Singly-periodic pointed pseudotriangulations have an expansive motion

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Abstract. We show that if a singly-periodic bar-joint framework in the Euclidean plane is derived from a pointed pseudotriangulation on the flexible flat cylinder, then it has a one-parameter deformation, which is expansive, i.e. the motion does not decrease the distance between any pair of joints. For the proof, we consider singly-periodic versions of Maxwell-Cremona liftings and adapt the proof of C. Borcea and I. Streinu for the doubly-periodic version of this statement.

1 Introduction

An expansive motion in a bar-joint framework is one that does not decrease the distance between any pair of joints. Applications of expansive motions include reconfigurable mechanisms and collision-free motion planning in robotics. In material science, they help design meta-materials with auxetic behaviour.

C. Borcea and I. Streinu previously showed that every doubly-periodic pointed pseudotriangulation in the plane has a one-parameter deformation (under a flexible lattice representation), which is expansive [2]. J. Cruickshank and B. Schulze [3] recently extended the concept of a pointed pseudotriangulation to other orbifolds beyond the flat torus, specifically the flat cylinder and flat cones. They noted that the Maxwell count of pointed pseudotriangulations on the flexible flat cylinder (whose liftings are singly-periodic frameworks in the plane) indicates exactly one non-trivial degree of freedom, and conjectured that such frameworks have an expansive motion. We verify this conjecture here.

We adapt Borcea and Streinu's proof for the doubly-periodic case to the singly-periodic case. This involves using singly-periodic versions of Maxwell-Cremona liftings. We show that a self-stress is translation-invariant (constant on each edge orbit) if and only if it is induced by a singly-periodic lifting. The proof differs from the doubly-periodic case due to the existence of two unbounded faces, which alters the arguments to show that the stress-weighted sum over a face-path from a bounded face to its translations is zero. Moreover, this result requires only translation-invariance of the self-stress, not periodicity (i.e. fulfillment of the extra condition arising from the flexibility of the cylinder), distinguishing it further from the doubly-periodic scenario.

We then demonstrate that pointed pseudotriangulations on the flexible flat cylinder have no non-trivial periodic liftings and thus no non-zero periodic self-stresses, implying a one-dimensional periodicity-preserving deformation space.

Finally, we adapt another argument from Borcea and Streinu to show that these deformations are expansive.

Throughout this paper, we will use standard notation from rigidity theory, see e.g. [5]. A d-dimensional *(bar-joint) framework* is a pair (G,p) consisting of a simple graph $G=(V,E)$ and an injective map $p:V\to\mathbb{R}^d$. When it's clear from the context we will use $p(v)$ and v interchangeably. An *infinitesimal motion* of a framework (G,p) is a map $u:V\to\mathbb{R}^d$ such that, for all $v_iv_j\in E$,

$$(p(v_i)-p(v_j))\cdot(u(v_i)-u(v_j))=0. \tag{1}$$

The infinitesimal motion u is *trivial* if it corresponds to a rigid body motion, and *non-trivial* otherwise. The matrix corresponding to (1) is called the *rigidity matrix*, denoted $R(G,p)$, and (G,p) is *infinitesimally rigid* if G is complete on at most $d+1$ vertices or $R(G,p)$ has maximum rank (equal to $d|V|-\binom{d+1}{2}$).

A *self-stress* of (G,p) is a map $s:E\to\mathbb{R}$ such that, for every vertex $v\in V$, the vector equation $\sum_{vv_i\in E}s_e(p(v_i)-p(v))=0$ holds. If a framework has no non-zero self-stress, then it is called *independent*. An independent and infinitesimally rigid framework is called *minimally rigid*.

