Homological and combinatorial properties of discrete cluster categories

Sofia Franchini

This thesis is submitted for the degree of Doctor of Philosophy

Lancaster University, School of Mathematical Sciences

October 2024

Homological and combinatorial properties of discrete cluster categories

Sofia Franchini

Abstract

In this thesis we work with combinatorial and homological aspects of Igusa–Todorov discrete cluster categories $\mathcal{C}_m$. Fix a positive integer m. The category $\mathcal{C}_m$ is an infinite discrete version of the classical cluster category of type A_n. This is a 2-Calabi–Yau triangulated category with cluster-tilting subcategories. The category $\mathcal{C}_m$ has a nice geometric model in terms of an ∞-gon, $\mathcal{Z}_m$, having m two-sided accumulation points, in which the indecomposable objects of $\mathcal{C}_m$ are in bijection with the arcs of $\mathcal{Z}_m$.

The Paquette–Yıldırım completion, $\overline{\mathcal{C}}_m$, of $\mathcal{C}_m$ has a geometric model where the indecomposable objects can be regarded as "limits of arcs" of $\mathcal{Z}_m$. The arc combinatorics of $\overline{\mathcal{C}}_m$ allows us to classify the torsion pairs, t-structures, co-t-structures, and recollements of $\overline{\mathcal{C}}_m$. We observe that the categories $\mathcal{C}_m$ and $\overline{\mathcal{C}}_m$, despite having similar combinatorics, have some relevant homological differences.

We also work on defining different Calabi–Yau versions of $\mathcal{C}_m$. We provide a candidate w-Calabi–Yau version for $w \geq 2$, $\mathcal{C}_{w,m}$, by taking the subcategory of w-admissible objects and morphisms of $\mathcal{C}_m$. We expect that, by restricting the triangulated structure of $\mathcal{C}_m$, we obtain a triangulated structure for $\mathcal{C}_{w,m}$. Under the assumption that $\mathcal{C}_{w,m}$ is triangulated, we classify its w-cluster tilting subcategories and torsion pairs.

We also define the category $\mathcal{C}_{-1,m}$, the (-1)-Calabi–Yau version of $\mathcal{C}_m$. To do so, we define an infinite discrete version of symmetric Nakayama representations using techniques from persistence theory. We obtain an abelian category which is Frobenius, uniserial, and symmetric. After stabilising, we obtain our desired (-1)-Calabi–Yau triangulated category. The category $\mathcal{C}_{-1,1}$ is additive equivalent to the Holm–Jørgensen category having (-1)-Calabi–Yau dimension, and we conjecture that the two categories are triangle equivalent.

Contents

Acknowledgments

I would like to thank my supervisor, David Pauksztello for the guidance, the encouragements, the generous support, and the attention always dedicated to me. Thank you for introducing me to the beautiful topics of this thesis and for all the mathematical discussions which expanded my views and understanding.

Thank you to Jan Grabowski and Raquel Coelho Simões for the advice and the support since the start of my PhD, and to Nicholas Williams for sharing mathematical interests during various seminars. I would also like to thank Eric Hanson for helpful mathematical conversation.

Thank you to my fellow PhD students in Lancaster for your enjoyable presence and the nice and welcoming environment, and to all my mathematical friends with whom I shared beautiful moments.

Thank you to my family for unconditional support.

Declaration

This thesis is my own work and has not been submitted for the award of a higher degree elsewhere. This thesis is not a result of a joint work. Most of the content of Chapter 3 and Chapter 4 is based on the preprint [20] which has been submitted for publication. This thesis does not exceed the maximum 80,000 word length.

Chapter 1

Introduction

The classical cluster category of type A_n, $\mathcal{C}(A_n)$ was defined by Buan, Marsh, Reineke, Reiten, and Todorov in [7] as the orbit category $\mathcal{D}^b(\mathbb{K}A_n)/\Sigma\tau^{-1}$. The geometric model of $\mathcal{C}(A_n)$ was described by Caldero, Chapoton, and Schiffler in [9], in terms of a $(n+3)$-gon: the isoclasses of indecomposable objects of $\mathcal{C}(A_n)$ are in bijection with the diagonals of the $(n+3)$-gon. Keller proved in [34] that $\mathcal{C}(A_n)$ is a 2-*Calabi–Yau* triangulated category, meaning that its Hom-spaces are somehow "symmetric". More precisely, for each pair of objects $a, b \in \mathcal{C}(A_n)$, there is a natural isomorphism

$$\operatorname{Hom}_{\mathcal{C}(A_n)}(a, b) \cong D \operatorname{Hom}_{\mathcal{C}(A_n)}(b, \Sigma^2 a)$$

where Σ is the shift functor on $\mathcal{C}(A_n)$ and $D = \operatorname{Hom}_{\mathbb{K}}(-, \mathbb{K})$ is the usual vector space duality. Another important property of $\mathcal{C}(A_n)$ is that it has *cluster-tilting subcategories*, which can be viewed as "projective-minded generating sets" and correspond to triangulations of the $(n+3)$-gon in the geometric model.

By considering the orbit categories of the infinite quivers A_∞ and A_∞^∞, Liu and Paquette obtained in [39] infinite versions of the category $\mathcal{C}(A_n)$. The orbit category of A_∞ coincides with the Holm-Jørgensen category $\mathcal{T}_2$, which first appeared in [26] and [32] as the finite derived category of $\mathbb{K}[T]$ viewed as a differential graded algebra with T in degree one. Equivalently, in [3] the category $\mathcal{T}_2$ was obtained by stabilising a certain subcategory of a Grassmannian category of infinite rank. By [35], the category $\mathcal{T}_2$ is also the unique algebraic triangulated category generated by a 2-spherical object, up to triangle equivalence. The Holm–Jørgensen and Liu–Paquette categories have geometric models which generalise the $(n+3)$-gon of $\mathcal{C}(A_n)$: the indecomposable objects of the Holm–Jørgensen category are in bijection with certain pairs of integers, while for the Liu–Paquette category they are in bijection with certain pairs of elements of an infinite strip, see [26] and [39] respectively.

Igusa–Todorov discrete cluster categories. Given a positive integer m, Igusa and Todorov defined the category $\mathcal{C}_m$ in [30]. This category is a generalisation of the Holm–Jørgensen and Liu–Paquette categories. The category $\mathcal{C}_m$ has a geometric model in terms of an ∞-gon $\mathcal{Z}_m$, which consists of an infinite discrete set of marked points on the circle

S^1. The marked points on S^1 accumulate to a finite number, m, of two-sided accumulation points. When $m = 1$ and $m = 2$, the ∞-gon $\mathcal{Z}_m$ coincides, respectively, with the set of integers and with the infinite strip mentioned above.

The category $\mathcal{C}_m$ is a Hom-finite, $\mathbb{K}$-linear, Krull–Schmidt, 2-Calabi–Yau triangulated category. Moreover, $\mathcal{C}_m$ has a nice geometric model. The indecomposable objects of $\mathcal{C}_m$ can be regarded as the arcs of $\mathcal{Z}_m$, while the Hom-spaces are at most one-dimensional and can be understood in terms of crossings of arcs. The geometric model of $\mathcal{C}_m$ allows us to use combinatorics for classifying some important classes of subcategories. For instance, the cluster-tilting subcategories of $\mathcal{C}_m$ were classified in [21] in terms of certain triangulations of the ∞-gon $\mathcal{Z}_m$. This classification generalises the results in [26] and [39] for the cases $m = 1$ and $m = 2$.

The torsion pairs of $\mathcal{C}_m$ were classified in [21] in terms of sets of arcs satisfying certain combinatorial conditions about arc crossings, the *Ptolemy condition*, and converging sequences of arcs, the *precovering conditions*. This classification generalises the results in [41] and [11] for the cases $m = 1$ and $m = 2$ respectively. Particular kinds of torsion pairs are the t-structures, which were classified in [22] with combinatorial objects called *decorated non-crossing partitions*. These consist of a non-crossing partition of the set $\{1, \dots, m\}$ decorated by elements of the closure of $\mathcal{Z}_m$. Similar combinatorial objects, the *non-exhaustive non-crossing partitions*, were used in [22] to describe the thick subcategories of $\mathcal{C}_m$.

The Paquette–Yıldırım completion. The *completion* of $\mathcal{C}_m$ can be viewed as closing $\mathcal{C}_m$ under "limits" of arcs of $\mathcal{Z}_m$. By completing $\mathcal{C}_m$, we obtain a new triangulated category which maintains some of the main features of $\mathcal{C}_m$. In [19] Fisher introduced the completion of the Holm–Jørgensen category $\mathcal{T}_2$ by taking the homotopy colimit closure of $\mathcal{T}_2$. In [17] Cummings and Gratz studied Neeman's completion of $\mathcal{C}_m$ using metrics coming from the t-structures in $\mathcal{C}_m$. In Chapter 4 we work with the Paquette–Yıldırım completion, $\overline{\mathcal{C}}_m$, of $\mathcal{C}_m$, introduced in [43]. The category $\overline{\mathcal{C}}_m$ was obtained by first considering the category $\mathcal{C}_{2m}$ obtained by doubling the accumulation points of $\mathcal{Z}_m$, and then localising $\mathcal{C}_{2m}$ with respect to a specific thick subcategory. In [3] the authors proved that $\overline{\mathcal{C}}_1$ is triangle equivalent to Fisher's completion of $\mathcal{T}_2$, and we expect that this holds in general for any $m \geq 1$.

The Paquette–Yıldırım completion $\overline{\mathcal{C}}_m$ is a Hom-finite, $\mathbb{K}$-linear, Krull–Schmidt triangulated category and has a geometric model similar to the one for $\overline{\mathcal{C}}_m$. The indecomposable objects of $\overline{\mathcal{C}}_m$ are in bijection with arcs, or limits of arcs, of $\mathcal{Z}_m$. The category $\overline{\mathcal{C}}_m$ inherits many properties from $\mathcal{C}_m$. For instance, $\overline{\mathcal{C}}_m$ also has cluster tilting subcategories which correspond to certain triangulations of the closure, $\overline{\mathcal{Z}}_m$, of $\mathcal{Z}_m$. In [10] the category $\overline{\mathcal{C}}_m$ was endowed with a specific extriangulated structure and its cluster-tilting subcategories, with respect to the new extriangulated structure, were classified in terms of a larger class of triangulations of $\overline{\mathcal{Z}}_m$.

Despite $\mathcal{C}_m$ and $\overline{\mathcal{C}}_m$ having many similarities, they also have relevant differences. One remarkable difference is that $\mathcal{C}_m$ is not 2-Calabi–Yau, although it is weakly 2-Calabi–Yau with respect to the extriangulated structure of [10]. Moreover, by a result in [49], $\mathcal{C}_m$ has only trivial co-t-structures because it is 2-Calabi–Yau, but both non-trivial t-structures and co-t-structures exist in $\overline{\mathcal{C}}_m$. In Chapter 4 we classify t-structures and co-t-structures in $\overline{\mathcal{C}}_m$ using combinatorial objects similar to the decorated non-crossing partitions of [22].

Theorem A (Theorem 4.7.4)**.** *There is a bijection between the aisles of t-structures in* $\mathcal{C}_m$ *and the half-decorated non-crossing partitions of* $\{1, \ldots, 2m\}$.

Theorem B (Theorem 4.8.2)**.** *There is a bijection between the aisles of the co-t-structures in* $\mathcal{C}_m$ *and the half-decorated half-non-crossing partitions of* $\{1, \ldots, 2m\}$.

In addition we prove that there exists a bijection between the functorially finite co-t-structures in $\overline{\mathcal{C}}_m$ and the t-structures in $\mathcal{C}_m$. Another important difference between $\mathcal{C}_m$ and $\overline{\mathcal{C}}_m$ is about *recollements*. These are the analogues of short exact sequences of triangulated categories. The thick subcategories of $\mathcal{C}_m$ and $\overline{\mathcal{C}}_m$ were classified in [22] and in [40] respectively. Recollements are in bijection with torsion torsion-free triples, as explained in [42], or equivalently with functorially finite thick subcategories. The only functorially finite thick subcatgories of $\mathcal{C}_m$ are 0 and $\mathcal{C}_m$ itself, but this is no longer true for $\overline{\mathcal{C}}_m$.

Theorem C (Theorem 4.8.25)**.** *There is a bijection between the functorially finite thick subcategories of* $\mathcal{C}_m$ *and certain half-decorated half-non-crossing partitions of* $\{1, \ldots, 2m\}$.

Different Calabi–Yau versions. There exist analogues of the classical cluster category $\mathcal{C}(A_n)$ and of the Holm–Jørgensen category having different Calabi–Yau parameters. It is therefore natural to search for different Calabi–Yau versions of the category $\mathcal{C}_m$. For an integer $w \geq 2$, in Chapter 5 we define the category $\mathcal{C}_{w,m}$ as a (not full) subcategory of $\mathcal{C}_m$. In [27] the authors introduced the definition of *w-admissible object* (or *arc*), where being w-admissible is determined combinatorially. We extend this concept to the category $\mathcal{C}_m$ and we give the notion of *w-admissible morphism* in $\mathcal{C}_m$. The category $\mathcal{C}_{w,m}$ consists of the w-admissible objects and morphisms of $\mathcal{C}_m$. We then consider the restriction of the triangulated structure of $\mathcal{C}_m$ to the subcategory $\mathcal{C}_{w,m}$ and we conjecture that this forms a triangulated structure for $\mathcal{C}_{w,m}$. We prove that this is true for some cases, and we expect that this holds in general, but the combinatorial nature of $\mathcal{C}_{w,m}$ does not allow us to provide a "natural" explanation for this fact.

Based on the assumption that $\mathcal{C}_{w,m}$ is triangulated, we prove that it has w-Calabi–Yau dimension. In [27] Holm and Jørgensen classified the w-cluster tilting subcategories of $\mathcal{T}_w$ in terms of certain w-admissibile $(w+1)$-angulations of $\mathcal{Z}_m$, which are maximal collections of non-crossing w-admissible arcs. In addition, in [14] Coelho Simões and Pauksztello classified the torsion pairs in $\mathcal{T}_w$ (for all values $w \in \mathbb{Z}$) as certain sets of arcs closed under taking w-admissible Ptolemy arcs. We extend these results to the category $\mathcal{C}_{w,m}$.

Proposition D (Proposition 5.6.2)**.** *The w-cluster tilting subcategories of* $\mathcal{C}_{w,m}$ *are in bijection with certain w-admissible* $(w+1)$*-angulations of* $\mathcal{Z}_m$.

Theorem E (Theorem 5.7.3)**.** *The torsion pairs in* $\mathcal{C}_{w,m}$ *are in bijection with certain sets of arcs of* $\mathcal{Z}_m$ *closed under taking w-admissible Ptolemy arcs.*

The natural counterpart of higher-Calabi–Yau triangulated categories are *negative Calabi–Yau* triangulated categories. The negative Calabi–Yau versions of the Holm–Jørgensen category were studied, for instance, in [12], [13], and [14], where the simple-minded systems and torsion pairs were classified. Moreover, in [28] the authors observed that for positive Calabi–Yau Holm–Jørgensen categories there exist non-trivial t-structures and only trivial co-t-structures, while for the negative Calabi–Yau cases the opposite holds. Later, in [49] Zhou and Zhu proved this fact for the more general setting of Calabi–Yau triangulated categories. Therefore, it is interesting to define also lower-Calabi–Yau versions of $\mathcal{C}_m$. To this end, we introduce the category $\mathcal{C}_{-1,m}$, which is (-1)-Calabi–Yau version of $\mathcal{C}_m$.

Infinite Nakayama representations. With the aim of defining $\mathcal{C}_{-1,m}$, in Chapter 6 we introduce infinite discrete versions of symmetric Nakayama representations. Given the finite oriented cycle C_n with n vertices, the projective and injective modules over the bound path algebra $\mathbb{K}C_n/\operatorname{rad}^{n+1}$ satisfy some important properties. Indeed, the category of finitely generated modules over $\mathbb{K}C_n/\operatorname{rad}^{n+1}$ is Frobenius and *symmetric*, i.e. at each vertex n of C_n the indecomposable projective and injective modules P_n and I_n coincide. By a well known result due to Happel, see for instance [24], such a category becomes triangulated after stabilising. Moreover, it also becomes (-1)-Calabi–Yau. We want to define the infinite discrete version of the bound path algebra $\mathbb{K}C_n/\operatorname{rad}^{n+1}$, which we can intuitively interpret as "$\mathbb{K}\mathcal{Z}_m/\operatorname{rad}^{\infty+1}$", and then stabilise the category of its representations.

To construct the category $\mathcal{C}_{-1,m}$ we use techniques from persistence theory. In [16] Crawley-Boevey provided a decomposition theorem for the pointwise finite dimensional representations of the real line $\mathbb{R}$ in terms of intervals. The pointwise finite dimensional representations of the circle S^1 were studied by Hanson and Rock in [23]. Rock and Zhu defined the category $\operatorname{rep}(S^1,\kappa)$ in [46], which is a continuous version of the category of representations over a Nakayama algebra. They considered the string representations of S^1, defined in [23] by "rolling up" bounded intervals of $\mathbb{R}$ around S^1, and introduced κ, the *Kupisch function*, whose role is to bound the length of the strings. The Kupisch function assigns at each point x of S^1 the length of the indecomposable projective representation P_x starting at x.

We introduce a Kupisch function, $\kappa_{\mathcal{Z}_m}$, which is specific to the ∞-gon $\mathcal{Z}_m$. Our category $\operatorname{rep}(\mathcal{Z}_m,\kappa_{\mathcal{Z}_m})$ of infinite discrete versions of the representations of symmetric Nakayama algebras, is an intermediate step between the categories $\operatorname{rep}(\mathbb{K}C_n/\operatorname{rad}^{n+1})$ and $\operatorname{rep}(S^1,\kappa_{\mathcal{Z}_m})$. The objects of $\operatorname{rep}(\mathcal{Z}_m,\kappa_{\mathcal{Z}_m})$ are the representations of $\operatorname{rep}(S^1,\kappa\mathcal{Z}_m)$ which are "constant" in between any two consecutive marked points of $\mathcal{Z}_m$. We obtain the following results in Chapter 6.

Theorem F (Theorem 6.3.1)**.** *The category* $\operatorname{rep}(\mathcal{Z}_m,\kappa_{\mathcal{Z}_m})$ *is abelian and Krull–Schmidt. The isoclasses of its indecomposable objects are in bijection with certain intervals of* $\mathbb{R}$.

Theorem G (Theorem 6.5.1, Theorem 6.7.2)**.** *The category* $\operatorname{rep}(\mathcal{Z}_m,\kappa_{\mathcal{Z}_m})$ *is Frobenius, and symmetric, i.e.* $P_z = I_z$ *for each* $z \in \mathcal{Z}_m$. *Moreover,* $\operatorname{rep}(\mathcal{Z}_m,\kappa_{\mathcal{Z}_m})$ *is uniserial, i.e.*

its indecomposable objects have a unique (possibly infinite) composition series.

We find that the triangulated category obtained by stabilising $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is (-1)-Calabi–Yau. Thus, we define $\mathcal{C}_{-1,m}$, the (-1)-Calabi–Yau version of $\mathcal{C}_m$, in this way. We expect that $\mathcal{C}_{-1,1}$ and the Holm–Jørgensen category $\mathcal{T}_{-1}$ are triangle equivalent because they are equivalent as additive categories.

Thesis outline. This thesis is organised as follows. In Chapter 2 we give some background about classical or well-known results. We discuss additive, abelian, Frobenius, and triangulated categories. In particular, we describe how the stable category of a Frobenius category becomes triangulated. We also discuss Serre functors, torsion pairs, and cluster tilting subcategories in a triangulated category. Finally, we describe localisation of triangulated categories.

Chapter 3 is mostly based on already known results of [21] and [22], except for part of Section 3.4. We introduce the geometric model and main properties of $\mathcal{C}_m$. Then we present the classifications of the torsion pairs and t-structures in $\mathcal{C}_m$.

Chapter 4 is mostly based on [20]. We discuss the geometric model and main properties of the Paquette–Yıldırım completion of $\mathcal{C}_m$, $\overline{\mathcal{C}}_m$. We prove the factorization properties of the morphisms of $\overline{\mathcal{C}}_m$ and describe its quiver. We classify the precovering and the extension-closed subcategories of $\overline{\mathcal{C}}_m$. Then, we classify the torsion pairs, t-structures, co-t-structures, and recollements in $\overline{\mathcal{C}}_m$. Finally, we prove that there exists a bijection between the functorially finite co-t-structures in $\overline{\mathcal{C}}_m$ and the t-structures in $\mathcal{C}_m$.

In Chapter 5 we define the candidate w-Calabi–Yau version of $\mathcal{C}_m$, $\mathcal{C}_{w,m}$, for $w \geq 2$. We observe that the restriction of the triangulated structure of $\mathcal{C}_m$ to $\mathcal{C}_{w,m}$ is a good candidate for being a triangulated structure for $\mathcal{C}_{w,m}$. Under the assumption that $\mathcal{C}_{w,m}$ is triangulated, we prove that it is w-Calabi–Yau and we describe its AR quiver. We then classify the w-cluster tilting subcategories and the torsion pairs in $\mathcal{C}_{w,m}$.

In Chapter 6 we define the category $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ of the infinite discrete symmetric Nakayama representations. We prove that $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is abelian and Krull–Schmidt, and we describe its indecomposable objects. We find the projective-injective objects and simple objects of $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ and we prove that each indecomposable object has a unique composition series. We describe the AR quiver of $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. Finally, we discuss the main properties of the stable category of $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, we give its AR quiver and geometric model, and we prove that it is (-1)-Calabi–Yau.

