CLOSED IDEALS OF OPERATORS ON THE BAERNSTEIN AND SCHREIER SPACES

NIELS JAKOB LAUSTSEN AND JAMES SMITH

Abstract. We study the lattice of closed ideals of bounded operators on two families of Banach spaces: the Baernstein spaces B_p for $1 < p < \infty$ and the Schreier spaces S_p for $1 \leqslant p < \infty$. Our main conclusion is that there are $2^{\mathfrak{c}}$ many closed ideals that lie between the ideals of compact and strictly singular operators on each of these spaces, and also $2^{\mathfrak{c}}$ many closed ideals that contain projections of infinite rank.

Counterparts of results of Gasparis and Leung using a numerical index to distinguish the isomorphism types of subspaces spanned by subsequences of the unit vector basis for the higher-order Schreier spaces play a key role in the proofs, as does the Johnson–Schechtman technique for constructing $2^{\mathfrak{c}}$ many closed ideals of operators on a Banach space.

1. Introduction

Recently, the lattice of closed ideals of the Banach algebra $\mathscr{B}(X)$ of bounded operators on a Banach space X has been studied intensively, in many cases leading to the conclusion that it has cardinality $2^{\mathfrak{c}}$, which is the largest possible value when X is separable, and an order structure that is at least as complex as the power set of $\mathbb{R}$. We add two families of Banach spaces to the list for which these conclusions can be drawn: the Baernstein spaces B_p for $1 < p < \infty$ and the Schreier spaces S_p for $1 \leqslant p < \infty$.

Their definitions rely on the family $\mathcal{S}_1$ of *Schreier sets,* that is, the finite subsets of the natural numbers whose minimum dominates their cardinality. We can now define the p^{th} *Schreier space* S_p, for $1 \leqslant p < \infty$, as the completion of the vector space c_{00} of finitely supported elements $x = (x(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ with respect to the norm

$$\|x\|_{S_p} = \sup\biggl\{ \Bigl(\sum_{n\in F} |x(n)|^p\Bigr)^{\frac{1}{p}} : F \in \mathcal{S}_1 \setminus \{\emptyset\}\biggr\}, \tag{1.1}$$

while the p^{th} *Baernstein space* B_p, for $1 < p < \infty$, is the completion of c_{00} with respect to the norm

$$\|x\|_{B_p} = \sup\biggl\{ \Bigl(\sum_{j=1}^{m}\Bigl(\sum_{n\in F_j} |x(n)|\Bigr)^p\Bigr)^{\frac{1}{p}} : m \in \mathbb{N},\ F_1,\dots,F_m \in \mathcal{S}_1 \setminus \{\emptyset\} \text{ and } \max F_j < \min F_{j+1} \text{ for } 1 \leqslant j < m\biggr\}. \tag{1.2}$$

(Note that (1.2) would simply define the ℓ_1-norm for $p = 1$, which is why there is no Baernstein space B_1.)

Having introduced these Banach spaces, let us summarize our main conclusions about them.

Theorem 1.1. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leqslant p < \infty$. Then:*

(i) *$\mathscr{B}(E)$ contains $2^{\mathfrak{c}}$ many closed ideals between the ideals of compact and strictly singular operators.*

(ii) *The ideals of strictly singular and inessential operators on E are equal. For $E = S_p$, these ideals are also equal to the ideal of weakly compact operators.*

2020 *Mathematics Subject Classification.* 46H10, 47L10, (primary); 46B03, 46B45, 47L20 (secondary).

Key words and phrases. Banach space, Baernstein space, Schreier space, bounded operator, closed operator ideal, ideal lattice, Gasparis–Leung index.

(iii) *$\mathscr{B}(E)$ contains $2^{\mathfrak{c}}$ many closed ideals which are larger than the ideal*

$$\{UV : U \in \mathscr{B}(D,E),\, V \in \mathscr{B}(E,D)\}$$

of operators factoring through D, where $D = \ell_p$ if $E = B_p$ and $D = c_0$ if $E = S_p$.

(iv) *$\mathscr{B}(E)$ contains at least continuum many maximal ideals.*

The Baernstein and Schreier spaces originate in Schreier's counterexample [24] from 1930, which showed that $C[0,1]$ does not have the Banach–Saks property. More precisely, Schreier defined what we call a Schreier set, but did not explicitly consider any of the Banach spaces we study. More than 40 years later, Baernstein [3] introduced the space B_2 to provide an example of a reflexive Banach space without the Banach–Saks property. Not long after, Seifert observed in his dissertation [25] that Baernstein's definition carries over to arbitrary $p \in (1,\infty)$.

Surprisingly, it appears that the Schreier space S_1 was not defined until seven years after Baernstein's work [3]. It is hard to imagine that Baernstein did not know S_1, but as far as we have been able to find out, Beauzamy [5] was the first person to define it explicitly, using it in combination with interpolation methods to obtain another example of a reflexive Banach space without the Banach–Saks property. Bird and the first author [6] studied the spaces S_p for $p > 1$.

A much more commonly researched variant of the Banach space S_1 is the family of *higher-order Schreier spaces* $X[\mathcal{S}_\xi]$, defined by Alspach and Argyros [2] for every countable ordinal ξ; the correspondence is that the space we denote S_1 is equal to $X[\mathcal{S}_1]$ (and $X[\mathcal{S}_0] = c_0$). We shall not add to the theory of these spaces for $\xi \geqslant 2$, but instead develop counterparts of some of the main results about them for the Baernstein and Schreier spaces, as we shall explain next.

Organization and overview of content. We begin by collecting some preliminary material in Section 2, most importantly a quantitative version of the fact that for every $1 < p < \infty$, the Baernstein space B_p is saturated with complemented copies of ℓ_p, while the Schreier spaces are saturated with complemented copies of c_0. Theorem 1.1(ii) follows easily from these results, as we shall show in Section 3.

The remaining parts of Theorem 1.1 are substantially harder to verify. Section 4 contains the main technical tool that we require: a counterpart for the Baernstein and Schreier spaces of some results of Gasparis and Leung [14] concerning the higher-order Schreier spaces. For every $n \in \mathbb{N}$, they introduced a numerical index which characterizes when two subspaces spanned by infinite subsequences of the unit vector basis for $X[\mathcal{S}_n]$ are isomorphic. Surprisingly, we find that their index for $n = 1$ works for the Baernstein spaces B_p and — perhaps less surprisingly — the Schreier spaces S_p for $p > 1$.

Beanland, Kania and the first author [4] used the results from [14] to demonstrate that the family of closed ideals of $\mathscr{B}(X[\mathcal{S}_n])$ that are singly generated by basis projections has a very rich structure for every $n \in \mathbb{N}$. In Section 5, we show that by referring to Section 4 instead of [14], we can transfer the arguments from [4] to the Baernstein and Schreier spaces; Theorem 5.4 states our main conclusions, which include Theorem 1.1(iv).

Answering a question raised in [4], Manoussakis and Pelczar-Barwacz [20] combined the results from [14] with the seminal idea of Johnson and Schechtman [16] to prove that $\mathscr{B}(X[\mathcal{S}_n])$ contains $2^{\mathfrak{c}}$ many closed ideals that lie between the ideals of compact and strictly singular operators for every $n \in \mathbb{N}$. Theorem 1.1(i) is the analogue of this result for the Baernstein and Schreier spaces. We prove it in Section 6, together with Theorem 1.1(iii), whose proof turns out to be the easier of the two. The reason is that Theorem 1.1(i) requires the non-trivial fact that the formal inclusion map from B_p into ℓ_p is strictly singular. We verify it using an inequality due to Jameson, who has generously allowed us to include his proof of it in Appendix A. In contrast to Manoussakis and Pelczar-Barwacz, we express our arguments in terms of the numerical index of Gasparis and Leung, thereby elucidating their combinatorial nature and providing a blueprint for other Banach spaces admitting a suitable index.

To provide additional context and background for our results, we conclude this introduction with a survey of separable Banach spaces X for which the Banach algebra $\mathscr{B}(X)$ contains $2^{\mathfrak{c}}$ many closed ideals. As far as we know, Gowers' hyperplane space X_{G} originally introduced in [15] is the first example of this kind; more precisely, the first author [17, Theorem 8.4] classified the maximal ideals of $\mathscr{B}(X_{\mathrm{G}})$ and noted that there are $2^{\mathfrak{c}}$ of them.

A major breakthrough occurred when Johnson and Schechtman [16] showed that $\mathscr{B}(L_p[0,1])$ contains $2^{\mathfrak{c}}$ many closed ideals for every $p \in (1,2)\cup(2,\infty)$. Their key technique has proved very versatile and spawned many new results. Theorem 6.1 states a variant of it, formulated by Freeman, Schlumprecht and Zsák [13], who used it to verify that $\mathscr{B}(X)$ contains $2^{\mathfrak{c}}$ many closed ideals for a number of direct sums of Banach spaces, notably $X = \ell_p \oplus \ell_q$, $X = \ell_q \oplus c_0$ and $X = \ell_q \oplus \ell_\infty$ for $1 \leqslant p < q < \infty$, as well as the Hardy space H_1 and its predual VMO.

Also building on the Johnson–Schechtman technique, Manoussakis and Pelczar-Barwacz [20] showed that $\mathscr{B}(X)$ contains $2^{\mathfrak{c}}$ many closed ideals for Schlumprecht's arbitrarily distortable Banach space [23] and the higher-order Schreier spaces $X[\mathcal{S}_n]$ for $n \in \mathbb{N}$, as already mentioned. In collaboration with Causey, Pelczar-Barwacz [8] has subsequently extended the latter result to the Schreier spaces $X[\mathcal{S}_\xi]$ of any countable order ξ, as well as their duals and biduals.

Finally, Chu and Schlumprecht [9] have shown that $\mathscr{B}(T[\mathcal{S}_\xi,\theta])$ contains $2^{\mathfrak{c}}$ many closed ideals for every countable ordinal ξ and $0<\theta<1$, where $T[\mathcal{S}_\xi,\theta]$ denotes the Tsirelson space of order ξ, as defined by Alspach and Argyros [2].

2. Preliminaries, including a saturation result for the Baernstein and Schreier spaces

We begin with some general conventions. All vector spaces are over the same scalar field $\mathbb{K}$, either the real or the complex numbers. We use the letters $X, Y, \ldots$ to denote generic Banach spaces, while we reserve the letter E for either the Baernstein space B_p or the Schreier space S_p and the letter D for either ℓ_p or c_0, in the same way as in the statement of Theorem 1.1. In line with these conventions, $(e_n)_{n\in\mathbb{N}}$ will always denote the unit vector basis for E (to be discussed in more detail below) and $(d_n)_{n\in\mathbb{N}}$ the unit vector basis for D.

The term "operator" means a bounded, linear map between two Banach spaces X and Y. We write $\mathscr{B}(X,Y)$ for the space of operators $X\to Y$ and abbreviate $\mathscr{B}(X,X)$ to $\mathscr{B}(X)$ in line with standard practice.

Let $(x_n)_{n\in\mathbb{N}}$ be a (Schauder) basis for a Banach space X. For every $n\in\mathbb{N}$, we denote the n^{th} coordinate functional by $x_n^* \in X^*$. Suppose that the basis (x_n) is unconditional. Then, for every subset N of $\mathbb{N}$, $P_N \in \mathscr{B}(X)$ denotes the basis projection given by $P_N x = \sum_{n\in N}\langle x, x_n^*\rangle x_n$ for $x\in X$. As usual, we abbreviate $P_{\{1,2,\ldots,n\}}$ to P_n for $n\in\mathbb{N}$.

We follow the convention that $\mathbb{N} = \{1,2,3,\ldots\}$ and write $[\mathbb{N}]$ and $[\mathbb{N}]^{<\infty}$ for the families of infinite and finite subsets of $\mathbb{N}$, respectively. As already mentioned,

$$\mathcal{S}_1 = \{F \in [\mathbb{N}]^{<\infty} : |F| \leqslant \min F\}$$

is the family of *Schreier sets*, where $|F|$ denotes the cardinality of the set F (and by convention $\min\emptyset = 0$, so $\emptyset\in\mathcal{S}_1$). Observe that $\mathcal{S}_1$ is closed under taking subsets, and is *spreading* in the following sense: suppose that $\{j_1 < j_2 < \cdots < j_n\} \in \mathcal{S}_1$ and $j_i \leqslant k_i$ for each $1\leqslant i\leqslant n$; then $\{k_1,\ldots,k_n\}\in\mathcal{S}_1$. A Schreier set is *maximal* if it is not contained in any strictly larger Schreier set. Clearly, this is equivalent to $F\neq\emptyset$ and $|F| = \min F$.

It will be convenient to express the p^{th} Schreier and Baernstein norms defined by (1.1) and (1.2) as the suprema over certain families of semi-norms. This is straightforward for the Schreier norms and was used extensively in [6]: for $1\leqslant p<\infty$ and $x = (x(n))_{n\in\mathbb{N}}\in\mathbb{K}^{\mathbb{N}}$, we can write

$$\|x\|_{S_p} = \sup\bigl\{\mu_p(x,F) : F\in\mathcal{S}_1\bigr\}\in[0,\infty], \quad \text{where} \quad \mu_p(x,F) = \begin{cases} 0 & \text{if } F=\emptyset \\ \Bigl(\sum_{n\in F}|x(n)|^p\Bigr)^{\frac{1}{p}} & \text{otherwise.}\end{cases}$$

As observed in [6, Lemma 3.3(iv)], $\|\cdot\|_{S_p}$ defines a complete norm on the subspace $Z_p = \{x\in\mathbb{K}^{\mathbb{N}} : \|x\|_{S_p}<\infty\}$ of $\mathbb{K}^{\mathbb{N}}$, and we can view S_p as the closed subspace of Z_p spanned by the "unit vector basis" $(e_n)_{n\in\mathbb{N}}$ given by $e_n(m) = 1$ if $m=n$ and $e_n(m)=0$ otherwise. Justifying its name, $(e_n)_{n\in\mathbb{N}}$ is a normalized basis for S_p that is 1-unconditional and shrinking, as shown in

[6, Propositions 3.5 and 3.10 and Corollary 3.12]. It is not hard to verify that S_p is a proper subspace of Z_p; in fact, Z_p is non-separable by [6, Corollary 5.6].

To analogously express the Baernstein norm as the supremum of a certain family of seminorms, we introduce the following notion: a *Schreier chain* is a non-empty, finite collection $\mathcal{C}$ of non-empty, consecutive Schreier sets; that is, $\mathcal{C} = \{F_1 < F_2 < \cdots < F_m\}$, where $m \in \mathbb{N}$, $F_1, \ldots, F_m \in \mathcal{S}_1 \setminus \{\emptyset\}$, and the notation $F_1 < F_2 < \cdots < F_m$ signifies that $\max F_j < \min F_{j+1}$ for each $1 \leqslant j < m$. We write SC for the collection of all Schreier chains. Then, for $1 < p < \infty$ and a Schreier chain $\mathcal{C}$, we can define a seminorm $\beta_p(\,\cdot\,, \mathcal{C})$ on $\mathbb{K}^{\mathbb{N}}$ by

$$\beta_p(x, \mathcal{C}) = \biggl(\sum_{F\in\mathcal{C}}\Bigl(\sum_{n\in F}|x(n)|\Bigr)^p\biggr)^{\frac{1}{p}} \qquad (x = (x(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}),$$

and

$$\|x\|_{B_p} = \sup\bigl\{\beta_p(x, \mathcal{C}) : \mathcal{C} \in \mathrm{SC}\bigr\} \in [0, \infty] \qquad (x \in \mathbb{K}^{\mathbb{N}}).$$

In contrast to the Schreier spaces, it turns out that the p^{th} Baernstein space (defined as the completion of c_{00} with respect to this norm) is precisely the collection of vectors $x \in \mathbb{K}^{\mathbb{N}}$ with finite Baernstein norm: $B_p = \{x \in \mathbb{K}^{\mathbb{N}} : \|x\|_{B_p} < \infty\}$; this follows by replacing the exponent 2 with p in Baernstein's argument given in [3, page 92, first paragraph of "Proof of (2)"]. As is the case for the Schreier spaces, the unit vector basis $(e_n)_{n\in\mathbb{N}}$ forms a 1-unconditional, normalized basis for B_p.

In line with standard practice, $\mathrm{supp}(x) = \{n \in \mathbb{N} : x(n) \neq 0\}$ denotes the *support* of an element $x = (x(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$. Clearly, when computing $\|x\|_{S_p}$, it suffices to consider $\mu_p(x, F)$ for $F \in \mathcal{S}_1$ with $F \subseteq \mathrm{supp}(x)$. Similarly, when computing $\|x\|_{B_p}$, it suffices to consider $\beta_p(x, \mathcal{C})$ for $\mathcal{C} \in \mathrm{SC}$ with $\bigcup\mathcal{C} \subseteq \mathrm{supp}(x)$.

Lemma 2.1. *Let $1 < p < \infty$, and suppose that the non-zero coordinates of $x \in c_{00}$ are decreasing in absolute value. Then x attains its Baernstein norm at some Schreier chain covering* $\mathrm{supp}(x)$*; that is, $\|x\|_{B_p} = \beta_p(x, \mathcal{C})$ for some $\mathcal{C} \in \mathrm{SC}$ with $\bigcup\mathcal{C} = \mathrm{supp}(x)$.*

Proof. Take $\mathcal{C} = \{F_1 < F_2 < \cdots < F_m\} \in \mathrm{SC}$ with $\bigcup\mathcal{C} \subsetneq \mathrm{supp}(x)$. For $1 \leqslant j \leqslant m$, let F_j' be the set which contains precisely the first $|F_j|$ points of $\mathrm{supp}(x) \cap [\min(F_j), \max(F_j)]$. Then $\mathcal{C}' = \{F_1' < F_2' < \cdots < F_m'\} \in \mathrm{SC}$ and

$$\sum_{n\in F_j}|\langle x, e_n^*\rangle| \leqslant \sum_{n\in F_j'}|\langle x, e_n^*\rangle| \qquad (1 \leqslant j \leqslant m)$$

because the non-zero coordinates of x are decreasing in absolute value. Hence $\beta_p(x, \mathcal{C}) \leqslant \beta_p(x, \mathcal{C}')$. The set $\mathcal{C}'' = \bigl\{\{n\} : n \in \mathrm{supp}(x) \setminus \bigcup\mathcal{C}'\bigr\}$ is non-empty because $\bigcup\mathcal{C} \subsetneq \mathrm{supp}(x)$. Clearly $\mathcal{D} = \mathcal{C}' \cup \mathcal{C}''$ is a Schreier chain, and

$$\beta_p(x, \mathcal{C}) \leqslant \beta_p(x, \mathcal{C}') < \beta_p(x, \mathcal{D}) \leqslant \|x\|_{B_p},$$

so $\|x\|_{B_p}$ is not attained at $\mathcal{C}$. However, it must be attained at some Schreier chain because x is finitely supported. □

For later reference, we now record an estimate for the norm of certain "flat" vectors.

