New Facets of the Clique Partitioning Polytope

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Abstract

The clique partitioning problem is a combinatorial optimisation problem which has many applications. At present, the most promising exact algorithms are those that are based on an understanding of the associated polytope. We present two new families of valid inequalities for that polytope, and show that the inequalities define facets under certain conditions.

Key Words: clique partitioning problem; combinatorial optimisation; polyhedral combinatorics

1 Introduction

In the *clique partitioning problem* (CPP), we are given a complete undirected graph $G = (V, E)$, where V is the set of vertices (or nodes) and E is the set of edges. We are also given a rational weight w_e for each edge $e \in E$. The task is to find a partition of V into cliques, such that the sum of the weights of the edges that have both end-nodes in the same clique is maximised [8, 13]. The CPP has applications in sociology, zoology and economics [8, 13], group technology [20], network analysis [7] and computational biology [2].

At present, the most promising exact algorithms are based on the following integer programming formulation, due to Marcotorchino [13]. For each edge $e = \{i, j\} \in E$, let x_e be a binary variable, taking the value 1 if and only if nodes i and j lie in the same set in the partition. We then have:

$$\begin{array}{lll} \max & \sum_{e \in E} w_e x_e & \\ \text{s.t.} & x_{ik} + x_{jk} - x_{ij} \le 1 & (\{i,j\} \in E,\ k \in V \setminus \{i,j\}) \quad (1) \\ & x_e \in \{0,1\} & (e \in E). \quad (2) \end{array}$$

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The constraints (1) are called *transitivity* inequalities. The convex hull of solutions to (1), (2) is called the *clique partitioning polytope*. Valid inequalities for this polytope can be used to good effect in exact algorithms for the CPP [8, 16, 17, 18, 19].

In this paper, we introduce two new families of valid inequalities for the clique partitioning polytope, and show that they define facets under certain conditions. The paper has a very simple structure. Section 2 reviews the relevant literature, and the remaining two sections present the two families of inequalities, along with the facet proofs.

Throughout the paper, we use the following notation. Given a set $S \subset V$, we let $E(S)$ denote the set of edges with both end-nodes in S. Given disjoint sets $S_1, S_2 \subset V$, we let $E(S_1 : S_2)$ denote the set of edges with one node in S_1 and the other in S_2. For brevity, we also write $E(v : S)$ for $E(\{v\} : S)$.

2 Literature Review

Let P_n denote the clique partitioning polytope of order n. The study of P_n was initiated by Grötschel & Wakabayashi [10]. They showed that the transitivity inequalities define facets, along with the non-negativity inequality $x_e \geq 0$ for all $e \in E$. They then presented four additional families of facet-defining inequalities, called *2-partition*, *2-chorded odd cycle*, *2-chorded even wheel* and *2-chorded path* inequalities. Among them, we will be particularly interested in the first two families of inequalities.

The 2-partition inequalities take the form:

$$\sum_{e \in E(S:T)} x_e - \sum_{e \in E(S)} x_e - \sum_{e \in E(T)} x_e \leq \min\{|S|, |T|\},$$

where S and T are disjoint and non-empty subsets of V. They define facets of P_n if and only if $|S| \neq |T|$.

The 2-chorded odd cycle (2-COC) inequalities take the form

$$\sum_{i=1}^{c} x(v_i, v_{i+1}) - \sum_{i=1}^{c} x(v_i, v_{i+2}) \leq \lfloor c/2 \rfloor, \tag{3}$$

where (a) $c \geq 5$ is an odd integer, (b) $v_1, \ldots, v_c$ are distinct nodes, and (c) indices are taken modulo c.

After the publication of [10], several other families of strong valid inequalities were quickly discovered [1, 3, 4, 6, 9, 14, 18]. Among those, we will be interested in the *odd wheel* (OW) inequalities of Chopra & Rao [4], which always define facets. They take the form:

$$\sum_{k=1}^{c} x(h, v_k) - \sum_{k=1}^{c} x(v_k, v_{k+1}) \leq \lfloor c/2 \rfloor, \tag{4}$$

where (a) $c \geq 3$ is an odd integer, (b) $v_1, \ldots, v_c$ and h are distinct nodes, and (c) indices are taken modulo c. The node h is called the 'hub'.

