STRONG ALMOST FINITENESS

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Abstract. A countable, bounded degree graph is almost finite if it has a tiling with isomorphic copies of finitely many Følner sets, and we call it strongly almost finite, if the tiling can be randomized so that the probability that a vertex is on the boundary of a tile is uniformly small. We give various equivalents for strong almost finiteness. In particular, we prove that Property A together with the Følner property is equivalent to strong almost finiteness. Using these characterizations, we show that graphs of subexponential growth and Schreier graphs of amenable groups are always strongly almost finite, generalizing the celebrated result of Downarowicz, Huczek and Zhang about amenable Cayley graphs, based on graph theoretic rather than group theoretic principles. We give various equivalents to Property A for graphs, and show that if a sequence of graphs of Property A (in a uniform sense) converges to a graph G in the neighborhood distance (a purely combinatorial analogue of the classical Benjamini-Schramm distance), then their Laplacian spectra converge to the Laplacian spectrum of G in the Hausdorff distance. We apply the previous theory to construct a new and rich class of classifiable $C^\star$-algebras. Namely, we show that for any minimal strong almost finite graph G there are naturally associated simple, nuclear, stably finite $C^\star$-algebras that are classifiable by their Elliott invariants.

Keywords. almost finiteness, Property A, amenability, spectra of graphs, classifiable C^*-algebras

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Contents

1. Introduction

1.1. **Motivations.** Amenability was first introduced by John von Neumann in 1929 [48] in the setting of discrete groups. A group is called amenable if it admits an invariant mean. In graph theoretic terms this is equivalent to the existence of Følner sets (sets of arbitrarily small relative boundary) in its Cayley graph [28]. Amenable groups play an important role in both the theory of von Neumann algebras and measurable equivalence relations. The concept of amenability was also developed for von Neumann algebras, where the analogue of the invariant mean is given by the conditional expectation onto the algebra. The group von Neumann algebra turned out to be amenable if and only if the group itself is amenable. Similarly, the measurable equivalence relation associated to a free probability measure-preserving (p.m.p.) action of a group is measurably amenable if and only if the group is amenable. A concept of finite approximability, later referred to as hyperfiniteness, was introduced for von Neumann algebras in 1943 by Murray and von Neumann [47]. One of the major breakthroughs of the 1970's was Connes' result [13] showing that for von Neumann algebras with separable preduals, amenability is equivalent to hyperfiniteness. Almost at the same time Connes, Feldman and Weiss [14] proved a very similar result for the measurale equivalence relation associated to probability measure preserving (p.m.p) group actions: measurable amenability is equivalent to measurable hyperfiniteness, where measurable hyperfiniteness means that the equivalence relation associated to the action can be approximated in measure by finite equivalence relations, as introduced by Dye. Weiss conjectured that amenability and hyperfiniteness are also equivalent in the Borel setting. However, this conjecture remains open; in particular, it is still unknown whether free Borel actions of amenable groups are hyperfinite—that is, whether the associated Borel equivalence relation is hyperfinite.

The topological setting (when the group is acting on a compact metric space instead of a probability space) is more subtle. Any free, continuous action of an amenable group on a compact metric space is topologically amenable and admits an invariant probability measure. Moreover, only amenable groups can admit such actions. However, the class of countable groups that have a free, topologically amenable action on a compact metric space, i.e. the so-called Property A (or exact) groups, also contains nonamenable groups. Thus in the topological setting, amenability appears in two forms: classical amenability and Property A. The first example of a group that is not Property A was constructed only in 2003 by Gromov [32] (see also [50]). The notion was introduced by Yu [62], who proved that the Novikov conjecture holds for compact manifolds with fundamental group of Property A. It was soon proved that amenable groups, hyperbolic groups [53], linear groups [33] are of Property A. Ozawa [51] proved that Property A is equivalent to the exactness of the reduced $C^\star$-algebras of the group. If the group is amenable then the reduced $C^\star$-algebra is even nuclear (that is, amenable [34]).

What is the topological analogue of hyperfiniteness? Suppose a countable group acts continuously on the Cantor set, with the property that the fixed point set of each group element is clopen (as in the case of free actions). For such topological actions (or more precisely, for the associated étale Cantor groupoid), Matui introduced the notion of almost finiteness [45]. If a group acts continuously on the Cantor set and the stabilizer map is also continuous, then the resulting groupoid is an ample étale groupoid. This perspective forms the basis for how we approach ample étale groupoids in our paper. As a matter of fact, all the groupoids in our paper are constructed in such a way. In particular, we can talk about their topological amenability through the respective definition for actions. Almost finiteness for these groupoids means that the Cantor set has a tiling by clopen sets that are Følner in each orbit graph, making this property a good candidate for hyperfiniteness in the topological context. This perspective is further supported by two queries posed in (Remark 3.7, [57]) of Suzuki, which suggest that such clopen Følner tilings could indicate a form of amenability:

- Is every minimal, almost finite, étale Cantor groupoid topologically amenable?
- Is every minimal, topologically amenable, étale Cantor groupoid that admits an invariant measure necessarily almost finite?

For groupoids arising from actions of amenable groups, the answer to the first question is affirmative; however, the first author provided a counterexample in the more general setting [23]. It was later observed that a slight strengthening of almost finiteness does, in fact, imply amenability. Specifically, suppose a groupoid is not only almost finite, but also satisfies the following condition: for every $\varepsilon > 0$, there exists a collection of almost finite tilings equipped with a probability distribution such that, for each point x in the Cantor set, the probability that x lies on the boundary of a tile is less than ε. Under this strengthened version, which we call strong almost finiteness, the groupoid is topologically amenable. The second question by Suzuki is still open.

One of the main motivations of our paper is to provide further evidence that strong almost finiteness is, in fact, the appropriate continuous analogue of hyperfiniteness. Here a concept of almost finiteness for infinite graphs will become important.

Let us assume that an étale Cantor groupoid arises from an action of a finitely generated group. If the groupoid is almost finite, then its orbit graphs must also exhibit almost finiteness in the sense of tileability by Følner sets, as defined in the influential paper of Downarowicz, Huczek, and Zhang [16], where they established the almost finiteness of Cayley graphs for countable amenable groups. Similarly, if the minimal groupoid is topologically amenable, then its orbit graphs possesses a graph theoretical version of Property A (which is equivalent to the group theoretical definition in case of Cayley graphs, and was defined by Higson and Roe [35]). These are straightforward consequences of the definitions and the theorem by Connes, Feldman and Weiss [14]. Moreover, if the groupoid admits an invariant measure, the graphs should contain Følner sets in a ubiquitous fashion—an idea previously explored by Ma under the term ubiquitous amenability [43]. Hence, in light of Suzuki's questions, it is reasonable to expect that Property A and the ubiquitous presence of Følner sets together are equivalent to strong almost finiteness in the context of bounded degree graphs. We will prove that this equivalence indeed holds, establishing the equivalence of the suitably defined concepts of amenability and hyperfiniteness for bounded degree graphs. Furthermore, we characterize the relationship among other reasonable candidates for the definition of these concepts in this setup.

An additional motivation for our work was the result of Ma and Wu ([44], Corollary 9.11) that the reduced C^*-algebra of an almost finite and amenable ample étale Cantor minimal groupoid is a simple, nuclear, unital, separable $\mathcal{Z}$-stable and quasidiagonal $C^\star$-algebra (that satisfies the Universal Coefficient Theorem by a theorem of Tu [59]). Hence, by the seminal result of Tikuisis, White and Winter [58], these $C^\star$-algebras can be classified by their Elliott invariants. Consequently, our results enable the construction of numerous new examples of classifiable, simple, nuclear $C^\star$-algebras arising built from strongly almost finite Schreier graphs.

1.2. **Around amenability. The key concepts of the paper.** Below, Gr_d will denote the set of all countable (not necessarily connected) graphs of vertex degree bound d.

(0) Let $G \in Gr_d$ be a graph, and let F be a subset of its vertex set $V(G)$. We denote by $\partial(F)$ the set of vertices in F that are adjacent to a vertex that is not in F. If $\frac{|\partial(F)|}{|F|} < \epsilon$ then F is called an **ϵ-Følner set.** We call the graph $G \in Gr_d$ **amenable** if it contains a Følner sequence, that is, a sequence of finite subsets $\{F_n\}_{n=1}^\infty$ such that $\frac{|\partial(F_n)|}{|F_n|} \to 0$. Amenable graphs were studied already in the eighties (see e.g. [15], earlier work of Kesten [40] is frequently viewed as the first instance where purely graph theoretical properties were used in the context of amenability) and various characterizations of amenability for graphs, similar to the one of von Neumann, were given in the eighties and nineties (see e.g. [6], [10][15], [19] or [37] for a survey).

(1) The graph G is **Følner** if for any $\epsilon > 0$ there exists an r such that any r-ball $B_r^G(x)$ of radius r centered around x contains an ϵ-Følner subset with respect to G. It is quite clear that the Cayley graphs of amenable groups are Følner graphs. Amenable, but non-Følner graphs can easily be constructed by attaching longer and longer paths to the vertices of a tree, or just taking the disjoint union of a 3-regular tree and an infinite path.

(2) A graph $G \in Gr_d$ is **setwise Følner** if for any $\epsilon > 0$ there is an $r > 1$ such that inside the r-neighborhood $B_r(L)$ of any finite set $L \subset V(G)$ there exists an ϵ-Følner set with respect to G that contains L. As far as we know, this definition is new.

(3) A graph $G \in Gr_d$ is **uniformly locally amenable** if for any $\epsilon > 0$ there exists $k > 0$ satisfying the following condition: For any finite subset $L \subset V(G)$ there exists a subset $M \subset L$ such that M is ϵ-Følner with respect to L (so M is not necessarily ϵ-Følner in the graph G) and $|M| \leq k$. The notion of uniform local amenability was introduced in [7].

(4) Building on Dye's definition for measure-preserving actions ([18], see also [38]), the first author extended the concept of hyperfiniteness to classes of finite graphs with bounded degree [20] in the following manner. A family of finite graphs $\mathcal{G} \subset Gr_d$ is said to be **hyperfinite** if for every $\varepsilon > 0$, there exists an integer $k > 0$ such that for every $G \in \mathcal{G}$, there exists a subset $L \subset V(G)$ with $|L| \leq \varepsilon|V(G)|$, such that removing L along with all incident edges results in a graph whose connected components each have at most k vertices. Note that the class of planar graphs, as well as any class of graphs with uniform subexponential growth, is hyperfinite. The notion of local hyperfiniteness was introduced in [24]. A graph $G \in Gr_d$ is **locally hyperfinite**, if the family of all its finite induced subgraphs is hyperfinite.

(5) A graph $G \in Gr_d$ is **weighted hyperfinite** if for any $\epsilon > 0$ there exists $k > 0$ satisfying the following condition: For any finitely supported non-negative function $w : V(G) \to \mathbb{R}$ there exists a subset $L \subset V(G)$ of total weight $w(L)$ that is at most $\epsilon w(V(G))$, such that if we delete L with all the adjacent edges then the size of the the remaining components are at most k. The notion of weighted hyperfiniteness was introduced by the authors of this paper in [21].

(6) A subset $Y \in V(G)$ is a k-separator of the graph $G \in Gr_d$ if deleting Y (with all the adjacent edges) the remaining components have size at most k. A graph $G \in Gr_d$ is **strongly hyperfinite** if for any $\epsilon > 0$ there exists $k > 0$ and a probability measure μ on the compact set of k-separators (with the compact subset topology on $V(G)$) such that for all $x \in V(G)$,

$$\mu(\{Y \,|\, x \in Y\}) < \epsilon\,.$$

The notion of strong hyperfiniteness appeared first in [54] in a somewhat restricted context.

(7) An (ϵ, r)-packing of a graph $G \in Gr_d$ is a family of disjoint ϵ-Følner sets of diameter at most r. A graph $G \in Gr_d$ is **strongly Følner hyperfinite** if for any $\epsilon > 0$ there exists $r > 0$ and a probability measure ν on the compact set of (ϵ, r)-Følner packings $P(\epsilon, r)$ (we give the precise definition of the topology on (ϵ, r)-Følner packings in Section 5) such that for all $x \in V(G)$,

$$\nu(\{\mathcal{P} \,|\, x \in \tilde{\mathcal{P}}\}) > 1 - \epsilon\,,$$

where $\tilde{\mathcal{P}}$ denotes the set of vertices contained in the elements of the packing $\mathcal{P}$. The notion of strong Følner hyperfiniteness is a crucial new notion of our paper.

(8) A graph G is **almost finite** if for any $\epsilon > 0$ there exists an $r > 1$ such that $V(G)$ can be tiled by ϵ-Følner sets of diameter at most r, in other words, there exists an (ϵ, r)-packing $\mathcal{P}$ such that $\tilde{\mathcal{P}} = V(G)$. As we mentioned earlier, Downarowicz, Huczek and Zhang [16] defined almost finiteness (under the name of "tileability") for groups and showed that the Cayley graph of a finitely generated amenable group is almost finite. Their work was motivated by the monotileability problem studied by Weiss [61]. The notion of almost finiteness (in the case of free continuous actions of amenable groups) and its relation to $C^\star$-algebras was further developed in the important paper of Kerr [39]. Almost finite graphs, as in our setup, were introduced by Ara et al. [3].

(9) A graph G is **strongly almost finite** if for any $\epsilon > 0$ there exists $r > 1$ and a probability measure on the (ϵ, r)-Følner tilings such that for any $x \in G$ the probability that x is on the boundary of the tile containing it is less than ϵ. This notion was introduced in a significantly weaker form by the first author in [23].

(10) For graphs (and even for more general metric spaces) Property A was introduced in [35] (see also [7]): a graph $G \in Gr_d$ is of **Property A** if for any $\epsilon > 0$ there exists $r > 1$ and a function $\Theta : V(G) \to \text{Prob}(G)$ satisfying the following conditions.
   - For every $x \in V(G)$ the support of $\Theta(x)$ is contained in the r-ball around x.
   - For every adjacent pair $x, y \in V(G)$ we have that

$$\|\Theta(x) - \Theta(y)\|_1 < \epsilon \,.$$

(11) For a graph G the finitely supported non-negative (non-zero) function $f : V(G) \to \mathbb{R}$ is an **ϵ-Følner function** if

$$\sum_{x \in V(G)} \sum_{y, x \sim y} |f(x) - f(y)| < \epsilon \sum_{x \in V(G)} f(x) \,.$$

If $\sum_{x \in V(G)} f(x) = 1$ we call such functions ϵ-Følner probability measures. The role of ϵ-Følner functions will be crucial in our paper. Note that in case of Cayley graphs of amenable groups the notions of Følner functions and Reiter functions (see e.g. Theorem 2.16 [37] for the definition of Reiter functions) are closely related. A graph G is of **Følner Property A** if it is of Property A and for all x , $\Theta(x)$ can be chosen as a ϵ-Følner function.

(12) The graph $G \in Gr_d$ is **fractionally almost finite** if for any $\epsilon > 0$ there exists $r \geq 1$ and a non-negative function $F : F_G(\epsilon, r) \to \mathbb{R}$, from the set of ϵ-Følner sets of diameter less than r such that that

(a) For any $x \in V(G)$
$$\sum_{x\in H, H\in F_G(\epsilon,r)} F(H) + c_x = 1\,,$$
where $0 \leq c_x < \epsilon$.

(b) For any $x \in V(G)$
$$\sum_{x\in \partial(H), H\in F_G(\epsilon,r)} F(H) < \epsilon.$$

This definition is motivated by Lovász's notion of fractional partition [42].

1.3. **The results.** Some of the above properties have long been central in group theory. When one goes beyond Cayley graphs, a more complex scene emerges. We fully investigate the relationships between properties (1) – (12). Figure 1 summarizes our results.

Theorem 1 (The Long Cycle Theorem)**.** *For graphs $G \in Gr_d$: Property A, Uniform Local Amenability, Local Hyperfiniteness, Weighted Hyperfiniteness and Strong Hyperfiniteness are equivalent.*

Using some results from [7] and [11], Sako [55] has already established the equivalence between Property A and weighted hyperfiniteness. Building on the results of [55], [7], and [54], the first author [24] has also shown that uniform local amenability is equivalent to Property A, confirming a conjecture proposed in [7]. Nonetheless, the proof of Theorem 1 is presented in a self-contained manner.

