx_3 \tag{5.1.43}$$

and noticing:

$$1- \frac{\kappa_2}{2v(\mathbf{x})} = \frac{1}{v(\mathbf{x})}\Big(\frac{|\tilde{z}_1|^2}{|\mathbf{X}(\mathbf{x})| - X_3(\mathbf{x})}\frac{|\mathbf{X}(\mathbf{x})| - X_3(\mathbf{x}) - \frac{1}{2}\kappa_2|\tilde{z}_1|^2}{|\tilde{z}_1|^2}\Big) = \frac{|\mathbf{X}(\mathbf{x})| - \kappa_2|\mathbf{X}^{ES}(\mathbf{x})|}{|\mathbf{X}(\mathbf{x})| - \Omega(\mathbf{x})X_3(\mathbf{x})}$$

$$= \frac{1 - \kappa_2 M(\mathbf{x})}{1 - \Omega(\mathbf{x})X_3(\mathbf{x})}.$$

and given that: $v(\mathbf{x}) = \frac{|\mathbf{X}(\mathbf{x})| - X_3(\mathbf{x})}{|\tilde{z}_1|^2} = \frac{|w|^2}{|\mathbf{X}(\mathbf{x})| + X_3(\mathbf{x})}$ the result is complete. □

Lemma 5.1.14. We have:

$$\frac{1}{4\Omega}[\sigma(\cdot), \sigma(d\Omega)] = \mathrm{i}\Omega^2 v(|\mathbf{X}| + vX_3)(2\zeta_1\sigma^3 - dx_3\Sigma^1) + \frac{\mathrm{i}}{2}\kappa_2^2\Omega^2(\mathbf{x})x_3|\mathbf{X}^{ES}|x_3 dx_3 \tag{5.1.44}$$

Proof. Starting with the fact that:

$$\Omega^{-1}(\mathbf{x})d\Omega(\mathbf{x}) = |\mathbf{X}(\mathbf{x})|d(|\mathbf{X}(\mathbf{x})|^{-1}) =$$

$$-\frac{1}{2}\Omega^2(\mathbf{x})d|\mathbf{X}(\mathbf{x})|^2 = -\Omega^2(\mathbf{x})(4(\kappa_1^2 - \kappa_2^2) + 2\kappa_2^2|\mathbf{X}^{ES}|(\mathbf{x}))\zeta^2 - 2\kappa_2^2\Omega^2(\mathbf{x})x_3|\mathbf{X}^{ES}(\mathbf{x})|dx_3$$

we get:

$$\sigma(d\Omega(\mathbf{x})) = -2\Omega^2(\mathbf{x})v(\mathbf{x})\sigma(\zeta^2) - 2\kappa_2^2\Omega^2(\mathbf{x})x_3|\mathbf{X}^{ES}(\mathbf{x})|\sigma_3.$$

Regarding any vector field, Y on $\mathbb{R}^3$, we have:

$$\sigma(\langle Y,\cdot\rangle) = \sigma(dx_1(Y)dx_1+dx_2(Y)dx_2+dx_3(Y)dx_3) = \begin{pmatrix} dx_3(Y) & (dx_1 - \mathrm{i}dx_2) \\ (dx_1+\mathrm{i}dx_2)(Y) & -dx_3(Y)\end{pmatrix},$$

$$\sigma(\langle Y,\cdot\rangle) = \frac{1}{2}d\Sigma_2(Y) + dx_3(Y)\sigma_3, \tag{5.1.45}$$

$$[\Sigma^2,\sigma_3] = \left[\begin{pmatrix} 0 & \overline{\tilde{z}_1} \\ \tilde{z}_1 & 0\end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right] = -2\mathrm{i}\begin{pmatrix} 0 & -\mathrm{i}\overline{\tilde{z}_1} \\ \mathrm{i}\tilde{z}_1 & 0\end{pmatrix} = -2\mathrm{i}\Sigma^1, \tag{5.1.46}$$

$\Longrightarrow$

$$[d\Sigma^2,\sigma_3] = -2\mathrm{i}d\Sigma^1.$$

Finally,

$$[d\Sigma^2,\Sigma^2] = \left[\begin{pmatrix} 0 & d\overline{\tilde{z}_1} \\ d\overline{\tilde{z}_1} & 0\end{pmatrix}, \begin{pmatrix} 0 & \overline{\tilde{z}_1} \\ \overline{\tilde{z}_1} & 0\end{pmatrix}\right] = \mathrm{i}\,\mathrm{Im}(\tilde{z}_1 d\overline{\tilde{z}_1})\sigma_3 = -4\mathrm{i}\zeta^1\sigma^3. \tag{5.1.47}$$

Completing the result. □

Now, we're ready to move on to the proof of (5.1.11).

Proof. We write, $\tilde{\psi} = \begin{pmatrix}\tilde{\psi}^+ \\ \tilde{\psi}^-\end{pmatrix}$ and using (3.3.9), we get:

$$\mathcal{P}_\mu(\tilde{\nabla}\tilde{\psi}) = U\begin{pmatrix} P_{\mu-}(d - \mathrm{i}(\tilde{a} - (\kappa_1+\kappa_2)\omega^c) \\ P_{\mu+}(d - \mathrm{i}(\tilde{a} + (\kappa_1+\kappa_2)\omega^c)\end{pmatrix}\begin{pmatrix}\tilde{\psi}^+ \\ \tilde{\psi}^-\end{pmatrix} =$$

$$U\big((d-\mathrm{i}\alpha)\otimes I_2 + \mathrm{i}(\kappa_1+\kappa_2)\alpha_c)\big)\begin{pmatrix} P_{\mu-}\tilde{\psi}^+ \\ P_{\mu+}\tilde{\psi}^-\end{pmatrix} =$$

$$\big((d-\mathrm{i}\alpha)\otimes I_2 - (dU)U^* + \mathrm{i}\kappa\alpha_c U\sigma_3 U^*\big)U\begin{pmatrix} P_{\mu-}\tilde{\psi}^+ \\ P_{\mu+}\tilde{\psi}^-\end{pmatrix} = (\nabla^\Omega - \mathcal{W})$$

where

$$\mathcal{W} = \mathcal{W}(\mathbf{x}) = \frac{1}{4\Omega}[\sigma(\cdot), \sigma(d\Omega)] + (dU_0(\mathbf{x}))U_0^*(\mathbf{x}) + \Omega(\mathbf{x})(\overline{\Gamma}(\mathbf{x})d\Gamma(\mathbf{x}) - \mathrm{i}\kappa\alpha_3)\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma} \tag{5.1.48}$$

utilizing lemma 5.1.7 as well as the fact that

$$(dU)U^* = (dU_0)U_0^* + U_0(dU_1)U_1^*U_0^* = (dU_0)U_0^* + \Omega(\overline{\Gamma}d\Gamma)\mathbf{X}.\boldsymbol{\sigma}. \tag{5.1.49}$$

However,

$$\mathbf{X}(\mathbf{x}).\sigma_3 = \kappa_1 W_1(\mathbf{x}).\boldsymbol{\sigma} + \kappa_2 W_2(\mathbf{x}).\boldsymbol{\sigma} + X_3(\mathbf{x})\sigma_3 d \tag{5.1.50}$$

and

$$\begin{gathered}\kappa\alpha_c = \kappa_1^2\big((1+\kappa_2 M) - (1-\kappa_2 M)\big)\mathbf{W}_1^b + (1-\kappa_2 M)\frac{|\mathbf{X}|^2}{|\mathbf{W}_1|^2}\mathbf{W}_1^b \\ +(1-\kappa_1 M)\frac{|\mathbf{X}|^2}{|\mathbf{W}_2|^2} + \kappa_2^2\big((1+\kappa_1 M) - (1-\kappa_1 M)\big)\mathbf{W}_2^b = \\ 2\kappa_1\kappa_2 M\alpha_b + \frac{|\mathbf{X}| - \kappa_2|\mathbf{X}^{ES}|}{|\tilde{z}_1|^2\Omega}\mathbf{W}_1^b + \frac{|\mathbf{X}| - \kappa_1|\mathbf{X}^{ES}|}{|\tilde{z}_2|^2\Omega}\end{gathered} \tag{5.1.51}$$

where $\alpha_b = \mathbf{X}^b = \kappa_1\mathbf{W}_1^b + \kappa_2\mathbf{W}_2^b$, and from the equation for α_c above and lemma 5.1.12 we get:

$$\begin{gathered}\Omega(\overline{\Gamma}d\Gamma - \mathrm{i}\kappa\alpha_c)\mathbf{X}.\boldsymbol{\sigma} = \\ \mathrm{i}\Big(\frac{|\mathbf{X}| - \kappa_1|\mathbf{X}^{ES}|}{|\tilde{z}_2|^2}\mathbf{W}_2^b + \kappa_1\Omega^2 d\big(1 - \frac{\kappa_2}{2v}\big)dx_3 - \\ 2\kappa_1\kappa_2\Omega M\alpha_b - \frac{|\mathbf{X}| - \kappa_2|\mathbf{X}^{ES}|}{|\tilde{z}_1|^2}\mathbf{W}_1^b - \frac{|\mathbf{X}| - \kappa_1|\mathbf{X}^{ES}|}{|\tilde{z}_2|^2}\mathbf{W}_2^b\Big)\mathbf{X}.\boldsymbol{\sigma} = \\ \mathrm{i}\Omega^2\big(1 - \frac{\kappa_2}{2v}\big)[\kappa_1 dx_3 - 2v\zeta^1](\kappa_1\Sigma^1 + \kappa_2 x_3\Sigma^2 + X_3\sigma_3) - 2\mathrm{i}\kappa_1\kappa_2\Omega M\alpha_b(\mathbf{X}.\boldsymbol{\sigma}).\end{gathered} \tag{5.1.52}$$

