Modeling green reputation decisions in a nonlinear Cournot duopoly of carbon emission abatement

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Abstract

In response to consumer environmental awareness and carbon taxation, we develop a nonlinear Cournot duopoly model to examine how firms make green reputation decisions. Using symbolic computation methods, we analyze equilibrium strategies in both static and dynamic settings. Our results show that a firm's emission abatement effort increases with its basic production cost, green efficiency, and carbon tax rate. Under nonlinear demand, less productive firms may exert greater effort to improve their green reputation, a phenomenon not observed under linear demand. Numerical simulations also show that excessive carbon taxes or low green efficiency can disrupt market equilibrium. Our study offers policymakers and business practitioners practical insights into how regulatory incentives and consumer preferences shape firms' green strategies. The proposed framework supports the development of environmental policies that foster green transformation and support long-term market stability.

Keywords: Green reputation; Carbon emission abatement; Nonlinear duopoly; Cournot competition; symbolic computation

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1 Introduction

The Intergovernmental Panel on Climate Change (IPCC) emphasizes the importance of limiting the rise in global average temperature to 2 °C to avoid the most dangerous consequences of climate change. In response to growing public concern about environmental issues, consumer attention to firm environmental practices and investments is increasing. This heightened awareness has been shown to influence consumer behavior, with more individuals willing to pay a premium for environmentally friendly products. For example, Laroche et al. (2001) investigated the demographic, psychological, and behavioral traits of environmentally conscious consumers and found a growing willingness to pay more for green products. Additional reports confirm this trend: in Europe, 75% of consumers expressed willingness to purchase green products at a premium in 2008, compared to only 31% in 2005[1]. In OECD countries, 27% of consumers are highly motivated to purchase environmentally friendly products and actively participate in environmental protection efforts[2].

Despite increasing pressure from consumers and regulators, firms' strategic decisions in balancing carbon abatement efforts with profitability remain complex, especially in oligopolistic markets where competition, cost heterogeneity, and nonlinear demand structures interact. In such settings, green reputation becomes a strategic asset: it not only affects consumer demand but also determines firms' market power and long-term viability. This paper models firms' green reputation and abatement strategies in a nonlinear Cournot duopoly framework, offering theoretical insights and policy implications.

This study develops a theoretical framework to investigate how firms strategically respond to consumers' green awareness under oligopolistic competition. Specifically, we investigate firm decision-making in terms of green reputation and carbon emission abatement in the presence of environmentally conscious consumers. Because many carbon-intensive industries have a small number of dominant players, we adopt an oligopoly structure, focusing on a Cournot duopoly. Economists generally believe that carbon pricing is a more effective mechanism for reducing greenhouse gas emissions than regulatory or command-and-control approaches. Carbon taxes and emissions trading are the two primary forms of carbon pricing that are currently used. As noted by Timilsina (2022), these mechanisms have comparable economic effects when designed similarly. Thus, our model uses carbon taxation as a representative policy tool. We propose a cost function that separates basic production costs from green production costs, the latter being associated with the use of environmentally friendly technologies. These two components are determined by the basic cost level and the green efficiency level, respectively.

The novelty of the models presented in this paper can be attributed to three major factors. First, we

[1]See European Commission (2008, 2009)
[2]See OECD (2002).

incorporate green reputation into firms' decision-making processes. The concept of green reputation is based on the work of André et al. (2011), who studied the effect of public disclosure of environmental records in a dynamic framework. Their findings revealed that disclosing a firm's environmental performance to the public has a significant impact on its reputation and public image, influencing market outcomes. Building on this insight, we consider green reputation a strategic variable that influences consumers' willingness to pay and, consequently, firms' competitive positioning in environmentally sensitive markets. Second, we construct a novel demand function that embeds green reputation alongside output levels, drawing from the structure of the widely used isoelastic demand function. This functional form, initially proposed by Puu (1995), has been widely applied in industrial organization and nonlinear dynamics literature (e.g., Ahmed and Agiza (1998); Bischi et al. (2007); Li et al. (2025); Li and Su (2024)) due to its analytical tractability and ability to capture nonlinear substitution effects. Our modified demand function takes into account the roles of both output and reputation in shaping consumer preferences under green awareness. Third, we develop a dynamic Cournot game based on the gradient adjustment mechanism to explore the green policies' long-term impact on market dynamics. Dynamic oligopoly games have been extensively used to study boundedly rational behavior and stability issues in industrial organizations (e.g., Bischi et al. (2010); Puu (2011)), but their application in the context of carbon emission abatement remains limited. By incorporating green reputation and policy instruments into the adjustment process, our model offers new insights into how environmental policies affect market stability and firm behavior over time.

To support our theoretical analysis, we employ symbolic computation methods, including the triangular decomposition method and the cylindrical algebraic decomposition (CAD) method. These methods are known for their precision and error-free nature, making them ideal for calculating and verifying economic results. We employ the triangular decomposition method to determine the Nash equilibrium of the proposed model and demonstrate that firms' equilibrium outputs are strictly positive. Furthermore, the CAD method allows us to conduct a thorough comparative static analysis, examining how equilibrium green reputation, output, market share, and profit react to changes in model parameters.

Beyond the novelty of the modeling framework, this paper provides several substantive economic findings. In the static analysis, we show that a firm's equilibrium green reputation (interpreted as its level of emission abatement effort) increases with both the basic production cost and the carbon tax rate, but decreases with the green efficiency level. Notably, our result that firms with lower production efficiency exert greater abatement effort under consumer environmental awareness is novel in the literature and stems specifically from the nonlinear demand setting. In contrast, studies using linear demand functions Buccella et al. (2021); Elsadany and Awad (2019); Wen et al. (2018); Xu et al. (2016) do not demonstrate this positive relationship.

This insight highlights an interesting mechanism: when consumers value green reputation, less efficient firms may face greater pressure to reduce emissions in order to remain competitive. Second, to examine the impact of green policy parameters on market stability, we develop a simple dynamic model. We analyze the dynamic system's Jacobian matrix under symmetry and find that Nash equilibrium stability can be lost when both the basic cost and the carbon tax rate are either too high or too low, or when green costs are excessively high. These theoretical results are supported by numerical simulations, which provide additional insight into how environmental policy influences market dynamics.

The economic modeling of consumers' green awareness has attracted substantial attention. Conrad (2005) analyzed how product differentiation affects markets when consumers care about environmental attributes. For instance, in a vertically differentiated framework, García-Gallego and Georgantzís (2010) found that environmental awareness campaigns might reduce overall welfare. Yakita and Yamauchi (2011) examined firms' R&D incentives under horizontal product differentiation and environmental R&D spillovers in a symmetric duopoly. Wen et al. (2018) studied pricing and carbon reduction strategies in a duopoly facing both emissions trading and consumer awareness. Other recent studies have incorporated green preferences into utility functions to explore the strategic behavior of firms under Cournot and Bertrand competition Xing and Lee (2024a); Xu and Lee (2023, 2024). For additional literature on consumer environmental awareness, see also Gori et al. (2024); Liu et al. (2012); Wen et al. (2018); Xing and Lee (2024b, 2025); Zhang et al. (2015).

A review of the literature reveals a plethora of studies that do not account for the possibility of consumers having green awareness but are still relevant to the field of emission abatement games. For example, Poyago-Theotoky (2007) examined two types of R&D organization regimes: independent R&D and environmental R&D cartels. This is pioneering work on carbon tax policies and competition policies related to quantity-setting duopolists with end-of-pipe technology. Ouchida and Goto (2014) revisited the model proposed by Poyago-Theotoky (2007) and conducted further analyses under revised environmental damage parameters. Meanwhile, Buccella et al. (2021) examined the strategic choices of firms in adopting abatement technologies in an environment of pollution externalities when the government imposed a carbon tax to incentivize firms to take abatement actions. Results indicate that when social awareness of a cleaner environment is relatively low (resp. high) and the index measuring the relative cost of abatement is relatively high (resp. low), the strategic interaction between two independent, competitive, and self-interested firms in the abatement game will lead to them not abating (resp. abating). In a related study, Xu et al. (2016) compared the Cournot and Bertrand models of competition in a differentiated mixed duopoly, incorporating both emissions taxes and privatization policies. The findings showed that the socially optimal combination of emissions tax and privatization resulted in the greatest environmental damage. They also stated that Cournot competition results in lower environmental damage and social welfare than Bertrand competition. Subsequently, Elsadany and Awad (2019) introduced naive and boundedly rational adjustment mechanisms into dynamic models, building on Xu et al. (2016). Regarding dynamic games, the work of Zeppini (2015) and Cavalli et al. (2023) should also be mentioned. They approached the environmental-economic problem from a new perspective, focusing on technologies rather than companies, and proposed a discrete choice model for the transition from dirty to clean technologies.

The remaining sections of this paper are organized as follows. Section 2 establishes a static duopoly game for the issue of carbon emission abatement. Section 3 compares the static analysis of the static game's Nash equilibrium. Section 4 introduces a dynamic model to examine the impact of green policies on market stability. Section 5 concludes the paper.

2 Static Model

Assume two firms produce homogeneous products and compete directly with one another. In this study, we utilize q_1 and q_2 to represent the output of the two firms, and e_1 and e_2 to denote the carbon emissions per unit of product produced by the two firms. The regulator will levy a carbon tax on firms, with a tax rate of $T_c \geq 0$. Consequently, firm i will be taxed $T_c e_i q_i$.

In a market featured by green consciousness, consumers prefer products with lower carbon emissions per unit. In particular, consumers are most willing to pay for products manufactured by firms with zero emissions (i.e., $e_i = 0$). Consumer's willingness to pay decreases as the level of emissions, represented by e_i, increases. In this context, we employ e_0 to represent the emissions per unit for which the willingness to pay is zero. In practice, questionnaires and other similar instruments can be used to estimate the value of e_0, which is subject to change as consumers' green awareness shifts. An increase in consumers' green awareness will lead to a decrease in e_0.

As a result, the *green reputation* of Firm i can be defined as

$$u_i = \frac{e_0 - e_i}{e_0}. \tag{1}$$

When $u_i = 0$, we have $e_i = e_0$, indicating that the firm's emissions per unit have reached the limit of consumers' willingness to pay. In this case, consumers are no longer willing to pay for the firm's product. When $u_i = 1$, we have $e_i = 0$, indicating that the company's production process is entirely free of carbon emissions. In practice, a company's green reputation will not be confined to the extremes of 0 or 1 but will occupy a position somewhere in the middle. Consequently, we assume that $u_i \in (0, 1)$ in what follows.

Suppose the representative consumer's utility function is

$$U_c = \ln(u_1 q_1 + u_2 q_2).$$

This utility function is derived by following the utility function, i.e., $\ln(q_1 + q_2)$, which represents the isoelastic demand. Several studies, including Ahmed and Agiza (1998); Gao et al. (2012); Li et al. (2025); Li and Su (2024); Puu (1991), have used the isoelastic demand function to describe the nonlinear nature of market prices.

Accordingly, the consumer surplus is defined as

$$CS = U_c - (p_1 q_1 + p_2 q_2), \tag{2}$$

where p_1 and p_2 are the prices of the two commodities, respectively.

Lemma 1. *The inverse demand functions for the products of the two firms are*

$$p_i = \frac{u_i}{u_i q_i + u_{-i} q_{-i}}, \quad i = 1, 2. \tag{3}$$

Proof. The first-order condition for maximizing (2) is

$$\frac{\partial U_c}{\partial q_i} = \frac{u_i}{u_i q_i + u_{-i} q_{-i}} - p_i = 0, \quad i = 1, 2,$$

which can be solved by (3). Moreover, the second-order condition always holds because

$$\frac{\partial^2 U_c}{\partial q_i^2} = -\frac{u_i^2}{(u_i q_i + u_{-i} q_{-i})^2} < 0, \quad i = 1, 2,$$

which completes the proof. □

Remark 1. From Eq. (3), the price of products is nonlinear with respect to u_i and q_i. Further calculations yield

$$\frac{\partial p_i}{\partial u_i} = \frac{u_{-i} q_{-i}}{(u_i q_i + u_{-i} q_{-i})^2} > 0.$$

Holding other factors constant, a rise in u_i will lead to a rise in p_i.

In addition, we have

$$\frac{p_1}{p_2} = \frac{u_1}{u_2}, \tag{4}$$

which implies that the ratio of firms' green reputation determines the ratio of their product prices. In particular, the product produced by the firm with a better green reputation is more expensive because consumers are willing to pay more for greener products. Notably, our model depicts a situation in which consumers are "perfectly" green, in the sense that, for a product with a green reputation close to zero, their willingness to pay is also close to zero.

Consider the production side. Given that $e_i = e_0(1 - u_i)$, it follows that Firm i is subject to a carbon tax of the form

$$T_c e_i q_i = T_c e_0 (1 - u_i) q_i.$$

For simplicity, we set $d = T_c e_0$. Consequently, the tax imposed on Firm i can be expressed as $d(1 - u_i)q_i$, where $d \geq 0$.

Furthermore, it is assumed that the production cost of Firm i is given by

$$C_i(q_i) = c_i q_i + r_i u_i^2 q_i,$$

where $c_i > 0$ and $r_i > 0$. In this context, the term "basic production cost" refers to the cost incurred by Firm i in the absence of any emission reduction measures, represented by $c_i q_i$. Meanwhile, the term "green production cost" refers to the additional cost incurred by Firm i when adopting green technologies, denoted by $r_i u_i^2 q_i$. The green efficiency parameter r_i captures the discrepancy between the green costs of different firms due to their varying emission abatement technology levels. A lower value of r_i indicates that Firm i can achieve a given level of green reputation u_i at a lower green cost, conferring a green cost advantage. We call c_i and r_i the basic cost level and the green efficiency level of Firm i, respectively.

The term $r_i u_i^2 q_i$ denotes the green cost and is quadratic in u_i, reflecting a widely accepted assumption in the environmental economics literature: the marginal cost of environmental improvement increases with effort. In other words, the quadratic form captures the idea that each additional unit of carbon reduction increases the cost of achieving it. This functional specification originates from the seminal work of Nordhaus (1991) and Moraga-González and Padrón-Fumero (2002), and has since been adopted in more recent studies such as Liu et al. (2012) and Wen et al. (2018).