Lemma 2.2. *Let $\{F_1 < F_2 < \cdots < F_m\}$ be a chain of maximal Schreier sets for some $m \in \mathbb{N}$. Then*

$$1 \leqslant \biggl\|\sum_{j=1}^m \frac{1}{|F_j|^{\frac{1}{p}}}\sum_{k\in F_j} e_k\biggr\|_{S_p} \leqslant 2^{\frac{1}{p}} \qquad (1 \leqslant p < \infty) \tag{2.1}$$

and

$$m^{\frac{1}{p}} \leqslant \biggl\|\sum_{j=1}^m \frac{1}{|F_j|}\sum_{k\in F_j} e_k\biggr\|_{B_p} \leqslant 2m^{\frac{1}{p}} \qquad (1 < p < \infty). \tag{2.2}$$

Proof. To prove (2.1), set $x = \sum_{j=1}^{m} |F_j|^{-\frac{1}{p}} \sum_{k\in F_j} e_k \in S_p$. By definition, we have $\mu_p(x, F_j) = 1$ for $1 \leqslant j \leqslant m$; the lower bound on the norm follows. On the other hand, since x is finitely supported and its non-zero coordinates are decreasing, $\|x\|_{S_p}$ is attained at some set $G \in \mathcal{S}_1$ that intersects at most two consecutive sets from $\{F_1, \ldots, F_m\}$; that is, $G \subseteq F_j \cup F_{j+1}$ for some $1 \leqslant j < m$, and we have $\mu_p(x, G)^p \leqslant \mu_p(x, F_j)^p + \mu_p(x, F_{j+1})^p = 2$, which proves the upper bound.

Turning our attention to (2.2), we define $x = \sum_{j=1}^{m} \frac{1}{|F_j|} \sum_{k\in F_j} e_k \in B_p$. The lower bound on the norm of x follows from the fact that $\mathcal{C} = \{F_1 < F_2 < \cdots < F_m\}$ is a Schreier chain for which $\beta_p(x, \mathcal{C}) = m^{\frac{1}{p}}$.

For the upper bound, Lemma 2.1 implies that we can take $\mathcal{C} = \{G_1 < \cdots < G_n\} \in \mathrm{SC}$ such that $\|x\|_{B_p} = \beta_p(x, \mathcal{C})$ and $\bigcup_{k=1}^{n} G_k = \bigcup_{j=1}^{m} F_j$. Define a map $\varphi\colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ by

$$\varphi(k) = \min\{j : G_k \cap F_j \neq \emptyset\}.$$

We observe that φ is surjective because otherwise we would have $F_j \subsetneq G_k$ for some j and k, contradicting that F_j is a maximal Schreier set.

Fix $j \in \{1, \ldots, m\}$, set $h(j) = \max \varphi^{-1}(\{j\})$, and note that $\bigcup\{G_k : \varphi(k) = j,\ k \neq h(j)\} \subseteq F_j$ because the sets $G_1, \ldots, G_n$ are successive. Since the ℓ_1-norm dominates the ℓ_p-norm, it follows that

$$\sum_{k\in\varphi^{-1}(\{j\})} \Big(\sum_{i\in G_k} |\langle x, e_i^*\rangle|\Big)^p \leqslant \Big(\sum_{k\in\varphi^{-1}(\{j\})} \sum_{i\in G_k} |\langle x, e_i^*\rangle|\Big)^p \leqslant \Big(1 + \frac{|G_{h(j)} \cap F_{j+1}|}{|F_{j+1}|}\Big)^p \leqslant 2^p.$$

Combining this with the fact that the sets $\{\varphi^{-1}(\{j\}) : 1 \leqslant j \leqslant m\}$ partition $\{1, \ldots, n\}$, we conclude that

$$\|x\|_{B_p}^p = \beta_p(x, \mathcal{C})^p = \sum_{j=1}^{m} \sum_{k\in\varphi^{-1}(\{j\})} \Big(\sum_{i\in G_k} |\langle x, e_i^*\rangle|\Big)^p \leqslant 2^p m,$$

which gives the upper bound. $\square$

We shall now present the main conclusion of this section, which is a quantitative version of two results in the literature: one by Seifert [25, Theorem 3] stating that B_p is saturated with complemented copies of ℓ_p for $1 < p < \infty$, the other by Bird and the first author [6, Corollary 5.4] stating that the Schreier spaces are saturated with copies of c_0 (which are automatically complemented by Sobczyk's Theorem). Our statement strengthens these results by providing explicit norm bounds on the projections and isomorphisms, using the following terminology.

Definition 2.3. Let X and Y be Banach spaces. We say that X is *C-uniformly saturated with complemented copies of Y* for some constant $C \geqslant 1$ if every closed, infinite-dimensional subspace of X contains a closed subspace Z for which

(i) there exists an isomorphism U of Y onto Z with $\|U\| \cdot \|U^{-1}\| \leqslant C$, and
(ii) there exists a projection P of X onto Z with $\|P\| \leqslant C$.

Theorem 2.4. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leqslant p < \infty$. Then E is C-uniformly saturated with complemented copies of D for every $C > 1$.*

We provide a full proof of this theorem, even though the norm bounds it provides may seem only a modest improvement of existing knowledge. However, Seifert never published his dissertation, and it contains an unfortunate error: [25, Lemma 2] claims that every semi-normalized block basic sequence $(w_n)_{n\in\mathbb{N}}$ of the unit vector basis for B_p admits a subsequence which is equivalent to the unit vector basis for ℓ_p. That is not true; for instance, no subsequence of the unit vector basis for B_p is equivalent to the ℓ_p-basis. The correct statement is that $(w_n)_{n\in\mathbb{N}}$ admits a block basic sequence which is equivalent to the ℓ_p-basis; this will follow from Lemma 2.7 and Proposition 2.14 below.

To compound this issue, Seifert's incorrect statement has been reproduced in [7, Theorem 0.15(c)–(d)], as well as [11, page 233] in a special case. This has caused at least one mistake in the published literature: citing [7], Flores *et al.* [12, page 334] deduce that the

Baernstein spaces are "disjointly homogeneous". However, that is impossible because by [12, Theorem 2.13], it would imply that every strictly singular operator on B_p is compact, which would contradict Theorem 1.1(i). (In more concrete terms, it would also contradict Proposition 6.6 below, which implies that the formal inclusion map of B_p into ℓ_p composed with any isomorphic embedding of ℓ_p into B_p is a strictly singular, non-compact operator on B_p.)

Having explained *why* Theorem 2.4 requires a detailed proof, we shall now present one, proceeding through a series of lemmas.

Definition 2.5. A *block subspace* of a Banach space X with a basis $(x_n)_{n\in\mathbb{N}}$ is a closed subspace of X of the form $\overline{\operatorname{span}}\,(w_n : n \in \mathbb{N})$ for some block basic sequence $(w_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$.

Lemma 2.6. *Let X and Y be Banach spaces, where X has a basis $(x_n)_{n\in\mathbb{N}}$. Suppose that there is a constant $C_1 \geqslant 1$ for which every block subspace W of X admits operators $U \in \mathscr{B}(Y, W)$ and $V \in \mathscr{B}(X, Y)$ such that $V|_W U = I_Y$ and $\|U\| \cdot \|V\| \leqslant C_1$. Then X is C_2-uniformly saturated with complemented copies of Y for every constant $C_2 > C_1$.*

Proof. Let K be the basis constant of $(x_n)_{n\in\mathbb{N}}$, and let $(P_n)_{n\in\mathbb{N}}$ be the corresponding basis projections. Given $C_2 > C_1$, choose $\varepsilon \in (0, 1)$ such that

$$\frac{7(C_2 - C_1)}{4(C_1 + C_2)} \geqslant \varepsilon. \tag{2.3}$$

Set $m_0 = 0$ and $P_0 = 0$, and let Z be a closed, infinite-dimensional subspace of X. By recursion, we can choose natural numbers $m_1 < m_2 < \cdots$ and unit vectors $z_n \in Z \cap \ker P_{m_{n-1}}$ such that $\|z_n - P_{m_n} z_n\| \leqslant \varepsilon/(2^{n+2}K)$ for every $n \in \mathbb{N}$. Set $w_n = P_{m_n} z_n \in X$, and note that

$$\|w_n\| = \|z_n - (z_n - w_n)\| \geqslant 1 - \frac{\varepsilon}{2^{n+2}K} \geqslant \frac{7}{8} \qquad (n \in \mathbb{N}). \tag{2.4}$$

In particular, $w_n \neq 0$, and since $z_n \in \ker P_{m_{n-1}}$, it follows that $(w_n)_{n\in\mathbb{N}}$ is a block basic sequence of $(x_n)_{n\in\mathbb{N}}$. Set $W = \overline{\operatorname{span}}\,(w_n : n \in \mathbb{N})$. By hypothesis, we can find operators $U_1 \in \mathscr{B}(Y, W)$ and $V_1 \in \mathscr{B}(X, Y)$ such that $V_1|_W U_1 = I_Y$ and $\|U_1\| \cdot \|V_1\| \leqslant C_1$.

For each $n \in \mathbb{N}$, choose a functional $f_n \in X^*$ of norm 1 such that $\langle w_n, f_n\rangle = \|w_n\|$. Then, using (2.4), we have

$$\sum_{n=1}^{\infty} \frac{\|(P_{m_n} - P_{m_{n-1}})^* f_n\|}{\|w_n\|} \cdot \|w_n - z_n\| \leqslant \sum_{n=1}^{\infty} \frac{2K}{7/8} \cdot \frac{\varepsilon}{2^{n+2}K} = \frac{4\varepsilon}{7},$$

so we can define an operator $R \in \mathscr{B}(X)$ by

$$R = \sum_{n=1}^{\infty} \frac{(P_{m_n} - P_{m_{n-1}})^* f_n}{\|w_n\|} \otimes (w_n - z_n),$$

where $f \otimes x$, for $f \in X^*$ and $x \in X$, denotes the rank-one operator $y \mapsto \langle y, f\rangle x$ as usual. Since $\|R\| \leqslant 4\varepsilon/7 < 1$, the Neumann series implies that the operator $S = I_X - R \in \mathscr{B}(X)$ is invertible, and $\|S^{-1}\| \leqslant (1 - 4\varepsilon/7)^{-1} = 7/(7 - 4\varepsilon)$. The definition of R shows that

$$Rw_j = \sum_{n=1}^{\infty} \frac{\langle (P_{m_n} - P_{m_{n-1}}) w_j, f_n\rangle}{\|w_n\|} (w_n - z_n) = \frac{\langle w_j, f_j\rangle}{\|w_j\|} (w_j - z_j) = w_j - z_j,$$

so $Sw_j = z_j$ for each $j \in \mathbb{N}$, and therefore $S[W] \subseteq Z$. It follows that $Z_0 = (S|_W U_1)[Y]$ is a subspace of Z, and the operators $U = S|_W U_1 \in \mathscr{B}(Y, Z_0)$ and $V = V_1 S^{-1} \in \mathscr{B}(X, Y)$ satisfy $V|_{Z_0} U = V_1|_W U_1 = I_Y$. Since U is surjective by definition, this implies that U is invertible with inverse $V|_{Z_0}$ and $P = UV$ is a projection of X onto Z_0. In particular, Z_0 is a closed subspace of Z, and the norm bounds on $\|U\| \cdot \|U^{-1}\|$ and $\|P\|$ specified in Definition 2.3(i)–(ii) follow from the fact that

$$\|U\| \cdot \|V\| \leqslant \|S\| \cdot \|U_1\| \cdot \|V_1\| \cdot \|S^{-1}\| \leqslant \Big(1 + \frac{4\varepsilon}{7}\Big) C_1 \cdot \frac{7}{7 - 4\varepsilon} \leqslant C_2,$$

where the final inequality is a direct consequence of (2.3). □

It turns out that the supremum norm $\|x\|_\infty = \sup_{n\in\mathbb{N}}|x(n)|$ for $x = (x(n))_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ plays an important auxiliary role in a number of results about the Baernstein and Schreier spaces. We shall sometimes use the coordinate functionals to express it in the alternative form $\|x\|_\infty = \sup_{n\in\mathbb{N}}|\langle x, e_n^*\rangle|$ for $x \in B_p$ or $x \in S_p$.

Lemma 2.7. *Every block basic sequence of the unit vector basis for B_p (for $1 < p < \infty$) or S_p (for $1 \leqslant p < \infty$) admits a normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ for which $\|u_n\|_\infty \to 0$ as $n \to \infty$.*

Proof. As usual, let $E = B_p$ or $E = S_p$, and let $(w_n)_{n\in\mathbb{N}}$ be a block basic sequence of the unit vector basis $(e_n)_{n\in\mathbb{N}}$ for E. Replacing $(w_n)_{n\in\mathbb{N}}$ with the block basic sequence $(w_n/\|w_n\|_E)_{n\in\mathbb{N}}$, we may suppose that $(w_n)_{n\in\mathbb{N}}$ is normalized in the E-norm. If $(w_n)_{n\in\mathbb{N}}$ admits a subsequence $(w_{n_j})_{j\in\mathbb{N}}$ such that $\|w_{n_j}\|_\infty \to 0$ as $j \to \infty$, there is nothing to prove.

Otherwise $\delta := \inf_{n\in\mathbb{N}}\|w_n\|_\infty > 0$, so for each $n \in \mathbb{N}$, we can choose $m_n \in \mathbb{N}$ such that $|\langle w_n, e_{m_n}^*\rangle| \geqslant \delta$. Since $(w_n)_{n\in\mathbb{N}}$ is a block basic sequence, we have $m_1 < m_2 < \cdots$ and $F_n = \{m_j : 2^{n-1} \leqslant j < 2^n\}$ is a Schreier set, being a spread of the interval $[2^{n-1}, 2^n) \cap \mathbb{N} \in \mathcal{S}_1$. This implies that the block basic sequence $(v_n)_{n\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$ defined by

$$v_n = \sum_{j=2^{n-1}}^{2^n-1} w_j \qquad (n \in \mathbb{N})$$

is unbounded because

$$\|v_n\|_E \geqslant \begin{cases} \mu_p(v_n, F_n) \geqslant 2^{(n-1)/p}\delta \to \infty & \text{as} \quad n \to \infty \quad \text{for} \quad E = S_p, \\ \beta_p(v_n, \{F_n\}) \geqslant 2^{n-1}\delta \to \infty & \text{as} \quad n \to \infty \quad \text{for} \quad E = B_p. \end{cases}$$

On the other hand, $\|v_n\|_\infty = \max\{\|w_j\|_\infty : 2^{n-1} \leqslant j < 2^n\} \leqslant 1$ because $(w_j)_{j\in\mathbb{N}}$ is a normalized block basic sequence of $(e_n)_{n\in\mathbb{N}}$, so $(u_n = v_n/\|v_n\|_E)_{n\in\mathbb{N}}$ is a normalized block basic sequence of $(w_n)_{n\in\mathbb{N}}$ such that

$$\|u_n\|_\infty = \frac{\|v_n\|_\infty}{\|v_n\|_E} \leqslant \frac{1}{\|v_n\|_E} \to 0 \quad \text{as} \quad n \to \infty. \qquad \square$$

Our next lemma involves the following standard piece of terminology. A basic sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X *dominates* a basic sequence $(y_n)_{n\in\mathbb{N}}$ in a Banach space Y if there is a constant $C > 0$ such that

$$\Big\|\sum_{n=1}^m \alpha_n y_n\Big\|_Y \leqslant C\Big\|\sum_{n=1}^m \alpha_n x_n\Big\|_X \qquad (m \in \mathbb{N},\ \alpha_1, \ldots, \alpha_m \in \mathbb{K}). \tag{2.5}$$

If we wish to record the value of the constant C, we say that $(x_n)_{n\in\mathbb{N}}$ *C-dominates* $(y_n)_{n\in\mathbb{N}}$.

Lemma 2.8. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leqslant p < \infty$, and suppose that $(u_n)_{n\in\mathbb{N}}$ is a normalized block basic sequence of the unit vector basis for E with $\inf_{n\in\mathbb{N}}\|u_n\|_\infty = 0$. Then, for every constant $C > 1$, $(u_n)_{n\in\mathbb{N}}$ admits a subsequence which is C-dominated by the unit vector basis for D.*

Proof. Set $\varepsilon = C^p - 1 > 0$. We begin with the easier case, which is the Schreier space; that is, $(E, D) = (S_p, c_0)$. Recursively, we can choose integers $1 = j_1 < j_2 < \cdots$ such that

$$\|u_{j_{k+1}}\|_\infty^p \leqslant \frac{\varepsilon}{\max(\operatorname{supp}(u_{j_k}))} \qquad (k \in \mathbb{N}). \tag{2.6}$$

In order to verify that the unit vector basis for c_0 C-dominates the basic sequence $(u_{j_k})_{k\in\mathbb{N}}$ in S_p, we must show that $\mu_p(x, F) \leqslant C$ whenever $x = \sum_{k=1}^n \alpha_k u_{j_k}$ for some $n \in \mathbb{N}$ and some $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ with $\max_{1\leqslant k\leqslant n}|\alpha_k| \leqslant 1$, and $F \in \mathcal{S}_1 \setminus \{\emptyset\}$ with $F \subseteq \operatorname{supp}(x)$. Set

$$m = \min\{k \in \mathbb{N} : F \cap \operatorname{supp}(u_{j_k}) \neq \emptyset\}.$$

Then we have $|F| \leqslant \min F \leqslant \max(\operatorname{supp}(u_{j_m}))$, so $|F| \cdot \|u_{j_k}\|_\infty^p \leqslant \varepsilon$ for $k > m$ by (2.6), and therefore

$$\begin{aligned}\mu_p(x,F)^p &= \mu_p(\alpha_m u_{j_m}, F)^p + \mu_p\Big(\sum_{k=m+1}^n \alpha_k u_{j_k}, F\Big)^p \\ &\leqslant |\alpha_m|^p\, \|u_{j_m}\|_{S_p}^p + |F|\Big(\max_{m<k\leqslant n} |\alpha_k|\, \|u_{j_k}\|_\infty\Big)^p \leqslant 1 + \varepsilon = C^p,\end{aligned}$$

as required.