2 Periodic frameworks

An infinite graph $\tilde{G}$ is called *singly-periodic* if there is a free action from $\mathbb{Z}$ to $\mathrm{Aut}(\tilde{G})$. A framework $(\tilde{G},\tilde{p})$ in $\mathbb{R}^2$ is *singly-periodic* if $\tilde{G}$ is a singly-periodic graph and there exists $\lambda\in\mathbb{R}^2\backslash\{(0,0)\}$ such that, for every vertex orbit representative v and $r\in\mathbb{Z}$, we have $\tilde{p}(v,r)=\tilde{p}(v,0)+r\lambda$. W.l.o.g. we assume in this paper that $\lambda=(1,0)$, so the periodicity is along the horizontal axis. Singly-periodic frameworks can alternatively be thought of as frameworks on the flat cylinder. It is important to note that, in our model, the flat cylinder is allowed to deform under a motion.

It is a common approach to represent frameworks with symmetry group Γ in terms of their group-labelled quotient graphs, also known as *Γ-gain graphs*. See [4, 1] for details. We use this here for $\Gamma=\mathbb{Z}$.

Let (G,m) be a $\mathbb{Z}$-gain graph with vertex set V, edge set E, and gain function $m:E\to\mathbb{Z}$. For brevity, for an edge $e=(v_i,v_j;m(e))$, we write $p(e)=p(v_i)-p(v_j)-(m(e),0)$, which is the vector along the bar derived by e.

Let (G,m) be a $\mathbb{Z}$-gain graph with vertex set V and edge set E. Let $(\tilde{G},\tilde{p})$ be a singly-periodic framework on the plane, derived from (G,m). Let $(s_{(e,k)})_{(e,k)\in E\times\mathbb{Z}}$ be a self-stress of $(\tilde{G},\tilde{p})$. The self-stress s is *translation-invariant* if, for every $e\in E$ and $k\in\mathbb{Z}$, we have $s_{(e,k)}=s_{(e,0)}$. When a self-stress is translation-invariant, $s_{(e,k)}$ can be abbreviated to s_e.

The self-stress s is *periodic* if it is translation-invariant and it satisfies

$$\sum_{e=(v_i,v_j;m(e))\in E} s_e m(e)p(e)=0. \tag{2}$$

Equivalently, a self-stress is periodic if it corresponds to an element of the cokernel of the flexible-lattice periodic rigidity matrix $R_{\text{fper}}(\tilde{G}, \tilde{p})$ [1, 2]. The row for an edge $e = (v_i, v_j; m(e))$ in this matrix has the form:

$$e \begin{bmatrix} 0 \dots 0 \overset{v_i}{(p(e))^T} 0 \dots 0 \overset{v_j}{-(p(e))^T} 0 \dots 0 \; m(e)[p(e)]_x \end{bmatrix},$$

where $[p(e)]_x$ denotes the x-coordinate of $p(e)$. Removing the final column corresponding to the flexibility of the cylinder yields the fixed-lattice periodic rigidity matrix whose co-kernel corresponds to the translation-invariant self-stresses.

Note that not all translation-invariant self-stresses are periodic. Consider, for example, a $\mathbb{Z}$-gain graph on K_1^1 (a single vertex with a loop), where the loop has gain 1. Any derived singly-periodic framework from this gain graph has a translation-invariant self-stress defined by assigning weight 1 to every bar. However, this is not periodic, as it does not satisfy Equation (2).

We say that a continuous motion of a (possibly infinite) framework (G, p) is *expansive* if it does not decrease the distance between any pair of joints. An infinitesimal motion u of (G, p) is *expansive* if it corresponds to an expansive continuous motion, that is, for all pairs $v_i, v_j \in V$, we have $(u(v_i) - u(v_j)) \cdot (p(v_i) - p(v_j)) \geq 0$. Similarly, u is *contractive* if, for all pairs $v_i, v_j \in V$, we have $(u(v_i) - u(v_j)) \cdot (p(v_i) - p(v_j)) \leq 0$. Clearly, an expansive infinitesimal motion exists if and only if a contractive one exists (via multiplication by -1).

3 Pointed pseudotriangulations

3.1 Preliminaries

Let (G, p) be a non-crossing framework in $\mathbb{R}^2$, i.e. $p : G \to \mathbb{R}^2$ is a planar embedding in $\mathbb{R}^2$, such that no edge-crossings are allowed. Note that p is uniquely determined by the configuration of V, so we may abuse notation. A *face* of (G, p) is a connected component of $\mathbb{R}^2 \backslash p(G)$.