Theorem 2. *For any $\epsilon > 0$ there exists a $\delta > 0$ such that if $G \in Gr_d$ and $p : V(G) \to \mathbb{R}$ is a δ-Følner probability measure, then there exists an ϵ-Følner subset $H \subset V(G)$ inside the support of p such that $p(H) > 1 - \epsilon$.*

Note (see Remark 1) that in the case of Cayley graphs this theorem is a quantitative strenghtening of Theorem 2.16 in [37]. By the triangle inequality, the finite sum of ϵ-Følner functions is always an ϵ-Følner function. So, they behave much better with respect to summation than ϵ-Følner sets do with respect to taking union. Using this advantage of the Følner functions, we prove that being a Følner graph implies the setwise Følner Property (Proposition 4.1).

Theorem 3 (The Short Cycle Theorem)**.** *For graphs $G \in Gr_d$ the following properties are equivalent: Property A plus Setwise Følner, Strong Følner Hyperfiniteness, Fractionally Almost Finiteness and Følner Property A.*

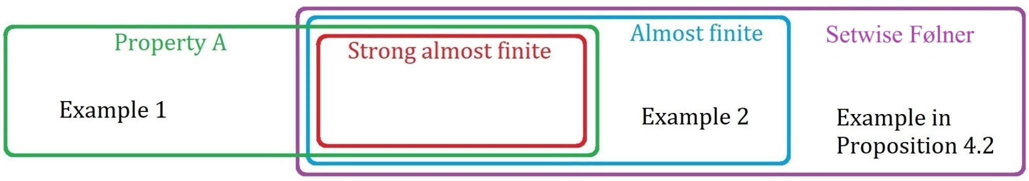
The properties in Theorem 1 are weaker than the ones in Theorem 3, since Følner Property A trivially implies Property A. The remaining strict inclusions are explained next, and are summarized on the diagram of Figure 1. First, there exist Følner graphs that are not almost finite (Proposition 4.3).

Example 1. The 3-regular tree is the simplest and earliest example of graphs that have Property A, but are not amenable, let alone almost finite. To see that it has Property A, pick an end and for each vertex take the averaged indicator function of the path of length n from the vertex towards the end.

Example 2. Almost finiteness does not imply Property A. Let $G \in Gr_d$ be a graph that is not of Property A, say it contains an embedded expander sequence or it is the Cayley graph of a non-exact group. Attach infinite paths to each vertex of G. Then, the resulting graph H is clearly almost finite, but it is not of Property A (see [3] or the unpublished result of the first author [23]). Observe that H is a Følner graph.

Theorem 4. *Strong Følner Hyperfiniteness and Strong Almost Finiteness are equivalent properties.*

As we mentioned earlier, Downarowicz et al. [16] proved that the Cayley graph of an amenable group is almost finite. The ingenious proof uses in a crucial way the fact that such graphs are based on groups. Putting together Theorem 3 and Theorem 4, we extend this result to much larger graph classes: for Schreier graphs of amenable groups (Proposition 7.4) and for graphs of subexponential growth (Proposition 7.3).



Lists of equivalent properties:

Property A
Uniform local amenable
Local hyperfinite
Weighted hyperfinite
Strong hyperfinite
Theorem 1

Strongly almost finite
Prop. A & setwise Følner
Strong Følner hyperfinite
Fractionally almost finite
Følner Property A
Theorems 3 and 4

Almost finite

Følner
Setwise Følner
Proposition 4.1.

FIGURE 1. The relationships of the properties we study.

We apply our results in spectral theory. Let $G \in Gr_d$ be a finite or infinite graph and $\mathcal{L}_G : l^2(V(G)) \to l^2(V(G))$ is its Laplacian. That is,

$$\mathcal{L}_G(f)(x) = \deg(x)f(x) - \sum_{x\sim y} f(y)\,. \tag{1}$$

It is well-known that $\mathcal{L}_G$ is a bounded, positive, self-adjoint operator and $\mathrm{Spec}(\mathcal{L}_G) \subset [0, 2d]$ (see [46]). It is well-known (see [40], [15] or [46]) that a graph $G \in Gr_d$ is amenable if and only if 0 is in the spectrum of G. Note that the spectrum of Cayley graphs of amenable groups can be rather complicated [30]. It is a well-studied question that if a sequence of finite graphs $\{G_n\}_{n=1}^\infty$ converges to an infinite graph (or some other limit object) in some metric, what sort of convergence we can guarantee for the spectra $\{\mathrm{Spec}(\mathcal{L}_{G_n})\}_{n=1}^\infty$. If the graphs $\{G_n\}_{n=1}^\infty$ are equipped with distinguished roots $\{x_n \in V(G_n)\}_{n=1}^\infty$ and the sequence of rooted graphs $\{(G_n, x_n)\}_{n=1}^\infty$ is convergent (see Proposition 8.4) then there exists a rooted graph (G, x) which is the limit of the sequence and the KNS-measures on $\{\mathrm{Spec}(\mathcal{L}_{G_n})\}_{n=1}^\infty$ converge to the KNS-measure of (G, x) in the weak topology (see [4]). Similar result is known ([1]) if $\{\mathrm{Spec}(\mathcal{L}_{G_n})\}_{n=1}^\infty$ is convergent in the sense of Benjamini and Schramm. We will define neighborhood convergence (Section 8), a purely combinatorial version of the Benjamini-Schramm convergence and we prove the following theorem.

Theorem 5. *Say that a countable collection of graphs has Property A if their disjoint union is a graph with Property A. Let $\{G_n\}_{n=1}^\infty \subset Gr_d$ be a countable set of graphs of Property A such that $\lim_{n\to\infty} G_n \to G$ in the neighborhood distance. Then, $\mathrm{Spec}(\mathcal{L}_{G_n}) \to \mathrm{Spec}(\mathcal{L}_G)$ in the Hausdorff distance.*

In the final section we establish a connection between our strong almost finiteness property and the Elliott Classification Program on simple, nuclear C^*-algebras. We will show that if a graph $G \in Gr_d$ is minimal (see Definition 10.1) and G is both of Property A and almost finite (that is G is strongly almost finite) then some of the étale groupoids (see Section 10 for the definition) which are naturally associated to G are minimal, topologically amenable and almost finite in the sense of Matui [45]. Consequently, by the results in [44] we have the following theorem which we explain in Section 10.

Theorem 6. *For every minimal, strongly almost finite graph M we can associate a stable action $\beta_E : \Gamma_{2d} \curvearrowright E$ so that all the orbit graphs are neighborhood equivalent to M and the simple, nuclear, tracial groupoid C^*-algebra $C^*_r(\mathcal{G}_{\beta_E})$ is classifiable by its Elliott invariants.*

These examples seem to be significantly different from the known ones.

Finally, we give a purely dynamical characterization of strong almost finiteness (Proposition 10.11) in the case of minimal graphs.

1.4. **An overview of the paper.** Let us finish the introduction with a short overview of the coming sections. In Section 2 we prove the five equivalents of Property A for bounded degree graphs, going along the "long cycle". Section 3 establishes the quantitative connection between Folner functions and Folner sets, while Section 4 proves that being a Følner graph and being a setwise Folner graph are the same. Property A together with the setwise Folner property can be characterized in four different ways, as shown in Section 5, going along the "short cycle". These are also equivalent to strong almost finiteness, as proved in Section 6. In Section 7 various classes of graphs are shown to be strong almost finite. In Section 8 we define neighborhood equivalence and a metric on the resulting equivalence classes of graphs, which will be the framework for Section 9, where the pointwise convergence of the spectrum is shown for convergent sequences of graphs in this topology. In Section 10, after the necessary preparations, we associate an *étale groupoid* to *minimal* graphs, and establish its topological amenability and almost finiteness under the assumption that the graph was strongly almost finite. This gives rise to a new and rich class of classifiable C^*-algebras.

2. The Long Cycle Theorem

The goal of this section is to prove Theorem 1. The way we prove the theorem is showing that: Property A $\Rightarrow$ Uniform local amenability $\Rightarrow$ local hyperfiniteness $\Rightarrow$ weighted hyperfiniteness $\Rightarrow$ strong hyperfiniteness $\Rightarrow$ Property A.

Proposition 2.1. *Property A implies Uniform Local Amenability.*

Proof. The proof is a simplified version of Lemma 7.2 in [26]. Let $G \in Gr_d$ be a countably infinite graph of Property A. Pick a $\delta > 0$ in such a way that we can find an ϵ-Følner set in the support of any $d\delta$-Følner function $\zeta : V(H) \to \mathbb{R}$, where H is an arbitrary finite induced subgraph of G. Such a choice is possible by Theorem 2. Since G is of Property A, there exists $r > 1$ and a function $\Theta : V(G) \to \text{Prob}(G)$ satisfying the following conditions.

- For any $x \in V(G)$ the support of $\Theta(x)$ is contained in the r-ball around the vertex x.
- For any adjacent pair $x, y \in V(G)$ we have that

$$\|\Theta(x) - \Theta(y)\|_1 < \delta\,.$$

Now, let H be an arbitrary finite induced subgraph of G. For $x \in V(G)$, pick $\tau(x) \in V(H)$ in such a way that $d_G(x, \tau(x)) = d_G(x, H)$. For $x \in V(H)$, let $\Omega(x)(z) = \sum_{t\in\tau^{-1}(z)} \Theta(x)(t)$. Note that $\tau^{-1}(z)$ denotes the set of vertices mapped to z by τ. Then by definition, $\operatorname{Supp}\Omega(x) \subset V(H)$ and for all $z \in V(H)$, $\Omega(x)(z) \geq 0$. Also,

$$\sum_{z\in V(H)} \Omega(x)(z) = \sum_{t\in V(G)} \Theta(x)(t) = 1\,,$$

hence $\Omega : V(H) \to \mathrm{Prob}(H)$. Also, if $x, y \in V(H)$ are adjacent vertices, then

$$\|\Omega(x) - \Omega(y)\|_1 \leq \delta.$$

Indeed,

$$\|\Omega(x) - \Omega(y)\|_1 = \sum_{z \in V(H)} |\Omega(x)(z) - \Omega(y)(z)| =$$

$$= \sum_{z \in V(H)} | \sum_{t \in \tau^{-1}(z)} \Theta(x)(t) - \sum_{t \in \tau^{-1}(z)} \Theta(y)(t)| \leq$$

$$\leq \sum_{z \in V(H)} \sum_{t \in \tau^{-1}(z)} |\Theta(x)(t) - \Theta(y)(t)| =$$

$$= \sum_{t \in V(G)} |\Theta(x)(t) - \Theta(y)(t)| = \|\Theta(x) - \Theta(y)\|_1 \leq \delta.$$

Observe that

$$\mathrm{Supp}(\Omega(x)) \subset B^G_{2r}(x), \tag{2}$$

where B^G_{2r} denotes the ball of radius $2r$ centered around x in the graph G. Indeed, if $\Omega(x)(z) \neq 0$, then there exists $t \in \tau^{-1}(z)$ such that $\Theta(x)(t) \neq 0$. Hence, $d_G(t, x) \leq r$ and also, $d_G(t, z) \leq r$, since $d_G(t, z) \leq d_G(t, x)$ by the definition of τ. That is, $d_G(x, z) \leq 2r$, so for any $x \in V(H)$ we have that (2) holds.

The following lemma finishes the proof of our proposition.

Lemma 2.2. *There exists a subset $L \subset V(H)$ such that $|\partial_H(L)| \leq \frac{d\delta}{2}|L|$ and $|L| \leq R_{2r}$, where R_{2r} is the maximal size of the $2r$-balls in G.*

Proof. By the inequalities above,

$$\sum_{x \in V(H)} \sum_{x \sim y} \|\Omega(x) - \Omega(y)\|_1 \leq \sum_{x \in V(H)} d\delta =$$

$$= \sum_{x \in V(H)} d\delta \|\Omega(x)\|_1 \,,$$

where here and going forward the summand y is required to be in $V(H)$. Hence,

$$\sum_{z \in V(H)} \sum_{x \in V(H)} \sum_{x \sim y} |\Omega(x)(z) - \Omega(y)(z)| \leq d\delta \sum_{z \in V(H)} \sum_{x \in V(H)} \Omega(x)(z)\,.$$

Hence, there exists $z_0 \in V(H)$ such that

$$\sum_{x \in V(H)} \sum_{x \sim y} |\Omega(x)(z_0) - \Omega(y)(z_0)| \leq d\delta \sum_{x \in V(H)} \Omega(x)(z_0)\,.$$

Thus, if we define the function $\zeta : V(H) \to \mathbb{R}$ by $\zeta(x) = \Omega(x)(z_0)$, we have that

$$\sum_{x \in V(H)} \sum_{x \sim y} |\zeta(x) - \zeta(y)| \leq d\delta \sum_{x \in V(H)} \zeta(x)\,. \tag{3}$$

That is ζ is a $d\delta$-Følner function on $V(H)$. So, by our assumption on δ, we can find a ϵ-Følner set $L \subset V(H)$ inside the support of ζ (that is, inside $B^G_{2r}(z_0)$). Hence, $|L| \leq R_{2r}$, thus our lemma follows. □

The proposition follows from the previous lemma right away. □

Proposition 2.3. *Uniformly locally amenable graphs are locally hyperfinite.*

Proof. First, let us remark that if G is uniformly locally amenable, then for any $\epsilon > 0$ there exists $k > 0$ such that all finite, induced subgraphs $H \subset G$ contain a *connected induced* subgraph L, $|V(L)| \leq k$ such that

$$|\partial_H(V(L)| \leq \epsilon|V(L)|. \tag{4}$$

Indeed, if for a subset $E \subset H$, $|\partial_H(E)| \leq \epsilon|E|$, $|E| \leq k$, then at least one of the induced graphs on the components of E satisfies (4).

So, let $\epsilon > 0$ and let $k > 0$ be as above. Set $H_1 := H$ and let L_1 be a connected subgraph of H_1 such that $|V(L_1)| \leq k$ and $|\partial_{H_1}(V(L_1))| \leq \epsilon|V(L_1)|$. Now let H_2 be the induced graph on $V(H_1)\backslash V(L_1)$. We pick a connected subgraph $L_2 \subset H_2$ such that $|V(L_2)| \leq k$ and $|\partial_{H_2}(V(L_2))| \leq \epsilon|V(L_2)|$. Inductively, we construct finite induced subgraphs $H_1 \supset H_2 \supset \dots$ and connected subgraphs $L_i \subset H_i$ such that $|V(L_i)| \leq k$ and $|\partial_{H_i}(V(L_i))| \leq \epsilon|V(L_i)|$ (of course, for large enough q, H_q and L_q are empty graphs).

Now, let $S := \cup_{i=1}^{\infty} \partial_{H_i}(V(L_i))$. Then, if remove S from H together with all the incident edges, the remaining components have size at most k. □

Proposition 2.4. *Locally hyperfinite graphs are weighted hyperfinite.*

Proof. Our proof is based on the one of Lemma 8.1 [54]. We call a finite graph H (δ, k)-hyperfinite if one can delete not more than $\delta|H|$ vertices of H together with all the incident edges such that the sizes of the remaining components are not greater than k. Also, we call a finite graph J (ϵ, k)-weighted hyperfinite, if for all positive weight function $w : V(J) \to \mathbb{R}$ one can delete a set of vertices $S \subset V(J)$ with total weight at most $\epsilon w(V(J))$ such that that the sizes of the remaining components is at most k. It is enough to prove that if $G \in Gr_d$, then for any $\epsilon > 0$ there exists $\delta > 0$ such that if all the finite induced subgraphs of G are (δ, k)-hyperfinite than they are (ϵ, k)-weighted hyperfinite as well.