Based on the outlined assumptions, the profit of Firm i is given by

$$\Pi_i = \frac{u_i q_i}{u_i q_i + u_{-i} q_{-i}} - d\,(1 - u_i)\,q_i - (c_i + r_i u_i^2) q_i. \tag{5}$$

In this analysis, u_i and q_i are treated as the firm's strategic decision variables, whereas d, c_i, and r_i are treated as exogenous parameters. In the static game framework, the two firms simultaneously choose their green reputation levels u_i and output quantities q_i. Consumers are assumed to observe the green reputations of both firms and make purchasing decisions based on the relative quantities and green reputations of the goods available on the market.

Firm i maximizes its profits by making decisions on u_i and q_i. We have

$$\frac{\partial \Pi_i}{\partial q_i} = \frac{u_i u_{-i} q_{-i}}{\left(u_i q_i + u_{-i} q_{-i}\right)^2} - d\left(1 - u_i\right) - \left(c_i + r_i u_i^2\right),$$

and the second-order partial derivative is

$$\frac{\partial^2 \Pi_i}{\partial q_i^2} = -\frac{2u_i^2 u_{-i} q_{-i}}{\left(u_i q_i + u_{-i} q_{-i}\right)^3} < 0.$$

Thus, Firm i determines its output by the first-order condition $\frac{\partial \Pi_i}{\partial q_i} = 0$.

Moreover, a few calculations show that

$$\frac{\partial \Pi_i}{\partial u_i} = \frac{q_i u_{-i} q_{-i}}{\left(u_i q_i + u_{-i} q_{-i}\right)^2} + d q_i - 2 r_i u_i q_i,$$

and the second-order partial derivative is

$$\frac{\partial^2 \Pi_i}{\partial u_i^2} = -\frac{2q_i\left(r_i u_i^3 q_i^3 + 3 r_i u_i^2 u_{-i} q_i^2 q_{-i} + \left(3 r_i u_i u_{-i}^2 q_{-i}^2 + u_{-i} q_{-i}\right) q_i + r_i u_{-i}^3 q_{-i}^3\right)}{\left(u_i q_i + u_{-i} q_{-i}\right)^3} < 0.$$

Thus, Firm i selects the desired level of green reputation by the first-order condition $\frac{\partial \Pi_i}{\partial u_i} = 0$.

Theorem 1. *The model has one unique Nash equilibrium $E_N = (u_1^*, u_2^*, q_1^*, q_2^*)$, where*

$$u_i^* = \sqrt{\frac{c_i + d}{r_i}}, \quad i = 1, 2, \tag{6}$$

and

$$q_i^* = \frac{r_i u_i^* (2 r_{-i} u_{-i}^* - d)}{4\left(c_i + d\right)\left(d^2 + \left(\left(1 - 2u_{-i}^*\right) r_{-i} + \left(1 - 2u_i^*\right) r_i\right) d + \left(2 r_i u_i^* u_{-i}^* + c_{-i}\right) r_{-i} + c_i r_i\right)}, \quad i = 1, 2. \tag{7}$$

Proof. The Nash equilibrium $E_N = (u_1^*, u_2^*, q_1^*, q_2^*)$ satisfies the first-order conditions

$$\left.\frac{\partial \Pi_i}{\partial u_i}\right|_{(u_i^*, u_{-i}^*, q_i^*, q_{-i}^*)} = 0, \quad \left.\frac{\partial \Pi_i}{\partial q_i}\right|_{(u_i^*, u_{-i}^*, q_i^*, q_{-i}^*)} = 0, \quad i = 1, 2.$$

As a result, a system of equilibrium equations is obtained as follows:

$$\begin{cases} \dfrac{u_1^* u_2^* q_2^*}{\left(u_1^* q_1^* + u_2^* q_2^*\right)^2} - d\left(1 - u_1^*\right) - \left(c_1 + r_1 u_1^{*2}\right) = 0, \\ \dfrac{u_2^* u_1^* q_1^*}{\left(u_1^* q_1^* + u_2^* q_2^*\right)^2} - d\left(1 - u_2^*\right) - \left(c_2 + r_2 u_2^{*2}\right) = 0, \\ \dfrac{q_1^* u_2^* q_2^*}{\left(u_1^* q_1^* + u_2^* q_2^*\right)^2} + d q_1^* - 2 r_1 u_1^* q_1^* = 0, \\ \dfrac{q_2^* u_1^* q_1^*}{\left(u_1^* q_1^* + u_2^* q_2^*\right)^2} + d q_2^* - 2 r_2 u_2^* q_2^* = 0. \end{cases}$$

The above system is nonlinear, and the triangular decomposition method[3] can help analyze the structure of its solution. We use u_i^* and q_i^* as variables and the remaining symbols as parameters. Using the triangular decomposition method[4], we can transform the equilibria in the equilibrium equations into the zeros of the triangular sets $[T_1, T_2, T_3, T_4]$, where

$$\begin{aligned} T_1 &= r_1 u_1^{*2} - c_1 - d, \\ T_2 &= r_2 u_2^{*2} - c_2 - d, \\ T_3 &= 4\left(c_1 + d\right)\left(d^2 + \left(\left(1 - 2\,u_2^*\right) r_2 + \left(1 - 2\,u_1^*\right) r_1\right) d + \left(2\, r_1 u_1^* u_2^* + c_2\right) r_2 + c_1 r_1\right) q_1^* \\ &\quad - r_1 u_1^* (2\, r_2 u_2^* - d), \\ T_4 &= \left(r_2 u_2^{*2} + c_2 + d - u_2^* d\right) q_2^* - \left(r_1 u_1^{*2} + c_1 + d - u_1^* d\right) q_1^*. \end{aligned} \tag{8}$$

T_1 contains only the variable u_1^*, T_2 contains only the variable u_2^*, T_3 contains the variables u_1^*, u_2^*, q_1^*, and T_4 contains the variables u_1^*, u_2^*, q_1^*, q_2^*. To put it differently, the number of variables gradually increases, and $[T_1, T_2, T_3, T_4]$ exhibits a triangular form. Such a special structure can greatly facilitate our analysis of the Nash equilibrium.

Solving u_1^*, u_2^* from $T_1 = 0$, $T_2 = 0$ and noting that $u_1^* > 0$ and $u_2^* > 0$, we obtain (6). From $T_3 = 0$, the expression for q_1^* can be derived immediately, and the expression for q_2^* follows by the symmetry. □

Assumption 1. According to Eq. (6), we assume $c_i + d < r_i$, $i = 1, 2$, because $u_i^* \in (0, 1)$.

Proposition 1. *For the Nash equilibrium* $E_N = (u_1^*, u_2^*, q_1^*, q_2^*)$*, there must be* $q_i^* > 0$*,* $i = 1, 2$*.*

[3]The triangular decomposition method can be viewed as an extension of the Gaussian elimination method. The primary concept behind both methods is to transform a system of equations into a triangular form. However, the triangular decomposition method is applicable to polynomial systems, whereas the Gaussian elimination method is only applicable to linear systems. For more details of triangular decomposition, see, e.g., (Jin et al., 2013; Li et al., 2010; Wang, 2001; Wu, 1986).

[4]The reader can directly use the Triangularize function from the RegularChains package of Maple 2022.

Proof. According to Eq. (7), to prove $q_i^* > 0$, we must show that $2r_{-i}u_{-i}^* - d > 0$ and that

$$d^2 + \left(\left(1 - 2u_{-i}^*\right) r_{-i} + \left(1 - 2u_i^*\right) r_i\right) d + \left(2r_i u_i^* u_{-i}^* + c_{-i}\right) r_{-i} + c_i r_i > 0. \tag{9}$$

Since $c_{-i} + d < r_{-i}$, we have

$$\frac{\partial\left(2r_{-i}u_{-i}^* - d\right)}{\partial d} = \sqrt{\frac{r_{-i}}{c_{-i} + d}} - 1 > 0.$$

This implies that $2r_{-i}u_{-i}^* - d$ is monotonically increasing with d, thus

$$2r_{-i}u_{-i}^* - d = 2\sqrt{r_{-i}(c_{-i} + d)} - d \geq 2\sqrt{r_{-i}(c_{-i} + 0)} - 0 > 0.$$

Meanwhile, the proof of the inequality (9) is more intricate and can be approached through the CAD method. In this demonstration, we prove (9) only for the case of $i = 1$ (the proof for the case of $i = 2$ is analogous). We know that $r_1 u_1^{*2} - c_1 - d = 0$ and $r_2 u_2^{*2} - c_2 - d = 0$, where $0 < u_1^* < 1$, $0 < u_2^* < 1$, and take into account the parameter constraints $c_1 > 0$, $c_2 > 0$, $r_1 > 0$, $r_2 > 0$, and $d \geq 0$. Then proving (9) for $i = 1$ is equivalent to showing that the following inequality does not hold:

$$d^2 + \left(\left(1 - 2u_2^*\right) r_2 + \left(1 - 2u_1^*\right) r_1\right) d + \left(2r_1 u_1^* u_2^* + c_2\right) r_2 + c_1 r_1 \leq 0.$$

In other words, we only need to prove that the following semi-algebraic system has no real solutions:

$$\begin{cases} r_1 u_1^{*2} - c_1 - d = 0, \;\; r_2 u_2^{*2} - c_2 - d = 0, \\ u_1^* > 0, \;\; u_2^* > 0, \;\; 1 - u_1^* > 0, \;\; 1 - u_2^* > 0, \\ c_1 > 0, \;\; c_2 > 0, \;\; r_1 > 0, \;\; r_2 > 0, \;\; d \geq 0, \\ d^2 + \left(\left(1 - 2u_2^*\right) r_2 + \left(1 - 2u_1^*\right) r_1\right) d + \left(2r_1 u_1^* u_2^* + c_2\right) r_2 + c_1 r_1 \leq 0. \end{cases}$$

Using the CAD method[5], we obtain computational results indicating no real solutions to the above system, thereby completing the proof. □

[5] The reader can directly use the SamplePoints function in the RegularChains package of Maple 2022. The CAD method is the first practical quantifier elimination algorithm proposed by Collins (1975). Therefore, it is also known as Collins' algorithm. This algorithm decomposes any semi-algebraic set in the n-dimensional space of real numbers into a finite number of disjoint semi-algebraic sets. The same set of polynomials defines all of the resulting semi-algebraic sets, and the sign of the polynomials defined on each semi-algebraic set remains constant. Specifically, the CAD method can compute the CAD and its sample points, ensuring that the signs of the given polynomials remain constant across decompositions. Note that the original CAD method was not efficient enough and was later improved by Collins and Hong (1991) and Brown (2001).

3 Comparative Static Analysis

This section presents a comparative static analysis of firms' equilibrium green reputation, output quantity, market share, and profit, as well as their equilibrium consumer surplus.

3.1 Green Reputation

Proposition 2. *The equilibrium green reputation of Firm i, i.e.,* u_i^**, will increase with* c_i *or* d*, but decrease with* r_i*.*

Proof. The conclusion can be immediately derived from Eq. (6). □

Eq. (1) calculates the equilibrium emissions per unit of Firm i as $e_i^* = e_0(1 - u_i^*)$. According to Proposition 2, the regulator can reduce the firm's per-unit emissions e_i^* by increasing the equilibrium green reputation u_i^*. This can be achieved by enhancing green efficiency—i.e., lowering the value of r_i—and by raising the carbon tax rate d. In addition, the regulator can reduce the baseline emissions parameter e_0 through initiatives such as green education and public awareness campaigns, thereby contributing to emissions reduction.

According to Eqs. (4) and (6), the equilibrium prices of the two firms satisfy

$$\frac{p_1^*}{p_2^*} = \frac{u_1^*}{u_2^*} = \sqrt{\frac{c_1 + d}{c_2 + d}}\sqrt{\frac{r_2}{r_1}}.$$

If $c_1 = c_2$, then $\frac{p_1^*}{p_2^*} = \frac{u_1^*}{u_2^*} = \sqrt{\frac{r_2}{r_1}}$. This indicates that, when basic costs are equal, the firm with a green efficiency advantage (i.e., a lower r_i) achieves a higher equilibrium green reputation, resulting in a higher equilibrium price. Alternatively, if $r_1 = r_2$, then $\frac{p_1^*}{p_2^*} = \frac{u_1^*}{u_2^*} = \sqrt{\frac{c_1+d}{c_2+d}}$. This implies that when green efficiency is constant, the equilibrium green reputation level u_i^* correlates positively with the basic cost c_i. From a profit-maximization perspective, a higher c_i incentivizes the firm to raise its price and imposes greater pressure to improve its green reputation to remain competitive in the face of consumer environmental awareness. To the best of our knowledge, this result has not been observed in existing studies under linear demand, and it highlights a unique mechanism specific to the nonlinear framework considered in this paper.

For analytical simplicity, most models of green oligopoly competition in the carbon emission abatement literature use linear demand functions. However, the linearity assumption may overlook potential links between firms' willingness to reduce emissions and their production efficiency. For example, Elsadany and Awad (2019) and Xu et al. (2016) demonstrated that, under both the Cournot and Bertrand frameworks, firms' emission abatement efforts depend solely on the carbon tax rate. The model by Buccella et al. (2021), which incorporates green costs, finds that abatement effort increases with the carbon tax rate but decreases with the level of green costs. Wen et al. (2018) introduced a carbon trading framework that incorporates green-aware consumers, demonstrating that firms' abatement incentives increase with the carbon price and consumer environmental awareness. Notably, all of these studies rely on linear demand and do not uncover any relationship between production efficiency and firms' abatement behavior.

In contrast, our model employs a nonlinear demand function and reveals a novel mechanism: firms with higher basic production costs may have greater incentives to improve their green reputation and reduce emissions. Although this result may appear counterintuitive, it emerges naturally from nonlinear consumer preferences and emphasizes the importance of modeling demand curvature in environmental competition. We acknowledge that this theoretical prediction requires empirical validation, which we leave for future research.