Let (G, p) be a (possibly infinite) non-crossing framework in $\mathbb{R}^2$ and let U_1 and U_2 be a pair of faces of this framework. A *face-path* $U_1 \to U_2$ consists of a sequence of directed edges that are crossed in a path through the plane from U_1 to U_2, with the standard orientation that edges are directed from left to right, relative to the direction of the path [2, Subsection 2.3].

In this paper, we will use techniques involving Maxwell-Cremona liftings. Let (G, p) be a non-crossing framework in the plane $\mathbb{R}^2$. A *lifting* of (G, p) is a continuous function $H : \mathbb{R}^2 \to \mathbb{R}$ where the restriction to any face of (G, p) is an affine function. A lifting is *trivial* if it is affine on $\mathbb{R}^2$. Let $(\tilde{G}, \tilde{p})$ be a singly-periodic framework in the plane $\mathbb{R}^2$ with periodicity vector $\lambda \in \mathbb{R}^2$. A lifting H of $(\tilde{G}, \tilde{p})$ is *periodic* if, for all $q \in \mathbb{R}^2$, $H(q + \lambda) = H(q)$. To discuss liftings further, we use the notation from [2]. Let H be a lifting of a non-crossing framework (G, p) in $\mathbb{R}^2$. Let $\mathcal{F}$ denote the set of faces of (G, p). Then the restriction of H to some face $U \in \mathcal{F}$ takes the form $H(q) = \nu_U \cdot q + C_U$, where $C_U \in \mathbb{R}$ and $\nu_U \in \mathbb{R}^2$ is the projection of the normal of the lifted face to the reference plane. With this notation, a lifting H can be defined in terms of the faces of the framework: $H = (\nu_U, C_U)_{U \in \mathcal{F}}$ [2].

Proposition 1. *[2, Proposition 1] Let (G, p) be a non-crossing framework in $\mathbb{R}^2$ with lifting $H = (\nu_U, C_U)_{U \in \mathcal{F}}$. Then there is a unique self-stress s_e of (G, p) such that, for every edge $e = (v_i, v_j)$ with arbitrary direction putting a face U_i on the right and U_j on the left,*

$$\nu_{U_j} - \nu_{U_i} = s_e(p(v_j) - p(v_i))^{\perp}.$$

The self-stress given by Prop. 1 is known as the self-stress *induced* by the lifting.

3.2 Pointed pseudotriangulations on the flat cylinder

Let F be a face of a non-crossing framework. We say that a vertex v in the boundary of F is *F-convex* (resp. *F-concave*) if the internal angle of v with respect to F is strictly smaller (resp. greater) than π. The face F is called a *pseudotriangle* if it has exactly 3 F-convex vertices. A joint $v \in V$ is said to be *pointed* if it is concave with respect to one of its incident faces. A framework in $\mathbb{R}^2$ is *pointed* if all of its joints are pointed.

A *pseudotriangulation* in a space X is a non-crossing connected framework in which every bounded face is a pseudotriangle, and each unbounded face has no convex vertices if X is the plane, and exactly one convex vertex if X is the flat cylinder [3]. A *pointed pseudotriangulation* in X is a pointed framework in X that is also a pseudotriangulation in X.

We aim to prove the following main result.

Theorem 1. *Every pointed pseudotriangulation on the flexible flat cylinder has a 1-parameter continuous deformation, and this deformation must be expansive.*

To begin, note that a pointed pseudotriangulation counts to have a one-dimensional space of non-trivial infinitesimal motions on the flexible flat cylinder, as pointed out in [3]. So if we can show that the pseudotriangulation has no non-zero periodic self-stress, then the configuration is regular (i.e. the flexible-lattice periodic rigidity matrix has maximum rank) and hence the single non-trivial infinitesimal motion must correspond to a continuous motion.