Fix $\epsilon > 0$ and let L be the smallest integer that is larger than $\frac{3}{\epsilon}$. Now, assume that the finite induced subgraphs of the countably infinite graph $G \in Gr_d$ are (δ, k)-hyperfinite, where

$$\delta = \left(\frac{\epsilon}{3d}\right)^{L-1} \frac{\epsilon}{3}\,. \tag{5}$$

Let H be a finite induced subgraph of G. We partition the vertices of H into subsets $B_i, i \in \mathbb{Z}$ such that

$$B_i = \{x \mid \left(\frac{\epsilon}{3d}\right)^{i+1} < w(x) \leq \left(\frac{\epsilon}{3d}\right)^{i}\}\,.$$

For $0 \leq q \leq L-1$ consider the subset

$$D_q = \cup_{i\in\mathbb{Z}} B_{iL+q}\,.$$

Since $L > \frac{3}{\epsilon}$, there exists some $0 \leq m \leq L-1$ such that

$$w(D_m) < \frac{\epsilon}{3} w(V(H)). \tag{6}$$

Now, for $i \in \mathbb{Z}$ set

$$C_i = \cup_{q=1}^{L-1} B_{iL+m+q}\,.$$

By our assumption, if $x \in C_i$ and $y \in C_j$, $i < j$, then

$$\left(\frac{\epsilon}{3d}\right) w(x) > w(y)\,. \tag{7}$$

For $i < j$ let $F_{i,j}$ be the set of vertices y in C_j such that there exists $x \in C_i$, $x \sim y$. By the vertex degree assumption, $|\cup_{i<j} F_{i,j}| \leq d|C_i|$. Also, if $y \in \cup_{i<j} F_{i,j}$ and $x \in C_i$, then $\frac{\epsilon}{3d} w(x) \geq w(y)$. Therefore, $w(\cup_{i<j} F_{i,j}) \leq \frac{\epsilon}{3} w(C_i)$. Consequently,

$$w(F) \leq \frac{\epsilon}{3} w(V(H))\,, \tag{8}$$

where $F = \cup_{i\in\mathbb{Z}} (\cup_{i<j} F_{i,j})$. Let us delete the subsets F and D_m from $V(H)$ together with all the incident edges. Then, each of the remaining components are inside of the subsets C_i. Let T be such a component. By our assumption, T is (δ, k)-hyperfinite, thus one can delete a subset $S \subset V(T)$ so that $|S| \leq \delta|V(T)|$ in such a way that all the remaining components are of size at most k. By the definition of C_i, if $y \in V(T)$,

$$\min_{x\in V(T)} w(x) \geq \left(\frac{\epsilon}{3d}\right)^{L-1} w(y)\,.$$

Hence,

$$w(S) \leq |S|(\min_{x\in V(T)} w(x)) \left(\frac{3d}{\epsilon}\right)^{L-1} \leq \delta w(V(T)) \left(\frac{3d}{\epsilon}\right)^{L-1} = \frac{\epsilon}{3} w(V(T))\,.$$

By (6) and (8),

$$w(F \cup D_m) < \frac{2\epsilon}{3} w(V(H))\,.$$

So, by deleting a set of vertices of weight less than $\epsilon|V(H)|$, we obtained a graph that has components of size at most k. □

Proposition 2.5. *If $G \in Gr_d$ is weighted hyperfinite then G is strongly hyperfinite.*

Proof. First we prove the proposition for finite graphs. This part is based on the proof of Lemma 4.1 in [24]. A finite graph G is (ϵ, k)-strongly hyperfinite, if there exists a probability measure μ on $\mathrm{Sep}(G, k)$ such that for each $x \in V(G)$

$$\mu(\{Y \mid x \in Y\}) \leq \epsilon\,.$$

Lemma 2.6. *If a finite graph G is (ϵ, k)-weighted hyperfinite, then it is (ϵ, k)-strongly hyperfinite as well.*

Proof. Assume that G is (ϵ, k)-weighted hyperfinite and $V(G) = n$. Let m be the number of k-separators. For a k-separator Y let $\{\underline{c}_Y : V(G) \to \{0,1\}\} \in \mathbb{R}^n$ be its characteristic vector. We define the hull $\mathcal{H}$ of the k-separators as the convex set of vectors $\underline{z} \in \mathbb{R}^n$ which can be written in the form

$$\underline{z} = \sum_{i=1}^{m} x_i \underline{c}_{Y_i} + \underline{y}\,,$$

where $\{x_i\}_{i=1}^m$ are non-negative real numbers, $\sum_{i=1}^m x_i = 1$ and $\underline{y}$ is a non-negative vector. Now, let $\underline{v} = \{\epsilon, \epsilon, \dots, \epsilon\}$. We have two cases.

Case 1. $\underline{v} \in \mathcal{H}$. Then, there exist non-negative real numbers $\{x_i\}_{i=1}^m$, $\sum_{i=1}^m x_i = 1$ such that all the absolute values of the coordinates of the vector $\sum_{i=1}^m x_i \underline{c}_{Y_i}$ are less than or equal to ϵ. That is, if the probability measure μ on the k-separators is given by $\mu(Y_i) = x_i$, the (ϵ, k)-strong hyperfiniteness follows.

Case 2. $\underline{v} \notin \mathcal{H}$. Since $\mathcal{H}$ is a closed convex set, there must exist a hyperplane $H \subset \mathbb{R}^n$ containing $\underline{v}$ such that $\mathcal{H}$ is entirely on one side of the hyperplane H. That is, there exists a vector $\underline{w} \in \mathbb{R}^n$ such that for any $\underline{c} \in \mathcal{H}$ we have

$$\langle \underline{w}, \underline{v} \rangle < \langle \underline{w}, \underline{c} \rangle\,.$$

Clearly, for any $1 \leq i \leq n$, $w_i \geq 0$, since the i-th coordinate of $\underline{c}$, can be increased arbitrarily, while keeping the other coordinates fixed. We can also assume that $\sum_{i=1}^n w_i = 1$. So,

$$\epsilon < \langle \underline{w}, \underline{c}_Y \rangle$$

holds for any k-separator Y, that is, G is not (ϵ, k)-weighted hyperfinite, leading to a contradiction. □

Lemma 2.7. *Let G be a countably infinite graph of vertex degree d. Suppose that for any $\epsilon > 0$ there exists $k > 0$ such that all the finite subgraphs of G are (ϵ, k)-strongly hyperfinite. Then, G is strongly hyperfinite.*

Proof. Let $\{H_m\}_{m=1}^\infty$ be finite induced subgraphs in G such that $V(H_1) \subset V(H_2) \subset \dots$ and $\cup_{m=1}^\infty V(H_n) = V(G)$. Let ν_n be a probability measure on $\mathrm{Sep}(H_n, k)$ such that for all $x \in V(H_n)$ we have

$$\nu_n(\{Y \in \mathrm{Sep}(H_n, k) \mid x \in Y\}) \leq \epsilon\,.$$

For all $n \geq 1$ we have an injective map

$$\varphi_n : \mathrm{Sep}(H_n, k) \to \mathrm{Sep}(G, k)$$

mapping the k-separator Y_n to $Y = Y_n \cup (V(G) \backslash V(H_n))$. Now, let $\mu_n = (\varphi_n)_\star(\nu_n)$. Let $\mu_{n_k} \to \mu$ be a weakly convergent subsequence. For all $x \in V(G)$, let $U_x \subset \mathrm{Sep}(G, k)$ be the set of k-separators containing x. Clearly, U_x is a closed-open subset of $\mathrm{Sep}(G, k)$. By our assumptions, for large enough k, $\mu_{n_k}(U_x) \leq \epsilon$. Therefore, $\mu(U_x) \leq \epsilon$. Hence, our lemma follows. □

This finishes the proof of our proposition. □

Proposition 2.8. *If $G \in Gr_d$ is strongly hyperfinite, then G is of Property A.*

Proof. Let $\epsilon > 0$. Since G is strongly hyperfinite, there exists $k > 0$ and a probability measure μ on $\mathrm{Sep}(G, k)$ such that for all $x \in V(G)$

$$\mu(Y \mid x \in Y) \leq \frac{\epsilon}{4}\,. \tag{9}$$

We define a non-negative function $F : I_G(k) \to \mathbb{R}$, where $I_G(k)$ is the set of induced, connected subgraphs of G having at most k vertices, in the following way. Let $F(H)$ be defined as the μ-measure of the set of k-separators Y such that H is a component in $G \backslash Y$.

Now, let p_H be the uniform probability measure on H. Then, let

$$\Theta_x = \sum_{H \in I_G(k), x \in H} F(H) p_H + c_x \delta_x\,,$$

where

$$c_x := 1 - \sum_{H \in I_G(k), x \in H} F(H) \leq \frac{\epsilon}{4}\,.$$

Then, for all $x \in V(G)$, $\|\Theta_x\|_1 = 1$ and $\mathrm{Supp}(\Theta_x) \subset B_k^G(x)$.

Now, let $x \sim y$ be adjacent vertices. Then,

$$\|\Theta_x - \Theta_y\|_1 \leq c_x + c_y + \sum_{H, x \in H, y \notin H} F(H) + \sum_{H, x \notin H, y \in H} F(H) \leq \epsilon\,.$$

Therefore, G is of Property A. □

Now, by Propositions 2.1, 2.3, 2.4, 2.5 and 2.8, the Long Cycle Theorem follows. □

3. Følner functions

The goal of this section is to prove Theorem 2.

Proof. We will prove the existence of δ for a fixed graph $G \in Gr_d$. However, this is enough to prove our theorem. Indeed, assume that for some $\epsilon > 0$ there is a sequence of graphs $\{G_n\}_{n=1}^\infty$ such that the largest δ's satisfying the condition of our theorem tend to zero. Then, for the disjoint union $\cup_{n=1}^\infty G_n$ one could not pick a δ that satisfies the condition of our theorem. So, we fix $G \in Gr_d$ and $\epsilon > 0$. As follows, let q be the smallest integer that is greater than $\frac{16}{\epsilon^2}$. Let $\rho > 0$ be such that $(1+\rho)^{2q} = 1.1$. Let us begin the proof with a technical lemma.

Lemma 3.1. *Let p be a probability measure on the finite set $\{x_i\}_{i=1}^n \subset V(G)$ such that for any $1 \leq i \leq n$ we have $0 < p(x_i) < \frac{1}{2}$. Then there exist positive real numbers $a_1 < a_2 < \cdots < a_m$ satisfying the following conditions:*

(1) $a_1 < \frac{\epsilon^2}{16n}$.

(2) *For any* $1 \leq i \leq m-1$, $a_{i+1} = 2a_i$.
(3) $\sum_{i=1}^{m-1} p(M_i) > 1 - \frac{\epsilon^2}{8}$,
where $M_i = \{x_j \,|\, (1+\rho)a_i < p(x_j) < (1+\rho)^{-1}a_{i+1}\}$.

Proof. Let $s < \frac{\epsilon^2}{32n}$ be a positive real number such that for all non-negative integers k, l, j, $2^k s(1+\rho)^j \neq p(x_l)$. Let m be the largest integer such that $s2^m < \frac{1}{2}$. We define the disjoint subsets $T_1, T_2, \dots, T_q$ by

$$T_j := \bigcup_{i=1}^{m-1} \left(2^{i-1}s(1+\rho)^{2j-2}, 2^{i-1}s(1+\rho)^{2j-1}\right) \cup \left(2^i s(1+\rho)^{2j-3}, 2^i s(1+\rho)^{2j-2}\right)$$

So, there exists $1 \leq l \leq q$ such that

$$p(T_l) \leq \frac{1}{q} < \frac{\epsilon^2}{16} .$$

Set $a_1 := s(1+\rho)^{l-1} < \frac{\epsilon^2}{16n}$, and define $a_i := 2^{i-1}s(1+\rho)^{l-1}$ as in (2) of the lemma. Observe that $\sum_{i|p(x_i)<a_1} p(x_i) < \frac{\epsilon^2}{16}$, hence we have that

$$\sum_{i=1}^{m-1} p(M_i) > 1 - p(T_l) - \sum_{i|p(x_i)<a_1} p(x_i) > 1 - \frac{\epsilon^2}{8} . \qquad \square$$

We define the constant δ in the following lemma.

Lemma 3.2. *Let* $\delta = \frac{\epsilon^2(\sqrt{1+\rho}-1)}{16}$ *and let* $p : V(G) \to \mathbb{R}$ *be a* δ*-Følner function, such that* $\sum_{x \in V(G)} p(x) = 1$. *Define* $B(G) \subset V(G)$ *as the set of vertices* x *in the support of* p *such that there exists* $y \sim x$, *so that* y *is not in the support of* p *or there exists* $y \sim x$, *such that either* $\frac{p(x)}{p(y)} > \sqrt{1+\rho}$ *or* $\frac{p(y)}{p(x)} > \sqrt{1+\rho}$. *Then,*

$$\sum_{x \in B(G)} p(x) < \frac{\epsilon^2}{8} . \tag{10}$$

Proof. If $\frac{p(x)}{p(y)} > \sqrt{1+\rho}$, then

$$|p(x) - p(y)| > (1 - \frac{1}{\sqrt{1+\rho}})p(x) > \frac{\sqrt{1+\rho}-1}{2}p(x) .$$

If $\frac{p(y)}{p(x)} > \sqrt{1+\rho}$, then

$$|p(x) - p(y)| > (\sqrt{1+\rho} - 1)p(x) .$$

If y is not in the support of p, then we also have that

$$|p(x) - p(y)| = p(x) > (\sqrt{1+\rho} - 1)p(x) .$$

Hence, we have that

$$\sum_{x \in B(G)} \frac{\sqrt{1+\rho}-1}{2}p(x) < \sum_{x \in B(G)} \sum_{x \sim y} |p(x) - p(y)| < \delta.$$

Therefore, our lemma follows. □

Now let us consider our δ-Følner probability measure p and let n be the size of support of p. We may assume that all the values of p are smaller than $\frac{1}{2}$. Pick the numbers $\{a_i\}_{i=1}^m$ to satisfy the conditions of Lemma 3.1. For $1 \leq i \leq m-1$, let $b_i = \sqrt{1+\rho}a_i$, $c_i = (\sqrt{1+\rho})^{-1}a_{i+1}$. Finally, let

$$S_i := \{x_j \mid b_i \leq p(x_j) \leq c_i\}\,.$$

Lemma 3.3. *Let $Q = \cup_{i|\, S_i \text{ is not } \epsilon\text{-Følner}} S_i$. Then, $p(Q) < \frac{\epsilon}{2}$.*

Proof. Observe that

$$p\left(\cup_{i=1}^{m-1}\partial(S_i)\right) < \frac{\epsilon^2}{4}. \tag{11}$$

Indeed,

$$\cup_{i=1}^{m-1}\partial(S_i) \subset L \cup B(G)\,,$$

where L is the complement of $\cup_{i=1}^{m-1} M_i$ from Lemma 3.1 and $B(G)$ is the set defined in Lemma 3.2. So, (11) follows from Lemma 3.1 and Lemma 3.2.

Now, if S_i is not ϵ-Følner, then we have that

- $|\partial(S_i)| \geq \epsilon |S_i|$.
- $a_i|\partial(S_i)| < p(\partial(S_i)) < 2a_i|\partial(S_i)|$.
- $a_i|S_i| < p(S_i) < 2a_i|S_i|$.

Thus, $p(\partial(S_i)) > a_i\epsilon|S_i| > \frac{\epsilon}{2}p(S_i)$. Hence,

$$\sum_{i|\, S_i \text{ is not } \epsilon\text{-Følner}} p(S_i) \leq \frac{2}{\epsilon}\frac{\epsilon^2}{4} < \frac{\epsilon}{2}. \qquad \square$$

That is,

$$\sum_{i|\, S_i \text{ is } \epsilon\text{-Følner}} p(S_i) > 1 - (\sum_{i|\, S_i \text{ is not } \epsilon\text{-Følner}} p(S_i)) - (1 - \sum_{i=1}^{m-1} p(M_i)) > 1-\epsilon.$$

Therefore, if we set $H = \cup_{i|\, S_i \text{ is } \epsilon\text{-Følner}} S_i$, our theorem follows. □

Remark 1. In the case of Cayley graphs, Theorem 2.16 of [37] entails the existence of an ϵ-Følner set E in the support of a δ-Følner probability measure p, without any control on the measure of E.

4. Følner Graphs are Setwise Følner

Proposition 4.1. *Følner Graphs are Setwise Følner.*

Proof. Assume that there exists a Følner graph G that is not setwise Følner. That is, there exists $\epsilon > 0$ such that for any $k \geq 1$ there is a finite subset $L \subset V(G)$ so that there is no ϵ-Følner set in $B_k(L)$ that contains L.

Given $\epsilon > 0$, let $\delta = \delta(\frac{\epsilon}{2})$ be as in Theorem 2. So, for each δ-Følner function f there exists an $\frac{\epsilon}{2}$-Følner subset H inside the support of f so that

$$\sum_{x \in H} f(x) > (1 - \frac{\epsilon}{2}) \sum_{x \in V(G)} f(x) \,.$$

Let r be a natural number such that for any $x \in V(G)$ the ball $B_r^G(x)$ contains a $\frac{\delta}{d}$-Følner set. Now, for all $k \geq 1$ we choose a natural number T_k satisfying the inequality

$$T_k \ln(1 + \frac{\epsilon}{d}) > 2\ln(k) \,,$$

that is,

$$(1 + \frac{\epsilon}{d})^{T_k} > k^2. \tag{12}$$

By our assumptions, for every large enough $k \geq 1$ there exists a finite subset $L_k \subset V(G)$ such that there is no ϵ-Følner set H_k so that

$$L_k \subset H_k \subset B_{T_k + r}(L_k) \,.$$

Lemma 4.2. *If k is large enough then we have*

$$|B_{T_k}(L_k)| > kR_r|L_k| \,, \tag{13}$$

where R_r is the size of the largest ball of radius r in the graph G.

