The stress-energy distributional multipole for both uncharged and charged dust

Jonathan Gratus[1,4,5], Spyridon Talaganis[2,4], and Willow Sparks[3,4,6]

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[1] j.gratus@lancaster.ac.uk. ORCID: 0000-0003-1597-6084
[2] s.talaganis@lancaster.ac.uk. ORCID: 0000-0003-0113-7546
[3] willow.sparks@stfc.ac.uk. ORCID:0009-0008-7630-9609
[4] Dept Physics, Lancaster University, Lancaster LA1 4YB, UK
[5] Cockcroft Institute of Accelerator Science, Daresbury Laboratory, Keckwick Lane, Daresbury, Warrington, WA4 4AD, UK
[6] Now based at the Scientific Computing Department, STFC, Daresbury Laboratory, Keckwick Lane, Daresbury, WA4 4AD, UK

Abstract

In this paper, we formulate the distributional uncharged and charged stress-energy tensors. These are integrals, along a worldline, of derivatives of the delta-function. These distributions are also multipoles and they are prescribed to any order. They represent an extended region of non-self-interacting uncharged or charged dust, shrunken to a single point in space. We show that the uncharged dust stress-energy multipole is divergence-free, while the divergence of the charged dust stress-energy multipole is given by the current and the external electromagnetic field. We discuss the constitutive relations which produce this multipole. We show that they can be obtained by squeezing a regular dust stress-energy tensor onto the worldine. We discuss the aforementioned calculations in a coordinate-free manner.

1 Introduction

There is much interest currently about distributional sources of gravity [1–7], in particular with reference to sources of gravitational waves. These are sources of gravity where all the mass is concentrated on a worldline. Hence the stress-energy tensor is the integral of a delta-function, and its derivatives are along the worldline. One may consider such distributional sources as approximations where the spatial extent of the source is small compared to the observers distance from such a source.

Since Einstein's equation are non-linear equations it is not possible to directly equate the Einstein tensor and a distributional source of a worldline. In [8], Geroch and Traschen show that the regular metrics can have distributional sources on three-dimensional surfaces. Worldlines and the 2–dimensional world-sheets of strings of cannot be sources of regular metrics. One approach to solving non-linear equations involving distributions is by using Colombeau algebra and is discussed in the conclusion. By contrast, it is possible to let this distribution be the source for the linearised

Einstein's equations. The solutions to the linearised equations can naturally be interpreted as gravitational waves. Hence we can interpret the distribution stress-energy tensor as sources for gravitational waves. In order to be a source of gravity or gravitational waves the stress-energy tensor must satisfy two conditions, namely being symmetric and divergence-free. These conditions can be relaxed if one is only considering a partial stress-energy tensor. For example, if the total stress-energy tensor has two components (one for matter and the other for the electromagnetic field) then it is only the total stress-energy tensor which needs to be symmetric and divergence-free.

In [1] the authors briefly look at the (uncharged) dust model for a quadrupole stress-energy tensor. In this article we extend the work. We posit the distributional dust stress-energy tensor. This tensor has not, as far as the authors are aware, been considered before, other than the brief mention in [1]. We look at this distributional dust stress-energy tensor in detail, showing it is symmetric and divergence-free for all orders.

We then consider the distributional stress-energy tensor for charge dust which is symmetric, but the divergence is not zero. As such this can only be a partial stress-energy tensor representing the matter in the model. It should be added to the stress-energy tensor of the electromagnetic field. This is achieved in [9] for many charged particles, at the monopole order, where each particle responds to the fields of the other particles. However, this is not possible here due to the rapid diverging of the electromagnetic fields as one approaches the worldlines. As a result, we only demand that the distribution interacts with an external electromagnetic field, and we derive the corresponding divergence equation that it must satisfy. Again we formulate an original distributional stress-energy tensor which satisfies this divergence condition.

Distributions can also be considered as multipoles. The order of the multipole is defined as the maximum number of derivatives of the delta-function used to define it. With respect to sources of gravitational waves, the most interesting case is that of the quadrupole. As a heuristic argument, one can say that the monopole and dipole do not give rise to any gravitational waves, whereas for orders above the quadrupole the corresponding gravitational waves fall off with distance at a faster rate. With current technology it is already challenging to detect the quadrupole contribution, so these higher moments are not relevant. Thus the dominant contribution to gravitational waves is the quadrupole moment. In the case when the background metric is Minkowski, there is an explicit formula for the components of the gravitational waves in terms of the moments of the quadrupole [3].

The monopole has no derivatives of the delta-functions. The symmetry and divergence-free conditions imply that the worldline must be a geodesic and that mass is conserved. In the charged case, it implies that the worldline satisfies the Lorentz force equation.

The dipole has a single derivative of the delta-function. If the worldline is prescribed, the components of the uncharged dipole satisfy the Mathisson–Papapetrou–Tulczyjew–Dixon equations. This is a well defined system and the dynamics of the components are completely determined by the initial values. By contrast, if the worldline is not prescribed then there is an under-determined system [1, 7] and additional equations are required to determine the motion of the worldline and the dynamics of the dipole. The same problem occurs if the dipole is charged, especially if it is constructed from multiple species.

The quadrupole has two derivatives of the delta-function. In this case, even for uncharged source with the worldline prescribed this is an under-determined system. There are 30 ordinary differential equations (ODEs), for 50 components. Thus one observes that for the most important case, namely the quadrupole, it is not possible to calculate the dynamics of the moments without additional information. These additional pieces of information are called *constitutive relations* as they are determined by the underlying constituents of the source. This is to be expected as the gravitational waves arising from two orbiting neutron stars, would be distinct from that of an asymmetric supernova. The multipole expansion for the stress-energy tensor due to two orbiting point masses in a Minkowski background is very well established [10] as are the corresponding gravitational waves. In principle it would be possible to find the corresponding constitutive relations for these moments.

The challenge addressed in this article is to derive the dynamics of multipoles representing either charged or uncharged dust. Here the uncharged dust can model a low density of matter which only interacts with an external gravitational field. It does not model a distribution of matter which is bound by its gravitational field such as orbiting neutron stars. The dust in this article is assumed to not be self-interacting, neither gravitationally or electromagnetically. One of the consequences of such a model, which we show here, is that it does not spin. This is in line with our intuition, as a distribution of non-interacting dust would fly apart instead of spinning. Thus it cannot be used to model orbiting neutron stars which are strongly gravitationally bound. In [1] we conjectured the constitutive relations for a dust model. This included a non zero spin component, and so does not correspond to the dust multipoles presented here. The constitutive relation of uncharged dust are discussed. We identify 20 algebraic relations and conjecture that these prescribe the uncharged dust quadrupole in a general coordinate system. However, these are insufficient to prescribe prescribe dust in the adapted coordinate system and require an additional 30 equations.

The distributional charged dust models dust which interacts with an external electromagnetic field, not its own internal field. Thus it cannot be used to model a body held together by its own electrostatic forces. The external electromagnetic field in interstellar space is only of the order a few microgauss. By contrast electromagnetic fields near planets are 10s of Gauss and those near a neutron star or black hole may be 1000s of Gauss. Thus the charged dust distribution can be used to model matter orbiting a neutron star or in the accretion disc of a black hole.

An example where the machinery discussed in this article can be used directly is to look at the gravitational waves generated from the charged dust in the accretion disc around an isolated black hole. It is true that the gravitation waves associated with such material is extremely small. However, since the black hole is isolated and therefore does not produce gravitational waves of its own, those produced by the dust could in principle be detected. We also need to assume that the charges on the dust only produce a weak electromagnetic, which is not strong enough to affect the motion on the dust. The modeling of particles around a black hole, using distributional multipoles, was examined in [14]. However, in that work the distributions were in phase-space, and further work is needed to compare the results to the dust model.

This article is arranged as follows. In section 2 we recap the Ellis representation [11] of a multipole. We state the dynamic equations for the quadrupole total stress-energy tensor. In [3] the authors compared the advantages of the Ellis representation which uses partial derivatives and the Dixon representation [12] which uses covariant derivatives. We look again at the number of constitutive relations needed. We outline the steps needed to give the formula for gravitational waves (29), and show the squeezing procedure which makes a distributional stress-energy tensor the limit of a regular stress-energy tensor.

In section 3 we look at the uncharged dust multipole. Using the Ellis representation, in a coordinate system adapted to a congruence of geodesics, allows us to greatly simplify the calculations. We present the uncharged dust multipole for any order and show that is it divergence-free. We also show that it automatically satisfies the dynamic equations for a total stress-energy tensor. We propose a conjecture for the constitutive relations for the uncharged dust. We also show that it arises when one squeezes a regular dust stress-energy tensor onto a worldline. This squeezing procedure depends on the coordinates system and the same regular dust stress-energy tensor can give rise to different distributional stress-energy tensors.

In section 4 we repeat the process for a charged dust multipole. In this case we use a coordinate system adapted to a congruence of worldlines which satisfy the Lorentz force equation (for the same species). We derive the formula for divergence of the stress-energy tensor for a charged distribution for which there is no self interaction. We present the charged dust multipole for any order and show that its divergence satisfies this formula. We then derive the dynamical equations for the quadrupole moments of an arbitrary charged quadrupole, and show that it is satisfied by the dust quadrupole.

In section 5 we show how the above calculations can be performed in a coordinate free manner, using the exterior covariant derivative. This enables us to express the conjecture about the constitutive relations, in a coordinate free manner. It is also useful when expressing distributional quantities in coordinate systems not adapted to the flow. Arbitrary uncharged and charged multipoles up to quadrupole order were considered, and the equations for the components were derived.

Finally, in chapter 6 we conclude and discuss future work.

2 The stress-energy tensor in adapted Ellis coordinates

Let (M, g) be spacetime with the Levi-Civia connection. We use Greek indices for the range $\mu, \nu, \ldots = 0, 1, 2, 3$ and Latin indices for $a, b, \ldots = 1, 2, 3$, with implicitly summation for repeated indexes. Round brackets in the indices mean the complete symmetric sum of these indices, for example $\chi^{\mu\nu(abc)} = \frac{1}{6}(\chi^{\mu\nu abc} + \chi^{\mu\nu acb} + \chi^{\mu\nu bac} + \chi^{\mu\nu bca} + \chi^{\mu\nu cab)} + \chi^{\mu\nu cba)})$.

Since we are dealing with distributions it is most convenient to consider $T^{\mu\nu}$ as a tensor density[1] of weight 1. Thus $\omega^{-1}T^{\mu\nu}$ is a tensor, where $\omega = \sqrt{-\det(g_{\mu\nu})}$. The definition of the covariant derivative of a tensor $S^{\mu\nu\cdots}$ density of weight 1 is given by

$$\nabla_\mu S^{\nu\rho\cdots} = \omega\nabla_\mu(\omega^{-1}S^{\nu\rho\cdots}) = -\Gamma^{\kappa}_{\mu\kappa} S^{\nu\rho\cdots} + \partial_\mu S^{\nu\rho\cdots} + \Gamma^{\nu}_{\mu\kappa} S^{\kappa\rho\cdots} + \Gamma^{\rho}_{\mu\kappa} S^{\nu\kappa\cdots} + \cdots \tag{1}$$

[1] An integral over $\mathcal{M}$ must contain the measure ω. There is therefore the following choice: one can choose $T^{\mu\nu}$ or $\phi_{\mu\nu}$ to be a density of weight 1, or put ω explicitly in the integrand. Here we have chosen to make $T^{\mu\nu}$ a density.

