Topological chaos in oligopoly competition

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Abstract

This letter demonstrates that oligopoly competition can exhibit topological chaos by applying Mitra's sufficient condition. To overcome the computational challenges arising from the lack of a simple closed-form expression for the modal point, we develop a resultant-based method. Our results show that, for a sufficiently large number of firms, there must exist parameter values that guarantee topological chaos.

Keywords: topological chaos; oligopoly game; Mitra's sufficient condition; resultant

JEL Classification: C60, C62, C63, D43, L13

1 Introduction

For unimodal maps, Mitra (2001) extended the result of Li and Yorke (1975), which revealed that period three implies chaos for one-dimensional continuous maps. Mitra's criterion (also known as Mitra's sufficient condition) is more verifiable, as it requires only the first three iterates of the modal point and avoids the need to eliminate existential quantifiers. This criterion was later generalized by Deng and Khan (2018) through the consideration of knife-edge parameter values.

This letter identifies a new instance of topological chaos in an oligopoly model, rather than the widely studied growth models (Day, 1982; Matsuyama, 1999). A key challenge in applying Mitra's criterion arises when the modal point lacks a simple closed-form expression, as is the case in our model. To address this, we develop a resultant-based method that bypasses direct substitution issues, extending the applicability of Mitra's sufficient condition to more complicated maps.

Unlike numerical approaches, our symbolic computation method yields exact results and provides analytical insights. Our main finding is that in oligopoly competition with the local monopolistic approximation (LMA)

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mechanism (Bischi et al., 2007), under decreasing returns to scale, a sufficiently large number of firms inevitably leads to parameter values for which the model exhibits topological chaos.

2 Model

We consider an oligopoly with N ($N \geq 2$) firms producing homogeneous goods under an isoelastic demand function. The inverse demand function is given by $P(Q) = \sigma/Q$, where $Q = \sum_{i=1}^{N} q_i$ is total market supply, and $\sigma > 0$ represents market size. Each firm i has a cost function $C_i(q_i) = f_i + c_i q_i + d q_i^2/2$, where $f_i \geq 0$, $c_i > 0$, and $d \geq 0$. The marginal cost is $C_i'(q_i) = c_i + d q_i$, with $d > 0$ corresponding to increasing marginal costs (decreasing returns to scale) and $d = 0$ to constant marginal costs (constant returns to scale).

Firms adjust output using the LMA, a boundedly rational adjustment mechanism traced back to the early idea of Silvestre (1977), and later formalized and applied in dynamic oligopoly models by Tuinstra (2004); Bischi et al. (2007), and others. Throughout, we denote the current period as $t+1$, and the previous period as t. Firms do not know the exact demand function[1] but can observe the previous period's price slope, i.e., $P'(Q(t)) = -\sigma/Q^2(t)$. Using this, firm i estimates the price at $t+1$ as

$$P_i^e(t+1) = \frac{\sigma}{Q(t)} - \frac{\sigma}{Q^2(t)}(Q_i^e(t+1) - Q(t)),$$

where $Q_i^e(t+1)$ is its expected market supply. Firms use naive expectations, assuming $Q_i^e(t+1) = q_i(t+1) + \sum_{j \neq i} q_j(t)$. Then, the expected profit function is

$$\Pi_i^e(t+1) = q_i(t+1)\left(\frac{\sigma}{Q(t)} - \frac{\sigma}{Q^2(t)}(q_i(t+1) - q_i(t))\right) - \left(f_i + c_i q_i(t+1) + \frac{d}{2} q_i^2(t+1)\right).$$

Maximizing $\Pi_i^e(t+1)$ gives the first-order condition

$$\frac{\partial \Pi_i^e(t+1)}{\partial q_i(t+1)} = \frac{\sigma}{Q(t)} - \frac{\sigma}{Q^2(t)}(q_i(t+1) - q_i(t)) - \frac{\sigma q_i(t+1)}{Q^2(t)} - c_i - d q_i(t+1) = 0.$$

The second-order condition always holds. Solving this first-order condition for $q_i(t+1)$ yields the best response function

$$q_i(t+1) = \frac{-c_i Q^2(t) + \sigma q_i(t) + \sigma Q(t)}{d Q^2(t) + 2\sigma}. \tag{1}$$

3 Existence of Chaos

Summing map (1) over $i = 1, ..., N$, we have the aggregate map

$$Q(t+1) = \frac{(N+1)Q(t) - sQ^2(t)}{vQ^2(t) + 2} \equiv h(Q(t)), \tag{2}$$

[1] A reasonable assumption under nonlinear demand.

where $s = \sum_{i=1}^{N} c_i/\sigma$ and $v = d/\sigma$.