3.2 Output Quantity

Proposition 3. *If $c_i < c_{-i}$, then q_i^* will decrease with d.*

Proof. By virtue of the symmetry, it is sufficient to demonstrate that $\frac{\partial q_1^*}{\partial d} < 0$ if $c_1 < c_2$. We have

$$\frac{\partial q_1^*}{\partial d} = -\frac{\begin{array}{l} 3r_1(((\frac{2}{3}r_2c_2 + \frac{2}{3}r_2d + \frac{1}{3}d^2)a_1 - \frac{d^3}{4} + (-\frac{r_1}{12} - \frac{c_1}{6} - \frac{13r_2}{12})d^2 + ((-\frac{2c_1}{3} - \frac{11c_2}{12})r_2 + \frac{r_1c_1}{12})d + \\ \frac{c_1(r_1c_1 - 3r_2c_2)}{6})a_2 + ((-\frac{7d^2}{6} + (-\frac{c_1}{6} - \frac{4c_2}{3})d - \frac{c_1c_2}{3})a_1 + d^3 + (\frac{2c_1}{3} + \frac{r_1}{3} + c_2 + \frac{r_2}{3})d^2 + \\ ((\frac{c_1}{6} + \frac{c_2}{2})r_2 + (\frac{2c_2}{3} + \frac{r_1}{6})c_1 + \frac{c_2r_1}{2})d + \frac{c_2(c_1+c_2)r_2}{6} - \frac{r_1c_1(c_1 - 3c_2)}{6})r_2) \end{array}}{2a_1a_2(c_1+d)((-2d+2a_1)a_2 - 2da_1 + d^2 + (r_1+r_2)d + r_1c_1 + r_2c_2)^2},$$

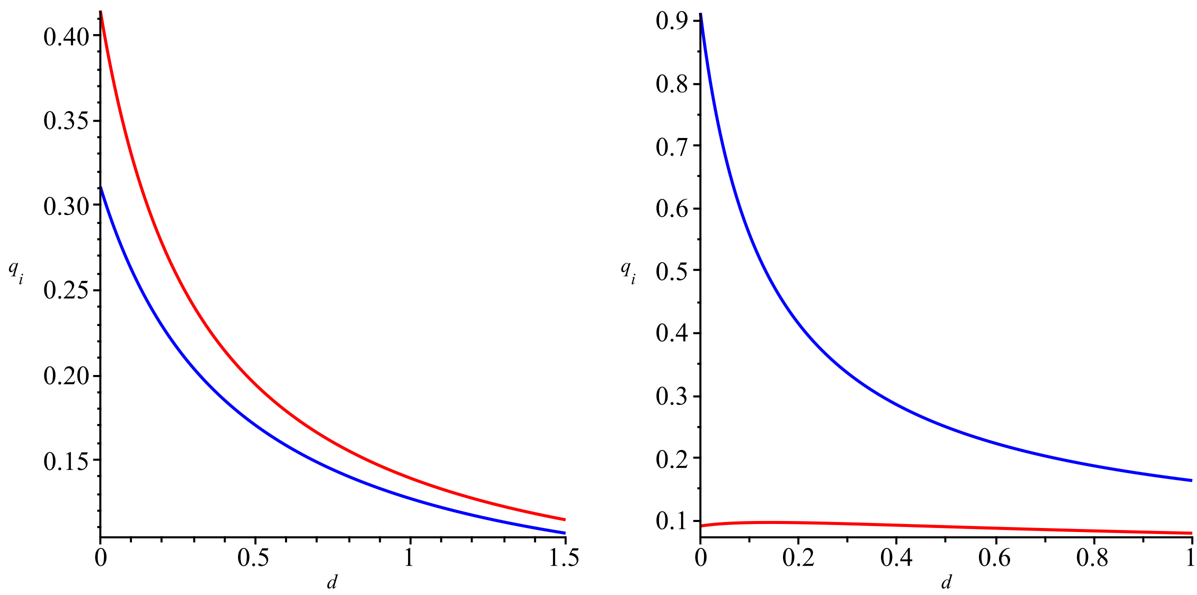
where $a_1 = \sqrt{r_1(c_1+d)}$, $a_2 = \sqrt{r_2(c_2+d)}$.

We use $\mathrm{Num}(\cdot)$ and $\mathrm{Den}(\cdot)$ to denote the numerator and denominator, respectively. Then, provided that $\mathrm{Den}\left(\frac{\partial q_1^*}{\partial d}\right) \neq 0$, it is evident that $\frac{\partial q_1^*}{\partial d}$ and $\mathrm{Num}\left(\frac{\partial q_1^*}{\partial d}\right) \cdot \mathrm{Den}\left(\frac{\partial q_1^*}{\partial d}\right)$ have same signs.

By using the CAD method, we derive that no real solutions exist for the following semi-algebraic system:

$$\begin{cases} a_1^2 - r_1(c_1+d) = 0,\ a_2^2 - r_2(c_2+d) = 0, \\ c_1 > 0,\ c_2 > 0,\ r_1 > 0,\ r_2 > 0,\ d \geq 0, \\ a_1 > 0,\ a_2 > 0,\ r_1 - (c_1+d) > 0,\ r_2 - (c_2+d) > 0, \\ \boldsymbol{c_2 - c_1 > 0}, \\ \mathrm{Num}\left(\frac{\partial q_1^*}{\partial d}\right) \cdot \mathrm{Den}\left(\frac{\partial q_1^*}{\partial d}\right) \geq 0. \end{cases}$$

It follows that if $c_1 < c_2$, then $\text{Num}\left(\frac{\partial q_1^*}{\partial d}\right) \cdot \text{Den}\left(\frac{\partial q_1^*}{\partial d}\right) < 0$. That is, $\frac{\partial q_1^*}{\partial d} < 0$, which completes the proof. □



(a) $c_1 = 0.3$, $c_2 = 0.4$, $r_1 = 2$, $r_2 = 2$

(b) $c_1 = 1$, $c_2 = 0.1$, $r_1 = 2$, $r_2 = 2$

Figure 1: Effect of the parameter d on q_i^*, where q_1^* and q_2^* are colored in red and blue, respectively.

Fig. 1 illustrates the effect of the carbon tax rate d on the equilibrium outputs q_i^*, where q_1^* and q_2^* are represented by red and blue lines, respectively. In Fig. 1(a), we set the other parameters to $c_1 = 0.3$, $c_2 = 0.4$, $r_1 = 2$, and $r_2 = 2$. The equilibrium output q_1^* of the more production efficient Firm 1 (i.e., the firm with the lower basic cost c_i) decreases as d increases, which is consistent with the result in Proposition 3. Moreover, we find that the output q_2^* of the less efficient Firm 2 (with higher c_i) decreases with an increase in d.

However, this pattern may not hold when basic cost levels differ significantly. In Fig. 1(b), we increase the basic cost of Firm 1 by ten times that of Firm 2. In this scenario, the response of q_1^* to changes in the carbon tax rate d becomes insignificant and non-monotonic. This observation suggests that the condition $c_i < c_{-i}$ in Proposition 3 is indeed necessary for the monotonicity result to hold.

Proposition 4. *The equilibrium output q_i^* will decrease with c_i. If $c_i < c_{-i}$ and $r_i < r_{-i}$, then q_i^* will decrease with c_{-i}.*

Proof. By the symmetry, it is sufficient to demonstrate that the proposition holds for $i = 1$. This entails proving that: (1) $\frac{\partial q_1^*}{\partial c_1} < 0$; (2) $\frac{\partial q_1^*}{\partial c_2} < 0$ if $c_1 < c_2$ and $r_1 < r_2$.

We have

$$\frac{\partial q_1^*}{\partial c_1} = \frac{(-2a_2+d)\left((4a_1-2d)\,a_2-4da_1+d^2+(3r_1+r_2)\,d+3r_1c_1+r_2c_2\right)r_1}{8a_1\,(c_1+d)\left((-2d+2a_1)\,a_2-2da_1+d^2+(r_1+r_2)\,d+r_1c_1+r_2c_2\right)^2},$$

where $a_1 = \sqrt{r_1(c_1+d)}$ and $a_2 = \sqrt{r_2(c_2+d)}$.

By using the CAD method, we derive no real solutions for the following semi-algebraic system:

$$\begin{cases} a_1^2 - r_1(c_1+d) = 0,\ a_2^2 - r_2(c_2+d) = 0, \\ c_1 > 0,\ c_2 > 0,\ r_1 > 0,\ r_2 > 0,\ d \geq 0, \\ a_1 > 0,\ a_2 > 0,\ r_1 - (c_1+d) > 0,\ r_2 - (c_2+d) > 0, \\ \mathrm{Num}\left(\frac{\partial q_1^*}{\partial c_1}\right) \cdot \mathrm{Den}\left(\frac{\partial q_1^*}{\partial c_1}\right) \geq 0. \end{cases}$$

It follows that $\mathrm{Num}\left(\frac{\partial q_1^*}{\partial c_1}\right) \cdot \mathrm{Den}\left(\frac{\partial q_1^*}{\partial c_1}\right) < 0$, and thus $\frac{\partial q_1^*}{\partial c_1} < 0$.

In addition, we have

$$\frac{\partial q_1^*}{\partial c_2} = \frac{(-da_1+da_2+(r_1-r_2)\,d+r_1c_1-r_2c_2)\,a_1r_2}{4a_2\left((2a_2-2d)\,a_1-2da_2+d^2+(r_1+r_2)\,d+r_1c_1+r_2c_2\right)^2(c_1+d)}.$$

By using the CAD method, we derive no real solutions for the following semi-algebraic system:

$$\begin{cases} a_1^2 - r_1(c_1+d) = 0,\ a_2^2 - r_2(c_2+d) = 0, \\ c_1 > 0,\ c_2 > 0,\ r_1 > 0,\ r_2 > 0,\ d \geq 0, \\ a_1 > 0,\ a_2 > 0,\ r_1 - (c_1+d) > 0,\ r_2 - (c_2+d) > 0, \\ \boldsymbol{c_2 - c_1 > 0,\ r_2 - r_1 > 0,} \\ \mathrm{Num}\left(\frac{\partial q_1^*}{\partial c_2}\right) \cdot \mathrm{Den}\left(\frac{\partial q_1^*}{\partial c_2}\right) \geq 0. \end{cases}$$

It follows that $\frac{\partial q_1^*}{\partial c_2} < 0$ if $c_1 < c_2$ and $r_1 < r_2$, which completes the proof. □

To illustrate Proposition 4, we show in Fig. 2 the effect of the basic cost parameter c_1 on the equilibrium outputs q_i^*. As Firm 1 becomes more efficient in production (i.e., as c_1 decreases), its equilibrium output q_1^* increases monotonically. This result supports Proposition 4 and is consistent with economic intuition: lower production costs lead to higher output. The blue curve shows that c_1 has a minor impact on Firm 2's equilibrium output q_2^*. Nonetheless, we observe that when $c_1 > c_2 = 0.4$, a reduction in c_1 leads to a moderate increase in q_2^*, suggesting some degree of strategic interdependence between firms.

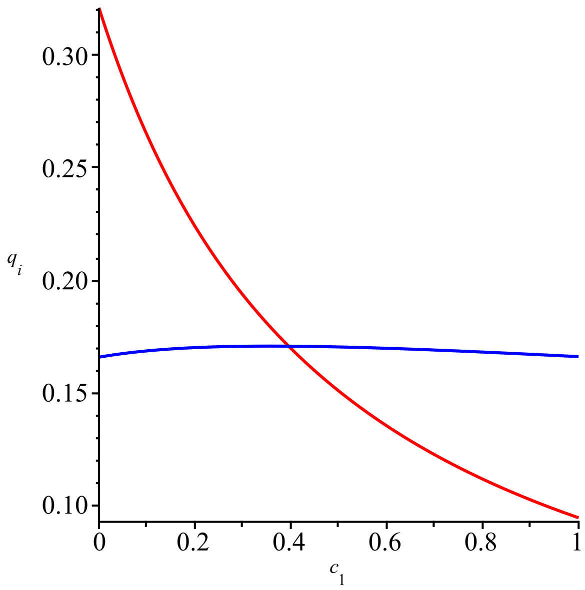


Figure 2: Effect of the parameter c_1 on q_i^*, where q_1^* and q_2^* are colored in red and blue, respectively. The other parameters are set to $c_2 = 0.4$, $r_1 = 2.1$, $r_2 = 2$, and $d = 0.5$.

Proposition 5. *If $c_i > c_{-i}$ and $r_i > r_{-i}$, then q_i^* will decrease with r_i. If $c_i < c_{-i}$ and $r_i < r_{-i}$, then q_i^* will decrease with r_{-i}.*

Proof. By the symmetry, we only need to show that the proposition holds for $i = 1$. This entails proving that: (1) $\frac{\partial q_1^*}{\partial r_1} < 0$ if $c_1 > c_2$ and $r_1 > r_2$; (2) $\frac{\partial q_1^*}{\partial r_2} < 0$ if $c_1 < c_2$ and $r_1 < r_2$.

It can be calculated that

$$\frac{\partial q_1^*}{\partial r_1} = \frac{(-2a_2 + d)\left(-2da_2 + d^2 + (-r_1 + r_2)\, d - r_1c_1 + r_2c_2\right)}{8a_1\left((-2d + 2a_1)\, a_2 - 2da_1 + d^2 + (r_1 + r_2)\, d + r_1c_1 + r_2c_2\right)^2},$$

where $a_1 = \sqrt{r_1(c_1 + d)}$, $a_2 = \sqrt{r_2(c_2 + d)}$.

By using the CAD method, we derive that there are no real solutions for the following semi-algebraic system:

$$\begin{cases} a_1^2 - r_1(c_1 + d) = 0,\ a_2^2 - r_2(c_2 + d) = 0, \\ c_1 > 0,\ c_2 > 0,\ r_1 > 0,\ r_2 > 0,\ d \geq 0, \\ a_1 > 0,\ a_2 > 0,\ r_1 - (c_1 + d) > 0,\ r_2 - (c_2 + d) > 0, \\ \boldsymbol{c_1 - c_2 > 0,\ r_1 - r_2 > 0,} \\ \text{Num}\left(\dfrac{\partial q_1^*}{\partial r_1}\right) \cdot \text{Den}\left(\dfrac{\partial q_1^*}{\partial r_1}\right) \geq 0. \end{cases}$$

Therefore, if $c_1 > c_2$ and $r_1 > r_2$, then $\frac{\partial q_1^*}{\partial r_1} < 0$.