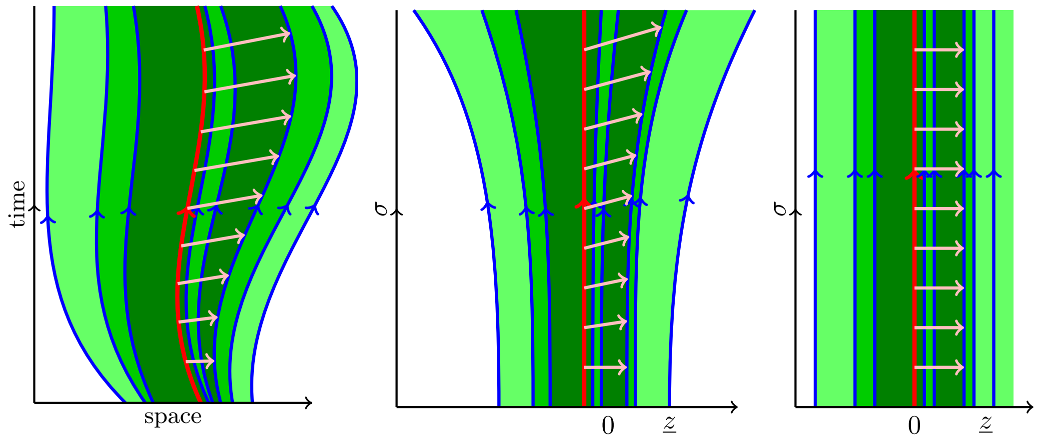

Figure 1: In all 3 diagrams the blue worldlines represent the flow lines of the material and also the contours of the density distribution. The red worldline is the particular flow line $C(\sigma)$. The left diagram is an arbitrary spacetime coordinates. In the middle diagram, the coordinates are adapted to the worldline $C(\sigma)$. In the right diagram, the coordinates are adapted to the flow. The pink arrow represent one of the components $\chi^{\mu\nu a}$. In the coordinate system adapted to the flow it is unchanged, whereas it changes in the other coordinate systems. When squeezing dust, the left or middle diagram would not give (40) below, whereas the left diagram will give (40).

so that if U^μ is a density of weight 1, then $\nabla_\mu U^\mu = \partial_\mu U^\mu$ In this article all distributions are considered to be Schwartz distributions. The stress-energy tensor $T^{\mu\nu}$ density distribution satisfies the symmetry condition

$$T^{\mu\nu} = T^{\nu\mu} \tag{2}$$

and the divergence-free condition

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{3}$$

It is defined by the way it acts on test tensors $\phi_{\mu\nu}$ of compact support via

$$\int_M T^{\mu\nu}\,\phi_{\mu\nu}\,d^4x\,. \tag{4}$$

There are several ways of writing the distributional stress-energy tensor. These include the Ellis representation in general coordinates, the Ellis representation in adapted coordinates, and the Dixon representation. There is also a coordinate free construction. For this work the Ellis representation in adapted coordinates greatly simplifies the calculations.

Let $C^\mu(\sigma)$ be the worldline which is the support of $T^{\mu\nu}$. We work in a coordinate system (σ, z^1, z^2, z^3) which is adapted to the worldline $C^\mu(\sigma)$, so that $C^\mu(\sigma) = (\sigma, 0, 0, 0)$ and $\dot{C}^\mu = \delta^\mu_0$. Let $\underline{z} = (z^1, z^2, z^3)$ denote the spatial coordinates. Different coordinate systems are depicted in Figure 1. In these adapted Ellis coordinates, the general multipole of order k can be written[2] as

$$T^{\mu\nu} = \sum_{r=0}^{k} \frac{1}{r!} \chi^{\mu\nu a_1\ldots a_r}(\sigma)\partial_{a_1}\ldots\partial_{a_r}\delta^{(3)}(\underline{z})\,, \tag{5}$$

[2] In this article we have slightly changed the notation compared to [1]. We have removed the trailing zeros in $\chi^{\mu\nu a_1\ldots a_r}$. This simplifies the notation when dealing with arbitrary order.

so that (4) becomes

$$\int_M T^{\mu\nu}\,\phi_{\mu\nu} d^4x = \sum_{r=0}^{k} \frac{(-1)^r}{r!} \int_{\mathbb{R}} \chi^{\mu\nu a_1\dots a_r}(\sigma)\;\partial_{a_1}\dots\partial_{a_r}\phi_{\mu\nu}(\sigma,\underline{0})\,. \tag{6}$$

From the symmetry (2) these components satisfy

$$\chi^{\mu\nu a_1\dots a_r} = \chi^{\nu\mu a_1\dots a_r}\,, \tag{7}$$

while from the commutation of partial derivatives we have

$$\chi^{\mu\nu a_1\dots a_r} = \chi^{\mu\nu(a_1\dots a_r)}\,. \tag{8}$$

By using squeezing, as we do below in section 3, we see that there is a relationship between the components $\chi^{\mu\nu a_1\dots a_r}$ and the moments of a regular stress-energy tensor.

At the quadrupole $k=2$ order (5) becomes

$$T^{\mu\nu} = \chi^{\mu\nu}(\sigma)\delta^{(3)}(\underline{z}) + \chi^{\mu\nu a}(\sigma)\partial_a\delta^{(3)}(\underline{z}) + \tfrac{1}{2}\chi^{\mu\nu ab}(\sigma)\partial_a\partial_b\delta^{(3)}(\underline{z})\,. \tag{9}$$

From the divergence-free condition (3) these components satisfy

$$\dot\chi^{\mu 0} = -\Gamma^{\mu}_{\nu\rho}\,\chi^{\rho\nu} + (\partial_a\Gamma^{\mu}_{\nu\rho})\,\chi^{\rho\nu a} - \tfrac{1}{2}\big(\partial_b\partial_a\Gamma^{\mu}_{\nu\rho}\big)\chi^{\rho\nu ab}\,, \tag{10}$$

$$\dot\chi^{\mu 0a} = -\chi^{\mu a} - \Gamma^{\mu}_{\nu\rho}\,\chi^{\rho\nu a} + (\partial_b\Gamma^{\mu}_{\nu\rho})\,\chi^{\rho\nu ba}\,, \tag{11}$$

$$\dot\chi^{\mu 0ab} = -2\chi^{\mu(ba)} - \Gamma^{\mu}_{\nu\rho}\,\chi^{\rho\nu ab} \tag{12}$$

and

$$\chi^{\mu(abc)} = 0\,. \tag{13}$$

This is proved in [1]. It is also a special case of theorem 7, when $F_{\mu\nu}=0$, which is proved below. In the case when the metric has Killing symmetries, then (10)-(12) leads to the conservation of corresponding current [1].

We observe that counting the number of independent components represented by these equations it a little more subtle than in the published literature [2–4]. This is due to the compatibility between (12) and (13). In (12), there are 6 ODEs involving $\dot\chi^{00ab}$ and 8 ODEs involving $\dot\chi^{0abc}$. This follows from setting $\mu=0$ in (13). Thus there are a total of 14 ODEs as opposed to the 24 ODEs given previously. The 8 independent ODEs may be chosen so that the left hand sides of (12) are

$$\dot\chi^{0112},\quad \dot\chi^{0113},\quad \dot\chi^{0122},\quad \dot\chi^{0133},\quad \dot\chi^{0223},\quad \dot\chi^{0233},\quad \dot\chi^{0123}\quad\text{and}\quad \dot\chi^{0231}\,. \tag{14}$$

As a result the right hand side of (11) is constrained by

$$2\chi^{(abc)} + \chi^{\rho\nu(bc}\,\Gamma^{a)}_{\nu\rho} = 0 \tag{15}$$

This gives a total of 30 ODEs as follows: the left hand sides of (10)-(12) are $\dot\chi^{\mu0}$ [4 equations], $\dot\chi^{\mu0a}$ [12 equations], $\dot\chi^{00ab}$ [6 equations], and the $\dot\chi^{\mu0ab}$ given in equation (14) [8 equations].

The number of independent components should also be corrected. This is given by $\chi^{\mu0}$ [4 components], χ^{ab} [6 components], $\chi^{\mu0a}$ [12 components], χ^{abc} (subject to

(15)) [8 components], χ^{00ab} [6 components], the χ^{0abc} given in (14) [8 components] and the χ^{abcd} subject to (13) [6 components]. Thus, there are a total of 50 components with 30 ODEs. Hence, there are 20 free components which must be determined by constitutive relations. These can now be specified as the χ^{ab}, χ^{abc} and χ^{abcd}. In section 3 we will give these constitutive relations.

As stated, the distributional stress-energy tensor, cannot be a source of gravity, but can be a source of gravitational waves. Let the perturbed metric be $\breve{g}_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h^{(1)}_{\mu\nu} + \cdots$ where $\kappa \ll 1$ is the small perturbation parameter and let $\mathcal{H}^{(1)}_{\mu\nu} = h^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(h^{(1)})^{\rho}{}_{\rho}$. The perturbation to the Einstein tensor $\breve{G}_{\mu\nu} = \kappa G^{(1)}_{\mu\nu} + \cdots$ and the stress-energy tensor $\breve{T}_{\mu\nu} = \kappa T^{(1)}_{\mu\nu} + \cdots$. The linearised Einstien's equations, in the Lorenz gauge, also called the de Donder gauge, become the gravitational wave equation $\partial_\rho \partial^\rho \mathcal{H}^{(1)}_{\mu\nu} = -16\pi T^{(1)}_{\mu\nu}$. In order to use the explicit formula [3] for $\mathcal{H}^{(1)}_{\mu\nu}$ it is necessary to express the stress-energy quadrupole $T^{\mu\nu}$ in the Dixon representation. Since we are in Minkowski spacetime and we are using Cartesian coordinates, the covariant derivatives are partial derivatives, and the stress-energy quadrupole becomes

$$T^{\mu\nu} = \int_{\mathcal{I}} \xi^{\mu\nu}\, \delta^{(4)}(z - C)\, d\sigma + \int_{\mathcal{I}} \xi^{\mu\nu\rho}\, \partial_\rho \delta^{(4)}(z - C)\, d\sigma + \tfrac{1}{2} \int_{\mathcal{I}} \xi^{\mu\nu\rho\sigma}\, \partial_\rho \partial_\sigma \delta^{(4)}(z - C)\, d\sigma, \, .$$

Here the Dixon components, $\xi^{\mu\nu\cdots}$, satisfy an orthogonality condition

$$N_\rho \xi^{\mu\nu\rho} = 0 \quad \text{and} \quad N_\rho \xi^{\mu\nu\rho\kappa} = 0 \tag{16}$$

where N_μ is called the Dixon Vector.

Lemma 1. *The Dixon components, $\xi^{\mu\nu\cdots}$ are given by the Ellis components $\chi^{\mu\nu\cdots}$ as*

$$\xi^{\hat\mu\hat\nu} = \gamma^{\hat\mu\hat\nu} + \ddot\beta_2^{\hat\mu\hat\nu} + \dot\beta_3^{\hat\mu\hat\nu} \tag{17}$$

$$\xi^{\hat\mu\hat\nu\hat\rho} = \gamma^{\hat\mu\hat\nu\hat\rho} + \dot\beta_1^{\hat\mu\hat\nu\hat\rho} + 2\dot\beta_2^{\hat\mu\hat\nu}\dot C^{\hat\rho} + \beta_2^{\hat\mu\hat\nu}\ddot C^{\hat\rho} + \beta_3^{\hat\mu\hat\nu}\dot C^{\hat\rho} \tag{18}$$

$$\xi^{\hat\mu\hat\nu\hat\rho\hat\sigma} = \gamma^{\hat\mu\hat\nu\hat\rho\hat\sigma} + \beta_1^{\hat\mu\hat\nu(\hat\rho}\dot C^{\hat\sigma)} + \beta_2^{\hat\mu\hat\nu}\dot C^{\hat\sigma}\dot C^{\hat\rho} \tag{19}$$

where

$$J_0^{\hat\mu} = \frac{\partial x^{\hat\mu}}{\partial\sigma}, \quad J_a^{\hat\mu} = \frac{\partial x^{\hat\mu}}{\partial z^a}, \quad J_{\mu\nu}^{\hat\mu\hat\nu} = J_\mu^{\hat\mu} J_\nu^{\hat\nu}, \quad \hat\partial_{\hat\rho} = \frac{\partial}{\partial x^{\hat\rho}} \tag{20}$$

$$\gamma^{\hat\mu\hat\nu} = \chi^{\mu\nu} J_{\mu\nu}^{\hat\mu\hat\nu} - \chi^{\mu\nu a} J_a^{\hat\rho}(\hat\partial_{\hat\rho} J_{\mu\nu}^{\hat\mu\hat\nu}) + \tfrac{1}{2}\chi^{\mu\nu ab} J_a^{\hat\rho}\big((\hat\partial_{\hat\rho} J_b^{\hat\sigma})(\hat\partial_{\hat\sigma} J_{\mu\nu}^{\hat\mu\hat\nu}) + J_b^{\hat\sigma}(\hat\partial_{\hat\rho}\hat\partial_{\hat\sigma} J_{\mu\nu}^{\hat\mu\hat\nu})\big)\,, \tag{21}$$