Proceeding to the case $(E,D) = (B_p, \ell_p)$, we use the fact that the function $t \mapsto t^p$ is uniformly continuous on $[0,2]$ to choose numbers $\delta_k \in (0,1)$ such that

$$(s+t)^p \leqslant s^p + \frac{\varepsilon}{2^k} \qquad (k \in \mathbb{N},\ s \in [0,1],\ t \in [0,\delta_k]). \tag{2.7}$$

After replacing $(u_n)_{n\in\mathbb{N}}$ with a suitable subsequence, we may suppose that $\|u_n\|_\infty \to 0$ as $n \to \infty$. We can then recursively choose integers $1 = j_1 < j_2 < \cdots$ such that

$$\|u_i\|_\infty \leqslant \frac{\delta_k}{\max(\operatorname{supp}(u_{j_k}))} \qquad (k \in \mathbb{N},\ i \geqslant j_{k+1}). \tag{2.8}$$

We seek to verify that the unit vector basis for ℓ_p C-dominates the basic sequence $(u_{j_k})_{k\in\mathbb{N}}$ in B_p. This amounts to showing that $\beta_p(x,\mathcal{C}) \leqslant C$ whenever $x = \sum_{k=1}^n \alpha_k u_{j_k}$ for some $n \in \mathbb{N}$ and some $\alpha_1,\ldots,\alpha_n \in \mathbb{K}$ with $\sum_{k=1}^n |\alpha_k|^p \leqslant 1$, and $\mathcal{C}$ is a Schreier chain contained in $\operatorname{supp}(x)$. Set

$$\mathcal{C}_k = \{F \in \mathcal{C} : \min F \in \operatorname{supp}(u_{j_k})\} \qquad (1 \leqslant k \leqslant n).$$

Then, defining $\beta_p(x,\emptyset) = 0$ to cover the case where $\mathcal{C}_k = \emptyset$ for some k, we can write

$$\beta_p(x,\mathcal{C})^p = \sum_{k=1}^n \sum_{F\in\mathcal{C}_k} \Big(\sum_{i\in F} |\langle x, e_i^*\rangle|\Big)^p = \sum_{k=1}^n \beta_p(x,\mathcal{C}_k)^p. \tag{2.9}$$

We claim that

$$\beta_p(x,\mathcal{C}_k)^p \leqslant |\alpha_k|^p + \frac{\varepsilon}{2^k} \qquad (1 \leqslant k \leqslant n), \tag{2.10}$$

from which the conclusion will follow because substituting (2.10) into (2.9), we obtain

$$\beta_p(x,\mathcal{C})^p \leqslant \sum_{k=1}^n \Big(|\alpha_k|^p + \frac{\varepsilon}{2^k}\Big) \leqslant 1 + \varepsilon = C^p.$$

It remains to prove (2.10). Take $k \in \{1,\ldots,n\}$ with $\mathcal{C}_k \neq \emptyset$, let G_k be the final set in $\mathcal{C}_k$ (in the sense that G_k is the set in $\mathcal{C}_k$ with the largest minimum), and define

$$G_k' = G_k \cap \operatorname{supp}(u_{j_k}) \qquad \text{and} \qquad G_k'' = G_k \setminus \operatorname{supp}(u_{j_k}).$$

Then we have

$$\begin{aligned}\beta_p(x,\mathcal{C}_k)^p &= \sum_{F\in\mathcal{C}_k\setminus\{G_k\}} \Big(\sum_{i\in F} |\langle x, e_i^*\rangle|\Big)^p + \Big(\sum_{i\in G_k} |\langle x, e_i^*\rangle|\Big)^p \\ &= |\alpha_k|^p \sum_{F\in\mathcal{C}_k\setminus\{G_k\}} \Big(\sum_{i\in F} |\langle u_{j_k}, e_i^*\rangle|\Big)^p + (s_k + t_k)^p,\end{aligned} \tag{2.11}$$

where we have introduced the quantities

$$s_k = \sum_{i\in G_k'} |\langle x, e_i^*\rangle| = |\alpha_k| \sum_{i\in G_k'} |\langle u_{j_k}, e_i^*\rangle| \qquad \text{and} \qquad t_k = \sum_{i\in G_k''} |\langle x, e_i^*\rangle|.$$

Now we observe that $0 \leqslant s_k \leqslant |\alpha_k|\, \|u_{j_k}\|_{B_p} \leqslant 1$ because $G_k' \in \mathcal{S}_1$, and

$$0 \leqslant t_k \leqslant |G_k''| \max_{k<i\leqslant n} |\alpha_i|\, \|u_{j_i}\|_\infty \leqslant \delta_k,$$

where we have used (2.8) together with the fact that $|G_k''| < |G_k| \leqslant \min(G_k) \leqslant \max(\operatorname{supp}(u_{j_k}))$. Hence (2.7) implies that $(s_k + t_k)^p \leqslant s_k^p + \varepsilon/2^k$. Substituting this into (2.11) and defining the Schreier chain $\mathcal{C}_k' = (\mathcal{C}_k \setminus \{G_k\}) \cup \{G_k'\}$, we obtain

$$\begin{aligned}\beta_p(x, \mathcal{C}_k)^p &\leqslant |\alpha_k|^p \sum_{F \in \mathcal{C}_k \setminus \{G_k\}} \Big(\sum_{i \in F} |\langle u_{j_k}, e_i^* \rangle|\Big)^p + |\alpha_k|^p \Big(\sum_{i \in G_k'} |\langle u_{j_k}, e_i^* \rangle|\Big)^p + \frac{\varepsilon}{2^k}\\ &= |\alpha_k|^p \beta_p(u_{j_k}, \mathcal{C}_k')^p + \frac{\varepsilon}{2^k} \leqslant |\alpha_k|^p + \frac{\varepsilon}{2^k}. \qquad \square\end{aligned}$$

Lemma 2.9. *Let $\mathcal{C} = \{F_1 < F_2 < \cdots\}$ be an infinite chain of successive Schreier sets, and take $1 < p < \infty$. Then, for each $x \in B_p$,*

$$\Sigma_{\mathcal{C}} x = \Big(\sum_{j \in F_n} \langle x, e_j^* \rangle\Big)_{n \in \mathbb{N}} \tag{2.12}$$

defines an element of ℓ_p with $\|\Sigma_{\mathcal{C}} x\|_{\ell_p} \leqslant \|x\|_{B_p}$. Hence (2.12) defines a map $\Sigma_{\mathcal{C}} \colon B_p \to \ell_p$, which is bounded and linear with norm 1.

Proof. For $x \in B_p$ and $m \in \mathbb{N}$, we have

$$\sum_{n=1}^m \Big|\sum_{j \in F_n} \langle x, e_j^* \rangle\Big|^p \leqslant \sum_{n=1}^m \Big(\sum_{j \in F_n} |\langle x, e_j^* \rangle|\Big)^p = \beta_p\big(x, \{F_1 < F_2 < \cdots < F_m\}\big)^p \leqslant \|x\|_{B_p}^p.$$

This shows that $\Sigma_{\mathcal{C}} x \in \ell_p$ with $\|\Sigma_{\mathcal{C}} x\|_{\ell_p} \leqslant \|x\|_{B_p}$ because the upper bound $\|x\|_{B_p}^p$ is independent of m. The remainder of the lemma is now straightforward to verify. $\square$

Lemma 2.10. *Let $(E, D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E, D) = (S_p, c_0)$ for some $1 \leqslant p < \infty$, and suppose that $(u_n)_{n \in \mathbb{N}}$ is a normalized block basic sequence of the unit vector basis for E. Then there exists an operator $V \in \mathscr{B}(E, D)$ of norm 1 such that $V u_n = d_n$ for every $n \in \mathbb{N}$.*

Proof. We begin with the case $(E, D) = (S_p, c_0)$. For $n \in \mathbb{N}$, set $m_n = \max(\operatorname{supp}(u_n))$, and use the Hahn–Banach Theorem to find a functional $f_n \in S_p^*$ such that $\langle u_n, f_n \rangle = 1 = \|f_n\|$. Then, for each $x \in S_p$, we can define

$$V x = \big(\langle (P_{m_n} - P_{m_{n-1}}) x, f_n \rangle\big)_{n \in \mathbb{N}} \in \ell_\infty, \tag{2.13}$$

where we have introduced $m_0 = 0$ and $P_0 = 0$ for notational convenience. We see that $Vx \in c_0$ because

$$\big|\langle (P_{m_n} - P_{m_{n-1}}) x, f_n \rangle\big| \leqslant \|(P_{m_n} - P_{m_{n-1}}) x\|_{S_p} \leqslant \|(I - P_{m_{n-1}}) x\|_{S_p} \to 0 \quad \text{as} \quad n \to \infty,$$

so (2.13) defines a map $V \colon S_p \to c_0$, which is clearly linear. Furthermore, V is bounded with $\|V\| \leqslant 1$ because $\|P_{m_n} - P_{m_{n-1}}\| = 1 = \|f_n\|$ for $n \in \mathbb{N}$, and we have $V u_n = d_n$ because $\operatorname{supp}(u_n) \subseteq (m_{n-1}, m_n]$ and $\langle u_n, f_n \rangle = 1$.

As before, the case $(E, D) = (B_p, \ell_p)$ is somewhat more involved. We begin by choosing a sequence of scalars $(\sigma_j)_{j \in \mathbb{N}}$ as follows. If $j \in \operatorname{supp}(u_n)$ for a (necessarily unique) $n \in \mathbb{N}$, we take $\sigma_j \in \mathbb{K}$ of modulus 1 such that $\sigma_j \cdot \langle u_n, e_j^* \rangle > 0$. Otherwise (that is, for $j \in \mathbb{N} \setminus \bigcup_{n=1}^\infty \operatorname{supp}(u_n)$), set $\sigma_j = 1$. The 1-unconditionality of the unit vector basis $(e_j)_{j \in \mathbb{N}}$ for B_p means that we can define an isometric isomorphism $\Delta \in \mathscr{B}(B_p)$ by $\Delta x = \sum_{j=1}^\infty \sigma_j \langle x, e_j^* \rangle e_j$. Our choice of the sequence $(\sigma_j)_{j \in \mathbb{N}}$ implies that

$$\Delta(u_n) = |u_n| \qquad (n \in \mathbb{N}), \tag{2.14}$$

where we have used the standard notion of *modulus* for an element of a Banach space with a 1-unconditional basis (justified by the fact that such a Banach space is a Banach lattice), that is, $\big|\sum_{j=1}^\infty \alpha_j e_j\big| = \sum_{j=1}^\infty |\alpha_j| e_j$.

For each $n \in \mathbb{N}$, take a Schreier chain $\mathcal{C}_n$ contained in $\operatorname{supp}(u_n)$ with $\beta_p(u_n, \mathcal{C}_n) = 1$, and set $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$. Defining $m_0 = 0$ and $m_n = \sum_{k=1}^n |\mathcal{C}_k|$ for $n \in \mathbb{N}$, we can enumerate $\mathcal{C}_n$ as

$$\mathcal{C}_n = \{F_{m_{n-1}+1} < F_{m_{n-1}+2} < \cdots < F_{m_n}\}.$$

Since $(u_n)_{n\in\mathbb{N}}$ is a block basic sequence, we have $F_{m_n} < F_{m_n+1}$ for $n \in \mathbb{N}$. Consequently $\mathcal{C} = \{F_1 < F_2 < \cdots\}$ is an infinite chain of successive Schreier sets, so it induces an operator $\Sigma_{\mathcal{C}} \in \mathscr{B}(B_p, \ell_p)$ of norm 1 by Lemma 2.9.

Set $D_n = \operatorname{span}(d_j : m_{n-1} < j \leqslant m_n) \subset \ell_p$, and let $Q_n \in \mathscr{B}(\ell_p, D_n)$ be the basis projection onto D_n; that is, $Q_n d_j = d_j$ for $m_{n-1} < j \leqslant m_n$ and $Q_n d_j = 0$ otherwise. Then we can define an isometric isomorphism $\Theta \in \mathscr{B}\big(\ell_p, \big(\bigoplus_{n\in\mathbb{N}} D_n\big)_{\ell_p}\big)$ by $\Theta x = (Q_n x)_{n\in\mathbb{N}}$.

In view of (2.14), the vector $y_n = \Sigma_{\mathcal{C}}\Delta(u_n) \in \ell_p$ satisfies

$$y_n = \Sigma_{\mathcal{C}}|u_n| = \sum_{j=m_{n-1}+1}^{m_n} \Big(\sum_{k\in F_j} |\langle u_n, e_k^*\rangle|\Big) d_j \in D_n.$$

In particular, we have

$$\|y_n\|_{\ell_p}^p = \sum_{j=m_{n-1}+1}^{m_n} \Big(\sum_{k\in F_j} |\langle u_n, e_k^*\rangle|\Big)^p = \beta_p(u_n, \mathcal{C}_n)^p = 1,$$

so by the Hahn–Banach Theorem, we can take $f_n \in D_n^*$ such that $\langle y_n, f_n\rangle = 1 = \|f_n\|_{\ell_p^*}$. This enables us to define an operator $\Gamma \in \mathscr{B}\big(\big(\bigoplus_{n\in\mathbb{N}} D_n\big)_{\ell_p}, \ell_p\big)$ of norm 1 by $\Gamma(x_n)_{n\in\mathbb{N}} = \big(\langle x_n, f_n\rangle\big)_{n\in\mathbb{N}}$.

Finally, we compose these operators to obtain an operator $V = \Gamma\Theta\Sigma_{\mathcal{C}}\Delta \in \mathscr{B}(B_p, \ell_p)$; that is,

$$B_p \xrightarrow{\quad\Delta\quad} B_p \xrightarrow{\quad\Sigma_{\mathcal{C}}\quad} \ell_p \xrightarrow{\quad\Theta\quad} \Big(\bigoplus_{n\in\mathbb{N}} D_n\Big)_{\ell_p} \xrightarrow{\quad\Gamma\quad} \ell_p.$$

Recalling that $y_n = \Sigma_{\mathcal{C}}\Delta(u_n) \in D_n$ and then using the definitions of the operators Θ and Γ, we conclude that

$$Vu_n = \Gamma\Theta y_n = \big(\langle Q_j y_n, f_j\rangle\big)_{j\in\mathbb{N}} = \langle y_n, f_n\rangle d_n = d_n \qquad (n \in \mathbb{N}).$$

In particular, since u_n and d_n are unit vectors, we have $1 \leqslant \|V\| \leqslant \|\Gamma\|\,\|\Theta\|\,\|\Sigma_{\mathcal{C}}\|\,\|\Delta\| = 1$, so V has norm 1. □

Proof of Theorem 2.4. By Lemma 2.6 (applied with $X = E$ and $Y = D$), it suffices to show that for every constant $C > 1$ and every block subspace $W = \overline{\operatorname{span}}\,(w_n : n \in \mathbb{N})$ of E, where $(w_n)_{n\in\mathbb{N}}$ is a block basic sequence of the unit vector basis for E, there are operators $U \in \mathscr{B}(D, W)$ and $V \in \mathscr{B}(E, D)$ such that $V|_W U = I_D$ and $\|U\|\,\|V\| \leqslant C$.

Lemma 2.7 implies that we can find a normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$ such that $\|u_n\|_\infty \to 0$ as $n \to \infty$. By Lemma 2.8, $(u_n)_{n\in\mathbb{N}}$ admits a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ which is C-dominated by the unit vector basis $(d_j)_{j\in\mathbb{N}}$ for D. Therefore we can define an operator $U \in \mathscr{B}(D, W)$ by $Ud_j = u_{n_j}$ for every $j \in \mathbb{N}$, and $\|U\| \leqslant C$. Lemma 2.10 shows that there exists an operator $V \in \mathscr{B}(E, D)$ of norm 1 such that $Vu_{n_j} = d_j$ for every $j \in \mathbb{N}$. It follows that $VUd_j = d_j$ for every $j \in \mathbb{N}$, so $V|_W U = I_D$, and $\|U\| \cdot \|V\| \leqslant C \cdot 1 = C$, as required. □

Definition 2.11. A Banach space X is *subprojective* if every closed, infinite-dimensional subspace of X contains a closed, infinite-dimensional subspace which is complemented in X.

Theorem 2.4 implies that the Baernstein and Schreier spaces have this property. We record this observation formally for later reference.

Corollary 2.12. *The Baernstein spaces B_p for $1 < p < \infty$ and the Schreier spaces S_p for $1 \leqslant p < \infty$ are subprojective.*

Remark 2.13. Originally, Baernstein [3] proved that the Banach space B_2 is reflexive by verifying that the unit vector basis is a shrinking and boundedly complete basis for it and then appealing to a well-known theorem of James. We can now give an alternative proof of this result using Theorem 2.4, valid for any $1 < p < \infty$: the fact that B_p is ℓ_p-saturated implies that it does not contain any subspace isomorphic to either c_0 or ℓ_1. Since B_p has an unconditional basis, it follows from another well-known theorem of James that B_p is reflexive.

With a small amount of extra effort, we can characterize the normalized block basic sequences of the unit vector basis for the Baernstein and Schreier spaces that admit a subsequence which is equivalent to the unit vector basis for ℓ_p or c_0, respectively.

Proposition 2.14. *Let $(E,D) = (B_p,\ell_p)$ for some $1 < p < \infty$ or $(E,D) = (S_p,c_0)$ for some $1 \leqslant p < \infty$. The following conditions are equivalent for a normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ of the unit vector basis for E:*

(a) $\inf_{n\in\mathbb{N}}\|u_n\|_\infty = 0$;
(b) $(u_n)_{n\in\mathbb{N}}$ *admits a subsequence which is C-equivalent to the unit vector basis for D, for every constant $C > 1$;*
(c) $(u_n)_{n\in\mathbb{N}}$ *admits a subsequence which is dominated by the unit vector basis for D.*

Proof. To see that (a) implies (b), suppose that $\inf_{n\in\mathbb{N}}\|u_n\|_\infty = 0$, and take $C > 1$. By Lemma 2.8, $(u_n)_{n\in\mathbb{N}}$ admits a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ that is C-dominated by $(d_j)_{j\in\mathbb{N}}$. On the other hand, $(u_{n_j})_{j\in\mathbb{N}}$ 1-dominates $(d_j)_{j\in\mathbb{N}}$ because Lemma 2.10 shows that there is an operator $V \in \mathscr{B}(E,D)$ with $\|V\| = 1$ such that $Vu_{n_j} = d_j$ for every $j \in \mathbb{N}$. Hence $(u_{n_j})_{j\in\mathbb{N}}$ and $(d_j)_{j\in\mathbb{N}}$ are C-equivalent.

The implication (b)⇒(c) is trivial.