Sørensen [18] found a family of facet-defining inequalities, called *odd clique wheel* (OCW) inequalities, that includes the OW inequalities as a special case. Let $c \geq 3$ be an odd integer, let h be a node, and let $S_1, \ldots, S_c \subset V \setminus \{h\}$ be disjoint and non-empty node sets. The OCW inequalities take the form:

$$\sum_{k=1}^{c} \sum_{e \in E(h:S_k)} x_e - \sum_{k=1}^{c} \sum_{e \in E(S_k)} x_e - \sum_{k=1}^{c} \sum_{e \in E\left(S_k:S_{k+1}\right)} x_e \leq \lfloor c/2 \rfloor, \tag{5}$$

where, as usual, indices are taken modulo c. Note that these reduce to the OW inequalities (4) when $|S_k| = 1$ for all k.

Still more valid and facet-defining inequalities for P_n can be found in [11, 12, 15, 16]. We do not go into details, for brevity. We will however need the following result. It was first stated explicitly in Section 2 of Bandelt *et al.* [1], but it is also an easy consequence of Theorem 2.9 in Deza *et al.* [6].

Theorem 1 (Trivial Lifting)**.** *If the inequality*

$$\sum_{1 \leq i < j \leq n} \alpha_{ij} x_{ij} \leq \beta$$

is valid (or facet-defining) for P_n, then it is valid (or facet-defining) for $P_{n'}$ for all $n' > n$.

3 Generalised Odd Wheel Inequalities

In this section and the next, we say that a clique is "trivial" if it contains only one node. The following theorem presents a family of valid inequalities that includes the OCW inequalities (5) as a special case.

Theorem 2. *Let $c \geq 3$ be an odd integer, and let H and $S_1, \ldots, S_c$ be disjoint and non-empty node sets. The "generalised odd wheel" (GOW) inequality*

$$\sum_{k=1}^{c} \sum_{e \in E(H:S_k)} x_e - \left\lfloor \frac{c}{2} \right\rfloor \sum_{e \in E(H)} x_e - \sum_{k=1}^{c} \sum_{e \in E(S_k)} x_e$$
$$- \sum_{k=1}^{c} \sum_{e \in E(S_k:S_{k+1})} x_e \leq |H| \left\lfloor \frac{c}{2} \right\rfloor \tag{6}$$

is valid for P_n.

Proof. Let $S' = \bigcup_{i=k}^{c} S_k$. By Theorem 1, we can assume without loss of generality that $V = H \cup S'$. Moreover, one can check that, if $c = 3$, then the inequality reduces to a 2-partition inequality. (It suffices to set S to H and T to S'.) So we can also assume $c \geq 5$.

Now, let x^1 be an extreme point of P_n, and let x^2 be the extreme point of P_{n+1} that one obtains from x^1 by setting $x_{i,n+1}$ to 0 for $i = 1, \ldots, n$ (or, equivalently, placing node $n+1$ in a trivial clique). Also consider any $k \in \{1, \ldots, c\}$. One can check that x^1 violates the GOW inequality (6) if and only if x^2 violates the modified GOW inequality that one would obtain by replacing S_k with $S_k \cup \{n+1\}$. Applying this procedure n times to each of the sets S_k, we obtain an extreme point of $P_{n(c+1)}$ that violates a GOW inequality, yet corresponds to a CPP solution in which at least n nodes of each set S_k lie in a trivial clique. We assume w.l.o.g. that we are working with such a CPP solution from now on.

Now suppose that there is a non-trivial clique C in the solution such that $C \subseteq H$. If we modify the solution by placing each node in C in its own trivial clique, the left-hand side of the GOW inequality increases. Thus, we can assume w.l.o.g. that no such clique exists. For a similar reason, we can assume w.l.o.g. that no non-trivial clique is a subset of S'. So, from now on, we assume that each non-trivial clique contains at least one node from H and at least one node from S'.