$$\gamma^{\hat\mu\hat\nu\hat\rho} = \chi^{\mu\nu a} J_a^{\hat\rho} J_{\mu\nu}^{\hat\mu\hat\nu} - \tfrac{1}{2}\chi^{\mu\nu ab}\big(J_a^{\hat\sigma}(\hat\partial_{\hat\sigma} J_b^{\hat\rho}) J_{\mu\nu}^{\hat\mu\hat\nu} + 2 J_a^{\hat\sigma} J_b^{\hat\rho}(\hat\partial_{\hat\sigma} J_{\mu\nu}^{\hat\mu\hat\nu})\big), \tag{22}$$

$$\gamma^{\hat\mu\hat\nu\hat\rho\hat\sigma} = \chi^{\mu\nu ab} J_a^{\hat\rho} J_b^{\hat\sigma} J_{\mu\nu}^{\hat\mu\hat\nu}\,, \tag{23}$$

$$\beta_1^{\hat\mu\hat\nu\hat\rho} = -\gamma^{\hat\mu\hat\nu\hat\rho\hat\sigma} N_{\hat\sigma}\beta_0^{-1}\,, \tag{24}$$

$$\beta_2^{\hat\mu\hat\nu} = \tfrac{1}{2}\gamma^{\hat\mu\hat\nu\hat\rho\hat\sigma} N_{\hat\rho} N_{\hat\sigma}\beta_0^{-2}\,, \tag{25}$$

$$\beta_3^{\hat\mu\hat\nu} = \beta_0^{-1}\gamma^{\hat\mu\hat\nu\hat\rho} N_{\hat\rho} - \beta_0^{-1}\dot\beta_1^{\hat\mu\hat\nu\hat\rho} N_{\hat\rho} - 2\dot\beta_2^{\hat\mu\hat\nu} - \beta_0^{-1}\beta_2^{\hat\mu\hat\nu}\ddot C^{\hat\rho} N_{\hat\rho}\,, \tag{26}$$

$$\beta_0 = \dot C^{\hat\rho} N_{\hat\rho} \tag{27}$$

Proof. Under the coordinate transformation

$$
\begin{aligned}
T^{\mu\nu}[\phi_{\mu\nu}] &= \int_{\mathcal{I}} \Big(\chi^{\mu\nu} J^{\hat\mu\hat\nu}_{\mu\nu} \hat\phi_{\hat\mu\hat\nu} - \chi^{\mu\nu a} J^{\hat\rho}_{a} \hat\partial_{\hat\rho} \big(J^{\hat\mu\hat\nu}_{\mu\nu} \hat\phi_{\hat\mu\hat\nu} \big) + \tfrac{1}{2} \chi^{\mu\nu ab} J^{\hat\rho}_{a} \hat\partial_{\hat\rho} \Big(J^{\hat\sigma}_{b} \hat\partial_{\hat\sigma} \big(J^{\hat\mu\hat\nu}_{\mu\nu} \hat\phi_{\hat\mu\hat\nu} \big) \Big) \Big) d\sigma \\
&= \int_{\mathcal{I}} \Big(\gamma^{\hat\mu\hat\nu} \hat\phi_{\hat\mu\hat\nu} - \gamma^{\hat\mu\hat\nu\hat\rho} \hat\partial_{\hat\rho} \hat\phi_{\hat\mu\hat\nu} + \tfrac{1}{2} \gamma^{\hat\mu\hat\nu\hat\rho\hat\sigma} \hat\partial_{\hat\rho} \hat\partial_{\hat\sigma} \hat\phi_{\hat\mu\hat\nu} \Big) d\sigma
\end{aligned}
\tag{28}
$$

where the $\gamma^{\hat\mu\hat\nu\cdots}$, (21)-(23), follow from expanding out the partial derivates. However these γ's do not satisfy the orthogonality condition (16) therefore we need to add some additional terms. Observe that since $\hat\phi_{\hat\mu\hat\nu}$ has compact support any total derivative vanishes under the integral. Thus we can write (28) as

$$
\begin{aligned}
T^{\mu\nu}[\phi_{\mu\nu}] = \int_{\mathcal{I}} \Big(& \gamma^{\hat\mu\hat\nu} \hat\phi_{\hat\mu\hat\nu} - \gamma^{\hat\mu\hat\nu\hat\rho} \hat\partial_{\hat\rho} \hat\phi_{\hat\mu\hat\nu} + \tfrac{1}{2} \gamma^{\hat\mu\hat\nu\hat\rho\hat\sigma} \hat\partial_{\hat\rho} \hat\partial_{\hat\sigma} \hat\phi_{\hat\mu\hat\nu} + \\
& \frac{d}{d\sigma} (\beta_1^{\hat\mu\hat\nu\hat\rho} \hat\partial_{\hat\rho} \hat\phi_{\hat\mu\hat\nu}) + \frac{d^2}{d\sigma^2} (\beta_2^{\hat\mu\hat\nu} \hat\phi_{\hat\mu\hat\nu}) + \frac{d}{d\sigma} (\beta_3^{\hat\mu\hat\nu} \hat\phi_{\hat\mu\hat\nu}) \Big) d\sigma
\end{aligned}
$$

and use the orthogonality condition to derive $\beta_1^{\hat\mu\hat\nu\hat\rho}$, $\beta_2^{\hat\mu\hat\nu}$, and $\beta_3^{\hat\mu\hat\nu}$. Now

$$
\begin{aligned}
\frac{d}{d\sigma} (\beta_1^{\hat\mu\hat\nu\hat\rho} \hat\partial_{\hat\rho} \hat\phi_{\hat\mu\hat\nu}) &= \dot\beta_1^{\hat\mu\hat\nu\hat\rho} \hat\partial_{\hat\rho} \hat\phi_{\hat\mu\hat\nu} + \beta_1^{\hat\mu\hat\nu\hat\rho} \dot C^{\hat\sigma} \hat\partial_{\hat\sigma} \hat\partial_{\hat\rho} \hat\phi_{\hat\mu\hat\nu} \\
\frac{d^2}{d\sigma^2} (\beta_2^{\hat\mu\hat\nu} \hat\phi_{\hat\mu\hat\nu}) &= \ddot\beta_2^{\hat\mu\hat\nu} \hat\phi_{\hat\mu\hat\nu} + 2 \dot\beta_2^{\hat\mu\hat\nu} \dot C^{\hat\rho} \hat\partial_{\hat\rho} \hat\phi_{\hat\mu\hat\nu} + \beta_2^{\hat\mu\hat\nu} \big(\ddot C^{\hat\sigma} \hat\partial_{\hat\sigma} \hat\phi_{\hat\mu\hat\nu} + \dot C^{\hat\sigma} \dot C^{\hat\rho} \hat\partial_{\hat\sigma} \hat\partial_{\hat\rho} \hat\phi_{\hat\mu\hat\nu} \big) \\
\frac{d}{d\sigma} (\beta_3^{\hat\mu\hat\nu} \hat\phi_{\hat\mu\hat\nu}) &= \dot\beta_3^{\hat\mu\hat\nu} \hat\phi_{\hat\mu\hat\nu} + \beta_3^{\hat\mu\hat\nu} \dot C^{\hat\rho} \hat\partial_{\hat\rho} \hat\phi_{\hat\mu\hat\nu}
\end{aligned}
$$

Gathering all the terms which have $\hat\partial_{\hat\sigma} \hat\partial_{\hat\rho} \hat\phi_{\hat\mu\hat\nu}$, we set $\xi^{\hat\mu\hat\nu\hat\rho\hat\sigma}$ by (19). We use the orthogonality condition to determine $\beta_1^{\hat\mu\hat\nu\hat\rho}$ and $\beta_2^{\hat\mu\hat\nu}$. From

$$
\begin{aligned}
0 = \tfrac{1}{2} \xi^{\hat\mu\hat\nu\hat\rho\hat\sigma} N_{\hat\sigma} &= (\tfrac{1}{2} \gamma^{\hat\mu\hat\nu\hat\rho\hat\sigma} + \tfrac{1}{2} \beta_1^{\hat\mu\hat\nu\hat\rho} \dot C^{\hat\sigma} + \tfrac{1}{2} \beta_1^{\hat\mu\hat\nu\hat\sigma} \dot C^{\hat\rho} + \beta_2^{\hat\mu\hat\nu} \dot C^{\hat\rho} \dot C^{\hat\sigma}) N_{\hat\sigma} \\
&= \tfrac{1}{2} \gamma^{\hat\mu\hat\nu\hat\rho\hat\sigma} N_{\hat\sigma} + \tfrac{1}{2} \beta_1^{\hat\mu\hat\nu\hat\rho} \beta_0 + \tfrac{1}{2} \beta_1^{\hat\mu\hat\nu\hat\sigma} N_{\hat\sigma} \dot C^{\hat\rho} + \beta_2^{\hat\mu\hat\nu} \dot C^{\hat\rho} \beta_0
\end{aligned}
$$

then after contracting again with $N_{\hat\rho}$ we get (24) and (25). Now gathering all the terms with $\hat\partial_{\hat\rho} \hat\phi_{\hat\mu\hat\nu}$ we have (18). Again contracting with $N_{\hat\rho}$ give (26). Finally gathering all the terms with just $\hat\phi_{\hat\mu\hat\nu}$ gives (17). □

It is necessary to choose N_μ to be parallel so that its components are constants. Using this N_μ we have the formula for the pertubation to the Minkowski metric, given by

$$
\begin{aligned}
\mathcal{H}^{\mu\nu}_{(1)}(t, \vec{x}) = \Bigg(& \frac{4\xi^{\mu\nu}}{r} + 4\xi^{\mu\nu\rho} \left(\frac{U_\rho}{\alpha^2 r} - \frac{\dot C_\rho}{r^2} \right) \\
& + 2\xi^{\mu\nu\rho\sigma} \left(\frac{2\dot C_\rho \dot C_\sigma}{r^3} - \frac{2U_\rho \dot C_\sigma}{r^2\,\alpha} + \frac{\eta_{\rho\sigma}}{r\alpha^2} + \frac{U_\sigma U_\rho}{r\alpha^3} \right) \Bigg) \Bigg|_{\sigma=\sigma_{\mathrm{R}}}
\end{aligned}
\tag{29}
$$

where $\sigma = \sigma_{\mathrm{R}}$ is the retarded time, $U^\mu = x^\mu - C^\mu(\sigma)$, $r = \dot C^\mu(\sigma) U_\mu$, and $\alpha = U^\mu N_\mu$.

In order to establish the dust multipole, discussed below, does correspond to dust, we compare it to the standard regular dust stress-energy tensor. This regular tensor is squeezed down to the worldline to become distributional [1]. In this, we start with a regular stress-energy tensor density $\mathcal{T}^{\mu\nu}(\sigma, \underline{z})$, of weight 1, which has compact support in the transverse $\underline{z}$–planes, and construct a one–parameter family of regular stress-energy tensor densities, of weight 1,

$$\mathcal{T}_\epsilon^{\mu\nu}(\sigma, \underline{z}) = \epsilon^{-3}\ \mathcal{T}^{\mu\nu}\big(\sigma, \epsilon^{-1}\underline{z}\big)\,. \tag{30}$$

In the weak limit $\mathcal{T}_\epsilon^{\mu\nu} \to T^{\mu\nu}$ at $\epsilon \to 0$ to order k,

$$\mathcal{T}_\varepsilon^{\mu\nu}(\sigma, \underline{z}) = \sum_{r=0}^{k} \frac{\epsilon^r}{r!} \chi^{\mu\nu a_1\cdots a_r}\, \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) + O(\epsilon^{k+1}) \tag{31}$$

where

$$\chi^{\mu\nu a_1\cdots a_r}(\sigma) = (-1)^r \int_{\mathbb{R}^3} d^3\underline{z}\ z^{a_1}\cdots z^{a_r}\mathcal{T}^{\mu\nu}\big(\sigma, \underline{z}\big)\,. \tag{32}$$

and the symbol $O(\epsilon^{k+1})$ means that any difference falls to zero as fast as ϵ^{k+1}. The proof is an easy generalisation of the proof given in [1] to arbitrary order. It should be noted that the components $\chi^{a_1\cdots a_r}$ are dependent on the adapted coordinate system. Changing from coordinates $(\sigma, \underline{z})$ to $(\sigma', \underline{z}')$ will result if different moments. In general it is not possible to perform a coordinate transformation for (32) as the surfaces of constant σ will be different from the surfaces of constant σ'. This is discussed in [14], where it highlights that usefulness of the distributional approach with regards to coordinate transformations.