We complete the proof by proving that (c) implies (a), arguing contrapositively. Suppose that $\delta := \inf_{n\in\mathbb{N}}\|u_n\|_\infty > 0$, and take a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$. To verify that $(d_j)_{j\in\mathbb{N}}$ does not dominate $(u_{n_j})_{j\in\mathbb{N}}$, it suffices to show that for every $C \geqslant 1$, there exists $k \in \mathbb{N}$ such that

$$\Big\|\sum_{j=k}^{2k-1} u_{n_j}\Big\|_E > C\Big\|\sum_{j=k}^{2k-1} d_j\Big\|_D = \begin{cases} C & \text{for } D = c_0, \\ Ck^{\frac{1}{p}} & \text{for } D = \ell_p. \end{cases} \tag{2.15}$$

Choose $k \in \mathbb{N}$ such that $k > (C/\delta)^p$ if $E = S_p$ and $k > (C/\delta)^{\frac{p}{p-1}}$ if $E = B_p$, and set $x = \sum_{j=k}^{2k-1} u_{n_j} \in E$. By hypothesis, we can find $m_j \in \operatorname{supp}(u_{n_j})$ such that $|\langle u_{n_j}, e^*_{m_j}\rangle| \geqslant \delta$ for each $j \in \{k,\ldots,2k-1\}$. Then $F = \{m_j : k \leqslant j < 2k\}$ is a Schreier set because $|F| = k \leqslant m_k = \min F$. Hence we have

$$\|x\|_{S_p} \geqslant \mu_p(x,F) = \Big(\sum_{j=k}^{2k-1} |\langle x, e^*_{m_j}\rangle|^p\Big)^{\frac{1}{p}} = \Big(\sum_{j=k}^{2k-1} |\langle u_{n_j}, e^*_{m_j}\rangle|^p\Big)^{\frac{1}{p}} \geqslant k^{\frac{1}{p}}\delta > C$$

and

$$\|x\|_{B_p} \geqslant \beta_p(x,\{F\}) = \sum_{j=k}^{2k-1} |\langle x, e^*_{m_j}\rangle| = \sum_{j=k}^{2k-1} |\langle u_{n_j}, e^*_{m_j}\rangle| \geqslant k\delta > Ck^{\frac{1}{p}},$$

where the final inequalities follow from the choice of k in both cases. This establishes (2.15). □

3. An application to operator ideals: the proof of Theorem 1.1(ii)

The main purpose of this short section is to use Theorem 2.4 to identify the ideals of strictly singular, inessential and weakly compact operators on the Baernstein and Schreier spaces. We begin by recalling the formal definitions of these ideals, as well as some other standard notions that we require.

Definition 3.1. An operator $T \in \mathscr{B}(X,Y)$ between Banach spaces X and Y is:

- *strictly singular* if the restriction of T to W is not an isomorphic embedding for any infinite-dimensional subspace W of X,
- *inessential* if $I_X + UT$ is a Fredholm operator (meaning that its kernel is finite-dimensional and its range has finite codimension in X) for every operator $U \in \mathscr{B}(Y,X)$,
- *weakly compact* if the image under T of the unit ball in X is relatively weakly compact,
- *unconditionally converging* if the series $\sum_{n=1}^\infty Tx_n$ converges unconditionally in norm for every series $\sum_{n=1}^\infty x_n$ in X which is weakly unconditionally Cauchy in the sense that the series $\sum_{n=1}^\infty \langle x_n, f\rangle$ converges absolutely for every functional $f \in X^*$.

Furthermore, we say that T *fixes a copy* of a Banach space Z if there is an operator $V \in \mathscr{B}(Z,X)$ such that the composition TV is an isomorphic embedding.

We write $\mathscr{S}(X,Y)$, $\mathscr{E}(X,Y)$ and $\mathscr{W}(X,Y)$ for the sets of strictly singular, inessential and weakly compact operators between X and Y, respectively, with the usual convention that $\mathscr{S}(X) = \mathscr{S}(X,X)$, *etc.* It is well known that $\mathscr{S}$, $\mathscr{E}$ and $\mathscr{W}$ are closed operator ideals in the sense of Pietsch.

The Banach–Alaoglu Theorem implies that every operator defined on a reflexive Banach space is weakly compact, so $\mathscr{W}(B_p) = \mathscr{B}(B_p)$ for every $1 < p < \infty$.

Pfaffenberger [22] has shown that $\mathscr{S}(X) = \mathscr{E}(X)$ for every subprojective Banach space X. Hence, in view of Corollary 2.12, we obtain:

Proposition 3.2. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leqslant p < \infty$. Then*

$$\mathscr{S}(E) = \mathscr{E}(E).$$

Consequently, to complete the proof of Theorem 1.1(ii), it remains only to show that $\mathscr{W}(S_p) = \mathscr{S}(S_p)$ for $1 \leqslant p < \infty$. This will follow from our next, considerably more general, result, which applies to $X = S_p$ by Theorem 2.4 and the fact that S_p has an unconditional basis.

Proposition 3.3. *Let $T \in \mathscr{B}(X,Y)$ be an operator, where X is a c_0-saturated Banach space that embeds into a Banach space with an unconditional basis and Y is a separable Banach space. The following conditions are equivalent:*

(a) *T is strictly singular;*
(b) *T is inessential;*
(c) *the identity operator on c_0 does not factor through T in the sense that there are no operators $U \in \mathscr{B}(Y,c_0)$ and $V \in \mathscr{B}(c_0,X)$ such that $UTV = I_{c_0}$;*
(d) *T does not fix a copy of c_0;*
(e) *T is unconditionally converging;*
(f) *T is weakly compact.*

The proof relies on a classical result of Pełczyński [21, Proposition 9, 1°].

Theorem 3.4. *Let X be a Banach space which embeds into a Banach space with an unconditional basis, and suppose that X does not contain any subspace which is isomorphic to ℓ_1. Then every unconditionally converging operator from X into a Banach space is weakly compact.*

Proof of Proposition 3.3. The proof has two parts. In part (i), we show that conditions (a)–(d) are equivalent, while part (ii) contains the proof that conditions (d)–(f) are equivalent.

(i). The implication (a)$\Rightarrow$(b) is always true, and (b) implies (c) because the identity operator on an infinite-dimensional Banach space cannot be inessential.

We prove that (c) implies (d) by contraposition. Suppose that we can find an operator $V \in \mathscr{B}(c_0,X)$ such that TV is an isomorphic embedding. Then $TV[c_0]$ is isomorphic to c_0, so it is complemented in Y by Sobczyk's Theorem. Therefore TV has a left inverse $U \in \mathscr{B}(Y,c_0)$; that is, $UTV = I_{c_0}$, as desired.

Finally, (d) implies (a) because X is c_0-saturated.

(ii). As observed in [10, Exercise 8(i), page 54], conditions (d) and (e) are equivalent in general (that is, without any restrictions on the Banach spaces X and Y). The hypothesis that X is c_0-saturated ensures that ℓ_1 does not embed into X, so Theorem 3.4 shows that (e) implies (f). Finally, the implication (f)$\Rightarrow$(d) follows from the fact that c_0 is not reflexive. $\square$

Remark 3.5. The condition that the codomain Y of the operator T in Proposition 3.3 is separable cannot be dropped because the embedding of c_0 into ℓ_∞ is an inessential opeator which obviously fixes a copy of c_0.

4. The Gasparis–Leung index and its applications

The aim of this section is to establish counterparts for the Baernstein and Schreier spaces of some important technical results of Gasparis and Leung [14]. They introduced a numerical index for each $n \in \mathbb{N}$ and every pair M, N of infinite subsets of $\mathbb{N}$ which characterizes when the subspaces spanned by the infinite subsequences of the unit vector basis for $X[\mathcal{S}_n]$ corresponding to M and N are isomorphic. As we mentioned in the introduction, it turns out that this index, for $n = 1$, also works for the Baernstein and Schreier spaces.

In order to define it, we must first introduce the *Schreier covering number* of a set $A \in [\mathbb{N}]^{<\infty}$, which according to [14, Definition 3.1] and using our notation from Section 2 is

$$\tau_1(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \min\Big\{|\mathcal{C}| : \mathcal{C} \in \mathrm{SC},\ A \subseteq \bigcup \mathcal{C}\Big\} & \text{otherwise.} \end{cases} \tag{4.1}$$

Unpacking the somewhat condensed notation for $A \neq \emptyset$, we can restate this definition as

$$\tau_1(A) = \min\bigg\{m \in \mathbb{N} : A \subseteq \bigcup_{j=1}^{m} F_j, \text{ where } F_1, \ldots, F_m \in \mathcal{S}_1 \text{ and } F_1 < F_2 < \cdots < F_m\bigg\}.$$

Furthermore, as observed in [4, Remark 4.2], we can refine it as follows. Let $m \in \mathbb{N}$. Then $\tau_1(A) = m$ if and only if there is a Schreier chain $\{F_1 < \cdots < F_m\}$ such that $A = \bigcup_{j=1}^{m} F_j$ and $F_1, \ldots, F_{m-1}$ are maximal Schreier sets; it is important to note that F_m need not be maximal.

As in [4], for a set $M = \{m_1 < m_2 < \cdots\} \in [\mathbb{N}]$ and $J \subseteq \mathbb{N}$, we define

$$M(J) = \{m_j : j \in J\}.$$

This piece of notation enables us to state [14, Definition 3.3] in the following compact form. For $M, N \in [\mathbb{N}]$, the *Gasparis–Leung index* of M with respect to N is

$$\Gamma\mathrm{L}_1(M, N) = \sup\{\tau_1(M(J)) : J \in [\mathbb{N}]^{<\infty},\ N(J) \in \mathcal{S}_1\}. \tag{4.2}$$

Gasparis and Leung denoted this quantity $d_1(M, N)$. We have chosen the more distinctive symbol $\Gamma\mathrm{L}_1(M, N)$ in their honour, noting that the Greek spelling of "Gasparis" begins with the letter Γ.

Before we state the first main result of this section, let us introduce a piece of notation that we shall use frequently. Given a Banach space X with a basis $(x_n)_{n\in\mathbb{N}}$, we set

$$X_N = \overline{\operatorname{span}}\,(x_n : n \in N) \qquad (N \subseteq \mathbb{N}). \tag{4.3}$$

Further, recall from Section 2 that we write $\|x\|_\infty = \sup_{n\in\mathbb{N}} |\langle x, e_n^*\rangle|$ for $x \in B_p$ or $x \in S_p$ in line with standard usage.

Theorem 4.1. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leqslant p < \infty$, equipped with the unit vector basis $(e_n)_{n\in\mathbb{N}}$. The following conditions are equivalent for $M, N \in [\mathbb{N}]$:*

(a) *The Gasparis–Leung index $\Gamma\mathrm{L}_1(M, N)$ is finite.*
(b) *The basic sequence $(e_m)_{m\in M}$ dominates $(e_n)_{n\in N}$.*
(c) *There exists an operator $T \in \mathscr{B}(E_M, E_N)$ for which $\inf_{m\in M}\|Te_m\|_\infty > 0$.*

We begin with a quantitative version of the implication (a)⇒(b).

Lemma 4.2. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leqslant p < \infty$, take $M, N \in [\mathbb{N}]$ for which $\Gamma\mathrm{L}_1(M, N) < \infty$, and define*

$$C = \begin{cases} \Gamma\mathrm{L}_1(M, N) & \text{for } E = B_p, \\ \Gamma\mathrm{L}_1(M, N)^{\frac{1}{p}} & \text{for } E = S_p. \end{cases} \tag{4.4}$$

Then the basic sequence $(e_m)_{m\in M}$ C-dominates $(e_n)_{n\in N}$.

Proof. Enumerate M and N as $M = \{m_1 < m_2 < \cdots\}$ and $N = \{n_1 < n_2 < \cdots\}$, respectively. Our aim is to show that $\|y\|_E \leqslant C\|x\|_E$ whenever $x = \sum_{j=1}^{k} \alpha_j e_{m_j}$ and $y = \sum_{j=1}^{k} \alpha_j e_{n_j}$ for some $k \in \mathbb{N}$ and some $\alpha_1, \ldots, \alpha_k \in \mathbb{K}$. We may of course suppose that $\alpha_1, \ldots, \alpha_k$ are not all 0.

We consider the Baernstein and Schreier spaces separately, but emphasize that the proofs follow similar strategies, originating in the proof of [14, Lemma 3.4]. For readability, we begin with the easier case, which is $E = S_p$. Since y is finitely supported, we can choose a Schreier set F such that $\|y\|_{S_p} = \mu_p(y, F)$ and $F \subseteq \operatorname{supp} y \subseteq \{n_i : 1 \leqslant i \leqslant k\}$. Take $J \subseteq \{1, \ldots, k\}$ such that $N(J) = F$. Then $\tau_1(M(J)) \leqslant \Gamma\mathrm{L}_1(M, N) = C^p$ by (4.2) and (4.4), so there is a Schreier chain $\{G_1 < \cdots < G_{C^p}\}$ such that $M(J) \subseteq \bigcup_{i=1}^{C^p} G_i$ by (4.1). Set $K_i = \{j \in J : m_j \in G_i\}$ for

$i \in \{1, \ldots, C^p\}$. Then we have $J = \bigcup_{i=1}^{C^p} K_i$ and $K_h \cap K_i = \emptyset$ for $h \neq i$, from which we deduce that

$$\|y\|_{S_p}^p = \mu_p(y, F)^p = \sum_{j \in J} |\alpha_j|^p = \sum_{i=1}^{C^p} \sum_{j \in K_i} |\alpha_j|^p = \sum_{i=1}^{C^p} \mu_p(x, G_i)^p \leqslant C^p \|x\|_{S_p}^p,$$

where the final inequality follows from the fact that $G_1, \ldots, G_{C^p} \in \mathcal{S}_1$.

Having completed the proof for $E = S_p$, we turn our attention to $E = B_p$. We begin in the same way as above: using that y is finitely supported, we can find a Schreier chain $\mathcal{C} = \{F_1 < \cdots < F_t\}$ such that $\|y\|_{B_p} = \beta_p(y, \mathcal{C})$ and $\bigcup_{r=1}^t F_r \subseteq \operatorname{supp} y \subseteq \{n_i : 1 \leqslant i \leqslant k\}$.

Fix $r \in \{1, \ldots, t\}$, and choose $J_r \subseteq \{1, \ldots, k\}$ such that $N(J_r) = F_r \in \mathcal{S}_1$. Then we have $\tau_1(M(J_r)) \leqslant \Gamma\mathrm{L}_1(M, N) = C$ by (4.2) and (4.4), so we can find a Schreier chain $\{G_1^r < \cdots < G_C^r\}$ such that $M(J_r) \subseteq \bigcup_{i=1}^C G_i^r$ by (4.1). Set

$$K_i^r = \{j \in J_r : m_j \in G_i^r\} \qquad \text{and} \qquad \gamma_i^r = \sum_{j \in K_i^r} |\alpha_j| \qquad (i \in \{1, \ldots, C\}),$$

and choose $\iota(r) \in \{1, \ldots, C\}$ for which $\gamma_{\iota(r)}^r = \max\{\gamma_i^r : 1 \leqslant i \leqslant C\}$. Since $J_r = \bigcup_{i=1}^C K_i^r$ and $K_h^r \cap K_i^r = \emptyset$ whenever $h \neq i$, we have

$$\sum_{j \in J_r} |\alpha_j| = \sum_{i=1}^C \gamma_i^r \leqslant C\gamma_{\iota(r)}^r = C \sum_{j \in K_{\iota(r)}^r} |\alpha_j|.$$

Furthermore, $\mathcal{D} = \{M(K_{\iota(r)}^r) : 1 \leqslant r \leqslant t\}$ is a Schreier chain, as the following two facts show:

- $M(K_{\iota(r)}^r) \in \mathcal{S}_1$ for each $1 \leqslant r \leqslant t$ because $M(K_{\iota(r)}^r) \subseteq G_{\iota(r)}^r \in \mathcal{S}_1$, and
- the sets $M(K_{\iota(1)}^1), M(K_{\iota(2)}^2), \ldots, M(K_{\iota(t)}^t)$ are successive because $M(K_{\iota(r)}^r) \subseteq M(J_r)$ for each $1 \leqslant r \leqslant t$ and $M(J_1) < M(J_2) < \cdots < M(J_t)$.

In conclusion, we have

$$\|y\|_{B_p}^p = \beta_p(y, \mathcal{C})^p = \sum_{r=1}^t \Bigl(\sum_{j \in J_r} |\alpha_j|\Bigr)^p \leqslant C^p \sum_{r=1}^t \Bigl(\sum_{j \in K_{\iota(r)}^r} |\alpha_j|\Bigr)^p = C^p \beta_p(x, \mathcal{D})^p \leqslant C^p \|x\|_{B_p}^p. \qquad \square$$

Proof of Theorem 4.1, (a)⇒(b)⇒(c). Lemma 4.2 shows that (a) implies (b).

To see that (b) implies (c), suppose that $(e_m)_{m \in M}$ dominates $(e_n)_{n \in N}$. By definition, this means that the linear map $T\colon \operatorname{span}(e_m : m \in M) \to E_N$ determined by $Te_{m_j} = e_{n_j}$ for every $j \in \mathbb{N}$ is bounded, where $M = \{m_1 < m_2 < \cdots\}$ and $N = \{n_1 < n_2 < \cdots\}$ are the increasing enumerations. Therefore T extends uniquely to an operator in $\mathscr{B}(E_M, E_N)$, also denoted T, which satisfies $\langle Te_{m_j}, e_{n_k}^* \rangle = \delta_{j,k}$ for every $j, k \in \mathbb{N}$. Hence $\inf_{m \in M} \|Te_m\|_\infty = 1 > 0$. $\square$

It remains to prove the implication (c)⇒(a) in Theorem 4.1. For this, we follow the approach of [14] closely, although we can shorten certain steps because we consider only the first Schreier family $\mathcal{S}_1$.

We begin by generalizing [14, Proposition 3.13], which Gasparis and Leung established for the higher-order Schreier spaces $X[\mathcal{S}_\xi]$ for $\xi < \omega_1$. However, as we shall show, it applies to a much larger class of Banach spaces, including the Baernstein and Schreier spaces that we are investigating. We provide a detailed proof for the reader's convenience.

Lemma 4.3. *Let X and Y be Banach spaces with unconditional, normalized bases $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$, respectively, and suppose that the basis $(x_n)_{n \in \mathbb{N}}$ for X is shrinking. The following conditions are equivalent:*

(a) *There is an operator $T \in \mathscr{B}(X, Y)$ for which*

$$\inf_{k \in \mathbb{N}} \sup_{j \in \mathbb{N}} |\langle Tx_k, y_j^* \rangle| > 0. \tag{4.5}$$

(b) *There is an operator $U \in \mathscr{B}(X, Y)$ for which $Ux_k \in \{y_j : j \in \mathbb{N}\}$ for every $k \in \mathbb{N}$.*

It is easy to see that (b) implies (a). The proof of the converse relies on a careful analysis of the matrix $(T_{j,k})_{j,k\in\mathbb{N}}$ associated with the operator T. We recall the standard definition of this matrix: for $j,k\in\mathbb{N}$, the $(j,k)^{\text{th}}$ coefficient of the matrix associated with an operator $T\in\mathscr{B}(X,Y)$ between Banach spaces X and Y with bases $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, respectively, is

$$T_{j,k}=\langle Tx_k,y_j^*\rangle; \tag{4.6}$$

that is, the k^{th} column of the matrix associated with T contains the coordinates with respect to the basis $(y_j)_{j\in\mathbb{N}}$ of the image under T of the k^{th} basis vector x_k. Dualizing, we have $T_{j,k}=\langle x_k,T^*y_j^*\rangle$, so if the basis $(x_n)_{n\in\mathbb{N}}$ for X is shrinking, then the j^{th} row of the matrix associated with T contains the coordinates with respect to the basis $(x_k^*)_{k\in\mathbb{N}}$ for X^* of the image under T^* of the j^{th} coordinate functional y_j^*.