Chapter 2

Background

We collect some relevant notions which will be used throughout the other chapters. We introduce abelian, Frobenius, and triangulated categories. We see how to obtain a triangulated category by stabilising a Frobenius category, and we describe how almost split sequeneces in a Frobenius category become almost split triangles after stabilising. We introduce torsion pairs, (co-)t-structures, and cluster-tilting subcategories for triangulated categories. Finally, we see how to obtain a new triangulated category from a given one, via the Verdier quotient.

2.1 Additive categories and Serre duality

In this section, based on [45, Section I.1], we introduce Serre functors. These functors are related to the existence of almost split sequences and the Calabi–Yau property in a triangulated category, see Section 2.3.1, but can be already defined in an additive setting. We start by recalling some definitions about additive categories.

An *additive subcategory* of an additive category $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ containing the zero object, and being closed under isomorphisms, direct sums and direct summands. Given an additive category $\mathcal{C}$ and an object $c \in \mathcal{C}$, we denote by $\operatorname{add}(c)$ the smallest additive subcategory of $\mathcal{C}$ containing c. The following are particular kinds of additive categories, whose objects and Hom-sets can be understood via "building blocks".

Definition 2.1.1. Let $\mathcal{C}$ be an additive category and $\mathbb{K}$ be a field.

- We say that $\mathcal{C}$ is $\mathbb{K}$-*linear* if its Hom-sets are $\mathbb{K}$-vector spaces. If the Hom-spaces are finite-dimensional over $\mathbb{K}$, we say that $\mathcal{C}$ is Hom-*finite.*
- We say that $\mathcal{C}$ is *Krull–Schmidt* if for each non-zero object $c \in \mathcal{C}$ there exist $c_1, \dots, c_n \in \mathcal{C}$ such that $c = c_1 \oplus \dots \oplus c_n$ and $\operatorname{End}_{\mathcal{C}}(c_i)$ is a local ring for each $i \in \{1, \dots, n\}$.

Given a Krull–Schmidt category $\mathcal{C}$, we denote by $\operatorname{ind}\mathcal{C}$ the class of its indecomposable objects.

Remark 2.1.2. In a Krull–Schmidt category, the decomposition of an object into indecomposable direct summands is unique, up to isomorphism and reordering the summands, see [37, Theorem 4.2]. Moreover, an object is indecomposable if and only if its endomorphism ring is local.

We now introduce Serre functors.

Definition 2.1.3 ([45, Section I.1])**.** Let $\mathcal{C}$ be an additive, Hom-finite, $\mathbb{K}$-linear category. A *right Serre functor* of $\mathcal{C}$ is a pair $(\mathbb{S}, \sigma)$ consisting of an additive functor $\mathbb{S}\colon \mathcal{C} \to \mathcal{C}$ and a collection of isomorphisms $\sigma = (\sigma_{a,b}\colon \operatorname{Hom}_{\mathcal{C}}(a,b) \to D\operatorname{Hom}_{\mathcal{C}}(b, \mathbb{S}a))_{a,b\in\mathcal{C}}$ which are natural in a and b, where $D(-) = \operatorname{Hom}_{\mathbb{K}}(-,\mathbb{K})$. A *Serre functor* is a right Serre functor which is essentially surjective. A *left Serre functor* is the dual of a right Serre functor.

Remark 2.1.4 ([45, Corollary I.1.2, Lemma I.1.3, Lemma I.1.5])**.** The following statements, and their dual versions, hold.

- A right Serre functor is full and faithful.
- A right Serre functor is unique up to natural equivalence.
- A Serre functor is a right Serre functor which is also a left Serre functor.

Serre functors are related to certain bilinear maps. We recall that, given $\mathbb{K}$-vector spaces U and V, a *non-degenerate pairing* is a bilinear map $\Phi\colon U \times V \to \mathbb{K}$ such that if $\Phi(u,v) = 0$ for all $u \in U$ then $v = 0$, and if $\Phi(u,v) = 0$ for all $v \in V$ then $u = 0$.

Theorem 2.1.5 ([45, Proposition I.1.4])**.** *Let $\mathcal{C}$ be an additive,* Hom*-finite, $\mathbb{K}$-linear category, $\mathbb{S}\colon \mathcal{C} \to \mathcal{C}$ be an additive functor, and $(\sigma_a\colon \operatorname{Hom}_{\mathcal{C}}(a, \mathbb{S}a) \to \mathbb{K})_{a\in\mathcal{C}}$ be a collection of $\mathbb{K}$-linear maps such that for each $a, b \in \mathcal{C}$ the composition*

$$\begin{array}{ccccc} \operatorname{Hom}_{\mathcal{C}}(a,b) \times \operatorname{Hom}_{\mathcal{C}}(b,\mathbb{S}a) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(a,\mathbb{S}a) & \xrightarrow{\sigma_a} & \mathbb{K} \\ (f,g) & \longmapsto & gf & \longmapsto & \sigma_a(gf) \end{array}$$

is a non-degenerate pairing. Then $(\mathbb{S}, \sigma)$ is a right Serre functor for some collection of isomorphisms $\sigma = (\sigma_{a,b}\colon \operatorname{Hom}_{\mathcal{C}}(a,b) \to D\operatorname{Hom}_{\mathcal{C}}(b,\mathbb{S}a))_{a,b\in\mathcal{C}}$ which are natural in a and b.

2.2 Abelian, exact, and Frobenius categories

An exact category consists of an additive category together with a collection of *exact sequences*, which play the role of short exact sequences in an abelian category. In the next chapters, our exact categories live inside an ambient abelian category. Before discussing exact categories, we recall the main features of abelian categories. We start with the following definition.

Definition 2.2.1. Let $f\colon a \to b$ be a morphism in an additive category.

- We say that f is a *monomorphism* if whenever $fg_1 = fg_2$ for some $g_1, g_2\colon c \to a$, we have that $g_1 = g_2$. Dually, f is a *epimorphism* if whenever $g_1 f = g_2 f$ for some $g_1, g_2\colon b \to c$, we have that $g_1 = g_2$.

- The *kernel* of f, if it exists, is a morphism $g\colon c \to a$ such that $fg = 0$, and for each such $g'\colon c' \to a$ there exists a unique $h\colon c' \to c$ such that $gh = g'$. We denote the object c by $\operatorname{Ker} f$.

- The *cokernel* of f, if it exists, is a morphism $g\colon b \to c$ such that $gf = 0$, and for each such $g'\colon b \to c'$ there exists a unique $h\colon c \to c'$ such that $hg = g'$. We denote the object c by $\operatorname{Coker} f$.

- The *image* of f, if it exists, is the kernel of the cokernel of f, and is denoted by $\mathrm{Im} f$.

Definition 2.2.2 (See for instance [50, Definition 3.3.4])**.** Let $\mathcal{A}$ be an additive category. We say that $\mathcal{A}$ is *abelian* if each morphism has kernel and cokernel, each monomorphism is the kernel of its cokernel, and each epimorphism is the cokernel of its kernel.

Examples of abelian categories are: the category of abelian groups, the category of modules over a ring, the category of finitely generated modules over a finite-dimensional algebra.

Remark 2.2.3. Let $\mathcal{A}$ be an abelian category and $f\colon a \to b$ be a morphism. It is straightforward to check that the following statements hold.

- The morphism f is a monomorphism if and only if $\operatorname{Ker} f = 0$. Dually, f is an epimorphism if and only if $\operatorname{Coker} f = 0$.

- The morphism f is an isomorphism if and only if f is a both a monomorphism and an epimorphism.

One of the main features of abelian categories, is that they have (short) exact sequences.

Definition 2.2.4. Let $\mathcal{A}$ be an abelian category. A sequence $\cdots \longrightarrow a_{n-1} \xrightarrow{f_{n-1}} a_n \xrightarrow{f_n} a_{n+1} \longrightarrow \cdots$ of objects and morphisms of $\mathcal{A}$ is called *exact* if $\mathrm{Im} f_{n-1} = \operatorname{Ker} f_n$ for each n. An exact sequence of the form $0 \longrightarrow a \longrightarrow e \longrightarrow b \longrightarrow 0$ is called a *short exact sequence.*

Now we can define some important additive subcategories of an abelian category.

Definition 2.2.5. Let $\mathcal{E}$ be an additive subcategory of $\mathcal{A}$.

- We say that $\mathcal{E}$ is *extension-closed* if for each short exact sequence $0 \longrightarrow a \longrightarrow e \longrightarrow b \longrightarrow 0$ with $a, b \in \mathcal{E}$ we have that $e \in \mathcal{E}$.

- Assume that $\mathcal{E}$ is extension-closed. We say that $\mathcal{E}$, together with the class of short exact sequences of $\mathcal{A}$ having terms in $\mathcal{E}$, is an *exact subcategory* of $\mathcal{A}$.

- We say that $\mathcal{E}$ is a *wide subcategory* of $\mathcal{A}$ if $\mathcal{E}$ is closed under kernels, cokernels, and extensions.

Abelian categories and wide subcategories are examples of exact categories. Moreover, it is straightforward to check that a wide subcategory is abelian.

Remark 2.2.6. Exact categories can also be defined without an ambient abelian category, see for instance [8, Definition 2.1].

In an exact category $\mathcal{E}$, a short exact sequence $0 \longrightarrow a \longrightarrow e \longrightarrow b \longrightarrow 0$ in the ambient abelian category $\mathcal{A}$ which restricts to $\mathcal{E}$, i.e. such that $a, e, b \in \mathcal{E}$, is called an *exact sequence* in $\mathcal{E}$. The collection of exact sequences form an *exact structure* in $\mathcal{E}$.

Definition 2.2.7. Let $\mathcal{E}$ be an exact category.

- A morphism $f\colon a \to e$ in $\mathcal{E}$ is a *proper monomorphism* if there exists an exact sequence in $\mathcal{E}$ of the form $0 \longrightarrow a \xrightarrow{f} e \longrightarrow b \longrightarrow 0$.
- A morphism $g\colon e \to b$ in $\mathcal{E}$ is a *proper epimorphism* if there exists an exact sequence in $\mathcal{E}$ of the form $0 \longrightarrow a \longrightarrow e \xrightarrow{g} b \longrightarrow 0$.

Some exact sequences are central for studying the Auslander–Reiten theory of an exact category. Before introducing them, we give the following definition.

Definition 2.2.8. Let $f\colon a \to b$ be a morphism in an additive category. We say that f is

- a *split monomorphism* if there exists $f'\colon b \to a$ such that $f'f = 1_a$,
- a *split epimorphism* if there exists $f'\colon b \to a$ such that $ff' = 1_b$,
- *left almost split* if f is not a split monomorphism and each $g\colon a \to c$ which is not a split monomorphism factors through f,
- is *right almost split* if f is not a split epimorphism and each $g\colon c \to b$ which is not a split epimorphism factors through f.

Definition 2.2.9. An exact sequence $0 \longrightarrow a \xrightarrow{f} e \xrightarrow{g} b \longrightarrow 0$ in an exact category is *split* if f is a split monomorphism, or equivalently g is a split epimorphism.

Definition 2.2.10 (See for instance [2, Definition 1.11, Theorem 1.13])**.** An exact sequence $0 \longrightarrow a \xrightarrow{f} e \xrightarrow{g} b \longrightarrow 0$ in an exact category is *almost split*, or an *Auslander–Reiten (AR) sequence*, if a and b are indecomposable and f is left almost split.

It is straightforward to check that, given $a \in \operatorname{ind}\mathcal{E}$, there exists at most one, up to equivalence, almost split sequence of the form $0 \longrightarrow a \longrightarrow e \longrightarrow b \longrightarrow 0$. We write $a = \tau b$ and call it the *Auslander–Reiten (AR) translate* of a.

Among exact categories, Frobenius categories are important because they become triangulated after stabilising, see Section 2.3.2.

Definition 2.2.11. Let $\mathcal{E}$ be an exact category.

- An object $x \in \mathcal{E}$ is *projective* if each short exact sequence in $\mathcal{E}$ of the form $0 \longrightarrow e_1 \longrightarrow e_2 \longrightarrow x \longrightarrow 0$ with $e_1, e_2 \in \mathcal{E}$ splits. We denote by $\operatorname{Proj}\mathcal{E}$ the class of projective objects of $\mathcal{E}$.
- An object $x \in \mathcal{E}$ is *injective* if each short exact sequence in $\mathcal{E}$ of the form $0 \longrightarrow x \longrightarrow e_1 \longrightarrow e_2 \longrightarrow 0$ splits. We denote by $\operatorname{Inj}\mathcal{E}$ the class of injective objects of $\mathcal{E}$.
- We say that $\mathcal{E}$ has *enough projectives* if for each $e \in \mathcal{E}$ there exists a proper epimorphismmorphism $x \to e$ with $x \in \operatorname{Proj}\mathcal{E}$.

- We say that $\mathcal{E}$ has *enough injectives* if for each $e \in \mathcal{E}$ there exists a proper monomorphism $e \to x$ with $x \in \operatorname{Inj}\mathcal{E}$.
- We say that $\mathcal{E}$ is *Frobenius* if $\mathcal{E}$ has enough projectives, enough injectives, and $\operatorname{Proj}\mathcal{E} = \operatorname{Inj}\mathcal{E}$.

2.3 Triangulated categories

In this section we introduce triangulated categories, as most of our work will be carried in a triangulated setting. After defining (algebraic) triangulated categories, we discuss certain important classes of subcategories (namely extension-closed subcategories, precovering and preenveloping subcategories), torsion pairs, cluster-tilting subcategories, and Verdier quotients.

2.3.1 Definition

Triangulated categories, introduced in [48], are analogues of exact categories: they are additive categories having triangles, playing the role of exact sequences for exact categories, which can be "rotated".

Consider an additive category $\mathcal{T}$ equipped with an automorphism $\Sigma\colon \mathcal{T} \to \mathcal{T}$ called a *shift functor*. A *sextuple* in $\mathcal{T}$ consists of a sequence of objects and morphisms of $\mathcal{T}$ of the form $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$. Given two sextuples $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ and $x \xrightarrow{u} y \xrightarrow{u} z \xrightarrow{w} \Sigma x$, a *morphism* of sextuples consists of a triple (α, β, γ) of morphisms $\alpha\colon a \to x$, $\beta\colon e \to y$, and $\gamma\colon b \to z$, such that the diagram below commutes.

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "\alpha"] & e \arrow[r, "g"] \arrow[d, "\beta"] & b \arrow[r, "h"] \arrow[d, "\gamma"] & \Sigma a \arrow[d, "\Sigma\alpha"] \\
x \arrow[r, "u"] & y \arrow[r, "v"] & z \arrow[r, "w"] & \Sigma x
\end{tikzcd}$$

If α, β and γ are isomorphisms, then (α, β, γ) is called an *isomorphism* of sextuples. We recall the following definition, see for instance [25, Definition 3.1].

Definition 2.3.1. An additive category $\mathcal{T}$ is called *pre-triangulated* if $\mathcal{T}$ has a shift functor Σ and a class of sextuples, called *triangles*, satisfying the following axioms.

(TR1)
1. The class of triangles is closed under isomorphism.
2. If $a \in \mathcal{T}$ then $a \xrightarrow{1} a \longrightarrow 0 \longrightarrow \Sigma a$ is a triangle.
3. Each morphism $h\colon b \to \Sigma a$ can be extended to a triangle $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$.

(TR2) The sextuple $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ is a triangle if and only if $e \xrightarrow{g} b \xrightarrow{h} \Sigma a \xrightarrow{-\Sigma f} \Sigma e$ is a triangle.

(TR3) Let $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ and $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x$ be triangles, and let $\alpha\colon a \to x$, $\beta\colon e \to y$ be morphisms such that $\beta f = u\alpha$. Then there exists a morphism $\gamma\colon b \to z$

such that the following diagram commutes.

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "\alpha"] & e \arrow[r, "g"] \arrow[d, "\beta"] & b \arrow[r, "h"] \arrow[d, dashed, "\gamma"] & \Sigma a \arrow[d, "\Sigma\alpha"] \\
x \arrow[r, "u"] & y \arrow[r, "v"] & z \arrow[r, "r"] & \Sigma x
\end{tikzcd}$$

If additionally $\mathcal{T}$ satisfies the following equivalent axioms, then we say that $\mathcal{T}$ is *triangulated.*

(TR4) (Octahedral axiom) Let $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$, $x \xrightarrow{u} e \xrightarrow{v} e' \xrightarrow{w} \Sigma x$, $a \xrightarrow{vf} e' \xrightarrow{g'} b' \xrightarrow{h'} \Sigma a$ be triangles. Then there exists a triangle $x \xrightarrow{u'} b \xrightarrow{v'} b' \xrightarrow{w'} \Sigma x$ such that the following diagram commutes.

$$\begin{tikzcd}
 & x \arrow[r, "1"] \arrow[d, "u"] & x \arrow[d, dashed, "u'"] & \\
a \arrow[r, "f"] \arrow[d, "1"] & e \arrow[r, "g"] \arrow[d, "v"] & b \arrow[r, "h"] \arrow[d, dashed, "v'"] & \Sigma a \arrow[d, "1"] \\
a \arrow[r, "vf"] & e' \arrow[r, "g'"] \arrow[d, "w"] & b' \arrow[r, "h'"] \arrow[d, dashed, "w'"] & \Sigma a \arrow[d, "\Sigma f"] \\
 & \Sigma x \arrow[r, "1"] & \Sigma x \arrow[r, "\Sigma u"] & \Sigma e
\end{tikzcd}$$

(TR4') Let $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$, $x \xrightarrow{u} e \xrightarrow{v} e' \xrightarrow{w} \Sigma x$, and $e \xrightarrow{\binom{-g}{v}} b \oplus e' \xrightarrow{(v'\ g')} b' \xrightarrow{\alpha} \Sigma e$ be triangles. Then there exist triangles $a \xrightarrow{f'} e' \xrightarrow{g'} b' \xrightarrow{h'} \Sigma a$ and $x \xrightarrow{u'} b \xrightarrow{v'} b' \xrightarrow{w'} \Sigma x$ such that the following diagram commutes and $\alpha = h'\Sigma f = (\Sigma u)w'$.

$$\begin{tikzcd}
 & x \arrow[r, "1"] \arrow[d, "u"] & x \arrow[d, dashed, "u'"] & \\
a \arrow[r, "f"] \arrow[d, "1"] & e \arrow[r, "g"] \arrow[d, "v"] & b \arrow[r, "h"] \arrow[d, dashed, "v'"] & \Sigma a \arrow[d, "1"] \\
a \arrow[r, dashed, "f'"] & e' \arrow[r, dashed, "g'"] \arrow[d, "w"] & b' \arrow[r, dashed, "h'"] \arrow[d, dashed, "w'"] & \Sigma a \arrow[d, "\Sigma f"] \\
 & \Sigma x \arrow[r, "1"] & \Sigma x \arrow[r, "\Sigma u"] & \Sigma e
\end{tikzcd}$$

(TR4") Let $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ and $a \xrightarrow{u} x \xrightarrow{v} y \xrightarrow{w} \Sigma a$ be triangles, and let $\alpha\colon b \to y$ be such that $w\alpha = h$. Then there exists a morphism $\beta\colon e \to x$ such that the following diagram commutes

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "1"] & e \arrow[r, "g"] \arrow[d, dashed, "\beta"] & b \arrow[r, "h"] \arrow[d, "\alpha"] & \Sigma a \arrow[d, "1"] \\
a \arrow[r, "u"] & x \arrow[r, "v"] & y \arrow[r, "w"] & \Sigma a
\end{tikzcd}$$

and $e \xrightarrow{\binom{\beta}{-g}} x \oplus b \xrightarrow{(v\ \alpha)} y \xrightarrow{(\Sigma f)w} \Sigma e$ is a triangle.

Remark 2.3.2. We refer to [29, Appendix A] for the equivalence between (TR4), (TR4'), and (TR4").

Classical examples of triangulated categories include: the homotopy category and the derived category of an abelian category, and the classical cluster category of a finite-dimensional hereditary algebra.

The following are the triangulated analogues of Definition 2.2.9 and Definition 2.2.10.

Definition 2.3.3. A triangle $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ is *split* if f is a split monomorphism, or equivalently g is a split epimorphism, or equivalently $h = 0$.

Definition 2.3.4 ([24, Chapter I Section 4.1])**.** A triangle $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ in a triangulated category is *almost split*, or an *Auslander–Reiten (AR) triangle*, if a and b are indecomposable and f is left almost split.

As for the exact setting, given $a \in \operatorname{ind} \mathcal{T}$, there exists at most one, up to equivalence, almost split triangle of the form $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$. We write $a = \tau b$ and we call it the *Auslander–Reiten (AR) translate* of b.

Definition 2.3.5. Let $\mathcal{T}$ be a $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt triangulated category. We say that $\mathcal{T}$ *has almost split triangles* if for each $t \in \operatorname{ind} \mathcal{T}$ there exist almost split triangles $\tau t \longrightarrow x \longrightarrow t \longrightarrow \Sigma \tau t$ and $t \longrightarrow y \longrightarrow \tau^{-1} t \longrightarrow \Sigma t$.

In a triangulated category, Serre duality is closely related to the existence of almost split triangles.

Proposition 2.3.6 ([45, Proposition I.2.4])**.** *Let $\mathcal{T}$ be a $\mathbb{K}$-linear,* Hom*-finite, Krull–Schmidt triangulated category. Then $\mathcal{T}$ has almost split triangles if and only if $\mathcal{T}$ has a Serre functor* $\mathbb{S}$.

By Proposition [45, Proposition I.2.3], if $\mathcal{T}$ has a Serre functor $\mathbb{S}$, then $\mathbb{S}$ acts as $\Sigma\tau$ on objects, up to isomorphism.

In the next chapters we will often work on triangulated categories whose Hom-spaces are somehow "symmetric".

Definition 2.3.7. Let $w \in \mathbb{Z}$. A Hom-finite, $\mathbb{K}$-linear triangulated category $\mathcal{T}$ is *w-Calabi–Yau*, w-CY for short, if Σ^w is a Serre functor, i.e. there is a natural isomorphism $\operatorname{Hom}_{\mathcal{T}}(x, y) \cong D \operatorname{Hom}_{\mathcal{T}}(y, \Sigma^w x)$ for each $x, y \in \mathcal{T}$.

