Lagrangian Klein bottles in $S^2 \times S^2$

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Abstract

We use Luttinger surgery to show that there are no Lagrangian Klein bottles in $S^2 \times S^2$ in the $\mathbb{Z}_2$-homology class of an S^2-factor if the symplectic area of that factor is at least twice that of the other.

1 Introduction

§1.1 Let $X = S^2 \times S^2$ and let ω_λ be the product symplectic form which gives the factors areas 1 and λ respectively. Define the homology classes

$$\alpha := [S^2 \times \{p\}]$$
$$\beta := [\{p\} \times S^2].$$

If $\lambda < 2$ then there is a Lagrangian Klein bottle in the homology class β; one can construct this as a visible Lagrangian submanifold [6] (see Figure 1 below[1]) or in several other ways [3, 4, 7].

The line over which the visible Lagrangian Klein bottle projects must have slope 2 and connect the bottom and top edges, so it can be drawn if and only if $\lambda < 2$. For this reason, the second author conjectured in [6] that there is no Lagrangian Klein bottle in the class β when $\lambda \geq 2$. We will prove this.

§1.2 Theorem. *If $L \subset X$ is a Lagrangian Klein bottle for ω_λ in the homology class β then $\lambda < 2$.*

[1] The picture in [6, Fig. 2] is wrong and should be rotated by 90 degrees otherwise the Klein bottle lives in the $\mathbb{Z}_2$-homology class α.

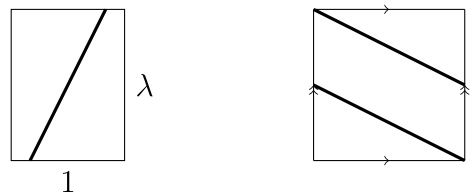

Figure 1: The base (left) and fibre (right) of the moment map $\mu\colon X \to \mathbb{R}^2$. A visible Lagrangian Klein bottle lives over the bold line in the base (slope 2) and intersects each fibre in the bold line shown on the right (slope $-1/2$).

§1.3 Indeed, there are no null-homologous Klein bottles in X by Shevchishin [13] and Nemirovski [11], and there are none in the $\mathbb{Z}_2$-homology class $\alpha + \beta$ because they would violate the Audin identity [1]:

$$\chi(L) = [L] \cdot [L] \mod 4$$

which holds for any totally real embedded submanifold L of an almost complex surface. Here χ is the Euler characteristic and $[L] \cdot [L] \mod 4$ denotes the Pontryagin square of the $\mathbb{Z}_2$-homology class. Therefore Theorem §1.2 gives a complete picture of which homology classes are inhabited by Lagrangian Klein bottles for which symplectic forms.

§1.4 There is also a complete understanding of which homology classes are inhabited by *totally real* Klein bottles, worked out by Derdzinski and Januszkiewicz [5, Proposition 29.1]. If we allow immersions then all homology classes can be represented, but there are restrictions on the Maslov classes. If we allow only embeddings then the class $\alpha + \beta$ cannot be represented but the others can. The restriction on Maslov classes will be vitally important to us, so we review this in §1.7 once we have established more notation.

§1.5 The usual toric diagrams for other Hirzebruch surfaces also carry visible Klein bottles. We can perform a sequence of almost toric mutations to get from such a diagram to either a rectangle (for an even Hirzebruch surface) or a triangle with its corner truncated (for an odd Hirzebruch surface), and this sequence of almost toric mutations happens away from the visible Klein bottles, so the families of visible Klein bottles coming from different Hirzebruch surfaces are symplectomorphic to each other. Note that in the case of odd Hirzebruch surfaces, the visible Klein bottles are also *real*, i.e. fixed loci of anti-symplectic involutions.

§1.6 Klein bottles. We will think of the Klein bottle as the quotient of $\mathbb{R}^2$ (coordinates (φ, ψ)) by the action generated by the transformations

$$(\varphi, \psi) \mapsto (\varphi + 1, -\psi), \qquad (\varphi, \psi) \mapsto (\varphi, \psi + 1).$$

We can represent this quotient as a square with its sides identified as in Figure 2; we write A for the homology class of the loop $t \mapsto (t, 0)$ and B for the homology class of the loop $t \mapsto (0, t)$. These generate the first homology and satisfy $2B = 0$, $A \cdot B = 1$, $B^2 = 0$, $A^2 = 1$.

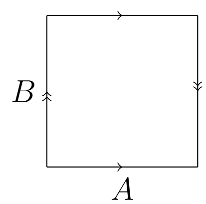

Figure 2: The Klein bottle as an identification space.

