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MAPPING FOOD INSECURITY IN THE BRAZILIAN AMAZON USING A SPATIAL ITEM FACTOR ANALYSIS MODEL

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Food insecurity, a latent construct defined as the lack of consistent access to sufficient and nutritious food, is a pressing global issue with serious health and social justice implications. Item factor analysis is commonly used to study such latent constructs, but it typically assumes independence between sampling units. In the context of food insecurity, this assumption is often unrealistic, as food access is linked to socio-economic conditions and social relations that are spatially structured. To address this, we propose a *spatial item factor analysis* model that captures spatial dependence, allowing us to predict latent factors at unsampled locations and identify food insecurity hotspots. We develop a Bayesian sampling scheme for inference and illustrate the explanatory strength of our model by analysing household perceptions of food insecurity in Ipixuna, a remote river-dependent urban centre in the Brazilian Amazon. Our approach is implemented in the R package `spifa`, with further details provided in the Supplementary Material. This spatial extension offers policymakers and researchers a stronger tool for understanding and addressing food insecurity to locate and prioritise areas in greatest need. Our proposed methodology can be applied more widely to other spatially structured latent constructs.

1. Introduction.

1.1. *Food insecurity and item response theory.* *Food security* is a latent construct that characterizes a 'situation that exists when all people, at all times, have physical, social and economic access to sufficient, safe and nutritious food that meets their dietary needs and food preferences for an active and healthy life' (FAO, 2003, pp 313). The latent construct *food insecurity* describes the opposite situation, in which individual or household access to sufficient, safe and nutritious food is unreliable or insufficient. Understanding the level of food insecurity in a region is crucial for designing appropriate public health and nutrition interventions.

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Food insecurity has been traditionally studied using *item response theory* (IRT), which is a family of statistical models used to relate *responses to items* to a *latent construct*. These models assume the latent construct (e.g. food insecurity) or *ability* is defined on a continuum. This allows us, for instance to score each individual's level or ability; to identify which items have the greatest capacity to *discriminate* between individuals of differing abilities or levels (i.e. how well each item identifies the trait of food insecurity); or to identify the *difficulty* or easiness associated to each item—more 'difficult' items in this context would tend to be endorsed by more food-insecure individuals, but less often by food-secure individuals (de Ayala, 2022).

Item response theory is widely applied across diverse fields of research beyond food insecurity analysis. In psychometrics, for example, it has been used to measure the theory of mind ability (Black, 2019), emotional intelligence (Fiori et al., 2014), and self-esteem (Crowe et al., 2018). In health and medicine, it is used to determine the health status of patients using self-reported outcomes (Proust-Lima et al., 2022), to measure individual scores of child developmental status (Drachler, Marshall and De Carvalho Leite, 2007) and to assess achievement and evaluation of clinical performance (De Champlain, 2010). In mental health research, it has been used to study disorders like psychopathy (Tsang et al., 2018), alcohol use (Saha, Chou and Grant, 2020) and depression (Stover et al., 2019). In e-learning, item response theory has been used to develop personalized intelligent tutoring systems that match learner ability and difficulty level (Pliakos et al., 2019). In computerized adaptive testing, it is used in tests like GMAT, GRE or TOEFL to dynamically select the most appropriate items for examinees according to individual abilities (Jia and Le, 2020). In marketing, it has been used to measure customer relationship satisfaction (Peng et al., 2019) and to measure extreme response styles (ERS) (de Jong et al., 2008). In criminology, it is applied to the analysis of the causes of crime and deviance using self-reporting measures of delinquency (Thomas, 2019) and to measure self-control (Manapat et al., 2021).

1.2. *Food insecurity in Ipixuna, Brazil.* Our motivating application concerns the assessment of household food insecurity which is mediated through a family's ability to access food and also through the supply of food potentially available. Both factors are relevant in the context of our study located in Ipixuna, shown in Figure 1, which is a 'jungle town' located on the banks of the River Juruá with a urban population of approximately twelve thousand residents at the time of fieldwork (IBGE, 2010, 2022); it is unconnected to the Brazilian road network, and is several thousand kilometers of upstream boat travel from the Amazonas State capital, Manaus. Being extremely remote and highly river-dependent, Ipixuna exhibits high social vulnerability to hydro-climatic shocks such as floods and droughts, which pose a serious risk of harm to the local population (Parry et al., 2018). This vulnerability is due to a combination of high social sensitivity (e.g. low levels of formal education, many households lack access to clean piped water and internal toilets), low adaptive capacity (e.g. relatively poor public services, proxied by educational delays and inadequate antenatal care), and high prices of imported food stuffs (e.g. frozen chicken) (Parry et al., 2018).

In these remote and roadless urban centres, accessible only by boat or plane, food insecurity is partly affected by seasonal variation in river levels. All river-dependent towns and cities in Amazonas State experience strong seasonal flood pulses (reflecting highly seasonal rainfall patterns, including in locations far upstream in neighbouring Amazonian countries; Chacón-Montalván et al. (2021)) which cause dramatic changes in river-levels and hence the relative ease or difficult of boats navigating between urban centres. However, Ipixuna's location far up the River Juruá means that this town tends to experience significant problems in boat-accessibility even in 'normal' dry seasons. During periods of extreme drought such as in 2023 (Santos de Lima et al., 2024), Ipixuna and other urban centres like it became inaccessible to larger boats for several months. Conversely, the annual high-water flood season carries

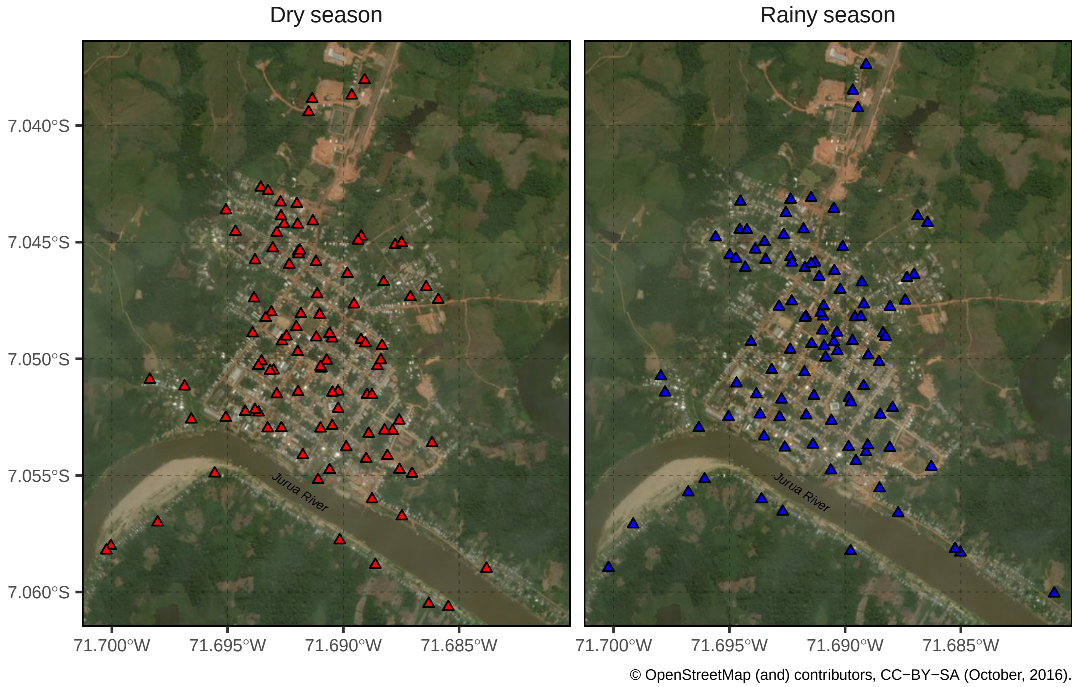

FIG 1. *Spatial distribution of sampled households (triangles) in the urban area of Ipixuna for each season. The locations have been jittered to preserve respondent privacy.*

risks of rising above the limits of the stilted houses at the urban periphery and even flood the urban centre. In extreme cases (such as early 2017) houses in Ipixuna's poorer riverine neighbourhoods flood and become uninhabitable, with associated risks for health (e.g. waterborne diseases) and income (i.e. disrupted livelihoods). These climate-health risks explain why prenatal exposure to periods of extremely deficient or intense rainfall in Amazonas lead to higher odds of pre-term birth and restricted intra-uterine growth (Chacón-Montalván et al., 2021). But there are other factors at play too: community, governmental and non-governmental support can bolster a family's access to food resources in difficult times (Garrett and Ruel, 1999; Battersby, 2011), and social inequities shape household's capacities for coping with environmental shocks. As is the case with cities in the Global North, neighbourhoods with certain characteristics tend to cluster together. In Ipixuna, the poorest neighbourhoods are at the urban margins, home to many rural-urban migrants. The arrival of such migrants to Amazonian towns from the 1980s onwards followed the collapse of the rubber economy and rural people's desires to access basic services, especially education (Parry et al., 2010). It is therefore unsurprising that Ipixuna has rapidly urbanized; its urban population expanded by 65% over the decade to 2010 and by another 58% by 2020 (IBGE, 2010, 2022).

We measured food insecurity items in August 2015 (low-water dry season) and March 2016 (high-water rainy season) with the male or female heads of 100 randomly sampled households per season (see Figure 1). We measured perceptions of household food insecurity using a questionnaire that we modified from the Brazilian Household Food Insecurity Scale (EBIA) (see Rivero et al., 2022, Supplementary Materials). The EBIA was developed and validated in Brazil in 2003, and built on the Household Food Security Survey Module (HFSSM) from the US Department of Agriculture (Pérez-Escamilla et al., 2004). The EBIA has similarities with the widely-used Food Insecurity Access Scale (HFIAS), which also originated from the HFSSM, and was designed by USAID to be adapted for diverse cultural contexts (Coates, Swindale and Bilinsky, 2007). In our study, we asked about experiences during the previous

30 days in order to obtain season-specific food insecurity measures, consistent with our sampling of peak rainy and dry seasons. The full questionnaire, which contains 18 items related to food insecurity, is available in the Supplementary Material (SM); see Chacón-Montalván et al. (2025, Section S3). Items in Section A referred to household as a whole, those in Section B referred to adults only, Section C concerned children (those under the age of 18), and Section D included items related to the regional context of our study. The regionally-specific questions in section D were designed to measure similar aspects as contained in the general scale but were adapted to reflect common coping strategies employed in this locality. It is common for observed data from household surveys to include missing data. This can be due to human error in recording information, but more importantly in our application, due to the presence of items in Section C that only apply to households that have children.

We revisited Ipixuna in May 2017 in order to share our provisional food insecurity results with diverse stakeholders, and use these findings (including maps and neighbourhood-scale summaries of food insecurity challenges) as a starting point for discussing food insecurity in this municipality in relation to key themes which included social marginalization, socio-spatial inequities, governance, seasonality (particularly in relation to local river levels, but also rainfall) and extreme hydro-climatic events. Our multi-disciplinary team had expertise in geography, epidemiology and nutrition, rural livelihoods, and hydrology, and citizen participants in our pan-Amazon 'Amazonas Citizens Network' (http://wp.lancs.ac.uk/rede-cidada-am/eventos/). Activities over our 12-day visit included a three-day workshop in the town (with representatives from diverse municipal and state-level institutions including Civil Defence, churches, etc), field-visits to neighbourhoods (especially more vulnerable peri-urban areas), and semi-structured interviews with key respondents (e.g. with the local head of the Amazonas Institute for Agricultural Development and Sustainable Forestry (IDAM)).

In our application, we are interested in estimating the difficulty and discrimination parameters in order to understand the relationship between the underlying latent factors with the items, using all the available data, including those households with missing data. We also aim to predict the latent factors not only in places where the observations were taken, but also in locations where we have no observations. Collecting our data was costly, difficult, and time-consuming, thus our method for predicting food insecurity at new locations is an important step for identifying particularly vulnerable areas that could be targeted for intervention. Since our method can also be used to map and predict the different dimensions of food insecurity, this information could be used to tailor specific interventions to specific regions.