2.3.2 Algebraic triangulated categories

Given a Frobenius category $\mathcal{E}$, we can obtain a triangulated category by stabilising $\mathcal{E}$. We describe this process by following [50, Chapter 5]. For each $a, b \in \mathcal{E}$, we denote by $\operatorname{Proj}(a, b)$ the set of morphism $f\colon a \to b$ of $\mathcal{E}$ such that $f = hg$ for some $g\colon a \to p$ and $h\colon p \to b$ with $p \in \operatorname{Proj} \mathcal{E}$. Then, we define the *stable category* $\underline{\mathcal{E}}$ of $\mathcal{E}$ as the category having as objects the same objects of $\mathcal{E}$, and $\operatorname{Hom}_{\underline{\mathcal{E}}}(a, b) = \operatorname{Hom}_{\mathcal{E}}(a, b)/\operatorname{Proj}(a, b)$ for each $a, b \in \mathcal{E}$. Note that $e \in \mathcal{E}$ is such that $e = 0$ in $\underline{\mathcal{E}}$ if and only if $e \in \operatorname{Proj} \mathcal{E}$. Given a morphism $f\colon a \to b$, we often denote by $[f]$ the equivalence class $f + \operatorname{Proj}(a, b)$, which is a morphism $[f]\colon a \to b$ in $\underline{\mathcal{E}}$.

The exact structure on $\mathcal{E}$ induces a triangulated structure on $\underline{\mathcal{E}}$. We describe the shift functor. Consider an object $a \in \mathcal{E}$. Since $\mathcal{E}$ has enough injectives, there exists a proper monomorphism $\iota\colon a \longrightarrow p$ with $p \in \operatorname{Inj}\mathcal{E} = \operatorname{Proj}\mathcal{E}$. Thus, ι can be extended to an exact sequence $0 \longrightarrow a \xrightarrow{\iota} p \xrightarrow{\pi} \Sigma a \longrightarrow 0$, where $\Sigma a = \operatorname{Coker}\iota$. By [50, Section 5.1.2], Σ induces an autoequivalence on $\underline{\mathcal{E}}$, and is independent of the choice of p.

Now we describe the triangles in $\underline{\mathcal{E}}$. Consider an exact sequence $0 \longrightarrow a \xrightarrow{f} e \xrightarrow{g} b \longrightarrow 0$, since p is injective, there exists $\varphi\colon e \to p$ such that $\varphi f = \iota$. Moreover, by the universal property of the cokernels, there exists $h\colon b \to \Sigma a$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & a \arrow[r, "f"] \arrow[d, "1"] & e \arrow[r, "g"] \arrow[d, "\varphi"] & b \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & a \arrow[r, "\iota"] & p \arrow[r, "\pi"] & \Sigma a \arrow[r] & 0
\end{tikzcd}$$

Finally, the sextuple $a \xrightarrow{[f]} e \xrightarrow{[g]} b \xrightarrow{[h]} \Sigma a$ is called a *standard triangle*.

Proposition 2.3.8 ([50, Proposition 5.1.10])**.** *Let $\mathcal{E}$ be a Frobenius category. Then the autoequivalence Σ together with the sextuples of $\underline{\mathcal{E}}$ which are equivalent to the standard triangles, form a triangulated structure on $\underline{\mathcal{E}}$.*

We recall that an equivalence between two triangulated categories is a *triangle equivalence* if it sends triangles to triangles and commutes with the shift functors.

Definition 2.3.9. A triangulated category $\mathcal{T}$ is called *algebraic* if there exists a Frobenius category $\mathcal{E}$ such that $\mathcal{T}$ is triangulated equivalent to $\underline{\mathcal{E}}$.

Given a Frobenius category $\mathcal{E}$, we want to see if the property of being Krull–Schmidt is preserved after stabilising. We cannot find references for this, thus we provide an argument for the convenience of the reader.

First note that an object a in a Krull–Schmidt Frobenius category $\mathcal{E}$ is indecomposable in $\underline{\mathcal{E}}$ if and only if $a \cong a' \oplus p$ in $\mathcal{E}$, for some $a' \in \operatorname{ind}\mathcal{E} \setminus \operatorname{Proj}\mathcal{E}$ and $p \in \operatorname{Proj}\mathcal{E}$.

Lemma 2.3.10. *Let $\mathcal{E}$ be a Frobenius category. If $\mathcal{E}$ is $\mathbb{K}$-linear,* Hom*-finite and Krull–Schmidt, then so is $\underline{\mathcal{E}}$.*

Proof. It is straightforward to check that $\underline{\mathcal{E}}$ is $\mathbb{K}$-linear and Hom-finite, we show that $\underline{\mathcal{E}}$ is Krull–Schmidt. For each object $a \in \mathcal{E}$ there exist $a_1, \dots, a_n \in \mathcal{E}$ such that $\operatorname{End}_{\mathcal{E}}(a_i)$ is local for each $i \in \{1, \dots, n\}$, and $a = a_1 \oplus \cdots \oplus a_n$ in $\mathcal{E}$. Consider all the non-projective-injective objects among $a_1, \dots, a_n$, and reindex them as $a_1, \dots, a_k$. We have that $\operatorname{End}_{\underline{\mathcal{E}}}(a_i) = \operatorname{End}_{\mathcal{E}}(a_i)/\operatorname{Proj}(a_i, a_i) \neq 0$ is local for each $i \in \{1, \dots, k\}$ because the quotient of a local ring is again local, and $a = a_1 \oplus \cdots \oplus a_k$ in $\underline{\mathcal{E}}$. This proves that $\underline{\mathcal{E}}$ is Krull–Schmidt. $\square$

Given a Frobenius category $\mathcal{E}$, each almost split triangle in $\underline{\mathcal{E}}$ is obtained by stabilising an almost split sequence in $\mathcal{E}$. Moreover, each irreducible morphism between indecomposable objects in $\underline{\mathcal{E}}$ is obtained by stabilising an irreducible morphism between indecomposable objects in $\mathcal{E}$, see Definition 2.3.12 for the notion of irreducible morphism. We believe that these facts are well known, but, since we cannot find references, we provide arguments in Proposition 2.3.13 and Corollary 2.3.14. We start with the following lemma.

Lemma 2.3.11. *Let $\mathcal{E}$ be a* Hom*-finite,* $\mathbb{K}$*-linear, Krull–Schmidt Frobenius category and $f\colon a \to b$ be a morphism of $\mathcal{E}$. The following statments hold.*

1. *If f is a split monomorphism in $\mathcal{E}$, then $[f]$ is a split monomorphism in $\underline{\mathcal{E}}$.*
2. *If f is a split epimorphism in $\mathcal{E}$, then $[f]$ is a split epimorphism in $\underline{\mathcal{E}}$.*
3. *If $a \in \operatorname{ind}\mathcal{E}\backslash\operatorname{Proj}\mathcal{E}$ and $[f]$ is a split monomorphism in $\underline{\mathcal{E}}$, then f is a split monomorphism in $\mathcal{E}$.*
4. *If $b \in \operatorname{ind}\mathcal{E}\backslash\operatorname{Proj}\mathcal{E}$ and $[f]$ is a split epimorphism in $\underline{\mathcal{E}}$, then f is a split epimorphism in $\mathcal{E}$.*

Proof. Statement (1) is straightforward, and statements (2) and (4) are the duals of statements (1) and (3) respectively. We prove statement (3). If $[f]\colon a \to b$ is a split monomorphism in $\underline{\mathcal{E}}$, then there exists $[f']\colon b \to a$ such that $[f'][f] = [1_a]$, i.e. $f'f - 1_a = p$ for some $p \in \operatorname{Proj}(a, a)$. Since a is indecomposable, $\operatorname{End}_{\mathcal{E}}(a)$ is local and then either $f'f$ is an isomorphism or $f'f - 1_a = p$ is an isomorphism. In the first case we obtain that f is a split monomorphism. Now assume that p is an isomorphism, we write $p = p_2p_1$ where $p_1\colon a \to q$, $p_2\colon q \to a$, and $q \in \operatorname{Proj}\mathcal{E}$. We have that p_1 is a split monomorphism, and as a consequence a is projective. This contradicts the assumption that a is not projective and concludes the argument. □

Definition 2.3.12. Let $f\colon a \to b$ be a morphism in an additive category. We say that f is *irreducible* if f is not a split monomorphism nor a split epimorphism, and whenever $f = hg$ for some $g\colon a \to c$ and $h\colon c \to b$, we have that g is a split monomorphism or h is a split epimorphism.

Proposition 2.3.13. *Let $\mathcal{E}$ be a* Hom*-finite,* $\mathbb{K}$*-linear, Krull–Schmidt Frobenius category and $f\colon a \to b$ be a morphism in $\mathcal{E}$. The following statements hold.*

1. *If $a, b \in \operatorname{ind}\mathcal{E} \setminus \operatorname{Proj}\mathcal{E}$, then f is irreducible in $\mathcal{E}$ if and only if $[f]$ is irreducible in $\underline{\mathcal{E}}$.*
2. *If f is a monomorphism and $a \in \operatorname{ind}\mathcal{E} \setminus \operatorname{Proj}\mathcal{E}$, then f is left almost split if and only if $[f]$ is left almost split.*
3. *If f is an epimorphism and $b \in \operatorname{ind}\mathcal{E} \setminus \operatorname{Proj}\mathcal{E}$, then f is right almost split if and only if $[f]$ is right almost split.*

Proof. First we prove (1). Assume that $f\colon a \to b$ is irreducible in $\mathcal{E}$. Since f is not a split monomorphism nor a split epimorphism, by Lemma 2.3.11, $[f]$ is not a split monomorphism nor a split epimorphism. Assume that $[f] = [h][g] = [hg]$ for some $g\colon a \to c$ and $h\colon c \to b$ in

$\mathcal{E}$. We have that $f - hg = p_2 p_1$ for some $p_1 \colon a \to q$ and $p_2 \colon q \to b$ where $q \in \operatorname{Proj} \mathcal{E} = \operatorname{Inj} \mathcal{E}$. Since f is irreducible, f is either a monomorphism or an epimorphism. Assume that f is a monomorphism, the other case is dual. Since $q \in \operatorname{Inj} \mathcal{E}$, there exists $\alpha \colon b \to q$ such that $\alpha f = p_1$. As a consequence, $hg = (p_2\alpha + 1_b)f$. Since $\operatorname{End}_{\mathcal{E}}(b)$ is local, $p_2\alpha + 1_b$ is an isomorphism because $p_2\alpha$ is not an isomorphism. Indeed, if $p_2\alpha$ is an isomorphism then α is a split monomorphism and $b \in \operatorname{Proj} \mathcal{E}$, giving a contradiction. Thus, $f = (p_2\alpha + 1_b)^{-1}hg$ and, from the fact that f is irreducible, g is a split monomorphism or $(p_2\alpha + 1_b)^{-1}h$ is a split epimorphism. We obtain that $[g]$ is a split mono or $[h]$ is a split epimorphism. This proves that $[f]$ is irreducible.

Now assume that $a, b \in \operatorname{ind} \mathcal{E} \setminus \operatorname{Proj} \mathcal{E}$ and that $[f] \colon a \to b$ is irreducible in $\underline{\mathcal{E}}$. We have that f is not a split monomorphism nor a split epimorphism. If $f = hg$ for some $g \colon a \to c$ and $h \colon c \to b$, then $[f] = [h][g]$. Therefore, $[g]$ is a split monomorphism or $[h]$ is a split epimorphism, and then g is a split monomorphism or h is a split epimorphism. We can conclude that f is irreducible in $\mathcal{E}$.

Now we prove (2), the proof of (3) is dual. If f is left almost split, then f is not a split monomorphism and therefore, by Lemma 2.3.11 $[f]$ is not a split monomorphism in $\underline{\mathcal{E}}$. Consider $[g] \colon a \to c$ in $\underline{\mathcal{E}}$ which is not a split monomorphism, and then $g \colon a \to c$ is not a split monomorphism in $\mathcal{E}$. Since f is left almost split, there exists $h \colon b \to c$ such that $hf = g$, and as a consequence $[h][f] = [g]$. This proves that $[f]$ is left almost split.

Now assume that $[f]$ is left almost split. We show that f is left almost split. Since $[f]$ is not a split monomorphism, f is not a split monomorphism. Now consider a morphism $g \colon a \to c$ which is not a split monomorphism in $\mathcal{E}$. Then $[g] \colon a \to c$ is not a split monomorphism, and then there exists $[h] \colon b \to c$ such that $[h][f] = [g]$, i.e. $hf = g + p_2 p_1$ for some $p_1 \colon a \to q$ and $p_2 \colon q \to c$ with $q \in \operatorname{Proj} \mathcal{E} = \operatorname{Inj} \mathcal{E}$. Since f is a monomorphism, there exists $\alpha \colon b \to q$ such that $\alpha f = p_1$, and then $(h - p_2\alpha)f = g$. We can conclude that f is left almost split. $\square$

The following is immediate from the proposition above.

Corollary 2.3.14. *Let $\mathcal{E}$ be a* Hom*-finite, $\mathbb{K}$-linear, Krull–Schmidt Frobenius category. If $0 \longrightarrow a \xrightarrow{f} e \xrightarrow{g} b \longrightarrow 0$ is an almost split sequence in $\mathcal{E}$, then $a \xrightarrow{[f]} e \xrightarrow{[g]} b \longrightarrow \Sigma a$ is an almost split triangle in $\underline{\mathcal{E}}$.*

2.3.3 Extension-closed subcategories

In this section we introduce extension-closed subcategories of triangulated categories. The following setup will be used in the next sections of this chapter.

Setup 2.3.15. The category $\mathcal{T}$ will be Hom-finite, $\mathbb{K}$-linear, Krull–Schmidt, and triangulated.

We start with some notation. Given additive subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{T}$, we write

$$\mathcal{X} * \mathcal{Y} = \{t \in \mathcal{T} \mid \text{there exists } x \longrightarrow t \longrightarrow y \longrightarrow \Sigma x \text{ for some } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}.$$

Moreover, we write $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X},\mathcal{Y})=0$ if $\operatorname{Hom}_{\mathcal{T}}(x,y)=0$ for each $x\in\mathcal{X}$ and $y\in\mathcal{Y}$. The following is a condition that ensures that $\mathcal{X}*\mathcal{Y}$ is still additive.

Proposition 2.3.16 ([31, Proposition 2.1])**.** *Let $\mathcal{X},\mathcal{Y}$ be additive subcategories of $\mathcal{T}$. If* $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X},\mathcal{Y})=0$ *then $\mathcal{X}*\mathcal{Y}$ is closed under direct summands, and is therefore an additive subcategory of $\mathcal{T}$.*

Remark 2.3.17. Let $\mathcal{X},\mathcal{Y},\mathcal{Z}$ be additive subcategories of $\mathcal{T}$. By the Octahedral Axiom we obtain that $(\mathcal{X}*\mathcal{Y})*\mathcal{Z}=\mathcal{X}*(\mathcal{Y}*\mathcal{Z})$, so we write: $\mathcal{X}*\mathcal{Y}*\mathcal{Z}$, see for instance [18, Lemma 2.1].

Now we give some definitions, which will be useful when discussing torsion pairs, cluster-tilting subcategories, and thick subcategories.

Definition 2.3.18. An additive subcategory $\mathcal{X}$ of $\mathcal{T}$ is called

- *extension-closed* if $\mathcal{X}*\mathcal{X}=\mathcal{X}$,
- *suspended* if $\mathcal{X}$ is extension-closed and $\Sigma\mathcal{X}\subseteq\mathcal{X}$,
- *co-suspended* if $\mathcal{X}$ is extension-closed and $\Sigma^{-1}\mathcal{X}\subseteq\mathcal{X}$,
- *thick* if $\mathcal{X}$ is suspended, and co-suspended.

2.3.4 Precovering and preenveloping subcategories

In this section we keep Setup 2.3.15. We discuss the properties of certain subcategories of $\mathcal{T}$ which are "approximating". Torsion classes and cluster-tilting subcategories are examples of subcategories with this property, see Section 2.3.5 and Section 2.3.6.

Definition 2.3.19. Let $\mathcal{X}$ be an additive subcategory of $\mathcal{T}$.

- We say that $f\colon a\to b$ is *right minimal* if for each $g\colon a\to a$ such that $fg=f$ we have that g is an isomorphism.
- We say that $f\colon a\to b$ is *left minimal* if for each $g\colon b\to b$ such that $gf=f$ we have that g is an isomorphism.
- A morphism $f\colon x\to t$ with $x\in\mathcal{X}$ is an *$\mathcal{X}$-precover*, or *right $\mathcal{X}$-approximation* of t if any $f'\colon x'\to t$ factors through f. An *$\mathcal{X}$-cover* is a right minimal $\mathcal{X}$-precover. The subcategory $\mathcal{X}$ is *precovering*, or *contravariantly finite*, if each object of $\mathcal{T}$ has an $\mathcal{X}$-precover.
- A morphism $g\colon t\to x$ with $x\in\mathcal{X}$ is an *$\mathcal{X}$-preenvelope*, or *left $\mathcal{X}$-approximation*, if any $g'\colon t\to x'$ factors through g. An *$\mathcal{X}$-envelope* is a left minimal $\mathcal{X}$-preenvelope. The subcategory $\mathcal{X}$ is *preenveloping*, or *covariantly finite*, if each object of $\mathcal{T}$ has an $\mathcal{X}$-preenvelope.
- The subcategory $\mathcal{X}$ is *functorially finite* if it is both precovering and preenveloping.

Remark 2.3.20. Being precovering can be checked at the level of the indecomposable objects. More precisely, $\mathcal{X}$ is a precovering subcategory of $\mathcal{T}$ if and only if for any $t\in\operatorname{ind}\mathcal{T}$

there exist $x \in \mathcal{X}$ and $f\colon x \to t$ such that any $g\colon x' \to t$ with $x' \in \operatorname{ind}\mathcal{X}$ factors through f, cf. [4, p. 81].

Lemma 2.3.21 ([33, Lemma 4.1])**.** *Let $\mathcal{X}$ be an additive subcategory of $\mathcal{T}$ and let $t \in \mathcal{T}$. If there exists an $\mathcal{X}$-precover $f\colon x \to t$, then there exists $x' \in \mathcal{X}$ such that $x \cong x' \oplus x''$ for some $x'' \in \mathcal{X}$, and the composition*

$$x' \xrightarrow{\binom{1}{0}} x' \oplus x'' \cong x \xrightarrow{f} t$$

is an $\mathcal{X}$-cover of t.

In [33] it is further assumed that $\mathcal{X}$ is extension-closed, but if we remove this assumption the same argument can be applied.

The following lemma will be useful in Section 2.3.6. It consists of [47, Lemma 5.3] for preenveloping subcategories, and its dual for precovering subcategories. The statement in the given reference has a different level of generality, therefore we give part of the proof for the convenience of the reader.

Lemma 2.3.22. *Let $\mathcal{X}$ and $\mathcal{Y}$ be additive subcategories of $\mathcal{T}$ such that* $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X},\mathcal{Y}) = 0$. *The following statements hold.*

1. *If $\mathcal{X}$ and $\mathcal{Y}$ are preenveloping in $\mathcal{T}$, then $\mathcal{X} * \mathcal{Y}$ is preenveloping.*
2. *If $\mathcal{X}$ and $\mathcal{Y}$ are precovering in $\mathcal{T}$, then $\mathcal{X} * \mathcal{Y}$ is precovering.*

Proof. We prove (1), the proof of (2) is dual. First note that, since $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X},\mathcal{Y}) = 0$, by Proposition 2.3.16, $\mathcal{X} * \mathcal{Y}$ is an additive subcategory of $\mathcal{T}$. Let $t \in \mathcal{T}$ and consider a $\mathcal{Y}$-preenvelope of t, denoted $h\colon t \to y$. We extend h to a triangle $\Sigma^{-1}y \xrightarrow{f} s \xrightarrow{g} t \xrightarrow{h} y$. Now consider an $\mathcal{X}$-preenvelope of s, denoted $v\colon s \to x$, and extend it to a triangle $r \xrightarrow{u} s \xrightarrow{v} x \xrightarrow{w} \Sigma r$.

We extend the morphism $\binom{g}{-v}\colon s \to t \oplus x$ to a triangle $s \xrightarrow{\binom{g}{-v}} t \oplus x \xrightarrow{(\alpha\ \beta)} z \longrightarrow \Sigma s$. By (TR4') of Definition 2.3.1, there exist triangles $\Sigma^{-1}y \longrightarrow x \xrightarrow{\beta} z \longrightarrow y$ and $r \longrightarrow t \xrightarrow{\alpha} z \longrightarrow \Sigma r$ such that the following diagram commutes.

$$\begin{tikzcd}
 & r \arrow[r, "1"] \arrow[d, "u"] & r \arrow[d] & \\
\Sigma^{-1}y \arrow[r, "f"] \arrow[d, "1"] & s \arrow[r, "g"] \arrow[d, "v"] & t \arrow[r, "h"] \arrow[d, "\alpha"] & y \arrow[d, "1"] \\
\Sigma^{-1}y \arrow[r] & x \arrow[r, "\beta"] \arrow[d, "w"] & z \arrow[r] \arrow[d] & y \\
 & \Sigma r \arrow[r, "1"] & \Sigma r &
\end{tikzcd}$$

Now, by [47, Lemma 5.3], we obtain that $\alpha\colon t \to z$ is an $\mathcal{X} * \mathcal{Y}$-preenvelope of t. Thus, $\mathcal{X} * \mathcal{Y}$ is preenveloping. $\square$

2.3.5 Torsion pairs

The notion of *torsion pair* in a triangulated setting is due to [31]. *T-structures*, introduced in [5], and *co-t-structures*, introduced in [44] and [6] where they are called *weight structures*, are particular kinds of torsion pairs. In this section we keep Setup 2.3.15.

