Influence of Price Elasticity of Demand on Monopoly Games under Different Returns to Scale

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Abstract

This paper examines a monopoly market featured by a general isoelastic demand function. With a quadratic cost function for the monopolist, we explore how the price elasticity of demand influences monopolistic behavior under different (decreasing, constant, and increasing) returns to scale. The combination of the general isoelastic demand and quadratic cost functions leads to a transcendental equilibrium equation, making closed-form solutions unattainable. To address this challenge, we develop an innovative approach that leverages the specific structure of marginal revenue and cost to conduct a comprehensive comparative static and stability analysis. Additionally, we introduce two dynamic models based on distinct adjustment mechanisms: gradient and local monopolistic approximation (LMA). Our findings reveal that the LMA model is more stable in both parameter and state spaces compared to the gradient model. Notably, we prove that the unique non-vanishing equilibrium of the LMA model is globally asymptotically stable.

Keywords: general isoelastic demand; quadratic cost; local asymptotical stability; global asymptotical stability

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1 Introduction

In classical economic theory, the assumption of linearity is often used to guarantee the uniqueness of a model's equilibrium. This assumption has been widely employed in economic modeling as it greatly simplifies analysis. However, nonlinearities are prevalent in real-world markets, and numerous empirical studies have challenged the linearity assumption. For example, Ng [34], using threshold autoregressive models, found evidence of nonlinearity in commodity price data. Moreover, Adrangi et al. [2] discovered strong evidence of nonlinear dependence in financial markets and suggested that well-known ARCH-type processes, with control for seasonal and contract-maturity effects, can largely account for the observed non-linearities.

Isoelastic demand is frequently proposed in the literature as an alternative to the widely adopted linear demand specification. However, the properties of this nonlinear demand function remain relatively unexplored. Despite this, numerous studies have incorporated isoelastic demand in modeling various economic phenomena. In particular, isoelastic demand has been applied to areas such as nonlinear monopoly (or oligopoly) theory [13] and new trade theory [9]. For instance, Cavalli and Naimzada [13] introduced a dynamic monopoly model where the demand function follows a general isoelastic form characterized by two parameters: elasticity and market size. In their model, the monopolist is assumed to adopt a gradient adjustment mechanism, and the firm's marginal costs remain constant.

However, many industries provide examples of firms operating with non-constant marginal costs. Digital products—often referred to as information goods, electronic information products, or virtual products—are bit-based objects distributed through electronic channels. As Goldfarb and Tucker [22] pointed out, the defining feature of digital goods is that certain costs decline significantly, potentially approaching zero. Compared to traditional products, digital goods have much lower variable costs, including lower search costs, replication costs, transportation costs, tracking costs, and verification costs. Building on this observation, this study extends the work of Cavalli and Naimzada [13] to incorporate scenarios with non-constant marginal costs. Specifically, we assume a quadratic cost function for the monopolist in order to examine the differences in monopolistic behavior under conditions of decreasing, constant, and increasing returns to scale.

First, we propose a static model and analyze the number of equilibria, finding that there is exactly one equilibrium in the case of decreasing or constant returns to scale, and at most two equilibria in the case of increasing returns to scale. We then conduct a comparative static analysis of the equilibrium output, price, and profit, arriving at more general conclusions compared to those of Cavalli and Naimzada [13]. Second, we introduce two dynamic models—Model G and Model L—based on the gradient mechanism and the local monopolistic approximation (LMA) mechanism, respectively. Our findings show that the non-vanishing equilibrium of Model L is locally asymptotically stable for all feasible parameter values and even globally asymptotically stable for all possible initial states. In other words, Model L is more stable than Model G in both the parameter and state spaces. We also establish the conditions under which Model G is locally asymptotically stable. Interestingly, the effect of the rate of diminishing returns to scale on the local stability of Model G, which represents a monopoly game, contrasts with the findings of Fisher [21] as well as McManus and Quandt [29], who examined oligopoly games with linear demand.

The paper is organized as follows. The next section reviews the relevant literature. In Section 3, we present a static Cournot game to model the monopoly market with general isoelastic demand under different returns to scale. Section 4 analyzes the equilibrium output, price, and profit of the static model. In Section 5, we introduce two dynamic games based on different adjustment mechanisms. Section 6 examines the stability of the equilibrium in these dynamic models. Section 7 presents numerical simulations to explore the complex dynamics that may arise when the dynamic models become destabilized. Section 8 concludes the paper with closing remarks.

2 Literature Review

This paper contributes to two strands of the research literature: the study of isoelastic demand functions and the exploration of dynamic monopoly models.

2.1 Isoelastic Demand Functions

Isoelastic demand functions play a critical role in addressing various economic issues, including international trade [9, 15, 33] and strategic delegation [14, 17]. Bandyopadhyay [9] examined trade policy within the framework of a duopoly game featuring a general isoelastic demand function, highlighting the importance of demand elasticity and cost asymmetry in determining strategic trade policy. The study finds that free trade is optimal in the symmetric case with isoelastic demand, while subsidies (resp. taxes) are optimal when the marginal cost of an exporting firm is lower (resp. higher) than that of its competitors. Neary [33] explored the use of general isoelastic demand in an oligopoly model but did not explicitly derive the Nash–Cournot equilibrium. In addition, Cieślik [15] employed a general isoelastic demand function to investigate the role of cost asymmetry in studying the effects of trade liberalization.

Chirco et al. [14] examined how optimal delegation schemes are affected by market concentration and demand elasticity within a strategic delegation framework. They derived that the distortion of the profit maximization rule decreases as market concentration decreases, and increases as demand elasticity rises. Similarly, Colombo et al. [17] showed that the degree of delegation is not necessarily monotonic with respect to the price elasticity of market demand and the number of firms. Moreover, Collie [16] adopted isoelastic functions to investigate how demand elasticity impacts the sustainability of collusion, a study that was later generalized by Beard [10] to incorporate the case of heterogeneous costs. Additionally, Waterson [42] derived a formula linking the elasticity of demand for a factor to the elasticity of final product demand, the elasticity of substitution, and the share of the factor in production costs. While these studies are based on the framework of static games, our research introduces two dynamic adjustment mechanisms into the monopoly game setting. We examine the impact of price elasticity across different scenarios of returns to scale and derive richer conclusions of the dynamic behavior resulting from these two adjustment mechanisms.

The study of dynamic games with isoelastic demand functions traces back to the pioneering work of Puu [35]. Specifically, Puu [35] investigated the possibility of complex dynamic behavior in the Cournot game [18] by introducing nonlinearity in the demand function and incorporating limited rationality (i.e., naive expectations) among firms in duopoly competitions. Since then, many economists, including Ahmed and Agiza [3], Askar and Alnowibet [8], Bischi et al. [11], and Tramontana et al. [40], have extended Puu's work and adopted the same isoelastic demand function. In a broader context, Zhang and Gao [43] studied the effects of the concavity and convexity of the demand function on the stability conditions of the dynamic Cournot game. Their findings can be directly applied to the isoelastic demand function introduced by Puu [35], leading to many interesting results.

It is worth noting that the isoelastic demand function used in Puu [35] assumes a fixed price elasticity. However, several studies have explored dynamic models with general isoelastic demand functions, where price elasticity can vary and is controlled by a parameter (see, e.g., [5, 6, 12, 13, 20, 24]). For instance, Fanti et al. [20] analyzed a nonlinear Cournot duopoly game with a general isoelastic demand, demonstrating that price elasticity not equal to one leads to both local and global dynamic behavior that cannot be observed in the case of unit-elastic demand and homogeneous players. Caravaggio and Sodini [12] studied a nonlinear model where a monopolist supplies a fixed quantity of an intermediate good, which is then used to produce two vertically differentiated final goods. Andaluz et al. [5] introduced a Cournot oligopoly model with n firms under general isoelastic demand, conducting a stability analysis of the Nash equilibrium and finding that the impact of price elasticity on stability depends on the firms' expectations. In a subsequent study, Andaluz et al. [6] examined the quantity competition between profit-maximizing firms and socially responsible firms, assuming market demand follows a general isoelastic form.

This paper contributes to the literature by using a quadratic cost function to investigate the effects of price elasticity on monopolistic behavior under decreasing, constant, and increasing returns to scale. We perform a comparative static analysis on the equilibrium output, price, and profit of the static model, deriving more general conclusions than those reached by Cavalli and Naimzada [13]. Furthermore, Model G, as presented in this paper, adopts the same gradient adjustment mechanism as in studies such as Andaluz et al. [5], Cavalli and Naimzada [13], and Caravaggio and Sodini [12]. However, we obtain new findings by focusing on non-constant marginal costs. For example, under diseconomies of scale, the equilibrium of Model G can only undergo period-doubling bifurcations, while under economies of scale, Model G may also experience fold bifurcations.

2.2 Dynamic Monopoly Models

Puu [36] introduced a dynamic monopoly game featuring an inverse demand function represented by a cubic function with an inflection point, alongside a quadratic marginal cost function. In this model, the monopolist is assumed to be a limited player. Puu demonstrated that the model can exhibit multiple equilibria (up to three) and that complex dynamics, such as chaos, can arise when the monopolist's responsiveness is sufficiently high. AlHdaibat et al. [4] revisited Puu's model, applying a numerical continuation method to compute solutions with different periods and identify their stability regions. Notably, they derived general formulas for 4-cycle orbits. However, their analysis of these orbits was incomplete, as pointed out by Li et al. [26], who corrected the shortcomings. Li et al. also established, for the first time, complete conditions for the local stability of Puu's model. The multiplicity of equilibria and the complex dynamics in Puu's model may be strictly dependent on the inverse demand function, particularly its inflection point.

In contrast, Naimzada and Ricchiuti [30] proposed a simpler monopoly model featuring a gradient adjustment firm, where the inverse demand function remains cubic but lacks inflection points. Their analysis revealed that complex dynamics can still emerge, particularly when the reaction coefficient to variation in profits is high. Building on this work, Askar [7] and Sarafopoulos [37] generalized the inverse demand function used by Naimzada and Ricchiuti [30] to a broader class, where the degree of the function can be any positive integer. The key difference between the two models lies in the cost structure: Askar's model assumes a linear cost function, while Sarafopoulos' model incorporates a quadratic cost function. Li et al. [25] compared the dynamics of two monopoly models. The first model was the original version introduced by Naimzada and Ricchiuti [30], while the second was a simplified adaptation of Puu's well-known monopoly model [36]. Their findings indicated that the topological structure of the parameter space in the second model is significantly more complex than in the first model.

Additionally, Elsadany and Awad [19] examined a monopoly game with delays, where the inverse demand function is log-concave. Their analysis revealed the presence of chaotic and multiple attractors, highlighting the complex behavior that can emerge from delayed adjustments in monopolistic settings. Tramontana [39] explored a model where a boundedly informed monopolist interacts with reference-dependent consumers, offering a behavioral explanation for price variability and the adoption of high-low pricing strategies. In this work, they used continuous dynamical systems to study monopolistic markets. Moreover, Matsumoto and Szidarovszky [28] proposed a continuous-time monopoly model that focuses on the effects of delays in obtaining and implementing output information.

In the literature on dynamic monopoly games (e.g., [7, 25, 30, 37]), the gradient mechanism for adjusting a monopolist's output has been extensively studied. Our research, however, examines two dynamic models: one employing the gradient mechanism and the other utilizing the LMA mechanism. While the LMA mechanism has been predominantly explored in the context of duopolistic competition, its application in monopolistic competition is less common. Despite the well-documented stability-enhancing effects of the LMA mechanism in existing literature, we find that Model L, which incorporates the LMA mechanism, demonstrates greater stability than Model G, which uses the gradient adjustment mechanism, across both the parameter and state spaces. Notably, we prove the global asymptotical stability of Model L—a strong convergence property that is rarely proven in dynamic economic models. Moreover, we discover that Model G exhibits distinct dynamical behavior under decreasing and increasing returns to scale.

3 Static Model

We consider a monopoly game in which only one firm supplies the entire market. This market is characterized by a general isoelastic demand function. Specifically, we assume that the inverse demand function is given by:

$$P(q) = \frac{a}{q^{1/e}},$$

where a and e are parameters, and q represents the output of the monopolist. Clearly, this inverse demand function is strictly decreasing and concave. The microfoundation of this demand function is detailed in [20, Appendix]. The parameter $a > 0$ controls the market size: a larger a implies a higher price for the same level of market supply, or equivalently, a higher demand for the same price. The price elasticity of demand is calculated as:

$$-\frac{\frac{P(q)}{q}}{\frac{\mathrm{d}P(q)}{\mathrm{d}q}} = -\frac{\frac{a}{q^{1/e-1}}}{-\frac{a}{eq^{1/e-1}}} = e.$$

The parameter e characterizes the price elasticity of demand. Cavalli and Naimzada [13] employed the same demand function, and pointed out that the case $e \leq 1$ is economically meaningless. Thus, we assume $e > 1$ throughout the remainder of the paper.