1.3. *Limitation of item response theory and item factor analysis.* One of the main limitations of classical IRT models is that they assume that the latent construct is unidimensional: this assumption may not be adequate for more complex latent constructs. For example, the items developed to study food insecurity capture a number of different concepts including: (i) the perception of reduction in the quality or quantity of food, (ii) an actual reduction in quality of food, (iii) an actual reduction in quantity of food, and (iv) a reduction in the quantity or quality of food for children in the household. Hence, the construct food insecurity has more than one dimension, and might also depend on characteristics of the population under study, Froelich and Jensen (2002) for example found a further dimension associated with the protection of children from hunger.

In this context, where unidimensional models are not appropriate, researchers have developed *Multidimensional Item Response Theory* (MIRT) or *Item Factor Analysis* (IFA), both approaches being conceptually similar (Bock, Gibbons and Muraki, 1988; Chalmers, 2012). These models extend the concept of standard multivariate factor analysis so it can be applied to binary or ordinal data and allows us to study the interaction between multiple items and a multi-dimensional latent construct. Although IFA overcomes the limitation of unidimensionality, it has other constraints that we seek to address in this paper.

Firstly, IFA assumes the latent construct of a particular subject to be independent of any other subject. In our subsequent example of food insecurity, this seems inadequate given that households near to each other are more likely to have similar socio-economic conditions and environmental exposures, share food between households, and thus a similar risk of food insecurity. This observation also applies to the analysis of latent constructs in other disciplines where spatial correlation is naturally expected, an example would be socio-economic status itself. Connected to this, an IFA model incorporating spatial random effects would allow us to map the latent factors at unobserved locations, which can be (and is in our case) of scientific interest. With respect to our own and other similar application(s), a complete map of the latent factors over the area under study will improve our understanding of the construct and help to better inform the decision-making process.

Secondly, IFA only relates items to the latent construct, but not to possible covariates that could help explain why certain individuals might have particularly high or low values of the latent construct. For example, our previous research in this area suggests that accessibility and political economic factors play an important role in determining social vulnerability in Amazonian towns and cities (Parry et al., 2018). In our case, therefore, understanding the relationship between the items, the latent construct and the covariates is highly desirable.

Finally, traditional methods often require removing profiles with missing data or applying simple imputation methods that do not account for the uncertainty associated with this imputation. Therefore, we need an approach that allows us to use the full data set while considering the uncertainty associated with the missing data.

1.4. *Spatial item factor analysis.* The above summarises our motivation for developing an extension to IFA which we here denominate *spatial item factor analysis* (SPIFA). Some extensions of factor analysis to the spatial domain have been undertaken, for example Wang and Wall (2003); Frichot et al. (2012); Thorson et al. (2015); however, they are not adequate for our study because the first considers only one latent factor; the second, includes the spatial structure only on the residual term; while the last, considers a spatial structure only on the latent factor. In comparison, our hierarchical framework allows the latent construct to be split into multiple latent factors, the number and composition of which are determined by initial exploratory analyses or expert knowledge. These latent factors are explained by observed covariates, spatially-correlated random effects and non-spatial multivariate random effects. The relationship between the latent factors and the item responses, in the case of binary outcomes, is mediated through a set of auxiliary variables which handle the conversion between continuous to discrete data forms.

Additionally, our model accounts for the presence of missing data and the uncertainty associated when imputing values by treating them as latent variables and sampling them when required (see Chacón-Montalván et al., 2025, Section S2). Although our model is motivated by our interest in mapping food insecurity, it can be applied more broadly in settings where IFA is used, particularly when the latent construct is expected to exhibit spatial structure—something that can also be explored through preliminary analysis.

1.5. *Structure.* Details of our proposed SPIFA model are presented in Section 2, and the model is implemented in our `R` package `spifa`. We then present the application of the model for predicting food insecurity in the urban town of Ipixuna, Brazil, in Section 3. Finally, the paper concludes with a discussion of the advantages, limitations, and possible extensions of our model in Section 4, along with the main findings from the analysis of food insecurity in Ipixuna.

2. Spatial Item Factor Analysis. In this section, we develop the modelling framework for SPIFA. We first introduce classical IFA in Section 2.1 and Section 2.2, and then present our new methods in Section 2.3. Solutions to identifiability issues are discussed in Section 2.4, while further flexibility in the multivariate spatial structure is explained in Section 2.5. We then provide details about the identifiable auxiliary variables in Section 2.6 and their matrix form in Section 2.7. We conclude with the specification of the likelihood function in Section 2.8, and information about our `spifa` R package in Section 2.9.

2.1. *Item factor analysis.* IFA can be seen as an extension of factor analysis for binary or ordinal data. In this article, we focus on binary outcomes and discuss possible extensions of the proposed framework to mixed data types—continuous, binary, and ordinal—in the SM (Chacón-Montalván et al., 2025, Section S5).

We begin by considering the response variable Y_{ij} for item $j = 1, 2, \dots, q$ from subject $i = 1, 2, \dots, n$ as a binarization around zero of a continuous but unobservable process Z_{ij}, explained by m latent factors θ_{ik}, also called latent abilities,

$$Z_{ij} = c_j + \sum_{k=1}^{m} a_{jk}\theta_{ki} + \epsilon_{ij}, \tag{1}$$

where $\epsilon_{ij} \sim \mathcal{N}(0, 1)$ and $\{c_j\}$ are intercept parameters that reflect difficulty of items. High positive (negative) values for c_j increase (reduce) the probability of endorsing j-th item, which is why they are also referred to as *easiness* parameters (Chalmers, 2015). The elements of the slope vector $\boldsymbol{a}_j = (a_{j1}, \dots, a_{jm})^\intercal$, commonly called *discrimination* parameters, indicate how well the j-th item can discriminate the k-th ability among the subjects under study. If $a_{jk} = 0$, then the k-th latent factor does not explain the variability of the j-th response item, meaning that this item does not help to discriminate the k-th latent ability among the subjects. In our work, we use this same parameterisation with intercepts and slopes. Further details on this model, including inference via the expectation-maximization algorithm, can be found in Bock, Gibbons and Muraki (1988).

Beyond estimating the easiness and discrimination parameters, another goal is to make inferences for the latent factor $\boldsymbol{\theta}_i = (\theta_{1i}, \dots, \theta_{mi})^\intercal$. This allows us to differentiate individuals with high or low levels of the construct under study. A practical application is in the area of *ideal point estimates*, where the objective is to estimate the ideological position of a legislator to predict whether they will vote in favour of a particular motion (Bafumi et al., 2005).

2.2. *Exploratory and confirmatory item factor analysis.* The IFA model defined by Equation (1) is not identifiable due to various types of aliasing (see Section 2.4). To achieve identifiability, restrictions must be imposed on certain parameters and the latent factors, which can be done in ways that lead to either exploratory or confirmatory analyses.

An exploratory item factor analysis (EIFA) is obtained when restrictions are imposed solely to enable inference; in this case, the restrictions are not tied to the specific construct or dataset under study. The usual restrictions are that the latent factors follow a standard multivariate normal distribution, $\boldsymbol{\theta}_i \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$, and that the matrix of discrimination parameters, $\boldsymbol{A} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_q)^\intercal$, is lower triangular (Cai, 2010a). Estimates can then be rotated according to the researcher's preference.

By contrast, a confirmatory item factor analysis (CIFA) applies restrictions in a semi-formal manner: the researcher draws on expert knowledge to define a plausible structure for the latent construct. The main restrictions are that the latent factors follow a zero-mean multivariate normal distribution with unit variances, $\boldsymbol{\theta}_i \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_\theta)$, where $\text{diag}(\boldsymbol{\Sigma}_\theta) = \boldsymbol{1}$, and that the researcher imposes at least $m(m-1)/2$ zero constraints on $\boldsymbol{A}$ (Cai, 2010b). In CIFA, the restrictions are tailored to the specific study and its context, whereas in EIFA they are general and not problem-specific. Where there is no consensus among experts about the structure, researchers can still compare candidate models using measures of fit (e.g., WAIC or DIC for Bayesian analysis).

2.3. *Extension to the spatial domain.* The extension of IFA to the spatial domain can be achieved by including a spatial process in Equation (1). Frichot et al. (2012), for example, proposed a model that includes a spatially correlated error term ϵ_{ij} to correct the principal components by modeling the residual spatial variation. Similarly to Wang and Wall (2003) and Thorson et al. (2015), our proposed SPIFA method allows the latent factors θ_{ki} to be spatially correlated because the nature of the particular construct under study suggests they should be treated in this way. For example, spatial patterns in food insecurity scores are expected across a municipality due to their association with socio-economic and environmental variables. Wang and Wall (2003) represent the latent factor as a uni-dimensional spatial process, while Thorson et al. (2015) represent it as a multi-dimensional spatial process with a lower-triangular structure for the discrimination parameters and employ integrated nested Laplace approximation for inference. In contrast, our model allows the multi-dimensional latent factor to be explained by predictors, a multivariate spatial process, and a multivariate non-spatial effect. The discrimination parameters in our model are flexible and can be defined by the researcher based on the area of analysis. Additionally, our approach employs full Bayesian inference using MCMC methods. The details of our model are explained below.

We model the binary response variables as discrete-state stochastic processes $\{Y_j(s) : s \in D\}$ where $D \subset \mathbb{R}^2$ and $Y_j(s)$ denotes the response to item j at spatial location s. The response variables take the values 0 or 1 according to the value of an underlying auxiliary spatial process $\{Z_j(s) : s \in D\}$:

$$Y_j(s) = \begin{cases} 1 \text{ if } Z_j(s) > 0, \\ 0 \quad \text{otherwise.} \end{cases} \tag{2}$$

Note that $Y_j(s)$ is deterministic conditional on $Z_j(s)$, which is defined as follows:

$$Z_j(s) = c_j + \boldsymbol{a}_j^\intercal \boldsymbol{\theta}(s) + \epsilon_j(s), \;\; \epsilon_j(s) \sim \mathcal{N}(0,1), \tag{3}$$

where c_j and $\boldsymbol{a}_j$ are the easiness and discrimination parameters, respectively. The m-dimensional latent factor $\boldsymbol{\theta}(s)$ is defined as a function of covariates, a multivariate spatial process, and a non-spatially correlated error term, as specified in Equation (4). The latent factor is the only source of spatial correlation in $Z_j(s)$ and hence in $Y_j(s)$: if the spatial variation is removed from $\boldsymbol{\theta}(s)$, then the model resembles a standard IFA.

Since one of our aims is to predict the latent factors at unobserved locations, part of the variability of $\boldsymbol{\theta}(s)$ is explained by a set of spatial covariates $\boldsymbol{x}(s) = (x_1(s), \ldots, x_p(s))^\intercal$ that are preferably also available at unobserved sites. This allows the model to help us understand why certain locations have high or low scores—similar in spirit to *multiple indicators, multiple causes models* (Tekwe et al., 2014). We also include a spatial m-dimensional process $\{\boldsymbol{w}(s) : s \in D\}$ to capture the remaining spatial structure in $\{\boldsymbol{\theta}(s)\}$ after accounting for the covariate effects; and a non-spatial m-dimensional term $\boldsymbol{v}(s) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_v)$ to represent additional variation. The latent factor is thus defined as:

$$\boldsymbol{\theta}(s) = \boldsymbol{\eta}(s) + \boldsymbol{w}(s) + \boldsymbol{v}(s), \tag{4}$$

where $\boldsymbol{\eta}(s) = \boldsymbol{B}^\intercal \boldsymbol{x}(s)$ represents the covariate effects, with $\boldsymbol{B}$ a $p \times m$ matrix of slopes associating the covariates $\boldsymbol{x}(s)$ with the latent factors $\boldsymbol{\theta}(s)$. Note that the covariates are assumed to be standardised; see Section 2.4 for further details.