Now suppose that there is a non-trivial clique C in the solution, and an index k, such that $C \cap S_k$ and $C \cap S_{k+1}$ are both non-empty. Let $r = |C \cap S_{k-1}|$ and $s = |C \cap S_{k+2}|$. One can check that, if $r > s$, then the left-hand side of the GOW inequality would increase if we modified the solution by (a) removing the nodes in $C \cap S_k$ from C and placing those nodes in trivial cliques, and (b) expanding C back to its original size by adding nodes from $S_{k+1} \setminus C$ that are currently in trivial cliques. Similarly, if $s > r$, then the left-hand side would increase if we removed the nodes in $C \cap S_{k+1}$ from C and replaced them with nodes from $S_k \setminus C$. If $s = r$, we could perform either operation without affecting the left-hand side.

Thus, we can assume w.l.o.g. that no non-trivial clique C intersects two consecutive sets S_k, S_{k+1}. This implies in particular that each non-trivial clique has a non-empty intersection with no more than $\lfloor c/2 \rfloor$ of the sets S_k. In other words, it suffices to show that the GOW inequality is satisfied by all CPP solutions for which each clique that intersects H has a non-empty intersection with no more than $\lfloor c/2 \rfloor$ of the sets S_k.

Now let C be a clique that intersects H. We define

$$T = \Big\{ k \in \{1, \ldots, c\} : \, C \cap S_k \neq \emptyset \Big\}.$$

Note that, from the previous paragraph, we can assume that $|T| \leq \lfloor c/2 \rfloor$.

For $k \in T$, consider the 2-partition inequality

$$\sum_{e \in E(C \cap H : C \cap S_k)} x_e - \sum_{e \in E(C \cap H)} x_e - \sum_{e \in E(C \cap S_k)} x_e \leq |C \cap H|.$$

Summing together the 2-partition inequalities over all $k \in T$, we obtain

$$\sum_{k \in T} \sum_{e \in E(C \cap H : C \cap S_k)} x_e - |T| \sum_{e \in E(C \cap H)} x_e - \sum_{k \in T} \sum_{e \in E(C \cap S_k)} x_e \leq |C \cap H| \, |T|.$$

Since $|T| \leq \lfloor c/2 \rfloor$, the contribution of the edges in $E(C)$ to the left-hand side of (6) cannot exceed $|C \cap H| \lfloor c/2 \rfloor$. Applying the same argument to all of the remaining cliques that intersect H, we find that the total contribution of those cliques to the left-hand side of (6) cannot exceed $|H| \lfloor c/2 \rfloor$. □

Note that the inequalities (6) reduce to the OCW inequalities (5) when $|H| = 1$. We have the following theorem:

Theorem 3. *If*

$$|S_k| > \lfloor |H|/2 \rfloor \qquad (k = 1, \ldots, c), \tag{7}$$

then the GOW inequality (6) defines a facet of P_n.

Proof. As before, let S' denote $\bigcup_{k=1}^{c} S_k$. From Theorem 1, it is enough to prove the result for the case in which $H \cup S' = V$. Moreover, we can assume that $c \geq 5$ since, for $c = 3$, the inequality reduces to a 2-partition inequality. We can also assume w.l.o.g. that $H = \{1, \ldots, h\}$.

Let us say that a CPP solution is a "root" if it satisfies the inequality (6) at equality. As usual in polyhedral proofs, we will suppose that all roots satisfy some linear equation, say $\alpha^T x = \beta$, and then show that this equation is a multiple of the equation that one obtains by changing (6) to an equation.

We start by constructing one specific root, which we will call R. This root will contain h non-trivial cliques, each of cardinality $\lceil c/2 \rceil$. For $i = 1, \ldots, h$, the i-th clique contains node i, together with one node from each of the sets $S_{2-i}, S_{4-i}, \ldots, S_{(c-1)-i}$. Any nodes that have not yet been placed into a clique are then placed into their own trivial clique.

Now, the condition (7) implies that at least $\lceil c/2 \rceil$ of the nodes in S' lie in trivial cliques. Moreover, due to the way that R was constructed, there exists an index k such that each of the sets $S_k, S_{k+2}, \ldots, S_{k+c-1}$ contains at least one trivial clique. This means in particular that, given any integer $t \in \{2, \ldots, \lfloor c/2 \rfloor\}$, we can find sets S_k and S_{k+t} that each contain a trivial clique. Let $\{u\}$ be a trivial clique in S_k and $\{v\}$ be a trivial clique in S_{k+t}. We can obtain a new root by taking R and replacing the cliques $\{u\}$ and $\{v\}$ with the single clique $\{u, v\}$. From this we deduce that $\alpha_{uv} = 0$. By symmetry, this implies that $\alpha_e = 0$ for all $e \in E(S_k : S_{k+t})$. Applying this argument for all t in the given range, we deduce that $\alpha_e = 0$ for all of the edges that have a coefficient of zero in the inequality (6).