Following Fisher [21], McManus and Quandt [29], we assume that the firm's cost function is given by:

$$C(q) = cq + \frac{1}{2}dq^2,$$

where $c > 0$ represents the initial marginal cost. The parameter d plays a key economic role: a positive value of d ($d > 0$) indicates decreasing returns to scale or diseconomies of scale, a zero value ($d = 0$) corresponds to constant returns to scale, and a negative value of d ($d < 0$) signifies increasing returns to scale or economies of scale.

Accordingly, the firm's revenue is given by $R(q) = q \cdot \frac{a}{q^{1/e}}$, and its profit function is expressed as:

$$\Pi(q) = R(q) - C(q) = q \cdot \frac{a}{q^{1/e}} - \left(cq + \frac{1}{2}dq^2\right).$$

Hence, the marginal profit is:

$$\Pi'(q) = R'(q) - C'(q) = (1 - 1/e)\frac{a}{q^{1/e}} - (c + dq). \tag{1}$$

In our static model, the monopolist determines its optimal output by maximizing the profit function $\Pi(q)$. This maximization is governed by the first-order condition, given by $\Pi'(q) = 0$, and the second-order condition:

$$\Pi''(q) = -\frac{e-1}{e^2}\frac{a}{q^{1+1/e}} - d < 0. \tag{2}$$

It is important to note that $\Pi'(q) = 0$ is a transcendental equation, making it generally impossible to derive a closed-form solution. For clarity, we define the output level q_* that satisfies the first-order condition $\Pi'(q_*) = 0$ as the *equilibrium* or *equilibrium output* of the static model.

From an economic perspective, $C'(q) = c + dq \geq 0$ should be satisfied since the cost function $C(q)$ must increase with the output q. One can see that $C'(q) > 0$ for any $q \in [0, +\infty)$ if $d \geq 0$. However, the domain of the cost function should be $q \in [0, -c/d]$ if $d < 0$. In addition, it can be derived that $C'(q_*) > 0$ for an equilibrium q_* since we have $R'(q_*) > 0$ and $R'(q_*) = C'(q_*)$.

4 Equilibrium Analysis

We begin by discussing the potential number of equilibria in the static model.

Proposition 1. *Under decreasing ($d > 0$) or constant ($d = 0$) returns to scale, the static model possesses a unique equilibrium.*

With decreasing returns to scale ($d > 0$) or constant returns to scale ($d = 0$), the profit function becomes concave, allowing the monopolist to achieve profit maximization at a unique equilibrium output level, q_*. Any deviation from this optimal output, whether by increasing or decreasing production, results in lower profits, reinforcing the firm's incentive to maintain output at this specific level.

The uniqueness of this equilibrium ensures that the monopolist has a single, well-defined output level for profit maximization, eliminating the possibility of multiple optimal output levels. As a result, the monopolist's decision-making process is straightforward, with q_* being the sole profit-maximizing choice.

Proposition 2. *Under increasing returns to scale ($d < 0$), the static model can have up to two equilibria. Specifically:*

1. *the model has two equilibria if and only if*

$$c + dq_c > (1 - 1/e)\frac{a}{q_c^{1/e}};$$

2. *the model has exactly one equilibrium if and only if*

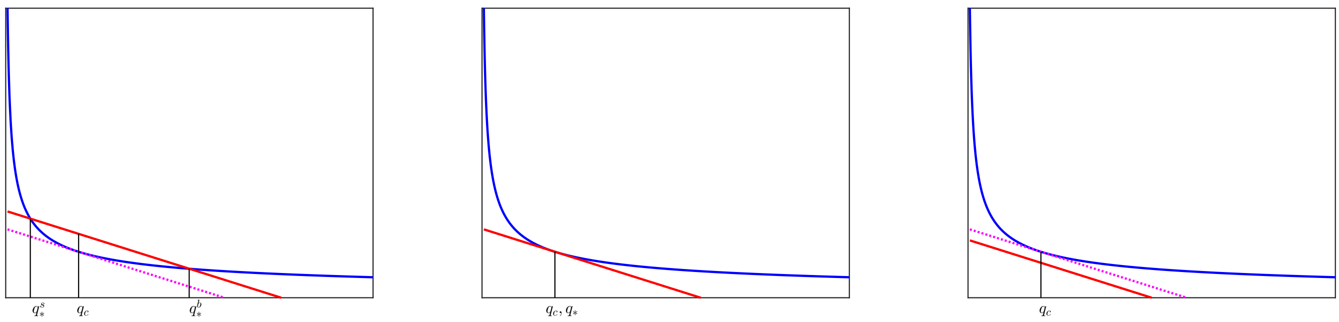
$$c + dq_c = (1 - 1/e)\frac{a}{q_c^{1/e}};$$

3. *the model has no equilibrium if and only if*

$$c + dq_c < (1 - 1/e)\frac{a}{q_c^{1/e}}.$$

Here, q_c represents the cut-off output level at which $\Pi''(q_c) = 0$. For $q < q_c$, we have $\Pi''(q) > 0$, while for $q > q_c$, we have $\Pi''(q) < 0$. The cut-off value q_c is given by:

$$q_c = \left[\frac{(e-1)a}{-e^2 d}\right]^{\frac{e}{e+1}}.$$



(a) $c + dq_c > (1 - 1/e)\frac{a}{q_c^{1/e}}$ (b) $c + dq_c = (1 - 1/e)\frac{a}{q_c^{1/e}}$ (c) $c + dq_c < (1 - 1/e)\frac{a}{q_c^{1/e}}$

Figure 1: The configuration of equilibrium points for three cases when $d < 0$, where the blue curve and the red line in each subfigure represent $R'(q)$ and $C'(q)$, while the pink dash line is the cut-off marginal cost $c + dq$ with $c = (1 - 1/e)\frac{a}{q_c^{1/e}} + dq_c$.

Figure 1 illustrates the configuration of equilibria for the three cases specified in Proposition 2, with parameters set to $a = 1$, $d = -1$, and $e = 2$. If the cost function lies above the curve representing the cut-off marginal cost, given by $c + dq$ with $c = (1 - 1/e)\frac{a}{q_c^{1/e}} + dq_c$, the static model has two equilibria. Conversely, if the cost function is below the cut-off marginal cost curve, the static model has no equilibrium.

Proposition 2 highlights the significant impact of cost structure on the stability of a monopolist's optimal output decision. It delineates the feasible range for the cost parameter c under conditions of increasing returns to scale ($d < 0$). Specifically, c must not fall below the threshold $\left(1 - \frac{1}{e}\right)\frac{a}{q_c^{1/e}} - dq_c$, as this would result in positive marginal profits at all output levels. In this scenario, profits would grow unboundedly with increasing output, ultimately implying that the static model lacks a feasible equilibrium. When c exactly equals this threshold, the model produces a "degenerate" equilibrium at q_c, where $\Pi''(q_c) = 0$. In this case, the marginal profit $\Pi'(\cdot)$ reaches zero at q_c, but the monopolist could achieve higher profits by choosing any output level greater than q_c, since $\Pi'(q) > 0$ for $q \neq q_c$. Therefore, a more economically meaningful range for c lies above this threshold, allowing the first-order condition $\Pi'(q) = 0$ to yield two equilibrium solutions. Proposition 3 further details the characteristics of these equilibrium solutions.

Since the second-order condition (2) is always satisfied when $d \geq 0$, ensuring that the firm achieves maximum profit at the equilibrium, we will focus exclusively on the case where $d < 0$ in the discussion below. By Proposition 2, when $d < 0$, the static model may have two equilibria. In such cases, we denote these two equilibria as q_*^s and q_*^b with $q_*^s < q_*^b$. This notation will be used consistently throughout the remainder of the paper.

Proposition 3. *Let $d < 0$. If the static model has two equilibria, q_*^s and q_*^b with $q_*^s < q_*^b$, then the following holds:*

$$\Pi''(q_*^s) < 0, \quad \Pi''(q_*^b) > 0.$$

Proposition 3 characterizes two possible equilibrium solutions for the monopolist within a reasonable range of the cost parameter c: the smaller equilibrium output q_*^s and the larger equilibrium output q_*^b. The monopolist achieves a local maximum profit at the smaller equilibrium q_*^s while at the larger output level q_*^b, the monopolist achieves a local minimum profit. Therefore, if the monopolist is a profit maximizer, it would naturally select the smaller output level q_*^s over q_*^b.

Moreover, the larger equilibrium q_*^b is economically impractical because the marginal profit at output levels greater than q_*^b remains positive ($\Pi'(q) > 0$ for $q > q_*^b$). This implies that the monopolist could continuously increase profits by expanding output beyond q_*^b, leading to unbounded profit growth, or a "profit explosion". Consequently, q_*^s is the profit-maximizing equilibrium, while q_*^b is unstable and undesirable from an economic standpoint. Therefore, in the subsequent analysis, we focus exclusively on the economically meaningful equilibrium solution, q_*^s.

Since the equilibrium condition $\Pi'(q_*) = 0$ establishes a relationship between the price elasticity of demand e and the equilibrium output q_*, we can perform total differentiation on both sides of the equation to derive the following lemma.

Lemma 1. *The differentials* $\mathrm{d}e$ *(price elasticity of demand) and* $\mathrm{d}q_*$ *(equilibrium output) satisfy the relation:*

$$\frac{\partial \Pi'(q_*)}{\partial e}\mathrm{d}e + \frac{\partial \Pi'(q_*)}{\partial q_*}\mathrm{d}q_* = 0, \tag{3}$$

where the partial derivatives are given by:

$$\frac{\partial \Pi'(q_*)}{\partial e} = \frac{\left((e-1)\ln q_* + e\right) a q_*^{-1/e}}{e^3}, \tag{4}$$

$$\frac{\partial \Pi'(q_*)}{\partial q_*} = -\frac{e-1}{e^2}\frac{a}{q_*^{1+1/e}} - d. \tag{5}$$

Lemma 1 is a zero-change condition or an equilibrium adjustment for marginal profit. It implies that any change in the price elasticity of demand e must be offset by a corresponding change in the equilibrium output quantity q_* to maintain the monopolist's marginal profit constant in equilibrium. The monopolist continuously adjusts its production in response to shifts in the price elasticity of demand, ensuring that marginal profit is preserved despite changing market conditions.

Lemma 1 indicates that the monopolist treats elasticity as a crucial factor in optimizing its production decisions. Since price elasticity of demand directly affects the marginal profit, the monopolist adjusts its output in response to changes in elasticity to maintain profit maximization. For example, if $\dfrac{\partial \Pi'(q_*)}{\partial e}\mathrm{d}e < 0$, this implies that an increase in elasticity e causes a negative change in marginal profit. The monopolist must reduce its equilibrium output

($\mathrm{d}q_* < 0$) to prevent profit erosion, provided that $\dfrac{\partial \Pi'(q_*)}{\partial q_*} < 0$. In this scenario, as demand becomes more elastic, consumers react more sensitively to price changes, making it harder for the monopolist to maintain high prices without losing a substantial portion of demand. To counteract this, the monopolist decreases output, thereby reducing supply-side pressure and attempting to stabilize marginal profit. The response is more pronounced when the returns to scale are decreasing ($d > 0$), as marginal costs rise more steeply with higher output levels.

Next, we analyze the effect of the price elasticity of demand e on the equilibrium output q_* that satisfies the second-order condition.

Proposition 4. *Let q_* denote the equilibrium that satisfies the second-order condition $\Pi''(q_*) < 0$. If $\ln q_* < -\dfrac{e}{e-1}$, then q_* decreases with e. Otherwise, if $\ln q_* > -\dfrac{e}{e-1}$, then q_* increases with e.*

When $\ln q_* < -\dfrac{e}{e-1}$, monopoly firms of this type are best suited for niche market strategies, characterized by high prices and low supply volumes. The defining characteristic of these firms is that their marginal cost decreases more slowly with increasing supply volume than the price does, or even increases with supply volume. For such firms, large supply volumes are a "burden" for their profit maximization. These firms primarily achieve profit maximization through higher average profit margins. Consumers' higher sensitivity to price changes benefits these firms by allowing them to reduce total demand more significantly while increasing prices, thereby "saving" more costs and enhancing overall profit.

On the contrary, when $\ln q_* > -\dfrac{e}{e-1}$, this proposition indicates that such type of monopoly firms are ideally positioned to adopt a high-volume, low-margin sales strategy. They maximize profits mainly through substantial sales volumes combined with lower average profit margins. Consumers' higher sensitivity to price changes benefits these firms by significantly boosting demand volumes with slight price reductions, thereby enhancing total profits.

Corollary 1. *Let $d = 0$. If $a/c \geq 1$, then q_* increases with e. If $a/c < 1$, q_* first increases with e, but then decreases with e.*

It is important to note that the conclusion in Corollary 1 aligns with the findings of Cavalli and Naimzada [13] regarding the effect of price elasticity of demand on equilibrium output. For further economic implications of Corollary 1, the reader is encouraged to refer to [13].