The m-dimensional spatial process $\{\boldsymbol{w}(s)\}$ is defined as a collection of m zero-mean, independent, stationary and isotropic Gaussian processes with variance σ_k^2 and correlation function Cor $[w_k(s), w_k(s')] = \rho_k(u)$, where u is the distance between two spatial locations s and s',

$$\{w_k(s)\} \sim \text{GP}(0, \sigma_k^2, \rho_k(u)), \quad k = 1, \ldots, m. \tag{5}$$

While this definition might appear restrictive, the independence assumption for these spatial processes is appropriate when the latent factors $\theta_k(s)$ are independent. This can also be extended to cases where the latent factors are correlated, as described in Section 2.5, where the multivariate notation $\{\boldsymbol{w}(s)\}$ is used to explore possible extensions to its structure. Consequently, we consider it a multivariate Gaussian process (MGP).

Equation (4) has the same structure as a multivariate geostatistical model. However, in our case, the dependent variable, $\boldsymbol{\theta}(s)$, is a low-dimensional latent process instead of a high-dimensional observed process as in Gelfand et al. (2004). A similar structure including fixed and random effects is also discussed in Chalmers (2015), but the author does not attempt to model unexplained spatial variation. In addition, the author mainly focuses on including covariates at the item level, whereas our emphasis is on the inclusion of spatial covariates which will then allow us to make predictions at unobserved locations.

Substituting the structure of the latent factors $\boldsymbol{\theta}(s)$ into Equation (3) results in

$$Z_j(s) = c_j + \boldsymbol{a}_j^\intercal [\boldsymbol{B}^\intercal \boldsymbol{x}(s) + \boldsymbol{w}(s) + \boldsymbol{v}(s)] + \epsilon_j(s). \tag{6}$$

We note that if $\boldsymbol{a}_j$ were known, then Equation (6) would be a multivariate geostatistical model, allowing for classical inference methods. However, the main challenges in performing inference for our proposed model come from the inclusion of the interaction between the latent variables with the (unknown) slopes, $\boldsymbol{a}_j$, as this interaction introduces identifiability issues, as discussed in Section 2.4.

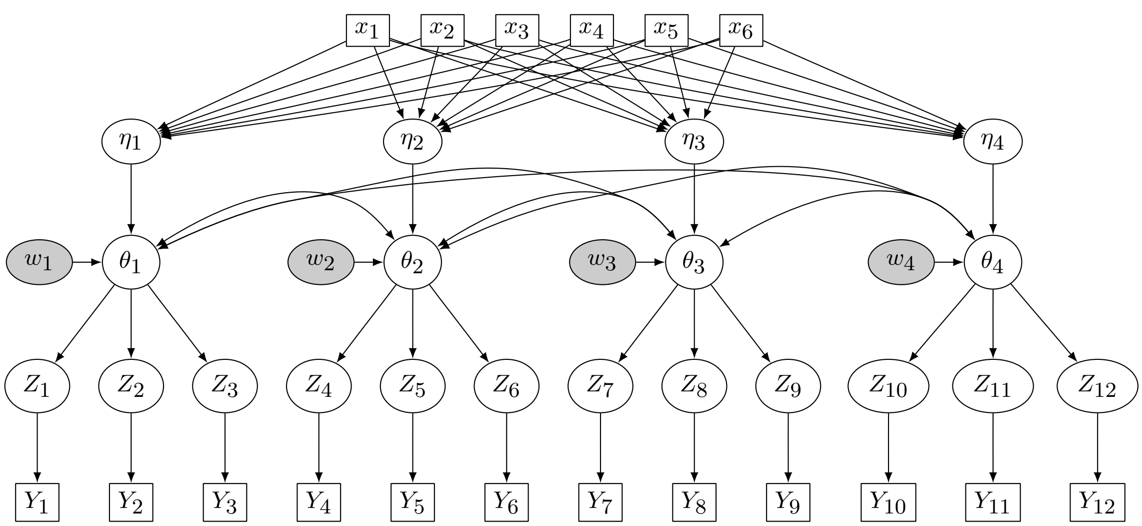

FIG 2. *Directed graph for a spatial item factor model with twelve response items (Y_j), twelve auxiliary variables (Z_j), four latent factors (θ_k), four Gaussian processes (w_k), four linear predictors (η_k), and six covariates (x_l).*

The relationship between covariates $\boldsymbol{x}(s)$, latent factors $\boldsymbol{\theta}(s)$, auxiliary latent variables $\boldsymbol{Z}(s)$ and response variables $\boldsymbol{Y}(s)$ can be seen more clearly through an example of confirmatory SPIFA, as shown in Figure 2. In this example some of the coefficients $\boldsymbol{a}_j$ are set to zero so that each factor is only explained by a subset of items. It can be seen that the 12-dimensional response vector $\boldsymbol{Y}(s)$ is reduced to a 4-dimensional space of factors $\boldsymbol{\theta}(s)$. These factors are allowed to be correlated with each other and also spatial correlation is permitted within factors. The top row in the figure shows how covariates $\boldsymbol{x}(s)$ are used to predict the latent factors $\boldsymbol{\theta}(s)$.

2.4. *Identifiability and restrictions.* The model presented above is subject to the same *identifiability* problems as those found in factor analysis and structural equation modelling. Identifiability issues arise when different sets of parameters lead to the same likelihood in a structured way - this leads to symmetry in the posterior distribution in a Bayesian framework, (or objective function in a classical approach), i.e. there are multiple modes. In our model, these identifiability issues could be due to *additive*, *scaling*, *rotational* or *reflection* aliasing, which will be discussed in detail below. Although sophisticated approaches can be taken to solve problems of identifiability like using the priors proposed in Bhattacharya and Dunson (2011), we use classical approaches as shown in Geweke and Zhou (1996) given that inference of our model is computationally expensive (Chacón-Montalván et al., 2025, Section S2).

Additive aliasing occurs when the item easiness parameter c_j and the product $\boldsymbol{a}_j^\intercal \boldsymbol{\theta}_i$ both have free means. In this situation, adding a constant to one term and subtracting it from the other leaves the probability density function unchanged. Similarly, if the item discrimination vector $\boldsymbol{a}_j^\intercal$ is multiplied by a constant and the latent factor $\boldsymbol{\theta}_i$ is divided by the same constant, the probability density function also remains unchanged—this is known as *scaling* aliasing. To address scaling and additive aliasing in exploratory IFA, as defined in Equation (1), it is common to assume that the latent factors follow a standard normal distribution, $\boldsymbol{\theta}_{ik} \sim \mathcal{N}(0,1)$ (Bafumi et al., 2005). More generally, we can assume $\boldsymbol{\theta}_i \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_\theta)$, and set $\boldsymbol{\Sigma}_\theta = \boldsymbol{I}$ for exploratory analysis, or $\text{diag}(\boldsymbol{\Sigma}_\theta) = \boldsymbol{1}$ for confirmatory analysis.

The SPIFA model presented in Section 2.3 does not suffer from additive aliasing, because the processes $\boldsymbol{w}(s)$ and $\boldsymbol{v}(s)$ are already assumed to be zero-mean. As mentioned above, we also assume that the covariates included in Equation (4) are standardised, which implies that the latent factor $\boldsymbol{\theta}(s)$ has mean zero. However, our model does suffer from scaling aliasing, so it is necessary to restrict the variances of the latent factors $\theta_k(s)$. This is complicated by the presence of covariates and spatial processes, because we cannot directly ensure that $\mathbb{V}\left[\boldsymbol{b}_k^\intercal \boldsymbol{x}(s) + w_k(s) + v_k(s)\right] = 1$. One simple way to achieve the required restriction is to fix the variance of one of the components in the latent factor structure given in Equation (4) (see Chacón-Montalván et al., 2025, Section S1). This restriction is usually applied to the multivariate error term, $\boldsymbol{v}(s)$, as in Tekwe et al. (2014). It is sufficient to fix a diagonal matrix, $\boldsymbol{D}$, containing the marginal standard deviations of $\boldsymbol{v}(s)$, that is, $\text{diag}(\boldsymbol{D}) = (\sigma_{v_1}, \ldots, \sigma_{v_m})^\intercal$, such that $\boldsymbol{\Sigma}_v = \boldsymbol{D}\boldsymbol{R}_v\boldsymbol{D}$, where $\boldsymbol{R}_v$ is a correlation matrix.

Although these restrictions change the interpretation of the discrimination parameters $\boldsymbol{a}_j$, because the latent factors are no longer on a unit scale, they are necessary only to make inference feasible. Therefore, a scaling transformation can be applied post-estimation to recover the usual interpretation of the discrimination parameters:

$$Z_j(s) = c_j + \boldsymbol{a}^\intercal \boldsymbol{Q}\boldsymbol{Q}^{-1}\boldsymbol{\theta}(s) + \epsilon_j(s), \tag{7}$$

where $\boldsymbol{Q}$ is a diagonal matrix containing the standard deviations of $\boldsymbol{\theta}(s)$. This transformation yields a new vector of latent abilities $\boldsymbol{Q}^{-1}\boldsymbol{\theta}(s)$ with unit variances and rescaled discrimination parameters $\boldsymbol{a}^\intercal\boldsymbol{Q}$, restoring the standard interpretation (see Chacón-Montalván et al., 2025, Section S2.4).

The other two types of aliasing, *rotational* and *reflection*, arise because linear transformations of the discrimination parameters, $\boldsymbol{a}_j^{*\intercal} = \boldsymbol{a}_j^\intercal \boldsymbol{\Lambda}^{-1}$, and of the latent factors, $\boldsymbol{\theta}_i^* = \boldsymbol{\Lambda}\boldsymbol{\theta}_i$, yield the same probability density function as the original parameters $\boldsymbol{a}_j$ and $\boldsymbol{\theta}_j$ (Erosheva and Curtis, 2011). To resolve rotational aliasing, $m(m-1)/2$ restrictions are needed to eliminate this non-uniqueness. The usual approach is to set the discrimination matrix $\boldsymbol{A}$ to be lower triangular in EIFA, or to fix at least $m(m-1)/2$ elements to zero in CIFA (Geweke and Zhou, 1996). For reflection aliasing, there are 2^m equivalent matrices $\mathbf{\Lambda}$, which arise by simultaneously changing the signs of both $\boldsymbol{a}_j$ and $\boldsymbol{\theta}_i$. Identifiability can therefore be ensured by fixing the signs of the diagonal elements of $\boldsymbol{A}$ (Geweke and Zhou, 1996).

For our SPIFA model, we require similar restrictions as described above. Focusing on the confirmatory SPIFA setting, it is sufficient to fix at least $m(m-1)/2$ entries of $\boldsymbol{A}$ to zero and to constrain one element in each column of $\boldsymbol{A}$ to be strictly positive (or negative). These restrictions can be imposed through a linear relationship between the constrained parameters $\boldsymbol{a}_j^*$ and the free parameters $\boldsymbol{a}_j$:

$$\boldsymbol{a}_j^* = \boldsymbol{u}_j + \boldsymbol{L}_j \boldsymbol{a}_j, \tag{8}$$

where the vector $\boldsymbol{u}_j$ contains the fixed values, and the matrix $\boldsymbol{L}_j$ indicates which elements of the free parameter $\boldsymbol{a}_j$ remain active (Cai, 2010b). In the example below, the third and fourth elements of the parameter vector are fixed to 0 and 1, respectively.

$$\begin{pmatrix} a_{j1}^* \\ a_{j2}^* \\ a_{j3}^* \\ a_{j4}^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{pmatrix} \begin{pmatrix} a_{j1} \\ a_{j2} \\ a_{j3} \\ a_{j4} \end{pmatrix} = \begin{pmatrix} a_{j1} \\ a_{j2} \\ 0 \\ 1 \end{pmatrix} \tag{9}$$

In practice, under a Bayesian framework, the required positivity (or negativity) constraints can be enforced by specifying appropriate marginal prior distributions (see Chacón-Montalván et al., 2025, Section S2.2).

In theory, the SPIFA model can be used for both exploratory and confirmatory factor analysis. However, we recommend applying it in a confirmatory setting, where no rotation of the latent factors is performed. In this way, the correlation parameters remain directly interpretable. By contrast, if the latent factors are rotated, as in exploratory analysis, interpreting the correlation parameters becomes more challenging.