Proof. For $k \geq 1$ we consider the subsets

$$L_k \subset B_1(L_k) \subset B_2(L_k) \subset \cdots \subset B_{T_k}(L_k) \,.$$

By our assumptions, for $0 \leq i \leq T_k$ $B_i(T_k)$ is not an ϵ-Følner set, hence $|B_{i+1}(L_k)| > (1 + \frac{\epsilon}{d})|B_i(L_k)|$, that is

$$|B_{T_k}(L_k)| > (1 + \frac{\epsilon}{d})^{T_k} |L_k| \,.$$

Hence by (12), for large enough k we have that

$$|B_{T_k}(L_k)| > kR_r|L_k| \,,$$

so our lemma follows. □

Now, for each $y \in B_{T_k}(L_k)$ consider the uniform probability measure p_y of a $\frac{\delta}{d}$-Følner set inside $B_r^G(y)$. Clearly, p_y is a δ-Følner function. Therefore, $g = \sum_{y \in B_{T_k}(L_k)} p_y$ is a δ-Følner function as well, supported in the set $B_{T_k + r}(L_k)$. So by our assumption, there exists an $\frac{\epsilon}{2}$-Følner subset $H_k \subset B_{T_k + r}(L_k)$ such that

$$\sum_{z \in H_k} g(z) > (1 - \frac{\epsilon}{2}) \sum_{z \in B_{T_k + r}(L_k)} g(z) \,.$$

By definition, for any $z \in V(G)$, $g(z) \leq R_r$. That is,

$$|H_k|R_r > (1 - \frac{\epsilon}{2}) \sum_{z \in B_{T_k + r}(L_k)} g(z) > (1 - \frac{\epsilon}{2}) kR_r|L_k| \,.$$

So, the inequality $|H_k| > \frac{k}{2}|L_k|$ holds provided that k is large enough. Therefore, for large k values the subset $H_k \cup L_k$ is an ϵ-Følner set inside the subset $B_{T_k+r}(L_k)$ containing L_k, leading to a contradiction. □

Proposition 4.3. *For sufficiently large d, there exist Følner graphs in Gr_d that are not almost finite.*

Proof. Let us pick a c-expander sequence of finite connected graphs $\{G_n\}_{n=1}^\infty$. That is, for some $0 < c < 1$ the following condition holds. If $n \geq 1$, $L_n \subset V(G_n)$ and $|L_n| \leq \frac{1}{2}|V(G_n)|$, then $|\partial(L_n)| \geq c|L_n|$. We also assume that the diameter of G_n is at least n. Now fix a sequence of integers $\{a_n\}_{n=1}^\infty$ such that

$$n2^{n+1} < a_n. \tag{14}$$

Now we pick pairs of vertices $x_k, y_k, d_{G_k}(x_k, y_k) = n$ and for each k connect y_k and x_{k+1} by a new edge. Let G be the resulting connected graph. Now for any $n \geq 1$ pick a maximal subset X_n of $V(G)$ satisfying the following conditions.

- If $x \neq y$ are elements of X_n then $d_G(x,y) > 2a_n$.
- If $a_n > \frac{\text{diam}(G_k)}{3}$, then $G_k \cap X_n$ is empty.

Now for each $n \geq 1$ attach a path of length n to all elements of X_n. If a vertex belongs to more than one set X_n, we attach only one path to it - the longest among them. Note that, by our conditions, a vertex can belong to only finitely many of the sets X_n. The resulting graph G^0 is Følner, since every vertex is at bounded distance from one of the attached paths of length n. For each $k \geq 1$, let Y_k be the set of vertices in $V(G^0)\backslash V(G)$ that are in a path that is attached to a vertex of G_k. Also, let $X_n^k := V(G_k) \cap X_n$.

Lemma 4.4. *For each $n \geq 1$ and $k \geq 1$, we have that*

$$a_n |X_n^k| \leq |V(G_k)|\,. \tag{15}$$

Proof. By our condition, the balls of radius a_n centered around the elements of X_n are disjoint. If $X_n^k \neq \emptyset$, then $\text{diam}(G_k) \geq 3a_n$ by definition of X_n^k. Also, if $x \in X_n^k$ then $|B_{a_n}^G(x) \cap V(G_k)| \geq a_n$. Indeed, let z be one of the elements in G_k that is farthest from x. Then, the shortest path in G_k from x to z contains at least a_n elements contained in $B_{a_n}^G(x)$. Therefore, $a_n|X_n^k| \leq |V(G_k)|$. □

Hence by (14) and (15), we have that $n|X_n^k| < \frac{1}{2^{n+1}}|V(G_k)|$. That is, we have that

$$|Y_k| < \frac{1}{4}|V(G_k)|. \tag{16}$$

Assume that G^0 is almost finite. Then, we have a tiling of $V(G^0)$ by $\frac{c}{10}$-Følner sets $\{T_i\}_{n=1}^\infty$ such that $\text{diam}(T_i) < r$ for some integer r. Clearly, if k is large enough there exists a tile T_i such that $T_i \cap V(G) = S$ is fully contained in $V(G_k)$ and $|T_i \cap Y_k| \leq \frac{|S|}{2}$. Then $|\partial S| \geq c|T_i| \geq \frac{c}{2}|S|$, in contradiction with the fact that T_i is a $\frac{c}{10}$-Følner set. □

5. The Short Cycle Theorem

The goal of this section is to prove Theorem 3. The way we prove the theorem is showing that: Property A + Setwise Følner ⇒ Strong Følner hyperfiniteness ⇒ Fractional almost finiteness ⇒ Følner Property A ⇒ Property A + Setwise Følner.

First, let us define a compact metric space structure on the space $P_G(\epsilon, r)$ of (ϵ, r)-Følner packings (see Introduction) in G. These packings $\mathcal{P}$ are special equivalence relations on $V(G)$. If $x, y \in V(G)$ are in the same elements of $\mathcal{P}$, then $x \equiv_{\mathcal{P}} y$. The vertices that are not covered by the elements of $\mathcal{P}$ form classes of size 1. Let us enumerate the vertices of G, $\{x_1, x_2, x_3, \dots\}$. Let the distance of two (ϵ, r)-Følner packings $\mathcal{P}_1$ and $\mathcal{P}_2$ be 2^{-n} if

- For $1 \leq i, j \leq n-1$ we have that $x_i \equiv_{\mathcal{P}_1} x_j$ if and only if $x_i \equiv_{\mathcal{P}_2} x_j$.
- For some $1 \leq i \leq n-1$, $x_i \equiv_{\mathcal{P}_1} x_n$ and $x_i \not\equiv_{\mathcal{P}_2} x_n$ or $x_i \not\equiv_{\mathcal{P}_1} x_n$ and $x_i \equiv_{\mathcal{P}_2} x_n$.

It is not hard to see that $P_G(\epsilon, r)$ a compact space with respect to the above metric.

We call the graph G **strongly Følner hyperfinite** if for any $\epsilon > 0$ there exist $r \geq 1$ and a Borel probability measure ν on the compact space $P_G(\epsilon, r)$ such that for any vertex $x \in V(G)$

$$\nu(\{\mathcal{P} \in P_G(\epsilon, r) \mid x \notin \tilde{\mathcal{P}}\}) < \epsilon \,,$$

where $\tilde{\mathcal{P}}$ denotes the set of vertices contained in the elements of the packing $\mathcal{P}$.

Proposition 5.1. *If the graph $G \in Gr_d$ is of Property A and is setwise Følner, then G is strongly Følner hyperfinite.*

Proof. Fix $\epsilon > 0$. Let $k > 0$ be an integer such that for any finite set $L \subset V(G)$ there exists an ϵ-Følner set H such that $L \subset H \subset B_k(L)$. Also, let R_{k+1} be the size of the largest $k+1$-ball in $V(G)$. By Theorem 1, we have an integer $l > 0$ such that there exists a Borel probability measure μ on the compact space $\mathrm{Sep}(G, l)$ of l-separators such that for any $x \in V(G)$:

$$\mu(\{Y \mid x \in Y\}) < \frac{\epsilon}{R_{k+1}} \,. \tag{17}$$

Then, for any $x \in V(G)$:

$$\mu(\{Y \mid B_{k+1}(x) \cap Y \neq \emptyset\}) < \epsilon. \tag{18}$$

We define the map $\Theta : \mathrm{Sep}(G, l) \to \mathrm{Sep}(G, l)$ by $\Theta(Y) = B_{k+1}(Y)$. Clearly, Θ is continuous. Now, let Y be an l-separator such that if we delete Y together with all the incident edges, the remaining components are $\{J_i^Y\}_{i=1}^{\infty}, |V(J_i^Y)| \leq l$.

Then, if we delete $\Theta(Y)$ from $V(G)$ the remaining vertices are in the disjoint union $\cup_{i=1}^{\infty} A_i^Y$, where $A_i^Y = V(J_i^Y) \setminus B_{k+1}(Y)$.

Now, for each $1 \leq i < \infty$ we pick the smallest ϵ-Følner set H_i such that $A_i^Y \subset H_i^Y \subset B_k(A_i^Y) \subset V(J_i^Y)$ in the following way. We enumerate the vertices of $V(G)$ and if there are more than one ϵ-Følner sets of the smallest size in $B_k(A_i^Y)$, then we pick the first one in the lexicographic ordering. Note that for every i, $\mathrm{diam}_G(H_i) \leq r$, where $r = 2k + l$.

Therefore, we have a continuous map

$$\Phi : \mathrm{Sep}(G, l) \to P_G(\epsilon, r)\,.$$

If $B_{k+1}(x) \cap Y = \emptyset$ then x is in some of the element of the packing $\Phi(Y)$. Hence if ν is the push-forward measure $\Phi_\star(\mathrm{Sep}(G, l))$, then by (18) for any $x \in V(G)$ we have that

$$\nu(\{\mathcal{P} \in P_G(\epsilon, r) \mid x \notin \tilde{\mathcal{P}}\}) < \epsilon\,.$$

Hence, G is strongly Følner hyperfinite. □

Proposition 5.2. *If G is strongly Følner hyperfinite then G is fractionally almost finite as well.*

Proof. Fix $\epsilon > 0$. Then there exists $r \geq 1$ and a Borel probability measure μ of $P_G(\epsilon, r)$ such that for every $x \in V(G)$

$$\mu(\{\mathcal{P} \mid y \in \tilde{\mathcal{P}} \text{ for all } y \text{ such that } d_G(x, y) \leq 1\}) > 1 - \epsilon. \tag{19}$$

For any (ϵ, r)-Følner set H let $F(H)$ be defined as the μ-measure of all (ϵ, r)-Følner packings $\mathcal{P}$ such that $H \in \mathcal{P}$. Then, F clearly satisfies the two conditions above. □

Proposition 5.3. *If the graph $G \in Gr_d$ is fractionally almost finite then G is of Følner Property A.*

Proof. For $\epsilon > 0$, let $r > 0$ be an integer and $F : \mathcal{F}_G(\epsilon, r) \to \mathbb{R}$, c_x be as in the definition of fractional hyperfiniteness. Since H is an ϵ-Følner set, the uniform probability measure p_H defined on H is a $2d\epsilon$-Følner function. Now, for $x \in V(G)$, let

$$P_x := \sum_{x \in H, H \in \mathcal{F}_G(\epsilon, r)} F(H) p_H + c_x \delta_x\,.$$

Then, P_x is a $2\epsilon d$-Følner function. If $y \sim x$ is an adjacent vertex then we have the following inequality.

$$\|P_x - P_y\|_1 \leq c_x + c_y + \sum_{x \in \partial(H), H \in \mathcal{F}_G(\epsilon, r)} F(H) + \sum_{y \in \partial(H), H \in \mathcal{F}_G(\epsilon, r)} F(H) \leq 4\epsilon.$$

Therefore, G has Følner Property A. □

By Theorem 2 and Proposition 4.1, we immediately have the following result.

Proposition 5.4. *If G is of Følner Property A, then G is of Property A and it is setwise Følner.*

By Propositions 5.1, 5.2, 5.3 and 5.4 the Short Cycle Theorem follows. □

6. Strong Følner Hyperfiniteness implies Strong Almost Finiteness

In this section we establish one more equivalent of Strong Følner Hyperfiniteness. First let us give a precise definition for strong almost finiteness.

Definition 6.1. The graph $G \in Gr_d$ is strongly almost finite if for any $\epsilon > 0$ there exists $r \geq 1$ and a probability measure ν on the space $P_G(\epsilon, r)$ of (ϵ, r)-Følner packings satisfying the following two conditions.

- ν is concentrated on tilings, that is, on packings $\mathcal{P}$ that fully covers $V(G)$.
- For each $x \in V(G)$ the ν-measure of tilings such that x in on the boundary of the tile containg x is less than ϵ.

The goal of this section is to prove Theorem 4.

Proof. The "if" part follows from the definition, we need to focus on the "only if" part. So, let $G \in Gr_d$ be a strongly Følner hyperfinite graph. The next proposition has analogues in Ornstein-Weiss theory, but the assumption of strong Følner hyperfiniteness in the present setting makes the proof simpler.

Proposition 6.2. *For any $\epsilon > 0$ there exists $\delta > 0$, $r \geq 1$ and an (ϵ, r)-Følner packing $\mathcal{P} = \{H_i\}_{i=1}^{\infty}$ such that for any δ-Følner set $T \subset V(G)$, the subsets H_i that are contained in T cover at least $(1-\epsilon)|T|$ vertices of T.*

Proof. Let $r \geq 1$ and ν be a Borel probability measure on the space $P_G(\epsilon, r)$ of (ϵ, r)-Følner packings such that for any vertex $x \in V(G)$

$$\nu(\{\mathcal{P} \mid x \notin \tilde{\mathcal{P}}\}) < \frac{\epsilon}{10}.$$

Pick $\delta > 0$ in such a way that if T is a δ-Følner set in $V(G)$ then

$$\frac{|T'|}{|T|} > 1 - \frac{\epsilon}{10},$$

where

$$T' := \{y \in T \mid d_G(y, \partial(T)) > 2r\}.$$

Observe that if $x \in T'$ for some δ-Følner set T, then $B^G_{2r+1}(x) \subset T$. Hence, we have the following lemma.

Lemma 6.3. *If H_1 and H_2 are both (ϵ, r)-Følner sets, H_1 intersects T' and H_2 intersects the complement of T, then H_1 and H_2 are disjoint sets.*

Now, for $x \in T'$ let ω_x be a random variable on the probability space $(P_G(\epsilon, r), \nu)$ such that $\omega_x(P) = 0$ if $x \in \tilde{P}$, $\omega_x(P) = 1$ if $x \notin \tilde{P}$. Then we have the following inequality for the expected value.

$$E(\omega_x) < \frac{\epsilon}{10}. \tag{20}$$

Lemma 6.4. *For any δ-Følner set T there exist an (ϵ, r)-Følner packing $\mathcal{P}$ such that $|T''| > (1-\frac{\epsilon}{5})|T|$, where T'' is the set of points in T that are covered by Følner-sets H in the packing $\mathcal{P}$ that intersects T'.*

Proof. By (20),

$$E(\sum_{x\in T'} \omega_x) < \frac{\epsilon}{10}|T'|\,.$$

Therefore, there exists a packing $\mathcal{P} \in P_G(\epsilon, r)$ that covers at least $(1-\frac{\epsilon}{10})|T'|$ vertices in T'. Hence,

$$|T''| > (1-\frac{\epsilon}{10})(1-\frac{\epsilon}{10})|T| > (1-\frac{\epsilon}{5})|T|\,. \tag{21}$$

□

Let us enumerate the δ-Følner sets $\{S_1, S_2, \dots\}$ in G. Let $\mathcal{P}_n = \{H_i^n\}_{i=1}^\infty$ be an (ϵ, r)-Følner packing such that it covers the maximal amount of vertices in $\cup_{j=1}^n S_j$.

Lemma 6.5. *For each $1 \le j \le n$, the set of H_i^n's that are contained in S_j covers at least $(1-\epsilon)|S_j|$ vertices in S_j.*

Proof. Assume that there exists $1 \le j \le n$ that does not satisfy the covering statement of the lemma. Let m be the number of vertices covered in $\cup_{q=1}^n S_j$ by $\mathcal{P}_n$. First, let us delete all the sets H_i^n from the packing $\mathcal{P}_n$ that are in S_j. Now the number of vertices covered in $\cup_{q=1}^n S_j$ remains at least $m-(1-\epsilon)|S_j|$. Using the previous lemma we can add (ϵ, r)-Følner sets in such a way that

- We increase the number of vertices covered in $\cup_{j=1}^n S_j$ by at least $(1-\frac{\epsilon}{5})|S_j|$.
- We still obtain an (ϵ, r)-Følner packing by Lemma 6.3.