Definition 2.3.23. Let $\mathcal{X}$ and $\mathcal{Y}$ be additive subcategories of $\mathcal{T}$. The pair $(\mathcal{X}, \mathcal{Y})$ is called

- a *torsion pair* if $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = 0$ and $\mathcal{T} = \mathcal{X} * \mathcal{Y}$,
- a *t-structure* if $(\mathcal{X}, \mathcal{Y})$ is a torsion pair and $\Sigma\mathcal{X} \subseteq \mathcal{X}$,
- a *co-t-structure* if $(\mathcal{X}, \mathcal{Y})$ is a torsion pair and $\Sigma^{-1}\mathcal{X} \subseteq \mathcal{X}$.

If $(\mathcal{X}, \mathcal{Y})$ is a torsion pair, then $\mathcal{X}$ is called a *torsion class* and $\mathcal{Y}$ is called a *torsion-free class*. If $(\mathcal{X}, \mathcal{Y})$ is a t-structure or a co-t-structure, $\mathcal{X}$ is called the *aisle* and $\mathcal{Y}$ is called the *co-aisle*. The *heart* of a t-structure $(\mathcal{X}, \mathcal{Y})$ is $\mathcal{X} \cap \Sigma\mathcal{Y}$. The *co-heart* of a co-t-structure $(\mathcal{X}, \mathcal{Y})$ is $\mathcal{X} \cap \Sigma^{-1}\mathcal{Y}$.

Definition 2.3.24. Let $(\mathcal{X}, \mathcal{Y})$ be a t-structure or a co-t-structure, we say that $(\mathcal{X}, \mathcal{Y})$ is

- *left bounded*, or *right bounded*, if $\mathcal{T} = \bigcup_{n\in\mathbb{Z}} \Sigma^n\mathcal{X}$ or $\mathcal{T} = \bigcup_{n\in\mathbb{Z}} \Sigma^n\mathcal{Y}$, respectively,
- *bounded* if $(\mathcal{X}, \mathcal{Y})$ is left bounded and right bounded,
- *left non-degenerate*, or *right non-degenerate*, if $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{X} = 0$ or $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{Y} = 0$, respectively, and
- *non-degenerate* if $(\mathcal{X}, \mathcal{Y})$ is left non-degenerate and right non-degenerate.

Remark 2.3.25. It is straightforward to check that if $(\mathcal{X}, \mathcal{Y})$ is left bounded then it is right non-degenerate, and if it is right bounded then it is left non-degenerate.

Given $\mathcal{X}$ and $\mathcal{Y}$ additive subcategories of $\mathcal{T}$, we denote

$$\mathcal{X}^{\perp} = \{t \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, t) = 0\} \quad \text{and} \quad {}^{\perp}\mathcal{X} = \{t \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(t, \mathcal{X}) = 0\}.$$

The following lemma is known as the triangulated version of Wakamatsu's Lemma, and is useful to prove Proposition 2.3.27, which characterises torsion pairs.

Lemma 2.3.26 (Triangulated Wakamtsu's Lemma, see for instance [33, Lemma 2.1])**.** *Let $\mathcal{X}$ be an extension-closed subcategory of $\mathcal{T}$ and $t \in \mathcal{T}$. Assume that there exists an $\mathcal{X}$-cover $f\colon x \to t$, and extend it to a triangle $x \xrightarrow{f} t \longrightarrow y \longrightarrow \Sigma x$. Then $y \in \mathcal{X}^{\perp}$.*

Proposition 2.3.27 ([31, Proposition 2.3])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be additive subcategories of $\mathcal{T}$. The following statements are equivalent.*

- *$(\mathcal{X}, \mathcal{Y})$ is a torsion pair.*
- *The subcategory $\mathcal{X}$ is extension-closed, precovering, and $\mathcal{Y} = \mathcal{X}^{\perp}$.*
- *The subcategory $\mathcal{Y}$ is extension-closed, preenveloping, and $\mathcal{X} = {}^{\perp}\mathcal{Y}$.*

We recall the following notion from [6].

Definition 2.3.28. Let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure in $\mathcal{T}$.

- If $\mathcal{X}$ is functorially finite, then $({}^{\perp}\mathcal{X}, \mathcal{X})$ is called its *left adjacent t-structure.*
- If $\mathcal{Y}$ is functorially finite, then $(\mathcal{Y}, \mathcal{Y}^{\perp})$ is called its *right adjacent t-structure.*
- If $\mathcal{X}$ and $\mathcal{Y}$ are functorially finite, then we say that $(\mathcal{X}, \mathcal{Y})$ is a *functorially finite co-t-structure.*

A triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of subcategories of $\mathcal{T}$ is called *torsion torsion-free triple* (*TTF triple* for short) if $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are t-structures in $\mathcal{T}$. By Proposition 2.3.27, $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple if and only if $\mathcal{Y}$ is a functorially finite thick subcategory of $\mathcal{T}$. TTF triples, and hence functorially finite thick subcategories, are in bijection with equivalence classes of *recollements*, see for instance [42, Section 2.2], which can be regarded as "exact sequences of triangulated categories". We refer to [42] for more details about recollements.

2.3.6 Cluster-tilting subcategories

Cluster-tilting subcategories are "projective-minded generating sets" for a triangulated category. Inspired by the work in [15] for simple-minded systems, we define *left* cluster-tilting and *right* cluster-tilting subcategories and give a characterization of cluster tilting subcategories. Throughout this section we keep Setup 2.3.15 and we fix an integer $w \geq 2$. We start with the follwing definition.

Definition 2.3.29. An additive subcategory $\mathcal{X}$ of $\mathcal{T}$ is called

- *weakly left w-cluster tilting* if $\mathcal{X} = \{t \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(\mathcal{X}, \Sigma^i t) = 0 \text{ for all } 1 \leq i \leq w-1\}$,
- *weakly right w-cluster tilting* if $\mathcal{X} = \{t \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(t, \Sigma^i \mathcal{X}) = 0 \text{ for all } 1 \leq i \leq w-1\}$,
- *weakly w-cluster tilting* if it is both left and right weakly w-cluster tilting,
- *w-cluster tilting* if it is weakly w-cluster tilting and functorially finite.

Remark 2.3.30. If $\mathcal{T}$ is w-CY, then an additive subcategory of $\mathcal{T}$ is (weak) left w-cluster tilting if and only if it is (weak) right w-cluster tilting.

The following theorem connects cluster-tilting subcategories with torsion pairs.

Theorem 2.3.31 ([31, Theorem 3.1])**.** *Let $\mathcal{X}$ be a w-cluster-tilting subcategory of $\mathcal{T}$. The following statements hold.*

- $\mathcal{T} = \mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^{w-1}\mathcal{X}$.
- $(\mathcal{X} * \cdots * \Sigma^k\mathcal{X}, \Sigma^{k+1}\mathcal{X} * \cdots * \Sigma^{w-1}\mathcal{X})$ *is a torsion pair for each* $0 \leq k \leq w-2$.

We want to prove the proposition below, which is the w-cluster tilting version of [15, Proposition 2.13].

Proposition 2.3.32. *Let $\mathcal{X}$ be an additive subcategory of $\mathcal{T}$. The following statements are equivalent.*

1. *The category $\mathcal{X}$ is w-cluster tilting.*

2. *The category $\mathcal{X}$ is such that $\mathcal{T} = \mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^{w-1}\mathcal{X}$ and $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, \Sigma^i\mathcal{X}) = 0$ for each $1 \leq i \leq w-1$.*

3. *The category $\mathcal{X}$ is weakly left w-cluster tilting and precovering.*

4. *The category $\mathcal{X}$ is weakly right w-cluster tilting and preenveloping.*

Before proceeding with an argument, we need some lemmas. The following is analogous to [15, Lemma 2.7].

Lemma 2.3.33. *Let $\mathcal{X}$ be an additive subcategory of $\mathcal{T}$ such that $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, \Sigma^i\mathcal{X}) = 0$ for each $1 \leq i \leq w-1$. Then $\Sigma^k\mathcal{X} * \mathcal{X} \subseteq \mathcal{X} * \Sigma^k\mathcal{X}$ for each $1 \leq k \leq w-2$.*

Proof. Let $t \in \Sigma^k\mathcal{X} * \mathcal{X}$, then there exists a triangle of the form $\Sigma^k x_1 \longrightarrow t \longrightarrow x_2 \longrightarrow \Sigma^{k+1}x_1$ with $x_1, x_2 \in \mathcal{X}$. Note that this is a split triangle beacuse $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, \Sigma^{k+1}\mathcal{X}) = 0$. Thus, $t \cong x_1 \oplus \Sigma^k x_2$ and there exists a triangle of the form $x_2 \longrightarrow t \longrightarrow \Sigma^k x_1 \xrightarrow{0} \Sigma x_2$, i.e. $t \in \mathcal{X} * \Sigma^k\mathcal{X}$. $\square$

Lemma 2.3.34. *Let $\mathcal{X}$ be an additive subcategory of $\mathcal{T}$ such that $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, \Sigma^i\mathcal{X}) = 0$ for all $1 \leq i \leq w-1$. The following statements hold.*

1. *If $\mathcal{X}$ is precovering then $\mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^{w-2}\mathcal{X}$ is an extension-closed precovering subcategory of $\mathcal{T}$.*

2. *If $\mathcal{X}$ is preenveloping then $\mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^{w-2}\mathcal{X}$ is an extension-closed preenveloping subcategory of $\mathcal{T}$.*

Proof. By Proposition 2.3.16 we have that $\mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^{w-2}\mathcal{X}$ is an additive subcategory of $\mathcal{T}$ and, by Lemma 2.3.22, it is precovering or preenveloping, if (1) or (2), respectively, holds. Since $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, \Sigma\mathcal{X}) = 0$, $\mathcal{X}$ is extension-closed. Moreover, by Lemma 2.3.33, we have that

$$
\begin{aligned}
(\mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^{w-2}\mathcal{X}) * (\mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^{w-2}\mathcal{X}) \subseteq& \\
(\mathcal{X} * \mathcal{X}) * \Sigma(\mathcal{X} * \mathcal{X}) * \cdots * \Sigma^{w-2}(\mathcal{X} * \mathcal{X}) &\subseteq \mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^{w-2}\mathcal{X}.
\end{aligned}
$$

Thus, we conclude that $\mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^{w-2}\mathcal{X}$ is extension-closed. $\square$

Proof of Proposition 2.3.32. From Definition 2.3.29 it follows that (1) implies (3) and (4). Moreover, by Theorem 2.3.31, (1) implies (2).

We prove that (2) implies (3) and (4). Since $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, \Sigma^i\mathcal{X}) = 0$ for each $1 \leq i \leq w-1$, by Proposition 2.3.16 we have that $\Sigma\mathcal{X} * \cdots * \Sigma^{w-1}\mathcal{X}$ and $\mathcal{X} * \cdots * \Sigma^{w-2}\mathcal{X}$ are additive subcategories of $\mathcal{T}$. Moreover, $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X}, \Sigma\mathcal{X} * \cdots * \Sigma^{w-1}\mathcal{X}) = 0$ and $\operatorname{Hom}_{\mathcal{T}}(\mathcal{X} * \cdots * \Sigma^{w-2}\mathcal{X}, \Sigma^{w-1}\mathcal{X}) = 0$. Since $\mathcal{T} = \mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^{w-1}\mathcal{X}$, it follows that $(\mathcal{X}, \Sigma\mathcal{X} * \cdots * \Sigma^{w-1}\mathcal{X})$ and $(\mathcal{X} * \cdots * \Sigma^{w-2}\mathcal{X}, \Sigma^{w-1}\mathcal{X})$ are torsion pairs. By Proposition 2.3.27, $\mathcal{X}$ is precovering and $\Sigma^{w-1}\mathcal{X}$ is preenveloping, i.e. $\mathcal{X}$ is functorially finite. Moreover, $\Sigma^{w-1}\mathcal{X} = (\mathcal{X} * \cdots *$

$\Sigma^{w-2}\mathcal{X})^{\perp}$ and $\mathcal{X} = {}^{\perp}(\Sigma\mathcal{X} * \cdots * \Sigma^{w-1}\mathcal{X})$. Then $\mathcal{X}$ is weakly left w-cluster tilting and weakly right w-cluster tilting. This proves that (2) implies (3) and (4). Moreover, (2) implies (1) because (2) and (3) together imply (1).

Now we prove that (3) implies (2). By Lemma 2.3.34, $\Sigma^{-w+1}(\mathcal{X}*\cdots*\Sigma^{w-2}\mathcal{X}) = \Sigma^{-w+1}\mathcal{X} * \cdots * \Sigma^{-1}\mathcal{X}$ is precovering and closed under extensions. Since $\mathcal{X}$ is weakly left w-cluster tilting, we also have that $\mathcal{X} = (\Sigma^{-w+1}\mathcal{X} * \cdots * \Sigma^{-1}\mathcal{X})^{\perp}$. Then $(\Sigma^{-w+1}\mathcal{X} * \cdots * \Sigma^{-1}\mathcal{X}, \mathcal{X})$ is a torsion pair and $\mathcal{T} = \Sigma^{w-1}(\Sigma^{-w+1}\mathcal{X} * \cdots * \Sigma^{-1}\mathcal{X} * \mathcal{X}) = \mathcal{X} * \cdots * \Sigma^{w-2}\mathcal{X} * \Sigma^{w-1}\mathcal{X}$. We conclude that (3) implies (2). The proof that (4) implies (2) is analogous. □

2.3.7 Verdier quotients

Given a triangulated category, we obtain a new triangulated category by localising with respect to a thick subcategory. This process is called *Verdier localisation* and was first defined in [48]. We describe this process following [36]. In this section we keep Setup 2.3.15, and we fix a thick subcategory $\mathcal{D}$ of $\mathcal{T}$.

Let us denote by S the class of morphisms $f\colon t_1 \to t_2$ of $\mathcal{T}$ which extend to triangles of the form $t_1 \xrightarrow{f} t_2 \longrightarrow d \longrightarrow \Sigma t_1$ with $d \in \mathcal{D}$. The class S forms a *multiplicative system* which is *compatible with the triangulation*, see [36, Section 3.1, Section 4.3].

Let $f\colon t_1 \to c$ and $\sigma\colon t_2 \to c$ be morphisms of $\mathcal{T}$ with $\sigma \in S$. We call a *left fraction* a pair of morphisms (f, σ) and we denote it by $t_1 \xrightarrow{f} c \xleftarrow{\sigma} t_2$. Two left fractions $t_1 \xrightarrow{f_1} c_1 \xleftarrow{\sigma_1} t_2$ and $t_1 \xrightarrow{f_2} c_2 \xleftarrow{\sigma_2} t_2$ are *equivalent* if there exists a left fraction $t_1 \xrightarrow{f_3} c_3 \xleftarrow{\sigma_3} t_2$ and morphisms $\alpha\colon c_1 \to c_3$ and $\beta\colon c_2 \to c_3$ such that the following diagram commutes.

$$\begin{tikzcd}
 & c_1 \arrow[d, dashed, "\alpha"] & \\
t_1 \arrow[ur, "f_1"] \arrow[r, dashed, "f_3"] \arrow[dr, "f_2"'] & c_3 & t_2 \arrow[ul, "g_1"'] \arrow[l, dashed, "g_3"'] \arrow[dl, "g_2"] \\
 & c_2 \arrow[u, dashed, "\beta"'] &
\end{tikzcd}$$

It is straightforward to check that this is indeed an equivalence relation, and we denote the equivalence class of (f, σ) by $[f, \sigma]$. Given two equivalence classes of left fractions $[t_1 \xrightarrow{f_1} c_1 \xleftarrow{\sigma_1} t_2]$ and $[t_2 \xrightarrow{f_2} c_2 \xleftarrow{\sigma_2} t_3]$, we define their composition as $[f_2, \sigma_2][f_1, \sigma_1] = [gf_1, \tau\sigma_2]$, where $c_1 \xrightarrow{g} c_3 \xleftarrow{\tau} c_2$ is a left fraction such that the following diagram commutes.

$$\begin{tikzcd}
 & & c_3 & & \\
 & c_1 \arrow[ur, dashed, "g"] & & c_2 \arrow[ul, dashed, "\tau"'] & \\
t_1 \arrow[ur, "f_1"] & & t_2 \arrow[ul, "\sigma_1"'] \arrow[ur, "f_2"] & & t_3 \arrow[ul, "\sigma_2"']
\end{tikzcd}$$

The existence of the left fraction (g, τ) is guaranteed by the Octahedral Axiom. Moreover, the composition is well defined, i.e. does not depend on the choice of representatives of the equivalence classes of left fractions.

Definition 2.3.35. We define the *Verdier quotient* as the category $\mathcal{T}/\mathcal{D}$ having as objects the same objects of $\mathcal{T}$ and as morphisms the equivalence classes of left fractions of morphisms of $\mathcal{T}$.

Remark 2.3.36. For any $f\colon t_1 \to c$ and $\sigma\colon t_2 \to c$ with $\sigma \in S$ it is straightforward to check that $[(f,\sigma)] = [(f,1)][(1,\sigma)]$, and that $[(1,\sigma)]$ and $[(\sigma,1)]$ are isomorphisms in $\mathcal{T}/\mathcal{D}$.

We define the functor $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{D}$ acting as the identity on objects, and sending morphisms $f\colon t_1 \to t_2$ to $[t_1 \xrightarrow{f} t_2 \xleftarrow{1} t_2]$. Then Q makes the morphisms in S invertible, and is universal with this property, i.e. Q is a *quotient functor*, see [36, Section 2.2].

The Verdier quotient $\mathcal{T}/\mathcal{D}$ has a triangulated structure, which consists of

- a shift functor $\Sigma\colon \mathcal{T}/\mathcal{D} \to \mathcal{T}/\mathcal{D}$ acting on objects as $\Sigma\colon \mathcal{T} \to \mathcal{T}$, and on morphisms as $\Sigma[f,\sigma] = [\Sigma f, \Sigma\sigma]$ for each equivalence class of left fractions $[f,\sigma]$,
- triangles in $\mathcal{T}/\mathcal{D}$ given by the image of the triangles in $\mathcal{T}$ under Q, up to isomorphism.

Proposition 2.3.37 ([36, Proposition 4.6.2])**.** *The following statements hold.*

- *There exists a unique triangulated structure on $\mathcal{T}/\mathcal{D}$ such that $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{D}$ is a triangulated functor, i.e. Q is an additive functor sending triangles to triangles and commuting with Σ.*
- *A morphism $f\colon t_1 \to t_2$ in $\mathcal{T}$ is such that $Q(f) = 0$ in $\mathcal{T}/\mathcal{D}$ if and only if $f = hg$ for some $g\colon t_1 \to d$ and $h\colon d \to t_2$ with $d \in \mathcal{D}$.*
- *An object $t \in \mathcal{T}$ is such that $Q(t) = 0$ in $\mathcal{T}/\mathcal{D}$ if and only if $t \in \mathcal{D}$.*

With Proposition 2.3.38 we prove that there exists a bijection between certain extension-closed subcategories of $\mathcal{T}$ and extension-closed subcategories of $\mathcal{T}/\mathcal{D}$. This is a generalisation of [48, Proposition 2.3.1] for thick subcategories. We introduce some terminology. Let $\mathcal{U} \subseteq \mathcal{T}$ and $\mathcal{X} \subseteq \mathcal{T}/\mathcal{D}$. The *essential image* of $\mathcal{U}$ under Q, and the *preimage* of $\mathcal{X}$ under Q, are respectively

$$\begin{aligned} Q(\mathcal{U}) &= \{x \in \mathcal{T}/\mathcal{D} \mid x \cong Q(u) \text{ in } \mathcal{T}/\mathcal{D} \text{ for some } u \in \mathcal{U}\} \text{ and} \\ Q^{-1}(\mathcal{X}) &= \{t \in \mathcal{T} \mid Q(t) \cong x \text{ in } \mathcal{T}/\mathcal{D} \text{ for some } x \in \mathcal{X}\}. \end{aligned}$$

Proposition 2.3.38. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{D}$ be a thick subcategory. The following is a bijection.*

$$\left\{ \begin{array}{c} \textit{Extension-closed subcategories} \\ \mathcal{U} \subseteq \mathcal{T} \textit{ such that } \mathcal{D} \subseteq \mathcal{U} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Extension-closed} \\ \textit{subcategories of } \mathcal{T}/\mathcal{D} \end{array} \right\}$$

$$\mathcal{U} \longmapsto Q(\mathcal{U})$$

$$Q^{-1}(\mathcal{X}) \longleftarrow\!\shortmid \mathcal{X}$$

The argument in [48] is in part not applicable with our assumptions when checking that the maps are well defined. Therefore, we provide an argument for this statement. Before doing so, we have the following lemma, which is included in the argument of [48, Proposition 2.3.1]. Our assumptions are more general than those of [48], therefore we give an argument for convenience of the reader.