Also,

$$\begin{gathered}-2v\big(1 - \frac{\kappa_2}{2v}\big)(-\kappa_2 x_3\Sigma^1 + \kappa_1\Sigma^2) - 2v\big(1 - \frac{\kappa_2}{2v}\big)\zeta^1(\kappa_1\Sigma^1 + \kappa_2 x_3\Sigma^2) = \\ 2v\big(1 - \frac{\kappa_2}{2v}(\kappa_2 x_3)(\zeta^2\Sigma^1 - \zeta^1\Sigma^2) - \kappa_1(\zeta^1\Sigma^1 + \zeta^2\Sigma^2)\big) = \\ \frac{1}{2}(|\mathbf{X}| - \kappa_2|\mathbf{X}^{ES}|)(\kappa_2 x_3 d\Sigma^1 - \kappa_1 d\Sigma^2).\end{gathered}$$

This, alongside the last two lemmas, gives us:

$$\mathcal{W} = \mathrm{i}\Omega^2 v(|\mathbf{X}| + \big(1 - \frac{\kappa_2}{2v}(\kappa_2 x_3\big) X_3)[2\zeta^1\sigma_3 - dx_3\Sigma^1] + \frac{\mathrm{i}}{2}\kappa_2^2\Omega^2 x_3|\mathbf{X}^{ES}|d\Sigma^1 +$$

$$\mathrm{i}\frac{\Omega}{2v}\kappa_1\kappa_2 dx_3\sigma^3 - 2\mathrm{i}\Omega v\zeta^1\sigma^3 + \mathrm{i}\frac{\Omega}{2}(\kappa_2 dx_3\Sigma^1 - \kappa_2 x_3 d\Sigma^1 + \kappa_1 d\Sigma^2) +$$

$$-\mathrm{i}\Omega^2\big(1 - \frac{\kappa_2}{2v}\big)(2v\zeta^2 + \kappa_2 x_3 dx_3)(-\kappa_2 x_3\Sigma^1 + \kappa_1\Sigma^2) +$$

$$\mathrm{i}\Omega^2\big(1 - \frac{\kappa_2}{2v}\big)(\kappa_1 dx_3 - 2v\zeta^1)(\kappa_1\Sigma^1 + \kappa_2 x_3\Sigma^2 + X_3\sigma_3) - 2\mathrm{i}\kappa_1\kappa_2\Omega M\alpha_b(\mathbf{X}.\boldsymbol{\sigma}) =$$

$$= 2\mathrm{i}\Omega^2 v\big((|\mathbf{X}| + \big(1 - \frac{\kappa_2}{2v}\big)X_3) - |\mathbf{X}| - \big(1 - \frac{\kappa_2}{2v}\big)X_3\big)\zeta^1\sigma_3 +$$

$$\mathrm{i}\Omega^2\big(- v(|\mathbf{X}| + \big(1 - \frac{\kappa_2}{2v}\big)X_3) + \frac{\kappa_2}{2}|\mathbf{X}| + \kappa_2^2 x_3^2\big(1 - \frac{\kappa_2}{2v} + \kappa_1^2\big(1 - \frac{\kappa_2}{2v}\big)dx_3\Sigma^1\big)\Sigma^1 dx_3 +$$

$$\mathrm{i}\Omega^2\big(\frac{\kappa_1\kappa_2|\mathbf{X}|}{2v}|\mathbf{X}| + \kappa_1\big(1 - \frac{\kappa_2}{2v}\big)X_3\big)dx_3\sigma_3 +$$

$$\frac{i}{2}\Omega^2\big(\kappa_2^2 x_3|\mathbf{X}^{ES}|d\Sigma^1 + (|\mathbf{X}| - (|\mathbf{X}| - \kappa_2|\mathbf{X}^{ES}|))(-\kappa_2 x_3 d\Sigma^1 + \kappa_1 d\Sigma^2)\big) - 2\mathrm{i}\kappa_1\kappa_2\Omega M\alpha_b(\mathbf{X}.\boldsymbol{\sigma})$$

$$= \mathrm{i}\kappa_1\kappa_2\Omega^2|\mathbf{X}|(\frac{1}{2}d\Sigma^2 + \sigma dx_3) - 2\mathrm{i}\kappa_1\kappa_2\Omega M\alpha_b(\mathbf{X}.\boldsymbol{\sigma}).$$

Noticing that

$$-v(|\mathbf{X}| + \big(1 - \frac{\kappa_2}{2v}\big)X_3) + \frac{\kappa_2}{2}|\mathbf{X}| + \kappa_2^2 x_3^2\big(1 - \frac{\kappa_2}{2v}\big) + \kappa_1^2\big(1 - \frac{\kappa_2}{2v}\big)$$

$$= -v\big(1 - \frac{\kappa_2}{2v}\big)(|\mathbf{X}| + X_3) + |w|^2\big(1 - \frac{\kappa_2}{2v}\big) = 0, \quad \text{and}$$

$$|\mathbf{X}|\frac{\kappa_2}{2v} + \big(1 - \frac{\kappa_2}{2v}\big)X_3 = |\mathbf{X}| - \big(1 - \frac{\kappa_2}{2v}\big)(|\mathbf{X}| - X_3) = \kappa_2|\mathbf{X}^{ES}|.$$

Lastly,

$$\frac{1}{2}d\Sigma^2 + dx_3\sigma_3 = \begin{pmatrix} dx_3 & dx_1 - \mathrm{i}dx_2 \\ dx_1 + \mathrm{i}dx_2 & -dx_3 \end{pmatrix} = (dx_1, dx_2, dx_3).\boldsymbol{\sigma} = \sigma(\cdot)$$

and $\Omega|\mathbf{X}^{ES}| = M$, the result follows. □

By the proof of this result, we get that conjugation identity, we have:

Corollary 5.1.15. Let $\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \in \Gamma(\mathbb{R}^3 \times \mathbb{C})$. The following equations holds:

$$U^*(\nabla^\Omega - \mathrm{i}\kappa_1\kappa_2\Omega M[\sigma(\cdot) - 2\alpha_b(\mathbf{X}.\boldsymbol{\sigma})])U\psi = \begin{pmatrix} (d - \mathrm{i}(\alpha - \kappa\alpha_c))\psi^+ \\ (d - \mathrm{i}(\alpha + \kappa\alpha_c))\psi^- \end{pmatrix} \tag{5.1.53}$$

Now, we're ready to see how Dirac operators on $\mathbb{R}^2$ and $\mathbb{R}^3$ interlace via the submersion F.

5.2 Dirac Operators on $\mathbb{R}^3$, $\mathbb{R}^2$ and $\mathbb{O}_\kappa$

Recall that Dirac operators on $\tilde{\mathcal{D}}_{\tilde{A}}$ and $\mathcal{D}_A$, on $\mathbb{R}^2$ and $\mathbb{R}^3$ change under conformal change of metric according to the following formulae respectively:

$$\tilde{\mathcal{D}}^{\omega}_{\tilde{A}}\cdot := \omega^{-3/2}\tilde{\mathcal{D}}_A(\omega^{1/2}\cdot) \tag{5.2.1}$$

and

$$\mathcal{D}^{\Omega}_{A}\cdot := \Omega^{-2}\mathcal{D}_{\tilde{A}}(\Omega\cdot) \tag{5.2.2}$$

where $\tilde{\boldsymbol{A}}$ and $\mathbf{A}$ be the corresponding magnetic fields.

$$\omega_1(|F_{\kappa,1}(\mathbf{x})||^2)|\boldsymbol{\nabla}\operatorname{Im}F_{\kappa,1}(\mathbf{x})| = \omega_2(|F_{\kappa^{-1},2}(\mathbf{x})|^2)|\boldsymbol{\nabla}\operatorname{Im}F_{\kappa^{-1},2}(\mathbf{x})| = \Omega(\mathbf{x}) \tag{5.2.3}$$

Recall that $|\operatorname{Im}F_{\kappa,1}(\mathbf{x})| = |F_{\kappa,1}(\mathbf{x})||\mathbf{Q}_1(\mathbf{x})| = |F_{\kappa,1}(\mathbf{x})||\mathbf{P}_1(\mathbf{x})| = |\operatorname{Re}F_{\kappa,1}(\mathbf{x})|$ and similarly for $|\operatorname{Im}F_{\kappa,1}(\mathbf{x})|$. Also, $\Omega(\mathbf{x}) = |\mathbf{X}(\mathbf{x})|^{-1}$.