Proposition 5. *Let q_* be the equilibrium that satisfies the second-order condition $\Pi''(q_*) < 0$. If $a > edq_*^{1+1/e} \ln q_*$, then the equilibrium price p_* decreases with e. Otherwise, if $a < edq_*^{1+1/e} \ln q_*$, p_* increases with e.*

If $d \leq 0$ and $\ln q_* > 0$ (i.e., $q_* > 1$), a condition commonly observed among monopolistic firms in the digital industry, the average production cost decreases as equilibrium output rises. In this scenario, regardless of the value of a, which represents the monopoly's pricing power or the market size, it follows that $a > edq_*^{1+1/e} \ln q_*$. Under these circumstances, an increase in consumer sensitivity to price changes enables the firm to achieve significant sales growth through slight price reductions. This, in turn, further reduces average costs and boosts total profits.

Under diseconomies of scale ($d > 0$), the average production cost increases as output rises. For a monopolistic firm engaged in mass production, where $\ln q_* > 0$, the firm's pricing power (or market scale index) significantly influences the equilibrium price dynamics. If the firm's pricing power a is sufficiently high (specifically, $a > edq_*^{1+1/e}\ln q_*$), the average profit margin is substantial enough to offset the rise in marginal costs driven by increased demand. In such a case, greater consumer price sensitivity allows the firm to maximize net profits through increased sales growth, achieved with only small price reductions. Conversely, when the firm's pricing power a is relatively low (precisely, $a < edq_*^{1+1/e}\ln q_*$), the firm's average profit margin is thinner, making it challenging to counterbalance the rising marginal costs brought by high demand. In this situation, as consumers become more sensitive to price changes, the firm must increase prices to suppress demand growth and offset rising marginal costs to maximize net profit.

For a monopolistic firm operating in a niche market, where $d \leq 0$ and $\ln q_* < 0$, as seen in many high-end customized product industries, the firm has dominating pricing power within this niche sector regardless of the absolute value of a. Therefore, when consumer price sensitivity increases, the firm can achieve significant sales growth through minimal price reductions, thereby maximizing net profit. Examples include luxury fashion brands and high-end electronics companies. For instance, Hermès, Louis Vuitton, and Chanel are prime examples of monopolistic firms dominating the high-end fashion market. These brands cater to niche markets by offering highly exclusive, customized products such as luxury handbags, watches, and apparel. During the 2008 financial crisis, some of these brands employed small price reductions or increased promotional activities. For example, Louis Vuitton experienced heightened demand as aspirational middle-class consumers responded favorably to marginally lower prices.[1] Similarly, Apple Inc. operates in a niche segment of high-end electronics, benefiting from constant or increasing returns to scale due to its ability to leverage economies of scale in production. In 2019, Apple reduced its prices in China in response to weakened demand, resulting in a significant surge in iPhone sales.[2]

When $d = 0$, it follows that $a > edq_*^{1+1/e}\ln q_*$. By Proposition 5, an increase in e results in a decrease in the equilibrium price p_*. This leads to the following corollary.

Corollary 2. *If $d = 0$, then the equilibrium price p_* decreases with e.*

Corollary 2 confirms the conclusions on equilibrium price presented by Cavalli and Naimzada [13]. Readers may refer to [13] for a detailed economic interpretation of Corollary 2. Moreover, Proposition 5 extends the findings of Cavalli and Naimzada [13] to the case where $d \neq 0$.

Proposition 6. *Let q_* be the equilibrium that satisfies the second-order condition $\Pi''(q_*) < 0$ and $\Pi_* = \Pi(q_*)$. If $q_* < 1$, then the equilibrium profit Π_* decreases with e. Otherwise, if $q_* > 1$, then Π_* increases with e.*

Proposition 6 implies that, when $q_* < 1$, monopoly firms of this type are constrained by factors such as market size, pricing power, or costs, and typically adopt a niche strategy characterized by high prices and low sales volumes. In this situation, when consumers' sensitivity to price changes increases, total demand decreases,

[1] The details of Louis Vuitton's pricing strategy during the 2008 financial crisis are discussed in The Wall Street Journal. See: https://www.wsj.com/articles/SB123387498708754283.

[2] Apple's pricing strategy in China during 2019 is reported in The Wall Street Journal. See: https://www.wsj.com/articles/apples-answer-to-slower-iphone-sales-getting-customers-to-trade-in-11546958562.

causing prices to drop. This leads to a situation where the firm's marginal revenue falls short of marginal cost, forcing the firm to significantly reduce production to mitigate the sharp decline in prices and maintain balance with marginal costs. Consequently, the firm's average revenue decreases along with the sales volume, resulting in a decline in total net profit.

Conversely, when $q_* > 1$, these firms benefit from economies of scale in terms of their market size or pricing power relative to their costs, making them well-suited for a high-volume, low-margin strategy. As consumers' sensitivity to price changes increases, the total market demand for the output also rises, leading to higher prices. In response, firms will expand their production capacity to meet the growing consumer demand. At the same time, higher price sensitivity means that even small price reductions by the firm can significantly boost market demand, thereby substantially increasing the firm's revenue and achieving profit growth.

When $d = 0$, the closed form of the equilibrium output q_* can be derived by solving the first-order condition $\Pi'(q_*) = 0$, which yields

$$q_* = \left[\frac{(e-1)a}{ec}\right]^e.$$

Hence, $q_* > 1$ if and only if $\dfrac{(e-1)a}{ec} > 1$, which is equivalent to $e > \dfrac{a}{a-c}$. From Proposition 6, we immediately obtain the following corollary.

Corollary 3. *Let $d = 0$. If $e < \dfrac{a}{a-c}$, then the equilibrium profit Π_* decreases with e. Otherwise, if $e > \dfrac{a}{a-c}$, then Π_* increases with e.*

5 Dynamic Models

In this section, we introduce two dynamic models, each utilizing a different mechanism for adjusting the output. Let $t+1$ denote the current period and t the previous period.

5.1 Model G

In this model, the monopolist adopts a gradient mechanism for output adjustment. Specifically, the firm determines its output in the current period $t+1$ based on the following rule:

$$q(t+1) = q(t) + k\Pi'(q(t)),$$

where $k > 0$ is a parameter that controls the adjustment speed. In other words, the firm adjusts its output based on the marginal profit observed in the previous period. This dynamic monopoly game can be represented by the following one-dimensional discrete map:

$$q(t+1) = F_G(q(t)) = q(t) + k\left[(1-1/e)\frac{a}{q^{1/e}(t)} - (c + dq(t))\right]. \tag{6}$$

The derivative of the reaction function is given by:

$$F_G'(q) = (1 - kd) - k\frac{a(e-1)}{e^2 q^{1/e+1}}.$$

Thus, if $1 - kd \leq 0$, the reaction function $F_G(q)$ is monotonically decreasing. Conversely, if $1 - kd > 0$, the function initially decreases and then increases as q increases. In this case, the minimum value of $F_G(q)$ is achieved when $F_G'(q_m) = 0$, where q_m can be calculated as:

$$q_m = \left[\frac{ka(e-1)}{(1-kd)e^2}\right]^{\frac{e}{e+1}}.$$

To ensure $F_G(q) > 0$, it suffices to assume $1 - kd > 0$ and $F_G(q_m) > 0$. In fact, the reaction function $F_G(q)$ can be rewritten as:

$$F_G(q) = (1 - kd)q + k\left[(1 - 1/e)\frac{a}{q^{1/e}} - c\right].$$

From this expression, it can be derived that as long as the initial marginal cost c is sufficiently small relative to the market size a, it is possible to guarantee $F_G(q) > 0$ under the condition $1 - kd > 0$. Therefore, in Model G, we assume that $1 - kd > 0$.

Setting $q(t+1) = q(t) = q_*$ in map (6) yields the equilibrium condition $\Pi'(q_*) = 0$, expressed as:

$$(1 - 1/e)\frac{a}{q_*^{1/e}} = c + dq_*.$$

Clearly, the equilibrium of Model G is consistent with that of the static model discussed in Section 3.

5.2 Model L

In Model L, the monopolist is also boundedly rational due to a lack of complete information about the demand function. The firm can only estimate future market demand and profits to decide its output. The boundedly rational adjustment mechanism in the model is referred to as the *local monopolistic approximation* (LMA), originally introduced by Silvestre [38]. This mechanism has been extensively applied in the literature by Tuinstra [41], Bischi et al. [11], Cavalli and Naimzada [32], Andaluz et al. [5], among others.

Specifically, we assume that the monopolist lacks precise knowledge of the market demand function's exact form. However, the firm can compute[3] the slope of the inverse demand function from the previous period t, which is given by:

$$\frac{\partial P(t)}{\partial q(t)} = \frac{-a/e}{q^{1/e+1}(t)}.$$

Based on this information, the firm can perform a linear estimation of the product price at period $t+1$, expressed

[3]This assumption is reasonable, as the slope can be obtained through standard business practices such as commercial experiments and market research.

as:

$$P^e(t+1) = \frac{a}{q^{1/e}(t)} + \frac{-a/e}{q^{1/e+1}(t)}(q(t+1) - q(t)).$$

Consequently, the firm's expected profit for the current period $t+1$ is given by:

$$\Pi^e(t+1) = q(t+1)\left[\frac{a}{q^{1/e}(t)} + \frac{-a/e}{q^{1/e+1}(t)}(q(t+1) - q(t))\right] - \left(f + cq(t+1) + \frac{1}{2}dq^2(t+1)\right).$$

To maximize $\Pi^e(t+1)$, we derive the first-order condition $\frac{\partial \Pi^e(t+1)}{\partial q(t+1)} = 0$, which simplifies to:

$$\left[\frac{a}{q^{1/e}(t)} + \frac{-a/e}{q^{1/e+1}(t)}(q(t+1) - q(t))\right] + q(t+1)\frac{-a/e}{q^{1/e+1}(t)} - c - dq(t+1) = 0,$$

and the second-order condition:

$$\frac{\partial^2 \Pi^e(t+1)}{\partial q(t+1)^2} = \frac{-2a/e}{q^{1/e+1}(t)} - d < 0.$$

However, this second-order condition may not hold when $d < 0$. Hence, we restrict our analysis to the case of $d \geq 0$ for this dynamic model.

If $d \geq 0$, the second-order condition is always satisfied. Hence, the firm's best response function can be obtained by solving the first-order condition $\frac{\partial \Pi^e(t+1)}{\partial q(t+1)} = 0$, resulting in:

$$q(t+1) = F_L(q(t)) = \frac{a(1/e+1)q(t) - cq^{1/e+1}(t)}{dq^{1/e+1}(t) + 2\,a/e}. \tag{7}$$

It is important to note that the reaction function $F_L(q)$ in map (7) may yield negative values. To ensure that $q(t+1) \geq 0$, we consider the map:

$$q(t+1) = \max\left(0, F_L(q(t))\right), \tag{8}$$

which transforms any state where $F_L(q(t)) < 0$ to the zero equilibrium. However, as discussed in Section 6.2, the zero equilibrium is unstable. Moreover, map (8) shares the same non-vanishing equilibrium with map (7). For simplicity, the subsequent analysis will focus on map (7).

Setting $q(t+1) = q(t) = q_*$ in map (7) yields the equilibrium condition:

$$q_* = \frac{a(1/e+1)q_* - cq_*^{1/e+1}}{dq_*^{1/e+1} + 2a/e}.$$

Evidently, $q_* = 0$ is one solution to this equilibrium equation. For the case where $q_* \neq 0$, it can be verified that:

$$1 = \frac{a(1/e+1) - cq_*^{1/e}}{dq_*^{1/e+1} + 2a/e},$$

which is equivalent to $\Pi'(q_*) = 0$. Thus, the non-vanishing equilibrium of Model L is characterized by $\Pi'(q_*) = 0$, meaning that it is the same as the equilibrium of the static model.

6 Local Stability and Bifurcation Analysis

In this section, we analyze the local (asymptotical) stability and local bifurcations of the two dynamic models introduced earlier. Our analysis reveals significant differences in the dynamics of Models G and L.

6.1 Model G

We begin by considering the case of decreasing or constant returns to scale, represented by $d \geq 0$.

Theorem 1. *Assume that returns to scale are decreasing or constant, i.e., $d \geq 0$.*

(1) The unique equilibrium of Model G, i.e., map (6)*, is locally stable if*

$$c < (1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1,$$

where

$$q_1 = \left[\frac{(e-1)a}{e^2\,(2/k - d)}\right]^{\frac{e}{e+1}}.$$

(2) Furthermore, a period-doubling bifurcation may occur if

$$c = (1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1.$$

We now provide a brief outline of the proof for Theorem 1. Details of the proof can be found in Appendix. We start the proof with the condition for the equilibrium q_* to be locally stable. Then we simplify the condition under the settings that $k > 0$, $e > 1$, $a > 0$, $q_* > 0$, $d \geq 0$, and $1 - kd > 0$. The period-doubling bifurcation can be viewed as the case where the boundary condition holds. The proofs of Theorems 2 and 3 are similar to that of the above theorem.