Lemma 2.3.39. *Let $\mathcal{D}$ be a thick subcategory of $\mathcal{T}$ and $\mathcal{U}$ be an extension-closed subcategory of $\mathcal{T}$ containing $\mathcal{D}$. If $t \in \mathcal{T}$ and $u \in \mathcal{U}$ are such that $Q(t) \cong Q(u)$ in $\mathcal{T}/\mathcal{D}$, then $t \in \mathcal{U}$.*

Proof. Consider $u \in \mathcal{U}$ and $t \in \mathcal{T}$ such that $Q(u) \cong Q(t)$ in $\mathcal{T}/\mathcal{D}$. This means that there is an isomorphism $\varphi = [t \xrightarrow{f} c \xleftarrow{\sigma} u]$ in $\mathcal{T}/\mathcal{D}$, with $\sigma \in S$. Since φ is an isomorphism, then $f \in S$, see Remark 2.3.36. We extend σ and f to triangles $u \xrightarrow{\sigma} c \longrightarrow d \longrightarrow \Sigma u$ and $\Sigma^{-1} d' \longrightarrow t \xrightarrow{f} c \longrightarrow d'$ in $\mathcal{T}$ with $d, d' \in \mathcal{D}$. Since $\mathcal{U}$ is extension-closed, and $\Sigma^{-1}\mathcal{D} = \mathcal{D} \subseteq \mathcal{U}$, we obtain that $c \in \mathcal{U}$ and then $t \in \mathcal{U}$. □

Proof of Proposition 2.3.38. We recall that extension-closed subcategories are assumed to be additive, see Definition 2.3.18. We check that the maps are well defined. To show that the two maps are mutually inverse we can proceed as in the argument of [48, Proposition 2.3.1]. Let $\mathcal{U}$ be an extension-closed subcategory of $\mathcal{T}$ containing $\mathcal{D}$. It is straightforward to see that $Q(\mathcal{U})$ is closed under isomorphism, $0 \in Q(\mathcal{U})$, and that $Q(\mathcal{U})$ is closed under direct sums. Moreover, by Lemma 2.3.39, it is straightforward to check that $Q(\mathcal{U})$ is closed under direct summands.

Now we show that $Q(\mathcal{U})$ is extension-closed. Consider a triangle in $\mathcal{T}/\mathcal{D}$

$$x_1 \longrightarrow y \longrightarrow x_2 \longrightarrow \Sigma x_1 \tag{T}$$

with $x_1, x_2 \in Q(\mathcal{U})$. Then there is a triangle $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ in $\mathcal{T}$ whose image under Q is isomorphic to the triangle (T) in $\mathcal{T}/\mathcal{D}$. Thus, in $\mathcal{T}/\mathcal{D}$ we have the isomorphisms $Q(a) \cong x_1$, $Q(b) \cong x_2$ and $Q(e) \cong y$. Since $x_1, x_2 \in Q(\mathcal{U})$, we have that there exist $u_1, u_2 \in \mathcal{U}$ such that $x_1 \cong Q(u_1)$ and $x_2 \cong Q(u_2)$. Then, by Lemma 2.3.39, we have that $a, b \in \mathcal{U}$. Since $\mathcal{U}$ is extension-closed, we obtain that $e \in \mathcal{U}$ and as a consequence $y \cong Q(e) \in Q(\mathcal{U})$. Thus, the map $\mathcal{U} \mapsto Q(\mathcal{U})$ is well defined.

Let $\mathcal{X}$ be an extension-closed subcategory of $\mathcal{T}/\mathcal{D}$. We check that $Q^{-1}(\mathcal{X})$ is an additive subcategory of $\mathcal{T}$. It is straightforward to see that $0 \in Q^{-1}(\mathcal{X})$, $Q^{-1}(\mathcal{X})$ is closed under isomorphisms, direct sums, direct summands, and that $\mathcal{D} \subseteq Q^{-1}(\mathcal{X})$. Now we show that $Q^{-1}(\mathcal{X})$ is extension-closed. Consider a triangle $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ in $\mathcal{T}$ with $a, b \in Q^{-1}(\mathcal{X})$. Then its image under Q is a triangle $Q(a) \longrightarrow Q(e) \longrightarrow Q(b) \longrightarrow \Sigma Q(a)$ in $\mathcal{T}/\mathcal{D}$ with $Q(a), Q(b) \in \mathcal{X}$. Since $\mathcal{X}$ is extension-closed, then $Q(e) \in \mathcal{X}$. As a consequence $e \in Q^{-1}(\mathcal{X})$. Hence, the map $\mathcal{X} \mapsto Q^{-1}(\mathcal{X})$ is well defined. □

Chapter 3

Discrete cluster categories

Given a positive integer m and a field $\mathbb{K}$, in [30] Igusa and Todorov defined a cluster category $\mathcal{C}_m$ which generalises the classical cluster category $\mathcal{C}(A_n)$ of type A_n introduced in [7] for finite-dimensional hereditary algebras. When $m = 1$ or $m = 2$, the category $\mathcal{C}_m$ is equivalent to the Holm–Jørgensen category defined in [26], or to the Liu–Paquette category defined in [39], respectively. In this chapter we introduce the category $\mathcal{C}_m$, its properties, geometric model, and AR quiver. We also discuss the classifications of the torsion pairs from [21] and [22]. These results will be useful in Chapter 4, Chapter 5, and Chapter 6. The material presented in this chapter consists of already existing results, except for part of Section 3.4.

3.1 The ∞-gon $\mathcal{Z}_m$

We consider the unit circle S^1 with anticlockwise orientation endowed with the usual topology. Given a positive integer m, the ∞-gon $\mathcal{Z}_m$ is an infinite discrete subset of S^1 consisting of m copies of $\mathbb{Z}$ embedded in S^1 with m two-sided accumulation points, see Figure 3.1. We denote the accumulation points of $\mathcal{Z}_m$ by $\{1, \ldots, m\} = [m]$. Given $p \in [m]$ we denote by $\mathbb{Z}^{(p)}$ all the elements of $\mathcal{Z}_m$ which belong to the p-th copy of $\mathbb{Z}$. The accumulation points are in cyclic order $1 < \cdots < m < 1$. If $p \in [m]$ is an accumulation point, we denote the successor and the predecessor of p with respect to the cyclic order by p^+ and p^-. We also regard $[m]$ as a totally ordered set $1 < \cdots < m$. This total order induces a total order $\leq$ on $\mathcal{Z}_m \cup [m]$. We can define intervals in $\mathcal{Z}_m$. Given $x, y \in \mathcal{Z}_m \cup [m]$ we denote

$$[x, y) = \begin{cases} \{z \in \mathcal{Z}_m \mid x \leq z < y\} & \text{if } x \leq y, \text{ and} \\ \{z \in \mathcal{Z}_m \mid z \leq x \text{ or } z > y\} & \text{otherwise.} \end{cases}$$

Similarly, we can define the intervals $(x, y]$, (x, y), and $[x, y]$. Since the set $\mathcal{Z}_m$ is discrete, for each $z \in \mathcal{Z}_m$ there exists a predecessor $z - 1$ and a successor $z + 1$.

When we write $x - y$ for some $x \in \mathbb{Z}^{(p)}$, $y \in \mathbb{Z}^{(q)}$, and $p, q \in [m]$, we mean the difference of x and y regarded as integers, and forgetting about what copy of $\mathbb{Z}$, namely $\mathbb{Z}^{(p)}$ or $\mathbb{Z}^{(q)}$, they belong to. More precisely, if we write $x = (z_1, p)$ and $y = (z_2, q)$ with $z_1, z_2 \in \mathbb{Z}$, then

$x - y = z_1 - z_2$ is an integer.

Definition 3.1.1. A pair $x = (x_1, x_2)$ of elements of $\mathcal{Z}_m$ is called an *arc* if $x_2 \geq x_1 + 2$, and in that case x_1 and x_2 are called *endpoints* or *coordinates* of x. Given two arcs $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of $\mathcal{Z}_m$, we say that x and y *cross* if $x_1 < y_1 < x_2 < y_2$ or $y_1 < x_1 < y_2 < x_2$. Given $p, q \in [m]$ with $p \leq q$, we define

$$\mathbb{Z}^{(p,q)} = \left\{ (x_1, x_2) \text{ is an arc of } \mathcal{Z}_m \;\middle|\; x_1 \in \mathbb{Z}^{(p)} \text{ and } x_2 \in \mathbb{Z}^{(q)} \right\}.$$

We introduce some notation which will be useful later. Given x and y both elements of $\mathcal{Z}_m$ with $x_2 \geq x_1 + 2$ or $x_1 \geq x_2 + 2$, we define

$$|x_1, x_2| = \begin{cases} (x_1, x_2) & \text{if } x_1 < x_2, \\ (x_2, x_1) & \text{if } x_2 < x_1. \end{cases}$$

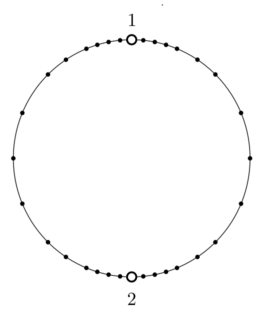


Figure 3.1: The ∞-gon $\mathcal{Z}_2$. The white circles denote the accumulation points.

3.2 The geometric model and AR quiver

Given the ∞-gon $\mathcal{Z}_m$ and a field $\mathbb{K}$, the category $\mathcal{C}_m$ was defined in [30]. This is a $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt triangulated category. We denote its shift functor by $\Sigma\colon \mathcal{C}_m \to \mathcal{C}_m$. Moreover, $\mathcal{C}_m$ is 2-Calabi–Yau, i.e. Σ^2 is a Serre functor. We recall some properties of $\mathcal{C}_m$.

- There is a bijection between the isoclasses of indecomposable objects of $\mathcal{C}_m$ and the arcs of $\mathcal{Z}_m$. We regard the indecomposable objects of $\mathcal{C}_m$ as arcs of $\mathcal{Z}_m$, see [30, Section 2.4.1].
- Given $x = (x_1, x_2) \in \operatorname{ind} \mathcal{C}_m$ we have that $\Sigma x = (x_1 - 1, x_2 - 1)$ by [30, Lemma 2.4.3].
- Given $x, y \in \operatorname{ind} \mathcal{C}_m$, by [30, Lemma 2.4.4] we have that

$$\operatorname{Hom}_{\mathcal{C}_m}(x, y) \cong \begin{cases} \mathbb{K} & \text{if } x \text{ and } \Sigma^{-1} y \text{ cross}, \\ 0 & \text{otherwise}. \end{cases}$$

Remark 3.2.1. We identify the indecomposable objects of $\mathcal{C}_m$ with the arcs of $\mathcal{Z}_m$, and the additive subcategories of $\mathcal{C}_m$ with sets of arcs.

From now on all the subcategories of $\mathcal{C}_m$ we refer to are assumed to be additive.

The following result describes the AR quiver of $\mathcal{C}_m$, and Figure 3.2 provides an illustration.

Theorem 3.2.2 ([30, Theorem 2.4.13])**.** *The AR quiver of $\mathcal{C}_m$ consists of*

- *m components of type $\mathbb{Z}A_\infty$, corresponding to the arcs of $\mathbb{Z}^{(p,p)}$ for $p \in [m]$,*
- *$\binom{m}{2}$ components of type $\mathbb{Z}A_\infty^\infty$, corresponding to the arcs of $\mathbb{Z}^{(p,q)}$ for $p, q \in [m]$ with $p < q$.*

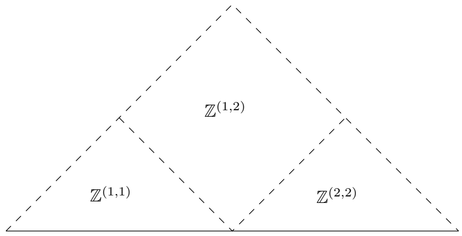


Figure 3.2: The AR quiver of $\mathcal{C}_2$.

We extend the definition of Hom-hammocks of $\mathcal{C}_m$, see [26, Definition 2.1], from $m = 1$ to the general case $m \geq 1$. We refer to Figure 3.3 for an illustration.

Definition 3.2.3. Let $a = (a_1, a_2) \in \operatorname{ind} \mathcal{C}_m$. We define

$$H^+(a) = \{(x_1, x_2) \in \operatorname{ind} \mathcal{C}_m \mid a_1 \leq x_1 \leq a_2 - 2 \text{ and } x_2 \geq a_2\} \text{ and}$$
$$H^-(a) = \{(x_1, x_2) \in \operatorname{ind} \mathcal{C}_m \mid x_1 \leq a_1 \text{ and } a_1 + 2 \leq x_2 \leq a_2\}.$$

Remark 3.2.4. For $a, b \in \operatorname{ind} \mathcal{C}_m$, by [30, Lemma 2.4.2] it follows that $\operatorname{Hom}_{\mathcal{C}_m}(a, b) \cong \mathbb{K}$ if and only if $b \in H^+(a) \cup H^-(\Sigma^2 a)$, or equivalently, $a \in H^+(\Sigma^{-2} b) \cup H^-(b)$.

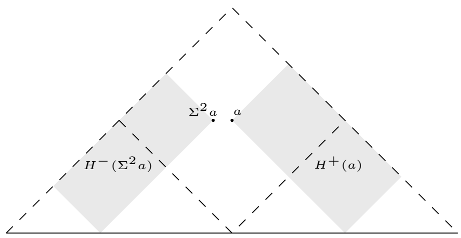


Figure 3.3: The hammocks $H^+(a)$ and $H^-(\Sigma^2 a)$ for some $a \in \operatorname{ind} \mathcal{C}_2$.

The following are the factorization properties of the morphisms in $\mathcal{C}_m$.

Lemma 3.2.5 ([30, Lemma 2.4.2])**.** *Let $a, b, c \in \operatorname{ind} \mathcal{C}_m$, $f \colon a \to b$, and $g \colon b \to c$ be non-zero morphisms. Assume that one of the following statements holds.*

1. *$b \in H^+(a)$ and $c \in H^+(a) \cap H^+(b)$.*
2. *$b \in H^+(a)$ and $c \in H^-(\Sigma^2 a) \cap H^-(\Sigma^2 b)$.*
3. *$b \in H^-(\Sigma^2 a)$ and $c \in H^-(\Sigma^2 a) \cap H^+(b)$.*

Then $gf \neq 0$.

Remark 3.2.6. We recall that, from [30, Lemma 2.4.11], for an object $(a_1, a_2) \in \operatorname{ind} \mathcal{C}_{2m}$, the irreducible morphisms of $\mathcal{C}_m$ are exactly the non-zero morphisms of the form $(a_1, a_2) \to (a_1, a_2{+}1)$ or $(a_1, a_2) \to (a_1{+}1, a_2)$, provided that $(a_1{+}1, a_2)$ is still an arc, i.e. $a_2 \geq a_1{+}3$.

The lemma below will be useful in Section 5.1 for defining $\mathcal{C}_{w,m}$, the higher-Calabi–Yau version of $\mathcal{C}_m$.

Lemma 3.2.7. *Let $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2) \in \operatorname{ind} \mathcal{C}_m$ be pairwise non-isomorphic objects, and let $f : a \to b, g : b \to c$ be non-zero morphisms. Assume that one of the following statements hold.*

1. *$b \in H^-(\Sigma^2 a)$ and $c \in H^-(\Sigma^2 b)$.*
2. *$b \in H^+(a)$ and $c \in H^+(a) \cap H^-(\Sigma^2 b)$.*
3. *$b \in H^-(\Sigma^2 a)$ and $c \in H^+(a) \cap H^+(b)$.*

Then $gf = 0$.

Proof. Assume that (1) holds. If $gf \neq 0$, then $c \in H^+(a) \cup H^-(\Sigma^2 a)$. Since $H^-(\Sigma^2 b) \cap H^+(a) = \varnothing$, we have that $c \in H^-(\Sigma^2 a) \cap H^-(\Sigma^2 b)$. Thus, $H^-(\Sigma^2 a) \cap H^-(\Sigma^2 b) \neq \varnothing$, and as a consequence $\Sigma^2 b \in H^-(\Sigma^2 a)$. Then, $\Sigma^2 a \in H^+(\Sigma^2 b)$, i.e. $a \in H^+(b)$. Since $a \in H^+(b)$ and $c \in H^-(\Sigma^2 a) \cap H^-(\Sigma^2 b)$, by Lemma 3.2.5, $g = \beta\alpha$ for some $\alpha : b \to a$ and $\beta : a \to c$. We obtain that $gf = \beta\alpha f \neq 0$, and then $\alpha f : a \to a$ is non-zero. Therefore, α is a split epimorphism and $a \cong b$, giving a contradiction. We can conclude that $gf = 0$.

Now assume that (2) holds, when (3) holds the proof is similar. If $gf \neq 0$, then $H^+(a) \cap H^-(\Sigma^2 b) \neq \varnothing$ because c belongs to both. As a consequence, $\Sigma^2 b \in H^+(a)$, i.e. $a \in H^-(\Sigma^2 b)$. Since $a \in H^-(\Sigma^2 b)$ and $c \in H^-(\Sigma^2 b) \cap H^+(a)$, by Lemma 3.2.5, $g = \beta\alpha$ for some $\alpha : b \to a$ and $\beta : a \to c$. We obtain a contradiction similarly as above. This concludes the proof. □

3.3 Precovering subcategories

The precovering subcategories of $\mathcal{C}_m$ were classified in [21], and in [41] for the case $m = 1$, in terms of converging sequences of arcs of $\mathcal{Z}_m$. The following definition corresponds to [21, Definition 0.7], but we use a different formulation which is more convenient for our purposes. For the notation $|x_1, x_2|$ we refer to Section 3.1. We recall that the accumulation points of $\mathcal{Z}_m$ are in cyclic order, i.e. $1 < 2 < \cdots < m < 1$, and, for each $p \in [m]$, p^+ denotes the next accumulation point of p with respect to the cyclic order.

Definition 3.3.1. Let $\mathcal{U}$ be a subcategory of $\mathcal{C}_m$. We say that $\mathcal{U}$ satisfies the *precovering conditions*, PC for short, if it satisfies the following combinatorial conditions.

(PC 1) If there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathcal{U} \cap \mathbb{Z}^{(p,q)}$ for some $p, q \in [m]$ such that $p \neq q$ and the sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are strictly increasing, then there exist strictly decreasing sequences $\{y_1^n\}_n \subseteq \mathbb{Z}^{(p^+)}$ and $\{y_2^n\}_n \subseteq \mathbb{Z}^{(q^+)}$ such that $\{|y_1^n, y_2^n|\}_n \subseteq \mathcal{U}$.

(PC 2) If there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathcal{U} \cap \mathbb{Z}^{(p,q)}$ for some $p, q \in [m]$ such that $p \neq q^+$ and the sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are respectively strictly decreasing and strictly increasing, then there exist strictly decreasing sequences $\{y_1^n\}_n \subseteq \mathbb{Z}^{(p)}$ and $\{y_2^n\}_n \subseteq \mathbb{Z}^{(q^+)}$ such that $\{|y_1^n, y_2^n|\}_n \subseteq \mathcal{U}$.

(PC 2′) If there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathcal{U} \cap \mathbb{Z}^{(p,q)}$ for some $p, q \in [m]$ such that $q \neq p^+$, $p \neq q$, and the sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are respectively strictly increasing and strictly decreasing, then there exist strictly decreasing sequences $\{y_1^n\}_n \subseteq \mathbb{Z}^{(p^+)}$ and $\{y_2^n\}_n \subseteq \mathbb{Z}^{(q)}$ such that $\{(y_1^n, y_2^n)\}_n \subseteq \mathcal{U}$.

(PC 3) If there exists a sequence $\{(x_1, x_2^n)\}_n \subseteq \mathcal{U} \cap \mathbb{Z}^{(p,q)}$ for some $p, q \in [m]$ such that the sequence $\{x_2^n\}_n$ is strictly increasing, then there exists a strictly decreasing sequence $\{y_2^n\}_n \subseteq \mathbb{Z}^{(q^+)}$ such that $\{|x_1, y_2^n|\}_n \subseteq \mathcal{U}$.

(PC 3′) If there exists a sequence $\{(x_1^n, x_2)\}_n \subseteq \mathcal{U} \cap \mathbb{Z}^{(p,q)}$ for some $p, q \in [m]$ such that $p \neq q$ and the sequence $\{x_1^n\}_n$ is strictly increasing, then there exists a strictly decreasing sequence $\{y_1^n\}_n \subseteq \mathbb{Z}^{(p^+)}$ such that $\{(y_1^n, x_2)\}_n \subseteq \mathcal{U}$.

The conditions (PC 1), (PC 3), and (PC 3′) correspond to condition (PC 1) in [21, Definition 0.7], and conditions (PC 2), (PC 2′), (PC 3), (PC 3′) correspond to (PC 2) in [21, Definition 0.7]. Figure 3.4 provides an illustration of the PC conditions.

Theorem 3.3.2 ([21, Theorem 3.1])**.** *A subcategory of $\mathcal{C}_m$ is precovering if and only if it satisfies the* PC *conditions.*

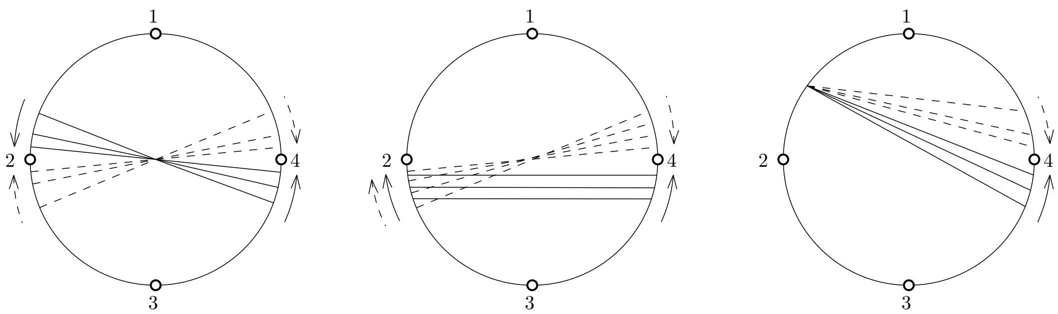


Figure 3.4: On the left (PC 1), in the middle (PC 2), on the right (PC 3).

The PC conditions were used in [21] for classifying torsion pairs, see Section 3.6, and cluster-tilting subcategories in $\mathcal{C}_m$. These correspond to some triangulations of $\mathcal{Z}_m$ satisfying certain conditions about convergence to the accumulation points. The classification of cluster-tilting subcategories in [21] generalises those in [26] and [39] for the cases $m = 1$ and $m = 2$.