In the proof of Lemma 4.3, we require a variant of a result due to Tong [26]. It involves the following notion: a matrix $\Gamma=(\gamma_{j,k})_{j,k\in\mathbb{N}}$ is a *block diagonal* of a matrix $A=(\alpha_{j,k})_{j,k\in\mathbb{N}}$ if there are increasing sequences $0\leqslant r_1<r_2<\cdots$ and $0\leqslant s_1<s_2<\cdots$ of integers for which

$$\gamma_{j,k}=\begin{cases}\alpha_{j,k} & \text{if } (j,k)\in\bigcup_{i=1}^\infty (r_i,r_{i+1}]\times(s_i,s_{i+1}]\\ 0 & \text{otherwise}\end{cases}\qquad (j,k\in\mathbb{N}).$$

Lemma 4.4. *Let $T\in\mathscr{B}(X,Y)$ be an operator between Banach spaces X and Y with unconditional bases $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, respectively, and suppose that $\Gamma=(\gamma_{j,k})_{j,k\in\mathbb{N}}$ is a block diagonal of the matrix associated with T. Then there is an operator $R\in\mathscr{B}(X,Y)$ whose matrix is Γ; that is,*

$$\langle Rx_k,y_j^*\rangle=\gamma_{j,k}\qquad (j,k\in\mathbb{N}).$$

Proof. As already mentioned, Tong proved a similar result in [26], using very different terminology. A simple proof of the above statement is outlined in the first remark after [19, Proposition 1.c.8]. Note, however, that the definition stated in the text above [19, Proposition 1.c.8] of the matrix associated with an operator $T\in\mathscr{B}(X,Y)$ produces the transpose of the matrix given by (4.6). Fortunately this difference does not matter, as the transpose of a block diagonal is again a block diagonal of the transposed matrix. □

Proof of Lemma 4.3. To see that (b) implies (a), suppose that $U\in\mathscr{B}(X,Y)$ is an operator for which $Ux_k\in\{y_j:j\in\mathbb{N}\}$ for every $k\in\mathbb{N}$. Then $\sup_{j\in\mathbb{N}}|\langle Ux_k,y_j^*\rangle|=1$ for every $k\in\mathbb{N}$, so $T=U$ satisfies (4.5).

We prove that (a) implies (b) by expanding on the approach Gasparis and Leung took in their proof of [14, Proposition 3.13]. In view of (4.5) and (4.6), we can choose $\delta>0$ such that, for every $k\in\mathbb{N}$, $|T_{j,k}|\geqslant\delta$ for some $j\in\mathbb{N}$. This allows us to define a map $\psi\colon\mathbb{N}\to\mathbb{N}$ by

$$\psi(k)=\min\{j\in\mathbb{N}:|T_{j,k}|\geqslant\delta\}.$$

Take $j\in\mathbb{N}$. Since the basis $(x_n)_{n\in\mathbb{N}}$ for X is shrinking, the series $\sum_{k=1}^\infty T_{j,k}x_k^*$ is convergent with sum $T^*y_j^*$, as explained in the text below (4.6). It follows that $|T_{j,k}|\to 0$ as $k\to\infty$, so every natural number has finite (possibly empty) pre-image under ψ. Therefore ψ has infinite image; let $\psi(\mathbb{N})=\{n_1<n_2<\cdots\}$ be its increasing enumeration. Then $\{\psi^{-1}(n_j):j\in\mathbb{N}\}$ partitions $\mathbb{N}$ into non-empty, finite, disjoint sets, so there is a unique permutation $\rho\colon\mathbb{N}\to\mathbb{N}$ such that

$$\rho(k)<\rho(m)\iff\begin{cases}\psi(k)<\psi(m), \text{ or}\\ \psi(k)=\psi(m)\text{ and } k<m\end{cases}\qquad (k,m\in\mathbb{N}).$$

In more concrete terms, we can define ρ as follows. Set $s_0=0$ and $s_j=\sum_{i=1}^j|\psi^{-1}(n_i)|$ for $j\in\mathbb{N}$. Then each $k\in\mathbb{N}$ belongs to the interval $(s_{j-1},s_j]$ for a unique $j\in\mathbb{N}$, and $\rho(k)$ is the $(k-s_{j-1})^{\text{th}}$ smallest element of the set $\psi^{-1}(n_j)$. In particular, we have $\psi(\rho(k))=n_j$, so the definition of ψ implies that $|T_{n_j,\rho(k)}|\geqslant\delta$, and therefore we can define a diagonal operator $\Delta\in\mathscr{B}(X)$ of norm at most K/δ by $\Delta x_m=T_{n_j,m}^{-1}x_m$ for each $m\in\mathbb{N}$, where $j\in\mathbb{N}$ is chosen such that $s_{j-1}<\rho^{-1}(m)\leqslant s_j$ and K denotes the unconditional basis constant of $(x_n)_{n\in\mathbb{N}}$.

The unconditionality of the basis $(x_n)_{n\in\mathbb{N}}$ means that any reordering of it is also a basis for X. Hence, viewing the composite operator $P_{\psi(\mathbb{N})}T\Delta$ as a map from X to $Y_{\psi(\mathbb{N})}=\overline{\operatorname{span}}\,(y_{n_j}:j\in\mathbb{N})$, we may consider its matrix with respect to the bases $(x_{\rho(k)})_{k\in\mathbb{N}}$ for X and $(y_{n_j})_{j\in\mathbb{N}}$ for $Y_{\psi(\mathbb{N})}$. Suppose that $j,k\in\mathbb{N}$ satisfy $s_{j-1}<k\leqslant s_j$. Then we have

$$(P_{\psi(\mathbb{N})}T\Delta)_{j,k}=\langle P_{\psi(\mathbb{N})}T\Delta x_{\rho(k)},y_{n_j}^*\rangle=\frac{\langle Tx_{\rho(k)},P_{\psi(\mathbb{N})}^*y_{n_j}^*\rangle}{T_{n_j,\rho(k)}}=\frac{\langle Tx_{\rho(k)},y_{n_j}^*\rangle}{T_{n_j,\rho(k)}}=1,$$

so the matrix $\Gamma=(\gamma_{j,k})_{j,k\in\mathbb{N}}$ defined by

$$\gamma_{j,k}=\begin{cases}1 & \text{if } s_{j-1}<k\leqslant s_j\\ 0 & \text{otherwise}\end{cases}\qquad (j,k\in\mathbb{N}) \tag{4.7}$$

is a block diagonal of the matrix associated with the operator $P_{\psi(\mathbb{N})}T\Delta$. Lemma 4.4 implies that there is an operator $R\in\mathscr{B}(X,Y_{\psi(\mathbb{N})})$ whose matrix is Γ; that is, $\langle Rx_{\rho(k)},y_{n_j}^*\rangle=\gamma_{j,k}$ for every $j,k\in\mathbb{N}$. In view of (4.7), this means that $Rx_{\rho(k)}=y_{n_j}$ for every $k\in\mathbb{N}$, where $j\in\mathbb{N}$ is the unique number such that $s_{j-1}<k\leqslant s_j$. Hence, writing $J\colon Y_{\psi(\mathbb{N})}\to Y$ for the inclusion map, we obtain an operator $U=JR\in\mathscr{B}(X,Y)$ which satisfies $Ux_m=Rx_m\in\{y_n:n\in\mathbb{N}\}$ for every $m\in\mathbb{N}$, as required. □

Lemma 4.5. *Let $x=\sum_{j=1}^k\alpha_je_{m_j}$, where $k\in\mathbb{N}$, $\alpha_1,\ldots,\alpha_k\in[0,\infty)$, and $m_1,\ldots,m_k\in\mathbb{N}$ are (not necessarily distinct) numbers for which $\{m_1,\ldots,m_k\}\in\mathcal{S}_1$. Then $\|x\|_E=\sum_{j=1}^k\alpha_j$ for $E=S_1$ and $E=B_p$, while $\|x\|_{S_p}\geqslant\big(\sum_{j=1}^k\alpha_j^p\big)^{1/p}$ for $1<p<\infty$.*

Proof. Take $J\subseteq\{1,\ldots,k\}$ such that $\{m_j:j\in J\}=\{m_1,\ldots,m_k\}$ and $m_i\neq m_j$ for distinct $i,j\in J$, and set $K_j=\{i\in\{1,\ldots,k\}:m_i=m_j\}$ for each $j\in J$. Then $\{K_j:j\in J\}$ partitions $\{1,\ldots,k\}$, and we have $x=\sum_{j\in J}\big(\sum_{i\in K_j}\alpha_i\big)e_{m_j}$. Since $\{m_j:j\in J\}$ is a Schreier set, we conclude that $\|x\|_E=\sum_{j\in J}\big(\sum_{i\in K_j}\alpha_i\big)=\sum_{j=1}^k\alpha_j$ for $E=S_1$ and $E=B_p$, while

$$\|x\|_{S_p}^p=\sum_{j\in J}\Big(\sum_{i\in K_j}\alpha_i\Big)^p\geqslant\sum_{j\in J}\sum_{i\in K_j}\alpha_i^p=\sum_{j=1}^k\alpha_j^p$$

for $1<p<\infty$, where the inequality follows from the fact that the ℓ_1-norm dominates the ℓ_p-norm. □

Lemma 4.6. *Let $E=B_p$ for some $1<p<\infty$ or $E=S_p$ for some $1\leqslant p<\infty$, let $M\in[\mathbb{N}]$, and suppose that $\theta\colon M\to\mathbb{N}$ is a map for which the linear map $U\colon\operatorname{span}(e_m:m\in M)\to E$ determined by $Ue_m=e_{\theta(m)}$ for $m\in M$ is bounded. Then*

$$\sup\{|\theta^{-1}(n)|:n\in\mathbb{N}\}<\infty\qquad\text{and}\qquad\sup\{\tau_1(\theta^{-1}(F)):F\in\mathcal{S}_1\}<\infty. \tag{4.8}$$

Proof. The hypothesis means that U extends uniquely to an operator in $\mathscr{B}(E_M,E)$, also denoted U. We begin by showing that the second supremum in (4.8) is finite. Note that this will include showing that the pre-image under θ of every Schreier set F is finite, as otherwise $\tau_1(\theta^{-1}(F))$ is not defined. Our strategy is as follows: given $F\in\mathcal{S}_1$, we take $m\in\mathbb{N}$ for which $\theta^{-1}(F)$ contains a chain $\{G_1<G_2<\cdots<G_m\}$ of maximal Schreier sets. As we shall verify below, m is then dominated by a constant times a power of the norm of the operator U; that is, $m\leqslant C\|U\|^t$ for some constants $C,t\in(0,\infty)$ that will depend only on p. This will give the desired conclusion because (i) if $\theta^{-1}(F)$ were infinite, it would contain arbitrarily long chains of maximal Schreier sets, contradicting the uniform bound on m; (ii) we can therefore use the characterization of τ_1 stated in the paragraph below its definition (4.1) to deduce that $\tau_1(\theta^{-1}(F))\leqslant C\|U\|^t+1$. Since the right-hand side of this inequality is independent of F, it provides an upper bound on the second supremum in (4.8).

It remains to establish the inequality $m\leqslant C\|U\|^t$. We consider the two types of spaces separately. For $E=B_p$, set $x=\sum_{k=1}^m|G_k|^{-1}\sum_{j\in G_k}e_j$, and recall from (2.2) that $\|x\|_{B_p}^p\leqslant 2^pm$. Consequently, we have

$$2^pm\|U\|^p\geqslant\|Ux\|_{B_p}^p=\bigg\|\sum_{k=1}^m\frac{1}{|G_k|}\sum_{j\in G_k}e_{\theta(j)}\bigg\|_{B_p}^p=\bigg(\sum_{k=1}^m\frac{|G_k|}{|G_k|}\bigg)^p=m^p, \tag{4.9}$$

where the penultimate step follows from Lemma 4.5 and the fact that $\bigcup_{k=1}^m \theta(G_k)$ is a Schreier set because it is contained in $F \in \mathcal{S}_1$. Rearranging (4.9), we obtain $m \leqslant (2\|U\|)^{\frac{p}{p-1}}$, which provides an upper bound on m of the desired form for $C = 2^{\frac{p}{p-1}}$ and $t = p/(p-1)$.

The argument for $E = S_p$ is very similar, except that we use the vector

$$x = \sum_{k=1}^m \frac{1}{|G_k|^{\frac{1}{p}}} \sum_{j \in G_k} e_j.$$

It has S_p-norm at most $2^{\frac{1}{p}}$ by (2.1), so

$$2\|U\|^p \geqslant \|Ux\|_{S_p}^p = \Big\|\sum_{k=1}^m \frac{1}{|G_k|^{\frac{1}{p}}} \sum_{j \in G_k} e_{\theta(j)}\Big\|_{S_p}^p \geqslant \sum_{k=1}^m \frac{|G_k|}{|G_k|} = m$$

by another application of Lemma 4.5. This establishes the desired upper bound on m for $C = 2$ and $t = p$, thereby completing our proof that the second supremum in (4.8) is finite.

We now turn our attention to the first supremum in (4.8). Assume towards a contradiction that the set $\{|\theta^{-1}(n)| : n \in \mathbb{N}\}$ is unbounded. Given a non-empty set $G \subseteq \mathbb{N}$, it will be convenient to introduce the notation $G^\dagger = G \setminus \{\min G\}$. Arguing as in the proof of [14, Proposition 3.11], we can recursively construct an increasing sequence of maximal Schreier sets $G_1 < G_2 < \cdots$, each contained in $M \cap [2, \infty)$, and a sequence $(n_j)_{j\in\mathbb{N}}$ of natural numbers such that

$$\theta(i) = n_j \qquad (j \in \mathbb{N},\, i \in G_j^\dagger). \tag{4.10}$$

We include the details of this recursion for the reader's convenience. Set $m_1 = \min M \cap [2, \infty)$. By hypothesis, we can choose a number $n_1 \in \mathbb{N}$ such that $|\theta^{-1}(n_1)| > m_1$, so we can find a subset $F_1 \subseteq \theta^{-1}(n_1) \setminus \{1, m_1\}$ of cardinality $m_1 - 1$. Then $G_1 = \{m_1\} \cup F_1 \subseteq M \cap [2, \infty)$ is a maximal Schreier set, and (4.10) is satisfied for $j = 1$ because $G_1^\dagger = F_1$.

Now assume recursively that $G_1 < \cdots < G_{j-1}$ have been chosen for some $j \geqslant 2$. Set $m_j = \min M \cap (\max G_{j-1}, \infty)$, choose $n_j \in \mathbb{N}$ such that $|\theta^{-1}(n_j)| \geqslant m_j + |M \cap [1, m_j)|$, and take a subset $F_j \subseteq \theta^{-1}(n_j) \cap (m_j, \infty)$ of cardinality $m_j - 1$. Then $G_j = \{m_j\} \cup F_j \subseteq M \cap [m_j, \infty)$ is a maximal Schreier set such that (4.10) is satisfied for the given value of j, and $G_j > G_{j-1}$ because $\min G_j = m_j > \max G_{j-1}$. Hence the recursion continues.

Choose an integer m such that $m > (4\|U\|)^{\frac{p}{p-1}}$ if $E = B_p$ and $m > 4\|U\|^p$ if $E = S_p$. We observe that the set $\{n_j : j \in \mathbb{N}\}$ is unbounded, or else we could take $n \in \mathbb{N}$ and $J \in [\mathbb{N}]$ such that $n_j = n$ for every $j \in J$, which would imply that $\bigcup_{j\in J} G_j^\dagger \subseteq \theta^{-1}(n)$, contradicting that $\theta^{-1}(n)$ is finite, as shown in the first part of the proof. Consequently, we can find a set $K \in [\mathbb{N}]^{<\infty}$ such that $|K| = m \leqslant \min\{n_k : k \in K\}$, and therefore $\{n_k : k \in K\} \in \mathcal{S}_1$.

For $E = B_p$, consider the vector $y = \sum_{k\in K} |G_k|^{-1} \sum_{j \in G_k^\dagger} e_j \in B_p$, which has norm at most $2m^{\frac{1}{p}}$ by (2.2) and the 1-unconditionality of the basis $(e_j)_{j\in\mathbb{N}}$. We can now argue as in (4.9) to obtain

$$2^p m\|U\|^p \geqslant \|Uy\|_{B_p}^p = \Big\|\sum_{k\in K} \frac{|G_k| - 1}{|G_k|} e_{n_k}\Big\|_{B_p}^p = \Big(\sum_{k\in K} \frac{|G_k| - 1}{|G_k|}\Big)^p \geqslant \Big(\frac{m}{2}\Big)^p,$$

where we have used (4.10), Lemma 4.5 and the fact that $|G_k| - 1 \geqslant |G_k|/2$ for every $k \in \mathbb{N}$. Rearranging this inequality, we find $4^p\|U\|^p \geqslant m^{p-1}$, which contradicts our choice of m.

Again, the argument for $E = S_p$ is very similar, just using the vector

$$y = \sum_{k\in K} \frac{1}{|G_k|^{\frac{1}{p}}} \sum_{j \in G_k^\dagger} e_j \in S_p,$$

whose norm is at most $2^{\frac{1}{p}}$. Following the same steps as above, we obtain

$$2\|U\|^p \geqslant \|Uy\|_{S_p}^p = \Big\|\sum_{k\in K} \frac{|G_k| - 1}{|G_k|^{\frac{1}{p}}} e_{n_k}\Big\|_{S_p}^p \geqslant \sum_{k\in K} \frac{(|G_k| - 1)^p}{|G_k|} \geqslant \frac{m}{2},$$

once again contradicting the choice of m. □

Proposition 4.7. *Let $M, N \in [\mathbb{N}]$, and suppose that there exists a map $\theta\colon M \to N$ for which*

$$\sup\{|\theta^{-1}(n)| : n \in N\} < \infty \qquad \textit{and} \qquad \sup\{\tau_1(\theta^{-1}(F)) : F \in \mathcal{S}_1 \cap [N]^{<\infty}\} < \infty. \tag{4.11}$$

Then $\Gamma\mathrm{L}_1(M, N) < \infty$.

Proof. This is a restatement of [14, Proposition 3.12] for $\xi = 1$, bearing in mind that

$$\sup\{|\theta^{-1}(n)| : n \in N\} = \sup\{\tau_0(\theta^{-1}(F)) : F \in \mathcal{S}_0 \cap [N]^{<\infty}\}$$

because $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ and $\tau_0(A) = |A|$ for every $A \in [\mathbb{N}]^{<\infty}$. □

Proof of Theorem 4.1, (c)⇒(a). Suppose that $T \in \mathscr{B}(E_M, E_N)$ is an operator for which

$$\inf_{m \in M} \|Te_m\|_\infty > 0.$$

Then T satisfies condition (4.5) with respect to the bases $(e_m)_{m\in M}$ and $(e_n)_{n\in N}$ for E_M and E_N, respectively, so Lemma 4.3 shows that there is an operator $U \in \mathscr{B}(E_M, E_N)$ for which $Ue_m = e_{\theta(m)}$ for every $m \in M$, where $\theta(m) \in N$ is a suitably chosen index. Regarding θ as a map of M into $\mathbb{N}$, we can apply Lemma 4.6 to deduce that both suprema in (4.8) are finite. However, they are equal to the suprema in (4.11) because $\theta(M) \subseteq N$, so Proposition 4.7 implies that $\Gamma\mathrm{L}_1(M, N) < \infty$, as required. □

With the proof of Theorem 4.1 complete, we state an important consequence of it that is the second main outcome of this section.