We provide further comments on Theorem 1 below.

Remark 1. Let $d \geq 0$. The smaller the value of c, the more stable Model G becomes. This behavior differs significantly from the case where $d < 0$, (as discussed in Remark 3). Furthermore, one can verify that $\lim_{e\to+\infty} q_1 = 0$ and $\lim_{e\to+\infty} q_1^{1/e} = 1$. Hence, we obtain:

$$\lim_{e\to+\infty}\left((1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1\right) = a.$$

This implies that, for sufficiently large price elasticity of demand, the stability of Model G can be ensured as long as $c < a$. Figure 2 illustrates the stable parameter region (e, c) for Model G with the parameters set to $a = 1/5$, $d = 1/2$, and $k = 1$. From Figure 2, we observe that when $c > a$, the model is either always unstable, or stable only for moderate values of e. In other words, excessively high or low values of e may result in a loss of stability for the model when $c > a$.

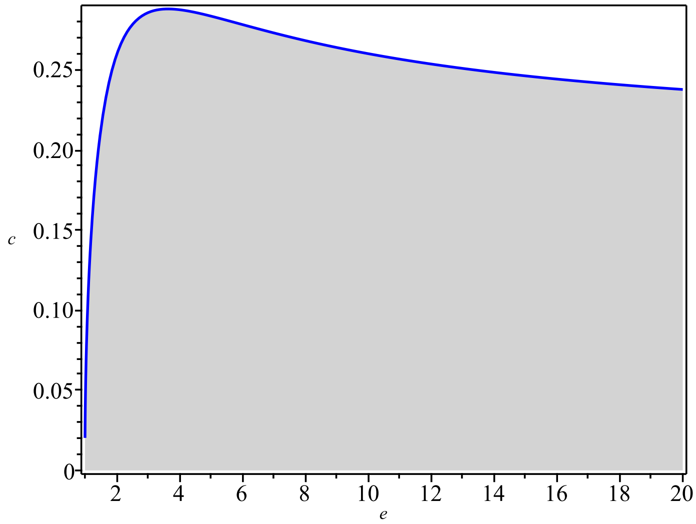


Figure 2: The region of parameters (e, c) for the stability of Model G when $d = 1/2$, with the other parameters set to $a = 1/5$ and $k = 1$. The stability region is marked in grey and the blue curve is defined by $c = (1-1/e)\dfrac{a}{q_1^{1/e}} - dq_1$.

Remark 2. Let $d \geq 0$. It can be verified that smaller values of k or d lead to a smaller q_1 in Theorem 1, which in turn increases $(1 - 1/e)\dfrac{a}{q_1^{1/e}} - dq_1$. Consequently, the condition

$$c < (1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1$$

is more easily satisfied when k or d is smaller. In other words, Model G becomes more stable with smaller values of k or d. The adjustment speed k acts as a destabilizing factor in the dynamic model with a gradient adjustment mechanism, aligning with findings in existing literature (e.g., [5, 31]). However, the impact of d on the model's stability presents a different outcome compared to the results of Fisher [21], and McManus and Quandt [29] in oligopolistic competition. Specifically, Fisher [21], McManus and Quandt [29] found that in oligopoly games with a linear demand function, a faster rate of decreasing returns to scale (i.e., a larger d) increases the equilibrium stability. This contrast sheds light on the intrinsic differences between monopoly models and oligopoly models.

We now turn to the case of economies of scale.

Theorem 2. *Assume economies of scale, i.e., $d < 0$.*

(1) Model G, represented by map (6)*, has a locally stable equilibrium (specifically, only the smaller of the two equilibria is stable) if*

$$(1 - 1/e)\frac{a}{q_c^{1/e}} - dq_c < c < (1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1,$$

where

$$q_c = \left[\frac{(e-1)a}{-e^2 d}\right]^{\frac{e}{e+1}}, \quad q_1 = \left[\frac{(e-1)a}{e^2\,(2/k - d)}\right]^{\frac{e}{e+1}}.$$

(2) A fold bifurcation may occur if

$$c = (1 - 1/e)\frac{a}{q_c^{1/e}} - dq_c.$$

(3) Furthermore, a period-doubling bifurcation may occur if

$$c = (1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1.$$

Remark 3. Let $d < 0$. Theorem 2 suggests that both excessively large or small values of c can destabilize Model G. Further calculations show that:

$$\lim_{e \to 1} \left((1 - 1/e)\frac{a}{q_c^{1/e}} - dq_c \right) = \lim_{e \to 1} \left((1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1 \right) = 0,$$

indicating that as $e \to 1$, the probability that locally stable equilibria exist in Model G tends towards zero if the other parameters a and d are set randomly. In addition, calculations also yield:

$$\lim_{e \to +\infty} \left((1 - 1/e)\frac{a}{q_c^{1/e}} - dq_c \right) = \lim_{e \to +\infty} \left((1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1 \right) = a,$$

implying that as $e \to +\infty$, the probability that locally stable equilibria exist also approaches zero if the parameters a and d are chosen randomly. In other words, under economies of scale, medium-sized values of price elasticity are more conducive to the stability of the model. Figure 3 depicts the stability region for parameters(e, c) in Model G, with the parameters set to $a = 1/5$, $d = -1/2$, and $k = 1$.

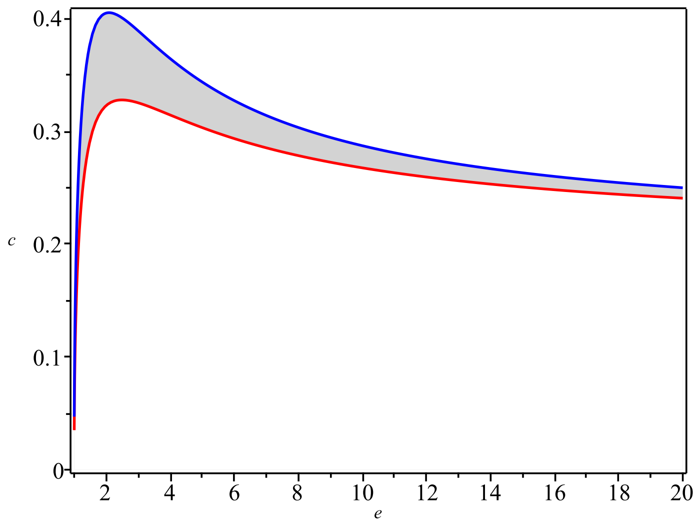


Figure 3: The region of parameters (e, c) for the stability of Model G when $d = -1/2$, with the other parameters set to $a = 1/5$ and $k = 1$. The stability region is marked in grey; the red and blue curves defined by $c = (1 - 1/e)\frac{a}{q_c^{1/e}} - dq_c$ and $c = (1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1$, respectively.

Model G exhibits distinct dynamical behavior in the two scenarios $d \geq 0$ and $d < 0$. When $d \geq 0$, the equilibrium can only undergo period-doubling bifurcations. In contrast, when $d < 0$, Model G may experience both period-doubling and fold bifurcations. The occurrence of fold bifurcations significantly diminishes the stability region of the model. This finding implies that economies of scale are detrimental to the stability of Model G, a result that contradicts the conclusions drawn from oligopoly games by Fisher [21] and McManus and Quandt [29].

6.2 Model L

Recall from Section 5.2 that we restrict our analysis to the case of $d \geq 0$ for Model L. Now, consider an equilibrium q_* of map (7), which is locally stable if $-1 < F_L'(q_*) < 1$. The derivative at the equilibrium can be calculated as:

$$F_L'(q_*) = -\frac{a\,(1+e)\left(dq_*^{\frac{1}{e}+1} + 2\,cq_*^{\frac{1}{e}} - 2\,a\right)}{\left(deq_*^{\frac{1}{e}+1} + 2\,a\right)^2}.$$

We first examine the zero equilibrium. Since $F_L(q)$ is defined for $q \in [0, +\infty)$, we evaluate its right derivative at $q = 0$:

$$\lim_{q \to 0^+} \frac{F_L(q) - F_L(0)}{q} = \frac{e+1}{2} > 1. \tag{9}$$

This result indicates that $q_* = 0$ is an unstable equilibrium of Model L.

Theorem 3. *Assume decreasing or constant returns to scale, i.e., $d \geq 0$. The unique non-vanishing equilibrium of Model L, represented by map* (8)*, is always locally stable.*

Theorem 3 implies that the non-vanishing equilibrium of Model L remains locally stable for all feasible parameter values (i.e., $a > 0$, $e > 1$, $c > 0$, and $d \geq 0$). This result stands in stark contrast to the behavior of Model G, where local stability depends more sensitively on parameter values (see Theorem 1 for details).

Theorem 4. *Assume decreasing or constant returns to scale, i.e., $d \geq 0$. The unique non-vanishing equilibrium of Model L, represented by map* (8)*, is globally asymptotically stable.*

The proof of Theorem 4 employs a divide-and-conquer strategy. More details of the proof can be found in Appendix. We begin by identifying the critical points, which are found by solving the equations $F_L(q) = 0$, $F_L'(q) = 0$ and $F_L(q) = q$. Next, we partition the interval $(0, +\infty)$ based on these critical points and assume that the initial state $q(0)$ falls within one of the resulting intervals. For each interval, we compare the values of $F_L(q)$ and q and analyze the model's dynamic behavior, which verifies the global asymptotic stability of the model.

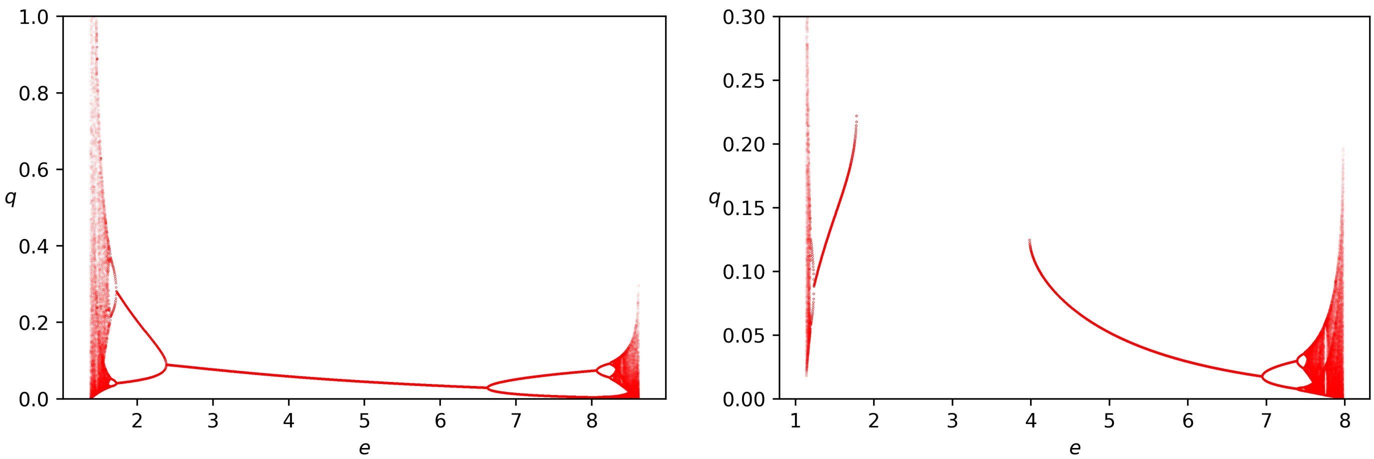
Theorem 4 reveals that the non-vanishing equilibrium of Model L is stable for all possible values of the initial state (i.e., $q(0) \in [0, \infty)$). In other words, the basin of attraction of the non-vanishing equilibrium covers all possible initial states. This is a strong convergence property concerning dynamic economic systems, which is not often proved for models in the existing literature. From an economic point of view, this property means that the trajectory will converge to the non-vanishing equilibrium regardless of the initial belief. By contrast, as discussed in the next section, the equilibrium of Model G is not globally asymptotically stable.

In summary, for a monopoly market characterized by a general isoelastic demand function, the LMA adjustment mechanism is more stable than the gradient adjustment mechanism in both the parameter and state spaces.

7 Numerical Simulations

In this section, we perform numerical simulations to validate the theoretical results derived in the previous section. As indicated Theorems 3 and 4, Model L exhibits strong stability in both the parameter space and the state space. Therefore, our focus here is on the complex dynamics of Model G when the stability conditions are violated due to changes in the price elasticity of demand e and the parameter d.