2.5. *Allowing further flexibility on the multivariate spatial structure.* In the discussion above, we proposed using a set of independent Gaussian processes in the structure of the latent factors $\boldsymbol{\theta}(s)$. It is adequate when the latent factors $\theta_k(s)$ are spatially structured and independent. However, in more general situations, it can be the case that some of the factors $\theta_k(s)$ are not spatially correlated, or that some of the unexplained variation in two or more factors may have a common (spatially-correlated) component. In this situation it will be desirable, respectively, to include spatial structure on only a subset of the factors, or to share the spatial structure across several factors.

In a similar way to how restrictions were imposed on the discrimination parameters, we can use an $m \times g$ transformation matrix $\boldsymbol{T}$ to convert g independent standard Gaussian processes in $\boldsymbol{w}(s)$ into an m-dimensional multivariate Gaussian process, $\boldsymbol{w}^*(s)$:

$$\boldsymbol{w}^*(s) = \boldsymbol{T}\boldsymbol{w}(s). \tag{10}$$

An example is given in Equation (11), where after transforming, $w_1(s)$ is common to the first and second factor and the second factor has an additional spatial structure, namely $w_2(s)$; $w_3(s)$ features in the third factor, and the last factor does not include any Gaussian process i.e. it is not spatially structured.

$$\begin{pmatrix} w_1^*(s) \\ w_2^*(s) \\ w_3^*(s) \\ w_4^*(s) \end{pmatrix} = \begin{pmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & t_{33} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1(s) \\ w_2(s) \\ w_3(s) \end{pmatrix} = \begin{pmatrix} t_{11}w_1(s) \\ t_{21}w_1(s) + t_{22}w_2(s) \\ t_{33}w_3(s) \\ 0 \end{pmatrix} \tag{11}$$

Notice that the variance of $\boldsymbol{w}^*(s)$ is controlled by $\boldsymbol{T}$. Using this stochastic process in Equation 4, we re-define the m-dimensional latent factor of our model as:

$$\boldsymbol{\theta}(s) = \boldsymbol{B}^\intercal \boldsymbol{x}(s) + \boldsymbol{w}^*(s) + \boldsymbol{v}(s). \tag{12}$$

The methods described in the section are closely connected to multivariate geostatistical models of coregionalization (Gelfand et al., 2004; Fanshawe and Diggle, 2012). The main difference is that, due to the sparse structure of $\boldsymbol{T}$, it is not ensured that $\boldsymbol{w}^*(s)$ has a positive definite covariance matrix, and that this structure is a way for the user to control the nature of interrelationships between factors (which would obviously change according to the problem and data under study), rather than allowing free reign estimating all the elements of $\boldsymbol{T}$. Although we propose $\boldsymbol{w}^*(s)$ because it is simple and easy to interpret, our model is not limited to this choice, it could be replaced with a more attractive multivariate stochastic process (e.g. see Gneiting, Kleiber and Schlather, 2010).

There is a sense in which the restrictions imposed can be thought of as prior specification. Provided the 'correct' overall sparse structure of $\boldsymbol{T}$ has been chosen, such restrictions are also beneficial; in particular if $m > g$ then inference becomes more tractable – both in terms of computation, and subsequently interpretation.

2.6. *Auxiliary variables in the identifiable spatial item factor analysis.* Using the restricted discrimination parameters $\boldsymbol{a}^*$ defined in Equation (8) and the new definition of the latent factor $\boldsymbol{\theta}(s)$ in Equation (12), we obtain an identifiable and flexible model for spatial item factor analysis where the auxiliary variables $Z_j(s)$ have the following structure

$$Z_j(s) = c_j + \boldsymbol{a}_j^{*\intercal}\boldsymbol{\theta}(s) + \epsilon_j(s) = c_j + \boldsymbol{a}_j^{*\intercal}[\boldsymbol{B}^\intercal \boldsymbol{x}(s) + \boldsymbol{w}^*(s) + \boldsymbol{v}(s)] + \epsilon_j(s). \tag{13}$$

We are assuming that the structure of the restricted discrimination parameters $\boldsymbol{a}_j^*$ and also the multivariate Gaussian process $\boldsymbol{w}^*(s)$ will be informed by expert opinion through direct involvement of researchers in the area of application and/or through consulting the academic literature in that area.

Doing this not only allows our model to be identifiable, but it also allows us to obtain interpretable latent factors which are practically useful to researchers in the field under consideration.

2.7. *Matrix form of the auxiliary variables.* Expressing the terms in our model at the individual level as above, and in Equation (13), is convenient for understanding the various components. However, in the following, we will use the matrix form of our model in order to define the likelihood function in Section 2.8 and later derive the conditional posterior distributions in the SM (Chacón-Montalván et al., 2025, Section S2).

Before proceeding with the matrix form of our model, we introduce some further notational conventions. Let $\boldsymbol{\alpha}(s)$ be a q-variate random variable at spatial location s. Then if $\boldsymbol{s} = (s_1, s_2, \dots, s_n)^\intercal$ is a set of locations, we will define the q-vector $\boldsymbol{\alpha}_i = \boldsymbol{\alpha}(s_i) = (\alpha_1(s_i), \dots, \alpha_q(s_i))^\intercal$ and the n-vector $\boldsymbol{\alpha}_{[j]} = \boldsymbol{\alpha}_j(\boldsymbol{s}) = (\alpha_j(s_1), \dots, \alpha_j(s_n))^\intercal$.

With the above conventions, the collection of auxiliary random variables $\boldsymbol{Z} = (\boldsymbol{Z}_{[1]}^\intercal, \dots, \boldsymbol{Z}_{[q]}^\intercal)^\intercal$ for q items at n locations can be expressed as

$$\boldsymbol{Z} = (\boldsymbol{I}_q \otimes \boldsymbol{1}_n)\boldsymbol{c} + (\boldsymbol{A}^* \otimes \boldsymbol{I}_n)\boldsymbol{\theta} + \boldsymbol{\epsilon} \tag{14}$$

where $\boldsymbol{I}_q$ and $\boldsymbol{I}_n$ are identity matrices of dimension q and n respectively, $\boldsymbol{1}_n$ is a n-dimensional vector with all elements equal to one, $\boldsymbol{c} = (c_1, \dots, c_q)^\intercal$ is a vector arrangement of the easiness parameters, $\boldsymbol{A}^*_{q\times m} = (\boldsymbol{a}_1^*, \dots, \boldsymbol{a}_q^*)^\intercal$ is a matrix arrangement of the restricted discrimination parameters, $\boldsymbol{\theta} = (\boldsymbol{\theta}_{[1]}^\intercal, \dots, \boldsymbol{\theta}_{[m]}^\intercal)^\intercal$ and $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_{[1]}^\intercal, \dots, \boldsymbol{\epsilon}_{[q]}^\intercal)^\intercal$ is a nq-vector of residual terms.

The vector of latent abilities $\boldsymbol{\theta}$ with respect to Equation (12) can be expressed as

$$\boldsymbol{\theta} = (\boldsymbol{I}_m \otimes \boldsymbol{X})\boldsymbol{\beta} + (\boldsymbol{T} \otimes \boldsymbol{I}_n)\boldsymbol{w} + \boldsymbol{v}, \tag{15}$$

where $\boldsymbol{\beta} = \text{vec}(\boldsymbol{B})$ is a column-vectorization of the multivariate fixed effects, $\boldsymbol{X}_{n\times p} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)^\intercal$ is the design matrix of the covariates, $\boldsymbol{w} = (\boldsymbol{w}_{[1]}^\intercal, \ldots, \boldsymbol{w}_{[m]}^\intercal)^\intercal$ is the collection of the multivariate Gaussian process and $\boldsymbol{v} = (\boldsymbol{v}_{[1]}^\intercal, \ldots, \boldsymbol{v}_{[m]}^\intercal)^\intercal$ is the collection of the multivariate residual terms. Substituting Equation (15) into Equation (14), we obtain:

$$\boldsymbol{Z} = (\boldsymbol{I}_q \otimes \boldsymbol{1}_n)\boldsymbol{c} + (\boldsymbol{A}^* \otimes \boldsymbol{X})\boldsymbol{\beta} + (\boldsymbol{A}^*\boldsymbol{T} \otimes \boldsymbol{I}_n)\boldsymbol{w} + (\boldsymbol{A}^* \otimes \boldsymbol{I}_n)\boldsymbol{v} + \boldsymbol{\epsilon}. \tag{16}$$

This matrix representation is useful for deriving the multivariate marginal and conditional distributions of $\boldsymbol{Z}$ in the following sections.

Alternatively, the collection of auxiliary variables $\boldsymbol{Z}$ can also be expressed as

$$\begin{aligned} \boldsymbol{Z} &= (\boldsymbol{I}_q \otimes \boldsymbol{1}_n)\boldsymbol{c} + (\boldsymbol{I}_q \otimes \boldsymbol{\Theta})\boldsymbol{a}^* + \boldsymbol{\epsilon} \\ &= (\boldsymbol{I}_q \otimes \boldsymbol{1}_n)\boldsymbol{c} + (\boldsymbol{I}_q \otimes \boldsymbol{\Theta})\boldsymbol{u} + (\boldsymbol{I}_q \otimes \boldsymbol{\Theta})\boldsymbol{L}\boldsymbol{a} + \boldsymbol{\epsilon}, \end{aligned} \tag{17}$$

where $\boldsymbol{\Theta}_{n\times m} = (\boldsymbol{\theta}_{[1]}, \ldots, \boldsymbol{\theta}_{[m]})$ is the matrix of latent abilities, $\boldsymbol{u} = (\boldsymbol{u}_1^\intercal, \ldots, \boldsymbol{u}_q^\intercal)^\intercal$ are the restrictions defined in Equation (8), $\boldsymbol{a} = (\boldsymbol{a}_1^\intercal, \ldots, \boldsymbol{a}_q^\intercal)^\intercal$ are the free discrimination parameters and $\boldsymbol{L} = \oplus_{j=1}^q \boldsymbol{L}_j$ is the direct sum of the activation matrices defined in Equation (8) (recall these link the free discrimination parameters $\boldsymbol{a}$ with the constrained discrimination parameters $\boldsymbol{a}^*$). We later use Equation (17) in the derivation of the conditional posterior distribution of the discrimination parameters $\boldsymbol{a}$.

2.8. *Likelihood function.* A challenging aspect of some motivating applications is the fact that not all items are observed for all subjects. More generally, it is common to have to deal with missing data (in this case item responses) in statistics, therefore in the present section we begin to introduce notation for observed and missing data; this will be revisited in Chacón-Montalván et al. (2025, Section S2) and is also connected with prediction.

Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)^\intercal$ be a set of locations at which data from q items has been collected. Let the random variable $Y_{ij} = Y_j(s_i)$ be the j-th item response at location s_i. Using notation introduced in Section 2.7, let $\boldsymbol{Y} = (\boldsymbol{Y}_{[1]}^\intercal, \ldots, \boldsymbol{Y}_{[q]}^\intercal)^\intercal$ be the collection of responses to all items. These can be divided into two groups; the set of observed variables $\boldsymbol{Y}_{obs}$ and the set of variables that were missing $\boldsymbol{Y}_{mis}$.

The marginal likelihood function for our spatial item factor analysis model is obtained by integrating the joint density of the observed variables $\boldsymbol{Y}_{obs}$, the associated auxiliary variables $\boldsymbol{Z}_{obs}$ and the collection of latent abilities $\boldsymbol{\theta} = (\boldsymbol{\theta}_{[1]}, \ldots, \boldsymbol{\theta}_{[m]})^\intercal$;

$$\mathcal{L}(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{B}, \boldsymbol{T}, \boldsymbol{\phi}, \boldsymbol{R}_v) = \int\int \Pr(\boldsymbol{y}_{obs} \mid \boldsymbol{z}_{obs}) \Pr(\boldsymbol{z}_{obs}, \boldsymbol{\theta} \mid \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{B}, \boldsymbol{T}, \boldsymbol{\phi}, \boldsymbol{R}_v)\, d\boldsymbol{z}_{obs}\, d\boldsymbol{\theta}, \tag{18}$$

where $\boldsymbol{a} = (\boldsymbol{a}_1^\intercal, \ldots, \boldsymbol{a}_q^\intercal)^\intercal$ is vector arrangement of all the discrimination parameters and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_g)^\intercal$ is the vector of scale parameters of the g-dimensional Gaussian process $\boldsymbol{w}(s)$. Note that the main computational cost inside the integral, $\mathcal{O}(n^3m^3)$, comes from evaluating the distribution associated with $\boldsymbol{\theta}$ which has a $mn \times mn$ covariance matrix.