Because $H_2(L; \mathbb{Z}) = H_1(X; \mathbb{Z}) = 0$, the long exact sequence of the pair (X, L) splits off a short exact sequence

$$0 \to H_2(X; \mathbb{Z}) = \mathbb{Z}^2 \to H_2(X, L; \mathbb{Z}) \to H_1(L; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2 \to 0.$$

In particular, for any class $H_1(L; \mathbb{Z})$ there exists a class $H_2(X, L; \mathbb{Z})$ which maps to it under the connecting homomorphism ∂. The set of such $H_2(X, L; \mathbb{Z})$-classes form a torsor over $H_2(X; \mathbb{Z})$.

§1.7 Maslov class constraint. If $\Sigma \subset X$ is an oriented surface with boundary on a totally real submanifold $L \subset X$ then it has a well-defined Maslov index (see §2.13 for a review). In fact, the reduction modulo 4 of this Maslov index depends only on the boundary $\partial\Sigma \in H_1(L; \mathbb{Z})$. Derdzinski and Januszkiewicz [5] call this mod 4 index $\boldsymbol{i}(\partial D)$.

Proof that $\boldsymbol{i}(\partial\Sigma)$ *is well-defined modulo* 4. Recall that $\partial^{-1}([\partial\Sigma])$ is a torsor over $H_2(X; \mathbb{Z})$. All classes in $H_2(X; \mathbb{Z})$ have Maslov index equal to zero modulo 4, since the Maslov index on a class $C \in H_2(X; \mathbb{Z})$ agrees with $2c_1(X) \cdot [C]$ and $c_1(X) = 2(\alpha + \beta)$. □

§1.8 We will be interested in surfaces with boundary homologous to the loop B. Derdzinski and Januszkiewicz show that for totally real immersions of the Klein bottle into $S^2 \times S^2$ in the $\mathbb{Z}_2$-classes 0 and $\alpha + \beta$, we must have $\boldsymbol{i}(B) = 0 \mod 4$, but that for totally real immersions in the $\mathbb{Z}_2$-classes α and β, we have $\boldsymbol{i}(B) = 2 \mod 4$. This comes from the characterisation of the set $\mathcal{Z}(\Sigma, M)$ of possible Maslov/homology class pairs that appears after [5, Theorem 2.2], which is spelled out for Klein bottles in $S^2 \times S^2$ in the paragraph before [5, Proposition 29.1]. We reproduce the specific parts of their argument we need in §2.17–§2.20 below for convenience.

§1.9 Example. To see that $\boldsymbol{i}(B) = 2 \mod 4$ for the visible Klein bottles, consider the visible symplectic (but not holomorphic) 2-sphere living over the same line ℓ as our visible bottle which intersects every (regular) fibre in a vertical loop as shown in Figure 3. This sphere lives in the homology class β and intersects L in a circle in the homology class B, which separates the sphere into two discs (one dotted, one dashed in Figure 3). These discs form part of a 1-parameter family, as the vertical line moves horizontally; in particular they have the same Maslov index. Since the sphere has Chern number 2 (and hence Maslov index 4), each disc has Maslov index 2. Again, we emphasise these are symplectic but not holomorphic discs.

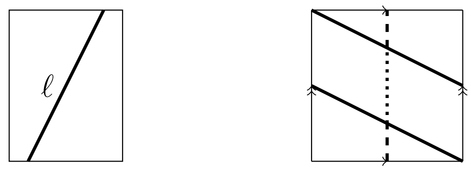

Figure 3: A visible symplectic sphere living over ℓ made up of two discs (dotted and dashed) with boundary on the visible Lagrangian L.

§1.10 Outline of the proof. We outline the proof of Theorem §1.2 here, leaving two facts to be established later (Section 3). Recall that given a Lagrangian Klein bottle $L \subset X$ there is a symplectic surgery called Luttinger surgery which excises a Weinstein neighbourhood of L and reglues it with a twist. This surgery is based on the surgery for Lagrangian tori introduced by Luttinger [10], modified for Klein bottles by Nemirovski [11]. We will write $(\tilde{X}, \tilde{\omega})$ for the result of this Luttinger surgery.

We will first prove (§3.1) that $\tilde{X}$ is diffeomorphic to the first Hirzebruch surface $\mathbb{F}_1 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. The second homology of $\mathbb{F}_1$ is generated by two classes H and E with squares 1 and -1 respectively. We next prove (§3.2) that

$$\int_E \tilde{\omega} = 1 - \frac{\lambda}{2}.$$

By a result of Li and Liu [8, Theorem A], building on the seminal work of Taubes (specifically [14, Proposition 4.5]), the fact that E is represented by a smoothly embedded sphere and has $E^2 = -1$ and $c_1(E) = 1$ implies that the class E is represented by a symplectic sphere. This then implies that $\int_E \tilde{\omega} > 0$, that is $1 - \lambda/2 > 0$, or $\lambda < 2$, as required.