A bounded trajectory is called to be *aperiodic* if it is neither periodic nor converging to a periodic orbit. Evidently, chaotic orbits are aperiodic.

Proposition 1. *If a trajectory of map (2) is aperiodic, then the corresponding trajectory of map (1) is aperiodic.*

When $v = 0$ ($d = 0$), setting $x = sQ/(N+1)$ transforms map (2) into $x(t+1) = \mu x(t)(1 - x(t))$ with $\mu = \frac{(N+1)^2}{2s}$. Hence, map (2) is topologically conjugate to the logistic map. From its well-known properties, map (2) exhibits chaos when $\mu > 3.57$.[2] This raises the natural question of whether chaos persists when $v > 0$ ($d > 0$), which is the central focus of our analysis. Throughout the rest of the paper, we assume $v > 0$.

The following lemma, introduced by Mitra (2001) and extended by Deng and Khan (2018), provides a sufficient condition for the existence of topological chaos.

Lemma 1. *Let $\mathcal{F}$ be the set of continuous maps from an interval $[a, b]$ to itself, with a generic element f, satisfying:*

1. *there exists $m \in (a, b)$ such that f is strictly increasing on $[a, m]$ and strictly decreasing on $[m, b]$;*
2. *$f(a) \geq a$, $f(b) < b$, and $f(x) > x$ for all $x \in (a, m]$.*

Provided that $f \in \mathcal{F}$ and z is the unique interior fixed point of f, if $f^2(m) < m$ and $f^3(m) \leq z$, then f^2 is turbulent and f exhibits topological chaos.[3]

To determine whether the map h defined in (2) belongs to the class $\mathcal{F}$, we present the following proposition.

Proposition 2. *If $N \geq 4$ and $A \equiv (s^2 - 8v)N - 7s^2 - 8v \leq 0$, then $h \in \mathcal{F}$, mapping $[0, (N+1)/s]$ to itself.*

From the proof of Proposition 2, $N \geq 4$ ensures $m < z$, while $A \leq 0$ ensures h maps $[0, (N+1)/s]$ to itself. Also, $A \leq 0$ is equivalent to $(r-8)N - 7r - 8 \leq 0$ with $r \equiv s^2/v$. Clearly, $r \leq 8$, i.e., $(\sum_{i=1}^{N} c_i/N)^2/(\sigma d N^2) \leq 8$, implies $A \leq 0$. In other words, if the average initial marginal cost $\sum_{i=1}^{N} c_i/N$ is fixed, then large enough σ, d, or N guarantees h mapping $[0, (N+1)/s]$ to itself.

Since the modal point m lacks a simple closed-form expression, directly verifying Mitra's sufficient condition by computing $h^2(m)$ and $h^3(m)$ is rather difficult. To overcome this challenge, we develop a novel method[4] based on the Sylvester resultant and derive the following theorem.

Theorem 1. *Let $N \geq 4$ and $A \leq 0$. Mitra's sufficient condition, $h^2(m) < m$ and $h^3(m) \leq z$, holds if and only if $R \equiv \alpha_6 r^3 + \alpha_4 r^2 + \alpha_2 r + \alpha_0 \leq 0$, where*

$$\alpha_6 = -256N^3 + 256N^2 + 5376N + 21248,$$

$$\alpha_4 = N^8 - 8N^7 - 36N^6 + 8N^5 + 486N^4 + 5064N^3 + 31132N^2 + 89784N + 96033,$$

$$\alpha_2 = -16N^8 - 40N^7 + 424N^6 + 3704N^5 + 17256N^4 + 61448N^3 + 152056N^2 + 222120N + 183048,$$

$$\alpha_0 = 64N^8 + 704N^7 + 4048N^6 + 15264N^5 + 43696N^4 + 94336N^3 + 154416N^2 + 176928N + 150544.$$

[2]By Proposition 1, the corresponding trajectory of map (1) is aperiodic.
[3]For the formal definitions of turbulence and topological chaos, the reader is referred to Mitra (2001).
[4]See Appendix for details.

Theorem 1 provides a condition for directly identifying chaos, allowing evaluation with given parameter values.