In Equation (18), the structure of our model implies

$$\Pr(\boldsymbol{y}_{obs} \mid \boldsymbol{z}_{obs}) = \prod_{o_{ij}=1} \Pr(y_{ij} \mid z_{ij}), \tag{19}$$

where o_{ij} is an indicator variable with value equals to one when the variable Y_{ij} has been observed (i.e. is not missing) and zero otherwise. We further have:

$$\Pr(\boldsymbol{z}_{obs}, \boldsymbol{\theta} \mid \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{B}, \boldsymbol{T}, \boldsymbol{\phi}, \boldsymbol{R}_v) = \prod_{o_{ij}=1} \Pr(z_{ij} \mid \boldsymbol{\theta}_i, c_j, \boldsymbol{a}_j, \boldsymbol{B}) \Pr(\boldsymbol{\theta} \mid \boldsymbol{T}, \boldsymbol{\phi}, \boldsymbol{R}_v), \tag{20}$$

and variables on the right hand side are normally distributed.

Note that the definition of the likelihood function through Equation (18), (19) and (20) does not depend on the missing observations. Therefore, if some items were not observed in some of the locations, inference will still be possible provided the missing data are *missing at random* (Merkle, 2011). Using this likelihood, inference from the model can proceed in a number of ways. Maximum likelihood estimation can be achieved by approximating the likelihood function in Equation (18) using a variety of Monte Carlo methods or via stochastic approximation (Cai, 2010b). However in the present article, we focus on a Bayesian approach because of the simplicity and reliability of uncertainty computation (Fox, 2010).

Our likelihood function can also be written using the auxiliary variables associated with both the observed and missing responses:

$$\mathcal{L}(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{B}, \boldsymbol{T}, \boldsymbol{\phi}, \boldsymbol{R}_v) = \int \Pr(\boldsymbol{y}_{obs} \mid \boldsymbol{z}_{obs}) \Pr(\boldsymbol{z} \mid \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{B}, \boldsymbol{T}, \boldsymbol{\phi}, \boldsymbol{R}_v)\, d\boldsymbol{z}. \tag{21}$$

The advantage of this representation is that the joint density of the auxiliary variables $\Pr(\boldsymbol{z} \mid \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{B}, \boldsymbol{T}, \boldsymbol{\phi}, \boldsymbol{R}_v)$ can be obtained in a straightforward manner using Equation (16). It is normally distributed with mean

$$\boldsymbol{\mu}_z = (\boldsymbol{I}_q \otimes \boldsymbol{1}_n)\boldsymbol{c} + (\boldsymbol{A}^* \otimes \boldsymbol{X})\boldsymbol{\beta} \tag{22}$$

and covariance matrix

$$\boldsymbol{\Sigma}_z = (\boldsymbol{A}^*\boldsymbol{T} \otimes \boldsymbol{I}_n)\boldsymbol{\Sigma}_w(\boldsymbol{T}^\intercal \boldsymbol{A}^{*\intercal} \otimes \boldsymbol{I}_n) + (\boldsymbol{A}^* \otimes \boldsymbol{I}_n)\boldsymbol{D}\boldsymbol{R}_v\boldsymbol{D}(\boldsymbol{A}^{*\intercal} \otimes \boldsymbol{I}_n) \tag{23}$$

where $\boldsymbol{\Sigma}_w = \oplus_{k=1}^{g} \boldsymbol{\Sigma}_{w_k}$ is the direct sum of the covariance matrices of the independent Gaussian processes. We prefer this last definition of the likelihood function, as it facilitates handling missing data via data augmentation (Chacón-Montalván et al., 2025, Section S2.3).

2.9. *R package.* Our SPIFA model is implemented in `C++` and can be used through our open-source `R` package, `spifa`, available at https://github.com/ErickChacon/spifa. The package implements the Bayesian inference method described in Chacón-Montalván et al. (2025, Section S2), allowing users to specify the structure of the multivariate Gaussian processes and prior hyperparameters; model selection is also available via the DIC. The package includes functions for summarising model output, performing MCMC diagnostics, and producing predictive maps using `sf` methods (Pebesma et al., 2018).

3. Results. In this section, we present results from our motivating application: the modelling and prediction of food insecurity in a remote urban centre, Ipixuna, located in Amazonas state, Brazil. The complete analysis, along with the underlying `R` code, is available at https://erickchacon.gitlab.io/food-insecurity-mapping. Following our analyses, we shared and discussed the results with local authorities and community members to interpret the findings collaboratively and gain additional qualitative insights. We also conducted site visits to neighbourhoods identified as particularly vulnerable to observe their characteristics. These interactions were highly valuable for interpreting the results.

3.1. *Data description.* The food insecurity questionnaire was conducted in August 2015 (low-water dry season) and March 2016 (high-water rainy season) with 100 randomly sampled households per season. Households were randomly selected, with sampling density adjusted based on the number of households per census sector (IBGE, 2010). Sampling points were geolocated using Open Street Map and Google Earth, and were restricted to the habitable area of Ipixuna, defined as a 20-meter radius from streets or river edges. We approached the nearest household to each sample location for interviews and recorded the GPS coordinates of all households. This research received ethical approval from Brazil's national health research ethics committee (CONEP-CNS; protocolo 45383215.5.0000.0005) and Lancaster University (S2014/126). Our questionnaire contains 18 items divided into 4 categories (Table 1). Section A refers to questions addressed to the households as a whole, Section B refers to adults only, Section C to households with children, and Section D contains items related to the regional context of Ipixuna. The full questionnaire is available in the SM (Chacón-Montalván et al., 2025, Section S3).

The items with higher endorsement probability were numbers 15, 3, 1, 18, and 2, see Table 1. In the present context, endorsement simply means 'answering with an affirmative'. This indicates that it is common that Ipixuna citizens have to buy food on credit (i.e. putting it on a 'tab' at a local shop), eat few food types, are worried that food will end before they have means to acquire more, reduce meat or fish consumption, or run out of food. Of the 200 surveyed households, 25 did not have children and this led to missing data on the 6 items of associated with food insecurity in children, see Section C in Table 1. This is treated as missing because it is desirable to obtain a joint model for all the population.

TABLE 1

Summary of the food insecurity items: i) the number of missing values (#NA) and the proportion of endorsement (π) are shown for the descriptive analysis, ii) the posterior median of the discrimination parameters $\{\hat{\boldsymbol{A}}_{\cdot 1}, \hat{\boldsymbol{A}}_{\cdot 2}, \hat{\boldsymbol{A}}_{\cdot 3}\}$ are shown for the confirmatory factor analysis (CIFA), and iii) the posterior median of the discrimination and easiness parameters $\{\hat{\boldsymbol{A}}_{\cdot 1}, \hat{\boldsymbol{A}}_{\cdot 2}, \hat{\boldsymbol{A}}_{\cdot 3}, \hat{\boldsymbol{c}}\}$ are shown for the spatial item factor analysis (SPIFA).

Item	**Section**	**Question**	**Descriptive**		**CIFA**			**SPIFA**			
			#NA	π	$\hat{\boldsymbol{A}}_{\cdot 1}$	$\hat{\boldsymbol{A}}_{\cdot 2}$	$\hat{\boldsymbol{A}}_{\cdot 3}$	$\hat{\boldsymbol{A}}_{\cdot 1}$	$\hat{\boldsymbol{A}}_{\cdot 2}$	$\hat{\boldsymbol{A}}_{\cdot 3}$	$\hat{\boldsymbol{c}}$
1	A	worried that food ends	0	0.56	·	1.63	·	·	1.77	·	0.44
2	A	run out of food	0	0.52	·	·	1.51	·	·	1.88	0.22
3	A	ate few food types	0	0.64	1.68	·	·	1.81	·	·	1.46
4	B	skipped a meal	0	0.30	1.47	·	1.02	1.92	·	1.00	-0.63
5	B	ate less than required	0	0.41	0.88	·	1.75	1.39	·	1.50	-0.07
6	B	hungry but did not eat	0	0.24	1.26	·	1.49	1.81	·	1.54	-1.44
7	B	at most one meal per day	0	0.26	1.57	·	·	1.80	·	·	-0.55
8	C	ate few food types	25	0.49	1.71	·	·	1.88	·	·	0.59
9	C	ate less than required	25	0.31	1.90	·	·	2.23	·	·	-0.36
10	C	decreased food quantity	25	0.36	2.17	·	·	2.53	·	·	-0.03
11	C	skipped a meal	25	0.23	1.99	·	·	2.52	·	·	-1.05
12	C	hungry but did not eat	25	0.20	2.11	·	·	2.59	·	·	-1.33
13	C	at most one meal per day	25	0.18	1.93	·	·	2.43	·	·	-1.51
14	D	food just with toasted manioc flour	0	0.17	0.34	·	1.25	0.63	·	1.28	-1.62
15	D	buying food on credit	0	0.68	·	0.72	·	·	0.80	·	0.63
16	D	borrowed food	0	0.14	·	1.44	·	·	1.61	·	-1.90
17	D	had meals at neighbors	0	0.17	·	0.98	·	·	1.04	·	-1.26
18	D	reduced meat or fish	0	0.54	1.29	·	·	1.43	·	·	0.75

3.2. *Confirmatory item factor analysis.* Before undertaking the CIFA, we first performed an EIFA to determine the number of dimensions and identify which items should be related to each factor. We compared models whose dimensions ranging from one to six and selected a model with three dimensions. A likelihood ratio test showed no significant improvement for models with more than three dimensions (p-value 0.594). Similarly, the Akaike Information Criterion (AIC) increased by 17 units, and the Bayesian Information Criterion (BIC) increased by 66 units when considering higher dimensions. Based on these three criteria, models with two and three dimensions were most suitable, and we choose to proceed with three dimensions. Finally, we applied a varimax rotation to the selected model to inform the structure of the CIFA model.

For the structure of the CIFA model, we decided to include only those items with a discrimination parameter greater than 0.5 in the EIFA model. This results in the following factor structure: the first factor is represented by items 3–14 and 18; the second factor, by items 1 and 15–17; and the third factor, by items 2, 4–6, and 14. To perform Bayesian inference, we used standard normal priors for the easiness parameters c_j; standard normal priors for the discrimination parameters with exception of $\{A_{11,1}, A_{13,1}A_{16,2}, A_{14,3}\}$ for which we used normal priors with mean $\mu = 1$ and standard deviation $\sigma = 0.45$. To address reflection aliasing and ensure the interpretability of our model, we employ a prior for these four parameters that constrains them, with high probability, to be positive. This approach, explained in Section 2.4, ensures that the latent factors are interpreted as measures of food insecurity rather than food security. We use a LKJ prior distribution, as proposed in Lewandowski, Kurowicka and Joe (2009, Section 3), with hyper-parameter $\eta = 1.5$ for the correlation matrix of the latent factors. The adaptive MCMC scheme had parameters $C = 0.7$ and $\alpha = 0.8$ with target acceptance probability of 0.234. This is a reference value which is an the optimal acceptance probability for the random walk Metropolis-Hastings algorithm assuming a target normal distribution to balance parameter space exploration and efficient convergence (Andrieu and Thoms, 2008). We ran the Metropolis-within-Gibbs algorithm described in Chacón-Montalván et al. (2025, Section S2.3) for 1.5 million iterations discarding the first 300,000 iterations and storing 1 in 200 of the remaining iterations. Convergence was assessed using the improved split potential scale reduction factor ($\hat{R}$) that compares the properties of the first half of a chain to the second half (Vehtari et al., 2021). All chains have $\hat{R}$ values lower then 1.05 indicating convergence (see Chacón-Montalván et al., 2025, Section S4.1).