So the new packing covers more than m vertices in $\cup_{j=1}^n S_j$ leading to a contradiction. □

Now we can finish the proof of our proposition. Let k be the maximal size of an (ϵ, r)-Følner set in G. Let $\{\mathcal{P}_{n_i}\}_{i=1}^\infty$ be a convergent subsequence in the compact space $P_G(\epsilon, r)$ converging to $\mathcal{P}$. By the definition of convergence and the previous lemma, $\mathcal{P}$ will satisfy the condition of our proposition. □

Proposition 6.6. *If the graph $G \in Gr_d$ is strongly Følner hyperfinite, then G is almost finite.*

Proof. Fix $0 < \epsilon < \frac{1}{3}$. Let $0 < \delta < \frac{1}{2}$, $r > 0$ so that by Proposition 6.2 there exists a (ϵ, r)-Følner packing $\mathcal{P} = \{H_i\}_{i=1}^\infty$ so that for each δ-Følner set T at least $(1-\epsilon)|T|$ vertices of T are covered by some $H_i \subset T$. Since G is fractionally almost finite by Theorem 3, there exists $k > 0$ and a non-negative function F on the space $\mathcal{F}_G(\delta, k)$ of δ-Følner sets of radius less than k, such that for any $x \in V(G)$ we have that

$$\sum_{x\in T,\, T\in\mathcal{F}_G(\delta,k)} F(T) + c_x = 1\,, \tag{22}$$

with $0 \le c_x < \delta$. Pick a subset $K_i \subset H_i$ such that $3\epsilon|H_i| < |K_i| < 4\epsilon|H_i|$. Let $A_1 \subset V(G)$ be the set of vertices not covered by any H_i and let $A_2 \subset V(G)$ be the set of vertices that are in some K_i. That is, for any δ-Følner set T we have that

$$2|T \cap A_1| < 2\epsilon|T| < 3\epsilon(1-\epsilon)|T| < |T \cap A_2|\,. \tag{23}$$

Let us construct a weighted, directed, bipartite graph $D(A_1, A_2)$ in the following way. For each (δ, k)-Følner set T so that $F(T) > 0$ and for each $x \in T \cap A_1$ we draw two outgoing edges towards $T \cap A_2$. One edge has weight $F(T)$, the other one has weight $c_x \frac{F(T)}{1-c_x}$. By (23), we can assume that for any T the endpoints of the drawn edges are different. Also by (22), for each vertex $x \in A_1$, the sum of the weights on the outgoing edges is 1 and for each vertex $y \in A_2$ the sum of the weights on the incoming edges is less than or equal to 1.

Therefore our directed graph satisfies the Hall condition, any finite subset M of A_1 has at least $|M|$ adjacent vertices in A_2. So by the Marriage Theorem, using a strategy somewhat similar to [16], there exists an injective map $\Phi : A_1 \to A_2$ such that for any $x \in A_1$, $d_G(x, \Phi(x)) < k$. Now, for each $1 \le i < \infty$ set $S_i = H_i \cup \Phi^{-1}(H_i)$. Then,

$$|\partial(S_i)| \le |\partial(H_i)| + |\Phi^{-1}(H_i)| \le 5\epsilon|S_i|.$$

Therefore, we have a partition $V(G) = \cup_{i=1}^{\infty} S_i$, where each S_i is a 5ϵ-Følner set and

$$\operatorname{diam}_G(S_i) \le 2k + r\,.$$

Hence, G is almost finite. $\square$.

Now let us finish the proof of our theorem. First fix $\epsilon > 0$. Since G is almost finite by Proposition 6.6, we have a partition $V(G) = \cup_{i=1}^{\infty} T_i$, where all the T_i's are ϵ-Følner having diameter at most t. Let us pick $\delta > 0$, $r > 0$ and a probability measure ν on $\mathcal{P}_G(\delta, r)$ in such a way that

- For any (δ, r)-Følner set F the set F' is ϵ-Følner whenever F' is the union of F and some of the sets T_i intersecting F.
- The measure ν is concentrated on packings, where the distance of two associated (δ, r)-Følner sets is at least $3t$.
- For any $x \in V(G)$, $\nu(\{\mathcal{P} \mid x \notin \tilde{\mathcal{P}}\}) < \delta$.

For each packing $\mathcal{P} = \{Q_j^{\mathcal{P}}\}_{j=1}^{\infty}$ we construct a tiling $\tau_{\mathcal{P}}$ in the following way. For $1 \le j < \infty$ let $R_j^{\mathcal{P}}$ be the union of $Q_j^{\mathcal{P}}$ and all the sets T_i intersecting $Q_j^{\mathcal{P}}$. Hence, by our condition $R_j^{\mathcal{P}}$ is ϵ-Følner. The remaining tiles in $\tau_{\mathcal{P}}$ are the sets T_i's that are not intersecting any $Q_j^{\mathcal{P}}$. By pushing-forward ν, we have a measure on the tilings $\tau_{\mathcal{P}}$ satisfying the definition of strong almost finiteness. $\square$

7. Examples of strongly almost finite graphs

In this section using the Short Cycle Theorem and Theorem 4, we extend the almost finiteness results of [16] about Cayley graphs to large classes of general graphs, to graphs of subexponential growth and to Schreier graphs of amenable groups.

First let us recall the definition of subexponential growth.

Definition 7.1. The graph $G \in Gr_d$ is of subexponential growth if

$$\lim_{r\to\infty} \sup_{x\in V(G)} \frac{\ln(|B^G_r(x)|}{r} = 0\,.$$

The following lemma is well-known, we provide the proof for completeness.

Lemma 7.2. *If $G \in Gr_d$ is a graph of subexponential growth, then for any $\epsilon > 0$ there exists $r > 0$ such that for all $x \in V(G)$ there is an $1 \leq i \leq r$ so that*

$$\frac{|B^G_{i+1}(x)|}{|B^G_i(x)|} < 1 + \epsilon\,.$$

.

Proof. Suppose that the statement of the lemma does not hold. Then, there exists an $\epsilon > 0$, a sequence of vertices $\{x_n \in V(G)\}_{n=1}^\infty$ and an increasing sequence of natural numbers $\{i_n\}_{n=1}^\infty$ such that for any $n \geq 1$, $\frac{\ln(|B^G_{i_n}(x_n)|}{i_n} \geq \ln(1+\epsilon)$, in contradiction with the definition of subexponentiality. □

Proposition 7.3. *Graphs $G \in Gr_d$ of subexponential growth are strongly almost finite.*

Proof. Let $G \in Gr_d$ be a graph of subexponential growth. By the Short Cycle Theorem and Theorem 4, it is enough to prove that G is Følner and it is of Property A. Observe that being a Følner graph follows immediately from Lemma 7.2. It has already been proved in [59] (Theorem 6.1) that graphs of subexponential growth are of Property A, nevertheless we give a very short proof of this fact using the Long Cycle Theorem. Let $H \in Gr_d$ be the graph obtained by taking the disjoint union of all induced subgraphs of G up to isomorphism. Clearly, H has subexponential growth, so H is Følner. However, the the Følnerness of H implies that G is uniformly locally amenable, hence by the Long Cycle Theorem, G is of Property A. □

Now Γ be a finitely generated group and Σ be a finite, symmetric generating set of Γ. Let $H \subset \Gamma$ be a subgroup. Recall that the Schreier graph $\mathrm{Sch}(\Gamma/H, \Sigma)$ is defined as follows.

- The vertex set of $\mathrm{Sch}(\Gamma/H, \Sigma)$ consists of the the right cosets $\{Ha\}_{a\in\Gamma}$.
- The coset Ha is adjacent to the coset $Hb \neq Ha$ if $Hb = Ha\sigma$ for some $\sigma \in \Sigma$.

If $H = e$ is the trivial subgroup, then the Cayley graph $\mathrm{Cay}(\Gamma, \Sigma)$ equals $\mathrm{Sch}(\Gamma/e, \Sigma)$.

Proposition 7.4. *For any amenable group* Γ *and symmetric generating system* Σ*, the graph* $\mathrm{Sch}(\Gamma/H, \Sigma)$ *is strongly almost finite.*

Proof. Pick $\epsilon > 0$. First, we will be working with $\mathrm{Cay}(\Gamma, \Sigma)$. Let F be a $\frac{\epsilon}{2|\Sigma|}$-Følner set in Γ containing the unit element. By the amenability of Γ, such subset exists. For $x \in \Gamma$ let P_{xF} be the uniform probability measure on the translate xF. By our condition on F, if $x \sim y$ we have that

$$|xF \triangle yF| \leq |F \triangle (x^{-1}yF \triangle yF)| + |yF \triangle F| < \epsilon |F| \,. \tag{24}$$

Therefore, $\|P_{xF} - P_{yF}\|_1 < \epsilon$. Also $\mathrm{Supp}(P_{xF}) \subset B^{\mathrm{Cay}(\Gamma,\Sigma)}_{\mathrm{diam}(F)}(x)$. Finally, we have that

$$\sum_{z \in \Gamma} \sum_{\sigma \in \Sigma} |P_{xF}(z) - P_{xF}(z\sigma)| \leq \frac{2|\partial(F)|}{|F|} |\Sigma| < \epsilon \,.$$

That is, P_{xF} is an ϵ-Følner function. By (24), $\mathrm{Cay}(\Gamma, \Sigma)$ is of Følner Property A. Net we will use the functions P_{xF} to prove that $\mathrm{Sch}(\Gamma/H, \Sigma)$ is of Følner Property A as well.

Let $f : \Gamma \to \mathbb{R}$ be a finitely supported non-negative function. Let us define $f^H : \Gamma/H \to \mathbb{R}$ by setting $f^H(Hz) = \sum_{x \in Hz} f(x)$.

Lemma 7.5. *We have that*

(a) $\|f^H\|_1 = \|f\|_1$.
(b) *If* $g : \Gamma \to \mathbb{R}$ *is another finitely supported non-negative function then* $\|f^H - g^H\|_1 \leq \|f - g\|_1$.
(c)

$$\sum_{a \in \Gamma/H} \sum_{\sigma \in \Sigma} |f^H(a) - f^H(a\sigma)| \leq \sum_{x \in \Gamma} \sum_{\sigma \in \Sigma} |f(x) - f(x\sigma)| \,.$$

Proof. First, we have that

$$\|f^H\|_1 = \sum_{a \in \Gamma/H} f^H(a) = \sum_{a \in \Gamma/H} \sum_{x \in Ha} f(x) = \sum_{x \in \Gamma} f(x) = \|f\|_1 \,.$$

Then,

$$\|f^H - g^H\|_1 = \sum_{a \in \Gamma/H} |f^H(a) - g^H(a)| = \sum_{a \in \Gamma/H} |(\sum_{x \in Ha} f(x)) - (\sum_{x \in Ha} g(x))| \leq$$

$$\leq \sum_{x \in \Gamma} |f(x) - g(x)| = \|f - g\|_1 \,.$$

Finally,

$$\sum_{a \in \Gamma/H} \sum_{\sigma \in \Sigma} |f^H(a) - f^H(a\sigma)| = \sum_{a \in \Gamma/H} \sum_{\sigma \in \Sigma} |(\sum_{x \in Ha} f(x)) - (\sum_{x \in Ha} f(x\sigma))| \leq$$

$$\sum_{x \in \Gamma} \sum_{\sigma \in \Sigma} |f(x) - f(x\sigma)| \,. \quad \square$$

Now, we finish the proof of our proposition. Let $\Theta : \Gamma/H \to \mathrm{Prob}(\Gamma/H)$ be defined by $\Theta(Hx) := P^H_{xF}$. Note that if $Hx = Hy$, then

$$P^H_{xF} = P^H_{yF}\,, \tag{25}$$

so, Θ is well-defined. Clearly, if $\mathrm{Supp}(f) \subset B_r^{\mathrm{Cay}(\Gamma,\Sigma)}(x)$, then $\mathrm{Supp}(f^H) \subset B_r^{\mathrm{Sch}(\Gamma/H,\Sigma)}(Hx)$. Therefore, for every $Hx \in \Gamma/H$ we have that $\mathrm{Supp}(\Theta(Hx)) \subset B_r^{\mathrm{Sch}(\Gamma/H,\Sigma)}(Hx)$, where $F \subset B_r^{\mathrm{Cay}(\Gamma,\Sigma)}(e)$.

By the previous lemma, for every $Hx \in \Gamma/H$, $\Theta(Hx)$ is an ϵ-Følner probability measure. Again, by the previous lemma, if $Hy = H\sigma x$ we have that

$$\|\Theta(Hx) - \Theta(Hy)\|_1 < \epsilon\,.$$

Therefore, $\mathrm{Sch}(\Gamma/H, \Sigma)$ is of Følner Property A. Hence, by the Short Cycle Theorem and Theorem 4, our proposition follows. □

Remark 2. Note that if N is a normal subgroup in a free group Γ and Γ/N is a nonexact group then the Cayley graph of Γ is of Property A, but the Cayley graph of Γ/N is not of Property A. One might wonder, why Lemma 7.5 does not imply that the Schreier graphs of Property A groups are of Property A themselves. The reason is that we used amenability in the proof of Proposition 7.4 in a crucial way. The functions $\{P_{xF}\}_{x\in\Gamma}$ form an automorphism invariant system, that is why we have (25). If the group Γ had such a canonical system of functions for every $\epsilon > 0$, then all of the continuous actions of Γ on the Cantor set would be topologically amenable (see Subsection 10.6), hence the group Γ would have to be amenable. Indeed, free continuous actions of nonamenable groups admitting invariant probability measures are never topologically amenable (see Proposition 10.11). Note that every countable group has free, minimal, continuous actions on the Cantor set that admit invariant probability measures [25].

Let Γ be an amenable group equipped with a generating system Σ, $H \subset \Gamma$ be a subgroup and let $\pi_H : \Gamma \to \Gamma/H$ be the factor map, mapping x into Hx. Then it is not true that for any $\epsilon > 0$ there is some $\delta > 0$ such that the image of a δ-Følner set is always an ϵ-Følner set. Indeed, let $\Gamma = \mathbb{Z} \times \mathbb{Z}$ and H be the first $\mathbb{Z}$-factor. Let $F_n = [0, n^2] \times [0, n]$. Now let J_n be a set of n elements in Γ such that the second coordinates of these elements are positive integers greater than n^2 and their pairwise difference is at least 2. For large n values both $G_n = F_n \cup J_n$ and F_n are δ-Følner sets with very small δ, however, $\pi_H(G_n)$ is not even $\frac{1}{3}$-Følner set. By removing J_n from G_n, we obtain a set such that its image is "very" Følner. The following proposition shows that this is always the case.

Proposition 7.6. *Let Γ be an amenable group with a symmetric generating set Σ. Then, for any $\epsilon > 0$ there is some $\delta = \delta(\epsilon) > 0$ as in Theorem 2 such that if $H \subset \Gamma$ is a subgroup and Q is a δ-Følner set in $\mathrm{Cay}(\Gamma/H, \Sigma)$, then we have a subset J, $|J| < \epsilon|Q|$ so that the subset $\pi_H(Q\backslash J)$ is an ϵ-Følner set in $\mathrm{Sch}(\Gamma/H, \Sigma)$.*

Proof. Fix $\epsilon > 0$ and let $\delta = \delta(\epsilon) > 0$ be as in Theorem 2 so that if p is a δ-Følner probability measure on the vertex set a graph $G \in Gr_d$, then there exists an ϵ-Følner set T inside the support of p so that the p-measure of $\mathrm{Supp}(p)\backslash T$ is less than ϵ. Let Q be a $\frac{\delta}{2d}$-Følner set in $\mathrm{Cay}(\Gamma, \Sigma)$. Then the uniform probability measure p_Q is a δ-Følner function. By Lemma 7.5, the function Q^H is a δ-Følner function supported on $\pi_H(Q) \subset \mathrm{Sch}(\Gamma/H, \Sigma)$. So by our condition, we have an ϵ-Følner set E inside $\pi_H(Q)$ such that the measure of $\pi_H(Q)\backslash E$ with respect to the probability measure Q^H is less than ϵ. That is,

$$|\pi_H^{-1}(\pi_H(Q)\backslash E)| \leq \epsilon |Q| \,.$$

Therefore by choosing $J = \pi_H^{-1}(\pi_H(Q)\backslash E)$, our proposition follows. □

8. Neighborhood convergence

Definition 8.1. The graphs $G, H \in Gr_d$ are called **neighborhood equivalent**, $G \equiv H$ if for any rooted ball $B_r^G(x) \subset G$ there exists a rooted ball $B_r^H(y) \subset H$ that is rooted-isomorphic to $B_r^G(x)$ and conversely, for any rooted ball $B_r^H(u) \subset H$ there exists a rooted ball $B_r^G(v) \subset G$ that is rooted-isomorphic to $B_r^H(u)$. So, if $\hat{G}$ denotes the set of all r-balls in G up to rooted isomorphism, then G and H are neighborhood equivalent if and only if $\hat{G} = \hat{H}$.