Returning to our original root R, observe that there must be a node $u \in H$, a set S_k and nodes $v, w \in S_k$ such that u and v are in the same clique, say C, but node w is in a trivial clique. From this, we can construct another root by removing node v from C, placing node v into a trivial clique, and putting node w into C. This shows that $\alpha_{uv} = \alpha_{uw}$. By symmetry, this also shows that $\alpha_{uv} = \alpha_{uw}$ for all $u \in H$, all $k \in \{1, \ldots, c\}$ and all $v, w \in S_k$.

We can also construct another root from R by placing node w into C. By symmetry, this shows that $\alpha_{vw} = -\alpha_{uw}$ for all $u \in H$, all $k \in \{1, \ldots, c\}$ and all $v, w \in S_k$.

Returning a second time to R, observe that there must be a node $u \in H$, an index k, a clique C, and nodes $v \in S_k$, $w \in S_{k+1}$ such that (a) u and v lie in C, (b) w lies in a trivial clique, and (c) $C \cap S_{k+2} = \emptyset$. From this, we can construct another root by removing node v from C, placing node v into a trivial clique, and putting node w into C. By symmetry, this shows that $\alpha_{uv} = \alpha_{uw}$ for all $u \in H$ and all $v, w \in S'$.

We can also construct another root from R by placing node w into C. By symmetry, this shows that $\alpha_{uw} = -\alpha_{vw}$ for all $u \in H$, all $k \in \{1, \ldots, c\}$, all $v \in S_k$ and all $w \in S_{k+1}$.

The restrictions on α that we have shown so far are already enough to prove the result for the case $h = 1$. To complete the proof, we must deal with the case $h \geq 2$.

Returning a third time to R, let u and v be distinct nodes in H, and let C_1 and C_2 be the cliques to which they belong. We can obtain another root by moving u to C_2 and moving v to C_1. From this we conclude that $\alpha_{uw} = \alpha_{vw}$ for every $w \in S'$. This in turn shows that α_{uw} is constant for all $u \in H$ and $w \in S'$. We can assume w.l.o.g. that the equation $\alpha^T x = \beta$ has been scaled so that this constant is 1.

Finally, observe that, when $h \geq 2$ and condition (7) holds, we have $S_k \geq 2$ for all k. Also recall that R has h non-trivial cliques. We construct one final root from R by (a) deleting the second non-trivial clique and putting all of its nodes into trivial cliques, and (b) enlarging the first non-trivial clique so that it contains nodes 1 and 2, together with *two* nodes from each of the sets $S_1, S_3, \ldots, S_{c-2}$. Comparing this root to R, we see that $\alpha_{12} = -\lfloor c/2 \rfloor$. By symmetry, we have $\alpha_{uv} = -\lfloor c/2 \rfloor$ for all $u, v \in H$. □

It was pointed out to us by an anonymous reviewer that the condition (7) is not necessary to obtain a facet. Indeed, as noted in the proofs of Theorems 2 and 3, the GOW inequalities reduce to 2-partition inequalities when $c = 3$. From that, one can show that the GOW inequalities with $c = 3$ define facets if and only if $|S_1| + |S_2| + |S_3| > |H|$. We leave the derivation of a necessary and sufficient condition for future research.

4 Generalised 2-Chorded Odd Cycle Inequalities

We now present a family of valid inequalities that includes the 2-COC inequalities (3) as a special case.

Theorem 4. *Let $c \geq 5$ be an integer, and let $S_1, \ldots, S_c$ be disjoint and non-empty node sets such that $\sum_{k=1}^{c} |S_k|$ is odd. The "generalised 2-chorded odd cycle" (G2COC) inequality*

$$\sum_{k=1}^{c} \sum_{e \in E\left(S_k : S_{k+1}\right)} x_e - \sum_{k=1}^{c} \sum_{e \in E(S_k)} x_e - \sum_{k=1}^{c} \sum_{e \in E\left(S_k : S_{k+2}\right)} x_e \leq \left\lfloor \frac{\sum_{k=1}^{c} |S_k|}{2} \right\rfloor \quad (8)$$

is valid for P_n.