Figure 4 presents one-dimensional bifurcation diagrams with respect to e, starting from an initial state of $q(0) = 0.1$. Figure 4a depicts the one-dimensional bifurcation diagram of Model G with respect to e under diseconomies of scale ($d = 0.5$), where the other parameters are set to $a = 0.2$, $k = 1$, and $c = 0.275$. In this scenario, the trajectory converges to a stable equilibrium when $2.391795898 < e < 6.616408204$. As e increases or decreases beyond this range, stability is lost through a cascade of period-doubling bifurcations. As e increases, the equilibrium bifurcates into a 2-cycle orbit at $e = 6.616408204$, further bifurcating into a 4-cycle orbit at $e = 8.068384192$, and eventually transitioning to an 8-cycle orbit at $e = 8.240970485$. Conversely, as e decreases, the equilibrium bifurcates into a 2-cycle orbit at $e = 2.391795898$, which further bifurcates into a 4-cycle orbit at $e = 1.720210105$, and then evolves into an 8-cycle orbit at $e = 1.645172586$. Furthermore, the system exhibits periodic orbits of odd orders. For example, a 3-cycle orbit emerges at $e = 1.472586293$, a 5-cycle orbit appears at $e = 1.468834417$, and a 9-cycle orbit can be observed at $e = 1.52136068$.



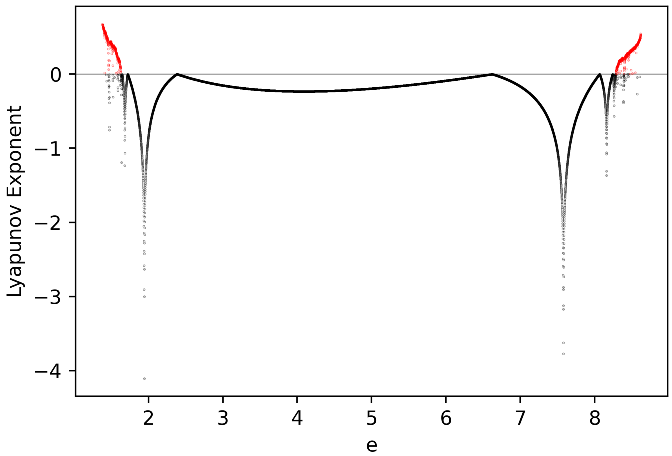
(a) $d = 0.5$

(b) $d = -0.5$

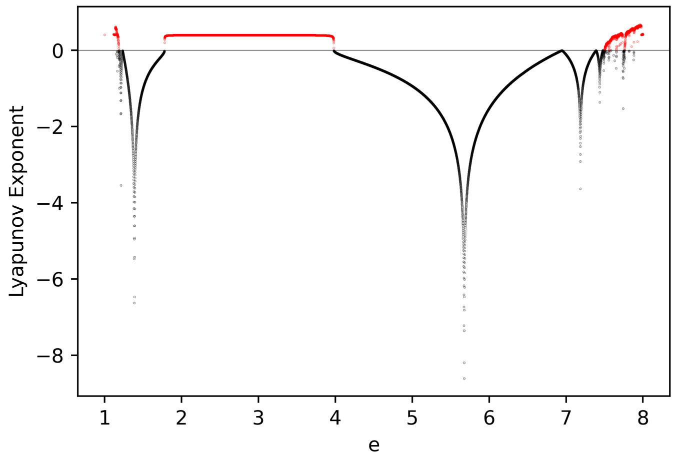
Figure 4: The one-dimensional bifurcation diagrams of Model G with respect to e, where the initial state is chosen as $q(0) = 0.1$. (a) Other parameters are set to $a = 0.2$, $k = 1$, and $c = 0.275$. (b) Other parameters are set to $a = 0.2$, $k = 1$, and $c = 0.315$.

The bifurcations of Model G under economies of scale $d = -0.5$ are illustrated in Figure 4b, where the price elasticity of demand e is the free parameter, while the other parameters are set to $a = 0.2$, $k = 1$, and $c = 0.315$. One can see that the trajectory diverges (approaches ∞) when e takes on medium-size values, i.e., $e \in (1.781927309, 3.973657886)$. Model G undergoes a fold bifurcation at $e = 3.973657886$, and an increase in e leads to the appearance of one stable equilibrium and one unstable equilibrium. As e increases further, Model G is destabilized through a series of period-doubling bifurcations. Specifically, the equilibrium bifurcates into a 2-cycle orbit at $e = 6.947315772$, a 4-cycle orbit at $e = 7.393131044$, and ultimately transitions to chaotic dynamics. During this process, cycles with odd orders also appear. For example, 3-cycle, 5-cycle, and 9-cycle orbits are observed at $e = 7.738579527$, $e = 7.652217406$, and $e = 7.817939313$, respectively. Conversely, Model G also undergoes a fold bifurcation at $e = 1.781927309$, followed by a series of period-doubling bifurcations as e continues to decrease.

Figure 5 plots the Lyapunov exponents of Model G with respect to e, corresponding to Figure 4. The Lyapunov exponent is useful for distinguishing among the various types of orbits as explained in [1, 23]. A negative Lyapunov exponent indicates attraction to a stable fixed point or stable periodic orbit, with greater negativity implying greater stability. In contrast, a positive Lyapunov exponent signifies chaotic behavior, where nearby points, regardless of their initial proximity, diverge to arbitrage separations, and all neighborhoods in the phase space are eventually visited. From Figure 5, we observe that if the parameter e becomes sufficiently large or small, the Lyapunov exponents turn positive, indicating the onset of chaotic dynamics.



(a) $d = 0.5$



(b) $d = -0.5$

Figure 5: The Lyapunov exponents of Model G with respect to e, where the initial state is chosen as $q(0) = 0.1$. (a) Other parameters are set to $a = 0.2$, $k = 1$, and $c = 0.275$. (b) Other parameters are set to $a = 0.2$, $k = 1$, and $c = 0.315$.

Figure 6 reveals more complex dynamics of Model G. This figure depicts a two-dimensional bifurcation diagram of Model G with respect to e and c, with other parameters set to $a = 0.2$, $d = 0.5$, and $k = 1$. The initial state is chosen as $q(0) = 0.1$. In Figure 6, parameter points corresponding to periodic orbits of different orders are represented by various colors. Dark blue points indicate convergence to the equilibrium (1-cycle), while light blue points correspond to convergence to 2-cycle orbits. Yellow points mark regions where trajectories converge to orbits with orders greater than or equal to 24. In addition, grey points indicate parameter values where the orbits diverge (approach ∞). For more information on two-dimensional bifurcation diagrams, readers are referred to [26, 27]. The numerical simulation results on the stability region reported in Figure 6 are consistent with the analytical results given by Theorem 1 (see also Figure 2).

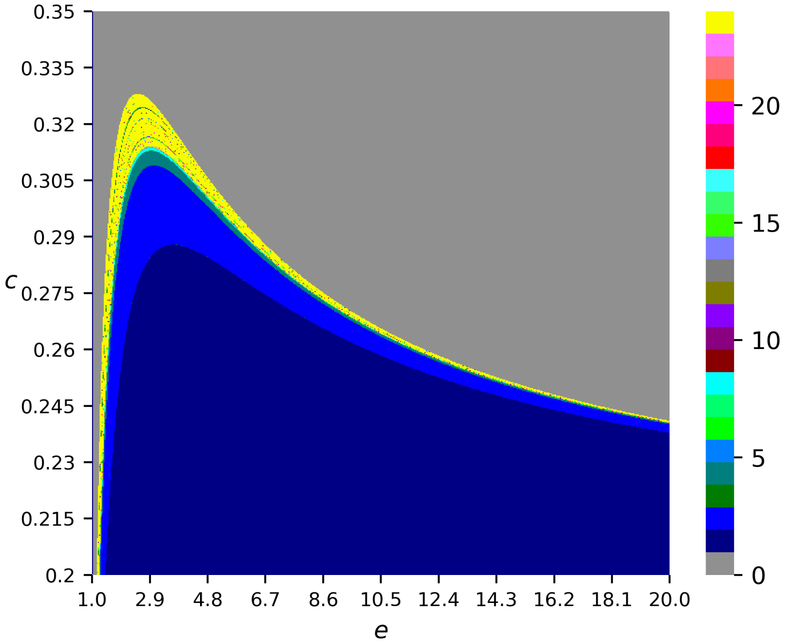


Figure 6: The two-dimensional bifurcation diagram of Model G with respect to e and c. Other parameters are set to $a = 0.2$, $d = 0.5$, and $k = 1$, and the initial state is chosen as $q(0) = 0.1$.

Figure 7 depicts the two-dimensional bifurcation diagram of Model G with respect to e and c under economies of scale, with other parameters set to $a = 0.2$, $d = -0.5$, and $k = 1$. We select the initial state as $q(0) = 0.1$. The numerical simulation results for the stability region, as shown in Figure 7, are in line with the analytical results given by Theorem 2 (see also Figure 3).

Additional numerical simulations for Model G with $q(0) = 0.1$ are reported in Figure 8. Specifically, Figure 8a plots the one-dimension bifurcation diagram of Model G with respect to the market size a, with parameters set to $e = 3$, $d = 0$, $c = 0.3$, and $k = 1$. It is observed that the output increases with a and may lose stability through a cascade of period-doubling bifurcations. Figure 8c shows the one-dimension bifurcation diagram of Model G with respect to the parameter d, where $e = 3$, $a = 0.2$, $c = 0.3$, and $k = 1$ are fixed. The trajectory converges to the equilibrium when d is at a moderate level, while fold bifurcations occur at sufficiently low values of d, and period-doubling bifurcations occur at sufficiently high values of d. Figure 8e illustrates the stability region of Model G in the (e, d) parameter space. The stability region is marked in grey, with fold bifurcation and period-doubling curves represented in red and blue, respectively. It is evident that extreme values of d (either very high or very low) can lead to the destabilization of the output equilibrium. In addition, Figure 8f depicts the two-dimensional bifurcation diagram of Model G with respect to the parameters e and d. This figure confirms the analytical results regarding the stability region given in Figure 8e. If returns to scale are decreasing or constant ($d \geq 0$), Model G can only be destabilized through period-doubling bifurcations as e varies. In contrast, if returns to scale are increasing ($d < 0$), Model G may be destabilized through two different routes: period-doubling bifurcations or fold bifurcations, depending on how e varies.

Moreover, numerical simulations reveal that the locally stable equilibrium of Model G is not necessarily globally asymptotically stable, which is in sharp contrast to the behavior of Model L. For example, when $e = 3$, $a = 0.2$,

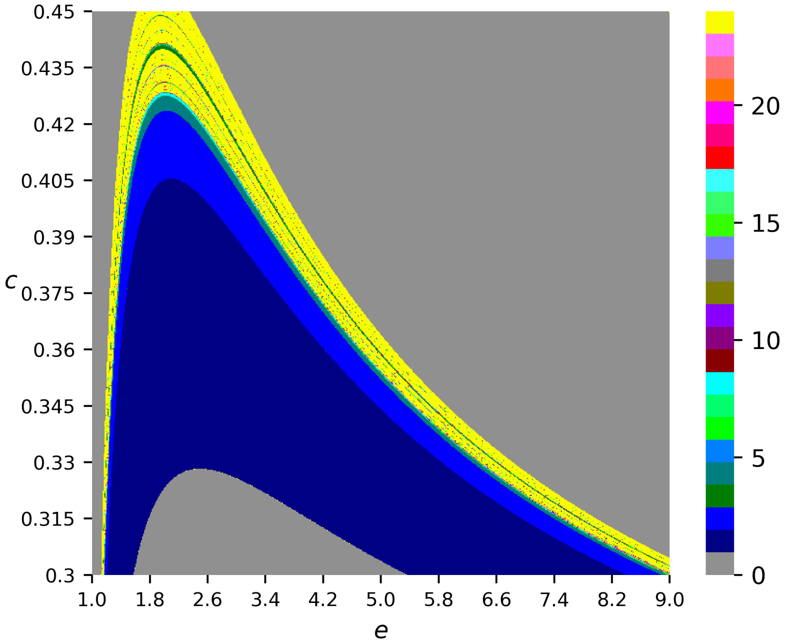


Figure 7: The two-dimensional bifurcation diagram of Model G with respect to e and c. Other parameters are set to $a = 0.2$, $d = -0.5$, and $k = 1$, and the initial state is chosen as $q(0) = 0.1$.