We call a graph property $\mathcal{P} \subset Gr_d$ a neighborhood equivalent property if for graphs $G \equiv H$, $G \in \mathcal{P}$ if and only if $H \in \mathcal{P}$.

Proposition 8.2. *Amenability, Property A, being a Følner graph, almost finiteness, q-colorability and having a perfect matching are all neighborhood equivalent properties.*

Proof. Let us assume that G is q-colorable for some $q \geq 2$ and $\alpha : V(G) \to \{1, 2, \dots, q\}$ is a proper q-coloring. It is enough to prove that any component of H is q-colorable. Let $x \in V(H)$ and for $n \geq 1$ $\beta_n : V(H) \to \{1, 2, \dots, q\}$ be labelings that are proper colorings restricted on the ball $B_n^H(x)$. By neighborhood equivalence, such labelings exist. Let $\beta_{n_k} \to \gamma$ be a convergent subsequence. Then γ is proper q-coloring of the component of H containing x. Similarly, we can prove that having perfect matching or being almost finite is neighborhood equivalent, since these properties can be described by colorings satisfying some local constraints. It is straightforward to prove that amenability, being a Følner graph and Property A (that is local hyperfiniteness) are neighborhood equivalent properties as well. □

Definition 8.3. Let $B_1, B_2, \dots$ be an enumeration of the finite rooted balls in Gr_d. We define a pseudo-metric on Gr_d in the following way. Let $\mathrm{dist}_{Gr_d}(G, H) = 2^{-n}$ if for $1 \leq i \leq n-1$ $B_i \in (\hat{G} \cap \hat{H})$ or $B_i \in (\hat{G} \cap \hat{H})^c$, and $B_n \in \hat{G} \triangle \hat{H}$. It is easy to see that dist_{Gr_d} defines a metric on the neighborhood equivalence classes of Gr_d. So, a sequence $\{G_n\}_{n=1}^\infty$ is a Cauchy-sequence in Gr_d if for any rooted ball B, either $B \in \hat{G}_n$ for finitely many n's or $B \in \hat{G}_n$ for all but finitely many n's.

Proposition 8.4. *The space of neighborhood equivalence is compact, or in other words, all Cauchy sequences are convergent.*

Proof. First, let RGr_d be the set of all rooted, connected graphs of vertex degree bound d up to rooted isomorphisms. Again, we can define a metric dist_{RGr_d} on RGr_d by setting

$$\mathrm{dist}_{RGr_d}((G,x),(H,y)) = 2^{-n},$$

where n is the largest integer for which the rooted n-balls around x resp. y are rooted isomorphic. It is easy to see that RGr_d is compact with respect to this metric. Now, let $\{G_n\}_{n=1}^\infty \subset Gr_d$ be a Cauchy sequence. Consider the set $\mathcal{A}$ of all rooted graphs (Q,x) that are limits of sequences in the form of $\{G_n, x_n\}_{n=1}^\infty$, where $x_n \in V(G_n)$. Clearly, if $(Q,x) \in \mathcal{A}$, then all the rooted balls in Q are rooted balls in all but finitely many G_n's. On the other hand, if B is a rooted ball in all but finitely many G_n's then there exists $(Q,x) \in \mathcal{A}$ so that B is a rooted ball in Q. Therefore, if $\{Q_n\}_{n=1}^\infty$, is a countable dense subset of $\mathcal{A}$, then for the graph G having components $\{Q_n\}_{n=1}^\infty$ we have that $\lim_{n\to\infty} G_n = G$. □

We say that a countable set of graphs $\{G_n\}_{n=1}^\infty$ possesses the graph property $\mathcal{P}$ if for the graph B having components $\{G_n\}_{n=1}^\infty$, $B \in \mathcal{P}$. The following proposition's proof is similar to the one of Proposition 8.2 and left to the reader.

Proposition 8.5. *Let $\lim_{n\to\infty} G_n = G$. Then, if the set $\{G_n\}_{n=1}^\infty$ possesses any of the properties listed in Proposition 8.2, except amenability, so does G.*

By definition, all finite graphs are amenable, and limits of finite graphs can easily be non-amenable, e.g the 3-regular tree is non-amenable and it is the limit of large girth 3-regular graphs. However, we can define the amenability of a countable set of graphs in the following way.

Definition 8.6. The countable set of graphs $\{G_n\}_{n=1}^\infty$ is amenable if for any $\epsilon > 0$ there exists $r \geq 1$ such that for any $n \geq 1$, the graph G_n contains an ϵ-Folner set of diameter at most r.

By the Long Cycle Theorem, any countable set of finite graphs having Property A is amenable. Obviously, this statement does not hold for infinite graphs.

9. Hausdorff limits of graph spectra

Let $G \in Gr_d$ be a finite or infinite graph and $\mathcal{L}_G : l^2(V(G)) \to l^2(V(G))$ be the Laplacian operator on G as in the Introduction.

Proposition 9.1. *If G and H are neighborhood equivalent, then* $\mathrm{Spec}(\mathcal{L}_G) = \mathrm{Spec}(\mathcal{L}_H)$.

Proof. First, we need a lemma.

Lemma 9.2. *Let P be a real polynomial, then $\|P(\mathcal{L}_G)\| = \|P(\mathcal{L}_H)\|$.*

Proof. Fix some $\epsilon > 0$. Let $f \in l^2(V(G))$ such that $\|f\| = 1$ and $\|P(\mathcal{L}_G)(f)\| \geq (1-\epsilon)\|P(\mathcal{L}_G)\|$. We can assume that f is supported on a ball $B_s^G(x)$ for some $s > 0$ and $x \in V(G)$. Let t be the degree of P. Then, $P(\mathcal{L}_G)(f)$ is supported in the ball $B_{s+t}^G(x)$. Since G and H are equivalent, there exists $y \in V(H)$ such that the ball $B_{s+t}^G(x)$ is rooted-isomorphic to the ball $B_{s+t}^H(y)$ under some rooted-isomorphism j. Then, $\|j_*(f)\| = 1$ and $\|P(\mathcal{L}_G)(f)\| = \|P(\mathcal{L}_H)(j_*(f))\|$, where $j_*(f)(z) = f(j^{-1}(z))$, for $z \in B_s^G(x)$. Therefore, $\|P(\mathcal{L}_H)\| \geq (1-\epsilon)\|P(\mathcal{L}_G)\|$ holds for any $\epsilon > 0$. Consequently, $\|P(\mathcal{L}_H)\| \geq \|P(\mathcal{L}_G)\|$. Similarly, $\|P(\mathcal{L}_G)\| \geq \|P(\mathcal{L}_H)\|$, thus our lemma follows. □

By Functional Calculus, we have that

$$\|\varphi(\mathcal{L}_G)\| = \|\varphi(\mathcal{L}_H)\| \tag{26}$$

holds for any real continuous function φ. Observe that $\lambda \in \operatorname{Spec}(\mathcal{L}_G)$ if and only if for any $n \geq 1$ $\|\varphi_n^\lambda(\mathcal{L}_G)\| \neq 0$, where φ_n^λ is a piecewise linear, continuous, non-negative function such that

- $\varphi_n^\lambda(x) = 1$ if $\lambda - \frac{1}{n} \leq x \leq \lambda + \frac{1}{n}$,
- $\varphi_n^\lambda(x) = 0$ if $x \geq \lambda + \frac{2}{n}$ or $x \leq \lambda - \frac{2}{n}$,
- and defined linearly otherwise.

Therefore, by (26) our proposition follows. □

The main goal of this section is to prove Theorem 5.

Proof. The following lemma shows how to test whether a certain value λ is near to the spectrum of the Laplacian.

Lemma 9.3. *Fix $\epsilon > 0$. Let $\varphi_{\lambda,\epsilon}$ be the following positive continuous function on the real line.*

- $\varphi_{\lambda,\epsilon}(x) = 0$ *if* $x \leq \lambda - \epsilon$ *or* $x \geq \lambda + \epsilon$.
- $\varphi_{\lambda,\epsilon}(x) = 1$ *if* $\lambda - \frac{\epsilon}{2} \leq x \leq \lambda + \frac{\epsilon}{2}$.
- $\varphi_{\lambda,\epsilon}$ *is linear on the intervals* $[\lambda - \epsilon, \lambda - \frac{\epsilon}{2}]$ *and* $[\lambda + \frac{\epsilon}{2}, \lambda + \epsilon]$.

Let $P_{\lambda,\epsilon}$ be a real polynomial such that $\sup_{x \in [0,2d]} |\varphi_{\lambda,\epsilon}(x) - P_{\lambda,\epsilon}(x)| \leq \epsilon$. If $\|P_{\lambda,\epsilon}(\mathcal{L}_H)\| > \epsilon$ for some $H \in Gr_d$, then there exists $\kappa \in \operatorname{Spec}(\mathcal{L}_H)$ such that $|\kappa - \lambda| < \epsilon$.

Proof. By Functional Calculus, we have that

$$\|\varphi_{\lambda,\epsilon}(\mathcal{L}_H)\| \geq \|P_{\lambda,\epsilon}(\mathcal{L}_H)\| - \epsilon\,.$$

Therefore, $\|\varphi_{\lambda,\epsilon}(\mathcal{L}_H)\| > 0$. Again by Functional Calculus, we can conclude that then there exists $\kappa \in \operatorname{Spec}(H)$ such that $|\kappa - \lambda| < \epsilon$. □

Proposition 9.4. *Let* $\lim_{n\to\infty} G_n = G$ *for some convergent sequence* $\{G_n\}_{n=1}^{\infty} \subset Gr_d$*. Suppose that* $\lambda \in \mathrm{Spec}(\mathcal{L}_G)$ *. Then, for any* $\epsilon > 0$ *there exists* $N_\epsilon > 1$ *such that if* $n \geq N_\epsilon$ *there exists* $\lambda_n \in \mathrm{Spec}(\mathcal{L}_{G_n})$ *so that* $|\lambda_n - \lambda| \leq \epsilon$*.*

Proof. Let $\varphi_{\lambda,\epsilon}$ and $P_{\lambda,\epsilon}$ be as in Lemma 9.3. By Functional Calculus, $\|\varphi_{\lambda,\epsilon}(\mathcal{L}_G)\| = 1$, so there exists a function $f \in l^2(G)$, $\|f\| = 1$ supported on some ball $B_s^G(x)$ such that

$$\|\varphi_{\lambda,\epsilon}(\mathcal{L}_G)(f)\| > \epsilon. \tag{27}$$

Let m be the degree of $P_{\lambda,\epsilon}$. Then $P_{\lambda,\epsilon}(f)$ is supported on $B_{s+m}^G(x)$. As in the proof of Lemma 9.2, we can see that if $\mathrm{dist}_{Gr_d}(G, H)$ is small enough, then we have some $g \in l^2(H)$, $\|g\| = 1$ supported on $B_s^H(y)$ such that

$$\|P_{\lambda,\epsilon}(\mathcal{L}_H)(g)\| = \|P_{\lambda,\epsilon}(\mathcal{L}_G)(f)\| > \epsilon\,.$$

Therefore $\|P_{\lambda,\epsilon}(\mathcal{L}_H)\| > \epsilon$, so our proposition follows from Lemma 9.3. □

Proposition 9.5. *Let* $\{G_n\}_{n=1}^{\infty} \subset Gr_d$ *be a countable set of graphs of Property A converging to* $G \in Gr_d$*. Suppose that for* $0 < \epsilon < \frac{1}{4}$ *and* $\lambda \geq 0$ *there exists* $N_\epsilon > 0$ *so that if* $n \geq N_\epsilon$*, then*

$$\mathrm{Spec}(\mathcal{L}_{G_n}) \cap (\lambda - \frac{\epsilon}{2}, \lambda + \frac{\epsilon}{2}) \neq \emptyset\,.$$

Then, $\mathrm{Spec}(\mathcal{L}_G) \cap (\lambda - \epsilon, \lambda + \epsilon) \neq \emptyset$*.*

Proof. First, fix $\epsilon > 0$. Denote by l the degree of the polynomial $P_{\lambda,\epsilon}$. By Proposition 8.5, the graph $\tilde{G} \in Gr_d$ whose components consist of $\{G_n\}_{n=1}^{\infty}$ and G is of Property A. Therefore by the Long Cycle Theorem, there exists an integer m and a probability measure μ on $\mathrm{Sep}(\tilde{G}, m)$ satisfying the following condition: For all $x \in V(\tilde{G})$,

$$\mu(\{Y \in \mathrm{Sep}(\tilde{G}, m) \mid x \in B_l^{\tilde{G}}(Y)\}) < \delta\,, \tag{28}$$

$1 - \epsilon - 6\sqrt[4]{\delta} > \epsilon$.

This condition can be fulfilled by the argument of the beginning of Proposition 5.1.

For $f \in l^2(\tilde{G})$, $\|f\|^2 = 1$ and $Y \in \mathrm{Sep}(\tilde{G}, m)$ we define f_Y by setting

- $f_Y(x) = f(x)$ if $x \notin B_l^{\tilde{G}}(Y)$.
- $f_Y = 0$ otherwise.

Lemma 9.6.

$$\mu(\{Y \in \mathrm{Sep}(\tilde{G}, m) \mid \|f_Y\|^2 < 1 - \sqrt{\delta}\}) < \delta\,.$$

Proof. By (28), we have that

$$\sum_{x \in V(\tilde{G})} \int_{\mathrm{Sep}(\tilde{G},m)} f_Y^2(x)\, d\mu(Y) \geq (1 - \delta)\,.$$

So by the Monotone Convergence Theorem,

$$\int_{\mathrm{Sep}(\tilde{G},m)} \|f_Y\|^2 \, d\mu(Y) \geq 1-\delta \,. \tag{29}$$

Let $A = \{Y \mid \|f_Y\|^2 < 1-\sqrt{\delta}\}$. Then by (29), we have that

$$\mu(A)(1-\sqrt{\delta}) + 1 - \mu(A) \geq 1-\delta \,.$$

Thus, $\sqrt{\delta} \geq \mu(A)$. □

Define

$$\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_{\diamondsuit} := \sup_g \frac{\|P_{\lambda,\epsilon}(\mathcal{L}_G)g\|}{\|g\|} \,,$$

where the supremum is taken for all nonzero functions $g \in l^2(G)$ which are supported on $(B_l^{\tilde{G}}(Y))^c \cap H$ for some $Y \in \mathrm{Sep}(\tilde{G}, m)$ and $H \subset V(G)$ is the vertex set of a component in Y^c. Note these functions do not form a vector space, so $\| \, \|_{\diamondsuit}$ is not a proper norm. Clearly, $\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_{\diamondsuit} \leq \|P_{\lambda,\epsilon}(\mathcal{L}_G)\|$. Let

$$\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_{\square} := \sup_g \frac{\|P_{\lambda,\epsilon}(\mathcal{L}_G)g\|}{\|g\|} \,,$$

where the supremum is taken for all nonzero functions $g \in l^2(G)$ such that there exists $Y \in \mathrm{Sep}(\tilde{G}, m)$ for which g is supported on the complement of $B_l^{\tilde{G}}(Y)$.

Lemma 9.7. $\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_{\diamondsuit} = \|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_{\square}$.

Proof. By definition, we have that $\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_{\diamondsuit} \leq \|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_{\square}$. Now let $Y \in \mathrm{Sep}(\tilde{G}, m)$ and $g \in l^2(G)$ such that g is supported on $\cup_{n=1}(B_l^G(Y))^c \cap H_n)$, where $\{H_n\}_{n=1}^{\infty}$ is an enumeration of the elements of the component of the complement of Y. Let g_n be the restriction of g onto $(B_l^G(Y))^c \cap H_n$. Clearly, the functions $\{g_n\}_{n=1}^{\infty}$ are pairwise orthogonal. Since l is the degree of $P_{\lambda,\epsilon}$, the function $P_{\lambda,\epsilon}(g_n)$ is supported on H_n. Indeed, the l-neigbourhood of $(B_l^G(Y))^c \cap H_n)$ is inside H_n. Hence, the functions $\{P_{\lambda,\epsilon}(g_n)\}_{n=1}^{\infty}$ are also pairwise orthogonal. Therefore, $\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_{\diamondsuit} \geq \|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_{\square}$. □

Similarly, we can define $\|P_{\lambda,\epsilon}(\mathcal{L}_{G_n})\|_{\diamondsuit}$ and $\|P_{\lambda,\epsilon}(\mathcal{L}_{G_n})\|_{\square}$. Then, $\|P_{\lambda,\epsilon}(\mathcal{L}_{G_n})\|_{\diamondsuit} = \|P_{\lambda,\epsilon}(\mathcal{L}_{G_n})\|_{\square}$.