3 Uncharged dust

We express the formula for uncharged dust in an adapted coordinate system $(\sigma, \underline{z})$. This coordinate system is adapted to a congruence of worldlines. Thus each curve by given $z^a = \text{const.}$ for $a = 1, 2, 3$ is a geodesic. Hence the Christoffel symbols satisfy

$$\Gamma^\mu_{00} = 0\,. \tag{33}$$

From (1), then setting $U^\mu = \delta^\mu_0$ to be a vector density of weight 1 we have $\nabla_\mu \delta^\mu_0 = \partial_\mu \delta^\mu_0 = 0\,..$

We can now formulate the dust multipole stress-energy tensor, in terms of this adapted coordinate system. As we stated in the introduction, this has not been considered previously in the literature, except for a brief mention in [1]. This is a tensor densities of order k and weight 1, given by

$$T^{\mu\nu} = m\, \delta^\mu_0 \delta^\nu_0 \sum_{r=0}^{k} \frac{1}{r!} Y^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z})\,, \tag{34}$$

where each $Y^{a_1\cdots a_r}$ is a constant and satisfies the symmetries (7) and (8), and $Y^\emptyset = 1$. Here $Y^\emptyset$ refers to case when there are no indices on Y, i.e. $r = 0$. This mass could be incorporated into the $Y^{a_1\cdots a_r}$. However it is needed in the charged case when we need the ratio q/m. We will show that this stress-energy tensor satisfies the divergence-free condition (3) and is also the limit of regular dust as it is squeezed onto the worldline.

Lemma 2. *The stress-energy distribution given in* (34) *satisfies the divergence-free condition* (3).

Proof. Since $T^{\mu\nu}$ is a tensor density of weight 1, we can choose any of the factors on the right hand side of (34) to carry the tensor density. We choose the factor δ_0^μ to have weight 1 and the rest of the factors to have weight 0. Thus $\nabla_\mu \delta_0^\mu = 0$. From (33)

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= m\delta_0^\mu (\nabla_\mu \delta_0^\nu) \sum_{r=0}^{k} \frac{1}{r!} Y^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \\
&\quad + m\delta_0^\mu \delta_0^\nu \sum_{r=0}^{k} \frac{1}{r!} \partial_\mu \big(Y^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \big) \\
&= m\Gamma_{00}^\nu \sum_{r=0}^{k} \frac{1}{r!} Y^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \\
&\quad + m\delta_0^\nu \sum_{r=0}^{k} \frac{1}{r!} \partial_0 \big(Y^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \big) \\
&= 0\,.
\end{aligned}$$

□

The uncharged dust quadrupole is obtained by setting $k = 2$ in (34),

$$T^{\mu\nu} = m\delta_0^\mu \delta_0^\nu \Big(\delta(\underline{z}) + Y^a \partial_a \delta(\underline{z}) + \tfrac{1}{2} Y^{ab} \partial_a \partial_b \delta(\underline{z}) \Big)\,. \tag{35}$$

Lemma 3. *As a check, we can show that at quadrupole order the dust stress-energy tensor* (34) *satisfies equation* (10)*–*(13)*.*

Proof. From (34) we see $\chi^{\rho\nu\sigma\cdots} = \delta_0^\rho \delta_0^\nu Y^{\sigma\cdots}$. Hence, from (33), it is trivial to see that the right hand sides of (10)-(12) vanish. Likewise for the left hand side of (13). Since $\chi^{\rho\nu\cdots}$ are constant the left hand side of (10)-(12) also vanish. □

We can now state necessary and sufficient conditions, which when combined with symmetry and divergence free imply the dust quadrupole.

Lemma 4. *Let $T^{\mu\nu}$ be an uncharged quadrupole satisfying,* (10)-(13) *in a coordinate system adapted to geodesic flow. Then $T^{\mu\nu}$ correspond to uncharged dust,* (35) *if and only if components $\chi^{\mu\nu\cdots}$ satisfy*

$$\chi^{a\mu} = 0, \quad \chi^{a\mu b} = 0 \quad \text{and} \quad \chi^{a\mu bc} = 0\,. \tag{36}$$

Proof. If $T^{\mu\nu}$ is given by (35), then trivially (36) holds. By contrast if (36) holds, then the constraints (13) and (15) are satisfied. The ODEs for dust are now given by $\dot{\chi}^{00} = 0$, $\dot{\chi}^{00a} = 0$ and $\dot{\chi}^{00ab} = 0$, which implies (35). □

We can express (36) in a coordinate invariant manner, with respect the a geodesic flow $V^\mu \nabla_\mu V^\nu = 0$ as $T^{\mu\nu} = V^\mu V^\nu \hat{\rho}$, for some distribution $\hat{\rho}$.

The number of equations in (36) is 50. This is too many for the constitutive relations. We conjecture that only a subset is needed. One possibility are the 20 equations $\chi^{ab} = 0$, $\chi^{abc} = 0$ and $\chi^{abcd} = 0$, which correspond to

$$T^{\mu\nu} = V^{(\mu} \hat{T}^{\nu)} \tag{37}$$

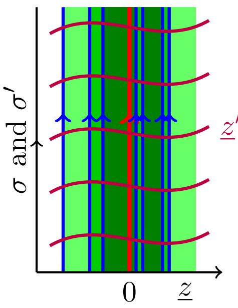

Figure 2: This figure depicts two different coordinate systems, both adapted to the flow: The rectangular coordinate system $(\sigma, \underline{z})$ and an alternative coordinate system $(\sigma', \underline{z}')$, The lines of constant σ' are depicted in purple. The moments generated by squeezing this dust will be different for the two coordinate systems.

for some distribution $\hat{T}^{\nu}$. The freedom to choose the remaining components may correspond to coordinate transformation of the existing dust model. If this is true that would imply that the dust constitutive relations (37) imply the dust stress-energy tensor. Thus we have the following conjecture:

Conjecture 5. *A stress-energy distribution $T^{\mu\nu}$ with support on a worldline $C(\sigma)$ satisfying* (2) *and* (3)*, with constitutive relations* (37)*, is the uncharged dust multipole given, which in adapted coordinates is given by* (34)*.*

In section 5 we show how to construct the dust multipole without coordinates, so that we can write coordinate free statement of this conjecture.

We can understand (34) as a model for dust by taking the squeezed limit of the regular dust stress-energy tensor, given by

$$\mathcal{T}^{\mu\nu} = \varrho(\underline{z})\delta_0^\mu \delta_0^\nu\,, \tag{38}$$

where ϱ is a scalar field (density of weight 0) and δ_0^μ is a vector density of weight 1. Thus $\nabla_\mu \delta_0^\mu = \partial_\mu\, \delta_0^\mu = 0$ and $\partial_0\big(\varrho(\underline{z})\big) = 0$. We see that

$$\begin{aligned}
\nabla_\mu \mathcal{T}^{\mu\nu} \quad &= \nabla_\mu\big(\varrho(\underline{z})\delta_0^\mu\delta_0^\nu\big)\\
&= \delta_0^\nu\delta_0^\mu\nabla_\mu\varrho(\underline{z}) + \delta_0^\nu\varrho(\underline{z})\nabla_\mu\delta_0^\mu + \varrho(\underline{z})\delta_0^\mu\nabla_\mu\big(\delta_0^\nu\big)\\
&= \delta_0^\nu\delta_0^\mu\partial_\mu\big(\varrho(\underline{z})\big) + \varrho(\underline{z})\nabla_0\big(\delta_0^\nu\big)\\
&= \delta_0^\nu\partial_0\big(\varrho(\underline{z})\big) + \varrho(\underline{z})\Gamma_{00}^\mu\\
&= 0\,.
\end{aligned}$$

From (30),

$$\mathcal{T}_\epsilon^{\mu\nu} = \epsilon^{-3}\varrho(\epsilon^{-1}z)\delta_0^\mu\delta_0^\nu\,. \tag{39}$$

Lemma 6. *The Taylor expansion about $\epsilon = 0$, to order k, is given by*

$$\mathcal{T}_\epsilon^{\mu\nu} = \hat{T}_\epsilon^{\mu\nu} + O(\epsilon^{k+1})\,, \tag{40}$$

where

$$\hat{T}_\epsilon^{\mu\nu} = m\,\delta_0^\mu\delta_0^\nu\sum_{r=0}^{k}\frac{\epsilon^r}{r!}Y^{a_1\cdots a_r}\partial_{a_1}\cdots\partial_{a_r}\delta^{(3)}(\underline{z})\,, \tag{41}$$

and

$$Y^{a_1\cdots a_r} = \frac{(-1)^r}{m}\int_{\mathbb{R}^3} z^{a_1}\cdots z^{a_r}\,\varrho(\underline{z})\,d^3\underline{z} \tag{42}$$

Proof. This follows from setting $w^a = z^a/\varepsilon$ and Taylor expanding around $\varepsilon = 0$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^4} \mathcal{T}_\varepsilon^{\mu\nu}(\sigma,\underline{z})\,\phi_{\mu\nu}(\sigma,\underline{z})\,d\sigma\,d^3z\\
&\quad= \int_{\mathbb{R}} d\sigma \int_{\mathbb{R}^3} d^3z\ \mathcal{T}_\varepsilon^{\mu\nu}(\sigma,\underline{z})\,\phi_{\mu\nu}(\sigma,\underline{z})\\
&\quad= \int_{\mathbb{R}} m\ d\sigma \int_{\mathbb{R}^3} d^3z\ \frac{1}{m}\epsilon^{-3}\varrho(\epsilon^{-1}z)\delta_0^\mu\delta_0^\nu\ \phi_{\mu\nu}(\sigma,\underline{z})\\
&\quad= \int_{\mathbb{R}} m\ d\sigma \int_{\mathbb{R}^3} d^3\underline{w}\ \frac{1}{m}\varrho(\underline{w})\,\phi_{00}(\sigma,\varepsilon\underline{w})\\
&\quad= \int_{\mathbb{R}} m\ d\sigma \int_{\mathbb{R}^3} d^3\underline{w}\ \frac{1}{m}\sum_{r=0}^{k}\frac{\epsilon^r}{r!}\varrho(\underline{w})\,w^{a_1}\cdots w^{a_r}\big(\partial_{a_1}\cdots\partial_{a_r}\phi_{00}(\sigma,\underline{0})\big) + O(\epsilon^{k+1})\\
&\quad= \sum_{r=0}^{k}\frac{\epsilon^r}{r!}\int_{\mathbb{R}} m\ d\sigma\big(\partial_{a_1}\cdots\partial_{a_r}\phi_{00}(\sigma,\underline{0})\big)\int_{\mathbb{R}^3} d^3\underline{w}\ \frac{1}{m}w^{a_1}\cdots w^{a_r}\ \varrho(\underline{w}) + O(\epsilon^{k+1})\\
&\quad= \sum_{r=0}^{k}\frac{\epsilon^r(-1)^r}{r!}\int_{\mathbb{R}} d\sigma\ mY^{a_1\cdots a_r}\big(\partial_{a_1}\cdots\partial_{a_r}\phi_{00}(\sigma,\underline{0})\big) + O(\epsilon^{k+1})\\
&\quad= \sum_{r=0}^{k}\frac{\epsilon^r}{r!}\int_{\mathbb{R}} m\ d\sigma\int_{\mathbb{R}^3} d^3\underline{z}Y^{a_1\cdots a_r}\big(\partial_{a_1}\cdots\partial_{a_r}\delta^{(3)}(\underline{z})\big)\big(\phi_{00}(\sigma,\underline{z})\big) + O(\epsilon^{k+1})\\
&\quad= \int_{\mathbb{R}} d\sigma\int_{\mathbb{R}^3} d^3\underline{z}\sum_{r=0}^{k}\frac{\epsilon^r}{r!}m\delta_0^\mu\delta_0^\nu Y^{a_1\cdots a_r}\big(\partial_{a_1}\cdots\partial_{a_r}\delta^{(3)}(\underline{z})\big)\big(\phi_{\mu\nu}(\sigma,\underline{z})\big) + O(\epsilon^{k+1})\\
&\quad= \int_{\mathbb{R}^4} d\sigma\,d^3\underline{z}\ \hat{T}_\epsilon^{\mu\nu}\ \phi_{\mu\nu}(\sigma,\underline{z}) + O(\epsilon^{k+1})\,.
\end{aligned}$$

□

Clearly setting $\epsilon = 1$ we have $\hat{T}_1^{\mu\nu} = T^{\mu\nu}$. However the nature of (40) is more subtle, since we cannot simply set $\epsilon = 1$. There are various interpretations. One option is to choose a total error $\mathcal{E}_{\max}$. Then from (40) there is a value of ϵ such that $|\mathcal{T}_\epsilon^{\mu\nu} - \hat{T}_\epsilon^{\mu\nu}| < \mathcal{E}_{\max}$, for all components. One can then redefine the $Y^{a_1\cdots a_r} \to \epsilon^r Y^{a_1\cdots a_r}$ to incorporate this value of ϵ. Then $|\mathcal{T}_\epsilon^{\mu\nu} - T^{\mu\nu}| < \mathcal{E}_{\max}$. Furthermore by replacing $\epsilon \to \epsilon/2$ we reduce the error by $\mathcal{E}_{\max} \to 2^{-k-1}\mathcal{E}_{\max}$.