Given weights ω_j on two copies $\mathbb{R}^2 \equiv \mathbb{C}$ respectively, we can write the associated Dirac operator(s) on L_μ, analytically as follows: Given $\mu \in \Sigma := \{(2k_1+1)\kappa_1 + (2k_2+1)\kappa_2 : k_{1,2} \in \mathbb{Z}\}$, $\mathbb{C}_j$ ($\equiv \mathbb{R}^2$ each for $j = 1,2$) two copies of the complex plane and $\tilde{\boldsymbol{e}}_{i,j}$ $\tilde{e}_{i,j}$, $i = 1,2$ a orthonormal basis of vector fields and their dual 1-forms on $(\mathbb{R}^2, \omega_j)$. Let $\tilde{\alpha}^j = \tilde{\alpha}'_j + \mu\gamma^j_z$ and $\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}$ the respective ("weighted") *Spin*^{c} connection on L_μ (as in (3.4.17)) and $\sigma_\omega \equiv \omega^{-1}\sigma$ the corresponding Clifford multiplication. We write the Dirac operator on L_μ as the pair of the Dirac operators:

$$\tilde{\mathcal{D}}^{\omega_j}_{\tilde{A}_j} = -\mathrm{i}\sum_{i=1}^{2}\tilde{\sigma}_{\omega_j}(\tilde{e}_{i,j})\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}_{\tilde{\boldsymbol{e}}_{i,j}} \tag{5.2.4}$$

This pair gives us well-defined mappings $\tilde{\mathcal{D}}^{\omega_j}_{\tilde{A}} : \Gamma(\mathbb{C}_j, \mathbb{C}^2) \to \Gamma(\mathbb{C}_j, \mathbb{C}^2)$. This is because, if we let $\tilde{\boldsymbol{e}}'_{i,j} = \sum_{k=1}^{2} v_{ij}\tilde{\boldsymbol{e}}_{i,j}$ another set of orthonormal vector fields (with respective forms $\tilde{e}'_{1,2}$). Then considering the matrix that defines this transformation $V := (v_{i,j})$, $i,j \in \{1,2\}$, it satisfies $VV^T = I_2$, i.e. $\sum_{k=1}^{2} v_{ik}v_{kj} = \sum_{k=1}^{2} v_{ki}v_{jk} = \delta_{ij}$ and so:

$$\sum_{i=1}^{2}\tilde{\sigma}_{\omega_j}(\tilde{e}_{i,j})\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}_{\tilde{\boldsymbol{e}}_{i,j}} = \sum_{k,l,i=1}^{2} v_{kl}v_{lk}\tilde{\sigma}_{\omega_j}(\tilde{e}_{i,j})\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}_{\tilde{\boldsymbol{e}}_{i,j}} \tag{5.2.5}$$

$$= \sum_{k,l,j=1}^{2}\delta_{kl}\tilde{\sigma}_{\omega_j}(\tilde{e}_{i,j})\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}_{\tilde{\boldsymbol{e}}_{i,j}} = \sum_{k,l,j=1}^{2}\tilde{\sigma}_{\omega_j}(\tilde{e}_{i,j})\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}_{\tilde{\boldsymbol{e}}_{i,j}}.$$

Now that we've defined Dirac Operators on sections on the orbit space, we may show that these mappings, are basically automorphisms. We have:

Lemma 5.2.1. Let $\psi = (\psi_{(1)}, \psi_{(2)}) \in \Gamma(E_\mu)$ for $\mu = (2k_1+1)\kappa_1 + (2k_2+1)\kappa_2$, $k_{1,2} \in \mathbb{Z}$. Set

$$\phi_{(j)} = \tilde{\mathcal{D}}^{\omega_j}_{\tilde{A}_j}\psi_{(j)}$$

Then $\phi = (\phi_{(1)}, \phi_{(2)}) \in \Gamma(L_\mu)$.

Proof. Given $j \in \{1,2\}$, consider open sets $\tilde{U}^j_{1,2} \subseteq \mathbb{C}_j$ such that $\tilde{U}^j_2 = R_{\kappa^{3-2j}}\tilde{U}^j_1$. Let $\tilde{e}^{j,2}_i$, $i = 1,2$, an orthonormal frame of 1-forms on U_2, and set $\tilde{e}^{j,1}_i = R^*_{\kappa^{3-2j}}\tilde{e}^{j,2}_i$ (again $i = 1,2$), an orthonormal frame of 1-forms on $\tilde{U}^j_1$. Similarly, consider the respective vector fields $\tilde{\boldsymbol{e}}^{j,2}_{1,2}$ and $\tilde{\boldsymbol{e}}^{j,1}_i = (R_{\kappa^{3-2j}})_*\tilde{\boldsymbol{e}}^{j,2}_i$, $i = 1,2$. We can easily see that

$$R^*_{\kappa^{3-2j}}(\mathcal{E}_{-k_j}\psi_{(j)}) = \mathcal{E}_{-k_j}\psi_{(j)} \tag{5.2.6}$$

recalling

$$\mathcal{E}_k(z) := \begin{pmatrix} \tilde{e}^k(z) & 0 \\ 0 & \tilde{e}^{k+1}(z) \end{pmatrix} \quad \text{for} \quad k \in \mathbb{Z} \tag{5.2.7}$$

This map satisfies (for $i = 1,2$):

$$R^*_{\kappa^{3-2j}}(\mathcal{E}_{-k_j}\psi_{(i)}) = \mathcal{E}_{-k_j}\psi_{(i)} \tag{5.2.8}$$

and

$$\tau^*(\mathcal{E}_{-k_2}\psi_{(2)}) = \mathcal{E}_{-k_1}\psi_{(1)} \tag{5.2.9}$$

As well as:

$$R^*_{\kappa^{3-2j}}(\mathcal{E}_{-k_j}\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}\mathcal{E}_{k_j}) = \mathcal{E}_{-k_j}\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}\mathcal{E}_{k_j} \tag{5.2.10}$$

and

$$\tau^*(\mathcal{E}_{-k_2}\tilde{\nabla}^{\omega_2,\tilde{\alpha}^2}\mathcal{E}_{k_2}) = \mathcal{E}_{-k_1}\tilde{\nabla}^{\omega_1,\tilde{\alpha}^1}\mathcal{E}_{k_1} \tag{5.2.11}$$

$$\implies R^*_{\kappa^{3-2j}}(\mathcal{E}_{-k_j}\tilde{\sigma}_{\omega_j}(\tilde{e}_{i,j})\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}_{\tilde{\mathbf{e}}_{i,j}}\psi_{(j)})$$

$$= R^*_{\kappa^{3-2j}}\big(\mathcal{E}_{-k_j}\tilde{\sigma}_{\omega_j}(\tilde{e}_i)\mathcal{E}_{k_j}\big)R^*_{\kappa^{3-2j}}\big(\mathcal{E}_{-k_j}\tilde{\nabla}^{\omega_j,\alpha^j}_{R^*_{\kappa^{3-2j}}\tilde{\mathbf{e}}_i}(\mathcal{E}_{-k_j}\mathcal{E}_{k_j}\psi_{(j)})\big) =$$

$$\mathcal{E}_{-k_j}\tilde{\sigma}_{\omega_j}(R^*_{\kappa^{3-2j}}\tilde{e}_i)\mathcal{E}_{k_j}R^*_{\kappa^{3-2j}}(\mathcal{E}_{-k_j}\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}\mathcal{E}_{k_j})_{\tilde{\mathbf{e}}_i}R^*_{\kappa^{3-2j}}(\mathcal{E}_{-k_j}\psi_{(j)}) =$$

$$\mathcal{E}_{-k_j}\tilde{\sigma}_{\omega_j}(\tilde{e}_i)\mathcal{E}_{k_j}\big(\mathcal{E}_{-k_j}\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}\mathcal{E}_{k_j}\big)_{\tilde{\mathbf{e}}_i}\mathcal{E}_{-k_j}\psi_j = \mathcal{E}_{-k_j}\tilde{\sigma}_{\omega_j}(\tilde{e}_i)\mathcal{E}_{k_j}\big(\mathcal{E}_{-k_j}\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}\big)$$

for $i = 1, 2$. In other words,

$$R_{3-2j}(\mathcal{E}_{-k_j}\phi_{(j)}) = \mathcal{E}_{-k_j}\phi_{(j)}. \tag{5.2.12}$$