Theorem 4.8. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leqslant p < \infty$. The following conditions are equivalent for $M, N \in [\mathbb{N}]$:*

(a) *The Gasparis–Leung indices $\Gamma\mathrm{L}_1(M, N)$ and $\Gamma\mathrm{L}_1(N, M)$ are both finite.*
(b) *The basic sequences $(e_m)_{m\in M}$ and $(e_n)_{n\in N}$ are equivalent.*
(c) *The subspaces E_M and E_N are isomorphic.*

As already indicated, we shall deduce this result from Theorem 4.1. However, the implication (c)⇒(a) requires one additional ingredient: every isomorphic embedding of E_M into E satisfies the technical condition (c) of Theorem 4.1.

Lemma 4.9. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leqslant p < \infty$, and suppose that $T \in \mathscr{B}(E_M, E)$ is an isomorphic embedding for some set $M \in [\mathbb{N}]$. Then $\inf_{m\in M} \|Te_m\|_\infty > 0$.*

Proof. Assume towards a contradiction that $\inf_{m\in M} \|Te_m\|_\infty = 0$ for some $M \in [\mathbb{N}]$ and some isomorphic embedding $T \in \mathscr{B}(E_M, E)$. Take $\eta > 0$ such that $\|Tx\| \geqslant \eta\|x\|$ for every $x \in E$, and set $k_0 = 0$, $P_0 = 0$ and $\varepsilon_j = \eta/(3 \cdot 2^j + 1)$ for $j \in \mathbb{N}$. We can then recursively choose increasing sequences $(m_j)_{j\in\mathbb{N}}$ in M and $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\|Te_{m_j}\|_\infty \leqslant \frac{\varepsilon_j}{2(k_{j-1} + 1)} \qquad \text{and} \qquad \|(I_E - P_{k_j})Te_{m_j}\|_E \leqslant \frac{\varepsilon_j}{2} \qquad (j \in \mathbb{N}).$$

This implies that for each $j \in \mathbb{N}$, the vector $v_j = (P_{k_j} - P_{k_{j-1}})Te_{m_j} \in E$ satisfies

$$\begin{aligned}\|Te_{m_j} - v_j\|_E &\leqslant \|(I_E - P_{k_j})Te_{m_j}\|_E + \|P_{k_{j-1}}Te_{m_j}\|_E \\ &\leqslant \frac{\varepsilon_j}{2} + k_{j-1} \cdot \max_{1\leqslant n\leqslant k_{j-1}} |\langle Te_{m_j}, e_n^*\rangle| \leqslant \frac{\varepsilon_j}{2} + \frac{\varepsilon_j}{2} = \varepsilon_j.\end{aligned}$$

In particular we have

$$\|T\| \geqslant \|v_j\|_E \geqslant \|Te_{m_j}\|_E - \|Te_{m_j} - v_j\|_E \geqslant \eta - \varepsilon_j \geqslant \frac{6\eta}{7},$$

so $(v_j)_{j\in\mathbb{N}}$ is a semi-normalized block basic sequence of $(e_n)_{n\in\mathbb{N}}$. Furthermore, since

$$\sum_{j=1}^\infty \frac{\|Te_{m_j} - v_j\|_E}{\|v_j\|_E} \leqslant \sum_{j=1}^\infty \frac{\varepsilon_j}{\eta - \varepsilon_j} = \sum_{j=1}^\infty \frac{1}{3 \cdot 2^j} = \frac{1}{3} < \frac{1}{2},$$

the Principle of Small Perturbations (see for instance [1, Theorem 1.3.9]) implies that $(Te_{m_j})_{j\in\mathbb{N}}$ is a basic sequence equivalent to $(v_j)_{j\in\mathbb{N}}$. (Here we have used the fact that the basis constant of $(v_j)_{j\in\mathbb{N}}$ is no greater than the basis constant of $(e_n)_{n\in\mathbb{N}}$, which is 1.)

Set $u_j = v_j/\|v_j\|_E$ for $j \in \mathbb{N}$. Being semi-normalized and unconditional, $(v_j)_{j\in\mathbb{N}}$ is equivalent to $(u_j)_{j\in\mathbb{N}}$, and

$$\|u_j\|_\infty = \frac{\|(P_{k_j} - P_{k_{j-1}})Te_{m_j}\|_\infty}{\|v_j\|_E} \leqslant \frac{\|Te_{m_j}\|_\infty}{6\eta/7} \leqslant \frac{7}{12(3\cdot 2^j + 1)(k_{j-1}+1)} \to 0 \quad \text{as} \quad j \to \infty,$$

so $\inf_{j\in\mathbb{N}}\|u_j\|_\infty = 0$. Hence Proposition 2.14 implies that $(u_j)_{j\in\mathbb{N}}$ admits a subsequence $(u_{j_n})_{n\in\mathbb{N}}$ which is equivalent to the unit vector basis $(d_n)_{n\in\mathbb{N}}$ for D, where $D = \ell_p$ if $E = B_p$ and $D = c_0$ if $E = S_p$, as usual.

In conclusion, we have shown that $(d_n)_{n\in\mathbb{N}}$ is equivalent to $(u_{j_n})_{n\in\mathbb{N}}$, which is equivalent to $(v_{j_n})_{n\in\mathbb{N}}$, which is equivalent to $(Te_{m_{j_n}})_{n\in\mathbb{N}}$, and therefore $(e_{m_{j_n}})_{n\in\mathbb{N}}$ is equivalent to $(d_n)_{n\in\mathbb{N}}$ because T is an isomorphic embedding. However, this is absurd: no subsequence of $(e_n)_{n\in\mathbb{N}}$ is dominated by $(d_n)_{n\in\mathbb{N}}$, as is easy to see (or alternatively this is a very special case of Proposition 2.14). ☐

Corollary 4.10. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leqslant p < \infty$, and suppose that E_M embeds isomorphically into E_N for some sets $M, N \in [\mathbb{N}]$. Then $\Gamma\mathrm{L}_1(M,N) < \infty$.*

Proof. Take an isomorphic embedding $T \in \mathscr{B}(E_M, E_N)$, and let $J\colon E_N \to E$ denote the inclusion map. Then we have $0 < \inf_{m\in M}\|JTe_m\|_\infty = \inf_{m\in M}\|Te_m\|_\infty$ by Lemma 4.9, so the implication (c)⇒(a) in Theorem 4.1 shows that $\Gamma\mathrm{L}_1(M,N) < \infty$. ☐

Proof of Theorem 4.8. The equivalence of conditions (a) and (b) in Theorem 4.1 implies that conditions (a) and (b) are also equivalent in Theorem 4.8. The implication (b)⇒(c) is clear, and finally Corollary 4.10 shows that (c) implies (a). ☐

We conclude this section with two applications of Corollary 4.10, both establishing counterparts for the Baernstein and Schreier spaces of results of Gasparis and Leung concerning the structure of the complemented subspaces of the higher-order Schreier spaces.

Definition 4.11. (i) A Banach space X is *primary* if the kernel or the range of P is isomorphic to X for every idempotent operator $P \in \mathscr{B}(X)$.
(ii) Two Banach spaces X and Y are *incomparable* if no subspace of X is isomorphic to Y and no subspace of Y is isomorphic to X.

Proposition 4.12. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leqslant p < \infty$. Then:*
(i) *The subspace E_N fails to be primary for every $N \in [\mathbb{N}]$.*
(ii) *There is a subset $\mathscr{A}$ of $[\mathbb{N}]$ of cardinality $\mathfrak{c}$ such that E_L and E_M are incomparable whenever $L, M \in \mathscr{A}$ are distinct.*

Proof. We follow the approach Gasparis and Leung took in their proofs of [14, Corollary 3.15 and Theorem 1.3], respectively. The common starting point is that for any set $N \in [\mathbb{N}]$, we can equip $[N]$ with the topology of pointwise convergence obtained by identifying the elements of $[N]$ with their indicator functions; this turns $[N]$ into a Polish space.

(i). The set

$$\mathscr{F} = \{(L,M) \in [N]\times[N] : L\cup M = N,\ L\cap M = \emptyset\}$$

is closed with respect to the product topology on $[N]\times[N]$, and

$$\mathscr{G} = \{(L,M) \in \mathscr{F} : \Gamma\mathrm{L}_1(N,L) = \Gamma\mathrm{L}_1(N,M) = \infty\}$$

is a dense G_δ-subset. In particular $\mathscr{G}$ is non-empty, so we can take $(L,M) \in \mathscr{G}$. The fact that $N = L\cup M$ and $L\cap M = \emptyset$ implies that $E_N = E_L \oplus E_M$, but E_N is neither isomorphic to E_L nor E_M by Corollary 4.10 because $\Gamma\mathrm{L}_1(N,L) = \Gamma\mathrm{L}_1(N,M) = \infty$. (In fact, E_L and E_M do not even contain subspaces which are isomorphic to E_N.) This proves that E_N is not primary.

(ii). By [14, Lemma 3.5],

$$\mathscr{D} = \{(L,M) \in [\mathbb{N}]\times[\mathbb{N}] : \Gamma\mathrm{L}_1(L,M) = \Gamma\mathrm{L}_1(M,L) = \infty\}$$

is a dense G_δ-subset of $[\mathbb{N}]\times[\mathbb{N}]$. Therefore, applying [14, Proposition 3.6], we can find a subset $\mathscr{A}$ of $[\mathbb{N}]$ which is homeomorphic to the Cantor set and satisfies $(L,M) \in \mathscr{D}$ whenever $L, M \in \mathscr{A}$ are distinct. In particular $\mathscr{A}$ has cardinality $\mathfrak{c}$, and Corollary 4.10 shows that E_L and E_M are incomparable for distinct $L, M \in \mathscr{A}$ because $\Gamma\mathrm{L}_1(L,M) = \Gamma\mathrm{L}_1(M,L) = \infty$. ☐

5. Spatial ideals of operators on the Baernstein and Schreier spaces

Let X be a Banach space. We write $\langle T\rangle$ for the (algebraic, two-sided) ideal of $\mathscr{B}(X)$ generated by an operator $T \in \mathscr{B}(X)$, that is,

$$\langle T\rangle = \left\{\sum_{j=1}^{k} U_j T V_j : k \in \mathbb{N},\, U_1,\ldots,U_k, V_1,\ldots,V_k \in \mathscr{B}(X)\right\}. \tag{5.1}$$

Since $\mathscr{B}(X)$ is a unital Banach algebra, the ideal $\langle T\rangle$ is proper if and only if its norm-closure $\overline{\langle T\rangle}$ is. Suppose that X has an unconditional basis. Following [4], we call the closed ideals of the form $\overline{\langle P_M\rangle}$ for some non-empty subset M of $\mathbb{N}$ *spatial*, where P_M denotes the basis projection, as usual.

The main aim of this section is to prove the following proposition, which is an extended counterpart of [4, Proposition 4.12] for the Baernstein and Schreier spaces. The key difference is the addition of a new quantitative condition, (d), that will play an essential role in the proofs of parts (i) and (iii) of Theorem 1.1 in the next section.

Proposition 5.1. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leqslant p < \infty$. The following conditions are equivalent for every pair of sets $M \subseteq \mathbb{N}$ and $N \in [\mathbb{N}]$:*

(a) $P_M \in \overline{\langle P_N\rangle}$,
(b) $\langle P_M\rangle \subseteq \langle P_N\rangle$,
(c) $\langle P_N\rangle = \langle P_{M\cup N}\rangle$,
(d) $\operatorname{dist}(P_M, \langle P_N\rangle) < 1$,
(e) $\Gamma\mathrm{L}_1(M\cup N, N) < \infty$,
(f) *E_N contains a subspace which is isomorphic to E_M,*
(g) *E_N contains a complemented subspace which is isomorphic to E_M,*
(h) *E_N is isomorphic to $E_{M\cup N}$,*
(i) *The basic sequences $(e_n)_{n\in M\cup N}$ and $(e_n)_{n\in N}$ are equivalent.*

We require two lemmas in the proof of this proposition. The first is a variant of the Neumann series, showing that every idempotent element which is close to an ideal of a Banach algebra must in fact belong to the ideal.

Lemma 5.2. *Let $\mathscr{I}$ be an ideal of a Banach algebra $\mathscr{A}$, and take a non-zero idempotent $p \in \mathscr{A}$. Then $p \in \mathscr{I}$ if (and only if) $\operatorname{dist}(p, \mathscr{I}) < \|p\|^{-2}$.*

Proof. The implication $\Rightarrow$ is obvious. Conversely, suppose that $\|p - a\| < \|p\|^{-2}$ for some $a \in \mathscr{I}$. Then $\|p - pap\| < 1$, so the series $\sum_{n=1}^{\infty}(p - pap)^n$ converges absolutely. Set $b = p + \sum_{n=1}^{\infty}(p - pap)^n \in \mathscr{A}$ and observe that

$$\begin{aligned}\mathscr{I} \ni bpap &= \left(p + \sum_{n=1}^{\infty}(p-pap)^n\right)\big(p - (p - pap)\big)\\ &= p - p(p - pap) + \sum_{n=1}^{\infty}(p-pap)^n p - \sum_{n=2}^{\infty}(p-pap)^n = p. \qquad \square\end{aligned}$$

The second lemma is the counterpart of [4, Proposition 4.6]. It will enable us to connect the first four conditions of Proposition 5.1 concerning ideals with the last four (or five) concerning subspaces.

Lemma 5.3. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leqslant p < \infty$. Then*

$$E_M \cong E_M \oplus E_M \qquad (M \in [\mathbb{N}]).$$

Proof. We take the same approach as in the proof of [4, Proposition 4.6]. Set $M' = 2M - 1$ and $M'' = 2M$. Since these sets are disjoint, we have

$$E_{M'\cup M''} = E_{M'} \oplus E_{M''}$$

by unconditionality, so it will suffice to show that each of these spaces is isomorphic to E_M, which in turn will follow from Theorem 4.8 provided that the appropriate Gasparis–Leung indices are finite. First, we have $\Gamma\mathrm{L}_1(M, M' \cup M'') \leqslant 3$ and $\Gamma\mathrm{L}_1(M' \cup M'', M) \leqslant 2$ by [4, Lemma 4.11], so $E_{M'\cup M''} \cong E_M$. Second, M'' is a spread of M and of M', so $\Gamma\mathrm{L}_1(M'', M) = 1 = \Gamma\mathrm{L}_1(M'', M')$. Third, we claim that

$$\Gamma\mathrm{L}_1(M, M'') \leqslant 2 \qquad \text{and} \qquad \Gamma\mathrm{L}_1(M', M'') \leqslant 2. \tag{5.2}$$

Indeed, suppose that $M''(J) \in \mathcal{S}_1$ for some non-empty $J \in [\mathbb{N}]^{<\infty}$. Then $2m_{j_1} \geqslant k$, where we have written $J = \{j_1 < \cdots < j_k\}$ and $M = \{m_1 < m_2 < \cdots\}$. This implies that we can partition J into two subsets, J_1 and J_2, each having at most m_{j_1} elements, and therefore $M(J_1), M(J_2), M'(J_1), M'(J_2) \in \mathcal{S}_1$. Hence we have $\tau_1(M(J)) \leqslant 2$ and $\tau_1(M'(J)) \leqslant 2$ because $M(J) = M(J_1) \cup M(J_2)$ and $M'(J) = M'(J_1) \cup M'(J_2)$. This proves (5.2), and consequently $E_M \cong E_{M''} \cong E_{M'}$. □

Proof of Proposition 5.1. Lemma 5.3 implies that $E_N \cong E_N \oplus E_N$ and $E_{M\cup N} \cong E_{M\cup N} \oplus E_{M\cup N}$ because N is infinite. Hence conditions (a), (b), (c), (g) and (h) are equivalent by [4, Lemma 2.3 and Corollary 2.5].

We have $\Gamma\mathrm{L}_1(N, M \cup N) = 1$ because N is a spread of $M \cup N$, so conditions (e), (h) and (i) are equivalent by Theorem 4.8.

The implications (a)⇒(d) and (g)⇒(f) are trivial, while Lemma 5.2 shows that (d) implies (b). We complete the proof by showing that (f) implies (e). Suppose that E_M embeds isomorphically into E_N. Then $E_{M\cup N} = E_M \oplus E_{N\setminus M}$ embeds isomorphically into $E_N \oplus E_N \cong E_N$, so $\Gamma\mathrm{L}_1(M \cup N, N) < \infty$ by Corollary 4.10. □

Proposition 5.1 enables us to establish a counterpart for the Baernstein and Schreier spaces of the main result of [4]. This requires one additional piece of terminology. Let X be a Banach space with an unconditional basis. The ideal $\mathscr{K}(X)$ of compact operators is always spatial because $\overline{\langle P_M \rangle} = \mathscr{K}(X)$ if (and only if) $M \in [\mathbb{N}]^{<\infty} \setminus \{\emptyset\}$. Following [4], we call a spatial ideal $\mathscr{I}$ *non-trivial* if $\mathscr{K}(X) \subsetneq \mathscr{I} \subsetneq \mathscr{B}(X)$.

Theorem 5.4. *Let $E = B_p$ for some $1 < p < \infty$ or $E = S_p$ for some $1 \leqslant p < \infty$. Then:*

- (i) *The family of non-trivial spatial ideals of $\mathscr{B}(E)$ is non-empty and has no minimal or maximal elements.*
- (ii) *Let $\mathscr{I} \subsetneq \mathscr{J}$ be spatial ideals of $\mathscr{B}(E)$. Then there is a family $\{\Gamma_L : L \in \Delta\}$ such that:*
  - (1) *the index set Δ has the cardinality of the continuum;*
  - (2) *for each $L \in \Delta, \Gamma_L$ is an uncountable chain of spatial ideals of $\mathscr{B}(E)$ such that*

    $$\mathscr{I} \subsetneq \mathscr{L} \subsetneq \mathscr{J} \qquad (\mathscr{L} \in \Gamma_L),$$

    and $\overline{\bigcup \Gamma_L}$ is a closed ideal that is not spatial;
  - (3) *$\mathscr{L} + \mathscr{M} = \mathscr{J}$ whenever $\mathscr{L} \in \Gamma_L$ and $\mathscr{M} \in \Gamma_M$ for distinct $L, M \in \Delta$.*
- (iii) *The Banach algebra $\mathscr{B}(E)$ contains at least continuum many maximal ideals.*
- (iv) *The ideal*

  $$\bigcap \{\mathscr{I} : \mathscr{I} \text{ is a non-trivial spatial ideal of } \mathscr{B}(E)\}$$

  is not contained in the ideal of strictly singular operators on E.

Proof. Clauses (i)–(iii) are the counterparts for E of [4, Theorem 1.1] and can be proved in exactly the same way; see [4, pages 10–11]. This requires that we establish the counterpart of [4, Lemma 2.8] for E, which we can do by copying the proof given in [4, pages 21–24] for $n = 1$, just referring to Proposition 5.1 instead of [4, Proposition 4.12] throughout.