Furthermore, we have

$$\frac{\partial q_1^*}{\partial r_2} = \frac{(-da_1 + da_2 + (r_1 - r_2)\,d + r_1c_1 - r_2c_2)\,(c_2 + d)\,a_1}{4a_2\,(c_1 + d)\,((2a_2 - 2d)\,a_1 - 2da_2 + d^2 + (r_1 + r_2)\,d + r_1c_1 + r_2c_2)^2}.$$

By using the CAD method, we prove no real solutions for the following semi-algebraic system:

$$\begin{cases} a_1^2 - r_1(c_1 + d) = 0,\ a_2^2 - r_2(c_2 + d) = 0, \\ c_1 > 0,\ c_2 > 0,\ r_1 > 0,\ r_2 > 0,\ d \geq 0, \\ a_1 > 0,\ a_2 > 0,\ r_1 - (c_1 + d) > 0,\ r_2 - (c_2 + d) > 0, \\ \boldsymbol{c_2 - c_1 > 0,\ r_2 - r_1 > 0,} \\ \mathrm{Num}\left(\frac{\partial q_1^*}{\partial r_2}\right) \cdot \mathrm{Den}\left(\frac{\partial q_1^*}{\partial r_2}\right) \geq 0. \end{cases}$$

It follows that $\frac{\partial q_1^*}{\partial r_2} < 0$ if $c_1 < c_2$ and $r_1 < r_2$, which completes the proof. □

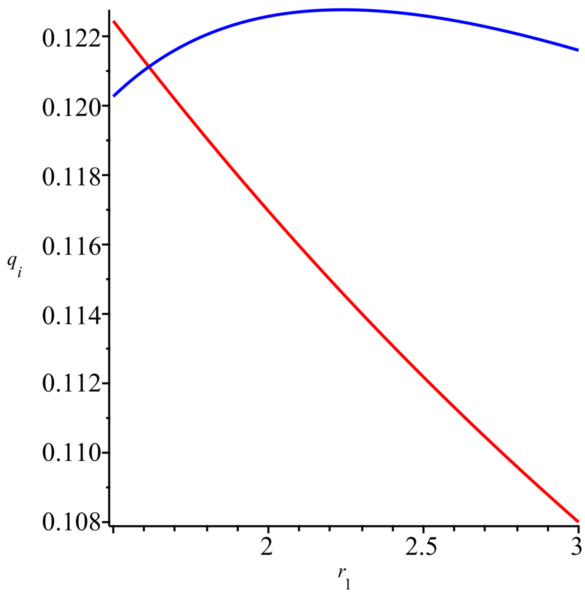


Figure 3: Effect of the parameter r_1 on q_i^*, where q_1^* and q_2^* are colored in red and blue, respectively. The other parameters are set to $c_1 = 0.5$, $c_2 = 0.4$, $r_2 = 2.4$, and $d = 1$.

Fig. 3 depicts the conclusion of Proposition 5. As shown in the figure, the equilibrium output of Firm 1 increases monotonically as its green efficiency improves—that is, as r_1 decreases. In contrast, r_1 has a non-monotonic effect on Firm 2's equilibrium output. However, when $r_1 > r_2 = 2.4$, we observe that a reduction in r_1 leads to an increase in q_2^*. This observation confirms the need for the conditions $c_i < c_{-i}$ and $r_i < r_{-i}$ in the second part of Proposition 5 to ensure the expected comparative statics.

3.3 Market Share

From an economic standpoint, it is important to examine how green policies affect firms' market shares. The market share of Firm i is defined as $\frac{q_i}{q_i+q_{-i}}$. It is easy to show that an increase in the output ratio $\frac{q_i}{q_{-i}}$ corresponds to an increase in Firm i's market share, and vice versa. Therefore, to analyze the impact of policy parameters on market shares, we must study their effects on the output ratio $\frac{q_i}{q_{-i}}$.

Proposition 6. *The market share of Firm i will decrease with r_i and increase with r_{-i}.*

Proof. By the symmetry, we only show that Firm 1's market share decreases with r_1 and increases with r_2. This is equivalent to proving that the output ratio $\frac{q_1^*}{q_2^*}$ decreases with r_1 and increases with r_2.

From T_4 in Eq. (8), it is known that

$$\frac{q_1^*}{q_2^*} = \frac{r_2 u_2^{*2} + c_2 + d - u_2^* d}{r_1 u_1^{*2} + c_1 + d - u_1^* d}.$$

Plugging Eq. (6) into the above yields

$$\frac{q_1^*}{q_2^*} = \frac{2\, c_2 + d\left(2 - \sqrt{\frac{c_2+d}{r_2}}\right)}{2\, c_1 + d\left(2 - \sqrt{\frac{c_1+d}{r_1}}\right)}. \tag{10}$$

Since $\sqrt{\frac{c_i+d}{r_i}} < 1$, it is evident that $2 - \sqrt{\frac{c_i+d}{r_i}} > 0$. If all other parameters are held constant, then one can readily derive that $\frac{q_1^*}{q_2^*}$ decreases with r_1 and increases with r_2. □

Proposition 6 suggests that a firm can increase its market share by improving its green efficiency—that is, by reducing its own green cost parameter r_i. If the rival firm follows a similar strategy, the firm's market share will decrease.

Proposition 7. *If $c_i \leq c_{-i}$ and $r_i > r_{-i}$, then the market share of Firm i will decrease with d.*

Proof. By the symmetry, we only examine the effect of the parameter d on Firm 1's market share. It is sufficient to show that if $c_1 \leq c_2$ and $r_1 > r_2$, then $\frac{\partial(q_1^*/q_2^*)}{\partial d} < 0$. We have

$$\frac{\partial\left(q_1^*/q_2^*\right)}{\partial d} = \frac{r_1^2\left(\left(\left(8c_1 - 8c_2\right)a_2 - 2d^2 - 6c_1 d - 4c_1 c_2\right)a_1 + \left(4c_1 c_2 + 6c_2 d + 2d^2\right)a_2 + d^2\left(c_1 - c_2\right)\right)}{8a_1 a_2\left(-\frac{da_1}{2} + r_1\left(c_1 + d\right)\right)^2},$$

where $a_1 = \sqrt{r_1(c_1+d)}$ and $a_2 = \sqrt{r_2(c_2+d)}$.

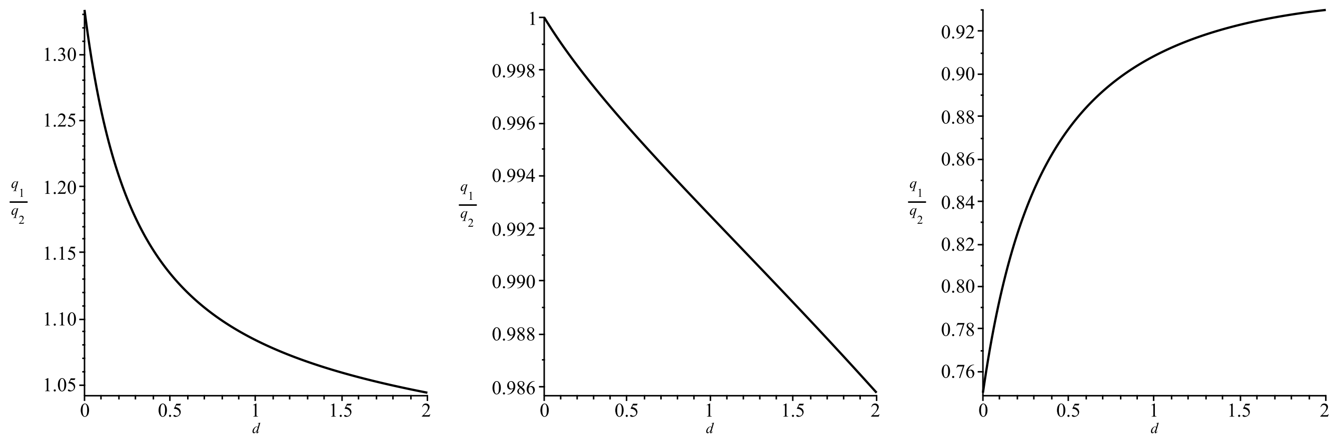
Using the CAD method, one can derive no real solutions for the following semi-algebraic system:

$$\begin{cases} a_1^2 - r_1(c_1 + d) = 0, \ a_2^2 - r_2(c_2 + d) = 0, \\ c_1 > 0, \ c_2 > 0, \ r_1 > 0, \ r_2 > 0, \ d \geq 0, \\ a_1 > 0, \ a_2 > 0, \ r_1 - (c_1 + d) > 0, \ r_2 - (c_2 + d) > 0, \\ \boldsymbol{c_2 - c_1 \geq 0, \ r_1 - r_2 > 0,} \\ \text{Num}\left(\dfrac{\partial\left(q_1^*/q_2^*\right)}{\partial d}\right) \cdot \text{Den}\left(\dfrac{\partial\left(q_1^*/q_2^*\right)}{\partial d}\right) \geq 0. \end{cases}$$

Therefore, $\frac{\partial\left(q_1^*/q_2^*\right)}{\partial d} < 0$ if $c_1 \leq c_2$ and $r_1 > r_2$, which completes the proof. □

Proposition 7 implies that when Firm i enjoys a productivity advantage (i.e., a lower basic cost c_i) but suffers from a green efficiency disadvantage (i.e., a higher r_i), its market share will decline as the carbon tax rate d increases. This result aligns well with economic intuition: stricter environmental policies amplify the competitive disadvantage of firms using less efficient green technology.

Furthermore, the special case of no carbon taxation ($d = 0$) deserves special attention, as it remains the norm in many countries today. In this case, Eq. (10) yields $\frac{q_1^*}{q_2^*} = \frac{c_2}{c_1}$, indicating that market shares are determined solely by production efficiency and unaffected by the green efficiency parameters r_i. This highlights the importance of carbon taxes in incentivizing greener technologies and reshaping competitive dynamics.



(a) $c_1 = 0.3, c_2 = 0.4, r_1 = 2.5, r_2 = 2.4$. (b) $c_1 = 0.4, c_2 = 0.4, r_1 = 2.5, r_2 = 2.4$. (c) $c_1 = 0.4, c_2 = 0.3, r_1 = 2.5, r_2 = 2.4$.

Figure 4: Effect of the parameter d on the output ratio $\frac{q_1^*}{q_2^*}$.

We conducted numerical simulations across a wide range of parameter combinations, and the results are consistent with the conclusion of Proposition 7. As shown in Fig. 4(a), the output ratio $\frac{q_1^*}{q_2^*}$ decreases as the carbon tax rate d increases when Firm 1 has a productivity advantage ($c_1 < c_2$) but a green efficiency disadvantage ($r_1 > r_2$). A similar pattern is observed in Fig. 4(b), where $c_1 = c_2$ and $r_1 > r_2$; again, $\frac{q_1^*}{q_2^*}$ declines as d increases. However, when Firm 1 has a productivity disadvantage ($c_1 > c_2$), the outcome may differ. For instance, Fig. 4(c) shows that $\frac{q_1^*}{q_2^*}$ may increase with d under the condition $r_1 > r_2$. This suggests that the relative magnitudes of the cost and green efficiency parameters jointly determine the direction of the effect.

3.4 Profit

For simplicity, we focus on the case where the carbon tax rate d is sufficiently small. Many countries, particularly developing economies, have low or near-zero carbon tax rates.

Proposition 8. *If d is small enough, then Π_i^* will decrease with c_i or r_i, but increase with c_{-i} or r_{-i}.*

Proof. By the symmetry, we only need to show that the conclusion holds for $i = 1$. Plugging Eqs. (7) and (6) into Eq. (5) in turn, we obtain

$$\lim_{d \to 0^+} \Pi_1^* = \frac{c_2 r_2 \left(\sqrt{c_1 r_1}\sqrt{c_2 r_2} + c_2 r_2\right)\sqrt{c_1 r_1}}{\left(2\sqrt{c_1 r_1}\sqrt{c_2 r_2} + c_1 r_1 + c_2 r_2\right)\left(c_2 r_2 \sqrt{c_1 r_1} + c_1 r_1 \sqrt{c_2 r_2}\right)}.$$

Some calculations yield that

$$\frac{\partial\left(\lim_{d\to 0^+} \Pi_1^*\right)}{\partial r_1} = -\frac{\left((c_1 r_1 + 3\, c_2 r_2)\sqrt{c_1 r_1} + (3\, c_1 r_1 + c_2 r_2)\sqrt{c_2 r_2}\right) c_2^2 r_2^2 c_1^2 r_1}{\sqrt{c_1 r_1}\left(2\sqrt{c_1 r_1}\sqrt{c_2 r_2} + c_1 r_1 + c_2 r_2\right)^2 \left(c_2 r_2 \sqrt{c_1 r_1} + c_1 r_1 \sqrt{c_2 r_2}\right)^2} < 0,$$

$$\frac{\partial\left(\lim_{d\to 0^+} \Pi_1^*\right)}{\partial c_1} = -\frac{\left((c_1 r_1 + 3\, c_2 r_2)\sqrt{c_1 r_1} + (3\, c_1 r_1 + c_2 r_2)\sqrt{c_2 r_2}\right) c_2^2 r_2^2 c_1 r_1^2}{\sqrt{c_1 r_1}\left(2\sqrt{c_1 r_1}\sqrt{c_2 r_2} + c_1 r_1 + c_2 r_2\right)^2 \left(c_2 r_2 \sqrt{c_1 r_1} + c_1 r_1 \sqrt{c_2 r_2}\right)^2} < 0.$$

However,

$$\frac{\partial\left(\lim_{d\to 0^+} \Pi_1^*\right)}{\partial r_2} = \frac{c_1 r_1 c_2^2 r_2 \sqrt{c_1 r_1}\left(3\, c_1 c_2 r_1 r_2 + c_1 r_1 \sqrt{c_1 r_1}\sqrt{c_2 r_2} + c_2^2 r_2^2 + 3\, c_2 r_2 \sqrt{c_1 r_1}\sqrt{c_2 r_2}\right)}{\left(2\sqrt{c_1 r_1}\sqrt{c_2 r_2} + c_1 r_1 + c_2 r_2\right)^2 \left(c_2 r_2 \sqrt{c_1 r_1} + c_1 r_1 \sqrt{c_2 r_2}\right)^2 \sqrt{c_2 r_2}} > 0,$$

$$\frac{\partial\left(\lim_{d\to 0^+} \Pi_1^*\right)}{\partial c_2} = \frac{c_1 r_1 c_2 r_2^2 \sqrt{c_1 r_1}\left(3\, c_1 c_2 r_1 r_2 + c_1 r_1 \sqrt{c_1 r_1}\sqrt{c_2 r_2} + c_2^2 r_2^2 + 3\, c_2 r_2 \sqrt{c_1 r_1}\sqrt{c_2 r_2}\right)}{\left(2\sqrt{c_1 r_1}\sqrt{c_2 r_2} + c_1 r_1 + c_2 r_2\right)^2 \left(c_2 r_2 \sqrt{c_1 r_1} + c_1 r_1 \sqrt{c_2 r_2}\right)^2 \sqrt{c_2 r_2}} > 0.$$

Therefore, the conclusion follows for $i = 1$. The proof is completed. □

Proposition 8 shows that in an economy with a sufficiently low carbon tax, a firm can enhance its profits by reducing basic production costs or improving its green efficiency. However, similar improvements by the rival firm may have a negative effect on the firm's profitability.