3.4 Extension-closed subcategories

Extension-closed subcategories of $\mathcal{C}_m$ were classified in [14, Theorem 7.2] for the case $m = 1$. The precovering extension-closed subcategories of $\mathcal{C}_m$, i.e. the torsion classes, were classified in [21, Theorem 4.7]. Here we classify the subcategories of $\mathcal{C}_m$ which are just extension-closed for all $m \geq 1$.

Recall that we identify the indecomposable objects of $\mathcal{C}_m$ with the arcs of $\mathcal{Z}_m$.

Definition 3.4.1. Let $a, b \in \operatorname{ind}\mathcal{C}_m$ be crossing arcs. The arcs of $\operatorname{ind}\mathcal{C}_m \setminus \{a, b\}$ which connect the endpoints of a and b are called *Ptolemy arcs*. We say that a subcategory $\mathcal{U}$ of $\mathcal{C}_m$ satisfies the *Ptolemy condition*, PT condition for short, if it is closed under taking Ptolemy arcs.

Figure 3.5 provides an illustration of Ptolemy arcs.

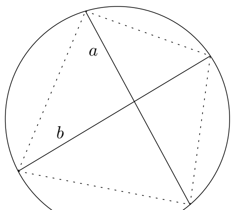


Figure 3.5: The dotted arcs are the Ptolemy arcs of a and b.

Consider a non-split triangle of the form $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ with $a, b \in \operatorname{ind}\mathcal{C}_m$ and $b \ncong \Sigma a$. The middle term e is determined by the Ptolemy arcs of a and b. More precisely

- if $a_1 < b_1 < a_2 < b_2$ then $e \cong e_1 \oplus e_2$ with $e_1 = (a_1, b_2)$ and $e_2 = (b_1, a_2)$, and
- if $b_1 < a_1 < b_2 < a_2$ then $e \cong e'_1 \oplus e'_2$ with $e'_1 = (b_1, a_1)$ and $e'_2 = (b_2, a_2)$.

In the first case, if $a_2 = b_2 + 1$ we interpret (b_1, a_2) as the zero object. In the second case, if $a_1 = b_1 + 1$ we interpret (b_1, a_1) as the zero object, and if $a_2 = b_2 + 1$ we interpret (b_2, a_2) as the zero object.

Now consider a triangle of the form $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$ with $a, b_1, \ldots, b_n \in \operatorname{ind}\mathcal{C}_n$, such that the objects $b_1, \ldots, b_n$ are pairwise Hom-orthogonal, i.e. $\operatorname{Hom}_{\mathcal{C}_m}(b_i, b_j) = 0$ for each $i \neq j$, and $h = \begin{pmatrix} h_1 & \cdots & h_n \end{pmatrix}$ has all non-zero entries, cf. [22, Lemma 3.2]. The middle term e of such a triangle was computed in [22, Lemma 4.16]. Their result generalises [14, Proposition 4.12] for the case $m = 1$. Now we show that computing the middle term of a triangle of that form is enough to obtain the middle term of a triangle of the form $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ with $a \in \operatorname{ind}\mathcal{C}_m$.

Lemma 3.4.2. *Let $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$ be a triangle in $\mathcal{C}_m$ with $a, b_1, \ldots, b_n \in \operatorname{ind}\mathcal{C}_m$, $b = \bigoplus_{i=1}^n b_i$, and $h = \begin{pmatrix} h_1 & \cdots & h_n \end{pmatrix}$. Then there exists $b' = \bigoplus_{i=1}^k b'_i$ a direct summand of b such that the objects $b'_1, \ldots, b'_k \in \operatorname{ind}\mathcal{C}_m$ are pairwise* Hom*-orthogonal, $b'_i \ncong \Sigma a$ for each i, all the entries of $h' = \begin{pmatrix} h'_1 & \cdots & h'_k \end{pmatrix} : b' \to \Sigma a$ are non-zero, and there is the following isomorphism of triangles.*

$$\begin{tikzcd}
a \arrow[r] \arrow[d, "1"] & e \arrow[r] \arrow[d, "\wr"] & b \arrow[r, "h"] \arrow[d, "\wr"] & \Sigma a \arrow[d, "1"] \\
a \arrow[r] & e' \oplus b'' \arrow[r] & b' \oplus b'' \arrow[r, "(h'\ 0)"] & \Sigma a
\end{tikzcd}$$

Proof. Without loss of generality, we can assume that $h_1, \ldots, h_n \neq 0$, see [14, Lemma 3.1]. Since the Hom-spaces are at most one dimensional, it is straightforward to check that

$h\colon b \to \Sigma a$ is an $\operatorname{add}\{b\}$-precover of Σa. Thus, by Lemma 2.3.21 there exists $b', b'' \in \operatorname{add}\{b\}$ and an isomorphism $\alpha\colon b' \oplus b'' \longrightarrow b$ such that the composition

$$b' \xrightarrow{\binom{1}{0}} b' \oplus b'' \xrightarrow{\alpha} b \xrightarrow{h} \Sigma a$$

is an $\operatorname{add}\{b\}$-cover of Σa, which we denote by $h'\colon b' \to \Sigma a$. We denote $h\alpha = \left(\begin{smallmatrix} h' & h'' \end{smallmatrix}\right) : b' \oplus b'' \to \Sigma a$. Since $h'\colon b' \to \Sigma a$ is an $\operatorname{add}\{b\}$-cover of Σa, there exists $\beta\colon b'' \to b'$ such that $h'\beta = h''$, and then

$$\begin{pmatrix} h' & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} h' & h'' \end{pmatrix} = h\alpha.$$

As a consequence, we obtain the isomorphism of triangles in the claim. Now we show that the object b' and the morphism h' satisfy the conditions of the claim.

We denote $b' = \bigoplus_{i=1}^{k} b'_i$ with $b'_1, \ldots, b'_k \in \operatorname{ind} \mathcal{C}_m$ and $h' = \left(\begin{smallmatrix} h'_1 & \cdots & h'_k \end{smallmatrix}\right)$. Since $h'\colon b' \to \Sigma a$ is a cover, it is right-miminal, and then we have that $h'_1, \ldots, h'_k \neq 0$. Moreover, for each $i \neq j$ we have that h'_i is not of the form $h'_j\beta$ for some $\beta\colon b'_j \to b'_i$. Indeed, assume that there exists $\beta\colon b'_j \to b'_i$ such that $h'_i = h'_j\beta$. Then we define the morphism $\gamma = (\gamma)_{s,t}\colon \bigoplus_{i=1}^{k} b'_i \to \bigoplus_{i=1}^{k} b'_i$ as

$$(\gamma)_{s,t} = \begin{cases} 1 & \text{if } s = t \neq i, \\ 0 & \text{if } s = t = i, \\ \beta & \text{if } s = j \text{ and } t = i, \\ 0 & \text{otherwise.} \end{cases}$$

We have that $h'\gamma = h'$. Since γ is not an isomorphism, we obtain a contradiction with the fact that h' is right minimal. Thus, in particular $b'_i \ncong \Sigma a$ for each i. Now we can apply the same argument of [14, Lemma 4.6] and we obtain that $\operatorname{Hom}_{\mathcal{C}_m}(b'_i, b'_j) = 0$ for each $i \neq j$. This concludes the argument. □

From [14, Theorem 4.1] we know that the middle terms of arbitrary triangles of $\mathcal{C}_m$ can be computed iteratively when $m = 1$. It is straightforward to check that the same holds for $m \geq 2$.

Let $\mathcal{U}$ be a subcategory of $\mathcal{C}_m$, and consider a triangle $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ in $\mathcal{C}_m$ with $a, b \in \mathcal{U}$. From [22, Lemma 3.4] the coordinates of the indecomposable summands of e belong to the set of coordinates of the indecomposable summands of a and b. Note that in general this does not imply that $e \in \mathcal{U}$. Now we discuss a necessary and sufficient condition for $\mathcal{U}$ to be extension-closed.

Lemma 3.4.3. *Let $\mathcal{U}$ be a subcategory of $\mathcal{C}_m$. If $\mathcal{U}$ is closed under extensions of the form $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ with $a, b \in \operatorname{ind} \mathcal{C}_m$, then $\mathcal{U}$ is extension-closed.*

Proof. We divide the proof into claims.

Claim 1. The subcategory $\mathcal{U}$ is closed under extensions of the form $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ with $a \in \operatorname{ind} \mathcal{U}$ and $b \in \mathcal{U}$.

Consider an extension $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$ where $b = \bigoplus_{i=1}^{n} b_i$, $a, b_1, \ldots, b_n \in \operatorname{ind}\mathcal{C}_m$, and $h = (\,h_1 \; \cdots \; h_n\,)$. We show by induction on n that $e \in \mathcal{U}$. If $n = 1$ the claim follows by assumption. If $n \geq 2$, by Lemma 3.4.2 we can further assume that $b_1, \ldots, b_n$ are pairwise Hom-orthogonal, and that $b_i \ncong \Sigma a$ and $h_i \neq 0$ for each i.

For each $i \in \{1, \ldots, n\}$ we have that $\operatorname{Hom}_{\mathcal{C}_m}(b_i, \Sigma a) \cong \mathbb{K}$. Since Σ^2 is a Serre functor for $\mathcal{C}_m$ and $\Sigma^2 \cong \Sigma\tau$, see Section 2.1 and Proposition 2.3.6, by Serre duality we have that $\operatorname{Hom}_{\mathcal{C}_m}(\tau^{-1}a, b_i) \cong D\operatorname{Hom}_{\mathcal{C}_m}(b_i, \Sigma a) \cong \mathbb{K}$. This is equivalent to: $b_i \in H^+(\tau^{-1}a) \cup H^-(\Sigma a)$ for each i. We have the following possibilities: $b_i \in H^-(\Sigma a)$ for each i, or there exists i such that $b_i \in H^+(\tau^{-1}a)$. In the first case, we rename $b_1, \ldots, b_n$ in such a way that the first coordinate of b_n is the minimum of the first coordinates of $b_1, \ldots, b_n$. In the second case, we rename $b_1, \ldots, b_n$ in such a way that the first coordinate of b_n is the maximum of the first coordinates of $b_1, \ldots, b_n$. We refer to Figure 3.6 for an illustration.

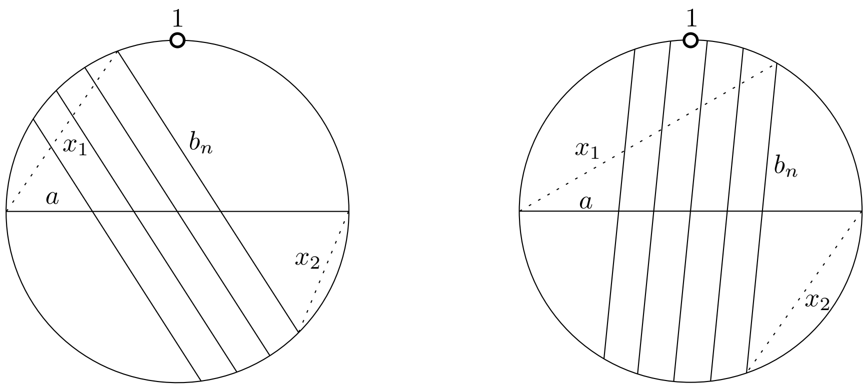


Figure 3.6: On the left when $b_i \in H^-(\Sigma a)$ for each $i \in \{1, \ldots, n\}$, on the right when there exists $i \in \{1, \ldots, n\}$ such that $b_i \in H^+(\tau^{-1}a)$.

Consider the following Octahedral Axiom diagram.

$$\begin{tikzcd}
 & b_n \arrow[r, "1"] \arrow[d] & b_n \arrow[d, dashed] & \\
e \arrow[r] \arrow[d, "1"'] & b \arrow[r, "h"] \arrow[d] & \Sigma a \arrow[r] \arrow[d, dashed] & \Sigma e \arrow[d, "1"] \\
e \arrow[r] & \bigoplus_{i=1}^{n-1} b_i \arrow[r, "g"] \arrow[d, "0"'] & \Sigma x \arrow[r] \arrow[d, dashed] & \Sigma e \\
 & \Sigma b_n \arrow[r, "1"] & \Sigma b_n &
\end{tikzcd}$$

Note that $x \in \mathcal{U}$ since it is the middle term of the triangle $a \longrightarrow x \longrightarrow b_n \longrightarrow \Sigma a$ and $a, b_n \in \operatorname{ind}\mathcal{U}$. The object x is either indecomposable or has two indecomposable direct summands. If x is indecomposable, then $e \in \mathcal{U}$ follows by hypothesis. Now we assume that x has two indecomposable direct summands $x \cong x_1 \oplus x_2$. We denote $b_n = (b_{n,1}, b_{n,2})$. If $b_n \in H^-(\Sigma a)$ then $x_2 = (b_{n,2}, a_2)$, and if $b_n \in H^+(\tau^{-1}a)$ then $x_2 = (b_{n,1}, a_2)$. In both cases x_2 does not cross any arc $b_1, \ldots, b_{n-1}$, see Figure 3.6. We define the object b' as

$b' = \bigoplus_{i=1}^{n-1} b_i$. We have the following isomorphism of triangles.

$$\begin{tikzcd}
x \arrow[r] \arrow[d, "\wr"] & e \arrow[r] \arrow[d, "\wr"] & b' \arrow[r, "g"] \arrow[d, "1"] & \Sigma x \arrow[d, "\wr"] \\
x_1 \oplus x_2 \arrow[r] & e' \oplus x_2 \arrow[r] & b' \arrow[r, "\binom{g_1}{0}"] & \Sigma x_1 \oplus \Sigma x_2
\end{tikzcd}$$

Consider the triangle $x_1 \longrightarrow e' \longrightarrow b' \xrightarrow{g_1} \Sigma x_1$. Since $x_1 \in \operatorname{ind}\mathcal{U}$ and $b' \in \mathcal{U}$, by induction hypothesis we obtain that $e' \in \mathcal{U}$. Moreover, since $x_2 \in \operatorname{ind}\mathcal{U}$, we have that $e \cong e' \oplus x_2 \in \mathcal{U}$. This concludes the argument of Claim 1.

Claim 2. The subcategory $\mathcal{U}$ is closed under extensions.

Consider a non-split extension $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ in $\mathcal{C}_m$ with $a = \bigoplus_{i=1}^{k} a_i$, $b = \bigoplus_{j=1}^{n} b_j$, and $a_i, b_j \in \operatorname{ind}\mathcal{U}$ for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$. We proceed by induction on k. If $k = 1$ then we have the statement by Claim 1. Assume that $k \geq 2$, and consider the following Octahedral Axiom diagram.

$$\begin{tikzcd}
 & a_k \arrow[r, "1"] \arrow[d] & a_k \arrow[d, dashed] & \\
\Sigma^{-1} b \arrow[r] \arrow[d, "1"'] & a \arrow[r] \arrow[d] & e \arrow[r] \arrow[d, dashed] & b \arrow[d, "1"'] \\
\Sigma^{-1} b \arrow[r] & \bigoplus_{i=1}^{k-1} a_i \arrow[r] \arrow[d, "0"'] & x \arrow[r] \arrow[d, dashed] & b \\
 & \Sigma a_k \arrow[r, "1"] & \Sigma a_k &
\end{tikzcd}$$

Consider the triangle $\bigoplus_{i=1}^{k-1} a_i \longrightarrow x \longrightarrow b \longrightarrow \bigoplus_{i=1}^{k-1} \Sigma a_i$. By induction hypothesis we obtain that $x \in \mathcal{U}$, and then $e \in \mathcal{X}$ from Claim 1. This concludes the argument. $\square$

Proposition 3.4.4. *Let $\mathcal{U}$ be a subcategory of $\mathcal{C}_m$. Then the following statements are equivalent.*

1. *The subcategory $\mathcal{U}$ satisfies the* PT *condition.*
2. *The subcategory $\mathcal{U}$ is closed under extensions of the form $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ where $a, b \in \operatorname{ind}\mathcal{C}_m$.*
3. *The subcategory $\mathcal{U}$ is closed under extensions.*

Proof. The equivalence of statements (1) and (2) follows from the computation of the middle term of an extension having indecomposable outer terms, and the equivalence between (2) and (3) is given by Lemma 3.4.3. $\square$

3.5 Non-crossing partitions

Non-crossing partitions, introduced in [38], are combinatorial objects used in [22] for classifying t-structures in $\mathcal{C}_m$. We will also use them in Section 4.7 and Section 4.8 for classifying (co-)t-structures in the Paquette Yıldırım completion of $\mathcal{C}_m$.

Let k be a positive integer. Consider the unit circle S^1 with anticlockwise orientation, and a finite set of elements of S^1, which we label as $\{1, \dots, k\} = [k]$, with the cyclic order $1 < 2 < \cdots < k < 1$.

Definition 3.5.1. A *non-crossing partition* of $[k]$ is a partition $\mathcal{P}$ of $[k]$ such that for any $i_1, i_2, j_1, j_2 \in [k]$ which are in cyclic order $i_1 < j_1 < i_2 < j_2 < i_1$, if $i_1, i_2 \in B$ and $j_1, j_2 \in C$ for some $B, C \in \mathcal{P}$, then $B = C$. If $\mathcal{P}$ is a non-crossing partition, its elements are called *blocks*.

The *Kreweras complement*, $\mathcal{P}^c$, of a non-crossing partition $\mathcal{P}$ of $[k]$ is obtained as follows, see Figure 3.7 for an illustration.

1. Double the elements of $[k]$ to get the set $[k^e] \cup [k^o] = \{1^e, 1^o, \dots, k^e, k^o\}$ with cyclic order $1^e < 1^o < \cdots < k^e < k^o < 1^e$.

2. Define $\mathcal{P}^e$ as the non-crossing partition of $[k^e]$ which consists of $\mathcal{P}$.

3. Complete $\mathcal{P}^e$ to a *serrée* (*dense*) non-crossing partition $\mathcal{P}^e \cup \mathcal{P}^o$ of $[k^e] \cup [k^o]$, see [38, p. 338].

4. Define $\mathcal{P}^c$ as $\mathcal{P}^o$ and relabel the elements of $[k^o]$ as $1, \dots, k$.

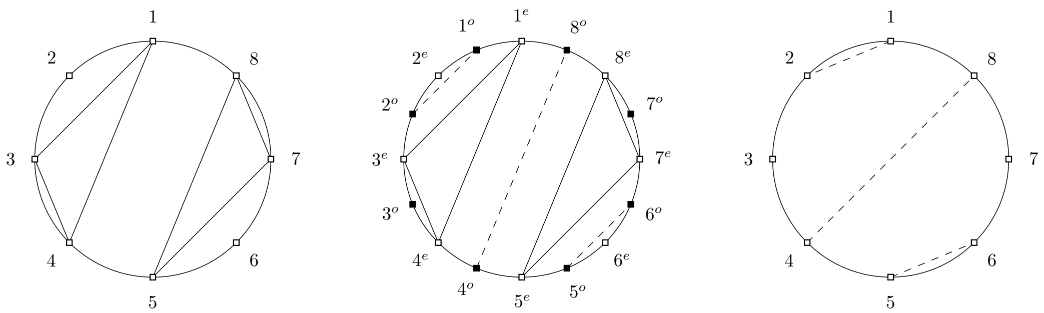


Figure 3.7: On the left $\mathcal{P} = \{\{1, 3, 4\}, \{2\}, \{5, 7, 8\}, \{6\}\}$ is a non-crossing partition of [8], and $\mathcal{P}^c = \{\{1, 2\}, \{3\}, \{4, 8\}, \{5, 6\}, \{7\}\}$ on the right is its Kreweras complement.

Now we introduce some notation which will be useful in Section 4.8.8. Given $i \in [k]$, by i^+ and i^- we denote respectively the successor and predecessor element of $[k]$ with respect to the cyclic order. Now consider a non-crossing partition $\mathcal{P}$ of $[k]$ and a block $B \in \mathcal{P}$. We denote $B^+ = \{i^+ \in [k] \mid i \in B\}$ and $B^- = \{i^- \in [k] \mid i \in B\}$. Then we define the non-crossing partitions $\mathcal{P}^+ = \{B^+ \mid B \in \mathcal{P}\}$ and $\mathcal{P}^- = \{B^- \mid B \in \mathcal{P}\}$ of $[k]$. Note that $\mathcal{P}^+$ consists of an anticlockwise rotation of $\mathcal{P}$, and $\mathcal{P}^-$ of a clockwise rotation of $\mathcal{P}$. It is straightforward to check that $(\mathcal{P}^c)^c$, which from now on we denote by $\mathcal{P}^{cc}$, coincides with $\mathcal{P}^+$. We define ${}^c\mathcal{P} = (\mathcal{P}^c)^+$, so that we have ${}^c(\mathcal{P}^c) = \mathcal{P} = ({}^c\mathcal{P})^c$.

3.6 Torsion pairs and t-structures

The torsion pairs in $\mathcal{C}_m$ were classified in [21], but also [41] and [11] for the cases $m = 1$ and $m = 2$ respectively.

Theorem 3.6.1 ([21, Theorem 4.7])**.** *Let $(\mathcal{U}, \mathcal{V})$ be a pair of subcategories in $\mathcal{C}_m$. Then $(\mathcal{U}, \mathcal{V})$ is a torsion pair in $\mathcal{C}_m$ if and only if $\mathcal{U}$ satisfies the* PC *conditions, the* PT *condition, and $\mathcal{V} = \mathcal{U}^{\perp}$.*

The t-structures in $\mathcal{C}_m$ were classified in [22], and in [41] and [11] for the cases $m = 1$ and $m = 2$ respectively. Now we introduce the combinatorial objects used in [22, Section 4] for classifying the aisles of the t-structures in $\mathcal{C}_m$. Let k be a positive integer and consider the ∞-gon $\mathcal{Z}_k$ whose set of accumulation points is $[k]$.