When transitioning via τ_a (recall, we're working on particular branches of the exponentials, indexed by a), on subsets $\tilde{U}, \tau_a(\tilde{U}) \subseteq \mathbb{C}^*$, we have:

$$\tau_a^*(\mathcal{E}_{-k_2}\tilde{\sigma}_{\omega_2}(\tilde{e}_i)\tilde{\nabla}^{\omega_2,\tilde{\alpha}^2}_{\tilde{\mathbf{e}}_i}\psi_{(2)}) = \tau_a^*(\mathcal{E}_{-k_2}\tilde{\sigma}_{\omega_2}(\tilde{e}_i)\mathcal{E}_{k_2}\mathcal{E}_{-k_2}\tilde{\nabla}^{\omega_2,\tilde{\alpha}^2}_{\tilde{\mathbf{e}}_i}\psi_{(2)})$$

$$= \tau_a^*(\mathcal{E}_{-k_2}\tilde{\sigma}_{\omega_2}(\tilde{e}_i)\mathcal{E}_{k_2})\tau_a^*(\mathcal{E}_{-k_2})\tilde{\nabla}^{\omega_2,\tilde{\alpha}^2}_{(\tau_a)_*\tilde{\mathbf{e}}_i}(\mathcal{E}_{k_2}\mathcal{E}_{-k_2}\psi_{(2)})$$

$$= \mathcal{E}_{-k_1}\tilde{\sigma}_{\omega_1}(\tau_a^*\tilde{e}_i)\mathcal{E}_{k_1}\tau_a^*\big(\mathcal{E}_{-k_2}\tilde{\nabla}^{\omega_2,\tilde{\alpha}_2}\mathcal{E}_2\big)_{\tilde{\mathbf{e}}_i}\tau_a^*(\mathcal{E}_{-k_2}\psi_{(2)})$$

$$= \mathcal{E}_{-k_1}\tilde{\sigma}_{\omega_1}(\tilde{e}_i)\mathcal{E}_{k_1}\big(\mathcal{E}_{-k_1}\tilde{\nabla}^{\omega_1,\tilde{\alpha}^1}\big)_{\tilde{\mathbf{e}}_i}\mathcal{E}_{-k_1}\psi_{(1)} = \mathcal{E}_{-k_1}\tilde{\sigma}_{\omega_1}(\tilde{e}_i)\tilde{\nabla}^{\omega_1,\tilde{\alpha}^1}\psi_{(1)}$$

for all $i = 1, 2$. Hence,

$$\tau_a^*(\mathcal{E}_{-k_2}\phi_{(2)}) = \mathcal{E}_{-k_1}\phi_{(1)} \tag{5.2.13}$$

In other words, for $u \in C^\infty(\mathbb{C}_j)$, $j = \{1, 2\}$ we have:

$$R_{\kappa^{3-2j}}(\tilde{\sigma}_{\omega_j}(\tilde{e}_i)\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}_{\tilde{\mathbf{e}}_i}u) = \mathcal{E}_{k_j}(e^{2\pi i\kappa^{3-2j}})\tilde{\sigma}_{\omega_j}(\tilde{e}_i)\tilde{\nabla}^{\omega_j,\tilde{\alpha}^j}_{\tilde{\mathbf{e}}_i}\big(\mathcal{E}_{-k_j}(e^{2\pi i\kappa^{3-2j}})R^*_{\kappa^{3-2j}}u\big) \tag{5.2.14}$$

on $\tau_a^*(\tilde{U})$. Now summing these over $i = 1, 2$ we get:

$$R^*_{\kappa^{3-2j}}(\tilde{\mathcal{D}}^{\omega_j}_{\tilde{A}_j}u) = \mathcal{E}_{k_j}(e^{2\pi i\kappa^{3-2j}})\tilde{\mathcal{D}}^{\omega_j}_{\tilde{A}_j}\big(\mathcal{E}_{-k_j}(e^{2\pi i\kappa^{3-2j}})R^*_{\kappa^{3-2j}}u\big) \tag{5.2.15}$$

□

Consider an $\alpha \in \Omega^1(\mathbb{R}^3)$ (a 1-form on $\mathbb{R}^3$ corresponding to the magnetic potential $\mathbf{A}(\mathbf{x})$) and set $\beta = d\alpha \in \Omega^2(\mathbb{R}^3)$ the respective 2-form corresponding to the magnetic field and let it be such that $d^2\alpha = 0$ and $\iota_{\mathbf{X}}(d\alpha) = 0$. The former is equivalent to $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$ while the latter is equivalent to $\mathbf{X} \times (\boldsymbol{\nabla} \times \mathbf{A}) = 0$ (the magnetic field $\boldsymbol{\nabla} \times \mathbf{A}$ is parallel to $\mathbf{X}$). We set the corresponding potentials $\tilde{\boldsymbol{A}}(\tilde{\mathbf{x}})$ (corresponding to the 1-form $\tilde{\alpha}$) in two dimensions and the respective Weyl-Dirac operators.

Definition 5.2.2. We define the following class of magnetic potentials:

$$\mathbf{A}^{\nu}(\mathbf{x}) = \frac{\nu}{|\mathbf{X}(x)|^2}\mathbf{X}(\mathbf{x}) + \mathbf{X}_0^{\perp}(\mathbf{x}) + \boldsymbol{\nabla}\phi(\mathbf{x}) \tag{5.2.16}$$

where $\mathbf{X}_0^{\perp}(\mathbf{x})$ is perpendicular to $\mathbf{X}(\mathbf{x})$ and satisfies $\mathcal{L}_{\mathbf{X}}\mathbf{X}_0^{\perp} = 0$ (it's Lie derivative with respect to $\mathbf{X}$ is equal to zero) and ϕ is some scalar function on $\mathbb{R}^3$.

These fields are the only ones of interest towards the purpose of this thesis. (5.2.16). Moreover, it can be shown that these fields are such that $\mathbf{X}(\mathbf{x}).\mathbf{A}^{\nu}(\mathbf{x})$ is bounded. When there is risk of confusion or particular need to emphasize on the parameter ν, we'll denote $\mathbf{A}^{\nu}$ simply as $\mathbf{A}$.

Theorem 5.2.3. Let a smooth $\mathbf{A}(\mathbf{x}) \in \mathbb{R}^3$ (pointwise) such that $\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x}) = //\mathbf{X}(\mathbf{x})$. Then $\mathbf{A}(\mathbf{x})$ is of the form (5.2.16).

The proof diverges for the purpose of this Thesis, so it will be omitted.

Without loss of generality, we can set $\mathbf{X}_0^{\perp}(\mathbf{x}) + \boldsymbol{\nabla}\phi(\mathbf{x}) = \boldsymbol{A}'(\mathbf{x})$ and re-write

$$\mathbf{A}^{\nu}(\mathbf{x}) := \frac{\nu}{|\mathbf{X}(\mathbf{x})|^2}\mathbf{X}(\mathbf{x}) + \mathbf{A}'(\mathbf{x}) \tag{5.2.17}$$

where $\mathbf{A}'(\mathbf{x}) = (\alpha'(e_1), \alpha'(e_2), \alpha'(e_3))$ for $\alpha' = F_{1,2}^*\tilde{\alpha}'_{1,2} \in \Omega^1(\mathbb{R}^3)$ (the pull-back of a 1-form on $\mathbb{O}_\kappa$), $\tilde{\alpha}'_{1,2} \in \Omega^1(\mathbb{R}^2_{1,2})$ (recall $\mathbb{R}^2$ is identified as $\mathbb{C}$). We can also define the respective class Weyl-Dirac operators as usual: $\mathcal{D}_{\mathbf{A}^{\nu}}$ (for the standard ones) and the respective "weighted" versions $\mathcal{D}^{\Omega}_{\mathbf{A}^{\nu}}$. Recall that our orbit space $\mathbb{O}_\kappa$, is defined as two copies $\mathbb{C}_{1,2}$ ($\equiv \mathbb{R}^2_{1,2}$) with points $z \neq 0$ identified via the map $\tau : \mathbb{C}_1^* \mapsto \mathbb{C}_2^*$, with $\tau(z) = C_\kappa z^{-1/\kappa}$ for $C_\kappa = \kappa(1+\kappa)^{-1-\frac{1}{\kappa}}$.

Now, we're ready to identify a proper choice of a local frame on $\mathbb{R}^3$, that's related to a suitable frame on $\mathbb{O}_\kappa$. In particular, we have the following lemma:

Lemma 5.2.4. Assume we have a smooth branch, indexed by a, $F^a_{1,2} : U_{1,2} \subset \mathbb{R}^3_{1,2} \mapsto \tilde{U}_{1,2} \subset \mathbb{C}_{1,2}$, where $\tilde{U}_{1,2} = F^a_{1,2}(U_{1,2})$, given as

$$F^a_{1,2} = |F^a_{1,2}|\tilde{e}(\tilde{z}_1)\epsilon_a \tag{5.2.18}$$

for ϵ_a being a branch of the exponential $e^{-\kappa^{3-2j}}(\tilde{z}_{j'})$ (again, $j \neq j' \in \{1,2\}$). Let $\tilde{e}^{1,2}$ a positively oriented frame of 1-forms on $\tilde{U}_{1,2}$. Set $\overline{e}_{1,2} = (F^a_{1,2})^*\tilde{e}_{1,2}$ and $\overline{e}_3 = \alpha_b|_{U_{1,2}}$. Then $\overline{e}_j$, $j = 1, 2, 3$ is a positively oriented frame of 1-forms on $U_{1,2}$ (with the weighted metric). Also, considering the aforementioned frames of 1-forms on $U_{1,2}$ and $\tilde{U}_{1,2}$ we get the respective dual vector fields on these sets; $\overline{\boldsymbol{e}}_{1,2} = (F^a_{1,2})_*\tilde{\boldsymbol{e}}_{1,2}$ and $\overline{\boldsymbol{e}}_3 = \mathbf{X}$ (on U).