We observe that in the results of lemmas 2 and 3 the use of the coordinate system adapted to geodesic flow is purely for convenience and the result is independent of the coordinate system. This can be seen via the coordinate independent approach, given in section 5 below. By contrast, the definition of $\mathcal{T}_\epsilon^{\mu\nu}$ (39) depends on the coordinate system as seen in figure 1. Furthermore, there are different coordinate systems which can be adapted to the geodesic flow (figure 2), and the moments generated by squeezing this dust depend on the choice of coordinate systems.

4 Charged Dust

In contrast to uncharged dust, the stress-energy tensor is not divergence-free. This is because it is not a total stress-energy tensor. Instead we have

$$T^{\mu\nu}_{\text{total}} = T^{\mu\nu}_{\text{mat}} + T^{\mu\nu}_{\text{EM}}\,, \tag{43}$$

where $T^{\mu\nu}_{\text{total}}$ is the total stress-energy tensor, and $T^{\mu\nu}_{\text{mat}}$ and $T^{\mu\nu}_{\text{EM}}$ are the contributions from the charged dust and the electromagnetic field. Since $\nabla_\mu T^{\mu\nu}_{\text{total}} = 0$, then $\nabla_\mu T^{\mu\nu}_{\text{mat}} = -\nabla_\mu T^{\mu\nu}_{\text{EM}}$. Since for regular charged dust and electromagnetic field which are both smooth

$$\nabla_\mu T^{\mu\nu}_{\text{EM}} = -g^{\nu\rho} F^{\text{reg}}_{\rho\mu}\, J^{\mu}_{\text{reg}}\ . \tag{44}$$

Thus, we have

$$\nabla_\mu T^{\mu\nu}_{\text{mat}} = g^{\nu\rho} F^{\text{reg}}_{\rho\mu}\, J^{\mu}_{\text{reg}}\ , \tag{45}$$

where the current J^{μ}_{reg} is given by Maxwell's equation

$$J^{\mu}_{\text{reg}} = \nabla_\nu F^{\nu\mu}_{\text{reg}}\,. \tag{46}$$

However, since we are dealing with Schwartz distributions, both $T^{\mu\nu}$ and J^μ are delta-functions which are infinite along the worldline. The problem is that, from (46) the components of $F_{\rho\mu}$ also diverge as one approaches the worldline and thus (44) is not defined at the worldline. This leads to all the questions about what is the correct equation of motion when a point charged particle responds to its own electromagnetic field. In this article, we avoid this problem by making the $F_{\mu\nu} = F^{\text{Ext}}_{\mu\nu}$ an external electromagnetic field which does not satisfy (46). That is

$$J^\mu \neq \nabla_\nu F^{\nu\mu}\,. \tag{47}$$

Thus, we demand that a distributional stress-energy tensor $T^{\mu\nu}$, with a corresponding distributional current J^μ, in the presence of an external electromagnetic $F^{\mu\nu}$ satisfies the divergence equation

$$\nabla_\mu T^{\mu\nu} = g^{\nu\rho} F_{\rho\mu}\, J^{\mu}\ . \tag{48}$$

In [9] the problem of self interaction was solved by making each particle respond to the electromagnetic field of all the other particles. However, this approach relied on the fact that the components $F_{\mu\nu} \sim R^{-2}$ as one approached the worldline, where $R = |\underline{z}|$ is the distance to the worldline. However, since we are dealing with higher order multipoles then we would have $F_{\mu\nu} \sim R^{-k-2}$. This diverges to quickly and this approach may no longer work.

In Ellis representation and adapted coordinates the current is given by [1]

$$J^\mu = \sum_{r=0}^{k} \frac{1}{r!} \gamma^{\mu a_1\cdots a_r}(\sigma) \partial_{a_1} \ldots \partial_{a_r} \delta^{(3)}(\underline{z})\,, \tag{49}$$

where $\gamma^{\mu a_1\cdots a_r} = \gamma^{\mu(a_1\cdots a_r)}$. These are subject to the constraint arising from the conservation of charge

$$\nabla_\mu J^\mu = 0\,. \tag{50}$$

At the octupole level

$$J^\mu = \gamma^\mu(\sigma)\delta^{(3)}(\underline{z}) + \gamma^{\mu a}(\sigma)\partial_a\delta^{(3)}(\underline{z}) + \frac{1}{2}\gamma^{\mu ab}(\sigma)\partial_a\partial_b\delta^{(3)}(\underline{z}) + \frac{1}{6}\gamma^{\mu abc}(\sigma)\partial_a\partial_b\partial_c\delta^{(3)}(\underline{z}) \tag{51}$$

and (49) gives rise to the conditions

$$\dot\gamma^0 = 0\,,\quad \dot\gamma^{0a} = -\gamma^a\,,\quad \dot\gamma^{0ab} = -2\gamma^{(ab)}\,,\quad \dot\gamma^{0abc} = -3\gamma^{(abc)}\,,\quad \gamma^{(abcd)} = 0\,. \tag{52}$$

The first of this equation implies the conservation of total charge $\gamma^0 = q$. This is a very underdetermined system. At this octupole order there are $4\times(1+3+6+10) = 80$ components with 15 algebraic equations, giving 65 unknowns. However, there are only 20 ODEs.

We wish to establish the general dynamic equations for the $\chi^{\mu\nu\cdots}$ components of an arbitrary quadrupole stress-energy tensor. That is to generalise (10)-(13). Since, in (44) we differentiate $T^{\mu\nu}$, to be most general we consider J^μ to be an octupole, $k = 3$.

Theorem 7. *The stress-energy quadrupole given by* (9) *satisfies the divergence condition* (48)*, with current given by* (51)*, if and only if*

$$\begin{aligned}
&\dot\chi^{\mu 0} + \Gamma^\mu_{\nu\rho}\,\chi^{\rho\nu} - \chi^{\rho\nu a}\,\partial_a\Gamma^\mu_{\nu\rho} + \tfrac{1}{2}\chi^{\rho\nu ab}\partial_b\partial_a\Gamma^\mu_{\nu\rho}\\
&\qquad = \gamma^\rho F^\mu{}_\rho - \gamma^{\rho a}\partial_a F^\mu{}_\rho + \tfrac{1}{2}\gamma^{\rho ab}\partial_b\partial_a F^\mu{}_\rho - \tfrac{1}{6}\gamma^{\rho abc}\partial_c\partial_b\partial_a F^\mu{}_\rho\,, && (53)\\
&\dot\chi^{\mu 0 a} + \chi^{\mu a} + \Gamma^\mu_{\nu\rho}\,\chi^{\rho\nu a} - (\partial_b\Gamma^\mu_{\nu\rho})\,\chi^{\rho\nu ba}\\
&\qquad = \gamma^{\rho a}F^\mu{}_\rho - \gamma^{\rho ab}\partial_b F^\mu{}_\rho + \frac{1}{2}\gamma^{\rho abc}\partial_c\partial_b F^\mu{}_\rho\,, && (54)\\
&\dot\chi^{\mu 0ab} + 2\chi^{\mu(ba)} + \Gamma^\mu_{\nu\rho}\,\chi^{\rho\nu ab} = \gamma^{\rho(ab)}F^\mu{}_\rho - \gamma^{\rho(ab)c}\partial_c F^\mu{}_\rho\,, && (55)\\
&\chi^{\mu(abc)} = \frac{1}{3}\gamma^{\rho(abc)}F^\mu{}_\rho\,. && (56)
\end{aligned}$$

Proof. We have that

$$\int_{\mathcal{M}} (\nabla_\mu T^{\mu\nu})\,\theta_\nu\, d^4x = \int_{\mathcal{M}} (g^{\nu\rho}F_{\rho\mu}\,J^\mu)\,\theta_\nu\,d^4x\,, \tag{57}$$

where θ^ν is a test vector. Then

$$\begin{aligned}
&\int_{\mathcal{M}} (\nabla_\mu T^{\mu\nu})\,\theta_\nu\, d^4x = \int_{\mathcal{M}} \big(\partial_\mu T^{\mu\nu} + \Gamma^\nu_{\mu\rho}T^{\mu\rho}\big)\,\theta_\nu\, d^4x = \int_{\mathcal{M}} T^{\mu\nu}\big(\Gamma^\rho_{\mu\nu}\,\theta_\rho - \partial_\mu\theta_\nu\big)\,d^4x\\
&= \int_{\mathcal{M}} \Big(\chi^{\mu\nu}\,\delta^{(3)}(\underline{z}) + \chi^{\mu\nu a}\,\partial_a\delta^{(3)}(\underline{z}) + \tfrac{1}{2}\chi^{\mu\nu ab}\,\partial_a\partial_b\delta^{(3)}(\underline{z})\Big)\big(\Gamma^\rho_{\mu\nu}\,\theta_\rho - \partial_\mu\theta_\nu\big)\,d^4x\\
&= \int_{\mathcal{I}} d\sigma\Big(\chi^{\mu\nu}\,\big(\Gamma^\rho_{\mu\nu}\,\theta_\rho - \partial_\mu\theta_\nu\big) - \chi^{\mu\nu a}\,\partial_a\big(\Gamma^\rho_{\mu\nu}\,\theta_\rho - \partial_\mu\theta_\nu\big) + \tfrac{1}{2}\chi^{\mu\nu ab}\partial_a\partial_b\big(\Gamma^\rho_{\mu\nu}\,\theta_\rho - \partial_\mu\theta_\nu\big)\Big)\\
&= \int_{\mathcal{I}} d\sigma\Big(\chi^{\mu\nu}\,\Gamma^\rho_{\mu\nu}\,\theta_\rho - \chi^{a\nu}\,\partial_a\theta_\nu + \dot\chi^{0\nu}\,\theta_\nu\\
&\qquad - \chi^{\mu\nu a}\,\partial_a\big(\Gamma^\rho_{\mu\nu}\,\theta_\rho\big) + \chi^{b\nu a}\,\partial_a\partial_b\theta_\nu - \dot\chi^{0\nu a}\,\partial_a\theta_\nu\\
&\qquad + \tfrac{1}{2}\chi^{\mu\nu ab}\partial_a\partial_b\big(\Gamma^\rho_{\mu\nu}\,\theta_\rho\big) - \tfrac{1}{2}\chi^{c\nu ab}\partial_a\partial_b\partial_c\theta_\nu + \tfrac{1}{2}\dot\chi^{0\nu ab}\partial_a\partial_b\theta_\nu\Big)\\
&= \int_{\mathcal{I}} d\sigma\Big(\chi^{\mu\nu}\,\Gamma^\rho_{\mu\nu}\,\theta_\rho - \chi^{a\nu}\,\partial_a\theta_\nu + \dot\chi^{0\rho}\,\theta_\rho
\end{aligned}$$