$d = -1$, $k = 1$, and $c = 0.39$, calculations show that the attractors of Model G and their corresponding basins of attraction are:

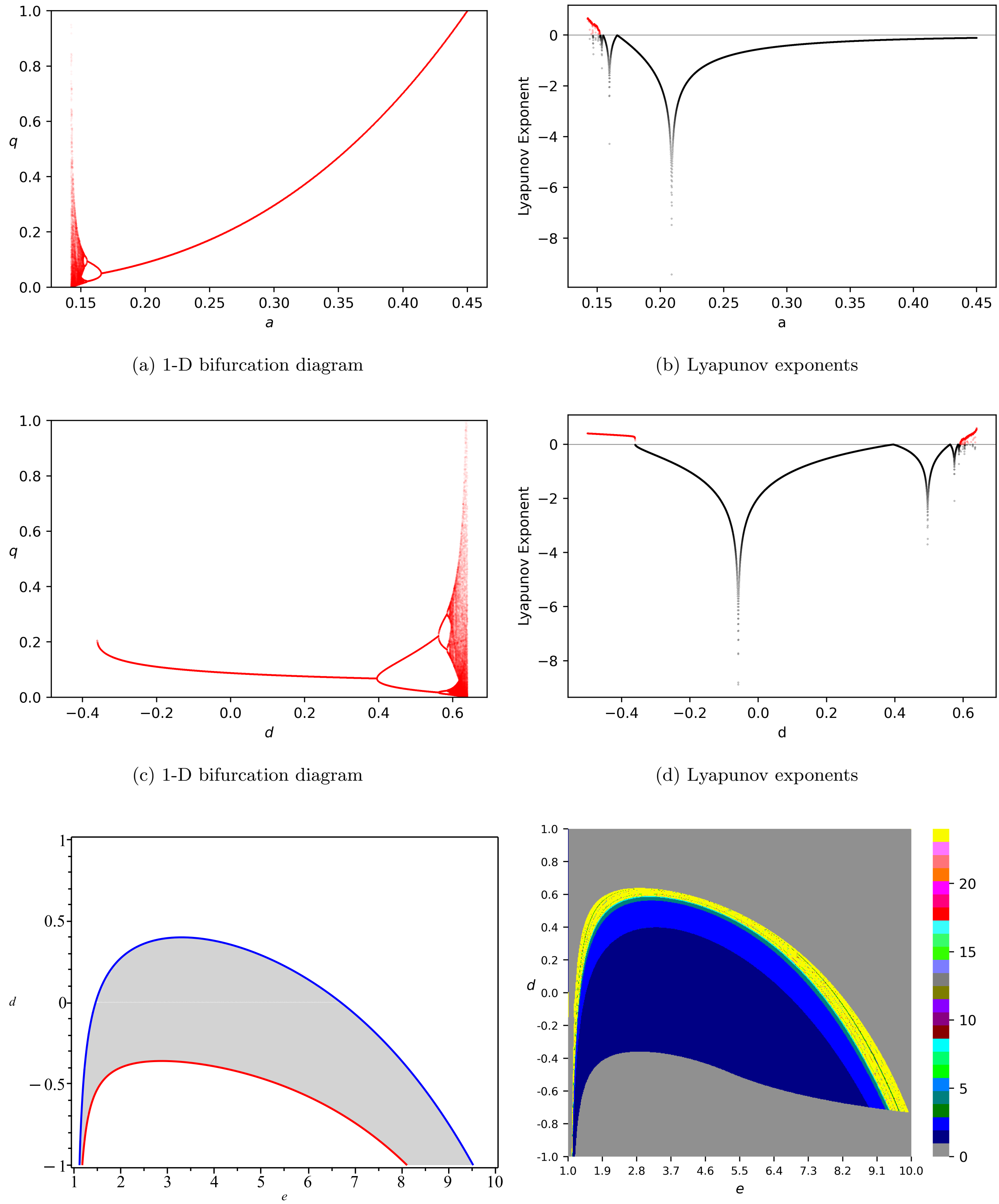
$$\mathcal{B}(0.0782) = (0.0244, 0.1187), \quad \mathcal{B}(\infty) = (0, 0.0244) \cup (0.1187, +\infty).$$

This demonstrates that the equilibrium $q_* = 0.0782$ of Model G is not globally asymptotically stable in this case.

8 Concluding Remarks

This paper examined a monopoly game featured by the general isoelastic demand function. Assuming a quadratic cost function for the monopolist, we studied the monopolistic behavior under decreasing, constant, and increasing returns to scale. We started with a static model, noting that the equilibrium equation is transcendental, which precludes closed-form solutions. To address this challenge, we leveraged the specific structure of the firm's marginal revenue and marginal cost to conduct an equilibrium analysis without relying on closed-form solutions. Our findings show that the static model has a unique equilibrium under decreasing or constant returns to scale (Proposition 1). In contrast, under increasing returns to scale, the model can have at most two equilibria, but only one satisfies the second-order condition (Propositions 2 and 3). We also conducted a comparative static analysis of the equilibrium output, price, and profit, obtaining more general results compared to Cavalli and Naimzada [13] (Propositions 4–6).

Furthermore, we introduced two dynamic models—Model G and Model L—each based on a distinct adjustment mechanism: the gradient adjustment for Model G and the LMA adjustment for Model L. Our analysis revealed that Model L is more stable than Model G in both the parameter and state spaces. More precisely, the non-vanishing equilibrium of Model L is locally asymptotically stable for all feasible parameter values (Theorem 3)



(a) 1-D bifurcation diagram

(b) Lyapunov exponents

(c) 1-D bifurcation diagram

(d) Lyapunov exponents

(e) stability region

(f) 2-D bifurcation diagram

Figure 8: Additional simulation results of Model G with the initial state as $q(0) = 0.1$. (a, b) The one-dimension bifurcation diagram and Lyapunov exponents with respect to a, where $e = 3$, $d = 0$, $c = 0.3$, and $k = 1$. (c, d) The one-dimension bifurcation diagram and Lyapunov exponents with respect to d, where $e = 3$, $a = 0.2$, $c = 0.3$, and $k = 1$. (e) The stability region in the (e, d) parameter space, where $a = 0.2$, $c = 0.3$, and $k = 1$. (f) The two-dimensional bifurcation diagram of Model G with respect to e and d, where $a = 0.2$, $c = 0.3$, and $k = 1$.

and is globally asymptotically stable for all possible initial states (Theorem 4). We also established parametric conditions for the local stability of Model G (Theorems 1 and 2). Interestingly, we found that the impact of the parameter d on the local stability of Model G contrasts with the results obtained in oligopoly games by Fisher [21] and McManus and Quandt [29]. This suggests that extending the current model to oligopoly games with N firms is a promising avenue for future research. An intriguing question is whether the LMA adjustment mechanism would continue to exhibit stronger stability than the gradient adjustment mechanism in such an oligopolistic setting.

Acknowledgements

The authors are grateful to the anonymous referees for their helpful comments. The first author is partially supported by Key Laboratory of Interdisciplinary Research of Computation and Economics (Shanghai University of Finance and Economics) and Innovation Team Project of Guangdong Colleges and Universities (Grant No. 2024WCXTD019). The second author is partially supported by Guangxi Science and Technology Program (Grant No. AD230223001).

Appendix

Proof of Proposition 1

Since

$$R''(q) = -\frac{e-1}{e^2}\frac{a}{q^{1+1/e}} < 0,$$

the marginal revenue $R'(q)$ is strictly decreasing with q. We can also deduce from $C''(q) = d$ that the marginal cost $C'(q)$ is strictly increasing with q when $d > 0$ and is invariant when $d = 0$. Thus, the curve of marginal revenue intersects with that of marginal cost at most once. Note that

$$\lim_{q\to 0^+} \big(R'(q) - C'(q)\big) = +\infty, \quad \lim_{q\to +\infty} \big(R'(q) - C'(q)\big) = \begin{cases} -\infty & \text{if } d > 0; \\ -c & \text{if } d = 0. \end{cases}$$

It indicates that $R'(q) - C'(q)$ has only one zero over $(0, +\infty)$, which is exactly the equilibrium of the static model. Thus, we have proved the existence and uniqueness of the equilibrium under decreasing ($d > 0$) or constant ($d = 0$) returns to scale.

Proof of Proposition 2

Recall (2). We have

$$\Pi''(q) = -\frac{e-1}{e^2}\frac{a}{q^{1+1/e}} - d \qquad \text{and} \qquad \Pi'''(q) = \frac{e-1}{e^2}\cdot\left(1+\frac{1}{e}\right)\cdot\frac{a}{q^{2+1/e}}.$$

Since $e > 1$ and $a > 0$, $\Pi'''(q) > 0$ for $q \in (0, +\infty)$, which implies that $\Pi''(q)$ increases in q over $(0, +\infty)$. Meanwhile, we note that

$$\lim_{q \to 0^+} \Pi''(q) = -\infty, \quad \lim_{q \to +\infty} \Pi''(q) = -d > 0.$$

Hence, $\Pi''(q)$ has a unique zero q_c over $(0, +\infty)$, which is the cut-off value of the output. More explicitly, we have

$$q_c = \left[\frac{(e-1)a}{-e^2 d} \right]^{\frac{e}{e+1}}.$$

When $q < q_c$, $\Pi''(q) < 0$ and thus $\Pi'(q)$ is decreasing. When $q > q_c$, $\Pi''(q) > 0$ and thus $\Pi'(q)$ is increasing. Therefore, the marginal profit function $\Pi'(q)$ achieves its minimum at $q = q_c$ and the minimum value is

$$\Pi'(q_c) = (1 - 1/e) \frac{a}{q_c^{1/e}} - (c + dq_c).$$

According to (1), we derive that $\lim_{q \to 0^+} \Pi'(q) = +\infty$ and $\lim_{q \to +\infty} \Pi'(q) = +\infty$. Now, we consider the following three cases.

- If $\Pi'(q_c) < 0$, i.e., $c + dq_c > (1 - 1/e) \frac{a}{q_c^{1/e}}$, then $\Pi'(q)$ has one zero over each of the intervals $(0, q_c)$ and $(q_c, +\infty)$. Thus, there are two equilibria.
- If $\Pi'(q_c) = 0$, i.e., $c + dq_c = (1 - 1/e) \frac{a}{q_c^{1/e}}$, then $\Pi'(q)$ has only one zero over $(0, +\infty)$, which is $q = q_c$. Thus, there is exactly one equilibrium.
- If $\Pi'(q_c) > 0$, i.e., $c + dq_c < (1 - 1/e) \frac{a}{q_c^{1/e}}$, then $\Pi'(q)$ is always positive over $(0, +\infty)$. Thus, there are no equilibria.

The proof is completed.

Proof of Proposition 3

Assume q_c is the output that minimizes the marginal profit when $d < 0$. According to the proof of Proposition 2, we conclude the following when the static model has two equilibria q_*^s and q_*^b:

- $q_*^s \in (0, q_c)$, $q_*^b \in (q_c, +\infty)$;
- when $q \in (0, q_c)$, $\Pi''(q) < 0$;
- when $q \in (q_c, +\infty)$, $\Pi''(q) > 0$.

In summary, we have $\Pi''(q_*^b) > 0$ and $\Pi''(q_*^s) < 0$.

Proof of Proposition 4

If $\ln q_* < -\frac{e}{e-1}$, by Eq. (4), we have $\frac{\partial \Pi'(q_*)}{\partial e} < 0$. Thus, when e increases, i.e., $\mathrm{d}e > 0$, we have $\frac{\partial \Pi'(q_*)}{\partial e} \mathrm{d}e < 0$. To make (3) an equality, it is required that $\frac{\partial \Pi'(q_*)}{\partial q_*} \mathrm{d}q_* > 0$. Since q_* satisfies the second-order condition, we have

$\dfrac{\partial \Pi'(q_*)}{\partial q_*} < 0$. Hence, we have $\mathrm{d}q_* < 0$, which implies that an increase in e will lead to a decrease in q_*.

Otherwise, if $\ln q_* > -\dfrac{e}{e-1}$, by Eq. (4), we know $\dfrac{\partial \Pi'(q_*)}{\partial e} > 0$. Hence, when e increases, $\dfrac{\partial \Pi'(q_*)}{\partial e}\mathrm{d}e > 0$. To make (3) an equality, it is required that $\dfrac{\partial \Pi'(q_*)}{\partial q_*}\mathrm{d}q_* < 0$. Since $\dfrac{\partial \Pi'(q_*)}{\partial q_*} < 0$, we have $\mathrm{d}q_* > 0$, which means that an increase in e will lead to an increase in q_*.

Proof of Corollary 1

When $d = 0$, one can obtain an analytical expression for the equilibrium q_* by solving the first-order condition $\Pi'(q_*) = 0$, which is

$$q_* = \left[\frac{(e-1)a}{ec}\right]^e.$$

Now, we consider the following two cases.

If $a/c \geq 1$, then $\ln \dfrac{a}{c} \geq 0$. Notice that the inequality $\ln(1+x) \leq x$ holds for all $x > -1$ (where "=" holds if and only if $x = 0$). Thereby,

$$\ln q_* = e\left(-\ln\frac{e}{e-1} + \ln\frac{a}{c}\right) \geq -e\ln\left(1 + \frac{1}{e-1}\right) > -\frac{e}{e-1}.$$

By Proposition 4, an increase in e will lead to an increase in q_*.