Definition 3.6.2 ([22, Definition 4.5])**.** A *decorated non-crossing partition* of $[k]$ is a pair $(\mathcal{P}, X)$ given by a non-crossing partition $\mathcal{P}$ of $[k]$ and an k-tuple $X = (x_p)_{p\in[k]}$ of elements of $\mathcal{Z}_k \cup [k]$, where for each $p \in [k]$

$$x_p \in \begin{cases} [p, p^+) & \text{if } \{p\} \in \mathcal{P}, \\ (p, p^+] & \text{if } p, p^+ \in B \text{ for some block } B \in \mathcal{P}, \\ (p, p^+) & \text{otherwise.} \end{cases}$$

The above definition will be applied to the cases $k = m$ and $k = 2m$. Figure 3.8 provides an example of decorated non-crossing partition.

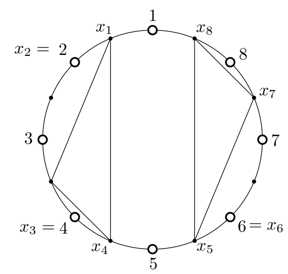


Figure 3.8: The decorated non-crossing partition $(\mathcal{P}, X)$ of [8], where $\mathcal{P}$ is as in Figure 3.7.

From a decorated non-crossing partition we can obtain the aisle of a t-structure, and conversely; we refer to Figure 3.9 for an illustration. Let $(\mathcal{P}, X)$ be a decorated non-crossing partition of $[m]$, we define

$$\mathcal{U}_{(\mathcal{P},X)} = \operatorname{add} \bigsqcup_{B\in\mathcal{P}} \left\{ (u_1, u_2) \in \operatorname{ind}\mathcal{C}_m \,\middle|\, u_1, u_2 \in \bigcup_{p\in B} (p, x_p] \right\}.$$

The subcategory $\mathcal{U}_{(\mathcal{P},X)}$ is the aisle of a t-structure, see [22, Proposition 4.8]. Now let $\mathcal{U}$ be the aisle of a t-structure in $\mathcal{C}_m$. The relation $\sim_{\mathcal{U}}$ on the set $[m]$ is defined as follows: for any $p, q \in [m]$ we have that $p \sim_{\mathcal{U}} q$ if and only if $p = q$ or there exists an arc of $\mathcal{U}$ with an endpoint in $\mathbb{Z}^{(p)}$ and the other in $\mathbb{Z}^{(q)}$. We define $\mathcal{P}_{\mathcal{U}}$ to be the partition of $[m]$ given by the equivalence classes of $\sim_{\mathcal{U}}$. For each $p \in [m]$ we define

$$x_p = \sup\{z \in \mathbb{Z}^{(p)} \mid \text{there exists an arc of } \mathcal{U} \text{ with an endpoint equal to } z\}.$$

We denote by $X_{\mathcal{U}}$ the m-tuple $X_{\mathcal{U}} = (x_p)_{p\in[m]}$. Then $(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}})$ is a decorated non-crossing partition of $[m]$, see [22, Lemma 4.12]. These assignments determine a bijection.

Theorem 3.6.3 ([22, Theorem 4.6])**.** *The following is a bijection.*

$$\{\ \textit{Decorated non-crossing partitions of } [m]\ \} \longleftrightarrow \{\ \textit{Aisles of t-structures in } \mathcal{C}_m\ \}$$
$$(\mathcal{P}, X) \longmapsto \mathcal{U}_{(\mathcal{P},X)}$$
$$(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}}) \longleftarrow\!\dashv \mathcal{U}$$

Given the aisle of a t-structure, we can compute its co-aisle via the Kreweras complement of the associated decorated non-crossing partition, see Figure 3.9 for an example. We now introduce some notation. Let $(\mathcal{P}, X)$ be a decorated non-crossing partition of $[m]$ with $X = (x_p)_{p\in[m]}$. By $(\mathcal{P}, X)^c$ we denote the pair $(\mathcal{Q}, Y)$ where $\mathcal{Q} = \mathcal{P}^c$ is the Kreweras complement of $\mathcal{P}$, and Y is the m-tuple $Y = (x_p - 1)_{p\in[m]}$. Now we define

$$\mathcal{V}_{(\mathcal{Q},Y)} = \operatorname{add} \bigsqcup_{B\in\mathcal{Q}} \left\{ (v_1, v_2) \in \operatorname{ind}\mathcal{C}_m \,\middle|\, v_1, v_2 \in \bigcup_{p\in B} [y_p, p^+) \right\}.$$

Corollary 3.6.4 ([22, Corollary 4.14])**.** *Let $(\mathcal{U}, \mathcal{V})$ be a t-structure in $\mathcal{C}_m$, $(\mathcal{P}, X)$ be the decorated non-crossing partition of $[m]$ associated to $\mathcal{U}$ with $X = (x_p)_{p\in[m]}$, and $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$. Then $\mathcal{V} = \mathcal{V}_{(\mathcal{Q},Y)}$.*

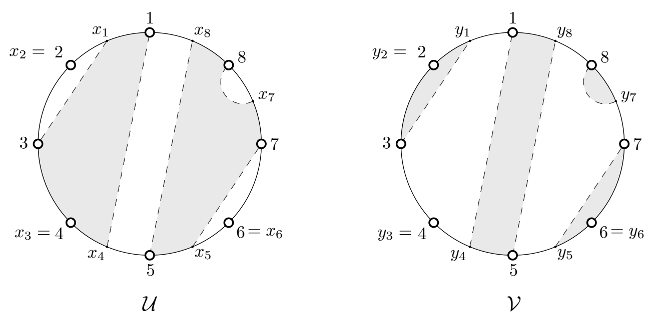


Figure 3.9: The t-structure $(\mathcal{U}, \mathcal{V})$ in $\mathcal{C}_8$ associated to the decorated non-crossing partition $(\mathcal{P}, X)$, where $\mathcal{P}$ is as in Figure 3.7.

In [22, Section 4], the authors classified important classes of t-structures in $\mathcal{C}_m$.

Proposition 3.6.5 ([22, Corollary 4.15, Corollary 4.19, Proposition 4.12, Corollary 4.22])**.** *Let $(\mathcal{U}, \mathcal{V})$ be a t-structure in $\mathcal{C}_m$ and consider its associated decorated non-crossing partition $(\mathcal{P}, X)$ of $[m]$ with $X = (x_p)_{p\in[m]}$. The following statements hold.*

- *The heart of $(\mathcal{U}, \mathcal{V})$ is given by $\mathcal{U} \cap \Sigma\mathcal{V} = \operatorname{add}\{(x_p - 2, x_p) \mid x_p \in \mathcal{Z}_m\}$.*

- *The t-structure* $(\mathcal{U}, \mathcal{V})$ *is left-bounded, or right-bounded, if and only if* $\mathcal{P} = \{1, \ldots, m\}$, *or* $\mathcal{P} = \{\{1\}, \ldots, \{m\}\}$ *respectively.*
- *If* $m \geq 2$*, there are no bounded t-structures in* $\mathcal{C}_m$*.*
- *The t-structure* $(\mathcal{U}, \mathcal{V})$ *is left non-degenerate, or right non-degenerate, if and only if* $x_p \neq p^+$*, or* $x_p \neq p$ *respectively, for each* $p \in [m]$*.*
- *The t-structure* $(\mathcal{U}, \mathcal{V})$ *is non-degenerate if and only if* $x_p \in \mathcal{Z}_m$ *for each* $p \in [m]$*.*

The thick subcategories of $\mathcal{C}_m$ were classified in [22, Section 3] via combinatorial objects similar to decorated non-crossing partitions, namely the *non-exhaustive non-crossing partitions*, see [22, Definition 3.5].

Remark 3.6.6. From [49, Proposition 4.6] we know that in the category $\mathcal{C}_m$ the only co-t-structures are $(\mathcal{C}_m, 0)$ and $(0, \mathcal{C}_m)$. As a consequence, there exist only trivial precovering or preenveloping, and thus functorially finite, thick subcategories of $\mathcal{C}_m$. Since functorially finite thick subcategories are in bijection with TTF triples and with recollements, see Section 2.3.5, we observe that $\mathcal{C}_m$ is *triangulated simple* (this term is inspired by the term *derived simple* of [1]).

Chapter 4

Completion of discrete cluster categories

The Paquette–Yıldırım completion $\overline{\mathcal{C}}_m$ of $\mathcal{C}_m$ was defined in [43] by taking the Verdier quotient of $\mathcal{C}_{2m}$ with respect to a specific thick subcategory. The completion was first defined by Fisher in [19] for the case $m = 1$, by closing the category $\mathcal{C}_1$ under homotopy colimits. In [3] the authors proved that the category $\overline{\mathcal{C}}_1$ is also equivalent to a stable Grassmannian category of infinite rank. In [17] was defined another completion of $\mathcal{C}_m$, namely the Neeman completion. In this chapter we introduce the definition and basic properties of $\overline{\mathcal{C}}_m$. After discussing its quiver and factorization properties of its morphisms, we classify the precovering, preenveloping, and extension-closed subcategories. Then we give a classification of torsion pairs, t-structures, co-t-structures, and recollements of $\overline{\mathcal{C}}_m$. To do so, we use combinatorial objects analogous to decorated non-crossing partitions, used in [22] for classifying t-structures in $\mathcal{C}_m$. Finally, we prove that there are bijections between the functorially finite co-t-structures in $\overline{\mathcal{C}}_m$ and the t-structures in $\mathcal{C}_m$.

4.1 The ∞-gons $\mathcal{Z}_{2m}$ and $\overline{\mathcal{Z}}_m$

We recall that in Section 3.1 we defined the ∞-gon $\mathcal{Z}_m$. From $\mathcal{Z}_m$ we define another ∞-gon $\overline{\mathcal{Z}}_m$, which will be useful to describe the geometric model of $\overline{\mathcal{C}}_m$. We start by taking an intermediate step and considering the ∞-gon $\mathcal{Z}_{2m}$. We re-label the accumulation points of $\mathcal{Z}_{2m}$ as $1', 1, \dots, m', m$, see Figure 4.1. The set of accumulation points $[m'] \cup [m]$ has cyclic order $1' < 1 < \cdots < m' < m < 1'$ and a total order $1' < 1 < \cdots < m' < m$, which induces a total order on $\mathcal{Z}_{2m}$. The notions of interval, successor, predecessor, arc, are the same as for the set $\mathcal{Z}_m$. On $\mathcal{Z}_{2m}$ we define an equivalence relation $\sim$ as follows. For each $x, y \in \mathcal{Z}_{2m}$ we have that

$$x \sim y \text{ if and only if } x = y \text{ or } x, y \in \mathbb{Z}^{(p)} \text{ for some } p \in [m'].$$

Consider $x \in \mathcal{Z}_{2m}$, we sometimes denote the equivalence class of x by $\overline{x}$. If $x \in \mathbb{Z}^{(p)}$ for some $p \in [m']$, we identify $\overline{x} = p$ with an abuse of notation. We define the set

$\overline{\mathcal{Z}}_m = \mathcal{Z}_{2m}/\sim$ and we observe that $\overline{\mathcal{Z}}_m$ can be regarded as the set $\mathcal{Z}_m \cup [m']$. The total order on $\mathcal{Z}_{2m}$ induces a total order on $\overline{\mathcal{Z}}_m$. Given a point $z \in \overline{\mathcal{Z}}_m = \mathcal{Z}_m \cup [m']$, we define the successor $z+1$ of z as

$$z+1 = \begin{cases} \text{the successor of } z \text{ in } \mathcal{Z}_m & \text{if } z \in \mathcal{Z}_m, \\ z & \text{if } z \in [m']. \end{cases}$$

We can define $z-1$ analogously. The notions of arc of $\overline{\mathcal{Z}}_m$ and of crossing arcs are the same of those for $\mathcal{Z}_m$, see Definition 3.1.1. Given $p, q \in [m'] \cup [m]$ we define the following sets

$$C^{(p)} = \begin{cases} \mathbb{Z}^{(p)} & \text{if } p \in [m], \\ \{p\} & \text{if } p \in [m'] \end{cases} \quad \text{and } C^{(p,q)} = \left\{ (x_1, x_2) \text{ arc of } \overline{\mathcal{Z}}_m \;\middle|\; x_1 \in C^{(p)}, x_2 \in C^{(q)} \right\}.$$

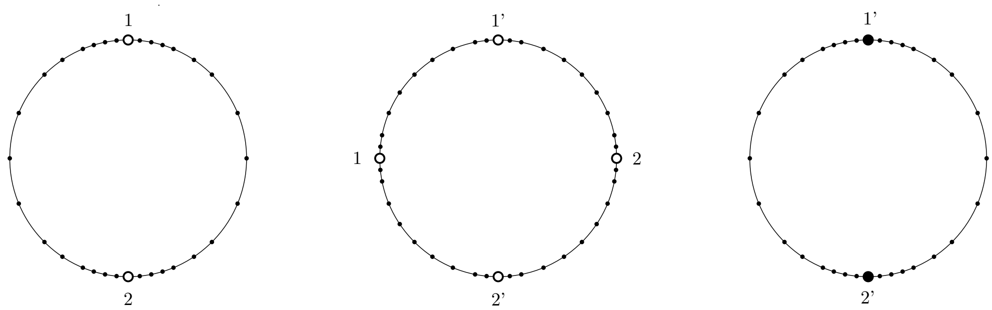


Figure 4.1: On the left the ∞-gon $\mathcal{Z}_2$, in the centre $\mathcal{Z}_4$, and on the right $\overline{\mathcal{Z}}_2$. The black circles denote the accumulation points of $\overline{\mathcal{Z}}_2$.

4.2 The geometric model

The completion $\overline{\mathcal{C}}_m$ of $\mathcal{C}_m$ was defined in [43], we recall its definition and basic properties. Consider the ∞-gon $\mathcal{Z}_{2m}$ defined in Section 4.1 and the associated category $\mathcal{C}_{2m}$. We define the subcategory $\mathcal{D}$ of $\mathcal{C}_{2m}$ as

$$\mathcal{D} = \operatorname{add} \left\{ \bigcup_{p \in [m']} \mathbb{Z}^{(p,p)} \right\}.$$

We recall that $\mathbb{Z}^{(p,p)}$ denotes the set of arcs of $\mathcal{C}_{2m}$ having both endpoints in $\mathbb{Z}^{(p)}$, see Section 3.1. It is straightforward to check that $\mathcal{D}$ is a thick subcategory of $\mathcal{C}_{2m}$.

The *Paquette–Yıldırım completion*, $\overline{\mathcal{C}}_m$, of $\mathcal{C}_m$ is defined as the Verdier quotient $\mathcal{C}_{2m}/\mathcal{D}$, and is a $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt triangulated category, see [43, Section 3] for more details. We denote the quotient functor as $\pi\colon \mathcal{C}_{2m} \to \mathcal{C}_{2m}/\mathcal{D} = \overline{\mathcal{C}}_m$ and the shift functor by $\Sigma\colon \overline{\mathcal{C}}_m \to \overline{\mathcal{C}}_m$, as for the shift functor of $\mathcal{C}_m$. We refer to Section 2.3.7 for some background about Verdier quotients.

We recall the following properties of $\overline{\mathcal{C}}_m$.

- The isoclasses of indecomposable objects of $\overline{\mathcal{C}}_m$ are in bijection with the arcs of $\overline{\mathcal{Z}}_m$, see [43, Corollary 3.11].

- For any $x = (x_1, x_2) \in \operatorname{ind}\mathcal{C}_{2m} \setminus \operatorname{ind}\mathcal{D}$ the object $\pi x \in \overline{\mathcal{C}}_m$ is indecomposable by [43, Proposition 3.10] and can be regarded as the arc $(\overline{x}_1, \overline{x}_2)$ of $\overline{\mathcal{Z}}_m$.

- Let $x \in \operatorname{ind}\overline{\mathcal{C}}_m$, then there exists $x' \in \operatorname{ind}\mathcal{C}_{2m}$ such that $\pi x' \cong x$. Indeed, if $x = (x_1, x_2)$ with $x_1, x_2 \in \overline{\mathcal{Z}}_m$, we can take $x'_1, x'_2 \in \mathcal{Z}_{2m}$ such that $\overline{x}'_1 = x_1$ and $\overline{x}'_2 = x_2$, we define $x' = (x'_1, x'_2) \in \operatorname{ind}\mathcal{C}_{2m}$.

- Given $x = (x_1, x_2) \in \operatorname{ind}\overline{\mathcal{C}}_m$ we have that $\Sigma x = (x_1 - 1, x_2 - 1)$.

- The Hom-spaces of $\overline{\mathcal{C}}_m$ between indecomposable objects are at most one-dimensional. More precisely, we have the following proposition.

Proposition 4.2.1 ([43, Proposition 3.14])**.** *Let $x, y \in \operatorname{ind}\overline{\mathcal{C}}_m$. Then $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(x, \Sigma y) \cong \mathbb{K}$ if and only if one of the following statements holds.*

- *The arcs x and y cross.*

- *The arcs x and y share exactly one endpoint $z \in [m']$, and we can reach y by rotating x in the anticlockwise direction about z.*

- *The arcs x and y share both endpoints $z_1, z_2 \in [m']$.*

Otherwise $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(x, \Sigma y) = 0$.

Remark 4.2.2. We identify the indecomposable objects of $\overline{\mathcal{C}}_m$ with the arcs of $\overline{\mathcal{Z}}_m$, and the additive subcategories of $\overline{\mathcal{C}}_m$ with sets of arcs.

From now on any subcategory of $\overline{\mathcal{C}}_m$ we refer to is assumed to be additive.

4.3 The quiver of $\overline{\mathcal{C}}_m$

In this section we describe the quiver of $\overline{\mathcal{C}}_m$ having as vertices the isoclasses of indecomposable objects of $\overline{\mathcal{C}}_m$ and as arrows the irreducible morphisms between them. We start by arranging the indecomposable objects of $\overline{\mathcal{C}}_m$ into a coordinate system, then we introduce the Hom-hammocks and we prove the factorization properties for the morphisms. Finally, we describe the irreducible morphisms and we prove that the coordinate system precisely determines the quiver of $\overline{\mathcal{C}}_m$.

4.3.1 The coordinate system

We can arrange the isoclasses of the indecomposable objects of $\overline{\mathcal{C}}_m$ into a coordinate system having

- m components of type $\mathbb{Z}A_\infty$, corresponding to the arcs of $C^{(p,p)}$ for $p \in [m]$,

- $\binom{m}{2}$ components of type $\mathbb{Z}A_\infty^\infty$, corresponding to the arcs of $C^{(p,q)}$ for $p, q \in [m]$ with $p < q$,

- $\binom{m}{2}$ components of type A_1, corresponding to the arcs of $C^{(p,q)}$ for $p, q \in [m']$ with $p < q$,
- m^2 components of type A_∞^∞, corresponding to the arcs of $C^{(p,q)}$ for $p, q \in [m'] \cup [m]$ such that either $p \in [m']$, $q \in [m]$, and $p < q$, or $p \in [m]$, $q \in [m']$, and $p < q$.

Figure 4.2 illustrates the coordinate system. With Proposition 4.3.8 we describe the irreducible morphisms of $\overline{\mathcal{C}}_m$ and thus show that the above describes the AR quiver of $\overline{\mathcal{C}}_m$.

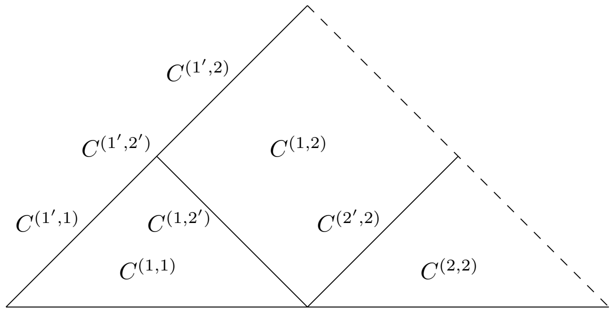


Figure 4.2: The AR quiver of $\overline{\mathcal{C}}_2$, cf. Figure 3.2.

4.3.2 Hom-hammocks

We define the Hom-hammocks for the category $\overline{\mathcal{C}}_m$ analogously to $\mathcal{C}_m$, cf. Definition 3.2.3. Figure 4.4 provides an illustration.

Definition 4.3.1. Let $a = (a_1, a_2) \in \operatorname{ind} \overline{\mathcal{C}}_m$, and let $p, q \in [m'] \cup [m]$ such that $a \in C^{(p,q)}$. We define the Hom-*hammocks* $\overline{H}^+(a)$ and $\overline{H}^-(a)$ as follows.

$$\overline{H}^+(a) = \begin{cases} \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid a_1 \leq x_1 \leq a_2 - 2 \text{ and } x_2 \geq a_2\} & \text{if } q \in [m], \\ \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid a_1 \leq x_1 < a_2 \text{ and } x_2 \geq a_2\} & \text{if } q \in [m']. \end{cases}$$

$$\overline{H}^-(a) = \begin{cases} \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid 1' \leq x_1 \leq a_1 \text{ and } a_1 + 2 \leq x_2 \leq a_2\} & \text{if } p, q \in [m], \\ \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid 1' \leq x_1 \leq a_1 \text{ and } a_1 + 2 \leq x_2 < a_2\} & \text{if } p \in [m] \text{ and } q \in [m'], \\ \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid 1' \leq x_1 < a_1 \text{ and } a_1 \leq x_2 \leq a_2\} & \text{if } p \in [m'] \text{ and } q \in [m], \\ \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid 1' \leq x_1 < a_1 \text{ and } a_1 \leq x_2 < a_2\} & \text{if } p, q \in [m']. \end{cases}$$

The following fact follows from Proposition 4.2.1.