The posterior median of the discrimination parameters $\{\hat{\boldsymbol{A}}_{\cdot 1}, \hat{\boldsymbol{A}}_{\cdot 2}, \hat{\boldsymbol{A}}_{\cdot 3}\}$ of the CIFA model is shown in Table 1. These values show that questions related to the reduction of quality and quantity of food in the diet of children (items 10–12) are the top three most important items for the first factor, labeled as *severe food insecurity including children*. The second factor, labeled as *anxiety and reliance on social relations*, includes three items relating to Amazonian coping strategies (15–17) and one concerning anxiety (1). Note that using credit (15), borrowing food (16), or relying on neighbors for meals (17) are likely sources of anxiety in their own right. Finally, the third factor, labeled as *adults eating less and experiencing hunger*, is related mainly to the reduction in quantity of food (2 and 4–6) and one item associated with the substitution of normal foods with only toasted manioc (farinha) flour, a staple carbohydrate in low-income households (14).

In order to evaluate the spatial correlation in the obtained factors, we use the empirical semi-variogram: see Figure 3. This exploratory tool for determining the extent and form of (spatial) correlation is defined as a function of the distance u between any two points,

$$\hat{\gamma}(u) = \frac{1}{2}\hat{\mathbb{E}}\left[(w(s) - w(s+u))^2\right].$$

The initial increasing behaviour of the semi-variogram, mainly, observed in the first and third factors is evidence for spatial correlation in these dimensions of food insecurity.

3.3. *Spatial confirmatory item factor analysis.* We placed the same restrictions on the discrimination parameters for the spatial item factor as we did for the confirmatory item factor analysis. Based on the empirical variograms shown in Figure 3, we proposed three models; model 1 includes a Gaussian process in the first latent factor (SPIFA I), model 2 includes Gaussian processes in the first and third factor (SPIFA II), and model 3 includes Gaussian processes in the three factors (SPIFA III). In Figure 3, we observe that the exponential model provides a good fit to the empirical semi-variogram. For instance, for the first latent factor, it had the lowest sum of squared errors (SSE $= 29.43$) compared to the Gaussian (SSE $= 29.5$) and spherical (SSE $= 63.15$) models, and similar results were observed

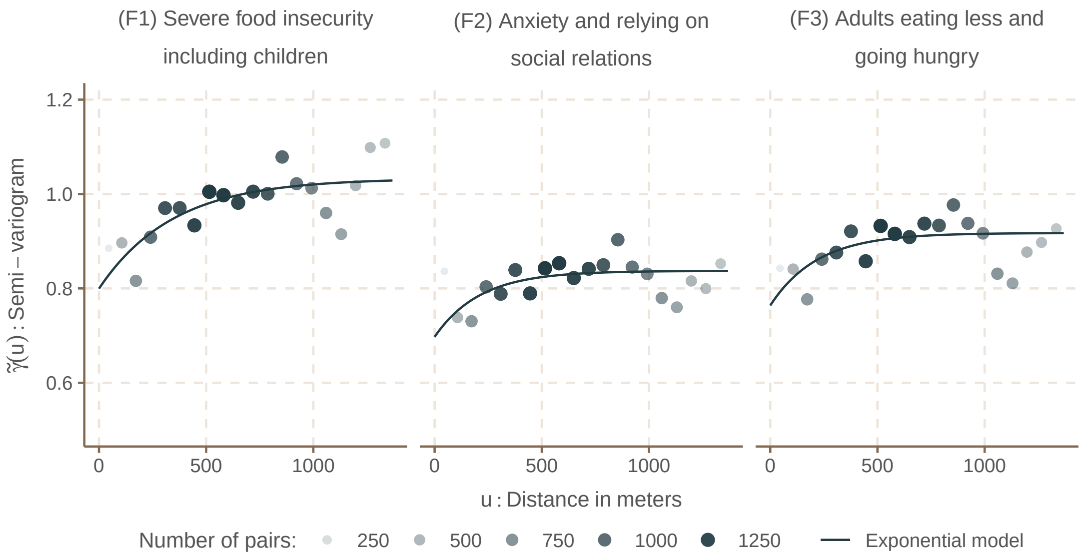

FIG 3. *Empirical semi-variogram $\tilde{\gamma}(u)$ for each latent factor: points represent the empirical values, while the lines indicate fitted theoretical exponential models.*

for the other factors. Due to its superior fit and simplicity, we chose the exponential correlation function, $\rho(u) = \exp(-u/\phi)$, to model the spatial structure of each Gaussian process in our study. Spatial predictors were not included in our model because these are insufficiently finely resolved in our study area. For instance, there are only eight urban census sectors in the 2010 demographic census covering Ipixuna (IBGE, 2010). In the future, we are planning a larger scale analysis in which spatial predictors *will* be included; our software package is already able to handle this case.

We used the same prior specifications as in the CIFA model for the easiness parameters c_j, discrimination parameters A_{jk} and correlation matrix $\boldsymbol{R}_v$. In addition, we used log-normal priors $\mathcal{LN}(\log(160), 0.3)$, $\mathcal{LN}(\log(80), 0.3)$ and $\mathcal{LN}(\log(80), 0.3)$ for the scale parameters $\{\phi_1, \phi_2, \phi_3\}$ of the Gaussian processes in factor 1, 2 and 3 respectively; and the log-normal prior distribution $\mathcal{LN}(\log(0.4), 0.4)$ for all the free elements of $\boldsymbol{T}$. The adaptive MCMC scheme had parameters $C = 0.7$ and $\alpha = 0.8$ with target acceptance probability of 0.234. We ran the Metropolis-within-Gibbs algorithm described in Chacón-Montalván et al. (2025, Section S2.3) for 1.5 million iterations discarding the first 300,000 iterations and storing 1 in every 200 iterations. Convergence was again assessed using the improved split potential scale reduction factor ($\hat{R}$). All chains of the three models have $\hat{R}$ values lower then 1.05 indicating convergence (see Chacón-Montalván et al., 2025, Section S.4.1)

We compared four models using the Deviance Information Criterion (DIC) proposed in Spiegelhalter et al. (2002), details of the computation for our model is provided in the SM (Chacón-Montalván et al., 2025, Section S2.5). In Table 2, we can see that the classical confirmatory model (CIFA) has lowest effective number of parameters (329.82); this model has independent random effects only. In contrast, the spatial models include both independent and spatial random effects as explained in Section 2. The DIC for the three spatial models is lower than that for CIFA, hence by this measure, it is statistically advantageous in terms of model fit to allow the factors to be spatially correlated. Among the three spatial models, SPIFA III, which incorporates Gaussian processes in all three factors, has the best performance in terms of DIC (2198.8). Consequently, the remainder of this section focuses on the results from this model when applied to the *full* data set, the low-water *dry* season data, and the high-water *rainy* season data.

TABLE 2
Deviance Information Criteria (DIC) for the Proposed Models: without spatial correlation (CIFA), with spatial correlation in factor 1 (SPIFA I), with spatial correlation in factor 1 and 3 (SPIFA II) and with spatial correlation in all factors (SPIFA III).

Model	**Diagnostics**		
	Posterior Mean Deviance	Effective Number of Parameters	DIC
CIFA	1894.31	329.82	2224.13
SPIFA I	1866.67	335.14	2201.81
SPIFA II	1861.22	338.64	2199.85
SPIFA III	1857.54	341.26	2198.80

Diagnostics such as the improved $\hat{R}$, the bulk effective sample size (bulk-ESS), and the tail effective sample size (tail-ESS) are presented in Figure 4 for the SPIFA model applied to the full, dry, and rainy season data sets. The bulk-ESS and tail-ESS measure the effective sample size concerning the center and the tail of the distribution, respectively (Vehtari et al., 2021). The histograms of $\hat{R}$ indicate that no chain had values greater than 1.05, demonstrating convergence. Traceplots for randomly selected chains can be seen in the SM (Chacón-Montalván et al., 2025, Section S4.2). Additionally, the sample sizes, around 5900, suggest a sufficiently large sample size for reliable estimates of the mean, median, and credible intervals.

3.3.1. *Estimation of the discrimination and easiness parameters.* The posterior medians of the discrimination parameters $\{\hat{\boldsymbol{A}}_{\cdot 1}, \hat{\boldsymbol{A}}_{\cdot 2}, \hat{\boldsymbol{A}}_{\cdot 3}\}$ for the selected model (SPIFA III) are shown in Table 1 under the column of SPIFA. We can see that the median of the obtained parameters have a broadly a similar structure as for the CIFA model, so their interpretation is as discussed in the previous section; notice that most of the discrimination parameters are higher for the SPIFA model. The last column of Table 1 shows the posterior median of the easiness parameters $\hat{\boldsymbol{c}}$; note the items with high easiness are those most frequently answered with an affirmative ('endorsed'). This column shows that eating few food types (item 3), obtaining credit for eating (item 15) and worrying that food will end (item 1) are the most common behaviours in the population of Ipixuna. Borrowing food (item 16), eating a meal containing only toasted manioc flour (item 14), only being able to feed your children one meal per day (item 13) and feeling hungry but do not eat (item 6 and 12) are less common.

The comparison of the credible intervals for the easiness and discrimination parameters for the full, rainy season, and dry season data is shown in Figures 5 and 6, respectively. The estimates for the easiness parameters are similar for most items, with some differences for item 12 ("hungry but did not eat") and item 16 ("borrowed food"). Item 12 is more commonly endorsed in the rainy season, while item 16 is more commonly endorsed in the dry season. On the other hand, the discrimination parameters also have similar values in both seasons, with the exception of item 4 ("skipped a meal") for the first latent factor, which has higher discrimination in the rainy season. Although not significant, there seems to be a difference in the discrimination of item 5 ("ate less than required") and item 9 ("children ate less than required") with respect to the first factor, showing higher discrimination in the rainy season.

3.3.2. *Spatial prediction of food insecurity.* The predicted median of the spatial component associated to each latent factor, when using the full data set, is shown in Figure 7. The left plot shows that the first factor has a strong spatial structure; the respective posterior median of the standard deviation and scale parameters of the associated Gaussian process are $\hat{T}_{1,1} = 0.468$ $(95\%\text{CI} : 0.29 - 0.7)$ and $\hat{\phi}_1 = 216$ meters $(95\%\text{CI} : 127 - 365)$. Examining the middle plot, for the second factor, it can be seen that the spatial structure is not as strong as the first factor. The respective parameters of the associated Gaussian process have posterior