2 Klein bottle Luttinger surgery

In this section, we will review the Klein bottle Luttinger surgery of Nemirovski [11] in a form closer to the exposition of Auroux, Donaldson and Katzarkov [2]. We will establish some basic but important properties about how Chern and Maslov classes are related before and after the surgery.

§2.1 Let D_r denote the square $[-r,r]\times[-r,r]$, with coordinates (p_1,p_2) and let T^2 be the 2-torus with 1-periodic coordinates (q_1,q_2). Consider the symplectic manifold $(D_{2r}\setminus D_r)\times T^2$ with symplectic form $\sum dp_i\wedge dq_i$ (this is a punctured subdomain in T^*T^2). Let $\chi\colon [-r,r]\to[0,1]$ be a smooth, monotonically increasing function satisfying

$$\chi(p)=\begin{cases}0 & \text{if } p<-r/3\\ 1 & \text{if } p>r/3.\end{cases}$$

To ensure compatibility with the Klein bottle case later, we will further assume that $\chi(p)-1/2$ is an odd function, that is

$$\chi(-p)=-\chi(p)+1.$$

As observed by Auroux, Donaldson and Katzarkov [2], the diffeomorphism

$$F(p_1,p_2,q_1,q_2)=\begin{cases}(p_1,p_2,q_1,q_2+\chi(p_2)) & \text{if } p_1>0\\ (p_1,p_2,q_1,q_2) & \text{if } p_1<0\end{cases}$$

preserves the symplectic form:

$$F^*\left(\sum dp_i\wedge dq_i\right)=dp_1\wedge dq_1+dp_2\wedge(dq_2+\chi'(p_2)dp_2)=\sum dp_i\wedge dq_i.$$

§2.2 We will think of T^2 as a double cover of the Klein bottle L, with deck transformation $(q_1,q_2)\mapsto(q_1+1/2,-q_2)$. This deck transformation induces a symplectomorphism of $D_{2r}\times T^2$:

$$\Psi(p_1,p_2,q_1,q_2)=(p_1,-p_2,q_1+1/2,-q_2)$$

which commutes with F. To see this, observe that

$$\begin{aligned}F(\Psi(p_1,p_2,q_1,q_2))&=(p_1,-p_2,q_1+1/2,-q_2+\chi(-p_2))\\ \Psi(F(p_1,p_2,q_1,q_2))&=(p_1,-p_2,q_1+1/2,-q_2-\chi(p_2)).\end{aligned}$$

Since by assumption $\chi(-p_2)=-\chi(p_2)+1$, we have

$$-q_2+\chi(-p_2)=-q_2-\chi(p_2)+1,$$

so the final coordinates of $F\circ\Psi$ and $\Psi\circ F$ coincide (recall that they are periodic modulo 1). Let us write U_r for the neighbourhood of the zero-section in T^*L given by quotienting $D_r\times T^2$ by Ψ and we continue to write F for the symplectomorphism of U_r induced by $F\colon D_r\times T^2\to D_r\times T^2$.

§2.3 If X is a symplectic manifold containing a Lagrangian Klein bottle L then we can find a symplectic embedding (Weinstein neighbourhood) $i\colon U_{2r}\to X$ for some r. Let $\mathcal{U}=i(U_{2r})$, let $\mathcal{V}=X\setminus i(U_r)$, and let $\mathcal{W}=\mathcal{U}\cap\mathcal{V}\cong U_{2r}\setminus U_r$. We can form the surgered manifold $\tilde{X}=\mathcal{U}\cup_{\mathcal{W}}\mathcal{V}$, where a point $x\in\mathcal{W}$ is identified with $x\in\mathcal{V}$ and with $F(x)\in\mathcal{U}$.

§2.4 Consider the hypersurface $N = i(\{p_1^2 + p_2^2 = 3r^2\}) \subset W$. Whilst the boundaries of $\mathcal{U}$ and $\mathcal{V}$ are only piecewise smooth, this is a smooth contact hypersurface isomorphic to the radius r cosphere bundle of L. The hypersurface N separates X into two closed sets U (containing L) and V which are deformation retracts of $\mathcal{U}$ and $\mathcal{V}$. We can think of V as a subset of both X and $\tilde{X}$ which is filled by two different symplectic fillings of N which we call U (before surgery) and $\tilde{U}$ (after surgery).