Lemma 9.8.

$$\|P_{\lambda,\epsilon}(\mathcal{L}_G)\| - \|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_{\diamondsuit} < 3\sqrt[4]{\delta} \,. \tag{30}$$

Proof. Let $f : V(G) \to \mathbb{R}$ be a function such that $\|f\| = 1$ and $\|P_{\lambda,\epsilon}(\mathcal{L}_G))f\| \geq \|P_{\lambda,\epsilon}(\mathcal{L}_G)\| - \sqrt[4]{\delta}$. Let f_Y be as above such that $\|f_Y\|^2 > (1-\sqrt{\delta})$. Observe that

$$1 = \|f\|^2 = \|f_Y\|^2 + \|f - f_Y\|^2.$$

Therefore, $\|f - f_Y\| \leq \sqrt[4]{\delta}$. By the triangle inequality,

$$\|P_{\lambda,\epsilon}(\mathcal{L}_G)f_Y\| \geq \|P_{\lambda,\epsilon}(\mathcal{L}_G)f\| - \|P_{\lambda,\epsilon}(\mathcal{L}_G)(f - f_Y)\| \,.$$

Since $\sup_{0\leq t\leq 2d}|P_{\lambda,\epsilon}(t)|\leq 1+\epsilon<2$ we have that

$$\|P_{\lambda,\epsilon}(\mathcal{L}_G)f_Y\|\geq\|P_{\lambda,\epsilon}(\mathcal{L}_G)f\|-2\|f-f_Y\|\geq\|P_{\lambda,\epsilon}(\mathcal{L}_G)f\|-2\sqrt[4]{\delta}\geq\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|-3\sqrt[4]{\delta}\,.$$

Since $\|f_Y\|\leq 1$ and f_Y is supported on the union of the subsets $\{(B_l^{\tilde{G}}(Y))^c\cap H_n)\}_{n=1}^\infty$ we have that

$$\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_\square\geq\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|-3\sqrt[4]{\delta}\,.$$

Thus, our lemma follows from Lemma 9.7. $\square$

Similarly, we have that

$$\|P_{\lambda,\epsilon}(\mathcal{L}_{G_n})\|-\|P_{\lambda,\epsilon}(\mathcal{L}_{G_n})\|_\diamondsuit<3\sqrt[4]{\delta}\,. \tag{31}$$

Lemma 9.9. *For large enough n, we have that*

$$\|P_{\lambda,\epsilon}(\mathcal{L}_{G_n})\|-\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|<6\sqrt[4]{\delta}\,. \tag{32}$$

Proof. By definition, $\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_\diamondsuit$ equals $\sup_g\frac{\|P_{\lambda,\epsilon}(\mathcal{L}_G)g\|}{\|g\|}$, where the supremum is taken for all g's such that g is supported on $H\cap(B_l^{\tilde{G}}(H^c))$, where $H\subset V(G)$ is a set of diameter at most m and its induced subgraph is connected. Indeed, H^c is an m-separator. For these functions g, $P_{\lambda,\epsilon}(\mathcal{L}_G)g$ is supported on H. Now, if n is large enough the set of induced subgraphs (up to isometry) on such H's are the same in G_n and in G. Therefore, $\|P_{\lambda,\epsilon}(\mathcal{L}_{G_n})\|_\diamondsuit=\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|_\diamondsuit$. Hence, our lemma follows from the the inequalities (30) and (31). $\square$

By our assumption on the spectra for large enough n, $\|\varphi_{\lambda,\epsilon}(\mathcal{L}_{G_n})\|=1$. Hence, $\|P_{\lambda,\epsilon}(\mathcal{L}_{G_n})\|\geq 1-\epsilon$, so

$$\|P_{\lambda,\epsilon}(\mathcal{L}_G)\|\geq 1-\epsilon-6\sqrt[4]{\delta}\,.$$

Thus, by our assumption on δ and by Lemma 9.3, our proposition follows. $\square$

Now we finish the proof of Theorem 5. Suppose that the compact sets $\mathrm{Spec}(\mathcal{L}_{G_n})$ do not converge to $\mathrm{Spec}(\mathcal{L}_G)$ in the Hausdorff distance.

Case 1. There exists $\delta>0$, a sequence of positive integers $k_1<k_2<\dots$ and $\{\lambda_n\}_{n=1}^\infty\subset\mathrm{Spec}(\mathcal{L}_G)$ such that $\inf_{\kappa\in\mathrm{Spec}(\mathcal{L}_{G_{k_n}})}|\kappa-\lambda_n|>2\delta$. Let $\lambda\in\mathrm{Spec}(\mathcal{L}_G)$ be a limit point of the sequence $\{\lambda_n\}_{n=1}^\infty$. Then, we cannot have elements κ_n in $\mathrm{Spec}(\mathcal{L}_{G_{k_n}})$ such that for large enough n, $|\kappa_n-\lambda|<\delta$, in contradiction with Proposition 9.4.

Case 2. There exists $\delta>0$, a sequence of positive integers $k_1<k_2<\dots$ and $\kappa_n\in\mathrm{Spec}(\mathcal{L}_{G_{k_n}})$ such that $\inf_{\lambda\in\mathrm{Spec}(\mathcal{L}_G)}|\kappa_n-\lambda|>2\delta$. Let κ be a limit point of the sequence $\{\kappa_n\}_{n=1}^\infty$. Then, for large enough n we have that

$$\mathrm{Spec}(\mathcal{L}_{G_{k_n}})\cap(\kappa-\frac{\delta}{2},\kappa+\frac{\delta}{2})\neq\emptyset\,.$$

However, $\mathrm{Spec}(\mathcal{L}_G)\cap(\kappa-\delta,\kappa+\delta)=\emptyset$, in contradiction with Proposition 9.5. Therefore, our theorem follows. $\square$

Remark 3. Let $\lim_{n\to\infty} G_n = G$, where $\{G_n\}_{n=1}^\infty$ is a large girth sequence of finite 3-regular graphs and G is the 3-regular tree. Then for all $n \geq 1$, $0 \in \mathrm{Spec}(\mathcal{L}_{G_n})$ and $0 \notin \mathrm{Spec}(\mathcal{L}_G)$. Also, if G is a large girth 3-regular expander graph, then its second smallest eigenvalue is away from zero. However, if G is a large girth 3-regular graph containing an ϵ-Følner set that is smaller than $\frac{3}{4}|V(G)|$, then the second smallest eigenvalue of G is very close to zero. That is, in general it is not true that the convergence of a finite graph sequence $\{G_n\}_{n=1}^\infty$ implies that $\{\mathrm{Spec}(\mathcal{L}_{G_n})\}_{n=1}^\infty$ converges in the Hausdorff distance.

We finish this section with a purely combinatorial application of Theorem 5.

Proposition 9.10. *For any positive integer d and $\epsilon > 0$ there exists $r > 0$ so that if the families of rooted r-balls (up to rooted-isomorphism) in two finite planar graphs $G, H \in Gr_d$ coincide, then the Hausdorff distance of their spectra is at most $\epsilon > 0$.*

Proof. A set of finite graphs $\mathcal{C} \subset Gr_d$ is called monotone if it is closed under taking induced subgraphs. By the Large Cycle Theorem, a monotone subset $\mathcal{C}$ is of Property A if and only if it is hyperfinite. The class of finite planar graphs (and the class of H-minor free graphs) are monotone hyperfinite ([5]), so they are Property A as well. Assume that our proposition does not hold. Then there exists some $\epsilon > 0$ and a sequence of pairs of planar graphs $\{(G_n, H_n)\}_{n=1}^\infty$ so that the families of the rooted n-balls in G_n and H_n coincide and

$$\mathrm{dist}_{\mathrm{Haus}}(\mathrm{Spec}(\mathcal{L}_{G_n}), \mathrm{Spec}(\mathcal{L}_{H_n})) > \epsilon\,. \tag{33}$$

Let us pick a subsequence $\{G_{n_k}\}_{k=1}^\infty$ such that $\lim_{k\to\infty} G_{n_k} = G$ for some infinite graph G. By our conditions, we have that $\lim_{k\to\infty} H_{n_k} = G$. So, by Theorem 5 we have that $\mathrm{Spec}(\mathcal{L}_{G_n}) \to \mathrm{Spec}(\mathcal{L}_G)$ and $\mathrm{Spec}(\mathcal{L}_{H_n}) \to \mathrm{Spec}(\mathcal{L}_G)$ in contradiction with (33). □

10. Strongly almost finite graphs and classifiable C^*-algebras

Arguably, one of the greatest achievements of operator algebras is the following result:

"Separable, unital, simple, nuclear, infinite dimensional C^*-algebras with finite nuclear dimension that satisfy the UCT are classified by their Elliott invariants."

The goal of this section is to show that starting from a minimal (see below), strongly almost finite graph $M \in Gr_d$ one can always construct tracial, classifiable C^*-algebras in a natural and almost canonical way. The construction is rather elementary and does not require any knowledge of C^*-algebras.

Somewhat similarly to the classification of finite simple groups, the theorem above was built on decades of work of C^*-algebrists that culminated in the paper [58]. A bit later it was proved [8], that for separable, unital, simple, nuclear C^*-algebras having finite nuclear dimension and being $\mathcal{Z}$-stable are equivalent. Many of the known classifiable C^*-algebras that have traces are associated to minimal free actions of countable amenable groups on compact metric spaces. Note that for a very large class of C^*-algebras that are traceless the classifiability had been proved in [41] more than twenty years ago. It is conjectured that the so-called crossed product C^*-algebras of free minimal actions of countable amenable groups on the Cantor set are always classifiable (see [17] and [12] for some interesting examples).

Our starting point is the following theorem:

Let $\mathcal{G}$ be a locally compact ample, minimal Cantor étale groupoid. Assume that $\mathcal{G}$ is almost finite and topologically amenable. Then, the simple, unital $C^\star$-algebra $C^*_r(\mathcal{G})$ is $\mathcal{Z}$-stable (Corollary 9.11 of [44], see also Corollary 5.8 of [9]). By the results of [60] and [8] $C^*_r(\mathcal{G})$ satisfies the UCT-condition and has finite nuclear dimension. ***Hence by Corollary D. of [58], $C^*_r(\mathcal{G})$ is classified by its Elliott invariants.***

The statement above looks a bit frightening, but the reader should look at the bright side. Namely, the quoted result completely eliminates C^*-algebras from the picture, by reducing the problem to groupoids. In constructing such groupoids, our strategy goes as follows.

- We introduce the notion of minimality for infinite graphs in a purely combinatorial fashion.
- We explain the notion of étale Cantor groupoids (the "unit space" of such a groupoid is the Cantor set) and show how very simple vertex and edge labeling constructions on a minimal graph $M \in Gr_d$ yield minimal étale Cantor groupoids.
- We recall the notion of topological amenability and almost finiteness for étale Cantor groupoids and show that if the minimal graph M is strongly almost finite, then appropriate labelings of M result in topologically amenable and almost finite étale Cantor groupoids. Hence, Corollary 9.11 of [44] can be invoked to conclude that our new C^*-algebras are classifiable.

10.1. **Minimal graphs.**

Definition 10.1. The connected graph $M \in Gr_d$ is **minimal** if for every rooted ball $B = (B^M_r(x), x)$ in M there exists a constant $r_B > 0$ such that for all $y \in V(M)$, the ball $B^M_{r_B}(y)$ contains a vertex z so that the rooted r-ball around z is rooted isomorphic to B.

Observe that a minimal graph is either Følner or nonamenable. Clearly, the Cayley graphs of finitely generated groups are minimal. Nevertheless, minimal graphs can be very different from Cayley graphs.

- For any $\alpha > 0$ real number there exists a minimal graph with asymptotic volume growth α (see Section 5.2 of [22]). On the other hand, the volume growth of a Cayley graph of polynomial growth is always an integer [31].

- A minimal graph might have n ends, where n is an arbitrary integer (see Section 10. in [22] for a very general fractal-like construction). Infinite Cayley graphs have 1, 2 or infinitely many ends.
- In fact, modifying somewhat the construction in [22] one can construct a minimal graph M for any bounded degree graph G by attaching suitable finite graphs to the vertices of G.

As we see soon, the notion of minimality is closely related to the concept of minimality for continuous group actions on compact spaces. Recall that the continuous action (that is, all elements are acting by homeomorphisms) of a countable group Γ on a compact metric space X, $\alpha : \Gamma \curvearrowright X$ is minimal if all the orbits of α are dense. Let $G \in Gr_d$ be a connected graph. Then, one can color the edges of G with colors $\{c_1, c_2, \dots, c_{2d}\}$ in such a way that if the edges $e \neq f$ have the same color then they do not have a joint vertex. This coloring defines the Schreier graph of an action of the $2d$-fold free product Γ_{2d} of cyclic groups of order 2 generated by the elements $\{c_1, c_2, \dots, c_{2d}\} = \Sigma_{2d}$ (Section 5.1 [22]). That is, every connected graph $G \in Gr_d$ is the underlying graph of a Schreier graph of Γ_{2d}. Let $RSch_d$ denote the space of all rooted Schreier graphs of Γ_{2d} with respect to the generating system Σ_{2d} such that the underlying graph G is in Gr_d. Similarly to the space RGr_d defined in the proof of Proposition 8.4, $RSch_d$ is a compact metric space (Section 2.1 [22]), also, $RSch_d$ is equipped with the natural, continuous (root-changing) action β of the group Γ_{2d}.

Lemma 10.2. *Let $\mathcal{M}$ be a closed, minimal Γ_{2d}-invariant subspace of $RSch_d$. Then, for all elements S of $\mathcal{M}$ the underlying graph M of S is a minimal graph.*

Proof. Let $(T, y) \in \mathcal{M}$, where y is the root. Suppose that the underlying graph M of (T, y) is not minimal. Then, there is a rooted ball B in M and there exist elements $\{g_n \in \Gamma_{2d}\}_{n=1}^{\infty}$ such that for $n \geq 1$ the rooted ball of radius n around $g_n(y)$ does not contain balls that are rooted-isomorphic to B. Take a subsequence of the sequence $\{\beta(g_n)(T, y)\}_{n=1}^{\infty}$ converging to some element $(S, z) \in RSch_d$. Then, the underlying graph of (S, z) does not contain a rooted ball isomorphic to B. Hence, the orbit closure of (S, z) does not contain (T, y), leading to a contradiction. □

It is not hard to see that the underlying graphs of the elements of $\mathcal{M}$ are neighborhood equivalent to the graph M and any connected graph that is neighborhood equivalent to M is the underlying graph of some elements in $\mathcal{M}$.

The idea above goes back to the paper of Glasner and Weiss [29]. They defined the **uniformly recurrent subgroups** (URS) of a countable group Γ as the closed, minimal, invariant subspaces in $\mathrm{Sub}(\Gamma)$, the compact space of subgroups of Γ. If H is an element of a URS, then the underlying graph of the Schreier graph $\mathrm{Sch}(\Gamma/H, \Sigma)$ (with respect to a symmetric generating system Σ) is a minimal graph. Conversely, let M be a minimal graph and consider an arbitrary element T of $RSch_d$ with M as its underlying graph. Let $\mathcal{T}$ be the orbit closure of T and $\mathcal{M}$ be a closed, minimal Γ_{2d}-invariant subspace in $\mathcal{T}$. Then, $\mathcal{M}$ is Γ_{2d}-isomorphic to a uniformly recurrent subgroup. Indeed, for each element of $\mathcal{M}$ consider the stabilizer of the root.

We extend the definition of minimality for Cantor-labeled rooted Γ_{2d}-Schreier graphs with underlying graphs in Gr_d. A graph $G \in Gr_d$ is said to be *Cantor-labeled* if there exists a labeling function $c : V(G) \to \mathcal{C}$ that assigns to each vertex of G a label from the Cantor set $\mathcal{C}$. Denote the set of such graphs by $R\mathcal{C}Sch_d$. Let us identify the Cantor set $\mathcal{C}$ with the product set $\{0,1\}^{\mathbb{N}}$. If $c \in \mathcal{C}$ and $c = \{a_1, a_2, \dots\}$ then we will refer to a_i as the i-th coordinate of c. Similarly to RGr_d and $RSch_d$ we have a natural compact metric structure on $R\mathcal{C}Sch_d$. Again, the group Γ_{2d} acts on $R\mathcal{C}Sch_d$ by changing the root. For a rooted edge-colored, Cantor-labeled ball B, we denote by $B_{(k)}$ the ball rooted-colored isomorphic to B labeled with the finite set $\{0,1\}^k$ in such a way that for all vertex x in $B_{(k)}$, the label of x is $\pi_k(c)$, where c is the Cantor label in B and $\pi_k : \{0,1\}^{\mathbb{N}} \to \{0,1\}^k$ is the projection onto the first k coordinates.