In the following lemma, we show that translation-invariant self-stresses on the flat cylinder also satisfy an additional condition regarding stress-weighted sums over face-paths. This condition is needed to make the connection between self-stresses and singly-periodic liftings.

Lemma 1. *Let $(\tilde{G}, \tilde{p})$ be a non-crossing framework on the flat cylinder with a translation-invariant self-stress $s = (s_e)_{e \in \tilde{E}}$. Let U be a bounded face of $(\tilde{G}, \tilde{p})$, let $t \in \mathbb{Z}$ and let $U + t$ be the face formed by a t-translation of U. Then, for every face-path $U \to U + t$,*

$$\sum_{e \in U \to U+t} s_e p(e) = 0. \tag{3}$$

Proof. By considering the self-stress constraints around each joint, it can be seen that the sum from Equation (3) will be the same for every choice of face-path from U to $U + t$. It is therefore only necessary to prove that Equation (3) holds for one such face-path. Let U_0 be one of the two unbounded faces of $(\tilde{G}, \tilde{p})$ and consider any face-path of the form $U \to U_0$. Reversing and translating this gives a corresponding face-path $U_0 \to U + t$. For $t = 2$, this is illustrated in Figure 1. Translating the face-path does not affect this sum and reversing the face-path merely changes the sign on each summand. Hence $\sum_{e \in U \to U_0} s_e p(e) = -\sum_{e \in U_0 \to U + \lambda} s_e p(e)$. It follows that

$$\sum_{e \in U \to U+t} s_e p(e) = \sum_{e \in U \to U_0} s_e p(e) + \sum_{e \in U_0 \to U+t} s_e p(e) = \sum_{e \in U \to U_0} (s_e p(e) - s_e p(e)) = 0.$$

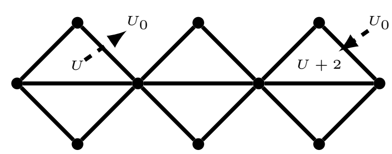

Fig. 1: An illustration of an example of an application of the proof of Lemma 1. The dashed line shows the face-path from U to $U + 2$ via U_0.

Proposition 2. *Every translation-invariant self-stress of a non-crossing framework on the flat cylinder is induced by a periodic lifting.*

Proof. Considering Lemma 1, the proof of [2, Proposition 5] can be applied directly to reach this result.

It is worth noting that Proposition 2 does not require the translation-invariant self-stress to be periodic. Note that not every periodic lifting induces a periodic self-stress, as demonstrated by the following example.

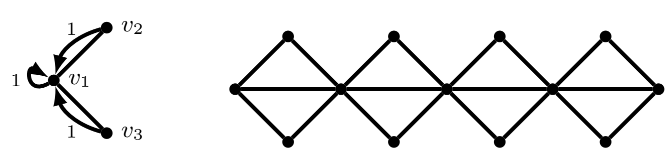

Fig. 2: The $\mathbb{Z}$-gain graph for Example 1 (left) and its derived graph (right).

Example 1. Let (G, m) be the $\mathbb{Z}$-gain graph seen in Figure 2 (left). Define a periodic configuration $\tilde{p}$ of the derived graph $\tilde{G}$ by setting $\tilde{p}(v_1, 0) = (0, 0)$, $\tilde{p}(v_2, 0) = (0.5, 0.5)$ and $\tilde{p}(v_3, 0) = (0.5, -0.5)$.

Define a periodic lifting $H : \mathbb{R}^2 \to \mathbb{R}$ of $(\tilde{G}, \tilde{p})$ as follows. For all $k \in \mathbb{Z}$, set $H(p(v_1, k)) = 0$, $H(p(v_2, k)) = 1$ and $H(p(v_3, k)) = 1$. Intuitively, this lifting can be thought of as "folding" the plane along the straight line through the edges derived by the loop at v_1. The self-stress induced by this lifting is translation-invariant but not periodic, since it is only non-zero on edges derived from the loop at v_1. So, it fails Equation (2) for periodicity. However, this self-stress still corresponds to a dependence in the fixed-lattice periodic rigidity matrix.