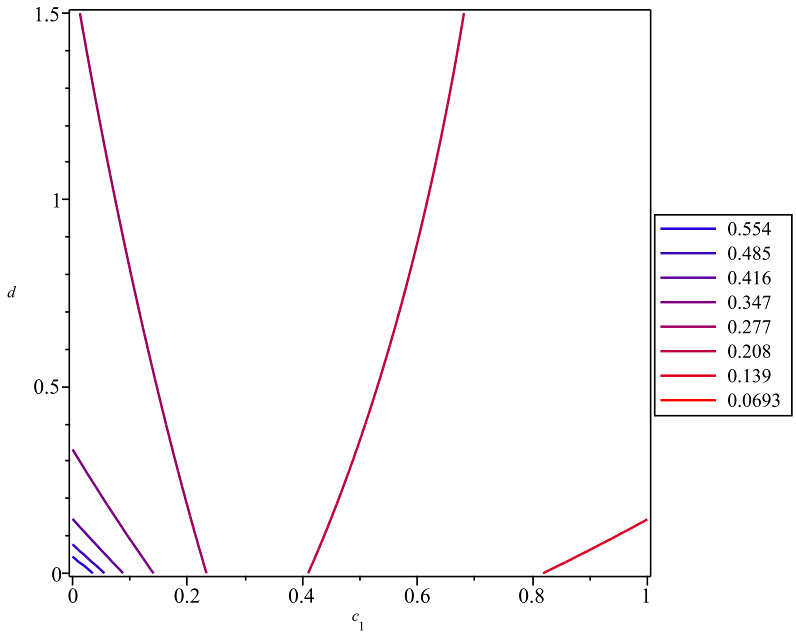


Figure 5: Contour plot of Π_1^* with respect to c_1 and d when $c_2 = 0.3$, $r_1 = 2.5$, and $r_2 = 2.4$.

Fig. 5 presents a contour plot of Π_1^* for the case where $c_2 = 0.3$, $r_1 = 2.5$, and $r_2 = 2.4$, illustrating how Firm 1's profits are affected by changes in c_1 and the carbon tax rate d, with $d > 0$. We observe that for a fixed value of d, the profit Π_1^* decreases as c_1 increases, which is consistent with the result in Proposition 8. However, the relationship between Π_1^* and d is not monotonic when c_1 is kept constant. When c_1 is low enough (e.g., $c_1 < 0.2$), Π_1^* decreases with increasing d. In contrast, when c_1 is relatively high (e.g., $c_1 > 0.4$), Π_1^* increases with d. This non-monotonic behavior highlights the interaction between production efficiency and carbon taxation in determining firm profitability.

Similarly, the contour plot in Fig. 6 depicts the effect of the parameters c_2 and d on Firm 1's equilibrium profit Π_1^*, with $c_1 = 0.4$, $r_1 = 2.5$, and $r_2 = 2.4$. It is observed that Π_1^* decreases as c_2, which is consistent with the result in Proposition 8. For a fixed value of c_2, the relationship between Π_1^* and the carbon tax rate d shows an opposite pattern to that shown in Fig. 5. When c_2 is low (e.g., $c_2 < 0.2$), Π_1^* increases with d. Conversely, when c_2 is high (e.g., $c_2 > 0.6$), Π_1^* decreases as d increases. This highlights the complex interplay between cost asymmetries and environmental regulation in shaping firm profitability.

In addition, Fig. 7 (respectively, Fig. 8) illustrates the effect of the parameters r_1 (respectively, r_2) and d on Firm 1's equilibrium profit Π_1^*. As shown in Fig. 7, for a fixed value of d, improving Firm 1's green efficiency (i.e., lowering r_1) leads to a higher profit Π_1^*. Conversely, Fig. 8 shows that for a given d, an increase in the rival's green efficiency level r_2 reduces Firm 1's profit. Furthermore, Fig. 8 shows that the impact of the carbon tax rate d on Π_1^* is not always monotonic. For instance, around $r_2 = 1.1$, Firm 1's profit initially increases and then decreases as d rises from zero. This again highlights the complex interplay

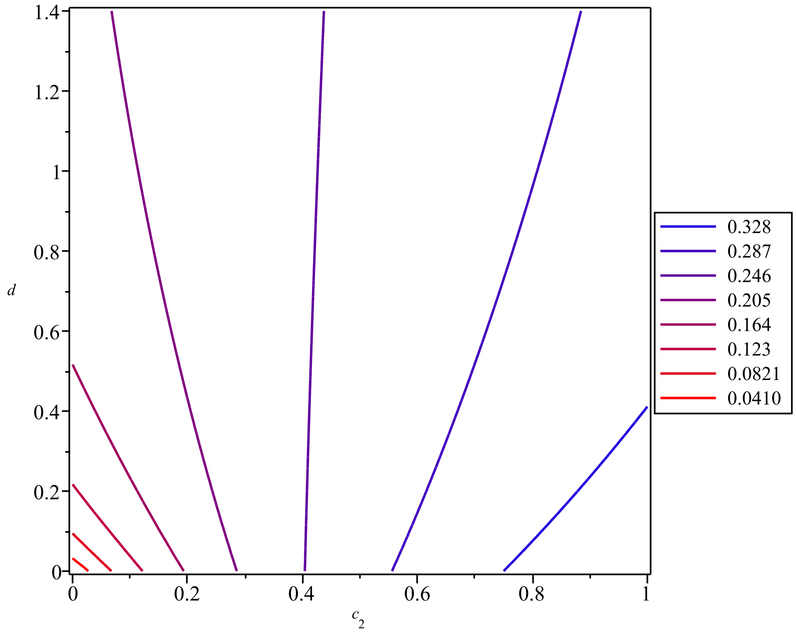


Figure 6: Contour plot of Π_1^* with respect to c_2 and d when $c_1 = 0.4$, $r_1 = 2.5$, and $r_2 = 2.4$.

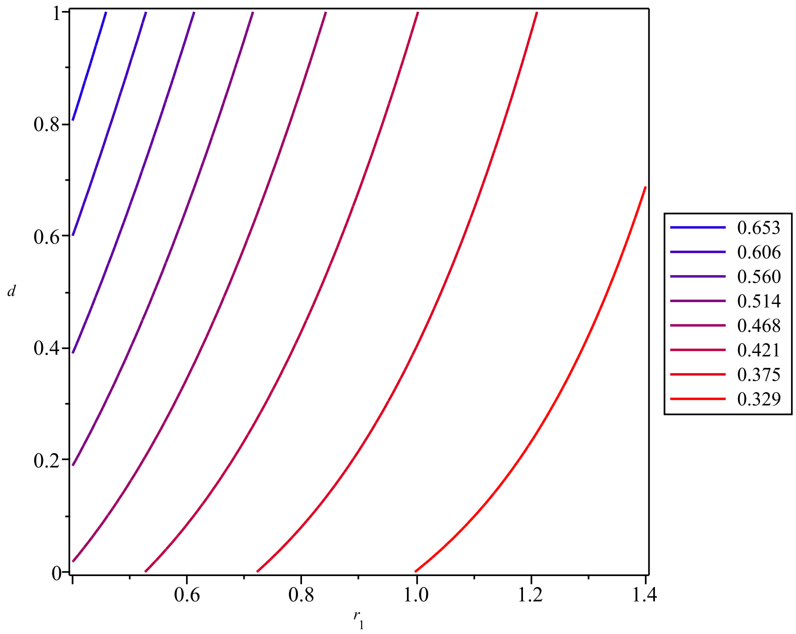


Figure 7: Contour plot of Π_1^* with respect to r_1 and d when $c_1 = 0.4$, $c_2 = 0.3$, and $r_2 = 2.4$.

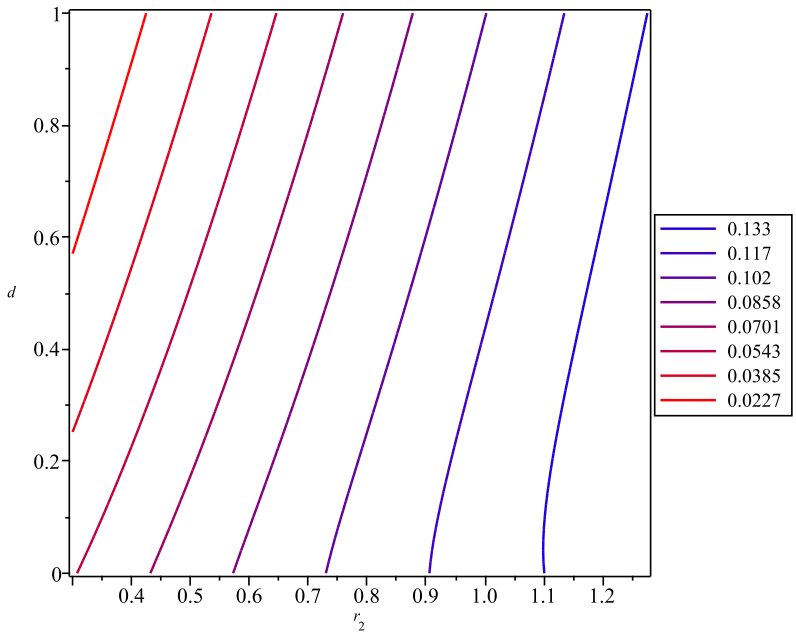


Figure 8: Contour plot of Π_1^* with respect to r_2 and d when $c_1 = 0.4$, $c_2 = 0.3$, and $r_1 = 2.5$.

between environmental regulation and inter-firm asymmetries.

An important question concerning producer surplus is how changes in the carbon tax rate d affect the total producer surplus, defined as $\Pi_1^* + \Pi_2^*$. The following proposition examines the special case where $c_1 = c_2$. It also provides some preliminary analytical results. The conclusion suggests that in markets with a high degree of environmental consciousness, an increase in the carbon tax rate may increase producer surplus, particularly when the existing carbon tax rate is low or close to zero.

Proposition 9. *If d is small enough and $c_1 = c_2$, then $\Pi_1^* + \Pi_2^*$ will not decrease with d.*

Proof. Let $c_1 = c_2 = c$. We have

$$\lim_{d \to 0^+} \frac{\partial \left(\Pi_1^* + \Pi_2^*\right)}{\partial d} = \frac{c^3 r_1 r_2 \left(r_1 - r_2\right)^2}{c\sqrt{r_1 r_2}\left(2c\sqrt{r_1 r_2} + c\left(r_1 + r_2\right)\right)^2 \left(r_1\sqrt{r_2 c} + r_2\sqrt{r_1 c}\right)} \geq 0.$$

This implies that $\Pi_1^* + \Pi_2^*$ will not decrease with d if d is sufficiently small. □

3.5 Consumer Surplus

For simplicity, we only consider the special case in which firms have identical basic production costs ($c_1 = c_2$) and identical green efficiency levels ($r_1 = r_2$).

Proposition 10. *Let $c_1 = c_2 = c$ and $r_1 = r_2 = r$. The equilibrium consumer surplus CS^* will decrease with r, c, and d.*

Proof. Plugging Eqs. (6) and (7) into Eq. (2) yields that

$$CS^* = -\ln\left(2\sqrt{r\,(c+d)} - d\right) - \ln(2) - 1.$$

Accordingly, we have

$$\frac{\partial CS^*}{\partial r} = -\frac{c+d}{\sqrt{r\,(c+d)}\left(2\sqrt{r\,(c+d)} - d\right)},$$

and

$$\frac{\partial CS^*}{\partial c} = -\frac{r}{\sqrt{r\,(c+d)}\left(2\sqrt{r\,(c+d)} - d\right)}.$$

Since $c + d < r$, it is readily derived that

$$\frac{\partial\left(2\sqrt{r\,(c+d)} - d\right)}{\partial d} = \sqrt{\frac{r}{c+d}} - 1 > 0,$$

which implies that $2\sqrt{r\,(c+d)} - d$ is monotonically increasing with d, thus

$$2\sqrt{r\,(c+d)} - d \geq 2\sqrt{r(c+0)} - 0 > 0.$$

It follows that $\frac{\partial CS^*}{\partial r} < 0$ and $\frac{\partial CS^*}{\partial c} < 0$.

Furthermore, one can see that

$$\frac{\partial CS^*}{\partial d} = -\frac{1}{2\sqrt{r\,(c+d)} - d}\left(\sqrt{\frac{r}{c+d}} - 1\right) < 0,$$

which completes the proof. □

Proposition 10 illustrates the impact of green policies, specifically raising the carbon tax rate and improving green efficiency, on the equilibrium consumer surplus CS^*, under the assumption that $c_1 = c_2$ and $r_1 = r_2$. The results show that improved green efficiency increases consumer surplus. In other words, if the regulator invests in or promotes the development of green technologies, consumer welfare can be improved. In contrast, Proposition 10 suggests that increasing the carbon tax rate reduces consumer surplus, despite potentially lowering per-unit carbon emissions. This reveals an important policy trade-off: although carbon taxes can promote environmental goals, regulatory authorities must consider the potential negative impact on consumer welfare when designing effective and balanced green policies.

To examine the case of heterogeneous basic cost levels and heterogeneous green efficiency levels, we

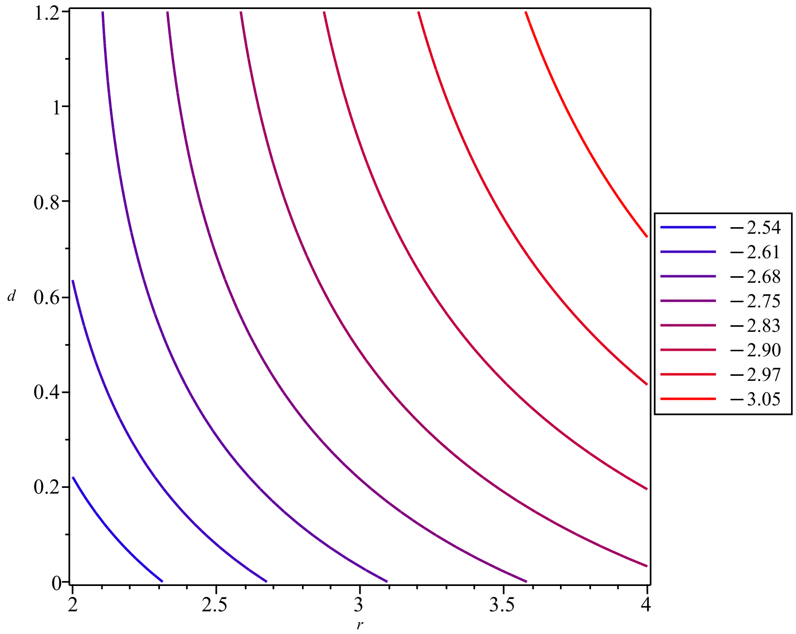


Figure 9: Contour plot of CS^* with respect to r ($r_1 = r_2 = r$) and d when $c_1 = 0.8$ and $c_2 = 0.4$.