Corollary 1. *If $r \le -\frac{5}{2} + \frac{\sqrt{377}}{2} \approx 7.208$, Mitra's sufficient condition, $h^2(m) < m$ and $h^3(m) \le z$, cannot hold.*

Proof. We rewrite $R = \beta_8 N^8 + \beta_7 N^7 + \beta_6 N^6 + \beta_5 N^5 + \beta_4 N^4 + \beta_3 N^3 + \beta_2 N^2 + \beta_1 N + \beta_0$, where

$$\beta_8 = (r-8)^2, \quad \beta_7 = -8r^2 - 40r + 704, \quad \beta_6 = -36r^2 + 424r + 4048,$$
$$\beta_5 = 8r^2 + 3704r + 15264, \quad \beta_4 = 486r^2 + 17256r + 43696,$$
$$\beta_3 = -256r^3 + 5064r^2 + 61448r + 94336, \quad \beta_2 = 256r^3 + 31132r^2 + 152056r + 154416,$$
$$\beta_1 = 5376r^3 + 89784r^2 + 222120r + 176928, \quad \beta_0 = 21248r^3 + 96033r^2 + 183048r + 150544.$$

It is evident that $\beta_0, ..., \beta_5 > 0$ for $r \in (0, +\infty)$. Furthermore, $\beta_7 \ge 0$ if $r \le -\frac{5}{2} + \frac{\sqrt{377}}{2} \approx 7.208$ and $\beta_6 \ge 0$ if $r \le \frac{53}{9} + \frac{\sqrt{11917}}{9} \approx 18.018$. Thus, if $r \le -\frac{5}{2} + \frac{\sqrt{377}}{2}$, then $\beta_6, \beta_7 \ge 0$, implying $R > 0$. The conclusion follows from Theorem 1. □

Corollary 2. *For $N = 4, 5, 6$, Mitra's sufficient condition cannot hold.*

Proof. All of $\alpha_0, \alpha_2, \alpha_4, \alpha_6$ in Theorem 1 are positive when $N = 4, 5, 6$. □

When $N = 7$, we have $R = -16384r^3 + 229376r^2 + 58654720r + 1827733504$ and $A = -64v$. It follows that $R \le 0$ if $r \ge 78.113$ and $A \le 0$ always holds, meaning that chaos emerges if $r \ge 78.113$ when $N = 7$. In addition, when $N = 8$, we have $R = -50432r^3 - 1784863r^2 - 5966264r + 4350193936$ and $A = (r - 72)v$, implying that $R \le 0$ if $r \ge 34.334$ and $A \le 0$ if $r \le 72$. That is, chaos occurs if $34.334 \le r \le 72$ when $N = 8$. A similar deduction shows that chaos emerges if $23.721 \le r \le 40$ when $N = 9$. In short, for $N = 7, 8, 9$, there exist values of r leading to topological chaos. This naturally raises the question of whether chaos persists for large N, which we address in the following proposition.

Proposition 3. *If N is sufficiently large, then values of r must exist such that the oligopoly game exhibits topological chaos.*

Proof. When $r = 8$, $A \le 0$ always holds, and

$$R = -128N^7 + 5136N^6 + 45408N^5 + 212848N^4 + 778944N^3 + 3494384N^2 + 10452576N + 18640016.$$

Thus, $R \le 0$ for sufficiently large N, proving the result via Theorem 1. □

4 Concluding Remarks

This study demonstrates topological chaos in an oligopoly game where the aggregate map is unimodal, akin to those in growth models (Deng et al., 2022; Gardini et al., 2008; Mukherji, 2005). The complexity of the modal point's analytical expression poses a challenge, particularly in comparing its third iterate with the fixed point. We addressed this challenge by employing resultant calculations, offering a methodological advance.

Our method also exhibits broad applicability to models with variable price elasticity. Consider the inverse demand function $P(Q) = \sigma/Q^{1/e}$ with elasticity $e > 0$, leading to the aggregate map $Q(t+1) = \frac{(N+1/e)Q(t)-sQ^{1/e+1}(t)}{vQ^{1/e+1}(t)+2/e}$. Setting $e = k/l$ with integers $k, l > 0$ and substituting $Q^{1/k} = H$ transforms the system into $H^k(t+1) = \frac{(N+l/k)H^k(t)-sH^{k+l}(t)}{vH^{k+l}(t)+2l/k}$. Despite the added complexity from elasticity heterogeneity, our polynomial-resultant method remains tractable and yields results analogous to Theorem 1 and Proposition 3, highlighting the robustness and versatility of our approach.

Appendix

Proof of Proposition 1

If the trajectory of map (1), say $(q_1(t), q_2(t), \ldots, q_N(t))$, converges to a p-periodic orbit, then for any $i = 1, 2, \ldots, N$ and $k = 0, 1, \ldots, p-1$, $\lim_{n\to+\infty} q_i(np+k)$ must exist. Denote $\lim_{n\to+\infty} q_i(np+k) = q^*_{i,k}$. For $k = 0, 1, \ldots, p-1$,

$$\lim_{n\to+\infty} Q(np+k) = \sum_{i=1}^{N} \lim_{n\to+\infty} q_i(np+k) = \sum_{i=1}^{N} q^*_{i,k},$$

which means that the trajectory of map (2) converges to the periodic orbit $\left\{\sum_{i=1}^N q^*_{i,0}, \sum_{i=1}^N q^*_{i,1}, \ldots, \sum_{i=1}^N q^*_{i,p-1}\right\}$. Therefore, if $(q_1(t), q_2(t), \ldots, q_N(t))$ is not aperiodic, then $Q(t)$ is not aperiodic, which completes the proof.