Note that such examples do not occur in the flat torus, as the analogous property to Equation (3) is only satisfied by periodic self-stresses. In particular, Equation (2) is equivalent to an analogue of Equation (3) in the flat torus, as seen in [2, Main Theorem].

A non-flat edge $e = vw$ is said to be a *mountain* (resp. *valley*) of a lifting $H : \mathbb{R}^2 \to \mathbb{R}$ if H is concave (resp. convex) on an open neighbourhood of $p(e)\backslash\{p(v), p(w)\}$. Equivalently, an edge is a mountain of the lifting if it has a positive stress value in the induced self-stress and a valley otherwise [2]. Mountains and valleys are important in the study of extrema of H. It's trivial to see that if H has a minimum (resp. maximum) at a single vertex v, then v is adjacent to three valley (resp. mountain) edges vw_1, vw_2, vw_3, and v is a non-pointed vertex on the framework induced by $\{v, w_1, w_2, w_3\}$. We say that the edges vw_1, vw_2, vw_3 form a *non-pointed triple of incident valley (resp. mountain) edges.*

Lemma 2. *Let (G, p) be an independent finite pointed non-crossing framework in $\mathbb{R}^2$ in which every bounded face is a pseudotriangle (the unbounded face need not be a pseudotriangle). Form a new graph G' from G by adding two new edges: e_a and e_b. Suppose that (G', p) has exactly a 1-dimensional space of self-stresses, spanned by $(s_e)_{e\in E'}$. Then the self-stress values s_{e_a} and s_{e_b} have opposite signs.*

Proof. Assume first that (G', p) is non-crossing. If the unbounded face of (G, p) has no convex vertices, then (G, p) is a pointed pseudotriangulation. By [6, Corollary 2.4 and Theorem 3.6], (G, p) is minimally rigid, so adding e_a and e_b would admit a 2-dimensional space of self-stresses. Hence the unbounded face F has a convex vertex, say v.

The non-zero self-stress on (G', p) is induced by a non-trivial lifting H [2, Proposition 2], so it is sufficient to show that one of e_a or e_b is a valley and that the other is a mountain for H. Since the restriction of H to any face of (G', p) is an affine function, each extremum must be at a single non-pointed vertex, or a single edge, or at all vertices on a particular face or union of adjacent faces. If the minimum (resp. maximum) is on a single vertex or a single edge, then it must have a non-pointed triple of incident valley (resp. mountain) edges. If the extremum is at all vertices on a union of adjacent bounded faces, we can remove edges, so that the extremum is on only one face that still has convex vertices. If the minimum (resp. maximum) is on a bounded face, then each of its convex vertices must have a non-pointed triple of incident valley (resp. mountain) edges. Since the original framework (G, p) is pointed, one of these valley (resp. mountain) edges must be e_a or e_b.

Consider now the case where the entire unbounded face F' is an extremum of H, w.l.o.g. say a minimum. Since H is non-trivial, it has a maximum on a vertex, edge or finite face that is not on the boundary of F'. In all cases, this gives a non-pointed triple of incident mountain edges, so either e_a or e_b is a mountain, say e_a. Assume that each edge on the boundary of F' has a non-zero stress value, as otherwise they can be removed to give a smaller framework that satisfies the same properties. Let v be the convex vertex of the unbounded face F in (G, p). If, after the edge additions, v is still on the boundary of F', then it has a non-pointed triple of incident valley edges, so e_b is one of these valley edges. If v is not incident to F', then e_b is on the boundary of F'. This is illustrated in Figure 3.

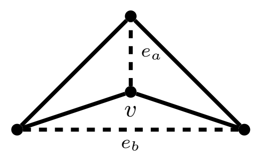

Fig. 3: An example where e_b is on the exterior face in the proof of Lemma 2.

Since F' is a minimum for H, all edges around it are valleys and therefore e_b is a valley.