If $a/c < 1$, we consider the difference between $\ln q_*$ and $-\dfrac{e}{e-1}$, namely

$$\ln q_* - \left(-\frac{e}{e-1}\right) = e\left[\frac{1}{e-1} - \ln\left(1 + \frac{1}{e-1}\right) + \ln\frac{a}{c}\right].$$

Let $f(x) = x - \ln(1+x) + \ln\dfrac{a}{c}$ where $x \in (0, +\infty)$. Thus,

$$\ln q_* - \left(-\frac{e}{e-1}\right) = ef\left(\frac{1}{e-1}\right).$$

It is obvious that $f(x)$ is monotonically increasing over $(0, +\infty)$ and

$$\lim_{x\to 0^+} f(x) = \ln\frac{a}{c} < 0, \qquad \lim_{x\to +\infty} f(x) = +\infty.$$

Hence, $f(x)$ must have a unique zero over $(0, +\infty)$, denoted by x_*. When $x < x_*$, $f(x) < 0$; when $x > x_*$, $f(x) > 0$. Let $\dfrac{1}{e_* - 1} = x_*$, which yields $e_* = 1 + 1/x_*$. Then, we have the following deduction.

- When $e < e_*$, $\dfrac{1}{e-1} > \dfrac{1}{e_*-1} = x_*$ and thus

$$\ln q_* - \left(-\frac{e}{e-1}\right) = ef\left(\frac{1}{e-1}\right) > 0,$$

  that is, $\ln q_* > -\dfrac{e}{e-1}$. By Proposition 4, q_* increases with e.

- When $e > e_*$, $\dfrac{1}{e-1} < \dfrac{1}{e_*-1} = x_*$, and thus

$$\ln q_* - \left(-\frac{e}{e-1}\right) = ef\left(\frac{1}{e_*-1}\right) < 0,$$

  that is, $\ln q_* < -\dfrac{e}{e-1}$. By Proposition 4, q_* decreases with e.

The proof is completed.

Proof of Proposition 5

We carry out the total differentiation on $P(q)$ and obtain

$$\mathrm{d}P(q) = \frac{\partial P(q)}{\partial e}\mathrm{d}e + \frac{\partial P(q)}{\partial q}\mathrm{d}q,$$

where

$$\frac{\partial P(q)}{\partial e} = \frac{a\ln q}{e^2 q^{1/e}}, \qquad \frac{\partial P(q)}{\partial q} = -\frac{a}{eq^{1+1/e}}.$$

Thus, when $q = q_*$,

$$\mathrm{d}P(q_*) = \frac{a\ln q_*}{e^2 q_*^{1/e}}\mathrm{d}e - \frac{a}{eq_*^{1+1/e}}\mathrm{d}q_*. \tag{10}$$

Recall (4) and (5). After taking their ratio, we have

$$\frac{\mathrm{d}q_*}{\mathrm{d}e} = -\frac{\dfrac{\partial \Pi'(q_*)}{\partial e}}{\dfrac{\partial \Pi'(q_*)}{\partial q_*}} = \frac{\left((e-1)\ln q_* + e\right)aq_*^{-1/e}}{e^3\left(\dfrac{e-1}{e^2}\dfrac{a}{q_*^{1+1/e}} + d\right)}. \tag{11}$$

Plugging the above equation into (10) yields

$$\mathrm{d}P(q_*) = -\frac{aq_*^{-1/e}\left(a - edq_*^{1+1/e}\ln q_*\right)}{e\left(e^2dq_*^{1+1/e} + a(e-1)\right)}\mathrm{d}e.$$

Notice that $e > 1$ and $q_*^{-1/e} > 0$. Moreover, since q_* satisfies the second-order condition, we have $e^2dq_*^{1+1/e} + a(e-1) > 0$. Hence, if $a > edq_*^{1+1/e}\ln q_*$, then $\mathrm{d}e > 0$ implies $\mathrm{d}P(q_*) < 0$, that is, an increase in e will lead to a decrease in the equilibrium price p_*. If $a < edq_*^{1+1/e}\ln q_*$, then $\mathrm{d}e > 0$ implies $\mathrm{d}P(q_*) > 0$, that is, an increase in e will lead to an increase in p_*. The proof is completed.

Proof of Proposition 6

We carry out the total differentiation on the profit function $\Pi(q)$ and obtain

$$\mathrm{d}\Pi(q) = \frac{\partial \Pi(q)}{\partial e}\mathrm{d}e + \frac{\partial \Pi(q)}{\partial q}\mathrm{d}q,$$

where

$$\frac{\partial \Pi(q)}{\partial e} = \frac{aq^{1-1/e}\ln q}{e^2}, \qquad \frac{\partial \Pi(q)}{\partial q} = -\frac{a(1-e)}{eq^{1/e}} - c - dq.$$

Hence, when $q = q_*$,

$$\mathrm{d}\Pi(q_*) = \frac{aq_*^{1-1/e}\ln q_*}{e^2}\mathrm{d}e - \left(\frac{a(1-e)}{eq_*^{1/e}} + c + dq_*\right)\mathrm{d}q_* \tag{12}$$

Recall (11). Plugging it into (12) yields

$$\mathrm{d}\Pi(q_*) = \frac{aq_*A}{e\left(e^2dq_*^{1+1/e} + a(e-1)\right)}\mathrm{d}e,$$

where

$$A = a\left(\ln q_* + 1\right)(e-1)\, q_*^{-1/e} - (c(e-1) - dq_*)\ln q_* - e(c + dq_*).$$

Thus, if $A < 0$, then $\frac{\mathrm{d}\Pi(q_*)}{\mathrm{d}e} < 0$; otherwise, if $A > 0$, then $\frac{\mathrm{d}\Pi(q_*)}{\mathrm{d}e} > 0$.

Notice that the equilibrium q_* satisfies the first-order condition, i.e.,

$$(1 - 1/e)\frac{a}{q_*^{1/e}} - (c + dq_*) = 0. \tag{13}$$

Thus, using the above condition to further simplify A, one can obtain

$$A = (edq_* + c + dq_*)\ln q_*.$$

Therefore, $d \geq 0$ implies $edq_* + c + dq_* > 0$. Below we show that if $d < 0$, the equilibrium q_*^s satisfies $edq_*^s + c + dq_*^s > 0$.

By Proposition 3,

$$\Pi''(q_*^s) = -\frac{e-1}{e}\frac{a}{(q_*^s)^{1+1/e}} - d < 0.$$

We use (13) to simplify the resulting expression above and obtain

$$\Pi''(q_*^s) = -\frac{c/q_*^s + d + ed}{e} < 0.$$

Thus, $c/q_*^s + d + ed > 0$. Since $q_*^s > 0$, we have $c + dq_*^s + edq_*^s > 0$.

To summarize, for any d, it is satisfied that $c + dq_* + edq_* > 0$. Thus, the sign of A is merely determined by that of $\ln q_*$. More explicitly, if $q_* < 1$, then $\ln q_* < 0$, which implies $A < 0$. In this case, an increase in e leads to a decrease in the equilibrium profit Π_*. Whereas, if $q_* > 1$, then $\ln q_* > 0$, which implies $A > 0$. In this case, an increase in e leads to an increase in the equilibrium profit Π_*. The proof is completed.

Proof of Theorem 1

The condition for the equilibrium q_* to be locally stable is $-1 < F'_G(q_*) < 1$, which can be written as

$$-1 < 1 + k\left[-\frac{e-1}{e^2}\frac{a}{q_*^{1+1/e}} - d\right] < 1.$$

Given that $k > 0$, $e > 1$, $a > 0$, $q_* > 0$, and $d \geq 0$, it is evident that

$$1 + k\left[-\frac{e-1}{e^2}\frac{a}{q_*^{1+1/e}} - d\right] < 1.$$

Thus, we only need to consider the condition

$$-1 < 1 + k\left[-\frac{e-1}{e^2}\frac{a}{q_*^{1+1/e}} - d\right],$$

which is equivalent to

$$\frac{e-1}{e^2}\frac{a}{q_*^{1+1/e}} < 2/k - d.$$

According to the assumption of $1 - kd > 0$, we have $2/k - d > 0$. Hence, the stability condition can be transformed into

$$q_* > \left[\frac{(e-1)a}{e^2\,(2/k-d)}\right]^{\frac{e}{e+1}}.$$

Let

$$q_1 = \left[\frac{(e-1)a}{e^2\,(2/k-d)}\right]^{\frac{e}{e+1}}.$$

Hence, the stability condition is simply $q_* > q_1$, meaning that the equilibrium must lie to the right of q_1, which is equivalent to $R'(q_1) > C'(q_1)$ (see Figure 9). In other words, the stability condition is

$$(1 - 1/e)\,\frac{a}{q_1^{1/e}} > c + dq_1,$$

or

$$c < (1 - 1/e)\,\frac{a}{q_1^{1/e}} - dq_1.$$

Furthermore, according to the classical bifurcation theory, a period-doubling bifurcation may occur at the equilibrium q_* of the model when $F'_G(q_*) = -1$, which is equivalent to

$$c = (1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1.$$

Thus, the theorem is proved.

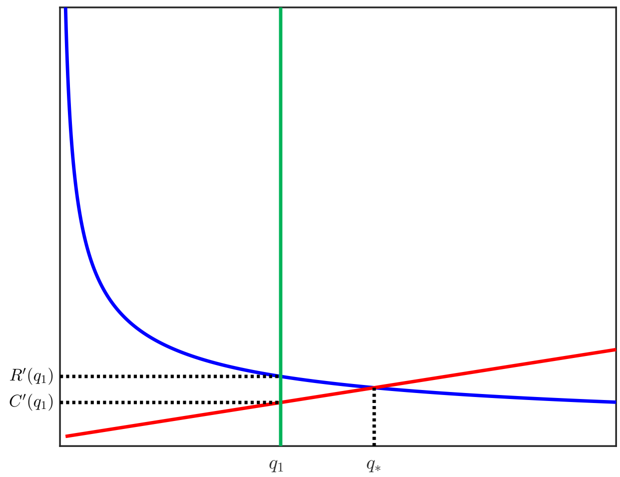


Figure 9: An illustrative diagram depicting the marginal revenue (blue curve) and the marginal cost (red line) in Model G when $d \geq 0$, where the green line represents the locus where $q = q_1$.

Proof of Theorem 2

The condition for the equilibrium q_* to be locally stable is $-1 < F'_G(q_*) < 1$, that is,

$$-1 < 1 + k\left[-\frac{e-1}{e^2}\frac{a}{q_*^{1+1/e}} - d\right] < 1.$$

By Proposition 2, the existence of an equilibrium requires the following condition to be fulfilled:

$$c \geq (1 - 1/e)\frac{a}{q_c^{1/e}} - dq_c.$$

In particular, when $c = (1 - 1/e)\frac{a}{q_c^{1/e}} - dq_c$, there exists a unique equilibrium $q_* = q_c = \left[\frac{(e-1)a}{-e^2 d}\right]^{\frac{e}{e+1}}$. One can verify that this equilibrium satisfies

$$1 + k\left[-\frac{e-1}{e^2}\frac{a}{q_*^{1+1/e}} - d\right] = 1,$$

which implies $F'_G(q_*) = 1$, and therefore Model G may have a fold bifurcation at $q = q_*$.

From now on, we assume that

$$c > (1 - 1/e)\frac{a}{q_c^{1/e}} - dq_c.$$

By Proposition 2, this implies the existence of two equilibria in the model. First, we consider the condition

$$1 + k\left[-\frac{e-1}{e^2}\frac{a}{q_*^{1+1/e}} - d\right] < 1,$$

which is equivalent to $q_* < q_c$, where

$$q_c = \left[\frac{(e-1)a}{-e^2 d}\right]^{\frac{e}{e+1}}.$$

From the proof of Proposition 2, it is known that the smaller equilibrium $q_*^s \in (0, q_c)$, while the larger equilibrium $q_*^b \in (q_c, +\infty)$. Consequently, only the smaller one q_*^s can be a potential locally stable equilibrium.

If the smaller equilibrium q_*^s is locally stable, it must satisfy the condition

$$-1 < 1 + k\left[-\frac{e-1}{e^2}\frac{a}{(q_*^s)^{1+1/e}} - d\right],$$

which simplifies to

$$\frac{e-1}{e^2}\frac{a}{(q_*^s)^{1+1/e}} < 2/k - d.$$

Given that $d < 0$, we have $2/k - d > 0$. Solving the above inequality yields $q_*^s > q_1$, where

$$q_1 = \left[\frac{(e-1)a}{e^2\,(2/k - d)}\right]^{\frac{e}{e+1}}.$$

Figure 10 demonstrates $q = q_1$ and $q = q_c$ with the green and brown lines, respectively. One can see that the smaller equilibrium satisfies $q_*^s > q_1$ if and only if $R'(q_1) > C'(q_1)$, which translates to

$$c < (1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1.$$

By the classical bifurcation theory, Model G may undergo a period-doubling bifurcation when $F_G'(q_*^s) = -1$, which is equivalent to

$$c = (1 - 1/e)\frac{a}{q_1^{1/e}} - dq_1.$$

The proof is completed.