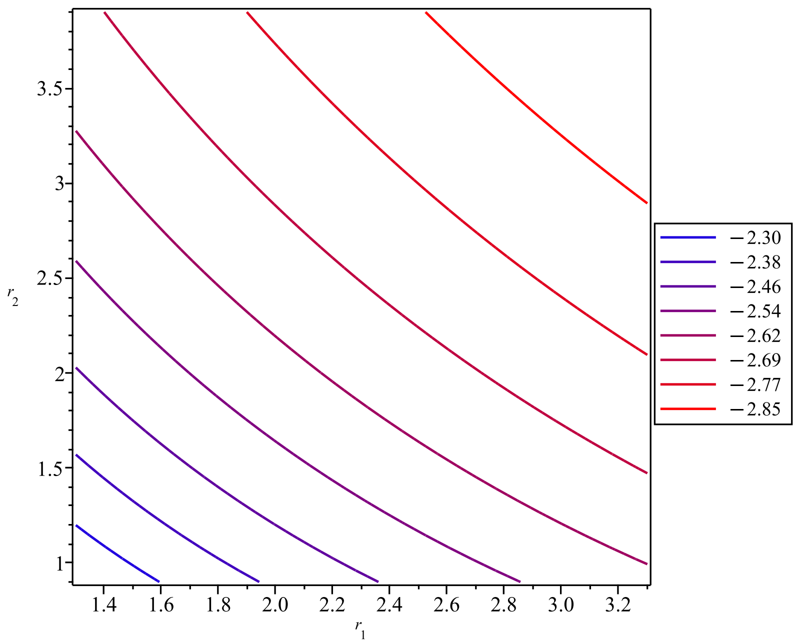


Figure 10: Contour plot of CS^* with respect to r_1 and r_2 when $c_1 = 0.8$, $c_2 = 0.4$, and $d = 0.5$.

conduct numerical simulations as shown in Figs. 9 and 10. For instance, Fig. 9 depicts the impact of the carbon tax rate and green efficiency on the equilibrium consumer surplus CS^* when the basic costs are asymmetric ($c_1 \neq c_2$). The results indicate that even under cost heterogeneity, the main conclusions of Proposition 10 continue to hold: increasing green efficiency enhances consumer surplus, whereas increasing the carbon tax rate tends to reduce it.

Furthermore, Fig. 10 illustrates the pattern of equilibrium consumer surplus CS^* under heterogeneous cost levels and green efficiency levels. It has been observed that consumer surplus decreases as either r_1 or r_2 increases, highlighting the importance of green technology efficiency in sustaining consumer welfare.

4 Dynamic Model

To examine the impact of green policies on market stability, we develop a simple dynamic model in this section. Let the current period be $t+1$ and the previous period be t. At time $t+1$, Firm i does not observe its competitor's green reputation decision $u_{-i}(t+1)$ or output decision $q_{-i}(t+1)$. However, it is assumed that Firm i can assess the marginal effect of its own green reputation on profits in the previous period (i.e., the partial derivative $\frac{\partial \Pi_i}{\partial u_i}(t)$) based on market research or business experimentation, while holding other strategic variables constant. Accordingly, Firm i updates its green reputation level at time $t+1$ with the following gradient adjustment mechanism:

$$u_i(t+1) = u_i(t) + k_i \frac{\partial \Pi_i}{\partial u_i}(t), \quad i = 1, 2,$$

where $k_i > 0$ is a parameter governing the speed of adjustment in green reputation.

Similarly, it is assumed that Firm i can observe the marginal effect of its own output on profits in the previous period, denoted by $\frac{\partial \Pi_i}{\partial q_i}(t)$. Based on this information, the firm adjusts its output level at time $t+1$ according to the following rule:

$$q_i(t+1) = q_i(t) + s_i q_i(t) \frac{\partial \Pi_i}{\partial q_i}(t), \quad i = 1, 2,$$

where $s_i > 0$ determines the output adjustment speed. Notably, the adjustment is not only governed by s_i, but also scaled by the firm's current output $q_i(t)$, reflecting size-dependent responsiveness.

For ease of presentation in the subsequent analysis, we define the following notation:

$$F_i(u_i(t),u_{-i}(t),q_i(t),q_{-i}(t)) \equiv u_i(t)+k_i\frac{\partial \Pi_i}{\partial u_i}(t)$$
$$= u_i(t)+k_i\left(\frac{q_i(t)u_{-i}(t)q_{-i}(t)}{(u_i(t)q_i+u_{-i}(t)q_{-i}(t))^2}+dq_i(t)-2\,r_iu_i(t)q_i(t)\right),\quad i=1,2,$$

and

$$G_i(u_i(t),u_{-i}(t),q_i(t),q_{-i}(t)) \equiv q_i(t)+s_iq_i(t)\frac{\partial \Pi_i}{\partial q_i}(t)$$
$$= q_i(t)+s_iq_i(t)\left(\frac{u_i(t)u_{-i}q_{-i}}{(u_i(t)q_i(t)+u_{-i}(t)q_{-i}(t))^2}-d\,(1-u_i(t))-\left(c_i+r_iu_i^2(t)\right)\right),\quad i=1,2.$$

In summary, the dynamic model presented above can be described by the following four-dimensional discrete dynamic system:

$$\begin{cases} u_1(t+1)=F_1(u_1(t),u_2(t),q_1(t),q_2(t)),\\ u_2(t+1)=F_2(u_2(t),u_1(t),q_2(t),q_1(t)),\\ q_1(t+1)=G_1(u_1(t),u_2(t),q_1(t),q_2(t)),\\ q_2(t+1)=G_2(u_2(t),u_1(t),q_2(t),q_1(t)), \end{cases}$$

where its Jacobian matrix is given by

$$J=\begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \frac{\partial F_1}{\partial q_1} & \frac{\partial F_1}{\partial q_2}\\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial q_1} & \frac{\partial F_2}{\partial q_2}\\ \frac{\partial G_1}{\partial u_1} & \frac{\partial G_1}{\partial u_2} & \frac{\partial G_1}{\partial q_1} & \frac{\partial G_1}{\partial q_2}\\ \frac{\partial G_2}{\partial u_1} & \frac{\partial G_2}{\partial u_2} & \frac{\partial G_2}{\partial q_1} & \frac{\partial G_2}{\partial q_2} \end{bmatrix}.$$

In general, analyzing the Jacobian matrix J is mathematically complex. To simplify the analysis, we limit our attention to the symmetric case by assuming $c_1=c_2=c$, $r_1=r_2=r$, $k_1=k_2=k$, and $s_1=s_2=s$. Under this symmetry, the Jacobian matrix evaluated at the Nash equilibrium can be simplified as follows:

$$J^* \equiv J(u_1^*,u_2^*,q_1^*,q_2^*)=\begin{bmatrix} A & 0 & B & 0\\ 0 & A & 0 & B\\ C & 0 & D & 0\\ 0 & C & 0 & D \end{bmatrix},$$

where

$$A = \frac{\begin{array}{c} k(d+1)(d+2)^2r^2 + (-4d^4 + (-4c-20)d^3 + (-12\sqrt{r(c+d)}\,k - 20c + 24k - 32)d^2 + ((-4\sqrt{r(c+d)}\,k + \\ 40k-32)c - 8\sqrt{r(c+d)}\,k + 8k - 16)d + 16c^2k + (8k-16)c)r + 32(d+1)(c+d)((\sqrt{r(c+d)}-1)d - c) \end{array}}{4\left(-\left(d+2\right)^2 r + 8\sqrt{r\left(c+d\right)}\,d - 8c - 8d\right)\left(d+1\right)\left(c+d\right)},$$

$$B = \left(-2\sqrt{r\left(c+d\right)} + d\right)k,$$

$$C = \frac{4\,s\sqrt{r\left(c+d\right)}\left(\sqrt{r\left(c+d\right)} - \frac{d}{2}\right)\left(-\frac{rd}{4} + \sqrt{r\left(c+d\right)}\right)}{\left(d+1\right)\left(-\left(d+2\right)^2 r + 8\sqrt{r\left(c+d\right)}\,d - 8c - 8d\right) r},$$

and

$$D = \frac{ds\sqrt{r\left(c+d\right)} - 2r\left(-\frac{1}{2} + \left(c+d\right)s\right)}{r}.$$

The local stability of the Nash equilibrium can be determined by examining the moduli of the eigenvalues of J^*. Specifically, the equilibrium is locally stable if all eigenvalues have moduli strictly less than 1. Conversely, if at least one eigenvalue has modulus greater than 1, the equilibrium is unstable. Our calculations show that the matrix J^* possesses four real eigenvalues, two of which are identical. We denote these as $e_1 = e_1'$ and $e_2 = e_2'$, respectively. Although we have derived closed-form expressions for both e_1 and e_2, they are algebraically complex, and each spans more than a page in length. Due to space limitations, we opt not to present the full expressions here. Instead, we provide a graphical representation of the eigenvalues to characterize their behavior and assess stability.

Fig. 11 depicts the two eigenvalues e_1 and e_2 of the Jacobian matrix J^*, with e_1 shown in red and e_2 in blue. It is observed that e_1 consistently lies within the interval $(-1, 1)$, indicating stability in that direction. Under certain parameter configurations, e_2 may fall below -1, indicating the dynamic system's possible instability. Fig. 11(a) shows that increasing the parameter c can result in $e_2 < -1$, whereas Fig. 11(d) demonstrates that even a small value of c can lead to instability when $d = 0$. Similarly, Fig. 11(b) reveals that increasing the carbon tax rate d may push e_2 below -1, whereas Fig. 11(e) demonstrates that a very small value of d can also lead to instability when $c = 0.01$. In other words, the model may become unstable when either c or d is too large, or when both are too small. Fig. 11(c) shows that an increase in the green cost parameter r may drive e_2 below -1, thereby destabilizing the equilibrium. However, as indicated in Fig. 11(f), even for relatively small values of r, the system may remain locally stable, as long as both c and d are not simultaneously small. These findings highlight the nontrivial role of model parameters in

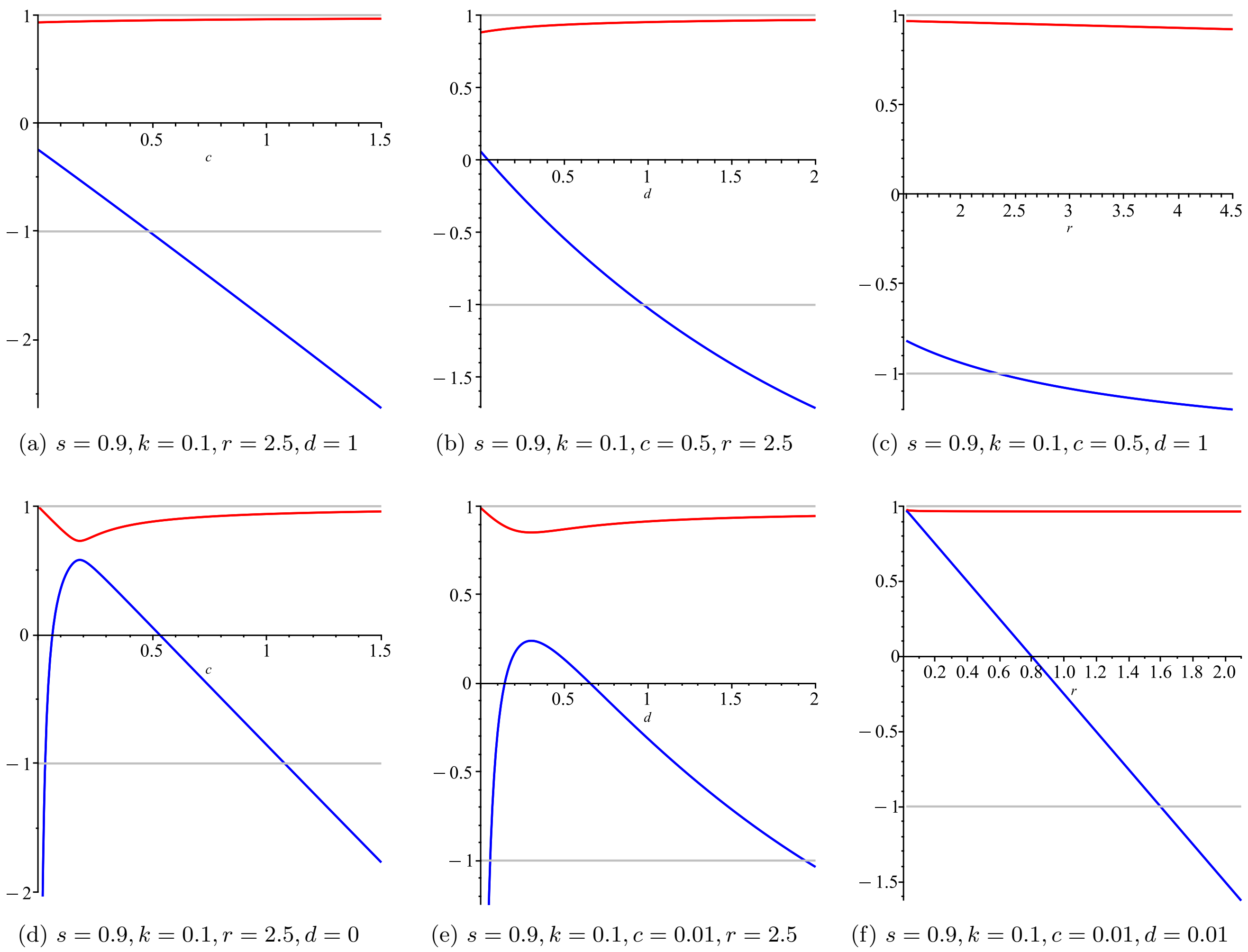
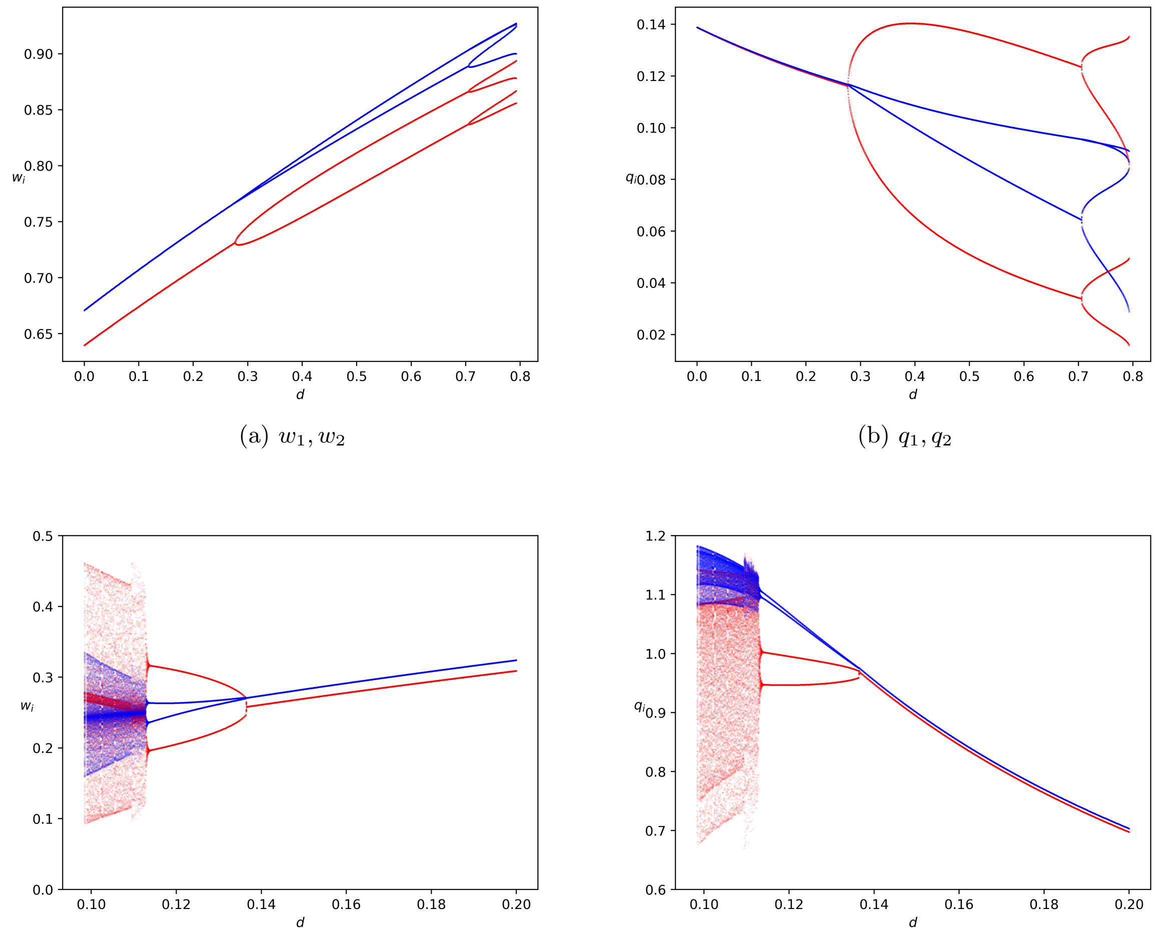