§2.5 Note that N is a circle bundle over L; we will use coordinates (q_1, q_2, θ) on N, where the coordinates are understood modulo the identifications

$$\begin{aligned}(q_1, q_2, \theta) &\mapsto (q_1 + 1/2, -q_2, -\theta)\\(q_1, q_2, \theta) &\mapsto (q_1, q_2 + 1, \theta)\\(q_1, q_2, \theta) &\mapsto (q_1, q_2, \theta + 1)\end{aligned}$$

shown in Figure 4.

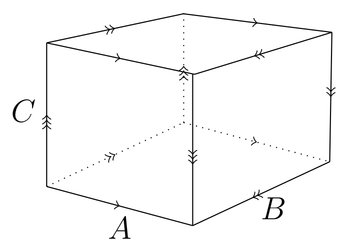

Figure 4: The space N as an identification space.

We write A, B and C for the homology classes of the three loops $t \mapsto (t/2, 0, 0)$, $t \mapsto (0, t, 0)$ and $t \mapsto (0, 0, t)$.

§2.6 Lemma. *From the cell decomposition inherited from the cube (respectively the square), we can easily compute:*

	N	L
$H_3(-;\mathbb{Z})$	$\mathbb{Z}$	0
$H_2(-;\mathbb{Z})$	$\mathbb{Z}$	0
$H_1(-;\mathbb{Z})$	$\langle A, B, C \mid 2B = 2C = 0\rangle$	$\langle A, B \mid 2B = 0\rangle$
$H^3(-;\mathbb{Z})$	$\mathbb{Z}$	0
$H^2(-;\mathbb{Z})$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}_2$
$H^1(-;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}$

and the inclusion $i \colon N \to U \simeq L$ induces the map $A \mapsto A$, $B \mapsto B$, $C \mapsto 0$ on homology. Over $\mathbb{R}$, the pullback i^* induces an isomorphism on H^1 and the zero map on H^2.

§2.7 Lemma. *Now suppose that* $H^1_{dR}(X) = H^3_{dR}(X) = 0$. *We have* $H^1_{dR}(V) \cong 0$ *and* $H^2_{dR}(V) \cong H^2_{dR}(X) \oplus \mathbb{R}$. *Moreover, the pullback map* $H^2_{dR}(X) \to H^2_{dR}(V)$ *is injective.*

Proof. For the purposes of the proof it is more convenient to work with $\mathcal{U}$ and $\mathcal{V}$ than U and V, but since they are deformation retracts the conclusions are the same. Since N is a deformation retract of $\mathcal{U} \cap \mathcal{V}$, the relevant part of the Mayer–Vietoris sequence for $X = \mathcal{U} \cup \mathcal{V}$ is:

$$\begin{aligned} \cdots \to H^1_{dR}(X) &\to H^1_{dR}(\mathcal{U}) \oplus H^1_{dR}(\mathcal{V}) \to H^1_{dR}(N) \to \\ \to H^2_{dR}(X) &\to H^2_{dR}(\mathcal{U}) \oplus H^2_{dR}(\mathcal{V}) \to H^2_{dR}(N) \to \\ \to H^3_{dR}(X) &\to \cdots \end{aligned} \tag{2.1}$$

Using the facts that $H^1_{dR}(X) = H^3_{dR}(X) = 0$, $H^2_{dR}(\mathcal{U}) = 0$, and that the map $H^1_{dR}(\mathcal{U}) \to H^1_{dR}(N)$ is an isomorphism, we deduce that $H^1_{dR}(\mathcal{V}) = 0$ and split off a short exact sequence

$$0 \to H^2_{dR}(X) \to H^2_{dR}(\mathcal{V}) \to H^2_{dR}(N) = \mathbb{R} \to 0,$$

which implies what is claimed. □

§2.8 Corollary. *The De Rham cohomology class of a 2-form on* X *is determined by its integrals over* 2*-cycles in* V.

§2.9 Lemma. *Let* $L \subset X$ *be a Lagrangian Klein bottle and let* $\tilde{X}$ *be the result of performing Luttinger surgery along* L. *Then* $H^1_{dR}(\tilde{X}) = H^3_{dR}(\tilde{X}) = 0$, $H^2_{dR}(\tilde{X}) \cong H^2_{dR}(X)$, *and the cohomology class of a 2-form on* $\tilde{X}$ *is determined by its integrals over* 2*-cycles in* V.