$$
\begin{aligned}
&\quad - \chi^{\mu\nu a}\,(\partial_a \Gamma^\rho_{\mu\nu})\,\theta_\rho - \chi^{\mu\nu a}\,\Gamma^\rho_{\mu\nu}\,\partial_a\theta_\rho + \chi^{b\nu a}\,\partial_a\partial_b\theta_\nu - \dot\chi^{0\nu a}\,\partial_a\theta_\nu \\
&\quad + \tfrac12 \chi^{\mu\nu ab}\big(\partial_a\partial_b\Gamma^\rho_{\mu\nu}\big)\theta_\rho + \chi^{\mu\nu ab}\big(\partial_a\Gamma^\rho_{\mu\nu}\big)\,\big(\partial_b\theta_\rho\big) + \tfrac12\chi^{\mu\nu ab}\Gamma^\rho_{\mu\nu}\partial_a\partial_b\theta_\rho \\
&\quad - \tfrac12\chi^{c\nu ab}\partial_a\partial_b\partial_c\theta_\nu + \tfrac12\dot\chi^{0\nu ab}\partial_a\partial_b\theta_\nu\Big) \\
&= \int_{\mathcal I} d\sigma\Big(\theta_\rho\Big(\chi^{\mu\nu}\,\Gamma^\rho_{\mu\nu} + \dot\chi^\rho - \chi^{\mu\nu a}\,(\partial_a\Gamma^\rho_{\mu\nu}) + \tfrac12\chi^{\mu\nu ab}\big(\partial_a\partial_b\Gamma^\rho_{\mu\nu}\big)\Big) \\
&\quad - \partial_a\theta_\rho\Big(\chi^{a\rho} + \chi^{\mu\nu a}\,\Gamma^\rho_{\mu\nu} + \dot\chi^{0\rho a} - \chi^{\mu\nu ba}\big(\partial_b\Gamma^\rho_{\mu\nu}\big)\Big) \\
&\quad + \partial_a\partial_b\theta_\rho\Big(\chi^{b\rho a} + \tfrac12\chi^{\mu\nu ab}\Gamma^\rho_{\mu\nu} + \tfrac12\dot\chi^{0\rho ab}\Big) - \tfrac12\chi^{c\nu ab}\partial_a\partial_b\partial_c\theta_\nu\Big)\,.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\int_{\mathcal M} g^{\nu\rho}F_{\rho\mu}\,J^\mu\,\theta_\nu\,d^4x \\
&\quad = \int_{\mathcal M} g^{\nu\rho}F_{\rho\mu}\,\Big(\gamma^\mu(\sigma)\delta^{(3)}(\underline z) + \gamma^{\mu a}(\sigma)\partial_a\delta^{(3)}(\underline z) + \frac12\gamma^{\mu ab}(\sigma)\partial_a\partial_b\delta^{(3)}(\underline z) \\
&\qquad\qquad + \frac16\gamma^{\mu abc}(\sigma)\partial_a\partial_b\partial_c\delta^{(3)}(\underline z)\Big)\,\theta_\nu\,d^4x \\
&\quad = \int_{\mathcal M}\,\Big(\gamma^\mu(\sigma)\delta^{(3)}(\underline z) + \gamma^{\mu a}(\sigma)\partial_a\delta^{(3)}(\underline z) + \frac12\gamma^{\mu ab}(\sigma)\partial_a\partial_b\delta^{(3)}(\underline z) \\
&\qquad + \frac16\gamma^{\mu abc}(\sigma)\partial_a\partial_b\partial_c\delta^{(3)}(\underline z)\Big)\,\Big(g^{\nu\rho}F_{\rho\mu}\theta_\nu\Big)\,d^4x \\
&\quad = \int_{\mathcal I} d\sigma\Big(\gamma^\mu g^{\nu\rho}F_{\rho\mu}\theta_\nu - g^{\nu\rho}\gamma^{\mu a}\partial_a(F_{\rho\mu})\theta_\nu - g^{\nu\rho}\gamma^{\mu a}F_{\rho\mu}\partial_a\theta_\nu + \frac12 g^{\nu\rho}\gamma^{\mu ab}\partial_a\partial_b(F_{\rho\mu})\theta_\nu \\
&\qquad + \frac12 g^{\nu\rho}\gamma^{\mu ab}F_{\rho\mu}\partial_a\partial_b\theta_\nu + \frac12 g^{\nu\rho}\gamma^{\mu ab}\partial_a(F_{\rho\mu})\partial_b\theta_\nu + \frac12 g^{\nu\rho}\gamma^{\mu ab}\partial_b(F_{\rho\mu})\partial_a\theta_\nu \\
&\qquad - \frac16 g^{\nu\rho}\gamma^{\mu abc}\partial_a\partial_b\partial_c(F_{\rho\mu})\theta_\nu - \frac16 g^{\nu\rho}\gamma^{\mu abc}F_{\rho\mu}\partial_a\partial_b\partial_c\theta_\nu - \frac16 g^{\nu\rho}\gamma^{\mu abc}\partial_a\partial_b(F_{\rho\mu})\partial_c\theta_\nu \\
&\qquad - \frac16 g^{\nu\rho}\gamma^{\mu abc}\partial_c(F_{\rho\mu})\partial_a\partial_b\theta_\nu - \frac16 g^{\nu\rho}\gamma^{\mu abc}\partial_b\partial_c(F_{\rho\mu})\partial_a\theta_\nu - \frac16 g^{\nu\rho}\gamma^{\mu abc}\partial_a(F_{\rho\mu})\partial_b\partial_c\theta_\nu \\
&\qquad - \frac16 g^{\nu\rho}\gamma^{\mu abc}\partial_a\partial_c(F_{\rho\mu})\partial_b\theta_\nu - \frac16 g^{\nu\rho}\gamma^{\mu abc}\partial_b(F_{\rho\mu})\partial_a\partial_c\theta_\nu\Big) \\
&\quad = \int_{\mathcal I} d\sigma\Big(\theta_\rho\Big(\gamma^\mu g^{\rho\nu}F_{\nu\mu} - g^{\rho\nu}\gamma^{\mu a}\partial_a(F_{\nu\mu}) + \frac12 g^{\rho\nu}\gamma^{\mu ab}\partial_a\partial_b(F_{\nu\mu}) \\
&\qquad - \frac16 g^{\rho\nu}\gamma^{\mu abc}\partial_a\partial_b\partial_c(F_{\nu\mu})\Big) - \partial_a\theta_\rho\Big(g^{\rho\nu}\gamma^{\mu a}F_{\nu\mu} - \frac12 g^{\rho\nu}\gamma^{\mu ab}\partial_b(F_{\nu\mu}) \\
&\qquad - \frac12 g^{\rho\nu}\gamma^{\mu ba}\partial_b(F_{\nu\mu}) + \frac16 g^{\rho\nu}\gamma^{\mu abc}\partial_b\partial_c(F_{\nu\mu}) + \frac16 g^{\rho\nu}\gamma^{\mu bac}\partial_b\partial_c(F_{\nu\mu}) \\
&\qquad + \frac16 g^{\rho\nu}\gamma^{\mu cba}\partial_b\partial_c(F_{\nu\mu})\Big) + \partial_a\partial_b\theta_\rho\Big(\frac12 g^{\rho\nu}\gamma^{\mu ab}F_{\nu\mu} - \frac16 g^{\rho\nu}\gamma^{\mu abc}\partial_c(F_{\nu\mu}) \\
&\qquad - \frac16 g^{\rho\nu}\gamma^{\mu bac}\partial_c(F_{\nu\mu}) - \frac16 g^{\rho\nu}\gamma^{\mu cba}\partial_c(F_{\nu\mu})\Big) - \frac16\partial_a\partial_b\partial_c\theta_\rho\Big(g^{\rho\nu}\gamma^{\mu abc}F_{\nu\mu}\Big)\Big) \\
&\quad = \int_{\mathcal I} d\sigma\Big(\theta_\rho\Big(\gamma^\mu F^\rho{}_\mu - \gamma^{\mu a}\partial_a F^\rho{}_\mu + \frac12\gamma^{\mu ab}\partial_a\partial_b F^\rho{}_\mu - \frac16\gamma^{\mu abc}\partial_a\partial_b\partial_c F^\rho{}_\mu\Big) \\
&\qquad - \partial_a\theta_\rho\Big(\gamma^{\mu a}F^\rho{}_\mu - \gamma^{\mu ab}\partial_b F^\rho{}_\mu + \frac12\gamma^{\mu abc}\partial_b\partial_c F^\rho{}_\mu\Big) \\
&\qquad + \partial_a\partial_b\theta_\rho\Big(\frac12\gamma^{\mu ab}F^\rho{}_\mu - \frac12\gamma^{\mu abc}\partial_c F^\rho{}_\mu\Big) - \frac16\partial_a\partial_b\partial_c\theta_\rho\Big(\gamma^{\mu abc}F^\rho{}_\mu\Big)\Big)\,.
\end{aligned}
$$

Thus, equating the two sides in (57), we obtain (53) - (56).

□

It is trivial to see that if $F_{\mu\nu} = 0$ in (53)-(56) then we recover (10)-(13). Thus there are 40 ODEs and 60 $\chi^{\mu\nu\cdots}$ components as in the uncharged case.

If both $\chi^{\mu\nu\cdots}$ and $\gamma^{\mu\cdots}$ are unknown, then combining with the conservation of charge we have 50 ODEs for 115 unknowns. Some of the additional constitutive relations would describe how the mass is related to the charge in the point source. This would correspond to how the mass and charge is distributed in the object. The charged dust considered below, has the distribution of mass coinciding with the distribution of charge. This happens because we are modelling charged dust of a single species.

The components $\chi^{\mu\nu\cdots}$, can be interpreted as moments using (32), and equations (53)-(56). They describe how the charged matter responds both to the curvature and the external electromagnetic field. In the case when the metric has Killing symmetries, then (53)-(56) corresponds to how energy or momentum is lost or gained from the matter $\xi^{\mu\nu\cdots}$, due to the electromagnetic field. However, as these are effectively test particles, this energy-momentum is not gained or lost by the background electromagnetic field. The first order moments $\gamma^{\mu a}$ correspond to a distribution which has more charge on one than on the other. These will naturally couple to the derivative electromatic field, as can be seen in (53). The interpretation of the higher order moments is similar, but clearly more complicated.

4.1 Charged dust stress-energy tensor and current

Here we formulate the charged dust multipole stress-energy tensor and current. We again work in an adapted coordinate system (σ, z^1, z^2, z^3). However, this time each curve given by $\underline{z} =$ constant is a solution to the Lorentz force equation with the same ratio q/m. Hence, the Christoffel symbols satisfy

$$\Gamma^{\mu}_{00} = \frac{q}{m} F^{\mu}{}_{0} \,. \tag{58}$$

The stress-energy tensor density (of weight 1) has the same structure as (34) but in this new coordinate system. That is

$$T^{\mu\nu} = m\, \delta^{\mu}_{0} \delta^{\nu}_{0} \sum_{r=0}^{k} \frac{1}{r!} Z^{a_1 \cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \,. \tag{59}$$

where $Z^{\emptyset} = 1$ and the $Z^{a_1\cdots a_r}$ are constants. In this model the distribution of charge is the same as the distribution of matter. Thus we are considering only a single species of charged particle. The current density (of weight 1) is given by

$$J^{\mu} = q\, \delta^{\mu}_{0} \sum_{r=0}^{k} \frac{1}{r!} Z^{a_1 \cdots a_r} \partial_{a_1} \ldots \partial_{a_r} \delta^{(3)}(\underline{z}) \,. \tag{60}$$

We can see that (60) trivially satisfies the conservation of charge (50), since using $\nabla_\mu \delta^{\mu}_{0} = 0$

$$\nabla_\mu J^{\mu} = q\delta^{\mu}_{0} \sum \partial_\mu \big(\tfrac{1}{r!} Z^{a_1\cdots a_r} \partial_{a_1} \ldots \partial_{a_r} \delta^{(3)}(\underline{z})\big)$$

$$
\begin{aligned}
&= q \sum \partial_0 (\tfrac{1}{r!} Z^{a_1\cdots a_r} \partial_{a_1} \ldots \partial_{a_r} \delta^{(3)}(\underline{z})) \\
&= 0 \,.
\end{aligned} \tag{61}
$$

Theorem 8. *The stress-energy tensor* (59) *satisfies the divergence condition* (45) *where the current is given by* (60)*.*

Proof. We assume that δ_0^μ is the factor with the weight 1, so that $\nabla_\mu \delta_0^\mu = 0$.