(iv) is the counterpart of [4, Theorem 1.2(ii), Equation (1.1)], and the proof is similar. Indeed, let $D = \ell_p$ if $E = B_p$ and $D = c_0$ if $E = S_p$, and take a projection $Q \in \mathscr{B}(E)$ whose range is isomorphic to D. Then Q is not strictly singular, but Theorem 2.4 implies that $Q \in \langle P_N \rangle$ for every $N \in [\mathbb{N}]$, and therefore Q belongs to every non-trivial spatial ideal of $\mathscr{B}(E)$. □

6. Finding $2^{\mathfrak{c}}$ many closed ideals of operators: the proofs of Theorem 1.1(i) and (iii)

In this section we combine our previous results about the Gasparis–Leung index and ideals generated by basis projections to prove the remaining two parts of Theorem 1.1; that is, taking $E = B_p$ for $1 < p < \infty$ or $E = S_p$ for $1 \leqslant p < \infty$, as usual, we shall show that $\mathscr{B}(E)$ contains $2^{\mathfrak{c}}$ many closed ideals that lie between the ideals of compact and strictly singular operators, as well as $2^{\mathfrak{c}}$ many closed ideals that are "large" in the sense that they contain projections of infinite rank. Note that $\mathscr{B}(E)$ cannot contain more than $2^{\mathfrak{c}}$ many closed ideals because E is separable.

Both results rely on a general theorem of Freeman, Schlumprecht and Zsák [13, Proposition 1] that extracts the key idea of the argument that Johnson and Schechtman [16] used to show that $\mathscr{B}(L_p[0,1])$ contains $2^{\mathfrak{c}}$ many closed ideals for every $p \in (1,2) \cup (2,\infty)$. Before we can state the said theorem of Freeman–Schlumprecht–Zsák precisely, we require two additional pieces of terminology. The first generalizes the classical notion of a (closed) ideal of a (Banach) algebra to the space of operators between two distinct Banach spaces X and Y: a *(closed) ideal* of $\mathscr{B}(X,Y)$ is a (norm-closed) subspace $\mathscr{J}$ of $\mathscr{B}(X,Y)$ such that $UTV \in \mathscr{J}$ whenever $V \in \mathscr{B}(X)$, $T \in \mathscr{J}$ and $U \in \mathscr{B}(Y)$. Extending (5.1), for a subset $\mathscr{T}$ of $\mathscr{B}(X,Y)$, we write $\langle \mathscr{T} \rangle$ for the ideal of $\mathscr{B}(X,Y)$ it generates.

The reason this generalization is useful for our purposes is that in the case where X contains a complemented subspace isomorphic to Y, the map

$$\mathscr{J} \mapsto \overline{\left\{ \sum_{j=1}^{n} U_j T_j : n \in \mathbb{N},\, U_1, \ldots, U_n \in \mathscr{B}(Y,X),\, T_1, \ldots, T_n \in \mathscr{J} \right\}} \tag{6.1}$$

is an injection from the lattice of closed ideals of $\mathscr{B}(X,Y)$ into the lattice of closed ideals of $\mathscr{B}(X)$. (This is a special case of an observation stated above [13, Proposition 2], and is also easy to verify directly.) Hence, to show that $\mathscr{B}(X)$ contains $2^{\mathfrak{c}}$ many closed ideals, it suffices to find a complemented subspace Y of X for which $\mathscr{B}(X,Y)$ contains $2^{\mathfrak{c}}$ many closed ideals.

The second notion that we require is that of a 1-*unconditional finite-dimensional decomposition,* or 1-*UFDD* for short, of a Banach space X; that is, a sequence $(X_n)_{n\in\mathbb{N}}$ of finite-dimensional subspaces of X such that every $x \in X$ has a unique decomposition of the form $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for every $n \in \mathbb{N}$, and the series $\sum_{n=1}^{\infty} \sigma_n x_n$ converges with $\|\sum_{n=1}^{\infty} \sigma_n x_n\| \leqslant \|x\|$ for every sequence $(\sigma_n)_{n\in\mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$. It follows that for every (non-empty) subset N of $\mathbb{N}$, we can define a projection $Q_N \in \mathscr{B}(X)$ of norm 1 by $Q_N x = \sum_{n\in N} x_n$.

In fact, we shall only consider 1-UFDDs of a very simple kind. Let X be a Banach space with a 1-unconditional basis $(x_n)_{n\in\mathbb{N}}$, and take a partition $J_1 < J_2 < \cdots$ of $\mathbb{N}$ into finite, successive intervals. Then the sequence of finite-dimensional subspaces given by

$$X_n = \operatorname{span}\{x_j : j \in J_n\} \qquad (n \in \mathbb{N}) \tag{6.2}$$

is a 1-UFDD for X. For later reference, we observe that in this case the projection Q_N, for $N \subseteq \mathbb{N}$, defined above is equal to the basis projection P_{L_N} induced by the set $L_N = \bigcup_{n\in N} J_n$.

Theorem 6.1 (Freeman, Schlumprecht and Zsák)**.** *Let $\mathcal{A} \subset [\mathbb{N}]$ be an almost disjoint family of cardinality $\mathfrak{c}$, let X and Y be Banach spaces with 1-UFDDs $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$, respectively, and suppose that $T \in \mathscr{B}(X,Y)$ is an operator which satisfies*

(i) *$T[X_n] \subseteq Y_n$ for every $n \in \mathbb{N}$;*
(ii) $\inf\{\operatorname{dist}(TQ_M, \langle TQ_N \rangle) : M, N \in [\mathbb{N}],\, |M \setminus N| = \infty\} > 0.$

Then the map

$$\mathcal{N} \mapsto \overline{\langle TQ_N : N \in \mathcal{N} \rangle} \tag{6.3}$$

defines an order-preserving injection from the power set of $\mathcal{A}$ into the lattice of closed ideals of $\mathscr{B}(X,Y)$.

To enable us to apply this theorem to the Baernstein and Schreier spaces, we present a variant not involving dyadic trees of the key construction that Manoussakis and Pelczar-Barwacz used in their proof of [20, Lemma 4.3].

Construction 6.2. Set $F_1 = \emptyset$. By recursion, we can partition $\mathbb{N}$ into finite, successive intervals $G_1 < F_2 < G_2 < F_3 < G_3 < \cdots$ with the following properties:

(i) G_n is the union of n successive maximal Schreier sets for each $n \in \mathbb{N}$ (so in particular $\tau_1(G_n) = n$),

(ii) $|F_n| = \sum_{m=1}^{n-1}(|F_m| + |G_m|)$ for each $n \geqslant 2$.

For brevity, we introduce the notation $J_n = F_n \cup G_n$ for $n \in \mathbb{N}$ and set

$$L_N = \bigcup_{n\in N} J_n \qquad (N \subseteq \mathbb{N}). \tag{6.4}$$

We observe for later reference that we can rewrite property (ii) as $|F_n| = \sum_{m=1}^{n-1}|J_m|$.

Lemma 6.3. *For $M, N \in [\mathbb{N}]$, the set $M \setminus N$ is bounded above by $\Gamma\mathrm{L}_1(L_M, L_N) \in \mathbb{N} \cup \{\infty\}$, where $L_M, L_N \in [\mathbb{N}]$ are defined by* (6.4).

Proof. Take $m \in M\setminus N$. We seek to prove that $m \leqslant \Gamma\mathrm{L}_1(L_M, L_N)$, which by the definition (4.2) of the Gasparis–Leung index means that we must find a set $K \in [\mathbb{N}]^{<\infty}$ such that

$$\tau_1(L_M(K)) \geqslant m \qquad \text{and} \qquad L_N(K) \in \mathcal{S}_1. \tag{6.5}$$

The case $m = 1$ is trivial, so we may suppose that $m \geqslant 2$. We have $J_m \subseteq L_M$ because $m \in M$, so we can find $H \in [\mathbb{N}]^{<\infty}$ such that $J_m = L_M(H)$. Note that H is an interval because J_m is, and the definition of J_m implies that we can split H in two subintervals $H_1 < H_2$ such that $L_M(H_1) = F_m$ and $L_M(H_2) = G_m$. We claim that $K = H_2$ satisfies (6.5).

The first part is immediate because $\tau_1(L_M(H_2)) = \tau_1(G_m) = m$, so it only remains to show that $L_N(H_2) \in \mathcal{S}_1$; that is, $|L_N(H_2)| \leqslant \min(L_N(H_2))$. Set $k = \min H_2$ and observe that $\min(L_N(H_2))$ is the k^{th} element of the set

$$L_N = \bigcup_{n\in N} J_n = \Bigl(\bigcup_{n\in N\cap[1,m)} J_n\Bigr) \cup \Bigl(\bigcup_{n\in N\cap(m,\infty)} J_n\Bigr)$$

because $m \notin N$. We have

$$\Bigl|\bigcup_{n\in N\cap[1,m)} J_n\Bigr| = \sum_{n\in N\cap[1,m)} |J_n| \leqslant \sum_{j=1}^{m-1} |J_j| = |F_m| = |L_M(H_1)| = |H_1| \leqslant \max H_1 < k,$$

so the k^{th} element of L_N must belong to the set $\bigcup_{n\in N\cap(m,\infty)} J_n$. Hence

$$\begin{aligned}\min(L_N(H_2)) &\geqslant \min\Bigl(\bigcup_{n\in N\cap(m,\infty)} J_n\Bigr) \geqslant \min J_{m+1} > \max J_m \\ &= \max G_m \geqslant |G_m| = |L_M(H_2)| = |H_2| = |L_N(H_2)|,\end{aligned}$$

and the conclusion follows. ☐

Corollary 6.4. *The following conditions are equivalent for $M, N \in [\mathbb{N}]$:*

(a) $\Gamma\mathrm{L}_1(L_M, L_N) < \infty$,

(b) $\Gamma\mathrm{L}_1(L_{M\cup N}, L_N) < \infty$,

(c) $|M \setminus N| < \infty$.

Proof. Lemma 6.3 shows that the set $M \setminus N$ is bounded above by $\Gamma\mathrm{L}_1(L_M, L_N)$, so (a) implies (c).

(c)⇒(b). Suppose that $M \setminus N$ is finite. Then $L_{M\cup N} \setminus L_N = L_{M\setminus N}$ is finite, too, so $\Gamma\mathrm{L}_1(L_{M\cup N}, L_N) < \infty$.

(b)⇒(a). This is a consequence of the fact that $\mathcal{S}_1$ is closed under spreading. ☐

Proof of Theorem 1.1(iii). We shall apply Theorem 6.1 with $X = Y = E$, $T = I_E$ and the 1-UFDDs given by $X_n = Y_n = \operatorname{span}\{e_j : j \in J_n\}$ for every $n \in \mathbb{N}$, where $J_1 < J_2 < \cdots$ are the intervals defined in Construction 6.2. These choices ensure that condition (i) of Theorem 6.1 is trivially satisfied. To verify condition (ii), we recall that for $N \subseteq \mathbb{N}$, the projection Q_N associated with the chosen 1-UFDDs is the basis projection P_{L_N}. Taking $M, N \in [\mathbb{N}]$ with $|M \setminus N| = \infty$, we have $\Gamma\mathrm{L}_1(L_M \cup L_N, L_N) = \Gamma\mathrm{L}_1(L_{M\cup N}, L_N) = \infty$ by Corollary 6.4, so

Proposition 5.1 implies that $\operatorname{dist}(Q_M, \langle Q_N\rangle) = \operatorname{dist}(P_{L_M}, \langle P_{L_N}\rangle) = 1$. Hence Theorem 6.1 shows that

$$\bigl\{\overline{\langle P_{L_N} : N \in \mathcal{N}\rangle} : \mathcal{N} \subseteq \mathcal{A}\bigr\}$$

is a collection of $2^{\mathfrak{c}}$ many distinct closed ideals of $\mathscr{B}(E)$ for any almost disjoint family $\mathcal{A} \subset [\mathbb{N}]$ of cardinality $\mathfrak{c}$.

The set $\{UV : U \in \mathscr{B}(D,E),\, V \in \mathscr{B}(E,D)\}$ is closed under addition and therefore an ideal of $\mathscr{B}(E)$ because $D \cong D \oplus D$, where we recall that $D = \ell_p$ if $E = B_p$ and $D = c_0$ if $E = S_p$, as usual. An easy standard argument shows that this ideal is equal to $\langle Q\rangle$ for any projection $Q \in \mathscr{B}(E)$ whose range is isomorphic to D. As we saw in the proof of Theorem 5.4(iv), $Q \in \langle P_N\rangle$ for every $N \in [\mathbb{N}]$, so $\langle Q\rangle \subseteq \overline{\langle P_{L_N} : N \in \mathcal{N}\rangle}$ for every non-empty subset $\mathcal{N}$ of $\mathcal{A}$. □

The proof of Theorem 1.1(i) follows a similar path, but some aspects require additional work. We begin with a standard characterization of strictly singular operators defined on a Banach space with a basis, and include a short proof for completeness.

Lemma 6.5. *Let $T \in \mathscr{B}(X,Y)$ be an operator between Banach spaces X and Y, and suppose that X has a basis. Then T is strictly singular if (and only if) the restriction of T to any block subspace of X fails to be an isomorphic embedding.*

Proof. The implication $\Rightarrow$ is trivial because block subspaces are infinite-dimensional.

Conversely, suppose that T fails to be strictly singular, so that its restriction to some closed, infinite-dimensional subspace Z of X is bounded below by some number $\eta > 0$. We use the same notation and approach as in the first part of the proof of Lemma 2.6; that is, $(x_n)_{n\in\mathbb{N}}$ denotes the basis of X, K is the basis constant, P_n is the n^{th} basis projection for $n \in \mathbb{N}$, and we set $m_0 = 0$ and $P_0 = 0$. By recursion, we choose natural numbers $m_1 < m_2 < \cdots$ and unit vectors $z_n \in Z \cap \ker P_{m_{n-1}}$ such that

$$\|z_n - w_n\| \leqslant \varepsilon_n, \quad \text{where} \quad w_n = P_{m_n} z_n \quad \text{and} \quad \varepsilon_n = \frac{\eta}{2^{n+2}K(\eta + \|T\|) + \eta} \qquad (n \in \mathbb{N}).$$

Then $(w_n)_{n\in\mathbb{N}}$ is a block basic sequence of $(x_n)_{n\in\mathbb{N}}$ because $\|w_n\| \geqslant 1 - \varepsilon_n > 0$ for every $n \in \mathbb{N}$.

We shall now complete the proof by showing that the restriction of T to the block subspace spanned by $(w_n)_{n\in\mathbb{N}}$ is bounded below by $\eta/2$. Take a unit vector $w = \sum_{n=1}^N \alpha_n w_n$ for some $N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_N \in \mathbb{K}$, and set $z = \sum_{n=1}^N \alpha_n z_n \in Z$. We have

$$\begin{aligned} \|Tw\| &\geqslant \|Tz\| - \|T(z-w)\| \geqslant \eta\|z\| - \|T\|\|z-w\| \\ &\geqslant \eta(\|w\| - \|z-w\|) - \|T\|\|z-w\| = \eta - (\eta + \|T\|)\|z-w\|. \end{aligned} \tag{6.6}$$

To find an upper bound on $\|z - w\|$, we observe that $\alpha_n w_n = (P_{m_n} - P_{m_{n-1}})w$, so

$$|\alpha_n| \leqslant \frac{2K}{\|w_n\|} \leqslant \frac{2K}{1-\varepsilon_n} \qquad (1 \leqslant n \leqslant N),$$

and therefore

$$\|z-w\| \leqslant \sum_{n=1}^N |\alpha_n|\|z_n - w_n\| \leqslant \sum_{n=1}^N \frac{2K\varepsilon_n}{1-\varepsilon_n} = \sum_{n=1}^N \frac{\eta}{(\eta + \|T\|)2^{n+1}} \leqslant \frac{\eta}{2(\eta + \|T\|)},$$

where the equality in the middle follows from the choice of ε_n. Substituting this estimate into (6.6), we conclude that $\|Tw\| \geqslant \eta/2$, which establishes the result. □

Proposition 6.6. *Let $(E,D) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(E,D) = (S_p, c_0)$ for some $1 \leqslant p < \infty$, and let $(e_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ denote the unit vector bases for E and D, respectively. Then the formal inclusion map given by $\iota\colon e_n \mapsto d_n$ for $n \in \mathbb{N}$ extends to a bounded linear injection $\iota\colon E \to D$ of norm 1. Furthermore, ι is strictly singular, but not compact.*

Proof. It is obvious that the formal inclusion map $\iota\colon S_p \to c_0$ is a bounded linear injection of norm 1, while the same conclusion for $\iota\colon B_p \to \ell_p$ is an easy consequence of the definition of the norm on B_p, or alternatively it follows by applying Lemma 2.9 to the chain $\mathcal{C} = \bigl\{\{n\} : n \in \mathbb{N}\bigr\}$. The non-compactness of ι is witnessed by its action on the unit vector basis in both cases, so it only remains to verify that ι is strictly singular.

Lemma 6.5 implies that it suffices to show that the restriction of ι to the closed subspace spanned by a block basic sequence $(w_n)_{n\in\mathbb{N}}$ of $(e_n)_{n\in\mathbb{N}}$ is not an isomorphic embedding. By Lemma 2.7, $(w_n)_{n\in\mathbb{N}}$ admits a normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ for which $\|u_n\|_\infty \to 0$ as $n\to\infty$. This completes the proof for the Schreier space S_p because $\|\iota(u_n)\| = \|u_n\|_\infty$ in this case. The argument for the Baernstein space B_p is more subtle, relying on an inequality due to Jameson that we shall establish in Appendix A below; it involves a constant $K_p > 0$ which depends only on p. Using the variant of Jameson's inequality stated in the last line of Theorem A.1, we obtain

$$\|\iota(u_n)\|^p = \|u_n\|_{\ell_p}^p \leqslant K_p\|u_n\|_\infty^{p-1}\|u_n\|_{B_p} = K_p\|u_n\|_\infty^{p-1} \to 0 \quad \text{as} \quad n\to\infty. \qquad \square$$

Proof of Theorem 1.1(i). We shall apply Theorem 6.1 with $(X, Y) = (E, D)$, that is, either $(X, Y) = (B_p, \ell_p)$ for some $1 < p < \infty$ or $(X, Y) = (S_p, c_0)$ for some $1 \leqslant p < \infty$, endowed with the 1-UFDDs obtained by blocking the unit vector bases as follows:

$$X_n = \operatorname{span}\{e_j : j \in J_n\} \qquad \text{and} \qquad Y_n = \operatorname{span}\{d_j : j\in J_n\} \qquad (n\in\mathbb{N}), \tag{6.7}$$

where $J_1 < J_2 < \cdots$ are the intervals defined in Construction 6.2, and $T = \iota \in \mathscr{B}(E, D)$ is the formal inclusion map.