Proof. As usual, we can assume w.l.o.g. that $V = S_1 \cup \cdots \cup S_c$. So let c and $S_1, \ldots, S_c$ be given. We say that a CPP solution is "bad" if the corresponding x vector violates (8).

Now suppose that we are given a bad solution. For a given non-trivial clique C that is used in the bad solution, let $G(C)$ be a graph with node set

$$V(C) = \Big\{ k \in \{1, \ldots, c\} : C \cap S_k \neq \emptyset \Big\},$$

in which the edge $\{k, k+1\}$ is present if and only if both k and $k+1$ are in $V(C)$. For example, if C has a non-empty intersection with S_1, S_2 and S_4, then $G(C)$ contains the nodes 1, 2 and 4 and the edge $\{1, 2\}$. We will call $G(C)$ the *signature* of the clique C. Note that the signature is a subgraph of a cycle on c nodes.

We will apply a series of operations to the bad solution, in order to obtain a bad solution such that each non-trivial clique that is used in the solution has a signature with special properties.

First, suppose that $G(C)$ has $q \geq 2$ connected components. We replace C with q smaller cliques as follows. For each component K of $G(C)$, we form the clique $\bigcup_{k \in K} (S_k \cap C)$. One can check that this transformation leaves the violation of the G2COC inequality unchanged. Thus, we can assume that each non-trivial clique has a connected signature.

Second, suppose that the signature consists of a single isolated node, say u. Since all edges in $E(S_u)$ have negative coefficients in (8), we obtain another bad solution by replacing the clique C with $|C|$ trivial cliques. Thus, we can assume that each non-trivial clique has a signature that contains between 2 and c nodes.

Third, suppose that the signature is a cycle on c nodes. Consider some $k \in \{1, \ldots, c\}$, and let v be some node in $C \cap S_k$. Suppose we removed v from C, and then placed v in its own trivial clique. One can check that this would cause the violation of (8) to increase by at least

$$\delta_k = |S_{k-2} \cap C| + |S_{k+2} \cap C| - |S_{k-1} \cap C| - |S_{k+1} \cap C|.$$

Now, since the sum of the δ_k values is zero, at least one of them must be non-negative. Thus, we are able to obtain another bad solution by removing a node from C. Repeating this operation, if necessary, we obtain a bad solution such that the signature of each non-trivial clique is a path that contains between 2 and $c-1$ nodes.

Fourth, suppose that the signature of a non-trivial clique is a path with 2 or more edges. We can assume w.l.o.g. that the path starts at node 1. That is, C intersects with $S_1, \ldots, S_3$, but does not intersect S_c. Let $u \in C \cap S_1$ and $v \in C \cap S_2$. Suppose we removed u and v from C, and placed u and v in their own clique. One can check that the violation of (8) would increase by $|S_4 \cap C| \geq 0$. By repeating this operation, if necessary, we can ensure that at least one of $C \cap S_1$ or $C \cap S_2$ becomes empty. Thus, we can assume that each non-trivial clique has a signature that contains a single edge.

Now, suppose that the signature of C contains 2 nodes, say k and $k+1$. Let $T_1 = S_k \cap C$ and $T_2 = S_{k+1} \cap C$. The contribution of C to the left-hand side of (8) is:

$$\sum_{e \in E(T_1:T_2)} x_e - \sum_{e \in E(T_1)} x_e - \sum_{e \in E(T_2)} x_e.$$

But there is a 2-partition inequality which states that this contribution cannot exceed $\min\{|T_1|, |T_2|\}$. This in turn implies that the contribution cannot exceed $\lfloor |C|/2 \rfloor$.

Thus, the contribution of any non-trivial clique C to the left-hand side of (8) cannot exceed $\lfloor |C|/2 \rfloor$. This implies that (8) is satisfied, which contradicts the assumption that the initial CPP solution was bad. □

Note that the G2COC inequalities (8) reduce to the 2-COC inequalities (3) when $|S_k| = 1$ for all k (which implies that c is odd).

The following theorem gives a simple sufficient (but not necessary) condition for a G2COC inequality to define a facet.