Proof. Write $\tilde{X}$ as $\tilde{U} \cup V$, where $\tilde{U}$ is the Weinstein neighbourhood of the Klein bottle after surgery. The map $N \to \tilde{U}$ still induces an isomorphism on H^1_{dR} and the maps $H^k_{dR}(V) \to H^k_{dR}(N)$ are unchanged, so the Mayer–Vietoris sequence (2.1) (with $\tilde{X}$ instead of X and $\tilde{U}$ instead of U) gives the conclusions. □

§2.10 Lemma. *We have* $c_1(X) \cdot [\omega_\lambda] = c_1(\tilde{X}) \cdot [\tilde{\omega}]$. (This observation is due to Nemirovski [11, Section 2.3]; we have given details here for convenience.)

Proof. Represent c_1 in each case by a Chern form $\rho, \tilde{\rho}$ given by the curvature of an Hermitian line bundle. By picking the Hermitian metrics to agree over V, we can ensure that $\rho|_V = \tilde{\rho}|_V$. We also know that $\omega_\lambda|_V = \tilde{\omega}|_V$, so that

$$\int_C \omega_\lambda = \int_C \tilde{\omega} \qquad \text{and} \qquad \int_C \rho = \int_C \tilde{\rho}$$

for any 2-cycle $C \subset V$. Since $H^2_{dR}(U) = 0$, both ω_λ and ρ are cohomologous to 2-forms ω'_λ and ρ' which vanish on $X \setminus V$. Their restrictions to V extend (by zero) over $\tilde{X} \setminus V$; let us write $\tilde{\omega}'$ and $\tilde{\rho}'$ for these extensions. Then, for any 2-cycle $C \subset V$, we have

$$\int_C \tilde{\omega} = \int_C \omega_\lambda = \int_C \omega'_\lambda = \int_C \tilde{\omega}',$$

and similarly $\int_C \tilde{\rho} = \int_C \tilde{\rho}'$. By §2.8, this shows that $[\tilde{\omega}] = [\tilde{\omega}']$ and $[\tilde{\rho}] = [\tilde{\rho}']$, so

$$c_1(X) \cdot [\omega_\lambda] = \int_X \rho \wedge \omega_\lambda = \int_V \rho' \wedge \omega'_\lambda = \int_{\tilde{X}} \tilde{\rho}' \wedge \tilde{\omega}' = \int_{\tilde{X}} \tilde{\rho} \wedge \tilde{\omega} = c_1(\tilde{X}) \cdot [\tilde{\omega}]. \qquad \square$$

§2.11 Corollary. *If $c_1(X) \cdot [\omega] > 0$ and $H^1_{dR}(X) = 0$ then $\tilde{X}$ is a rational symplectic 4-manifold.*

Proof. By Lemma §2.10, $c_1(\tilde{X}) \cdot [\tilde{\omega}] > 0$, so by the Liu–Ohta–Ono Theorem ([9, Theorem B], [12, Theorem 1.2]) the 4-manifold $\tilde{X}$ is either rational or ruled. But by Lemma §2.9, $H^1_{dR}(\tilde{X}) = 0$, so $\tilde{X}$ cannot be (a blow-up of) an irrational ruled surface. Therefore it is rational. $\square$

§2.12 Chern and Maslov classes. Let X be an almost complex manifold of real dimension $2n$. Consider the complex line bundle $V := (\Lambda^n_{\mathbb{C}} TX)^{\otimes 2}$ and let $E \subset V$ be its unit circle bundle. The first Chern class of $E \to X$ is $2c_1(X)$, and if $S \subset X$ is a closed, oriented surface then $2c_1(X) \cdot [S]$ is the obstruction to finding a section of $E|_S$; in other words, $2c_1(X) \cdot [S]$ is the signed count of zeros of a generic section of $V|_S$.

§2.13 If we have a surface Σ with boundary and a nowhere-vanishing section σ of V defined along the boundary $\partial\Sigma$ then the Maslov index $\mu(\Sigma, \sigma)$ of the pair (Σ, σ) is the number of zeros of an extension of σ over Σ. One way of obtaining a nowhere-vanishing section of V over a subset $Y \subset X$ is from a field of unoriented totally real n-planes ζ on Y: namely, we pick a section of $(\Lambda^n_{\mathbb{R}} \zeta)^{\otimes 2}$ by squaring a local orientation of ζ. Since ζ is totally real, this section is nowhere-vanishing when considered as a section of V.