$$
\begin{aligned}
\nabla_\mu T^{\mu\nu} &= m(\nabla_\mu \delta_0^\mu)\delta_0^\nu \sum_{r=0}^{k} \frac{1}{r!} Z^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \\
&\quad + m\delta_0^\mu (\nabla_\mu \delta_0^\nu) \sum_{r=0}^{k} \frac{1}{r!} Z^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \\
&\quad + m\delta_0^\mu \delta_0^\nu \sum_{r=0}^{k} \frac{1}{r!} \partial_\mu \big(Z^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \big) \\
&= m\Gamma^\nu_{00} \sum_{r=0}^{k} \frac{1}{r!} Z^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \\
&= m(q/m) F^\nu{}_0 \sum_{r=0}^{k} \frac{1}{r!} Z^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \\
&= q F^\nu{}_\mu \delta_0^\mu \sum_{r=0}^{k} \frac{1}{r!} Z^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \\
&= q g^{\nu\rho} F_{\rho\mu} \delta_0^\mu \sum_{r=0}^{k} \frac{1}{r!} Z^{a_1\cdots a_r} \partial_{a_1} \cdots \partial_{a_r} \delta^{(3)}(\underline{z}) \\
&= g^{\nu\rho} F_{\rho\mu} J^\mu \,.
\end{aligned}
$$

□

The dust stress-energy tensor (59) and current (60) to quadrupole order are given by

$$
T^{\mu\nu} = m\delta_0^\mu \delta_0^\nu \Big(\delta(\underline{z}) + Z^a \partial_a \delta(\underline{z}) + Z^{ab} \partial_a \partial_b \delta(\underline{z}) \Big) \tag{62}
$$

and

$$
J^\mu = q\delta_0^\mu \Big(\delta(\underline{z}) + Z^a \partial_a \delta(\underline{z}) + Z^{ab} \partial_a \partial_b \delta(\underline{z}) \Big) \,. \tag{63}
$$

Lemma 9. *The charged dust stress-energy quadrupole* (62) *and current quadrupole* (63) *satisfy the ODEs* (53)*-*(56)*.*

Proof. We wish to verify that the equations (53) - (56) hold. The $\dot{\chi}^{\mu\nu\cdots}$ terms vanish while the $\chi^{\mu\nu\cdots}$ terms are non-zero when the first two indices are equal to zero. The condition on the Christoffel symbols for charged dust is (58). As one can see in equations (53) and (54), there are exact cancellations between the gravitational and electromagnetic terms in the case of a dust model. Note the for a given multipole order, the χ and γ terms encompass the same X constants since $\chi^{\mu\nu a_1\cdots a_r} = m\delta_0^\mu \delta_0^\nu Z^{a_1\cdots a_r}$ and $\gamma^{\mu a_1\cdots a_r} = q\delta_0^\mu Z^{a_1\cdots a_r}$.

Moreover, differentiating (58) with respect to a spatial component gives $\partial_b(\Gamma^\mu_{00}) = (q/m)\partial_b(F^\mu{}_0)$ and $\partial_b\partial_c(\Gamma^\mu_{00}) = (q/m)\partial_b\partial_c(F^\mu{}_0)$. In (55), exact cancellations also arise. Both sides of equation (56) vanish as well. □

For charged dust, we need to provide the constitutive relations for both the stress-energy tensor and the current. As before these will need to be augmented by additional equations to get the form (62) and (63). A possible set of conditions

$$\chi^{\mu a} = 0, \quad \chi^{\mu ab} = 0, \quad \chi^{\mu abc} = 0, \quad \gamma^a = 0, \quad \gamma^{ab} = 0, \quad \gamma^{abc} = 0 \quad \text{and} \quad \gamma^{\mu abc} = 0 \tag{64}$$

However even these are not sufficient as χ^{00} and J^0 are related. Thus we would have to include the following:

$$\chi^{00} = (q/m)J^0, \quad \chi^{00a} = (q/m)J^{0a} \quad \text{and} \quad \chi^{00ab} = (q/m)J^{0ab} \tag{65}$$

Only once we combine (64), (65) and (53)-(56) do we deduce (62) and (63).

To demonstrate that this is charged dust, one can repeat the calculation in lemma 6. In addition to confirm the current one can squeeze the regular current given by

$$\mathcal{J}^{\mu\nu}_\epsilon = \epsilon^{-3}\varrho(\epsilon^{-1}z)\delta^\nu_0\omega\,. \tag{66}$$

As a simple example of a source of gravitational waves, consider negatively charged dust, orbiting a positive charge at non-relativistic velocities, in a Minkowski background. The Minkowski background is chosen so we can use (29) for the gravitational waves. We expand around the point $C^0(\sigma) = \sigma$, $C^a(\sigma) = 0$. In general it would be necessary to work out the orbits and covert this into a coordinate system. This would encode how the dust deforms due to different orbiting speeds. For the moment lets assume that the radial spread of the dust is small. Since the central positive charge is very high we assume it does not respond the the orbiting dust. In this case we can have a monopole term given by $\chi^{00} = 1$ and a dipole term, Given by $\chi^{001} = q_1$ is a constant. Everything else we set to zero, effectively truncating at dipole. We can now use lemma 1 to convert this into Minkowski coordinates. Choosing $N_\mu = \delta^0_\mu$, we have $\xi^{\hat\mu\hat\nu} = q\delta^{\hat\mu}_0\delta^{\hat\nu}_0$, $\xi^{\hat\mu\hat\nu 1} = q_1\cos(\omega\sigma)$ and $\xi^{\hat\mu\hat\nu 2} = q_1\sin(\omega\sigma)$, with other ξ's being. From (29) we have

$$\mathcal{H}^{\hat\mu\hat\nu}_{(1)} = \delta^{\hat\mu}_0\delta^{\hat\nu}_0(4qr^{-1} + 4q_1\cos(\omega t)r^{-3}) \tag{67}$$

5 The coordinate free de Rham formulation of the stress-energy tensor

The results in this article can all be reproduced in a coordinate free notation using the language of differential geometry and de Rham currents. This is very useful when one needs to express distributional quantities such as the current and stress-energy tensor in a coordinate system which is not adapted to the flow (e.g. a coordinate system adapted to the observer, rather than the source). The transformation of the components for these quantities under change of coordinates is complicated [1, 4], involving higher order derivatives and integrals. By using a coordinate free notation, the components in a preferred coordinate system can then be extracted. In section 2 we showed how to transform from an adapted coordinate system to Cartesian coordinates on Minkowski spacetime, in order to express the gravitational wave formula (29). However, if one were interested in the gravitation waves from matter orbiting a black hole, one may need to consider several coordinate systems, including the Schwarzschild coordinates, the Eddington-Finkelstein coordinates, the coordinate system adapted to flow of the dust, and the retarded time coordinates associated with the observer. In such a scenario, it is useful having a construction of the dust stress-energy tensor in a coordinate free way. Another advantage is being able to express the conjecture 5, without reference to any coordinate system.

The detail of how to construct the stress-energy distribution is given in [1, Section 6] and summarised here. Even though all the work can be repeated in this language, here we only reproduce the key result, theorem 8.

Given that $C : \mathcal{I} \to \mathcal{M}$, is a closed embedding, the DeRham push forward with respect to C of a p–form, $\alpha \in \Gamma\Lambda^p\mathcal{I}$ is given by the distribution $C_\varsigma(\alpha)$, given by

$$\big(C_\varsigma(\alpha)\big)[\varphi] = \int_{\mathcal{I}} C^\star(\varphi) \wedge \alpha\,. \tag{68}$$

where φ is a test form of degree 0 or 1 and $C^\star(\varphi)$ is the pullback of $\varphi \in \Gamma\Lambda^q\mathcal{M}$ to $\Gamma\Lambda^q\mathcal{I}$. This has degree $\deg\big(C_\varsigma(\alpha)\big) = 3 + p$. A general p–form distribution is then given by applying an arbitrary number of sums, wedge products, exterior derivatives, internal contraction and Lie derivatives, to $C_\varsigma(\alpha)$ using the rules $\Psi_1 + \Psi_2)[\varphi] = \Psi_1[\varphi] + \Psi_2[\varphi]$, $(\beta \wedge \Psi)[\varphi] = \Psi[\varphi \wedge \beta]$, $(d\Psi)[\varphi] = (-1)^{(3-p)}\Psi[d\varphi]$, $(i_v\Psi)[\varphi] = (-1)^{(3-p)}\Psi[i_v\varphi]$, and $(L_v\Psi)[\varphi] = -\Psi[L_v\varphi]$ where Ψ is an arbitrary distribution.

This formulation is sufficient to construct the electromagnetic current 3–form [4]. However, the stress-energy distribution τ acts on a test tensor of type $(0, 2)$. The general such tensor distribution is constructed via

$$(\Psi \otimes X)[\varphi \otimes \alpha] = \Psi[(\alpha : X)\,\varphi]\,. \tag{69}$$

where $\alpha : Y$ is the internal product between the 1–form α and the vector X.

The stress-energy tensor distribution τ can be used to define the stress-energy 3–form [13] distribution τ_α via $\tau_\alpha[\theta] = \tau[\theta \otimes \alpha]$, so that $\tau = \tau^\mu \otimes \partial_\mu$ where $\tau^\mu = \tau_{dx^\mu}$.

The symmetry condition (2) is given by

$$\tau[\beta \otimes \alpha] = \tau[\alpha \otimes \beta]\,, \tag{70}$$

and the divergenceless condition (3) is given by

$$D\tau = 0\,, \tag{71}$$

where

$$(D\tau)[\theta] = -\tau[D\theta] \tag{72}$$

and

$$(D\theta)(X, Y) = (\boldsymbol{\nabla}_Y\theta) : X\,. \tag{73}$$

Here $\boldsymbol{\nabla}_Y$ is the coordinate free covariant derivative. This covariant derivative knows about the tensor structure, but not the indices, so that $\boldsymbol{\nabla}_X Y^\mu = X^\nu\,\partial_\nu Y^\mu$. It is related to the Christoffel symbols via $\Gamma^\mu_{\nu\rho}\,\partial_\mu = \boldsymbol{\nabla}_{\partial_\nu}\partial_\rho$.

The relationship between the stress-energy forms and the tensor density $T^{\mu\nu}$ is given by

$$\int_{\mathcal{I}} T^{\mu\nu}\,\phi_{\mu\nu}\,d^4x = \tau^\mu[\phi_{\mu\nu}\,dx^\nu] = \tau[\phi_{\mu\nu}\,dx^\nu \otimes dx^\mu]\,. \tag{74}$$

Using this coordinate system, (70) becomes

$$dx^\mu \wedge \tau^\nu = dx^\nu \wedge \tau^\mu\,, \tag{75}$$

and (71) becomes

$$d(\tau^\mu) + \Gamma^\mu_{\nu\rho}\,dx^\rho \wedge \tau^\nu = 0\,. \tag{76}$$

We apply this to the charged dust stress-energy tensor. The right hand side of (48) is written $\mathcal{F}\wedge J$ where J is a current distribution and $\mathcal{F}$ encodes the Maxwell 2–form F. For a test 1–form α

$$(\mathcal{F}\wedge J)[\alpha] = -J[i_{\widetilde{\alpha}}F]\,, \tag{77}$$

where $\widetilde{\alpha}$ is the metric dual of α. As discussed above in section 4, F is the external electromagnetic field so, from (47), $dF \neq J$. Equation (48) becomes

$$D\tau = \mathcal{F}\wedge J\,. \tag{78}$$

In order to construct symmetric stress-energy tensors we introduce the symmetry operator, Sym, where

$$\mathrm{Sym}(\alpha\otimes\beta) = \tfrac{1}{2}\alpha\otimes\beta + \tfrac{1}{2}\beta\otimes\alpha\,. \tag{79}$$

Lemma 10. *Let θ be a 1–form, then*

$$\big(\mathrm{Sym}\,D(\theta)\big)(Y,X) = \big(\boldsymbol{\nabla}_V\theta - \tfrac{1}{2}i_V d\theta\big) : X\,, \tag{80}$$

where i_V is the internal contraction.