Figure 11: In the symmetric case of $c_1 = c_2 = c$, $r_1 = r_2 = r$, $k_1 = k_2 = k$, and $s_1 = s_2 = s$, the eigenvalues e_1 and e_2 of the Jacobian matrix J^*, which are colored in red and blue, respectively. (Left) The effect of c. (Center) The effect of d. (Right) The effect of r.

determining the system's local stability.

In addition, we conduct numerical simulations to explore the dynamic behavior of the model in the asymmetric case. Throughout all simulations, the initial conditions are set to $w_1(0) = w_2(0) = 0.1$ and $q_1(0) = q_2(0) = 0.1$. The simulation results confirm that the conclusions obtained from the eigenvalue analysis of the Jacobian matrix J^* in the symmetric case remain qualitatively consistent under asymmetry. Furthermore, the simulations reveal that the model's trajectories may converge to various types of attractors, including periodic cycles, quasi-periodic orbits (invariant closed curves), and chaotic trajectories. Moreover, different bifurcation phenomena, such as period-doubling and Neimark–Sacker bifurcations, are observed. These findings underscore the model's rich dynamic structure and sensitivity to parameter variation. For related studies on numerical simulations in dynamic oligopoly games, see, for example, (Yu and Yu, 2014a,b).



(a) w_1, w_2 (b) q_1, q_2

(c) w_1, w_2 (d) q_1, q_2

Figure 12: One-dimensional bifurcation diagrams with respect to d, where the other parameters are set to $r_1 = 2.2$, $r_2 = 2$, $s_1 = 0.9$, $s_2 = 0.5$, $k_1 = 0.24$, and $k_2 = 0.2$. (Top) $c_1 = c_2 = 0.9$. (Bottom) $c_1 = c_2 = 0.01$. The diagrams of Firms 1 and 2 are marked in red and blue, respectively.

Figure 12 depicts one-dimensional bifurcation diagrams of the model with respect to the carbon tax rate

d. In the top two panels, we set $c_1 = c_2 = 0.9$, whereas in the bottom two panels, we use $c_1 = c_2 = 0.01$. For visual clarity, the left-hand panels show the trajectories of w_1 and w_2, whereas the right-hand panels display the trajectories of q_1 and q_2. Firms 1 and 2 are represented in red and blue, respectively.

Fig. 12 shows that when the trajectories converge to an equilibrium, we observe that $w_1^* < w_2^*$ (both increasing in d) and $q_1^* < q_2^*$ (both decreasing in d), which is consistent with the intuition that $r_1 > r_2$. Panels (a) and (b) demonstrate that, for $c_1 = c_2 = 0.9$, increasing the carbon tax rate d can destabilize the Nash equilibrium through a period-doubling bifurcation: as d increases, the equilibrium shifts to a 2-cycle orbit and then to a 4-cycle orbit. In contrast, panels (c) and (d) show that when $c_1 = c_2 = 0.01$, the equilibrium can become unstable if the carbon tax rate is too low. Specifically, as d approaches zero, the system converges to a stable 2-cycle orbit, which then transitions into two stable invariant closed curves as d decreases.

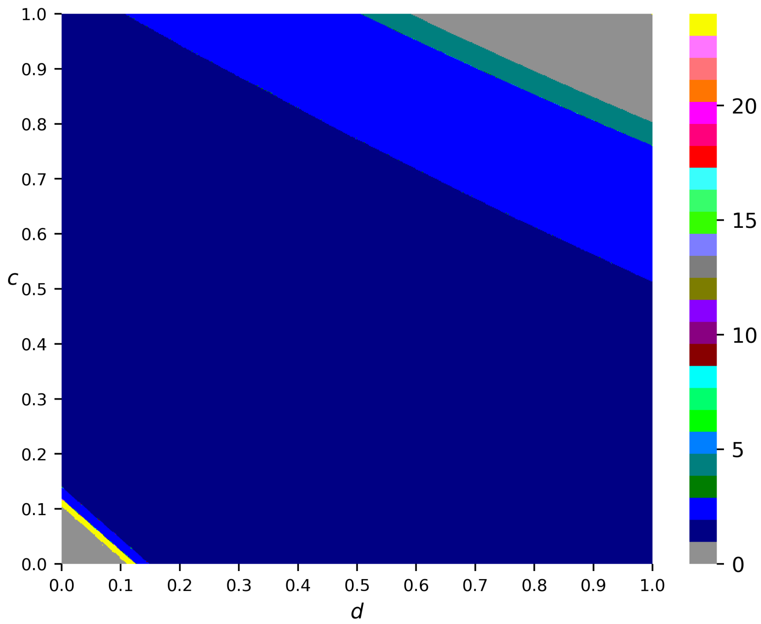
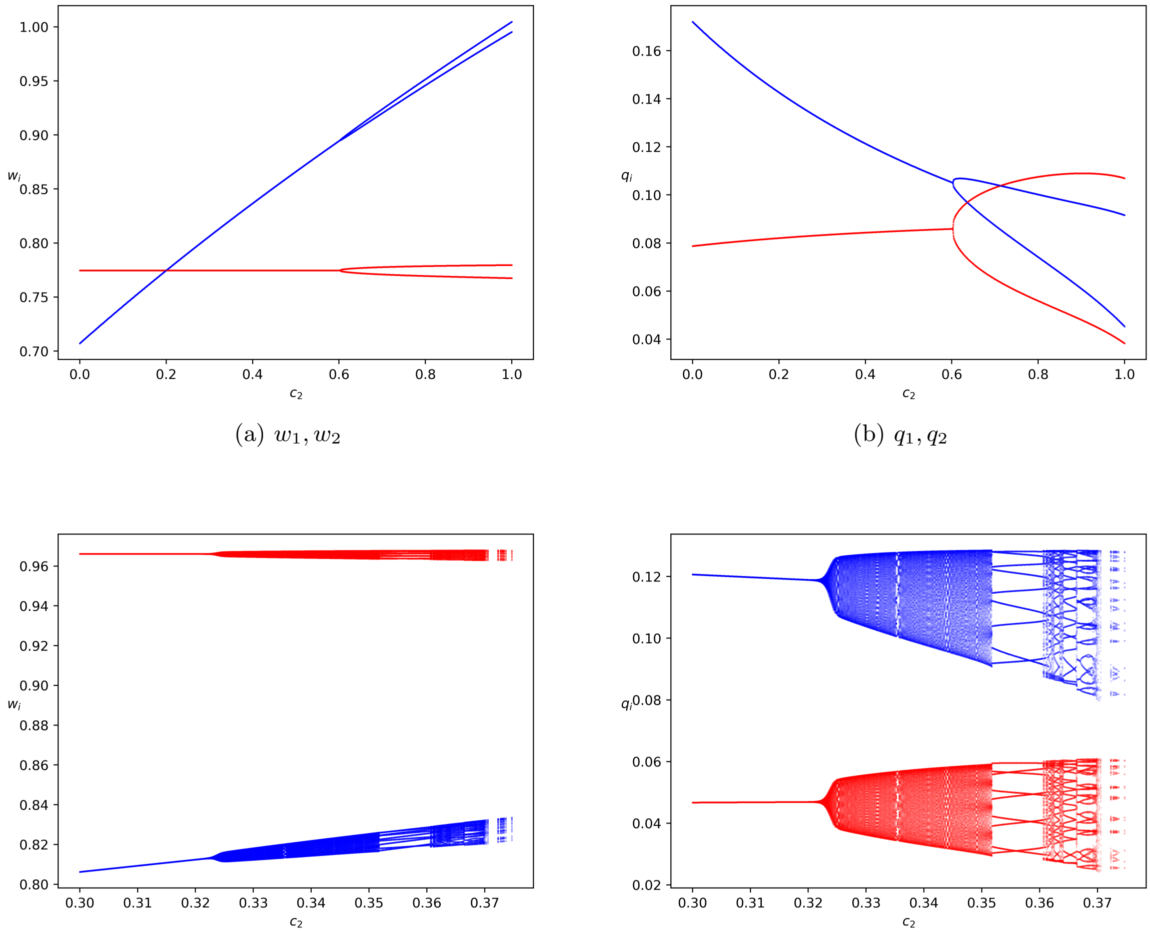


Figure 13: Two-dimensional bifurcation diagram with respect to d and $c = c_1 = c_2$, where the other parameters are set to $r_1 = 2.2$, $r_2 = 2$, $s_1 = 0.9$, $s_2 = 0.5$, $k_1 = 0.24$, and $k_2 = 0.2$.

To gain a broader understanding of the system's global dynamics, we present a two-dimensional bifurcation diagram with respect to the carbon tax rate d and the symmetric basic cost level $c = c_1 = c_2$ in Fig. 13. For a detailed explanation of two-dimensional bifurcation diagrams, see Marszalek et al. (2019). In such diagrams, different colors are used to distinguish parameter regions that correspond to different types of attractors. Dark blue points indicate convergence to a fixed point (i.e., a 1-cycle orbit), whereas light blue points represent convergence to a 2-cycle orbit. Yellow points represent parameter values for which trajectories converge to attractors with order 24 or higher, indicating the presence of high-order periodic cycles, quasi-periodic orbits, or chaotic (complex) dynamics. Grey points indicate divergence, where trajectories tend to infinity. Fig. 13 shows that the system exhibits similar bifurcation behavior to that observed in Fig. 12. In particular, when both c and d are small enough, the Nash equilibrium may experience a period-doubling bifurcation, causing instability. Similarly, when both parameters are large enough, a similar route to instability via period-doubling is observed. These findings reinforce the system's sensitivity to the joint effects of cost structure and environmental regulation.



(a) w_1, w_2 (b) q_1, q_2

(c) w_1, w_2 (d) q_1, q_2

Figure 14: One-dimensional bifurcation diagrams with respect to c_2, where the other parameters are set to $d = 1$, $r_1 = 3$, $r_2 = 2$, $s_1 = 0.9$, $s_2 = 0.5$, $k_1 = 0.12$, and $k_2 = 0.1$. (Top) $c_1 = 0.8$. (Bottom) $c_1 = 1.8$. The diagrams of Firms 1 and 2 are marked in red and blue, respectively.

In contrast to the previous bifurcation patterns driven by the carbon tax rate, Fig. 14 illustrates how variations in the parameter c_i can induce model instability through different bifurcation mechanisms. Specifically, the figure presents a series of one-dimensional bifurcation diagrams with respect to c_2, with $c_1 = 0.8$ used in the top two panels and $c_1 = 1.8$ in the bottom two panels. As before, the left and right panels display the trajectories of w_1, w_2 and q_1, q_2, respectively. The trajectories for Firms 1 and 2 are marked in red and blue, respectively.

Figs. 14(a) and (c) show that when the trajectories converge to a Nash equilibrium, the value of w_1^* remains constant, whereas an increase in c_2 results in a noticeable rise in w_2^*. When $c_1 = 0.8$, increasing c_2 destabilizes the equilibrium through a period-doubling bifurcation. The system transitions from a fixed point to a 2-cycle orbit, with the amplitude of oscillation growing as c_2 increases. In contrast, when $c_1 = 1.8$, a rise in c_2 causes the equilibrium to lose stability via a Neimark–Sacker bifurcation. The system transitions to a quasi-periodic orbit, and the amplitude of this orbit also increases progressively. Additionally, for $c_1 = 1.8$, we observe an intermittency phenomenon in which the system alternates between quasi-periodic and periodic dynamics as the value of c_2 varies, indicating rich and complex transitions in the underlying dynamic structure.