Condition (i) of Theorem 6.1 is trivially satisfied because $\iota(e_j) = d_j$ for $j\in\mathbb{N}$. We claim that the infimum in condition (ii) equals 1. To prove that, we begin by recalling that $Q_N = P_{L_N}$ for every $N\subseteq\mathbb{N}$, where $Q_N \in \mathscr{B}(E)$ denotes the projection associated with the 1-UFDD $(X_n)_{n\in\mathbb{N}}$ of E, the set L_N is given by (6.4), and $P_{L_N}\in\mathscr{B}(E)$ is the corresponding basis projection, as usual. Hence the claim will follow provided that we show that

$$\operatorname{dist}(\iota P_{L_M}, \langle \iota P_{L_N}\rangle) = 1 \qquad (M, N \in [\mathbb{N}],\ |M\setminus N| = \infty).$$

The inequality $\leqslant$ is trivial because $\|\iota P_{L_M}\| = 1$. We shall verify the opposite inequality by showing that if $\operatorname{dist}(\iota P_{L_M}, \langle\iota P_{L_N}\rangle) < 1$ for some $M, N\in[\mathbb{N}]$, then $|M\setminus N| < \infty$. Hence, suppose that $\|\iota P_{L_M} - R\| < 1$ for some operator $R \in \langle\iota P_{L_N}\rangle$, say $R = \sum_{j=1}^k U_j\iota P_{L_N}V_j$, where $k\in\mathbb{N}$, $U_1,\ldots,U_k\in\mathscr{B}(D)$ and $V_1,\ldots,V_k\in\mathscr{B}(E)$. By replacing U_j with $\|V_j\|U_j$ and V_j with $\frac{V_j}{\|V_j\|}$ if $\|V_j\| > 0$, we may suppose that $\|V_j\|\leqslant 1$ for each $j\in\{1,\ldots,k\}$.

Take $m\in L_M$. Since e_m and $d_m = \iota P_{L_M}e_m$ are unit vectors, we have

$$\|\iota P_{L_M} - R\| \geqslant \|(\iota P_{L_M} - R)e_m\|_D \geqslant \|\iota P_{L_M}e_m\|_D - \|Re_m\|_D = 1 - \|Re_m\|_D,$$

so

$$1 - \|\iota P_{L_M} - R\| \leqslant \|Re_m\|_D \leqslant \sum_{j=1}^k \|U_j\|\,\|\iota P_{L_N}V_je_m\|_D \leqslant k\cdot \max_{1\leqslant j\leqslant k}\|U_j\|\cdot\|\iota P_{L_N}V_{\varphi(m)}e_m\|_D, \tag{6.8}$$

where we have chosen $\varphi(m)\in\{1,\ldots,k\}$ such that

$$\max_{1\leqslant j\leqslant k}\|\iota P_{L_N}V_je_m\|_D = \|\iota P_{L_N}V_{\varphi(m)}e_m\|_D.$$

This defines a map $\varphi\colon L_M\to\{1,\ldots,k\}$ which in view of (6.8) satisfies

$$\|\iota P_{L_N}V_{\varphi(m)}e_m\|_D \geqslant \eta \qquad (m\in L_M), \qquad \text{where} \qquad \eta = \frac{1 - \|\iota P_{L_M} - R\|}{k\cdot\max_{1\leqslant j\leqslant k}\|U_j\|} > 0. \tag{6.9}$$

We use this map to introduce a new operator

$$W = P_{L_N}\sum_{j\in\varphi(L_M)} V_jP_{\varphi^{-1}(\{j\})}|_{E_{L_M}} \in \mathscr{B}(E_{L_M}, E_{L_N}).$$

Our aim is to show that it satisfies

$$\inf_{m\in L_M}\|We_m\|_\infty \geqslant \begin{cases} \eta & \text{for } E = S_p,\\ \Big(\dfrac{\eta^p}{K_p}\Big)^{\frac{1}{p-1}} & \text{for } E = B_p,\end{cases} \tag{6.10}$$

where $K_p > 0$ denotes the constant from Theorem A.1. Take $m \in L_M$, and observe that $\|We_m\|_\infty = \|P_{L_N} V_{\varphi(m)} e_m\|_\infty$ because

$$P_{\varphi^{-1}(\{j\})} e_m = \begin{cases} e_m & \text{if } j = \varphi(m), \\ 0 & \text{otherwise.} \end{cases}$$

If $E = S_p$, then $D = c_0$, so $\|P_{L_N} V_{\varphi(m)} e_m\|_\infty = \|\iota P_{L_N} V_{\varphi(m)} e_m\|_D \geqslant \eta$ by (6.9), which establishes (6.10) in the first case. Otherwise $E = B_p$ and $D = \ell_p$; combining (6.9) with Jameson's inequality stated in the last line of Theorem A.1, we obtain

$$\eta^p \leqslant \|\iota P_{L_N} V_{\varphi(m)} e_m\|_{\ell_p}^p \leqslant K_p \|P_{L_N} V_{\varphi(m)} e_m\|_\infty^{p-1} \|P_{L_N} V_{\varphi(m)} e_m\|_{B_p} \leqslant K_p \|We_m\|_\infty^{p-1}, \quad (6.11)$$

where the simple estimate $\|P_{L_N} V_{\varphi(m)} e_m\|_{B_p} \leqslant \|P_{L_N}\| \, \|V_{\varphi(m)}\| \, \|e_m\|_{B_p} \leqslant 1$ justifies the final inequality. The second case of (6.10) follows by rearranging (6.11).

Hence the operator W satisfies condition (c) of Theorem 4.1, so $\Gamma\mathrm{L}_1(L_M, L_N) < \infty$, and therefore $|M \setminus N| < \infty$ by Corollary 6.4, as required.

We have thus verified both conditions of Theorem 6.1. It follows that the map (6.3) is injective. Composing it with the injection (6.1), we obtain $2^{\mathfrak{c}}$ many closed ideals of $\mathscr{B}(E)$. They are contained in the ideal of strictly singular operators because the operator ι is strictly singular, as we showed in Proposition 6.6. ☐

Appendix A. Jameson's inequality for the Schreier and Baernstein norms

The aim of this appendix is to establish an inequality which relates the ℓ_p-norm, the first Schreier (or p^{th} Baernstein) norm and the ℓ_∞-norm. (Recall that we denote the latter by $\|\cdot\|_\infty$.) Its proof is due to Graham Jameson; we are very grateful for his permission to include it here. The inequality plays a key role in the proof of Theorem 1.1(i) that we gave in Section 6.

Theorem A.1 (Jameson)**.** *For every $1 < p < \infty$, there is a constant $K_p \in \big[\frac{2^p-1}{2^{p-1}-1}, \frac{3\cdot 2^{p-1}-2}{2^{p-1}-1}\big]$ such that*

$$\|x\|_{\ell_p}^p \leqslant K_p \|x\|_\infty^{p-1} \|x\|_{S_1} \qquad (x \in \mathbb{K}^{\mathbb{N}}). \quad (A.1)$$

Consequently, $\|x\|_{\ell_p}^p \leqslant K_p \|x\|_\infty^{p-1} \|x\|_{B_p}$ for every $x \in B_p$.

We begin with a lemma that will help us reduce to the case of decreasing sequences.

Lemma A.2. *Let $x\colon \mathbb{N} \to [0, \infty)$ be decreasing with limit 0. Then $\|x\|_{S_p} \leqslant \|x \circ \sigma\|_{S_p}$ for every $1 \leqslant p < \infty$ and every permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$.*

Proof. Take $F \in \mathcal{S}_1 \setminus \{\emptyset\}$, and let $k = \min F$. Since σ is surjective, the set $\sigma^{-1}([1, 2k) \cap \mathbb{N}) \setminus [1, k)$ contains a subset G of cardinality k. Then $G \in \mathcal{S}_1$, and therefore

$$\|x \circ \sigma\|_{S_p}^p \geqslant \sum_{n \in G} x(\sigma(n))^p \geqslant \sum_{j=k}^{2k-1} x(j)^p \geqslant \mu_p(x, F)^p,$$

where the second inequality follows because $\sigma(G)$ is a k-element subset of $[1, 2k) \cap \mathbb{N}$ and x is decreasing. Now the conclusion follows by taking the supremum over F. ☐

Proof of Theorem A.1. Since all three norms in (A.1) depend only on the moduli of the coordinates of x, it suffices to consider non-negative x. We may also suppose that $\|x\|_{S_1} < \infty$, as otherwise the inequality is trivial. This implies that $x \in c_0$, which in turn means that we can find a permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $x \circ \sigma$ is decreasing. Therefore we can apply Lemma A.2 to $x \circ \sigma$ and the permutation σ^{-1} to obtain that $\|x \circ \sigma\|_{S_1} \leqslant \|(x \circ \sigma) \circ \sigma^{-1}\|_{S_1} = \|x\|_{S_1}$, while $\|x \circ \sigma\|_{\ell_p} = \|x\|_{\ell_p}$ and $\|x \circ \sigma\|_\infty = \|x\|_\infty$. In conclusion, this shows that it suffices to consider the case where $x\colon \mathbb{N} \to [0, \infty)$ is decreasing, and after scaling, we may suppose that $\|x\|_{S_1} = 1$.

Take $n \in \mathbb{N}_0$, and let $F_n = [2^n, 2^{n+1}) \cap \mathbb{N} \in \mathcal{S}_1$. Since x is non-negative and decreasing, we have $\|x\|_\infty = x(1)$ and $x(2^{n+1}) \leqslant x(j) \leqslant x(2^n)$ for $j \in F_n$. Combining this with the fact that $\mu_1(x, F_n) \leqslant \|x\|_{S_1} = 1$, we obtain

$$x(2^{n+1}) \leqslant \frac{1}{2^n} \qquad \text{and} \qquad \mu_p(x, F_n)^p \leqslant x(2^n)^{p-1} \mu_1(x, F_n) \leqslant x(2^n)^{p-1}. \quad (A.2)$$

Write $x(1) = \frac{\theta}{2^k}$, where $k \in \mathbb{N}$ and $1 \leqslant \theta \leqslant 2$. Then we have

$$\sum_{n=0}^{k-1} \mu_p(x, F_n)^p = \sum_{j=1}^{2^k-1} x(j)^p \leqslant (2^k - 1)x(1)^p \leqslant \theta x(1)^{p-1}.$$

Furthermore, (A.2) implies that $\mu_p(x, F_k)^p \leqslant x(1)^{p-1}$ and

$$\sum_{n=k+1}^{\infty} \mu_p(x, F_n)^p \leqslant \sum_{n=k+1}^{\infty} \frac{1}{2^{(n-1)(p-1)}} = \Big(\frac{1}{2^k}\Big)^{p-1} \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{x(1)^{p-1}2^{p-1}}{\theta^{p-1}(2^{p-1} - 1)}.$$

Since $(F_n)_{n=0}^{\infty}$ is a partition of $\mathbb{N}$, we conclude that

$$\|x\|_{\ell_p}^p = \sum_{n=0}^{\infty} \mu_p(x, F_n)^p \leqslant f(\theta)x(1)^{p-1} = f(\theta)\|x\|_\infty^{p-1}, \quad \text{where} \quad f(\theta) = \theta + 1 + \frac{2^{p-1}}{(2^{p-1} - 1)\theta^{p-1}}.$$

This defines a smooth function $f\colon (0, \infty) \to (1, \infty)$ whose second derivative $f''(\theta) = \frac{2^{p-1}p(p-1)}{(2^{p-1}-1)\theta^{p+1}}$ is positive. Hence f is convex, so $\max\{f(\theta) : 1 \leqslant \theta \leqslant 2\} = \max\{f(1), f(2)\}$. We find that

$$f(1) = f(2) = \frac{3 \cdot 2^{p-1} - 2}{2^{p-1} - 1},$$

and therefore the inequality (A.1) is satisfied for some constant $K_p \leqslant \frac{3\cdot 2^{p-1}-2}{2^{p-1}-1}$.

To verify that this constant is at least $\frac{2^p-1}{2^{p-1}-1}$, take $k \in \mathbb{N}$ and define $x\colon \mathbb{N} \to (0, \infty)$ by

$$x(j) = \begin{cases} \dfrac{1}{2^k} & \text{for } 1 \leqslant j < 2^{k+1}, \\ \dfrac{1}{2^n} & \text{for } 2^n \leqslant j < 2^{n+1}, \text{ where } n \in (k, \infty) \cap \mathbb{N}. \end{cases}$$

Then $\|x\|_\infty = \frac{1}{2^k}$ and

$$\|x\|_{\ell_p}^p = \frac{2^{k+1} - 1}{2^{kp}} + \sum_{n=k+1}^{\infty} \frac{2^n}{2^{np}} = \frac{2}{2^{k(p-1)}} - \frac{1}{2^{kp}} + \frac{1}{2^{k(p-1)}(2^{p-1} - 1)}.$$

We claim that $\|x\|_{S_1} = 1$. Since x is decreasing, it suffices to consider Schreier sets of the form $[j, 2j) \cap \mathbb{N}$ for $j \in \mathbb{N}$ when computing $\|x\|_{S_1}$. Clearly $\mu_1(x, [j, 2j) \cap \mathbb{N}) = j/2^k \leqslant 1$ for $1 \leqslant j \leqslant 2^k$. Otherwise $j = 2^n + m$ for some $n \geqslant k$ and $1 \leqslant m \leqslant 2^n$, and we have

$$\mu_1(x, [j, 2j) \cap \mathbb{N}) = \frac{2^n - m}{2^n} + \frac{2m}{2^{n+1}} = 1.$$

This proves the claim. Hence

$$\begin{aligned} K_p \geqslant \frac{\|x\|_{\ell_p}^p}{\|x\|_\infty^{p-1}\|x\|_{S_1}} &= \Big(\frac{2}{2^{k(p-1)}} - \frac{1}{2^{kp}} + \frac{1}{2^{k(p-1)}(2^{p-1} - 1)}\Big)2^{k(p-1)} \\ &= 2 - \frac{1}{2^k} + \frac{1}{2^{p-1} - 1} \to 2 + \frac{1}{2^{p-1} - 1} = \frac{2^p - 1}{2^{p-1} - 1} \qquad \text{as} \qquad k \to \infty. \end{aligned}$$

The inequality stated in the last line of the theorem follows immediately from (A.1) because the p^{th} Baernstein norm 1-dominates the first Schreier norm due to the fact that

$$\mu_1(x, F) = \beta_p(x, \{F\}) \qquad (x \in \mathbb{K}^{\mathbb{N}},\ F \in \mathcal{S}_1). \qquad \square$$

References

[1] F. Albiac and N.J. Kalton, *Topics in Banach space theory.* Graduate Texts in Mathematics 233, Springer, New York, 2006.

[2] D.E. Alspach and S. Argyros, Complexity of weakly null sequences, *Diss. Math. (Rozprawy Mat.)* **321** (1992).

[3] A. Baernstein II, On reflexivity and summability, *Studia Math.* **42** (1972), 91–94.

[4] K. Beanland, N.J. Laustsen and T. Kania, Closed ideals of operators on the Tsirelson and Schreier spaces, *J. Funct. Anal.* **279** (2020), 108668.

[5] B. Beauzamy, Banach–Saks properties and spreading models, *Math. Scand.* **44** (1979), 357–384.

[6] A. Bird and N.J. Laustsen, An amalgamation of the Banach spaces associated with James and Schreier, Part I: Banach-space structure, in *Proceedings of the 19th International Conference on Banach Algebras* (ed. R.J. Loy, V. Runde and A. Sołtysiak), *Banach Center Publications* **91** (2010), 45–76.
[7] P.G. Casazza and T.J. Shura, *Tsirelson's space.* Springer Lecture Notes in Mathematics 1363, Springer-Verlag, Berlin, 1989.
[8] R.M. Causey and A. Pelczar-Barwacz, Equivalence of block sequences in Schreier spaces and their duals, preprint available at `https://arxiv.org/abs/2311.00761`
[9] H.V. Chu and Th. Schlumprecht, Higher order Tsirelson spaces and their modified versions are isomorphic, preprint available at `https://arxiv.org/abs/2401.16491`
[10] J. Diestel, *Sequences and series in Banach spaces.* Springer-Verlag, New York, 1984.
[11] H. Fetter and B. Gamboa de Buen, *The James forest.* LMS Lecture Notes 236, Cambridge University Press, 1997.
[12] J. Flores, F. Hernandez, E. Semenov and P. Tradacete, Strictly singular and power compact operators on Banach lattices, *Israel J. Math.* **188** (2012), 323–352.
[13] D. Freeman, Th. Schlumprecht and A. Zsák, Banach spaces for which the space of operators has $2^{\mathfrak{c}}$ closed ideals, *Forum Math., Sigma* **9** (2021), 1–20.
[14] I. Gasparis and D.H. Leung, On the complemented subspaces of the Schreier spaces, *Studia Math.* **141** (2000), 273–300.
[15] W.T. Gowers, A solution to Banach's hyperplane problem, *Bull. London Math. Soc.* **26** (1994), 523–530.
[16] W.B. Johnson and G. Schechtman, The number of closed ideals in $L(L_p)$, *Acta Math.* **227** (2021), 103–113.
[17] N.J. Laustsen, Maximal ideals in the algebra of operators on certain Banach spaces, *Proc. Edinb. Math. Soc.* **45**(2002), 523–546.
[18] N.J. Laustsen and R.J. Loy, Closed ideals in the Banach algebra of operators on a Banach space, in *Proceedings of the Conference on Topological Algebras, their Applications, and Related Topics* (ed. K. Jarosz and A. Sołtysiak), *Banach Center Publications* **67** (2005), 245–264.
[19] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces* I. Springer-Verlag, Berlin, 1977.
[20] A. Manoussakis and A. Pelczar-Barwacz, Small operator ideals on the Schlumprecht and Schreier spaces, *J. Funct. Anal.* **281** (2021), 109156.
[21] A. Pełczyński, Banach spaces on which every unconditionally converging operator is weakly compact, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **10** (1962), 641–648.
[22] W. Pfaffenberger, On the ideals of strictly singular and inessential operators, *Proc. Amer. Math. Soc.* **25** (1970), 603–607.
[23] Th. Schlumprecht, An arbitrarily distortable Banach space, *Israel J. Math.* **76** (1991), 81–95.
[24] J. Schreier, Ein Gegenbeispiel zur Theorie der schwachen Konvergentz, *Studia Math.* **2** (1930), 58–62.
[25] C.J. Seifert, *Averaging in Banach spaces.* Kent State University Graduate School Dissertations, 1977.
[26] A.E. Tong, Diagonal submatrices of matrix maps, *Pac. J. Math.* **32** (1970), 551–559.

(N.J. Laustsen) School of Mathematical Sciences, Fylde College, Lancaster University, Lancaster LA1 4YF, United Kingdom
Email address: `n.laustsen@lancaster.ac.uk`

(J. Smith) School of Mathematical Sciences, Fylde College, Lancaster University, Lancaster LA1 4YF, United Kingdom
Email address: `j.smith43@lancaster.ac.uk`