Proof. We have:

$$\langle \overline{\boldsymbol{e}}_i, \overline{\boldsymbol{e}}_j \rangle_\Omega = (F^a_l)_* \langle \overline{\boldsymbol{e}}_i, \overline{\boldsymbol{e}}_j \rangle_{\omega_l} = (F^a_l)\delta_{ij} = \delta_{ij} \tag{5.2.19}$$

for $i, j = 1, 2$. Also for $\overline{e}^3 = \mathbf{X}^b$ we have:

$$\langle \overline{e}_3, \overline{e}_l \rangle_\Omega = \overline{e}_l(\mathbf{X}) = \begin{cases} \overline{e}^l((F^a_{1,2})_*\mathbf{X}) & \text{if l=1,2} \\ \alpha_b(\mathbf{X}) & \text{if l=3} \end{cases} \tag{5.2.20}$$

In other words, $\langle \overline{e}^3, \overline{e}^l \rangle_\Omega = \delta_{l3}$. Therefore, $\overline{e}^j$, $j = 1, 2, 3$ is an orthonormal frame of 1-forms on U. Now since for i,j = 1,2 we have:

$$\delta_{ij} = \overline{e}_i(\overline{\boldsymbol{e}}_i) = \tilde{e}_i((F^*_{1,2})_*\overline{\boldsymbol{e}}_i)$$

we get that $(F^a_{1,2})_*\overline{e}_j = \tilde{e}_j$ pointwise on $F^a_{1,2}(U) = \tilde{U}_{1,2}$. We also have

$$\boldsymbol{\nabla} F^a_{1,2} = F^a_{1,2}(\mathbf{P}_{1,2} + \mathrm{i}\mathbf{Q}_{1,2})$$

by (3.2.10), from which we get:

$$(F^a_{1,2})_*\mathbf{P}_{1,2} = \mathbf{P}_{1,2}(\boldsymbol{\nabla} F^a_{1,2}\partial_z + \boldsymbol{\nabla}\overline{F}^a_{1,2}\partial_{\overline{z}})|\mathbf{Q}_{1,2}|^2(F^a_{1,2}\partial_z + \overline{F}^a_{1,2}\partial_{\overline{z}}) \tag{5.2.21}$$

and

$$(F^a_{1,2})_*\mathbf{Q}_{1,2} = \mathbf{Q}_{1,2}(\boldsymbol{\nabla} F^a_{1,2}\partial_z + \boldsymbol{\nabla}\overline{F}^a_{1,2}\partial_{\overline{z}}) = \mathrm{i}|\mathbf{Q}_{1,2}|^2(F^a_{1,2}\partial_z - \overline{F}^a_{1,2}\partial_{\overline{z}}) \tag{5.2.22}$$

However $\tilde{e}^1 \wedge \tilde{e}^2 = \mathrm{VOL}^2_{\omega_{1,2}}$ and so

$$\overline{e}^1 \wedge \overline{e}^2(\mathbf{P}_{1,2}, \mathbf{Q}_{1,2}) = \tilde{e}^1 \wedge \tilde{e}^2(F^a_{1,2}\mathbf{P}_{1,2}, F_{1,2^a}\mathbf{Q}_{1,2}) =$$

$$\mathrm{i}\omega^2_{1,2}(F^a_{1,2})|\mathbf{Q}_{1,2}|^4\Big(\frac{1}{2\mathrm{i}}\Big)dz \wedge d\overline{z}(F^a_{1,2}\partial_z + \boldsymbol{\nabla}\overline{F}^a_{1,2}\partial_{\overline{z}}, F^a_{1,2}\partial_z - \overline{F}^a_{1,2}\partial_{\overline{z}}) =$$

$$\omega^2_{1,2}(F^a_{1,2})|\mathbf{Q}_{1,2}|^4|F^a_{1,2}|^2 = \frac{1}{2}\omega^2_{1,2}(F^a_{1,2})|\mathbf{Q}_{1,2}|^2|\boldsymbol{\nabla} F^a_{1,2}|^2 = |\mathbf{Q}_{1,2}|^2\Omega^2.$$

We also note that

$$\overline{e}_3(\mathbf{P}_{1,2}) = \Omega^2\mathbf{X}.\mathbf{P} = 0 = \Omega^2\mathbf{X}.\mathbf{Q} = \overline{e}_3(\mathbf{Q}_{1,2})\overline{e}^1 \wedge \overline{e}^2(\mathbf{P}_{1,2}, \mathbf{Q}_{1,2}) \tag{5.2.23}$$

while $\overline{e}^3(\mathbf{X}) = 1$. Hence,

$$(\overline{e}^1 \wedge \overline{e}^2 \wedge \overline{e}^3)(\Omega^{-1}\mathbf{P}_{1,2}, \Omega^{-1}|\mathbf{Q}_{1,2}|, \mathbf{X}) = \Omega^{-2}|\mathbf{Q}_{1,2}|^{-2}(\overline{e}^1 \wedge \overline{e}^2)(\mathbf{P}_{1,2}, \mathbf{Q}_{1,2})\overline{e}^3(\mathbf{X}) = 1. \tag{5.2.24}$$

It can easily be shown that $\mathbf{P}_{1,2}, \mathbf{Q}_{1,2}, \mathbf{X}$ is a right handed frame, so $\overline{e}^1 \wedge \overline{e}^2 \wedge \overline{e}^3 = \mathrm{vol}^3_\Omega$ and $\overline{e}^1, \overline{e}^2, \overline{e}^3$ is also right-handed. □

Consider $\mu \in \Sigma := \{(2k_1+1)\kappa_1+(2k_2+1)\kappa_2 : \kappa_{1,2} \in \mathbb{Z}\}$ (for $\kappa_{1,2} \in \mathbb{R}_{>0}$); $\tilde{\alpha}' \in \Omega^1(\mathbb{O}_\kappa)$ and $\tilde{\alpha} = \tilde{\alpha}' + \mu\tilde{\gamma}$ where $\tilde{\gamma} = l_{1,2}(|z|^2)\tilde{\zeta}$ (see (4.2.26) and the equations above).

For the associated Dirac operators, with weights Ω and $\omega_{1,2}$ on $\mathbb{R}^3$ and $\mathbb{R}^2_{1,2}$, we have:

Theorem 5.2.5. *Let $\mu \in \Sigma$ and the Weyl-Dirac operators $\mathcal{D}^\Omega_{\boldsymbol{A}^\nu}$ and $\mathcal{D}^\omega_{\tilde{\boldsymbol{A}}}$. Then:*

$$\mathcal{D}^\Omega_{\boldsymbol{A}^\nu}(\mathcal{P}_\mu\tilde{\psi}) = \mathcal{P}_\mu\big((\tilde{\mathcal{D}}^\omega_{\tilde{A}} + \kappa_1\kappa_2 m I_2 + (\mu-\nu)\sigma_3)\tilde{\psi}\big). \tag{5.2.25}$$

Proof. Since we can't define Dirac operators on $\mathbb{O}_{\boldsymbol{\kappa}}$ globally, we'll show this result locally. Consider $\tilde{\boldsymbol{e}}_{1,2}$ a basis of vector fields on $\mathbb{R}^2$, and set $\overline{\boldsymbol{e}}_i = F^*\tilde{\boldsymbol{e}}_i$ for $i = 1,2$ and $\overline{\boldsymbol{e}}_3 = \mathbf{X}(\mathbf{x})$. By 5.1.10 we have: $\mathcal{P}_\mu(\sigma_\omega(\tilde{\alpha})\tilde{\psi}) = \sigma_\Omega(F^*\tilde{\alpha})(\mathcal{P}_\mu\tilde{\psi})$ which for $\tilde{\alpha} = \tilde{e}_{1,2}$ implies: $\mathcal{P}_\mu(\sigma_\omega(\tilde{e}_{1,2})\tilde{\psi}) = \sigma_\Omega(\overline{e}_{1,2})(\mathcal{P}_\mu\tilde{\psi})$. However, we also have: $\sigma_\Omega(\overline{e}_3)\mathcal{P}_\mu\tilde{\psi} = \mathcal{P}_\mu(\sigma_\Omega(\overline{e}_3)\tilde{\psi})$