Proof of Proposition 2

First, we know

$$h'(x) = \frac{-v\,(N+1)\,x^2 - 4sx + 2(N+1)}{(vx^2+2)^2},$$

which equals zero at $m \equiv \frac{-2s+\sqrt{2N^2v+4Nv+4s^2+2v}}{v(N+1)}$. It is easy to verify that $m \in (0, (N+1)/s)$, $h'(x) > 0$ for $x \in [0, m)$ and $h'(x) < 0$ for $x \in (m, (N+1)/s]$. Hence, h is strictly increasing on $[0, m]$ and strictly decreasing on $[m, (N+1)/s]$. Some tedious calculations show that $h(m) \leq (N+1)/s$ if and only if $A \leq 0$.

Then, we have $h(0) = 0$ and $h((N+1)/s) = 0$. It is readily derived that $h(x) > x$ for $x \in (0, z)$, where z is the unique interior fixed point, satisfying $h(z) = z$ or simply $\frac{N-1}{z} = vz + s$. The rest is to prove $m < z$. One can see that $m < z$ if and only if $\frac{N-1}{m} > vm + s$, i.e., $vm^2 + sm - (N-1) < 0$. Since $h'(m) = 0$, i.e., $-v\,(N+1)\,m^2 - 4sm + 2(N+1) = 0$, it follows that

$$vm^2 + sm - (N-1) = sm + vm^2 - (N-1) + \frac{1}{4}(-v\,(N+1)\,m^2 - 4sm + 2(N+1)) = \frac{3-N}{4}(vm^2+2).$$

Therefore, $m < z$ if $N \geq 4$, which completes the proof.

Proof of Theorem 1

To prove this theorem, we need the following notation.

Definition 1. Let $A = \sum_{i=0}^{m} a_i\, x^i$ and $B = \sum_{j=0}^{l} b_j\, x^j$ be two univariate polynomials in x with coefficients a_i, b_j, and $a_m, b_l \neq 0$. The determinant

$$\left|\begin{array}{cccccc} a_m & a_{m-1} & \cdots & a_0 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & a_m & a_{m-1} & \cdots & a_0 \\ b_l & b_{l-1} & \cdots & b_0 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & b_l & b_{l-1} & \cdots & b_0 \end{array}\right| \begin{array}{l} \left.\vphantom{\begin{array}{c} a\\ \ddots \\ a\end{array}}\right\} l \\ \left.\vphantom{\begin{array}{c} b\\ \ddots \\ b\end{array}}\right\} m \end{array}$$

is called the *Sylvester resultant* (or simply *resultant*) of A and B with respect to x, and denoted by $\mathrm{res}(A, B, x)$.

The following lemma reveals the main property of the resultant, which can be found in Mishra (1993).

Lemma 2. *Let A and B be two univariate polynomials in x. There exist two polynomials F and G in x such that*

$$FA + GB = \mathrm{res}(A, B, x).$$

Furthermore, A and B have common zeros in the field of complex numbers if and only if $\mathrm{res}(A, B, x) = 0$.

Noting that the resultant is defined for polynomials, we need to transform our problem into that involving polynomial equations. Assume $m_1 = h(m_0)$, $m_2 = h(m_1) = h^2(m_0)$, $m_3 = h(m_2) = h^3(m_0)$, where $h'(m_0) = 0$. We use h_N and h_D to represent the numerator and denominator of h. Accordingly, five polynomial equations are obtained: $g_m \equiv f'(m_0) = 0$, $g_z \equiv h_D(z)z - h_N(z) = 0$, and $g_i \equiv h_D(m_{i-1})m_i - h_N(m_{i-1}) = 0$, $i = 1, 2, 3$.

To establish the parameter conditions for $m_0 - m_2 > 0$ and $z - m_3 \geq 0$, we first compute the boundary conditions under which the sign-switching of $m_0 - m_2$ and $z - m_3$ may occur, i.e., the parameter conditions corresponding to $m_0 - m_2 = 0$ and $z - m_3 = 0$.