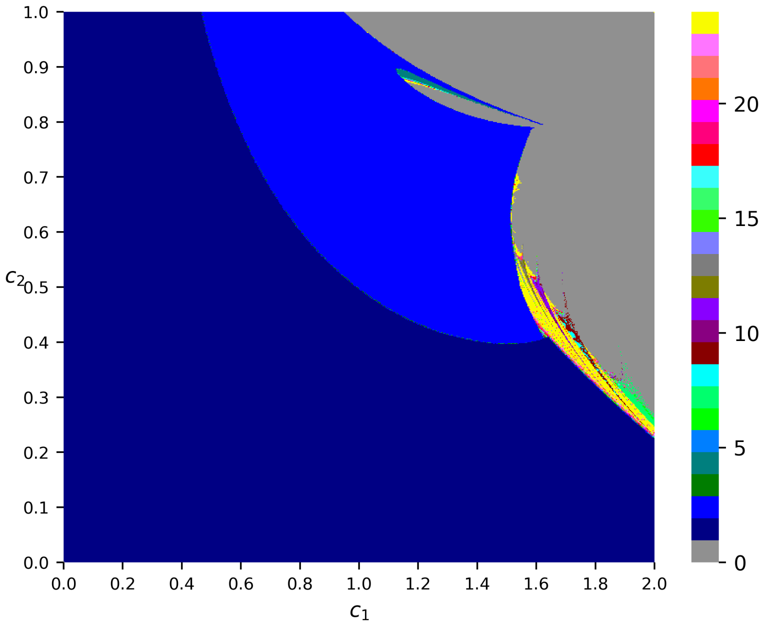
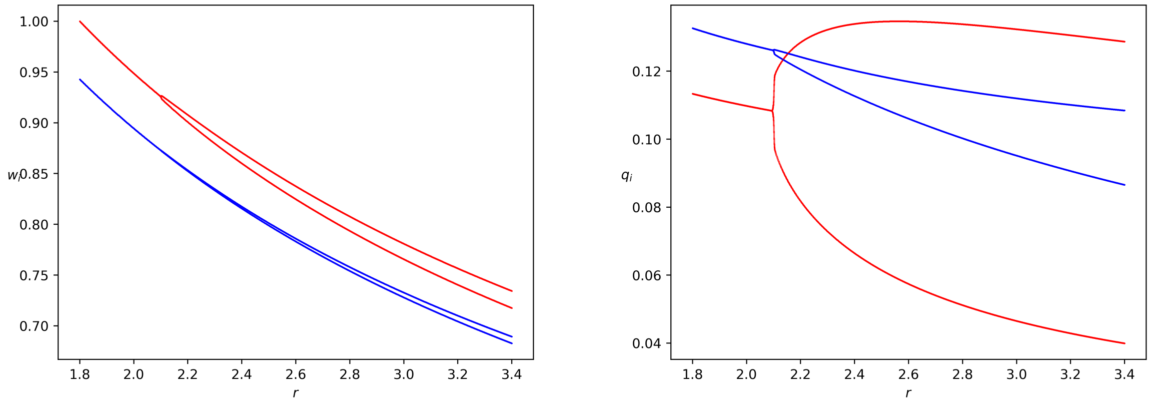


Figure 15: Two-dimensional bifurcation diagram with respect to c_1 and c_2, where the other parameters are set to $d = 1$, $r_1 = 3$, $r_2 = 2$, $s_1 = 0.9$, $s_2 = 0.5$, $k_1 = 0.12$, and $k_2 = 0.1$.

Fig. 15 reports a two-dimensional bifurcation diagram for the parameters c_1 and c_2, offering a more comprehensive view of the model's dynamic behavior. The diagram reveals that if c_1 (respectively, c_2) is less than a certain threshold, variations in c_2 (respectively, c_1) do not disrupt the Nash equilibrium. When c_1 is moderate, increasing c_2 may result in a period-doubling bifurcation. However, if c_1 is large enough, an increase in c_2 may cause a Neimark–Sacker bifurcation. Conversely, when c_2 is moderate, raising c_1 can lead to a Neimark–Sacker bifurcation. Meanwhile, for larger values of c_2, the system may undergo a period-doubling bifurcation as c_1 increases. Notably, the diagram also reveals the presence of Arnold tongues, which are regions corresponding to periodic attractors of different orders, highlighting the model's rich and intricate bifurcation structure.



(a) w_1, w_2

(b) q_1, q_2

Figure 16: One-dimensional bifurcation diagrams with respect to $r = r_1 = r_2$, where the other parameters are set to $d = 1.4$, $c_1 = 0.4$, $c_2 = 0.2$, $s_1 = 0.9$, $s_2 = 0.5$, $k_1 = 0.12$, and $k_2 = 0.1$. The diagrams of Firms 1 and 2 are marked in red and blue, respectively.

Fig. 16 depicts the effect of green efficiency on the model's dynamics. Assuming symmetry in green efficiency parameters, i.e., $r = r_1 = r_2$, the figure shows one-dimensional bifurcation diagrams with respect to r. Panels (a) and (b) show trajectories for w_1, w_2 and q_1, q_2, respectively. As shown in Fig. 16(a) (respectively, (b)), when the system converges to the Nash equilibrium, both w_1^*, w_2^* (respectively, q_1^*, q_2^*) increase as r decreases, aligning with economic intuition: improved green efficiency promotes higher equilibrium green reputations and output levels. Furthermore, as r increases, the Nash equilibrium becomes unstable through a period-doubling bifurcation, indicating the destabilizing effect of rising green costs on market dynamics.

Fig. 17 shows a two-dimensional bifurcation diagram of the model based on the carbon tax rate d and the (symmetric) green efficiency parameter $r = r_1 = r_2$. The figure shows that, under homogeneous green efficiency, increasing r can destabilize the Nash equilibrium exclusively through a period-doubling bifurcation. No Neimark–Sacker bifurcations are observed in this setting.

In the case of heterogeneous green efficiency levels ($r_1 \neq r_2$), Fig. 18 displays one-dimensional bifurcation diagrams with respect to r_1, fixing $r_2 = 1.9$. The results show that a decrease in r_1 may cause instability via a period-doubling bifurcation, whereas an increase in r_1 may trigger a Neimark–Sacker bifurcation. We observe intermittent switching between quasi-periodic and periodic dynamics when the system undergoes a Neimark–Sacker bifurcation, reflecting rich and complex behavior near the bifurcation boundary.

To gain a more comprehensive understanding, Fig. 19 shows the two-dimensional bifurcation diagram

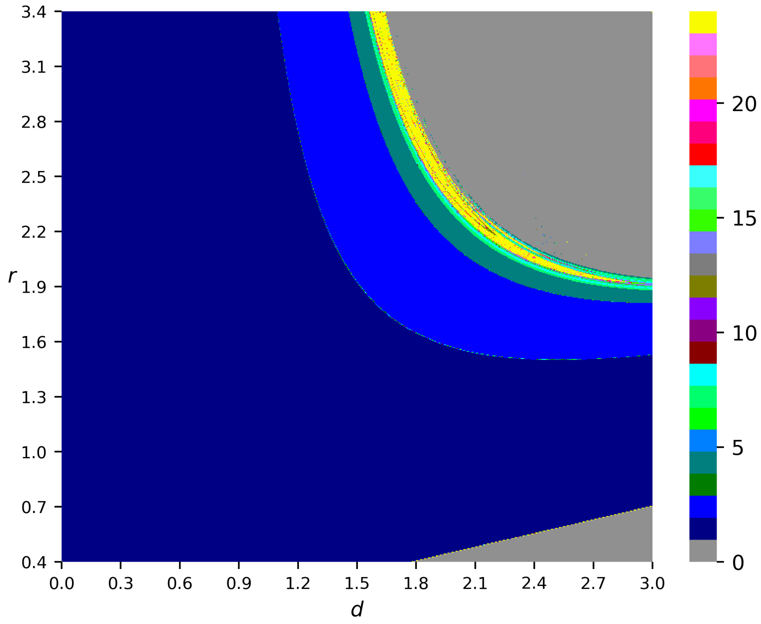
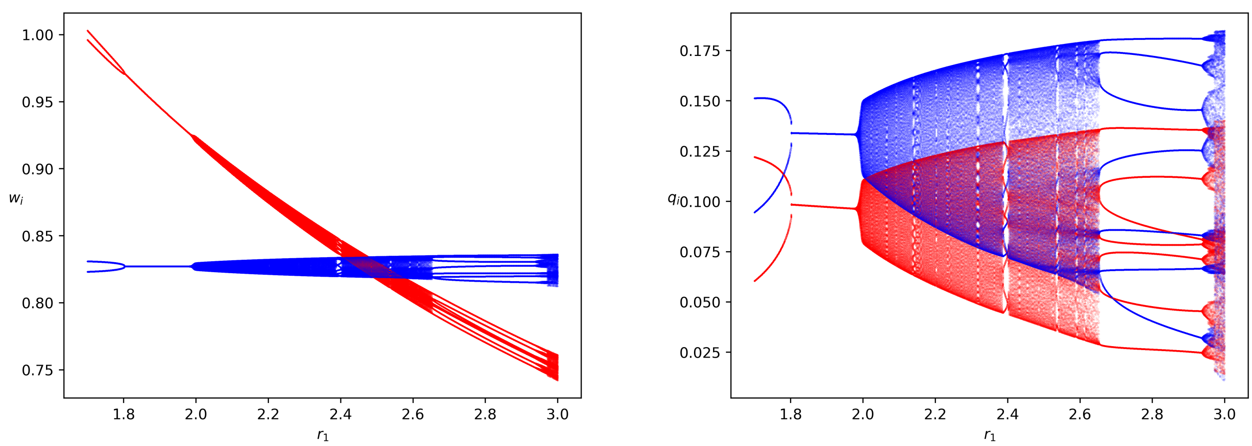


Figure 17: Two-dimensional bifurcation diagram with respect to d and $r = r_1 = r_2$, where the other parameters are set to $c_1 = 0.4$, $c_2 = 0.2$, $s_1 = 0.9$, $s_2 = 0.5$, $k_1 = 0.12$, and $k_2 = 0.1$.



(a) u_1, u_2

(b) q_1, q_2

Figure 18: One-dimensional bifurcation diagrams with respect to r_1, where the other parameters are set to $r_2 = 1.9$, $c_1 = 0.8$, $c_2 = 0.4$, $d = 0.9$, $s_1 = s_2 = 0.9$, and $k_1 = k_2 = 0.1$. The diagrams of Firms 1 and 2 are marked in red and blue, respectively.

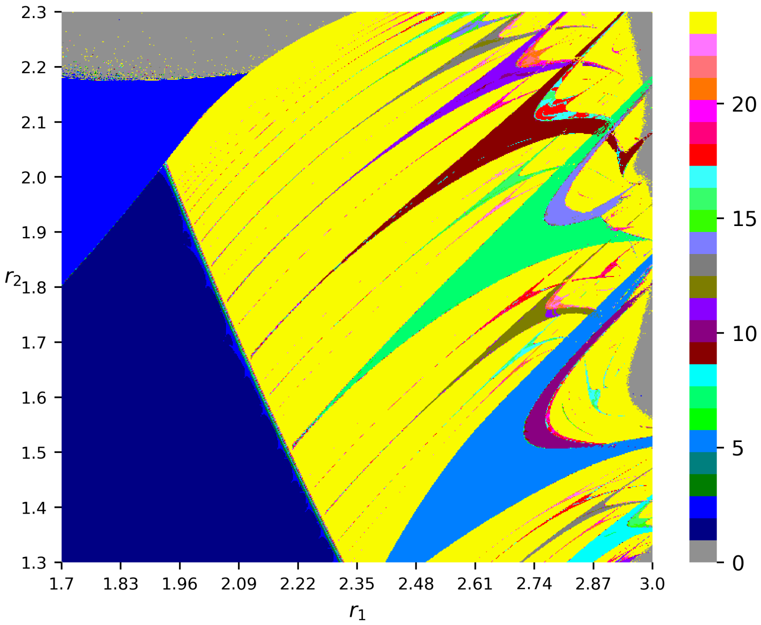


Figure 19: Two-dimensional bifurcation diagram with respect to r_1 and r_2, where the other parameters are set to $c_1 = 0.8$, $c_2 = 0.4$, $d = 0.9$, $s_1 = s_2 = 0.9$, and $k_1 = k_2 = 0.1$.

with respect to r_1 and r_2. The figure shows that the Nash equilibrium can lose stability due to period-doubling and Neimark–Sacker bifurcations in different regions of the parameter space. Furthermore, the presence of Arnold tongues of various orders demonstrates the intricate structure of bifurcation dynamics in the asymmetric green efficiency case.

5 Concluding Remarks

To examine how firms make green reputation decisions under consumer environmental awareness and carbon taxation, this paper develops a nonlinear Cournot duopoly model. Our analysis highlights how carbon tax rates and green efficiency affect firms' production, pricing, and emission strategies. Our model, which links green reputation with demand and cost, provides new insights into strategic environmental behavior under policy incentives.

The study establishes a strong foundation for future research. One particularly relevant extension is to incorporate green managerial delegation into the existing framework. For example, CSR-oriented firms often delegate to managers whose goals include environmental performance in addition to profit. Exploring such scenarios may reveal how delegation affects firms' green investment and competitive strategies. Future research could focus on related topics such as consumer environmental awareness in green managerial delegation contracts with common ownership, the theoretical foundations of green managerial delegation, and cross-ownership and environmental delegation models in mixed oligopolies. Future research could analyze firms' decisions under optimal carbon taxation, providing useful implications for designing effective environmental tax policies.

Our model focuses on Cournot competition, but it would be valuable to extend the analysis to Bertrand or Cournot–Bertrand competition in differentiated markets, or to endogenize the competition mode following Singh and Vives (1984). These comparisons may provide additional insights, but are beyond the scope of this paper and should be left for future work.

Lastly, the model can be adapted to investigate broader environmental policy instruments, such as green subsidies or green transformation initiatives, which are receiving growing attention in recent energy economics research. We hope that these directions will inspire further research into the role of strategic behavior in sustainability and industrial policy.

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