$\mathcal{P}_\mu(\sigma_\omega(\tilde{e}_{1,2})\tilde{\psi}) = \sigma_\Omega(\overline{e}_{1,2})(\mathcal{P}_\mu\tilde{\psi})$ and $\sigma(\overline{e}_3)(\mathcal{P}_\mu\tilde{\psi}) = \sigma_\Omega(\mathbf{X}^b_\mathbf{x})\mathcal{P}_\mu\tilde{\psi} = \mathcal{P}_\mu(\sigma(\overline{e}^3)\tilde{\psi})$ where the *flat* co-vector $\mathbf{X}^b_\mathbf{x}$ of $\mathbf{X}(\mathbf{x})$ corresponds to the weighted metric $(\cdot,\cdot)_\Omega$. By 5.1.11 we have:

$$(\nabla^{\Omega,\alpha_\mu} - \mathrm{i}\kappa_1\kappa_2\frac{|\mathbf{X}^{ES}(\mathbf{x})|}{|\mathbf{X}(\mathbf{x})|^2}(\sigma(\cdot) - 2(\mathbf{X}(\mathbf{x}).\boldsymbol{\sigma})))_{\overline{\boldsymbol{e}}_i}(\mathcal{P}_\mu\psi) = \mathcal{P}_\mu(\tilde{\nabla}^{\omega,\tilde{\alpha}}\tilde{\psi}) \tag{5.2.26}$$

for $i = 1, 2$ and " $\cdot$ " in $\sigma(\cdot)$ denoting the dual 1-form of the vector field $\overline{\boldsymbol{e}}^i$. We also have:

$$(\nabla^{\Omega,\alpha_\mu} - \mathrm{i}\kappa_1\kappa_2\frac{|\mathbf{X}^{ES}(\mathbf{x})|}{|\mathbf{X}(\mathbf{x})|^2}(\sigma(\cdot) - 2\mathbf{X}(\mathbf{x}).d\mathbf{x})_{\overline{\boldsymbol{e}}_3})\mathcal{P}_\mu\tilde{\psi} = 0 \tag{5.2.27}$$

and plugging the matrices σ_i, $i = 1, 2, 3$ respectively we get:

$$(\mathcal{D}^\Omega_{\mathbf{A}^\nu} - \kappa_1\kappa_2\mathcal{M})\mathcal{P}_a u = \mathcal{P}_a\tilde{\mathcal{D}}^\omega_{\tilde{\mathbf{A}}}\tilde{\psi} \tag{5.2.28}$$

for all $\psi \in \Gamma(\tilde{U} \times \mathbb{C}^2)$ for some open subset $\tilde{U} \in \mathbb{C}$ where:

$$\mathcal{M} = \Omega M \sum_{i=1}^{3} \sigma_\Omega(\sigma(\overline{e}_i))(\sigma(\cdot) - 2\alpha_b(\mathbf{X}.\boldsymbol{\sigma}))_{\overline{\boldsymbol{e}}_i}.$$

Now, for $i = 1, 2$ we have $\alpha_b(\overline{\boldsymbol{e}}_i) = \langle \mathbf{X}, \overline{\boldsymbol{e}}_i\rangle_\Omega$ and $\langle \overline{e}_i, \cdot\rangle = \Omega^{-2}\overline{e}^i$, where the $\langle\cdot,\cdot\rangle$ denotes the inner product on the space of vector fields and 1-forms respectively, so

$$\sigma_\Omega(\overline{e}_i)(\sigma(\cdot) - 2\alpha_b(\mathbf{X}.\boldsymbol{\sigma}))_{\overline{\boldsymbol{e}}_i} = \sigma_\Omega(\overline{e}_i)\Omega^{-2}\sigma(\overline{e}_i) = \Omega^{-1}(\sigma_\Omega(\overline{e}_i))^2 = \Omega^{-1}I_2$$

Moreover, $\alpha_b(\overline{\boldsymbol{e}}_3) = \alpha(\mathbf{X}) = \langle \mathbf{X}, \mathbf{X}\rangle_\Omega = 1$ and

$$\mathbf{X}.\boldsymbol{\sigma} = \sigma(\langle \overline{e}_3, \cdot\rangle) = \Omega^{-2}\sigma(\overline{e}_3) = \Omega^{-1}\sigma_\Omega(\overline{e}_3) \implies$$

$$\sigma_\Omega(\overline{e}_i)(\sigma(\cdot) - 2\alpha_b(\mathbf{X}.\boldsymbol{\sigma}))_{\overline{e}_3} = \sigma_\Omega(\overline{e}_3)(\Omega^{-1}\sigma_\Omega(\overline{e}_3) - 2\Omega^{-1}\sigma_\Omega(\overline{e}_3)) = -\Omega^{-1}I_2.$$

Hence, $\mathcal{M} = \Omega M\Omega^{-1}I_2 = MI_2 = F^*_{1,2}m_{1,2}$ ((3.2.23)-given a particular choice of value/branch of the exponential). This, alongside the first three equations on this page, gives:

$$\mathcal{D}^{\Omega}_{\mathbf{A}^\nu}(\mathcal{P}_{\mu,a}\tilde{\psi}) = (\mathcal{D}^{\Omega}_{\mathbf{A}^\nu} - \kappa_1\kappa_2\mathcal{M})(\mathcal{P}_{\mu,a}\tilde{\psi}) + (\kappa_1\kappa_2\mathcal{M} + (\mu-\nu)\sigma_\Omega(\alpha_b))\mathcal{P}_{\mu,a}\tilde{\psi} =$$

$$\mathcal{P}_{\mu,a}(\tilde{\mathcal{D}}^{\omega_i}_{\tilde{A}_i}\tilde{\psi}) + \mathcal{P}_{\mu,a}\big((\kappa_1\kappa_2 m_i I_2 + (\mu-\nu)\sigma_3)\tilde{\psi}\big)$$

where $\mathcal{P}_{\mu,a}$ refers to the operator $\mathcal{P}_\mu$ for a particular choice of branch (of the exponentials involved in the maps $F_{1,2}$) indexed by a. □

This concludes the main results of this Thesis.

Chapter 6: Supplementary results

6.1 Some useful & interesting calculations

In this section, we'll prove some useful lemmas and propositions regarding the quantities introduced previously.

6.1.1 Orthonogonal frames

Lemma 6.1.1. The vectors $\mathbf{P}_{1,2}(\mathbf{x})$, $\mathbf{Q}_{1,2}(\mathbf{x})$ and $\mathbf{X}(\mathbf{x})$ satisfy:

$$|\mathbf{P}_{1,2}(\mathbf{x})| = |\mathbf{Q}_{1,2}(\mathbf{x})| \tag{6.1.1}$$

$$\mathbf{P}_{1,2}(\mathbf{x}) \times \mathbf{Q}_{1,2}(\mathbf{x}) = |\mathbf{Q}_{1,2}(\mathbf{x})|^2 \Omega(\mathbf{x})\mathbf{X}(\mathbf{x}) \tag{6.1.2}$$

Proof. The first property is derived via very simple calculations from the definitions of these vectors. Regarding the second property, we calculate:

$$\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x}) = (-2x_2, 2x_1, 0) \times (2x_1x_3, 2x_2x_3, 1 - |\mathbf{x}|^2 + 2x_3^2)$$

$$= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ -2x_2 & 2x_1 & 0 \\ 2x_1x_3 & 2x_2x_3 & 1 - |\mathbf{x}|^2 + 2x_3^2 \end{vmatrix}$$

$$= 2x_1(1 - |\mathbf{x}|^2 + 2x_3^2)\boldsymbol{e}_1 - (-2x_2)(1 - |\mathbf{x}|^2 + 2x_3^2)\boldsymbol{e}_2 - 4(x_1^2 + x_2^2)x_3\boldsymbol{e}_3$$

$$= 2(1 - |\mathbf{x}|^2 + 2x_3^2)(x_1, x_2, 0) - 4(x_1^2 + x_2^2)x_3(0, 0, 1) = 2X_3^{ES}(\mathbf{x})(x_1, x_2, 0) - 4(x_1^2 + x_2^2)x_3\mathbf{e}_3$$

so

$$\frac{|\mathbf{W}_1(\mathbf{x})|^2|\mathbf{W}_2(\mathbf{x})|^2}{2|\mathbf{X}_\kappa(\mathbf{x})|}\mathbf{P}_1(\mathbf{x}) \times \mathbf{Q}_1(\mathbf{x}) =$$

$$(\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x})) \times \Big(\frac{2}{|\mathbf{W}_1(\mathbf{x})|^2}\mathbf{W}_1(\mathbf{x}) - \frac{2\kappa}{|\mathbf{W}_2(\mathbf{x})|^2}\mathbf{W}_2(\mathbf{x})\Big)$$
$$= \frac{2}{|\mathbf{W}_1(\mathbf{x})|^2}(\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x})) \times \mathbf{W}_1(\mathbf{x}) - \frac{2\kappa}{|\mathbf{W}_2(\mathbf{x})|^2}(\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x})) \times \mathbf{W}_2(\mathbf{x}).$$

Now we have:

$$(x_1, x_2, 0) \times (-x_2, x_1, 0) = (x_1^2 + x_2^2)\boldsymbol{e}_3, \tag{6.1.3}$$

and

$$(0, 0, 1) \times (-x_2, x_1, 0) = (-x_1, x_2, 0). \tag{6.1.4}$$

So

$$(\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x})) \times \mathbf{W}_1(\mathbf{x}) = (2X_3^{ES}(\mathbf{x})(-x_1, x_2, 0) - 4(x_1^2 + x_2^2)x_3\mathbf{e}_3) \times \mathbf{W}_1(\mathbf{x})$$
$$= 4(x_1^2 + x_2^2)X_3^{ES}(\mathbf{x})\boldsymbol{e}_3 + 8(x_1^2 + x_2^2)x_3(-x_1, x_2, 0) \; = |\mathbf{W}_1(\mathbf{x})|^2\mathbf{W}_2(\mathbf{x})$$
$$\implies \frac{2}{|\mathbf{W}_1(\mathbf{x})|^2}(\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x})) \times \mathbf{W}_1(\mathbf{x}) = 2x_3(-x_1, x_2, 0) + 2X_3^{ES}(\mathbf{x})\boldsymbol{e}_3 = 2\mathbf{W}_2(\mathbf{x}).$$