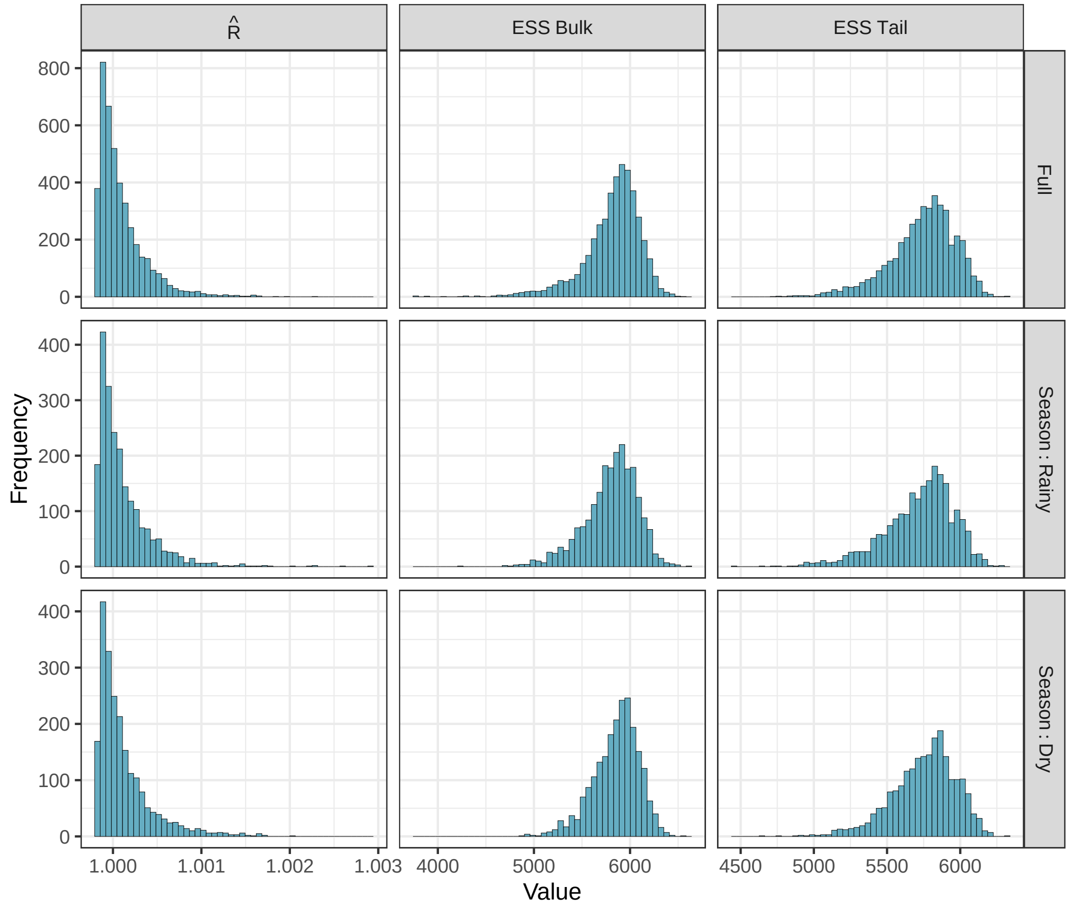

FIG 4. *Histograms of the improved $\hat{R}$, bulk effective sample size (ESS Bulk), and tail effective sample size (ESS Tail) for assessing convergence across all MCMC chains for the SPIFA model applied to the full data, rainy season, and dry season.*

medians $\hat{T}_{2,2} = 0.20$ $(\mathrm{CI} : 0.09 - 0.41)$ and $\hat{\phi}_2 = 83$ $(\mathrm{CI} : 45.1 - 148)$. The right hand plot, referring to the third factor, shows moderate spatial structure with similar median posterior estimates as the second factor: $\hat{T}_{3,3} = 0.29$ $(\mathrm{CI} : 0.15 - 0.49)$ and $\hat{\phi}_3 = 76.8$ $(\mathrm{CI} : 43.9 - 136)$.

Figure 8 shows the predicted median of the spatial component associated with each latent factor when using the dry and rainy season data separately. For the rainy season, the first component exhibits higher variability ($\hat{T}_{1,1} = 0.487$) and a larger correlation scale parameter ($\hat{\phi}_1 = 197$) compared to the second ($\hat{T}2,2 = 0.165, \hat{\phi}_2 = 80.7$) and third factors ($\hat{T}_{3,3} = 0.295, \hat{\phi}_3 = 83.8$). Similarly, for the dry season, the first component shows greater variability ($\hat{T}_{1,1} = 0.435$) and a higher correlation scale parameter ($\hat{\phi}_1 = 178$) than the second ($\hat{T}_{2,2} = 0.184, \hat{\phi}_2 = 79.6$) and third factors ($\hat{T}_{3,3} = 0.276, \hat{\phi}_3 = 76.2$).

Spatial food insecure hotspots were identified as regions where the probability that the spatial effect is positive is greater than 0.7. Maps of these exceedance probabilities are provided in Chacón-Montalván et al. (2025, Section S4.4). Using all data, we identified four hotspots (A, B, C, D), as shown in Figure 7. Under the same criteria, in the dry season, only one hotspot (A) was identified, while in the rainy season, four hotspots (B, E, F, G) were identified, as shown Figure 8.

Examining the obtained maps of food insecurity for the first factor, we observe lower levels of food insecurity around the center of the study area during both the rainy and dry season, as well as when combining both seasons. When combining both seasons, more severe food

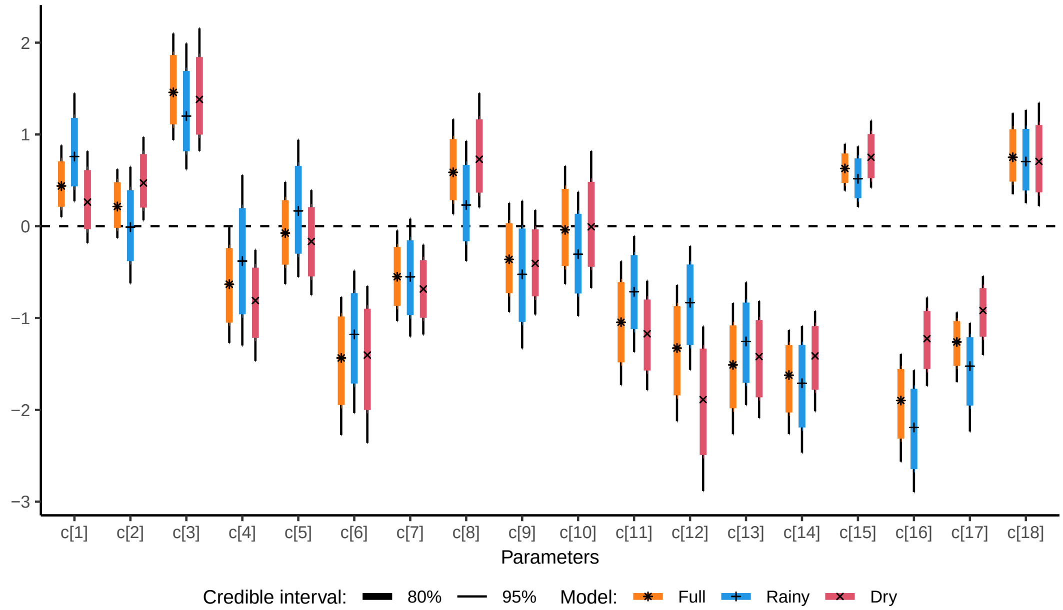

FIG 5. *Credible intervals of the easiness parameters under three SPIFA models: using all data (Full), high-water season (Rainy) and low-water season (Dry). Intervals above (or below) zero for the easiness parameter indicate that the items are more (or less) likely then average to be endorsed.*

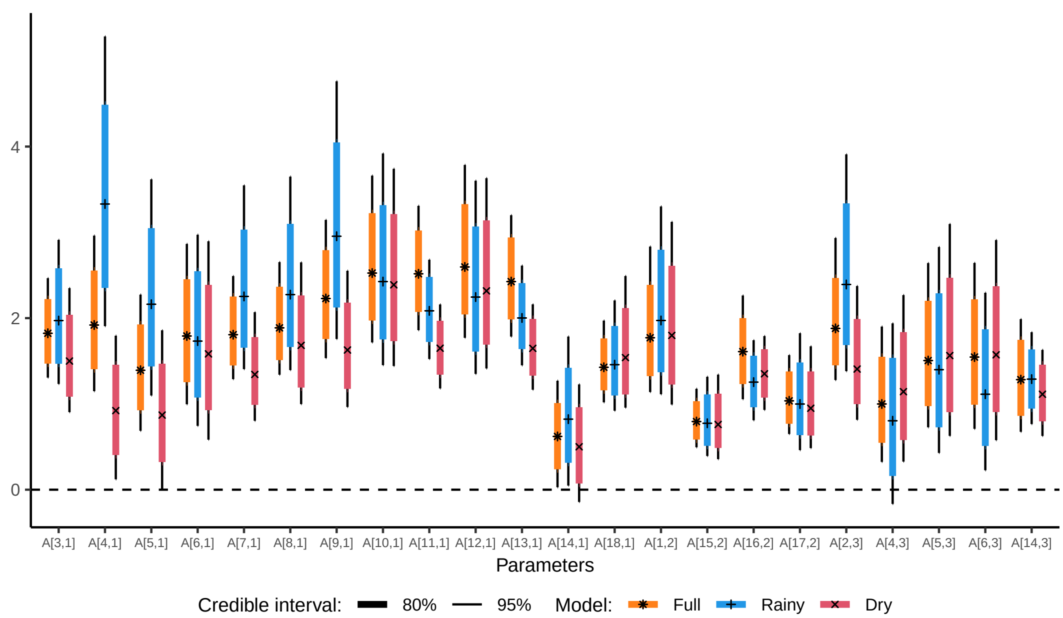

FIG 6. *Credible intervals of the discrimination parameters $A[i, j]$ for item i with respect to factor j under three SPIFA models: all data (Full), high-water season (Rainy), and low-water season (Dry). Intervals above (or below) zero for the discrimination parameter indicates that the items are positively (or negatively) related with the factor of interest.*

insecurity regions are found around locations A ($71.684°$ W, $7.059°$ S) and B ($71.698°$ W, $7.052°$ S), as shown in Figure 7. Location A is also classified as hotspot in the dry season, while location B is detected in the rainy season as well, as shown in Figure 8. Additionally,

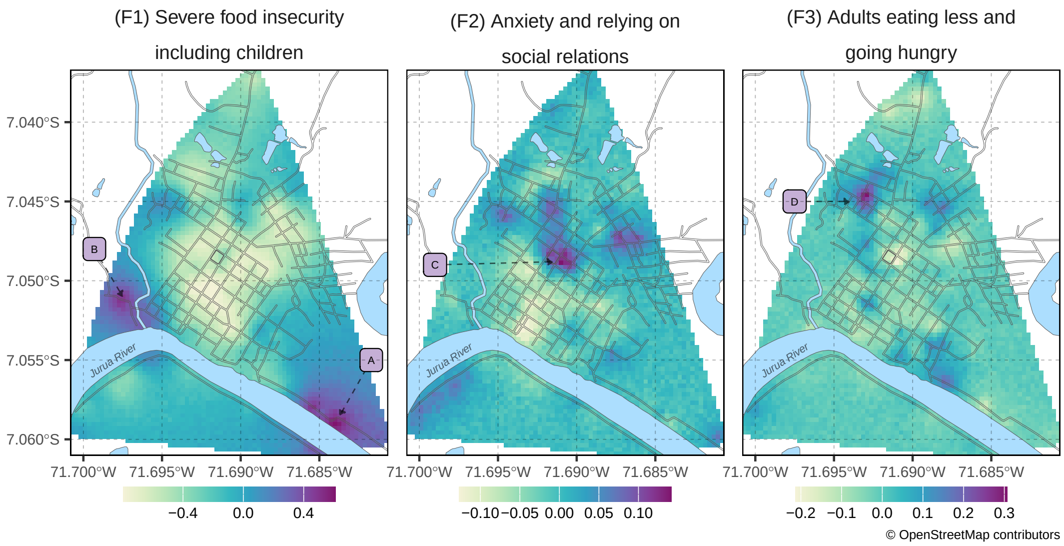

FIG 7. *Median of the predicted spatial component of latent food insecurity using all data. Labeled regions indicate hotspots high levels of food insecurity.*

locations E (71.686° W, 7.058° S) and F (71.698° W, 7.059° S) are only identified in the rainy season.

In this city, location A is at the edge of the River Juruá. It is a relatively new and very poor neighbourhood called Bairro da Várzea. Location B refers to the flood-prone, politically marginalized and poor neighbourhood of Turrufão. Housing is on stilts, and there is no sanitation and poor provision of public services. Location E is close to location A, also at the edge of the River Juruá, and is highly flood-prone. Area F is another flood-prone peri-urban neighbourhood on the other side of the River Juruá, by the name of Bairro da Ressaca. The common characteristic among these locations is that they are poor, marginalized, and mostly flood-prone neighbourhoods on the peri-urban fringe. Most of the heads of households in these neighbourhoods are rural-urban migrants (often relatively recent), and many of their livelihoods are still based in rural areas. The high levels of food insecurity in region A might not be due to flood-risk but rather river-dependent livelihoods. All these relatively large areas, for the first factor, indicate relatively severe food insecurity, yet without apparent anxiety and a distinct absence of some coping strategies: borrowing food, eating in other households or accessing credit.

With respect to the maps of the second factor, higher levels of food insecurity can be identified around location C (71.691° W, 7.048° S) when combining both seasons (Figure 7). This location covers a large and older area of the town, including portions of two neighbourhoods: Bairro do Cemetério and Multirão Velho. These areas are not flood-prone and are not as marginalized or poor, though they are certainly not wealthy. It is plausible that this factor captures more moderate food insecurity and coping strategies associated with higher levels of horizontal social capital and access to credit.