Proof.

$$\begin{aligned}
2\,\mathrm{Sym}\,D(\theta)(X,Y) &= D(\theta)(X,Y) + D(\theta)(Y,X) = \boldsymbol{\nabla}_X\theta : Y + \boldsymbol{\nabla}_V\theta : X \\
&= X(\theta : Y) - \theta : \boldsymbol{\nabla}_X Y + \boldsymbol{\nabla}_V\theta : X \\
&= L_X\theta : Y - \theta : (\boldsymbol{\nabla}_X Y + [X,Y]) + \boldsymbol{\nabla}_V\theta : X \\
&= L_X\theta : Y - \theta : \boldsymbol{\nabla}_V X + \boldsymbol{\nabla}_V\theta : X = i_V L_X\theta + Y(\theta : X) + 2\boldsymbol{\nabla}_V\theta : X \\
&= i_V i_X d\theta + i_V d i_X\theta - Y(\theta : X) + 2\boldsymbol{\nabla}_V\theta : X = (-i_V d\theta + 2\boldsymbol{\nabla}_V\theta) : X\,.
\end{aligned}$$

□

We can write (80) without the arbitrary vector X using the slot notation as

$$\big(\mathrm{Sym}\, D(\theta)\big)(Y,-) = \boldsymbol{\nabla}_Y\theta - \tfrac{1}{2} i_Y d\theta\,. \tag{81}$$

If κ is a distribution which acts on test tensors of type (0,2) then we define its symmetry as

$$\mathrm{Sym}(\kappa)[\phi] = \kappa[\mathrm{Sym}(\phi)]\,. \tag{82}$$

Also if κ is a distribution which acts on test 1–forms then we define

We can now define the stress-energy and current distribution for dust in a coordinate free manner. Let V be the vector field such that the multipole trajectory C is an integral curve and it is the flow under the Lorentz force,

$$\nabla_V V = \widetilde{i_V F}\,. \tag{83}$$

Let $W_1,\ldots,W_k \in \Gamma TM$ be a set of vector fields such that

$$[W_j, V] = 0\,. \tag{84}$$

Let the current for the dust multipole be given by

$$J = q L_{W_1}\cdots L_{W_K} C_\varsigma(1) \tag{85}$$

and the corresponding stress-energy multipole

$$\tau = \mathrm{Sym}\big(m L_{W_1}\cdots L_{W_k} C_\varsigma(1)\otimes V\big)\,. \tag{86}$$

Here $C_\varsigma(1)$ is the de Rham push forward, given in [1, Section 6]. By definition τ is symmetric. From linearity we can construct any current and stress-energy tensor by adding together an arbitary number of J and τ. To compare (85) and (86) with (60) and (59), we observe that we set $V=\partial_0$ and the W_j as the coordinate vectors ∂_{a_i}. We then act on a test 2–form. We show here it also satisfies the divergence property (78).

Theorem 11. *The stress-energy multipole τ and current multipole J given by* (86) *and* (85) *satisfy the divergence condition* (78)*.*

Proof. Let θ be a test 1–form

$$\begin{aligned}
D\tau[\theta] &= -\tau[D(\theta)] = -\mathrm{Sym}\big(m L_{W_1}\cdots L_{W_k} C_\varsigma(1)\otimes V\big)[D(\theta)]\\
&= -m\big(L_{W_1}\cdots L_{W_k} C_\varsigma(1)\otimes V\big)[\mathrm{Sym}D(\theta)]\\
&= -m L_{W_1}\cdots L_{W_k} C_\varsigma(1)[\mathrm{Sym}D(\theta)(V,-)]\\
&= -m L_{W_1}\cdots L_{W_k} C_\varsigma(1)[\nabla_V\theta - \tfrac{1}{2} i_V d\theta]\\
&= (-1)^{k+1} m C_\varsigma(1)[L_{W_k}\cdots L_{W_1}(\nabla_V\theta - \tfrac{1}{2} i_V d\theta)]\\
&= (-1)^{k+1} m \int C^\star\big(L_{W_k}\cdots L_{W_1}(\nabla_V\theta - \tfrac{1}{2} i_V d\theta)\big)\\
&= (-1)^{k+1} m \int d\sigma\, C^\star\big(i_{\dot C} L_{W_k}\cdots L_{W_1}(\nabla_V\theta - \tfrac{1}{2} i_V d\theta)\big)\\
&= (-1)^{k+1} m \int d\sigma\, C^\star\big(i_V L_{W_k}\cdots L_{W_1}(\nabla_V\theta - \tfrac{1}{2} i_V d\theta)\big)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{k+1} m \int d\sigma\, C^\star\big(L_{W_k}\cdots L_{W_1}(i_V\nabla_V\theta - \tfrac{1}{2} i_V i_V d\theta)\big) \\
&= (-1)^{k+1} m \int d\sigma\, C^\star\big(L_{W_k}\cdots L_{W_1}(i_V\nabla_V\theta)\big) \\
&= (-1)^{k+1} m \int d\sigma\, C^\star\big(L_{W_k}\cdots L_{W_1}(L_V(\theta : V) - \theta : \nabla_V V)\big) \\
&= (-1)^{k+1} m \int d\sigma\, C^\star\big(L_V L_{W_k}\cdots L_{W_1}(\theta : V)\big) \\
&\qquad + (-1)^{k} q \int d\sigma\, C^\star\big(L_{W_k}\cdots L_{W_1}(\theta : \widetilde{i_V F})\big) \\
&= (-1)^{k+1} m \int dC^\star\big(L_{W_k}\cdots L_{W_1}(\theta : V)\big) \\
&\qquad + (-1)^{k+1} q \int d\sigma\, C^\star\big(L_{W_k}\cdots L_{W_1}(i_V i_{\widetilde{\theta}} F)\big) \\
&= (-1)^{k+1} q \int d\sigma\, C^\star\big(i_V L_{W_k}\cdots L_{W_1}(i_{\widetilde{\theta}} F)\big) \\
&= (-1)^{k+1} q \int C^\star\big(L_{W_k}\cdots L_{W_1}(i_{\widetilde{\theta}} F)\big) \\
&= (-1)^{k+1} q C_\varsigma(1)[L_{W_k}\cdots L_{W_1}(i_{\widetilde{\theta}} F)] \\
&= -q L_{W_1}\cdots L_{W_k} C_\varsigma(1)[i_{\widetilde{\theta}} F] = -J[i_{\widetilde{\theta}} F] = \big(\mathcal{F}\wedge J\big)[\theta]\,.
\end{aligned}
$$

□

We can now express the conjecture 5 for uncharged dust (setting $q = 0$) in a coordinate free manner.

Conjecture 12. *A stress-energy distribution τ with support on a worldline $C(\sigma)$ satisfying* (70) *and* (71)*, with constitutive relations*

$$
\tau = \mathrm{Sym}(\hat{\tau} \otimes V) \tag{87}
$$

for some 3–form distribution $\hat{\tau}$, is the uncharged dust multipole given by (86).

There should also exist an equivalent conjecture for the charged dust multipole

6 Conclusion and discussion

In this paper we consider the stress-energy multipole for both charged and uncharged dust. These are distributions which have support on a worldline. We demand that both are symmetric and that uncharged dust satisfies the divergence-free condition, whilst the divergence of charged dust is related to the current and the external electromagnetic field.

The required divergence of the charged multipole (44) is subtle. Since the electromagnetic field of the generated by a multipole would diverge on the worldline (and this divergence is very fast), we cannot simply equate the divergence of the multipole stress-energy tensor with the divergence of the electromagnetic stress-energy tensor. Instead, inspired by the divergence of the electromagnetic stress-energy tensor, we posit the required equation (34).

We formulate both the charged and uncharged stress-energy multipoles to arbitrary order. We show how they satisfy the required conditions and also how they arise naturally in the limit as one squeezes regular dust onto the worldline. These are particularly simple in the Ellis representation of multipoles, with coordinates adapted to a flow of geodesics or the Lorentz force equation. In this case the components are constants.

Although the multipoles are simple in the adapted coordinate system, and therefore their properties hold in all coordinate systems, the formula for transformation between coordinate systems is complicated. They are not tensorial as they involve both higher derivatives and integration [1, 4]. For this reason, in section 5, we also show how the general results can be demonstrated in a coordinate free language.

We consider arbitrary uncharged and charged multipoles up to quadrupole order and derive the equations for the components. Then as a sanity check, we confirm that the components of the charged and uncharged dust multipoles, when truncated to quadrupoles, do indeed satisfy these equations.

In [1], we observed that, at the quadrupole order, the divergence equations, are not sufficient to determine the dynamics of the components. For uncharged dust, there are 30 equations for 50 variables, assuming the worldline is prescribed. This corrects an error in [1–3]. Thus there is a need for constitutive relations to fully describe the dynamics. To date, there is very little known about different constitutive relations. In section 3 we conjecture the constitutive relations for uncharged dust. The constraint in (37) is necessary, as shown in lemma 4, however it is not clear if they are sufficient. Assuming it is sufficient, conjectures 5 and 12, then this could form the template for other constitutive relations. The long term goal would therefore be to have constitutive relations for orbiting masses, rotating asymmetric objects, plasmas, etc. Once we have these constitutive relations, they can be combined with (10)-(13), to give the dynamics of the multipole and then, in the case of a Minkowski background, combined with (29) to give the gravitational waves. In theory, with extremely precise gravitational wave detection at multiple locations, one could use (29) to measure the dynamics of components of $T^{\mu\nu}$. This could then be compared to the various models.

In section 5, we noted that if one wished to model dust in a strong gravitational field, such as a black hole, there would be several useful coordinate systems. As well as coordinate system adapted to the solutions of the geodesic or Lorentz force equation, one could also use coordinates adapted to the black hole. Thus it was useful to have a coordinate free definition of the multipole. For the observation of gravitational waves, a useful coordinate system would be backward lightcone coordinates, adapted to the observer. In general these are challenging to derive, as it involves solving the lightlike geodesic equation, in every direction from the observer. If, in addition, one had the retarded Greens function for this metric, then one could combine this with the source give a formula for the gravitation waves generated by the dust multipole.

As noted in the introduction the distributional stress-energy tensor on a worldline cannot be a source of Einstein's equation [8]. Several approaches have been investigated for handling non-linear differential equations which involve distributions, especially with applications to Einstein's equations. One of these [15–17] is to use the Colombeau algebra of generalised functions. In this approach there is a parameter $\epsilon > 0$. As $\epsilon \to 0$ the generalised functions become distributions. This is usually just a formal parameter with no obvious physical interpretation. It would be natural to ask if one could promote the regular dust stress-energy tensors, given in (30) to generalised functions and consider the squeezing parameter ϵ in (30) to be Colombeau parameter.

As stated in the introduction, this work can be applied to various branches of physics. The uncharged dust is a good model nebula, or even galactic systems. The dynamics of the quadrupole components are directly related to gravitational waves [3]. Thus, one can compare the detected gravitational waves with those by the dust model. For example, when we are able to detect primordial gravitational waves we could ask if these are consistent with dust quadrupoles. One mathematical calculation which would need to be performed is to express the components in a coordinate system adapted to us as observers, instead of the geodesic flow of the source. In this context the coordinate free language will be invaluable.

There are many other stress-energy multipoles one could consider. Examples include kinetic, pressure and spin.

In a kinetic model, there is a range of velocities at each event in spacetime, and one must work in 7-dimensional time-phase space. This range of velocities is incorporated into the kinetic "distribution"[3] scalar field on 7–dimensional phase-space-time space. Collisionless charged particles obey the Vlasov equation [18], which describes the time evolution of the kinetic distribution function of plasma, consisting of charged particles (electrons and ions) with long-range interaction. In [14,19,20], the dynamics of the components of a Vlasov multipole on phase-space-time space are given. Using the Ellis representation of the de Rham current representation of the moments, coordinate transformations were derived [14] between frames that mix the space and time coordinates. The results were confirmed numerically for the case of particles orbiting a black hole. The current and stress-energy distributions, corresponding to the Vlasov distribution, can be derived by projecting the distribution onto spacetime using the de Rham push forward.

Another possibility is to consider a fluid with a pressure. This model should arise in the limit as one squeezes a fluid with pressure onto the worldline. However, this cannot be done naively as, unlike dust, the pressure, directly opposes such squeezing. We conjecture that it would be possible if in (40), the pressure acts at order ϵ^2 and higher.

As noted in the introduction, cosmic dust does not possess any total spin. In order to introduce spin, one could look into the Weyssenhoff dust model [21, 22], which includes a factor of spacetime torsion to model spin. Employing the adapted coordinate system, one may be able to compute the dynamics of the moments of the Weyssenhoff dust multipole.

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[3] A different use of the word *distribution*.

Author Contributions

All authors did the research regarding the uncharged dust and the coordinate free calculations. JG and ST did the research regarding charged dust. WS reviewed the Weyssenhoff dust model. JG and ST wrote the manuscript.

Conflict of interest statement

There are no conflicts of interest.

Ethics statement

This work did not involve human participants, animal subjects, or sensitive data, and therefore did not require ethical approval. No ethical issues were identified in the course of this research.

Data Access Statement

No additional data was created for this article.

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