Also, we have:

$$(x_1, x_2, 0) \times \mathbf{W}_2(\mathbf{x}) = (x_1, x_2, 0) \times (2x_1x_3, 2x_2x_3, 1 - |\mathbf{x}|^2 + 2x_3^2) =$$
$$\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & 0 \\ 2x_1x_3 & 2x_2x_3 & X_3^{ES}(\mathbf{x}) \end{vmatrix} = x_2X_3^{ES}(\mathbf{x})\mathbf{e}_1 - x_1X_3^{ES}(\mathbf{x})\mathbf{e}_2 = -X_3^{ES}(\mathbf{x})\mathbf{W}_1(\mathbf{x}), \tag{6.1.5}$$

and

$$\boldsymbol{e}_3 \times \mathbf{W}_2(\mathbf{x}) = (0, 0, 1) \times (2x_1x_3, 2x_2x_3, 1 - |\mathbf{x}|^2 + 2x_3^2)$$
$$= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 0 & 0 & 1 \\ 2x_1x_3 & 2x_2x_3 & 1 - |\mathbf{x}|^2 + 2x_3^2 \end{vmatrix} = \; - 2x_2x_3\mathbf{e}_1 + 2x_1x_3\boldsymbol{e}_2 = x_3\mathbf{W}_1(\mathbf{x})$$

So

$$(\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x})) \times \mathbf{W}_2(\mathbf{x}) = (2X_3^{ES}(\mathbf{x})(x_1, x_2, 0) - 4(x_1^2 + x_2^2)x_3\mathbf{e}_3) \times \mathbf{W}_2(\mathbf{x})$$

$$= 2X_3^{ES}(\mathbf{x})((x_1, x_2, 0) \times \mathbf{W}_2(\mathbf{x})) - 8(x_1^2 + x_2^2)x_3(\boldsymbol{e}_3 \times \mathbf{W}_2(\mathbf{x}))$$

$$= -2X_3^2(\mathbf{x})\mathbf{W}_1(\mathbf{x}) - 8(x_1^2 + x_2^2)x_3^2\mathbf{W}_1(\mathbf{x}) = -|\mathbf{W}_2(\mathbf{x})|^2\mathbf{W}_1(\mathbf{x})$$

$\Longrightarrow$

$$\frac{2\kappa}{|\mathbf{W}_2(\mathbf{x})|^2}(\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x})) \times \mathbf{W}_2(\mathbf{x}) = -2\kappa\mathbf{W}_1(\mathbf{x}). \tag{6.1.6}$$

Consequently,

$$\frac{2}{|\mathbf{W}_1(\mathbf{x})|^2}(\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x})) \times \mathbf{W}_1(\mathbf{x}) - \frac{2\kappa}{|\mathbf{W}_2(\mathbf{x})|^2}(\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x})) \times \mathbf{W}_2(\mathbf{x})$$

$$= 2\kappa\mathbf{W}_1(\mathbf{x}) + 2\mathbf{W}_2(\mathbf{x})$$

$$\Longrightarrow \frac{|\mathbf{W}_1(\mathbf{x})|^2|\mathbf{W}_2(\mathbf{x})|^2}{2|\mathbf{X}_\kappa(\mathbf{x})|}\mathbf{P}_1(\mathbf{x}) \times \mathbf{Q}_1(\mathbf{x}) = (2\kappa\mathbf{W}_1(\mathbf{x}) + 2\mathbf{W}_2(\mathbf{x}))$$

$\Longrightarrow$

$$\begin{aligned}\mathbf{P}_1(\mathbf{x}) \times \mathbf{Q}_1(\mathbf{x}) &= \frac{4\kappa_2^{-1}|\mathbf{X}_\kappa(\mathbf{x})|}{|\mathbf{W}_1(\mathbf{x})|^2|\mathbf{W}_2(\mathbf{x})|^2}\mathbf{X}(\mathbf{x}) = \frac{4\kappa_2^{-2}|\mathbf{X}(\mathbf{x})|}{|\mathbf{W}_1(\mathbf{x})|^2|\mathbf{W}_2(\mathbf{x})|^2}\mathbf{X}(\mathbf{x}) \\ &= \frac{4\kappa_2^{-2}}{\Omega(\mathbf{x})|\mathbf{W}_1(\mathbf{x})|^2|\mathbf{W}_2(\mathbf{x})|^2}\mathbf{X}(\mathbf{x}).\end{aligned} \tag{6.1.7}$$

Using the equations (3.2.14) and (3.2.12) we have:

$$\begin{aligned}\mathbf{P}_2(\mathbf{x}) \times \mathbf{Q}_2(\mathbf{x}) &= \frac{1}{\kappa^2}\frac{4\kappa_2^{-1}|\mathbf{X}_\kappa(\mathbf{x})|}{|\mathbf{W}_1(\mathbf{x})|^2|\mathbf{W}_2(\mathbf{x})|^2}\mathbf{X}(\mathbf{x}) = \frac{1}{\kappa^2}\frac{4\kappa_2^{-2}|\mathbf{X}(\mathbf{x})|}{|\mathbf{W}_1(\mathbf{x})|^2|\mathbf{W}_2(\mathbf{x})|^2}\mathbf{X}(\mathbf{x}) \\ &= \frac{4\kappa_1^{-2}}{\sqrt{\Omega(\mathbf{x})}|\mathbf{W}_1(\mathbf{x})|^2|\mathbf{W}_2(\mathbf{x})|^2}\mathbf{X}(\mathbf{x}) = \frac{4\kappa_1^{-2}|\mathbf{X}(\mathbf{x})|^2}{|\mathbf{W}_1(\mathbf{x})|^2|\mathbf{W}_2(\mathbf{x})|^2}\Omega(\mathbf{x})\mathbf{X}(\mathbf{x}).\end{aligned} \tag{6.1.8}$$

Finally, we have:

$$|\mathbf{Q}_1(\mathbf{x})| = \frac{2}{|\mathbf{W}_1(\mathbf{x})|^2|\mathbf{W}_2(\mathbf{x})|^2}\sqrt{|\mathbf{W}_2(\mathbf{x})|^4|\mathbf{W}_1(\mathbf{x})|^2 + \kappa^2|\mathbf{W}_1(\mathbf{x})|^4|\mathbf{W}_2(\mathbf{x})|^2}$$

$\Longrightarrow$

$$|\mathbf{Q}_1(\mathbf{x})| = \frac{2}{|\mathbf{W}_1(\mathbf{x})||\mathbf{W}_2(\mathbf{x})|}\sqrt{|\mathbf{W}_2(\mathbf{x})|^2| + \kappa^2|\mathbf{W}_1(\mathbf{x})|^2|} = \frac{2}{|\mathbf{W}_1(\mathbf{x})||\mathbf{W}_2(\mathbf{x})|}|\mathbf{X}_\kappa(\mathbf{x})|$$
$$= \frac{2\kappa_2^{-1}}{|\mathbf{W}_1(\mathbf{x})||\mathbf{W}_2(\mathbf{x})|}|\mathbf{X}(\mathbf{x})|, \tag{6.1.9}$$

and

$$|\mathbf{Q}_2(\mathbf{x})| = \frac{1}{\kappa}|\mathbf{Q}_1(\mathbf{x})| = \frac{2\kappa_1^{-1}}{|\mathbf{W}_1(\mathbf{x})||\mathbf{W}_2(\mathbf{x})|}|\mathbf{X}(\mathbf{x})|. \tag{6.1.10}$$

The last two equations alongside with (6.1.7) and (6.1.8) give

$$\mathbf{P}_{1,2}(\mathbf{x}) \times \mathbf{Q}_{1,2}(\mathbf{x}) = |\mathbf{Q}_{1,2}(\mathbf{x})|^2\Omega(\mathbf{x})\mathbf{X}(\mathbf{x}). \tag{6.1.11}$$

□

As a consequence of the above, we have:

Corollary 6.1.2. The vectors $\mathbf{P}_{1,2}(\mathbf{x})$, $\mathbf{Q}_{1,2}(\mathbf{x})$ and $\mathbf{X}(\mathbf{x})$ are orthogonal and they satisfy:

$$\mathbf{Q}_{1,2}(\mathbf{x}) \times \mathbf{X}(\mathbf{x}) = \Omega^{-1}(\mathbf{x})\mathbf{P}_{1,2}(\mathbf{x}), \quad \mathbf{X}(\mathbf{x}) \times \mathbf{P}_{1,2}(\mathbf{x}) = \Omega^{-1}(\mathbf{x})\mathbf{Q}_{1,2}(\mathbf{x}) \tag{6.1.12}$$