For example, if Σ has boundary on a totally real submanifold L then there is a *canonical*[2] nowhere-vanishing section σ_{can} defined along $\partial\Sigma$ coming from the field of tangent planes TL. The Maslov index $\mu(\Sigma)$ is then defined to be the Maslov index of the pair $\mu(\Sigma, \sigma_{can})$.

§2.14 If two boundary conditions σ_0 and σ_1 are homotopic through sections of $E|_{\partial\Sigma}$ then $\mu(\Sigma, \sigma_0) = \mu(\Sigma, \sigma_1)$. If L is a Lagrangian submanifold and we choose a Weinstein neighbourhood then one obvious choice for an alternative boundary condition equivalent

[2] Canonical up to a positive real scalar.

to σ_{can} is the vertical distribution of tangent spaces to cotangent fibres. This actually gives us a section of E over the whole of the Weinstein neighbourhood. We call this section ξ_{can}.

§2.15 Lemma. *Let F be the diffeomorphism of a punctured Weinstein neighbourhood of a Lagrangian Klein bottle defined in §2.1–§2.2. The field ξ_{can} is homotopic through sections of E to $F_*\xi_{can}$.*

Proof. The vertical distribution on the cotangent bundle is spanned by ∂_{p_1} and ∂_{p_2}. We have $F_*\partial_{p_1} = \partial_{p_1}$ and $F_*\partial_{p_2} = \partial_{p_2} + \chi'(p_2)\partial_{q_2}$. The homotopy between these is given by taking ∂_{p_1} and $\partial_{p_2} + t\chi'(p_2)\partial_{q_2}$, which are complex linearly-independent for all t. □

§2.16 Lemma. *Let X be a symplectic 4-manifold and $L \subset X$ a Lagrangian Klein bottle. Let U be a closed Weinstein neighbourhood of L with boundary N and let $V = \overline{X \setminus U}$. Let $\Sigma \subset X$ be a smooth, oriented surface with boundary on L; by making a small perturbation, assume that Σ is transverse to N. Let $\Sigma' = V \cap \Sigma$; this is a smooth surface with boundary on N. Suppose that $F(\partial\Sigma)$ is nullhomologous in $\tilde{U}$, pick an oriented smooth surface $T \subset \tilde{U}$ with $\partial T = -F(\partial\Sigma)$, and let $\tilde{S} = \Sigma' \cup T$. Then*

$$2c_1(\tilde{X}) \cdot [\tilde{S}] = \mu(\Sigma).$$

Proof. Using ξ_{can} on U and on $\tilde{U}$ produces homotopic nowhere-vanishing boundary conditions for Σ', regardless of whether the rest of the surface is $\Sigma \cap U$ or $\tilde{S} \cap \tilde{U}$. Therefore the obstruction to extending these boundary conditions is the same; in the one case it is $\mu(\Sigma)$ and in the other it is $2c_1(\tilde{X}) \cdot [\tilde{S}]$. □

§2.17 For the convenience of the reader, we now give a summary of how Derdzinski and Januszkiewicz prove that a surface Σ in $S^2 \times S^2$ with boundary on a Lagrangian Klein bottle L in the $\mathbb{Z}_2$-homology class α or β must have Maslov index $\mu(\Sigma) = 2 \mod 4$.

§2.18 Lemma. *For $X = S^2 \times S^2$ we have $\pi_1(E) = \mathbb{Z}/4$.*

Proof. This follows from the homotopy long exact sequence of the circle bundle $E \to X$:

$$0 \to \pi_2(E) \to \pi_2(X) \to \mathbb{Z} \to \pi_1(E) \to 0.$$

Under the identification $H^2(X;\mathbb{Z}) \cong \mathrm{Hom}(\pi_2(X), \mathbb{Z})$, the map $\pi_2(X) = \mathbb{Z}^2 \to \pi_1(S^1) = \mathbb{Z}$ is the first Chern class of the circle bundle E, that is $2c_1(X) = 4(\alpha + \beta)$. Therefore its image is the subgroup $4\mathbb{Z} \subset \mathbb{Z}$ and its cokernel is $\pi_1(E) = \mathbb{Z}/4$. □

§2.19 Lemma. *Let Σ be an oriented surface with boundary on a totally real surface $L \subset X = S^2 \times S^2$. Let ℓ be the loop in E given by restricting the section ξ_{can} to $\partial\Sigma$. Then $\mu(\Sigma) \mod 4 = [\ell] \in \pi_1(E)$.*