Definition 10.3. The Cantor-labeled rooted Schreier graph $S \in R\mathcal{C}Sch_d$ is minimal if for all rooted labeled-colored ball $B = (B_r^M(x))$ in S there exists a constant $r_B > 0$ such that for all $y \in V(S)$, the ball $B_{r_B}^M(y)$ contains a vertex z so that $(B_r^S(z))_{(k)}$ is rooted labeled-colored isomorphic to $B_{(k)}$. Clearly, the underlying graph of a minimal graph $S \in R\mathcal{C}Sch_d$ is minimal in Gr_d.

The following lemma can be proved in the same way as Lemma 10.2.

Lemma 10.4. *Let $\mathcal{M}$ be a closed, minimal, Γ_{2d}-invariant subspace of $R\mathcal{C}Sch_d$. Then, every element $S \in \mathcal{M}$ is minimal.*

Lemma 10.5. *Any minimal graph $M \in Gr_d$ can be edge-colored properly and equipped with a Cantor-labeling in such a way that the resulting labeled-colored graph is minimal in $R\mathcal{C}Sch_d$.*

Proof. Let $\tilde{M}$ be an arbitrary element of $R\mathcal{C}Sch_d$ such that its underlying graph is isomorphic to M. Let $\mathcal{L} \subset R\mathcal{C}Sch_d$ be a closed, invariant, minimal subspace in the orbit closure of $\tilde{M}$. Let $x \in V(M)$. By the minimality of M, for any $n \geq 1$ there exists $S_n \in \mathcal{L}$ such that $B_n^{S_n}(\mathrm{root}\,(S_n))$ is rooted isomorphic to $B_n^M(x)$. Therefore, if S is a limit point of the sequence $\{S_n\}_{n=1}^\infty$, the underlying graph of S is isomorphic to M. Thus, by Lemma 10.4, S is minimal. □

10.2. **Étale Cantor groupoids and infinite graphs.** Let $\alpha : \Gamma \curvearrowright \mathcal{C}$ be a continuous action of a countable group Γ on the Cantor set $\mathcal{C}$. Assume that the action is **stable**, that is, for every $g \in \Gamma$ there exists $r_g > 0$ such that if $x \in \mathcal{C}$ and $\alpha(g)(x) \neq x$ then $\mathrm{dist}_{\mathcal{C}}(x, \alpha(g)(x)) \geq r_g$. Here, $\mathrm{dist}_{\mathcal{C}}$ is a metric on $\mathcal{C}$ that metrizes the compact topology. Note that an action is stable if and only if the stabilizer map $\mathrm{Stab} : \mathcal{C} \to \mathrm{Sub}(\Gamma)$ (the space of subgroups of Γ, [29]) is continuous. Of course, free continuous actions are always stable.

Now we define an equivalence relation on $\mathcal{C}$ and the groupoid of α. We set $x \equiv_\alpha y$ if for some $g \in \Gamma$ $\alpha(g)(x) = y$. The groupoid [56] $\mathcal{G}_\alpha \subset \mathcal{C} \times \mathcal{C}$ is the set of pairs (x, y) such that $x \equiv_\alpha y$. The product is given by $(x, y)(y, z) = (x, z)$. The range map is given by $r : (x, y) \to y$ and the source map is given by $s : (x, y) \to x$. The inverse map γ is given by $\gamma : (x, y) = (y, x)$. The **unit space** of the groupoid $\mathcal{G}^0_\alpha$ is the set of pairs $(x, x), x \in \mathcal{C}$. Then by the stability of the action α,

- for every $(x, y) \in \mathcal{G}_\alpha$ such that $\alpha(g)(x) = y$ for some $g \in \Gamma$, there exist clopen sets U, V in the Cantor set, $x \in U$ and $y \in V$, such that $\alpha(g) : U \to V$ is a homeomorphism,
- and if also $\alpha(h)(x) = y$, then there exists a clopen set $x \in W \subset \mathcal{C}$ such that
$$\alpha(g) \mid_W = \alpha(h) \mid_W \ .$$

The topology on $\mathcal{G}_\alpha$ is defined in the following way. The base neighbourhoods of the element (x, y) are in the form of (U, V, g, x, y), where $\alpha(g) : U \to V$ as above, and $(a, b) \in (U, V, g, x, y)$ if $a \in U$ and $\alpha(g)(a) = b$. Now, we can easily check that $r : \mathcal{G}^0_\alpha \to \mathcal{C}$ is in fact a homeomorphism and for any pair (x, y) such that $\alpha(g)(x) = y$, $r : (U, g, x, y) \to U$ is a homeomorphism, that is, r is a **local homeomorphism.** Consequently, $\mathcal{G}_\alpha$ is a locally compact Hausdorff **étale groupoid** with unit space isomorphic to the Cantor set ([56]), as required in Corollary 5.8 in [9]. The étale groupoid $\mathcal{G}_\alpha$ is called minimal if the action α is minimal [56].

Now, let us consider the Γ_{2d}-action $\beta_d : \Gamma_{2d} \curvearrowright RCSch_d$. Recall that we connect $x \neq y \in RCSch_d$ with an edge if for some generator c_i we have $\Gamma_{2d}(c_i)(x) = y$. For $x \in RCSch_d$, the **orbit graph** of x is the connected component of the graph above containing x, and the **rooted orbit graph** of x is the orbit graph of x with root x. We have two minor problems to settle.

- The action β_d is not stable. Consider the rooted Cayley graph K of Γ_{2d} generated by the elements $c_1, c_2, \dots, c_{2d}$ such that all vertices of K are labeled with the same element $t \in \mathcal{C}$. Then, K is an element of $RCSch_d$ fixed by all elements of Γ_{2d}. In every neighborhood U of K there are graphs that are not fixed by any element of Γ_{2d}.
- We might expect that the rooted orbit graph of an element $N \in RCSch_d$ is rooted isomorphic to the underlying rooted graph of N. Unfortunately, the rooted orbit graph of the above K is a graph of no edges and has one single vertex.

We will put more restrictions on the Cantor labelings to get rid of these inconveniences. Take an arbitrary rooted Schreier graph $G \in RSch_d$ and label the vertices of the underlying rooted graph by elements of $\mathcal{C}$ in the following way: For any $n \geq 1$ there exists an $\epsilon > 0$ such that if $0 < d_G(x, y) \leq n$, we have $\text{dist}_\mathcal{C}(l(x), l(y)) \geq \epsilon$. Here, l is the labeling function. These *proper* labelings exist and the action of Γ_{2d} on the orbit closure of the labeled version of G is stable. Also, if K is an element of the orbit closure then the rooted orbit graph of K is rooted isomorphic to K (see [22], [25] for details). So, let us start with a minimal graph $M \in Gr_d$ and equip it with a proper Cantor vertex labeling and a proper Σ_{2d}-edge labeling to obtain $\tilde{M} \in R\mathcal{C}Sch_d$. Let $E \subset R\mathcal{C}Sch_d$ be a minimal, closed invariant subspace in the orbit closure of $\tilde{M}$. Then, E is a stable, minimal, closed invariant subspace E of $R\mathcal{C}Sch_d$ such that each element of E is rooted isomorphic to its own rooted orbit graph and the underlying graphs are neighborhood equivalent to M. Since the action on E is minimal, E cannot have isolated points, so E is homeomorphic to $\mathcal{C}$. If β_E is the restriction of β_d on such a space E, then we call $\mathcal{G}_{\beta_E}$ the minimal étale Cantor groupoid associated to E.

10.3. **Topologically amenable and almost finite étale groupoids.** Graph properties such as almost finiteness, finite asymptotic dimension, paradoxicality or Property A have been defined for free Cantor actions of finitely generated groups. Informally speaking, the meaning in this context is that the property holds for every orbit in a continuous way. This idea has already been extended to étale Cantor groupoids as well.

The continuous version of Property A is called **topological amenability** for historical reasons. It was introduced in [2] more than twenty years ago. In the definition of Property A we have probability measures concentrated on r-balls around vertices for certain values r. First note that for a fixed r and a fixed set of colors there exist only finitely many rooted edge-colored r-balls (up to rooted colored isomorphisms) in graphs in $RSch_d$. Of course, for each such ball B there exist uncountably many probability measures supported on the vertices of B. Nevertheless, for any $\epsilon > 0$ there exists a finite set of probability measures $\{p_i^{B,\epsilon}\}_{i=1}^{N(B,\epsilon)}$ supported on B, such that for every probability measure p supported on B there exists $1 \leq i \leq N_{(B,\epsilon)}$ such that $\|p - p_i^{B,\epsilon}\|_1 < \frac{\epsilon}{2}$. Hence, we can suppose that in the definition of Property A, the functions $\Theta(x)$ are all in the form of $p_i^{B,\epsilon}$, for some rooted edge-colored ball B and $\epsilon > 0$. So, we have the following simple version of topological amenability in the case of our groupoids $\mathcal{G}_{\beta_E}$.

Definition 10.6 (**Topological amenability**). The groupoid $\mathcal{G}_{\beta_E}$ is topologically amenable if for any $\epsilon > 0$ there exists $r > 0$ and a partition $\mathcal{P}$ of the totally disconnected space E into finitely many clopen sets $\{U_\kappa\}_{\kappa \in P_{C_\epsilon}^r}$, and for each $x \in E$ there exists a probability measure p_x on the orbit graph of x supported on the r-ball around x such that

- if $x \in U_\kappa$, then the probability measure p_x has type κ, for some $\kappa \in P_{C_\epsilon}^r$. Here $P_{C_\epsilon}^r$ is the finite set of all probability measures in the form of $p_i^{B,\epsilon}$, where B is a rooted edge-colored r-ball. We call the elements of $P_{C_\epsilon}^r$ "types".
- For any $x \in E$ and generator $c_j \in \Sigma_{2d}$, $\|p_x - p_{\beta(c_j)(x)}\|_1 < \epsilon$.

Similar definition can be given for any stable action of a finitely generated group with a given symmetric generating set.

Proposition 10.7. *For every minimal graph $M \in Gr_d$ of Property A, we have an invariant subspace E as above, so that the associated étale groupoid $\mathcal{G}_{\beta_E}$ is topologically amenable.*

Proof. Since M is of Property A, there exists r_n and for each $x \in V(M)$ a probability measure $\Theta(x)$ of type in $P^{r_n}_{C_\epsilon}$ such that if x and y are adjacent vertices, then $\|\Theta(x) - \Theta(y)\| < \epsilon$. Now let us denote by Q_n the set $P^{r_n}_{C_\epsilon}$. For each $n \geq 1$ we have a vertex labeling by Q_n of $V(M)$, where the label of x is the type of $\Theta(x)$. Altogether, we have a labeling of $V(M)$ by the product set $\prod_{n=1}^\infty Q_n$ that is isomorphic to the Cantor set. Also, we add edge-colors and a Cantor labeling in the way described above to ensure stability. So, we have a $\mathcal{C} \times \mathcal{C}$-labeling of $V(M)$, however, $\mathcal{C} \times \mathcal{C}$ is still homeomorphic to $\mathcal{C}$. Add an arbitrary root to the resulting graph to obtain a rooted colored-labeled graph $S \in R\mathcal{C}Sch_d$. Now, find a closed, minimal invariant subspace E in its orbit closure in $R\mathcal{C}Sch_d$. Since the $\prod_{n=1}^\infty Q_n$-labelings encode the witnessing of Property A for M, it also witnesses Property A for any other graph in the orbit closure, that is E satisfies the conditions of Definition 10.6, so E is topologically amenable. □

The notion of **almost finiteness for étale Cantor groupoids** was defined by Matui (Definition 6.2 [45]). We will need this concept only for the case of étale groupoids associated to stable actions of Γ_{2d}. One can check that the following simple definition that is analogous to Definition 10.6 applies to the case of our étale groupoids E, hence they are almost finite in the sense of Matui.

Definition 10.8 (**Almost finiteness for groupoids**)**.** The groupoid $\mathcal{G}_{\beta_E}$ is almost finite if for any $\epsilon > 0$ there exists $r > 0$, K_ϵ and a partition of the totally disconnected space E into finitely many clopen sets $\{W_i\}_{i=1}^n$ such that

(1) if $x, y \in E$, $\alpha(g)(x) = y$ for some $g \in \Gamma$, and x, y are in the same clopen set, then either $d_E(x, y) \leq K_\epsilon$ or $d_E(x, y) \geq 3K_\epsilon$, where d_E is the graph distance on the orbit graphs.
(2) For any $x \in E$, the set H_x is ϵ-Følner, where

$$H_x = \{z \in E\,, x, z \,\text{are in the same clopen set}\, W_k \,\text{and}\, d_E(x, z) \leq K_\epsilon\,\}.$$

One should note that a definition of almost finiteness was given by Kerr [39] for free actions of amenable groups on compact metric spaces. If the amenable group Γ acts on the Cantor set freely then the almost finiteness of the associated étale groupoid in the sense of Matui is equivalent to the almost finiteness of the free action in the sense of Kerr. The definition of Kerr was extended to non-free actions by Joseph [36]. It is important to note that for such non-free actions the almost finiteness of the associated étale groupoid in the sense of Matui does not necessarily imply the almost finiteness of the action in the sense of Joseph.

The following proposition can be proved in the same way as Proposition 10.7.

Proposition 10.9. *For any minimal almost finite graph $M \in Gr_d$, we have an invariant subspace E as above, so that the associated étale groupoid $\mathcal{G}_{\beta_E}$ is almost finite.*

Corollary 10.10. *For every minimal strongly almost finite graph $M \in Gr_d$ we have an invariant subspace E as above, so that the associated étale groupoid $\mathcal{G}_{\beta_E}$ is both topologically amenable and almost finite.*

Proof. By Theorem 3 and Theorem 4, the graph M is Property A and almost finite. Label the vertices of M by the product of the labelings in Proposition 10.7 (that encodes Property A) and the labelings in Proposition 10.9 (that encodes almost finiteness). Let S be the new labeled graph. Now, find a minimal, closed invariant subspace E in the orbit closure of S. Putting together Proposition 10.7 and Proposition 10.9, we obtain the corollary. □

Question 1. Is it true that a minimal étale groupoid is almost finite if all of its orbit graphs are strongly almost finite?

Note that if the answer for this question is yes, then the groupoids associated to stable Cantor actions of amenable groups are always almost finite.

Remark 4. We could start with any finitely generated amenable group Γ (with some finite generating system Σ) and an element H of an arbitrary uniformly recurrent subgroup of Γ. By Proposition 7.4, the underlying graph of $\mathrm{Sch}(\Gamma/H, \Sigma)$ is a strongly almost finite minimal graph. Using this Schreier graph, we can repeat the construction above to obtain a stable, minimal, Γ-action α such that all orbit graphs are isomorphic to S and the associated étale groupoid is both topologically amenable and almost finite.

Now we are ready to prove the main result of this section.

Proof of Theorem 6. The algebras associated to the E's in Corollary 10.10 are always tracial (see Section 9 in [22]) due to the existence of invariant probability measures on E. The existence of such invariant measures follows from the amenability of the orbit graphs of E (a consequence of M being almost finite) in the same way as one proves that continuous actions of amenable groups on compact metric spaces always admit invariant probability measures (Theorem 6 [22]). Hence, by Corollary 9.11 in [44] cited at the beginning of the section, we finish the proof of our theorem. □

We also obtain a dynamical characterization of strong almost finiteness in the case of minimal graphs.

Proposition 10.11. *A minimal graph $M \in Gr_d$ is strongly almost finite if and only if there exists a stable action $\alpha : \Gamma \curvearrowright \mathcal{C}$, for a finitely generated group Γ with a finite generating set Σ with the following properties.*

- *All the orbit graphs of α are neighborhood equivalent to M.*

- *The action α is topologically amenable, admitting an invariant probability measure.*

Proof. As we have seen in the proof of Theorem 6, if M is strongly almost finite, actions as above always exist. Now, assume that M is not strongly almost finite. If M is not of Property A, then the required action cannot be topologically amenable. If M is of Property A, but not strongly almost finite, then by Theorem 3 and Theorem 4, M is not Følner. Hence by minimality, M must be nonamenable. If the action were topologically amenable, the action would be hyperfinite with respect to any invariant probability measure μ, and μ-almost all of its orbit graphs would be amenable [38]. This leads to a contradiction. □

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