Proof. From (6.1.11), and by the *triple vector product formula* we have:

$$\mathbf{Q}(\mathbf{x}) \times \mathbf{X}(\mathbf{x}) = |\mathbf{Q}_{1,2}(\mathbf{x})|^{-2}\Omega^{-1}(\mathbf{x})(\mathbf{Q}(\mathbf{x}) \times (\mathbf{P}_{1,2}(\mathbf{x}) \times \mathbf{Q}_{1,2}(\mathbf{x})))$$

$$= |\mathbf{Q}_{1,2}(\mathbf{x})|^{-2}\Omega^{-1}(\mathbf{x})\big((\mathbf{Q}_{1,2}(\mathbf{x})).\mathbf{Q}_{1,2}(\mathbf{x}))\mathbf{P}_{1,2}(\mathbf{x}) - (\mathbf{Q}_{1,2}(\mathbf{x}).\mathbf{P}_{1,2}(\mathbf{x}))\mathbf{Q}_{1,2}(\mathbf{x})\big)$$
$$= \Omega^{-1}(\mathbf{x})\mathbf{P}_{1,2}(\mathbf{x}).$$

Also: $\mathbf{X}(\mathbf{x}) \times \mathbf{P}_{1,2}(\mathbf{x}) = \Omega^{-1}|\mathbf{Q}_{1,2}(\mathbf{x})|^{-2}|\big((\mathbf{P}_{1,2}(\mathbf{x}) \times \mathbf{Q}_{1,2}(\mathbf{x})\big) \times \mathbf{P}_{1,2}(\mathbf{x})\big)$

$$= \Omega^{-1}(\mathbf{x})|\mathbf{Q}_{1,2}(\mathbf{x})|^{-2}|\big((\mathbf{P}_{1,2}(\mathbf{x}).\mathbf{Q}_{1,2}(\mathbf{x})\big)\mathbf{P}_{1,2}(\mathbf{x}) - (\mathbf{P}_{1,2}(\mathbf{x}).\mathbf{P}_{1,2}(\mathbf{x}))\mathbf{Q}_{1,2}(\mathbf{x})\big)$$

$$= \Omega^{-1}(\mathbf{x})|\mathbf{Q}_{1,2}(\mathbf{x})|^{-2}|\mathbf{P}_{1,2}(\mathbf{x})|^2\mathbf{Q}_{1,2}(\mathbf{x}) = \Omega^{-1}(\mathbf{x})\mathbf{Q}_{1,2}(\mathbf{x}).$$

□

Moreover, from (3.2.13), (3.2.14) and (6.1.9), (3.2.12), the respective normalized version of $\mathbf{P}_{1,2}(\mathbf{x})$ satisfy

$$\begin{aligned}\frac{\mathbf{P}_j(\mathbf{x})}{|\mathbf{Q}_j(\mathbf{x})|} &= -\frac{(-1)^j}{|\mathbf{W}_1(\mathbf{x})||\mathbf{W}_2(\mathbf{x})|}\mathbf{W}_1(\mathbf{x}) \times \mathbf{W}_2(\mathbf{x}) \\ = -\frac{(-1)^j}{|\mathbf{W}_1(\mathbf{x})||\mathbf{W}_2(\mathbf{x})|}&\big(2X_3^{ES}(\mathbf{x})(x_1, x_2, 0) - (x_1^2 + x_2^2)\boldsymbol{e}_3\big)\end{aligned} \tag{6.1.13}$$

6.1.2 Compatibility condition invariance

As mentioned in *Chapter 3*; in particular in the parts where the transition map τ, (3.3.1), is defined and discussed, the transition between the two charts $\mathbb{O}_{\kappa^{3-2j}}$ $(j = 1, 2)$ is not trivial. Instead, it depends on which value we give to the exponential $z^{-1/\kappa}$. Now, we prove that the definition of this transition map is well-posed, given (3.3.6), as it's consistent with that "rotation" property of section of the line bundle, regardless of the choice of value we pick for the exponential $z^{-1/\kappa} = e^{-\frac{1}{\kappa}\ln(z)}$

Proposition 6.1.3. The compatibility condition (3.3.6), with $z_2 = \tau(z_1)$ is invariant under any choice of value for $\tau(z)$.

Proof. Let $m \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}$, we have:

$$\tau(e^{2m\pi\mathrm{i}}z_1) = C_\kappa(e^{2m\pi\mathrm{i}}z_1)^{-1/\kappa} = C_\kappa z_1^{-1/\kappa}e^{-\frac{2\pi m\mathrm{i}}{\kappa}} \implies$$

$$e^{-k_2}(\tau(e^{2m\pi\mathrm{i}}z_1)) = e^{-k_2}(C_\kappa z_1^{-1/\kappa}e^{-\frac{2\pi m\mathrm{i}}{\kappa}}) = e^{\frac{2k_2\pi m\mathrm{i}}{\kappa}}e^{k_2\mathrm{i}Arg(z_1)/\kappa} = e^{\frac{2k_2\pi m\mathrm{i}}{\kappa}}e^{-k_2}(\tau(z_1))$$

However, $u_2(\tau(e^{2m\pi\mathrm{i}}z_1)) =$

$$u_2(e^{-\frac{2m\pi\mathrm{i}}{\kappa}}\tau(z_1)) = u_2(e^{-\frac{2m\pi\mathrm{i}}{\kappa}}\tau(z_1)) = u_2(R_{-\frac{m}{\kappa}}\tau(z_1)) = e^{-2m\frac{k_2\pi}{\kappa}\mathrm{i}}u_2(\tau(z_1)).$$

So (3.3.6) remains the same. □

This shows that given any choice of $\tau(z_1)$, (3.3.6) still holds. In other words, it is well-defined as it's consistent with any possible choice of the value of $\tau(z_1)$ $(z_1 \in \mathbb{C}_1)$.

6.2 The Lichnerowicz formula

As mentioned in the introduction, an essential tool for advancements in this research area is the Lichnerowicz formula (a formula for $\boldsymbol{\mathcal{D}}_{\mathbf{A}}^2$), which is used (in its original forms or variants); in Elton, 2018 a variation of this is used. This is particularly useful as in the case of a self-adjoint Dirac operator $\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}$ on some *Hilbert space* $\mathcal{H}$, we obtain

$$\langle \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}\psi, \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}\psi\rangle_{\mathcal{H}} = \langle \psi, \boldsymbol{\mathcal{D}}_{\mathbf{A}}^2\psi\rangle_{\mathcal{H}},$$

This can potentially help us get investigate existence and/or growth estimates for solutions of the Weyl-Dirac equation. We have:

$$\boldsymbol{\mathcal{D}}_{\mathbf{A}}^2 = \boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}) = (\boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}} - \boldsymbol{A}))(\boldsymbol{\sigma}.(\boldsymbol{\mathcal{D}} - \mathbf{A})) = (\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}} - \boldsymbol{\sigma}.\boldsymbol{A}).(\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}} - \boldsymbol{\sigma}.\mathbf{A})$$

$$\Longrightarrow$$

$$\boldsymbol{\mathcal{D}}_{\mathbf{A}}^2 = (\sigma_i\mathcal{D}^i - \sigma_iA^i).(\sigma_j\mathcal{D}^j - \sigma_jA^j) = \sigma_i\sigma_j\mathcal{D}^i\mathcal{D}^j - \sigma_i\sigma_jA^i\mathcal{D}^j - \sigma_i\sigma_j\mathcal{D}^i(A^j\cdot) + \sigma_i\sigma_jA^iA^j.$$

Taking into account that the Dirac operators $\mathcal{D}_i$ act on twice continuously differentiable spinors, at least on the space(s) we're concerned with now, we have that $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$, as well as $A_iA_j = A_jA_i$, and since $\sigma_i\sigma_j = -\sigma_j\sigma_i$, $\forall i \neq j \in \{1, 2, 3\}$. Also, we note that:

$$\sigma_i\sigma_j\mathcal{D}^i(A^j\cdot) = \sigma_i\sigma_j(\mathcal{D}_iA^j)\cdot + \sigma_i\sigma_jA^j\mathcal{D}_i\cdot$$

and so finally we get: $\boldsymbol{\mathcal{D}}_{\mathbf{A}}^2 = \sigma_i^2(\mathcal{D}^i)^2 - 2\sigma_i^2A^i\mathcal{D}^i + \sigma_i^2(A^i)^2 - \sigma_i\sigma_j\mathcal{D}^iA^j \implies$

$$\boldsymbol{\mathcal{D}}_{\mathbf{A}}^2 = -\Delta - 2\mathrm{i}A_i\nabla^i + \textstyle\sum_i A_i^2 + \mathrm{i}\boldsymbol{\nabla}\cdot\mathbf{A} - \boldsymbol{\sigma}.\boldsymbol{\nabla}\times\mathbf{A}.$$

In our case, the Weyl-Dirac operator is equipped with a weight $\Omega(\mathbf{x}) = |\mathbf{X}(\mathbf{x})|^{-1}$, and so we recall (2.3.7):

$$(\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}(\Omega\cdot))^2 = \Omega^2\boldsymbol{\mathcal{D}}_{\boldsymbol{A}}^2 + \Omega(\boldsymbol{\sigma}.\boldsymbol{\mathcal{D}}(\Omega))\boldsymbol{\mathcal{D}}_{\boldsymbol{A}} + 2\Omega(\boldsymbol{\mathcal{D}}\Omega.\boldsymbol{\mathcal{D}}) + \Omega(\Delta\Omega) - \sum_{i=1}^{3}(\nabla_i\Omega)^2.$$

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