Proof. The bundles $E|_{\partial\Sigma}$ and $E|_{\Sigma}$ are trivial, and only one of the trivialisations of $E|_{\partial\Sigma}$ is compatible with a trivialisation of $E|_{\Sigma}$. The Maslov index $\mu(\Sigma)$ can be interpreted as the winding number of the boundary condition ξ_{can} with respect to this trivialisation, that is the projection of $\xi_{can}|_{\partial\Sigma}$ to the second factor in $\pi_1(E|_{\partial\Sigma}) = \pi_1(\partial\Sigma) \times \mathbb{Z}$. The inclusion map $\partial\Sigma \to E$ induces a map $\pi_1(\partial\Sigma) \times \mathbb{Z} \to \pi_1(E) = \mathbb{Z}/4$, which coincides with reduction modulo 4 on the second factor, and $\xi_{can}|_{\partial\Sigma}$ maps to $[\ell]$. □

§2.20 Lemma. *Let $L \subset X = S^2 \times S^2$ be a totally real Klein bottle (or, more generally, nonorientable surface with $\chi(L) = 0 \mod 4$) in the $\mathbb{Z}_2$-homology class α or β, let $B \subset L$ be the meridian loop (or, more generally, a 2-sided simple closed curve representing the unique torsion class in $H_1(L;\mathbb{Z})$) and let $\Sigma \subset X$ be a surface with $\partial\Sigma = B$. Then $\mu(\Sigma) = 2 \mod 4$.*

Proof. Let $\ell \subset E$ be the loop given by the section $\xi_{can}|_B$. Since B is 2-sided, the section $\xi_{can}|_B$ admits a square root (that is, a lift to $\Lambda^n_{\mathbb{C}} TX$), which means that $[\ell] \in \{0, 2\} \subset \pi_1(E) = \mathbb{Z}/4$. Suppose that $[\ell] = 0 \in \pi_1(E)$; we will derive a contradiction. We can cut open L along B to obtain an *orientable* surface L' with two circular boundary components; let L'' be the abstract orientable surface obtained by capping these circles off with discs. Since $[\ell] = 0 \in \pi_1(E)$, the canonical section $\sigma_L|_{L'} \colon L' \to E$ extends to a map $L'' \to E$ which sends the two capping discs to a single disc Δ which is a nullhomotopy of ℓ. We can then project L'' to X to obtain a continuous map $f \colon L'' \to X$. Since f^*E admits a section (by construction) this means that $c_1(f^*E) = 0$. Since $c_1(f^*E) = 2c_1(X) \cdot f_*[L'']$ and $2c_1(X) = 4(\alpha + \beta)$, this tells us that $f_*[L''] = k(\alpha - \beta)$ for some $k \in \mathbb{Z}$, which implies that $f_*[L''] \mod 2$ is either 0 or $\alpha + \beta$. But since the two capping discs project to the same disc (with opposite orientations), $f_*[L''] = [L] \mod 2$, which contradicts the fact that $[L]$ is either $\alpha \mod 2$ or $\beta \mod 2$. □

3 Finishing the proof

We now provide the details that were missing from the sketch proof.

§3.1 Proposition. *Let $X = S^2 \times S^2$ with the symplectic form ω_λ and suppose that $L \subset X$ is a Lagrangian Klein bottle in the $\mathbb{Z}_2$-homology class β. Let $\tilde{X}$ be the result of Luttinger surgery. Then $\tilde{X}$ is diffeomorphic to $\mathbb{F}_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.*

Proof. By Corollary §2.11, $\tilde{X}$ is a rational symplectic 4-manifold and by Lemma §2.9, it has $b_2 = 2$. Therefore $\tilde{X}$ is diffeomorphic to either $S^2 \times S^2$ or to $\mathbb{F}_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. We will exhibit a surface $\tilde{\Sigma}$ in $\tilde{X}$ with odd Chern number, which is only possible for $\mathbb{F}_1$ since $c_1(S^2 \times S^2)$ is divisible by 2 in integral cohomology.

To find $\tilde{\Sigma}$, we start with a surface D with boundary on L such that $[\partial D] = B$ and $\mu(D) = 2 \mod 4$. It is possible that D intersects L at interior points of D; if so, we perturb D so that it intersects L transversely on its interior and open out these intersections to form small nullhomologous boundary components on L. We continue to call the resulting surface D. Since the boundary of D still lies in the class B (because the punctures gave only nullhomologous boundary components), the intersection $D \cap N$ lies either in the class $B + C$ or the class B. In the former case, we define $\Sigma = D$. In the latter case, we take $\Sigma = S \cup D$ where S is a sphere in the homology class α, perturbed to be transverse to L, again with these intersections opened out into boundary components on L. Since S must intersect L an odd number of times, this has the effect of changing the intersection $N \cap (D \cup S)$ so that it lives in the class $B + C$.