Regarding $m_0 - m_2$, we compute the following resultants iteratively:

$$\tau_1 \equiv \mathrm{res}(m_0 - m_2, g_2, m_2) = (-vm_0 - s)m_1^2 + (N+1)m_1 - 2m_0,$$

$$\tau_2 \equiv \mathrm{res}(\tau_1, g_1, m_1) = -m_0\left((-vm_0 - s)m_1^2 + N - 1\right)\left(\left(-s^2 - 2v\right)m_0^2 + s\,(N+3)\,m_0 - 2N - 6\right),$$

$$\tau_3 \equiv \mathrm{res}(\tau_2, g_m, m_0) = -4v\,(N+1)\,(N-3)^2\left(N^2 v + 2Nv + v + 2s^2\right)^2 R_1,$$

where

$$R_1 = \left(s^2 - 2v\right)N^2 + \left(-2s^2 - 20d\right)N - 19s^2 - 50d.$$

Denote the above process obtaining τ_3 as $\text{res}(m_0 - m_3, [g_2, g_1, g_m])$. Among the factors of τ_3, only R_1 may be zero. That is, the sign of $m_0 - m_2$ may change at $R_1 = 0$.

Similarly, regarding $z - m_3$, we have

$$\begin{aligned}\text{res}(z - m_3, [g_3, g_2, g_1, g_m, g_z]) = {} & 4096AR_2R_3s^4(N-3)^8(N^2v + 2Nv + v + 2s^2)^{12} \\ & \left(N^4s + 8N^3v + 38N^2v + \left(16s^2 + 88v\right)N + 80s^2 + 121v\right),\end{aligned} \tag{3}$$

where $R_3 = v^3R$ and

$$\begin{aligned}R_2 = {} & N^5v^2 + \left(2s^2v + 5v^2\right)N^4 + \left(s^4 - 8s^2v + 42v^2\right)N^3 + \left(-5s^4 - 4s^2v + 106v^2\right)N^2 \\ & + \left(-13s^4 + 280s^2v + 357v^2\right)N + 121s^4 + 274s^2v + 289v^2.\end{aligned}$$

Among the factors of (3), only R_2 and R_3 may be zero. Thus, the sign of $z - m_3$ may change at $R_2 = 0$ and $R_3 = 0$.

From Lemma 2, one can see that if we continuously vary values of the parameters N, s, v but do not cross the algebraic varieties $R_1 = 0$, $R_2 = 0$, and $R_3 = 0$, then the signs of $m_0 - m_2$ and $z - m_3$ will not change. Therefore, the set $\{(N, s, v) \mid A \leq 0, N \geq 4, N \text{ is a real number}\}$ is divided by $R_1 = 0$, $R_2 = 0$, $R_3 = 0$ into regions. In a given region, the signs of $m_0 - m_2$ and $z - m_3$ are invariant.

Accordingly, we just need to select one sample point from each region and determine the signs of $m_0 - m_2$ and $z - m_3$ on it. The cylindrical algebraic decomposition (CAD) method (Collins and Hong, 1991) can be used to generate at least one sample point from each region. Table 1 lists these sample points and reports the signs of $m_0 - m_2$, $z - m_3$ and R_1, R_2, R_3 at these sample points. From this Table, it is readily seen that $m_0 - m_2 > 0$ and $z - m_3 \geq 0$ are simultaneously satisfied if and only if $R_3 \leq 0$, i.e., $R \leq 0$ since $R_3 = v^3R$. The proof is completed.

Table 1: Sample points generated by the CAD method

sample points	$m - m_2$	$z - m_3$	R_1	R_2	R_3
$N = 303/64, s = 1/2, v = 1/2$	−	−	−	+	+
$N = 757/128, s = 1/2, v = 71/32768$	−	−	+	+	+
$N = 757/128, s = 1/2, v = 1033/2048$	−	−	−	+	+
$N = 855/128, s = 1/2, v = 109/131072$	+	+	+	+	−
$N = 855/128, s = 1/2, v = 423/65536$	+	−	+	+	+
$N = 855/128, s = 1/2, v = 4189/8192$	−	−	−	+	+
$N = 1245/128, s = 1/2, v = 21/2048$	+	+	+	+	−
$N = 1245/128, s = 1/2, v = 367/16384$	+	−	+	+	+
$N = 1245/128, s = 1/2, v = 1091/2048$	−	−	−	+	+
$N = 1659/128, s = 1/2, v = 131/8192$	+	+	+	+	−
$N = 1659/128, s = 1/2, v = 271/8192$	+	−	+	+	+
$N = 1659/128, s = 1/2, v = 561/1024$	−	−	−	+	+
$N = 4, s = 1/2, v = 1/2$	−	−	−	+	+

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