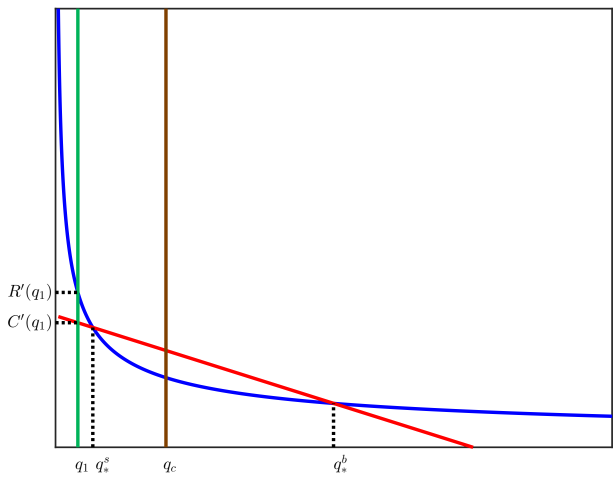


Figure 10: An illustrative diagram depicting the marginal revenue (blue curve) and the marginal cost (red line) in Model G when $d < 0$, where the green and brown lines are defined by $q = q_1$ and $q = q_c$, respectively.

Proof of Theorem 3

The equilibrium q_* is locally stable if and only if

$$-1 < -\frac{a\,(1+e)\left(dq_*^{\frac{1}{e}+1} + 2\,cq_*^{\frac{1}{e}} - 2\,a\right)}{\left(deq_*^{\frac{1}{e}+1} + 2\,a\right)^2} < 1. \tag{14}$$

Since q_* is a non-vanishing equilibrium of Model L, it must satisfy

$$(1-1/e)\frac{a}{q_*^{1/e}} = c + dq_*,$$

which implies

$$q_*^{1/e} = (1-1/e)\frac{a}{c+dq_*}.$$

Substituting the above expression into (14), we obtain the condition for the local stability of q_*:

$$-1 < -\frac{a\,(1+e)\left(d(1-1/e)\dfrac{aq_*}{c+dq_*} + 2\,c(1-1/e)\dfrac{a}{c+dq_*} - 2\,a\right)}{\left(de(1-1/e)\dfrac{aq_*}{c+dq_*} + 2\,a\right)^2} < 1,$$

which simplifies to

$$-1 < (1+1/e)\frac{dq_* + c}{(e+1)dq_* + 2\,c} < 1.$$

Given that $d \geq 0$, $c > 0$, $e > 1$, and $q_* > 0$, we immediately derive that

$$(1+1/e)\frac{dq_* + c}{(e+1)dq_* + 2\,c} > 0 > -1.$$

Next, we examine the condition

$$(1+1/e)\frac{dq_* + c}{(e+1)dq_* + 2\,c} < 1,$$

which is equivalent to

$$(e-1/e)dq_* > (1/e-1)c.$$

Since $e > 1$, we have $e - 1/e > 0$ and $1/e - 1 < 0$, thus the above inequality always holds.

To sum up, the equilibrium always satisfies the condition (14) and thus it is always locally stable.

Proof of Theorem 4

One can verify that $F_L(q) = 0$ when $q = 0$ or $q = \hat{q} \equiv \left[\frac{a(e+1)}{ce}\right]^e$, and $F_L(q) > 0$ when $q \in (0, \hat{q})$. The maximum point q_M of F_L can be obtained by solving $F_L'(q) = 0$. Since

$$F_L'(q) = -\frac{a(e+1)\left(dq^{1/e+1} + 2cq^{1/e} - 2a\right)}{(edq^{1/e+1} + 2a)^2},$$

the maximum point q_M of F_L satisfies

$$dq_M^{1/e+1} + 2\,cq_M^{1/e} - 2\,a = 0. \tag{15}$$

We now prove that the non-vanishing equilibrium q_* of Model L is less than q_M, which is equivalent to proving

$$(1 - 1/e)\frac{a}{q_M^{1/e}} < c + dq_M,$$

or

$$cq_M^{1/e} + dq_M^{1/e+1} - (1 - 1/e)a > 0.$$

From (15), we have

$$\begin{aligned} & cq_M^{1/e} + dq_M^{1/e+1} - (1 - 1/e)a \\ = & cq_M^{1/e} + dq_M^{1/e+1} - (1 - 1/e)a - \frac{1}{2}\left(dq_M^{1/e+1} + 2cq_M^{1/e} - 2a\right) \\ = & \frac{d}{2}q_M^{1/e+1} + \frac{a}{e} > 0, \end{aligned}$$

which implies that $q_* < q_M$.

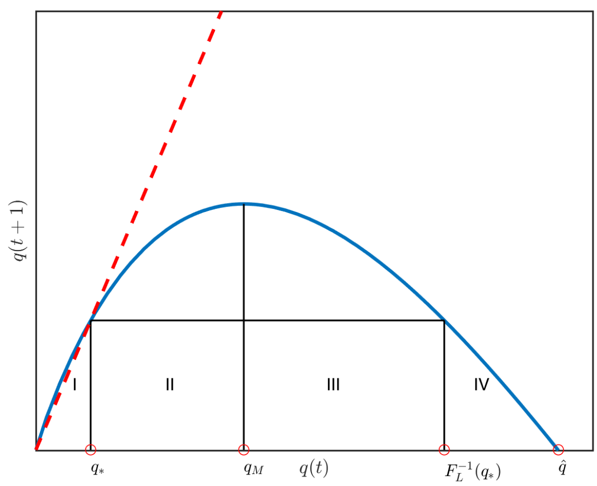


Figure 11: The plot of the reaction function of Model L, where the blue solid curve and red dashed line are defined by $y = F_L(q)$ and $y = q$, respectively.

See Figure 11, where the two intersection points of the curve $y = F_L(q)$ and the line $y = q$ correspond to the zero equilibrium 0 and the non-vanishing equilibrium q_* of Model L. Moreover, we have

$$\lim_{q\to 0^+} \frac{F_L(q)-F_L(0)}{q} = \frac{e+1}{2} > 1,$$

which indicates that the curve $y = F_L(q)$ lies above the straight line $y = q$ near the right side of 0. Therefore, for any $q \in (0, q_*)$, we have $F_L(q) > q$. Notice that $\hat{q}$ is the positive root of $F_L(q)$. In addition, any initial value $q(0)$ for map (8) belongs to one of the following cases:

$$q(0) = 0, \quad q(0) \in (0, q_*), \quad q(0) = q_*, \quad q(0) \in \left(q_*, F_L^{-1}(q_*)\right),$$
$$q(0) = F_L^{-1}(q_*), \quad q(0) \in \left(F_L^{-1}(q_*), \hat{q}\right), \quad q(0) \in [\hat{q}, +\infty).$$

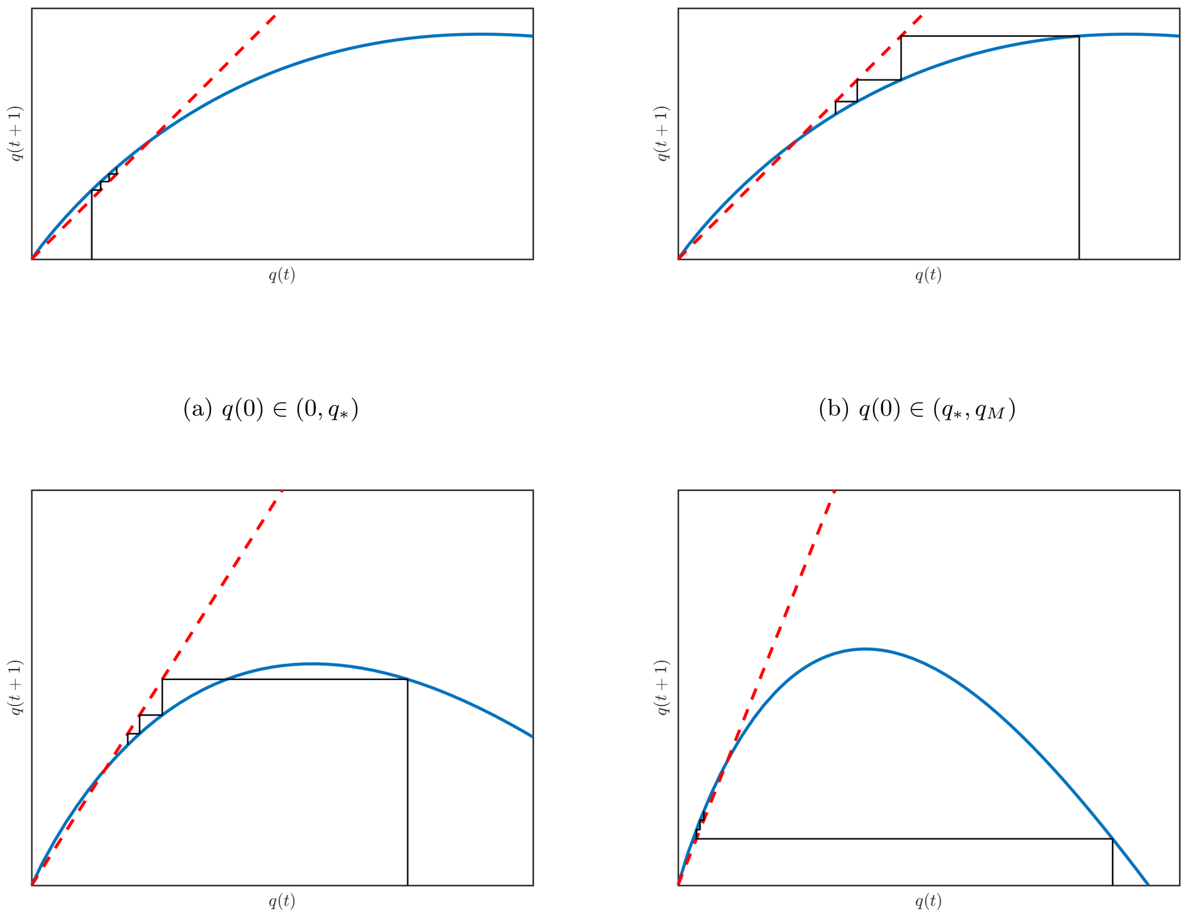
In the sequel, we first address the three boundary cases when $q(0) = 0$, q_*, and $F_L^{-1}(q_*)$, respectively.

- When $q(0) = 0$, it is obvious that 0 is an unstable equilibrium of map (8), and a small perturbation will cause its iteration sequence to fall into the interval $(0, q_*)$.
- When $q(0) = q_*$, by Theorem 3, q_* is a locally stable equilibrium of map (8), thus $q(t) = q_*$ for any $t > 0$.
- When $q(0) = F_L^{-1}(q_*)$, we have $q(1) = q_*$, which means $q(t) = q_*$ for any $t > 1$.

Furthermore, if $q \in [\hat{q}, +\infty)$, then $q(1) = 0$, and the subsequent iteration sequence will be the same as the first case mentioned above. Next, we analyze the stability of the model when $q(0)$ is located in the intervals $(0, q_*)$ (i.e., Region I), (q_*, q_M) (i.e., Region II), $(q_M, F_L^{-1}(q_*))$ (i.e., Region III), and $(F_L^{-1}(q_*), \hat{q})$ (i.e., Region IV), respectively.

- When $q(0) \in (0, q_*)$, $F_L'(q) > 0$, hence $F_L(q) < F_L(q_*) = q_*$. Furthermore, for $q \in (0, q_*)$, $F_L(q) > q$, which implies $q_* > q(t+1) > q(t)$. This indicates that the iteration sequence $q(t)$ is monotonically increasing and bounded above by q_*. Thus, $q(t)$ converges monotonically to q_* (see Figure 12a).
- When $q(0) \in (q_*, q_M)$, $F_L'(q) > 0$, hence $F_L(q) > F_L(q_*) = q_*$. Furthermore, for $q \in (q_*, q_M)$, we have $F_L(q) < q$, which implies $q_* < q(t+1) < q(t)$. This indicates that the iteration sequence $q(t)$ is monotonically decreasing and bounded below by q_*. Thus, $q(t)$ converges monotonically to q_* (see Figure 12b).
- When $q(0) \in (q_M, F_L^{-1}(q_*))$, $q(1) \in (q_*, q_M)$, and the subsequent iteration sequence will behave similarly to the second case mentioned above, thus converging monotonically to q_* (see Figure 12c).
- When $q(0) \in (F_L^{-1}(q_*), \hat{q})$, $q(1) \in (0, q_*)$, and the subsequent iteration sequence will behave similarly to the first case mentioned above, thus converging monotonically to q_* (see Figure 12d).

The proof is completed.



(a) $q(0) \in (0, q_*)$

(b) $q(0) \in (q_*, q_M)$

(c) $q(0) \in (q_M, F_L^{-1}(q_*))$

(d) $q(0) \in (F_L^{-1}(q_*), \hat{q})$

Figure 12: The cobweb charts of Model L for different initial states of $q(0)$, where the blue solid curve and red dashed line are defined by $y = F_L(q)$ and $y = q$, respectively.

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