It remains to remove the assumption that (G', p) is non-crossing. For this, we follow the approach described in [2]. For the case where a pair of bars cross at a position that is not incident to a joint, a new framework can be formed by inserting a new joint at the position of the crossing and replacing the crossing bars with new bars incident to the new joint. This new framework inherits a self-stress from the previous one, which satisfies all of the required properties to use the same method as before to show that either s_{e_a} and s_{e_b} have opposite signs or one of them is 0. A similar process can be performed in the case where a bar crosses the position of a joint. This completes the proof.

Proof (Proof of Theorem 1). Let $(\tilde{G}, \tilde{p})$ be a pointed pseudotriangulation on the flat cylinder and let (G, m) be its underlying $\mathbb{Z}$-gain graph. We first show that $(\tilde{G}, \tilde{p})$ has only one non-trivial infinitesimal motion (up to scale). For this, we just need to show it has no non-zero periodic self-stress. To see this, note that any non-zero periodic self-stress is induced by a non-trivial periodic lifting. Such a lifting must have a maximum and a minimum. If both extrema are either at single vertices, at single edges or at faces with convex vertices, then, by the same argument used in the proof of Lemma 2, there must be a non-pointed joint. If an extremum is formed by multiple coplanar faces that form an unbounded face when put together, then it may be that this combined face has no convex vertices. However, this can only occur for one extremum, so the other must be formed of either a single vertex or a bounded flat face. Hence, there must be a non-pointed joint. Since the configuration is pointed, this is not possible.

Now, let u be a non-trivial periodic infinitesimal motion of $(\tilde{G}, \tilde{p})$. We aim to show that u is either expansive or contractive. For two non-adjacent vertex pairs in $(\tilde{G}, \tilde{p})$, add an edge to the gain graph that corresponds to a hypothetical edge between the pair. Call these e_a and e_b. Form the gain graph (G', m) from (G, m) by adding e_a and e_b. Suppose that (G', m) derives the singly-periodic framework $(\tilde{G}', \tilde{p})$. We aim to show that the entries of $R_{\text{fper}}(\tilde{G}', \tilde{p})u$ that correspond to e_a and e_b either have the same sign or one of them is 0.

If adding e_a alone admits a non-zero self-stress, then it is clear that the entry of $R_{\text{fper}}(\tilde{G}', \tilde{p})u$ that corresponds to e_a is 0, so the desired conclusion is reached. The same applies in the case where adding e_b alone admits a non-zero self-stress. Hence, it remains to consider the case where neither new edge addition alone admits a non-zero self-stress. Since we added two edges to the gain graph, the derived framework $(\tilde{G}', \tilde{p})$ admits a non-trivial periodic self-stress $s = (s_e)_{e \in E'}$. Suppose that neither e_a nor e_b admits a self-stress alone, so $s_{e_a} \neq 0$ and $s_{e_b} \neq 0$. Note that the self-stress s must be orthogonal to $R_{\text{fper}}(\tilde{G}', \tilde{p})u$. This means that the entries of $R_{\text{fper}}(\tilde{G}', \tilde{p})u$ that correspond to e_a and e_b have the same sign if and only if the stress values s_{e_a} and s_{e_b} have opposite signs. Hence, the objective is now to show that s_{e_a} and s_{e_b} have opposite signs.

By the same method used for Lemma 2, it can be assumed that $(\tilde{G}', \tilde{p}')$ is non-crossing. If the periodic self-stress s is non-zero on only a balanced subgraph of G', then applying Lemma 2 with this balanced subgraph shows that the signs of s_{e_a} and s_{e_b} are different. It remains to consider the case where s is non-zero on all edges in some unbalanced subgraph of G'. While ignoring any edges that have stress value 0, it can be seen that every face of $(\tilde{G}', \tilde{p})$ has a convex vertex. By Proposition 2, the periodic self-stress s is induced by a non-trivial periodic lifting. Clearly, this lifting must have both a minimum and a maximum. The same idea from the proof of Lemma 2 and [2, Theorem 1] can then be used to show that one of e_a or e_b is a valley and that the other is a mountain. Hence, s_{e_a} and s_{e_b} have opposite signs, as desired.

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