Theorem 5. *If all of the S_k have the same cardinality, then the G2COC inequality (8) defines a facet of P_n.*

Proof. Let $|S_k| = d$ for $k = 1, \ldots, c$. We can assume that $d \geq 3$ and odd, since the inequality reduces to a 2-COC inequality when $d = 1$, and $\sum_{k=1}^{c} |S_k|$ must be odd. (This implies of course that $c \geq 3$ and odd as well.) Moreover, from Theorem 1, we can assume that every node in V lies in one of the S_k.

As in the proof of Theorem 3, we say that a CPP solution is a "root" if it satisfies (8) at equality, and we will suppose that all roots satisfy the equation $\alpha^T x = \beta$.

Now, let $E' = \bigcup_{k=1}^{c} E(S_k : S_{k+1})$ and let $G' = (V, E')$. Our first claim is that, given any $u \in V$, there exists a matching in G' that touches every node except u. To see why, suppose w.l.o.g. that $u \in S_1$. Match $(d-1)/2$

nodes from S_1 (not including node u) with $(d-1)/2$ nodes in S_2. Then take the $(d+1)/2$ unmatched nodes in S_2 and match them with $(d+1)/2$ nodes in S_3. Proceed in the same way, matching $(d-1)/2$ nodes from each S_k with k odd to S_{k+1} and matching $(d+1)/2$ nodes from each S_k with k even to S_{k+1}. This procedure ends by taking the final $(d-1)/2$ unmatched nodes in S_c and matching them with the remaining unmatched nodes in S_1.

Given a node $u \in V$ and a matching of the given type, we can construct a root by treating each edge $\{v, w\}$ in the matching as a clique of cardinality 2, and putting u into its own trivial clique. We call such roots "u-roots".

Now, let u and v be nodes in S_1 and let w be a node in S_2. Also let R be a u-root that contains the edge $\{v, w\}$. We can construct another root by putting u and w into a clique and putting v into a trivial clique. This shows that $\alpha_{uw} = \alpha_{vw}$. Moreover, we can construct a different root by putting u, v and w into a single clique. This shows that $\alpha_{uv} = -\alpha_{uw}$. By symmetry, this shows that $\alpha_e = -\alpha_f$ for all $e \in \bigcup_{k=1}^{c} E(S_k)$ and all $f \in E'$. Thus, we can assume w.l.o.g. that the equation $\alpha^T x = \beta$ has been scaled so that $\alpha_e = -1$ for all $e \in \bigcup_{k=1}^{c} E(S_k)$ and $\alpha_e = 1$ for all $e \in E'$.

Now let $u \in S_1$, $v \in S_2$ and $w \in S_3$. Also let R be a u-root that contains the cliques $\{u\}$ and $\{v, w\}$. We can construct another root by replacing those two cliques with the single clique $\{u, v, w\}$. This shows that $\alpha_{uw} = -\alpha_{uv} = -1$. By symmetry, this shows that $\alpha_e = -1$ for all $e \in \bigcup_{k=1}^{c} E(S_k : S_{k+2})$.

Finally, suppose that $c \geq 7$. Let k be any integer between 4 and $c-3$, and let $u \in S_1$, $v \in S_k$ and $w \in S_{k+1}$. Also let R be a u-root that contains the clique $\{v, w\}$. We can construct another root by inserting u into the clique. This shows that $\alpha_{uv} + \alpha_{uw} = 0$. By symmetry, this shows that $\alpha_e = 0$ for all of the edges that have a coefficient of zero in (8). □

It turns out that G2COC inequalities can define facets of P_n even if not all of the S_k have the same cardinality. Using a software package such as `PORTA` [5], one can verify that the inequality remains facet-defining, for example, if $5 \leq c \leq 7$ and $|S_k| \in \{3, 4\}$ for all k. After experimenting with several such examples, we make the following conjecture:

Conjecture. *Suppose the following holds for $k = 1, \ldots, c$ and $p = 1, \ldots, \lfloor c/2 \rfloor$:*

$$\sum_{q=0}^{p-1} |S_{k+2q}| \leq \sum_{q=0}^{p} |S_{k+2q-1}|.$$

Then the G2COC inequality (8) defines a facet of P_n.

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