Now $F_*(B + C) = C$, which is nullhomologous in $\tilde{U}$, so we can apply Lemma §2.16: note that $\mu(\Sigma) = \mu(D) + k\mu(\alpha)$ for some k, but $\mu(\alpha) = 4$, so $\mu(\Sigma) = \mu(D) \mod 4$ and by Lemma §2.20, $\mu(D) = 2 \mod 4$. This implies that $2c_1(\tilde{\Sigma}) = 2 \mod 4$, so $c_1(\tilde{\Sigma})$ is odd. □

§3.2 Proposition. *Let H and E be the classes in $\mathbb{F}_1$ coming from a line in $\mathbb{CP}^2$ and the exceptional curve. If $L \subset X$ is a Lagrangian Klein bottle in the $\mathbb{Z}_2$-class β in $X = S^2 \times S^2$ with symplectic form ω_λ then the Luttinger surgery $(\tilde{X}, \tilde{\omega})$ satisfies*

$$\int_E \tilde{\omega} = 1 - \frac{\lambda}{2}.$$

Proof. The rational homology of V fits into a Mayer–Vietoris sequence

$$H_2(N; \mathbb{Z}) \to H_2(U; \mathbb{Z}) \oplus H_2(V; \mathbb{Z}) \to H_2(X; \mathbb{Z}) \to H_1(N; \mathbb{Z}) \to H_1(U; \mathbb{Z}).$$

Since $[L] = \beta \mod 2$, a generic sphere in the class β intersects L transversely an even number of times. Thus $\partial\beta \in H_1(N; \mathbb{Z})$ is an even multiple of C; since $2C = 0$, this means that β lives in the image of $H_2(V; \mathbb{Z})$. Similarly, 2α lies in the image of $H_2(V; \mathbb{Z})$. In particular, this means that 2α and β can be represented respectively by surfaces $S_{2\alpha}$ and S_β in V with

$$S_{2\alpha}^2 = S_\beta^2 = 0, \qquad c_1(X) \cdot S_{2\alpha} = 4 \qquad \int_{S_{2\alpha}} \omega_\lambda = 2,$$
$$S_{2\alpha} \cdot S_\beta = 2, \qquad c_1(X) \cdot S_\beta = 2, \qquad \int_{S_\beta} \omega_\lambda = \lambda.$$

After surgery, since these surfaces were in V, they can be thought of as surfaces $\tilde{S}_{2\alpha}$ and $\tilde{S}_\beta$ in $\tilde{X}$. We still have

$$\begin{aligned} \tilde{S}_{2\alpha}^2 = \tilde{S}_\beta^2 = 0, \qquad & c_1(\tilde{X}) \cdot \tilde{S}_{2\alpha} = 4, \qquad & \int_{\tilde{S}_{2\alpha}} \tilde{\omega} = 2, \\ \tilde{S}_{2\alpha} \cdot \tilde{S}_\beta = 2, \qquad & c_1(\tilde{X}) \cdot \tilde{S}_\beta = 2, \qquad & \int_{\tilde{S}_\beta} \tilde{\omega} = \lambda. \end{aligned}$$

Suppose that $[S_\beta] = aH + bE$. Then $S_\beta^2 = a^2 - b^2 = 0$, so $a = \pm b$, and $c_1(\tilde{X}) \cdot S_\beta = 3a \pm a = 2$. Since $a \in \mathbb{Z}$, we must have $a = -b = 1$, so $[S_\beta] = H - E$.

Suppose that $S_{2\alpha} = cH + dE$. In the same way, from $S_{2\alpha}^2 = 0$ we have $c = \pm d$, and from $c_1(\tilde{X}) \cdot [S_{2\alpha}] = 4$ we have $3c \pm d = 4$, so either $c = d = 1$ or $c = -d = 2$, that is $S_{2\alpha} = H + E$ or $2H - 2E$. Since $S_\alpha \cdot S_\beta = 2$, we must have $S_{2\alpha} = H + E$.

But now

$$\int_{S_{2\alpha}} \tilde{\omega} = 2 = \int_H \tilde{\omega} + \int_E \tilde{\omega} \qquad \text{and} \qquad \int_{S_\beta} \tilde{\omega} = \lambda = \int_H \tilde{\omega} - \int_E \tilde{\omega},$$

so

$$2 - \lambda = 2 \int_E \tilde{\omega} \qquad \text{or} \qquad \int_E \tilde{\omega} = 1 - \frac{\lambda}{2}. \qquad \square$$

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