In the maps of the third factor of food insecurity, we observe areas of severe food insecurity around location D (71.693° W, 7.045° S) when combining both seasons, and around G (71.691° W, 7.048° S) in the rainy season. Area D covers the border between two poor, peri-urban neighbourhoods: Bairro da Liberdade and Multirão Novo. Region G is similarly situated as location C, between the neighborhoods Bairro do Cemetério and Multirão Velho, and shares similar characteristics to C as explained above.

In general, the parameter estimates of our model do not vary drastically between the dry and rainy seasons. However, we observe different patterns of food insecurity when spatially

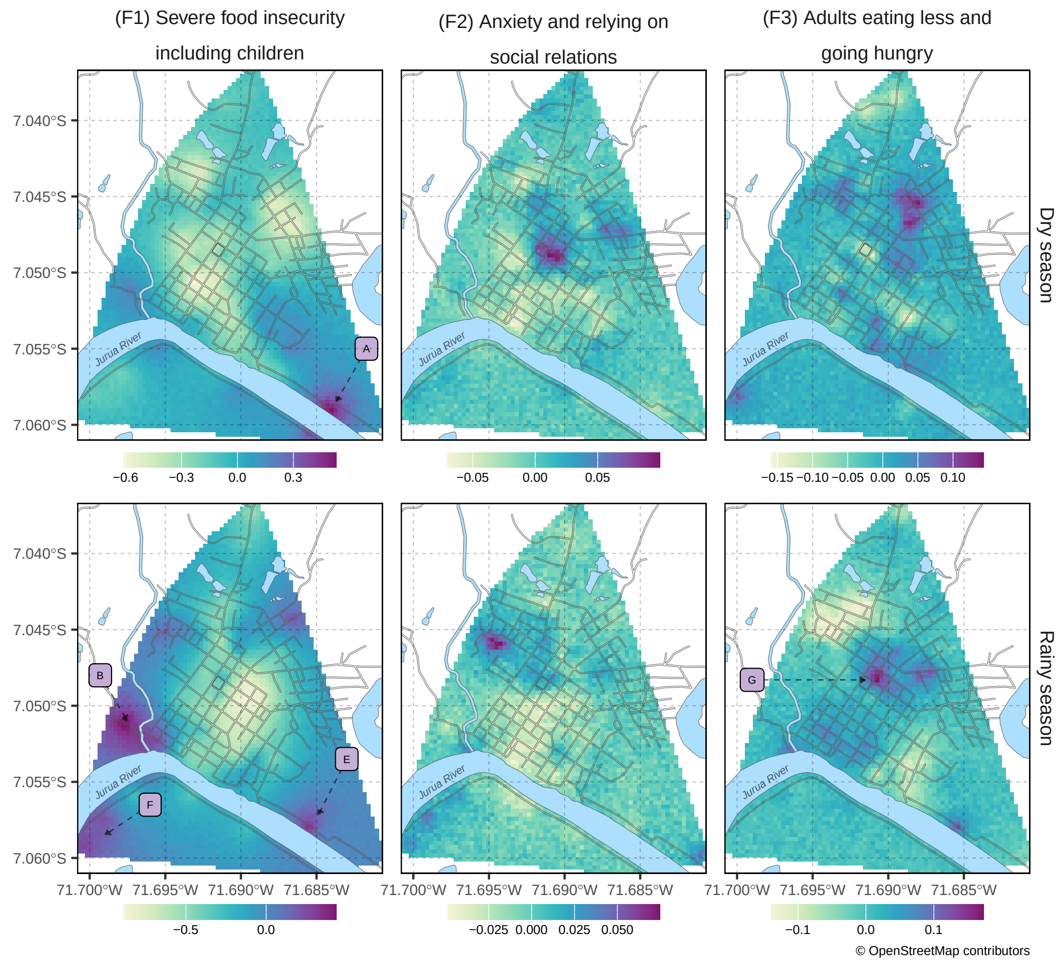

FIG 8. *Median of the predicted spatial component of latent food insecurity for the high-water season (Rainy) and low-water season (Dry). Labeled regions are hotspots high levels of food insecurity.*

predicting the latent factors. This indicates that it is relevant to acknowledge that the levels of food insecurity in the population will vary depending on the season. This is particularly important in cities like Ipixuna, where livelihoods are river-dependent not only due to the risk of flooding but also because access to food through fishing might be affected.

Identifying the areas of high (and also low) food insecurity is of relevance for future research in this area, for example: exploring the social and environmental (e.g. household flood risk due to elevation) determinants of vulnerability to food insecurity. Understanding the spatial-variation of food insecurity at local (e.g. neighbourhood or street) scales will also allow us to continue our dialogue with local government and other stakeholders around which are the priority areas for intervention and what type(s) of intervention should be deployed in order to reduce the risk of food insecurity.

4. Discussion.

4.1. *Spatial Item Factor Analysis.* In this work we have developed a new extension of item factor analysis to the spatial domain, where the latent factors are allowed to be spatially correlated. Our model allows for the inclusion of predictors to help explain the variability of the factors. These developments allow us to make prediction of the latent factors at unobserved locations as shown in our case of study of food insecurity in the Brazilian Amazon.

We solved the issues of identifiability and interpretability by employing a similar strategy as for confirmatory item factor analysis in order to obtain an identifiable model, and by standardizing the resulting factors after inference. Our model has been successfully implemented in the `spifa` open source R package. Since item factor analysis is used across such a wide range of scientific disciplines, we believe that our model and method of inference will be important for generating and investigating many new hypotheses. For instance, it could be used to model socio-economic status.

Computationally, our model is more efficient compared to a model where the spatial structure is used at the level of the response variables. By including spatial structure at the level of the factors, we reduce the computational cost from $\mathcal{O}(q^3n^3)$ to $\mathcal{O}(m^3n^3)$ where the number of items (q) is usually much greater than the number of latent factors (m). For larger datasets, we can reduce the computational burden by using alternatives to the Gaussian process. For example, we could use spatial basis functions (Fahrmeir, Kneib and Lang, 2004), nearest neighbour Gaussian processes (Datta et al., 2016) or stochastic partial differential equations (Lindgren, Rue and Lindström, 2011) to reduce the cost. This is not so obvious because some of the nice properties of these processes can be lost when working with multivariate models.

Our work can be extended in several directions. First, note that in our application we adopted a two-stage modeling procedure: one stage to determine the sparse structure between the latent factors and the response items, and a second stage to fit the SPIFA model given that structure. In other applications, researchers may have prior knowledge to define this structure directly, or the SPIFA model could be extended to stochastically infer the sparse structure, thereby accounting for the associated uncertainty (Ročková and George, 2016). In addition, our model can be extended to the spatio-temporal domain, though again with increased computational expense, depending on the chosen parameterisation of the spatio-temporal correlation. A more complex extension of our model would allow the use of binary, ordinal and continuous items and would also allow predictors to be related in a non-linear way to the latent factors. These extensions would allow us to answer more complex research questions and would also improve prediction of the latent factors (see Chacón-Montalván et al., 2025, Section S5). Extensions to other distributional assumptions (e.g. heavier tailed densities) are also possible if one desires to trade the convenience conjugacy for realism; the Gaussian model fitted our particular dataset well.

4.2. *Food insecurity mapping in Ipixuna.* In our study of food insecurity in Ipixuna, Brazil, we demonstrated that the proposed SPIFA models are more adequate than the classical CIFA models because food insecurity contains a spatial component that may be explained by socio-economic conditions or common environmental risks. In particular, the SPIFA model that introduces spatial correlation in all the latent factors is more appropriate.

By fitting our model to the full data, rainy season data, and dry season data, we show that the estimation of the parameters is consistent across both seasons. However, the spatial variability differs between seasons. In the dry season, we identify a highly food-insecure location (A) for the first factor at the edge of the River Juruá, which is a relatively new and very poor neighbourhood called Bairro da Várzea. On the other hand, in the rainy season, we identified three flood-prone regions with high levels of food insecurity in the first factor: (B) the politically marginalized and poor neighbourhood of Turrufão, (E) a region close to hotspot A at the edge of the River Juruá, and (F) a peri-urban neighbourhood on the other side of the River Juruá called Bairro da Ressaca. Additionally, region G, situated between the neighborhoods Bairro do Cemetério and Multirão Velho, was identified as highly food-insecure for the third factor in the rainy season. It is plausible that this factor captures more moderate food insecurity and coping strategies associated with higher levels of horizontal social capital and access to credit.

Our results reinforce the idea that urban sub-populations in remote parts of Amazonas are highly disadvantaged and experience food and nutrition insecurity likely due to the following reasons: (i) In Ipixuna and other small river-dependent towns in Amazonas, fishing enables poor urban households to eat more fish and may facilitate a modest reduction in their food insecurity risks. However, fishing is only partially effective in compensating for the socio-economic disadvantages experienced by these households (Rivero et al., 2022). (ii) The food security of urban households in Ipixuna is relatively dependent on what households can harvest or grow themselves in surrounding rural areas, or obtain through barter or market exchange with people in their social networks. For instance, someone in 45 percent of urban households in Ipixuna had been fishing in the previous 30 days (Rivero et al., 2022) and fish is the main animal-sourced food in such towns (Carignano Torres et al., 2022). Our data also show that 86 percent of urban households in Ipixuna had consumed bushmeat (terrestrial forest animals) in the previous year (Torres, Morsello and Parry, 2022). The accessibility of these agricultural products and wildfoods is highly seasonal and hence reliance on them - which is greater among the poorer peri-urban households – is likely to be associated with certain food insecurity risks. (iii) Overall, the nutritional status of children in Ipixuna and similar towns is precarious, with iron-deficiency anemia prevalence of 45 percent in children-under-five years old (Carignano Torres et al., 2022). (iv) The health inequities in Ipixuna and places like it are not the outcome of environmental variability. Instead, they reflect historical marginalization of Amazonia's riverine populations which has led to high vulnerability in these neglected remote cities (Parry et al., 2018) and the invisibilization of climate-health risks facing marginalized populations (Parry et al., 2019). The food security challenges of people in Ipixuna's poorest, peripheral neighbourhoods are, of course, influenced by measurable household metrics such as monetary income and family size. However, fundamentally, inequitable access to nutritious food across the urban population reflects power imbalances, politics, and policies. The precarious situation of household's in Turrufao is a consequence of multiple factors including: Statal neglect of rural populations (in-part because a lack of schools and healthcare drove a rural exodus and rapid urbanization); a failure of urban planning (in which the poorest people live in the most flood-prone areas, poorly served by basic amenities such as clean water and sanitation); the on-going neglect of the neighbourhood by urban elites (in some cases, the descendants of the former rubber bosses, Parry et al., 2019) inter-mixed with corruption and governance failures; and deep multi-dimensional poverty, and social structures which perpetuate stigma and racism against indigenous people and non-tribal riberinhos of mixed ancestry.

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SUPPLEMENTARY MATERIAL

Supplement to "Mapping food insecurity in the Brazilian Amazon using a spatial item factor analysis model"

Additional details about our spatial item factor analysis model. Section S1 discusses the identifiability of the model. Section S2 covers the theory behind Bayesian inference for our model, including model specification, prior distributions, sampling scheme, post-sampling scaling, model selection, and spatial prediction. Section S3 presents the food insecurity questionnaire applied in Ipixuna. Section S4 offers additional results related to our food insecurity mapping application in Ipixuna, including diagnostics for all models, trace plots of the posterior distributions for the selected model, missing data comparison, and exceedance probabilities of food insecurity. Finally, Section S5 discusses an extension of our model for mixed outcome types, encompassing binary, ordinal, and continuous items.

Code for the analysis of "Mapping food insecurity in the Brazilian Amazon using a spatial item factor analysis model"
All analyses for mapping food insecurity in Ipixuna were conducted using the `R` language and our `spifa` package. Although the data cannot be shared for privacy reasons, this supplement provides all the scripts, and the rendered outputs can be viewed at https://erickchacon.gitlab.io/food-insecurity-mapping.

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