Proposition 4.3.2. *Let $a, b \in \operatorname{ind} \overline{\mathcal{C}}_m$. Then $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(a, b) \cong \mathbb{K}$ if and only if $b \in \overline{H}^+(a) \cup \overline{H}^-(\Sigma^2 a)$.*

Since $\overline{\mathcal{C}}_m$ is not 2-Calabi-Yau, for $a, b \in \operatorname{ind} \overline{\mathcal{C}}_m$ in general $b \in \overline{H}^+(a) \cup \overline{H}^-(\Sigma^2 a)$ is not equivalent to $a \in \overline{H}^+(\Sigma^{-2} b) \cup \overline{H}^-(b)$. Therefore, we also define the *reverse* Hom-*hammocks* $\overline{I}^+$ and $\overline{I}^-$, for which $b \in \overline{H}^+(a) \cup \overline{H}^-(\Sigma^2 a)$ if and only if $a \in \overline{I}^+(\Sigma^{-2} b) \cup \overline{I}^-(b)$.

Definition 4.3.3. Let $a \in \operatorname{ind} \overline{\mathcal{C}}_m$, and let $p, q \in [m'] \cup [m]$ such that $a \in C^{(p,q)}$. We define the *reverse* Hom-*hammocks* $\overline{I}^+(a)$ and $\overline{I}^-(a)$ as follows.

$$\overline{I}^+(a) = \begin{cases} \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid a_1 \le x_1 \le a_2 - 2 \text{ and } x_2 \ge a_2\} & \text{if } p, q \in [m], \\ \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid a_1 \le x_1 \le a_2 \text{ and } x_2 > a_2\} & \text{if } p \in [m] \text{ and } q \in [m'], \\ \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid a_1 < x_1 \le a_2 - 2 \text{ and } x_2 \ge a_2\} & \text{if } p \in [m'] \text{ and } q \in [m], \\ \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid a_1 < x_1 \le a_2 \text{ and } x_2 > a_2\} & \text{if } p, q \in [m']. \end{cases}$$

$$\overline{I}^-(a) = \begin{cases} \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid 1' \le x_1 \le a_1 \text{ and } a_1 + 2 \le x_2 \le a_2\} & \text{if } p \in [m], \\ \{(x_1, x_2) \in \operatorname{ind} \overline{\mathcal{C}}_m \mid 1' \le x_1 \le a_1 \text{ and } a_1 < x_2 \le a_2\} & \text{if } p \in [m']. \end{cases}$$

Note that in general $\overline{H}^+(a) \neq \overline{I}^+(a)$ and $\overline{H}^-(a) \neq \overline{I}^-(a)$. The following fact follows from Proposition 4.2.1.

Proposition 4.3.4. *Let $a, b \in \operatorname{ind} \overline{\mathcal{C}}_m$. Then $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(a, b) \cong \mathbb{K}$ if and only if $a \in \overline{I}^+(\Sigma^{-2}b) \cup \overline{I}^-(b)$.*

4.3.3 Factorization properties

Here we study the factorization properties of the morphisms of $\overline{\mathcal{C}}_m$. We say that a morphism $f : a \to b$ in $\mathcal{C}_m$ *factors through an object* $c \in \operatorname{ind} \mathcal{C}_m$ if there exist $g : a \to c$ and $h : d \to c$ such that $f = hg$. We say that a morphism $f : a \to b$ in $\mathcal{C}_m$ *factors through $\mathcal{D}$* if it factors through some $d \in \operatorname{ind} \mathcal{D}$.

Lemma 4.3.5. *Let $a, b \in \operatorname{ind} \overline{\mathcal{C}}_m$ be such that $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(a, b) \cong \mathbb{K}$, and let $a' \in \operatorname{ind} \mathcal{C}_m$ be such that $\pi a' \cong a$. The following statements hold.*

1. *If $b \in \overline{H}^+(a)$, then there exists $b' \in \operatorname{ind} \mathcal{C}_{2m}$ such that $\pi b' \cong b$, $b' \in H^+(a')$, and any non-zero morphism $a' \to b'$ in $\mathcal{C}_{2m}$ does not factor through $\mathcal{D}$.*
2. *If $b \in \overline{H}^-(\Sigma^2 a)$, then there exists $b' \in \operatorname{ind} \mathcal{C}_{2m}$ such that $\pi b' \cong b$, $b' \in H^-(\Sigma^2 a')$, and any non-zero morphism $f' : a' \to b'$ in $\mathcal{C}_{2m}$ does not factor through $\mathcal{D}$.*

Proof. We show statement (1), statement (2) is analogous. Assume that $a = (a_1, a_2) \in C^{(p,q)}$ with $p \in [m]$ and $q \in [m']$, the other cases are similar. We write $a' = (a'_1, a'_2) \in \mathbb{Z}^{(p,q)}$. Since $b = (b_1, b_2) \in \overline{H}^+(a)$, we have that $a_1 \le b_1 < a_2$ and $b_2 \ge a_2$, see Figure 4.3.

It is straightforward to check that there exists b'_1 such that $\overline{b}'_1 = b_1$ and $a'_1 \le b'_1 < q$, and there exists b'_2 such that $\overline{b}'_2 = b_2$ and $b'_2 \ge a'_2$. Therefore, $b' \in H^+(a')$ and, since $b'_1 \notin \mathbb{Z}^{(q)}$, any non-zero morphism $a' \to b'$ does not factor through $\mathcal{D}$, see Figure 4.4. $\square$

The following lemma is dual to the lemma above, and will be useful for proving Lemma 4.4.4.

Lemma 4.3.6. *Let $a, b \in \operatorname{ind} \overline{\mathcal{C}}_m$ be such that $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(b, a) \cong \mathbb{K}$, and let $a' \in \operatorname{ind} \mathcal{C}_m$ be such that $\pi a' \cong a$. The following statements hold.*

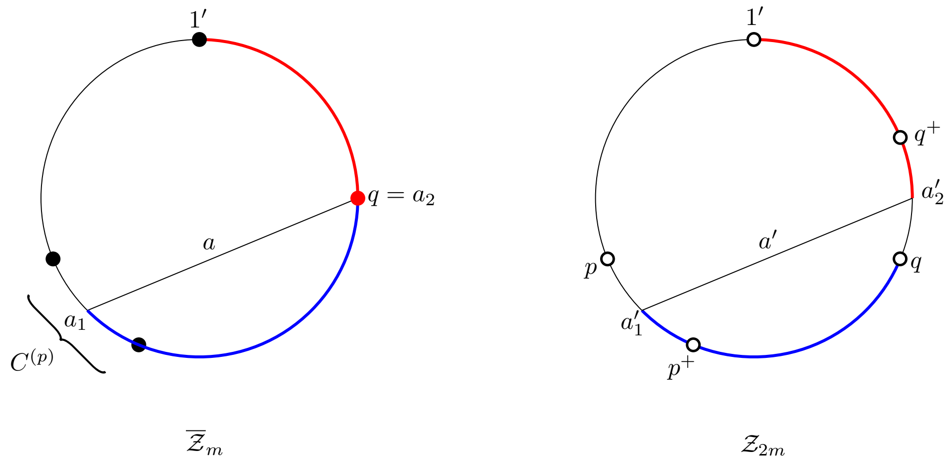


Figure 4.3: The element b_1 belongs to the blue interval of $\overline{\mathcal{Z}}_m$, and the element b_2 belongs to the red interval of $\overline{\mathcal{Z}}_m$. We can find b'_1 such that $\overline{b'_1} = b_1$ belonging to the blue interval of $\mathcal{Z}_{2m}$. The same holds for b'_2 in the red interval of $\mathcal{Z}_{2m}$.

1. *If $b \in \overline{I}^-(a)$, then there exists $b' \in \operatorname{ind} \mathcal{C}_{2m}$ such that $\pi b' \cong b$, $b' \in H^-(a')$, and any non-zero morphism $b' \to a'$ in $\mathcal{C}_{2m}$ does not factor through $\mathcal{D}$.*

2. *If $b \in \overline{I}^+(\Sigma^{-2}a)$, then there exists $b' \in \operatorname{ind} \mathcal{C}_{2m}$ such that $\pi b' \cong b$, $b' \in H^+(\Sigma^{-2}a')$, and any non-zero morphism $f' \colon b' \to a'$ in $\mathcal{C}_{2m}$ does not factor through $\mathcal{D}$.*

Now we have the factorization properties of $\overline{\mathcal{C}}_m$, cf. Lemma 3.2.5.

Proposition 4.3.7. *Let $a, b, c \in \operatorname{ind} \overline{\mathcal{C}}_m$. Assume that one of the following statements holds.*

1. *$b \in \overline{H}^+(a)$ and $c \in \overline{H}^+(a) \cap \overline{H}^+(b)$.*

2. *$b \in \overline{H}^+(a)$ and $c \in \overline{H}^-(\Sigma^2 a) \cap \overline{H}^-(\Sigma^2 b)$.*

3. *$b \in \overline{H}^-(\Sigma^2 a)$ and $c \in \overline{H}^-(\Sigma^2 a) \cap \overline{H}^+(b)$.*

Then any morphism $a \to c$ in $\overline{\mathcal{C}}_m$ factors through b.

Proof. We prove statement (1), statements (2) and (3) are analogous. Fix $a' \in \operatorname{ind} \mathcal{C}_{2m}$ such that $\pi a' \cong a$. Since $b \in \overline{H}^+(a)$, by Lemma 4.3.5 there exists $b' \in \operatorname{ind} \mathcal{C}_{2m}$ such that $\pi b' \cong b$, $b' \in H^+(a')$, and any non-zero morphism $a' \to b'$ does not factor through $\mathcal{D}$. Fix such b', since $c \in \overline{H}^+(b)$, then there exists $c' \in \operatorname{ind} \mathcal{C}_{2m}$ such that $\pi c' \cong c$, $c' \in H^+(b')$, and any non-zero morphism $b' \to c'$ does not factor through $\mathcal{D}$. We show that $c' \in H^+(a') \cap H^+(b')$.

We denote $a = (a_1, a_2)$, $a' = (a'_1, a'_2)$, $c = (c_1, c_2)$, and $c' = (c'_1, c'_2)$. Assume that $c' \notin H^+(a')$, then $c'_1 \geq a'_2 - 1$. It is straightforward to check that as a consequence $c_1 \geq a_2 - 1$. Then $c \notin \overline{H}^+(a)$ and we have a contradiction. Therefore $c' \in H^+(a') \cap H^+(b')$. Now, if there exists a non-zero morphism $f' \colon a' \to c'$ which factors through $\mathcal{D}$, then $a'_2, c'_1 \in \mathbb{Z}^{(q)}$, see Figure 4.4. This implies that $c_1 = q = a_2$, and then $c \notin \overline{H}^+(a)$ giving a contradiction. Then any non-zero morphism $f' \colon a' \to c'$ does not factor through $\mathcal{D}$.

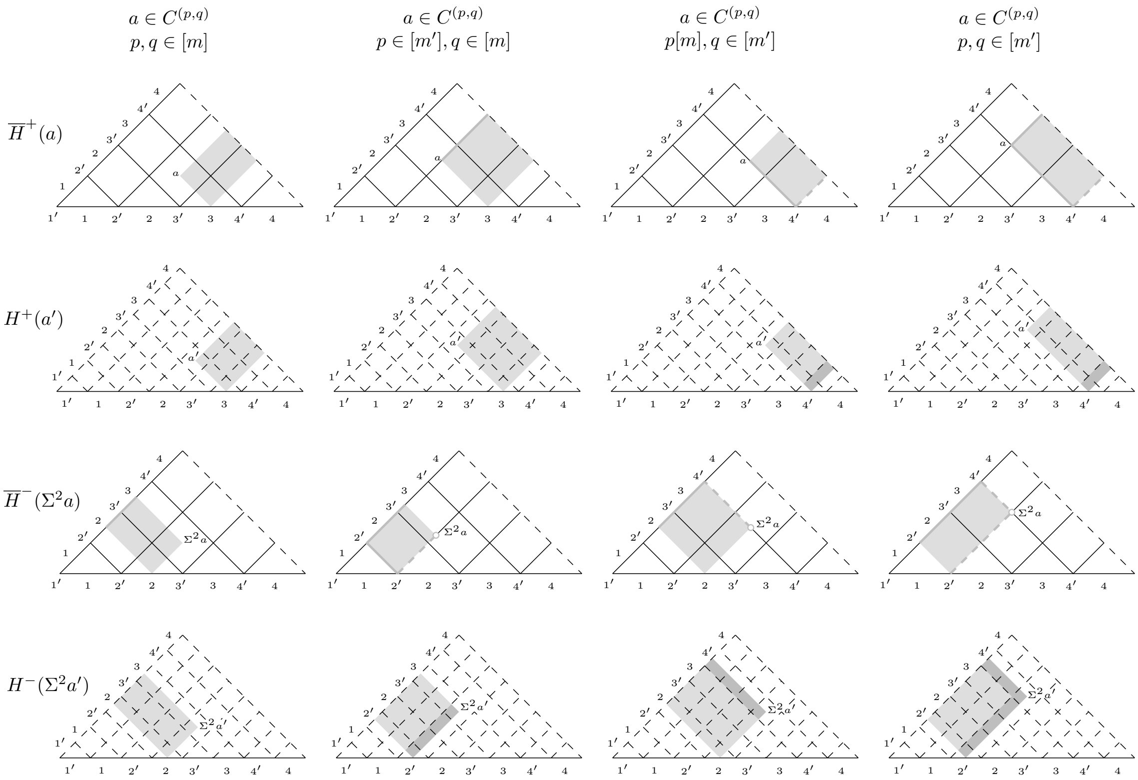


Figure 4.4: Illustration of Lemma 4.3.5. The darker area in $H^+(a')$ or in $H^+(\Sigma^2 a')$, if present, denotes the objects $b' \in H^+(a') \cup H^-(\Sigma^2 a')$ such that any non-zero morphism $a' \to b'$ factors through $\mathcal{D}$. Whenever the darker area is not present there are no such objects in $H^+(a') \cup H^-(\Sigma^2 a')$.

Since $b' \in H^+(a')$ and $c' \in H^+(a') \cap H^+(b')$, by Lemma 3.2.5 there exist $h' \colon a' \to b'$ and $g' \colon b' \to c'$ such that $f' = g'h'$, and then $\pi f' = \pi(g')\pi(h')$. Since f' does not factor through $\mathcal{D}$, we have that $\pi f' \neq 0$. Now consider a non-zero morphism $f \colon a \to c$ in $\overline{\mathcal{C}}_m$. Since $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(a, c) \cong \mathbb{K}$, we have that $f = \lambda \pi f$ for some $\lambda \in \mathbb{K}^*$, and then $f = \lambda \pi f' = \lambda \pi(g')\pi(h')$. This concludes the argument. $\square$

4.3.4 Irreducible morphisms

In this section we describe the irreducible morphisms of $\overline{\mathcal{C}}_m$. From Section 4.3.1 we already know that the isoclasses of indecomposable objects of $\overline{\mathcal{C}}_m$ are in bijection with the arcs of $\overline{\mathcal{Z}}_m$ and that they can be arranged in a coordinate system.

Proposition 4.3.8. *Let $a = (a_1, a_2), b = (b_1, b_2) \in \operatorname{ind} \overline{\mathcal{C}}_m$. Assume that $a, b \in C^{(p,q)}$ for some $p, q \in [m'] \cup [m]$ and that one of the following conditions holds.*

1. *$p \in [m']$, $q \in [m]$ and $(b_1, b_2) = (a_1, a_2 + 1)$.*
2. *$p \in [m]$, $q \in [m']$ and $(b_1, b_2) = (a_1 + 1, a_2)$.*
3. *$p, q \in [m]$ and $(b_1, b_2) = (a_1, a_2 + 1)$ or $(b_1, b_2) = (a_1 + 1, a_2)$.*

Then any non-zero morphism $a \to b$ is irreducible. Moreover, there are no other irreducible morphisms in $\overline{\mathcal{C}}_m$ between indecomposable objects.

Proof. First we show that if any of the conditions (1), (2) and (3) holds, then any non-zero morphism $f\colon a \to b$ is irreducible. Assume that condition (1) holds, for the other cases we can proceed analogously. Consider a non-zero morphism $f\colon a \to b$ and note that, since $(a_1, a_2+1) \ncong (a_1, a_2)$, f is not a split mono nor a split epi. Assume that $f = hg$ with $g\colon a \to c$ and $h\colon c \to b$ for some object $c \in \overline{\mathcal{C}}_m$. Since the Hom-spaces are one dimensional, we can assume that $c \in \operatorname{ind}\overline{\mathcal{C}}_m$. We show that g is a split mono or h is a split epi. Note that

$$c \in \left(\overline{H}^+(a) \cup \overline{H}^-(\Sigma^2 a)\right) \cap \left(\overline{I}^-(b) \cup \overline{I}^+(\Sigma^{-2} b)\right).$$

Assume that $c \ncong a$ and $c \ncong b$, then $g\colon a \to c$ factors as $g = lf$ with $l\colon b \to c$, see Figure 4.5. From the fact that $0 \neq f = hg = hlf$, it follows that $hl\colon b \to b$ is non-zero and $hl = \lambda 1_b$ for some $\lambda \in \mathbb{K}^*$. This implies that $b \cong c$, which gives a contradiction with our assumption. We conclude that $c \cong a$ or $c \cong b$, i.e. $f\colon a \to b$ is irreducible.

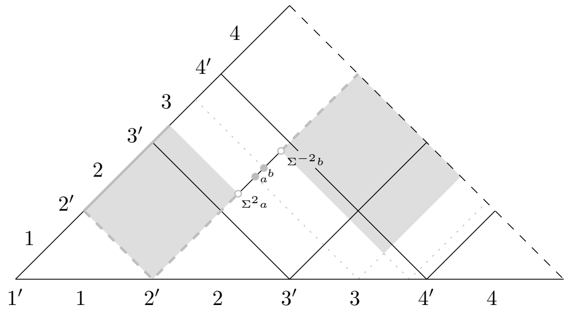


Figure 4.5: The object c is isomorphic to a or b, or belongs to the grey area.

Now, consider $a = (a_1, a_2), b = (b_1, b_2) \in \operatorname{ind}\overline{\mathcal{C}}_m$ and a non-zero morphism $f\colon a \to b$. We show that if f is irreducible then it has to be of the form listed in the statement. Let $p, q \in [m'] \cup [m]$ be such that $a \in C^{(p,q)}$. Assume that $p \in [m']$ and $q \in [m]$, the other cases are analogous. Note that if $b_2 \neq a_2$ then, from Proposition 4.3.7, f factors through the irreducible morphism $a \to (a_1, a_2+1)$, and then f is not irreducible unless $(b_1, b_2) = (a_1, a_2+1)$. If $b_2 = a_2$, then consider the object $c = (b_2 - 1, a_2)$ and the non-zero morphisms $g\colon a \to c$ and $h\colon c \to b$. From Proposition 4.3.7 we have that $f = hg$, and then f is not irreducible. We can conclude that if f is irreducible then $(b_1, b_2) = (a_1, a_2+1)$. □

4.4 Precovering and preenveloping subcategories

In this section we classify the precovering and preenveloping subcategories of $\overline{\mathcal{C}}_m$ using arc combinatorics. We also relate the precovering or preenveloping subcategories in $\overline{\mathcal{C}}_m$ to their preimages in $\mathcal{C}_{2m}$ under the localisation functor $\pi\colon \mathcal{C}_{2m} \to \overline{\mathcal{C}}_m$. In [43] the authors classified the functorially finite weak cluster-tilting subcategories of $\overline{\mathcal{C}}_m$, i.e. the cluster-tilting subcategories, generalising [19] for the case $m = 1$. After endowing $\overline{\mathcal{C}}_m$ with a specific extriangulated structure, the cluster-tilting subcategories were also classified in [10] in terms of a larger class of triangulations of $\overline{\mathcal{Z}}_m$. Here we classify subcategories of $\overline{\mathcal{C}}_m$ which are just precovering or preenveloping.

4.4.1 Precovering subcategories of $\overline{\mathcal{C}}_m$

Now we classify the precovering subcategories of $\overline{\mathcal{C}}_m$. Our approach is to relate the precovering subcategories of $\overline{\mathcal{C}}_m$ to some subcategories of $\mathcal{C}_{2m}$ which are "almost precovering". To do so, we need to introduce an auxiliary subcategory of $\mathcal{C}_{2m}$. Fix $z^0 \in \mathbb{Z}$. For each $p \in [m']$ we denote by $z^0_p \in \mathcal{Z}_{2m}$ the copy of z^0 belonging to $\mathbb{Z}^{(p)}$.

Definition 4.4.1. We define the subcategory $\mathcal{A}$ of $\mathcal{C}_{2m}$ as

$$\mathcal{A} = \operatorname{add}\left\{ (a_1, a_2) \in \operatorname{ind}\mathcal{C}_{2m} \,\middle|\, a_1, a_2 \in \bigcup_{p\in[m]} (p, z^0_{p^+}] \right\}.$$

Figure 4.6 illustrates the subcategory $\mathcal{A}$.

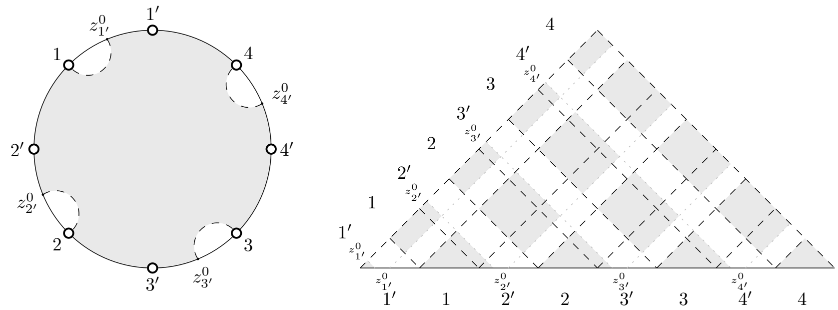


Figure 4.6: The category $\mathcal{A}$.

Now we define the completed versions of the PC conditions, cf. Definition 3.3.1.

Definition 4.4.2. Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. We say that $\mathcal{X}$ satisfies the *completed precovering conditions*, $\overline{\mathrm{PC}}$ for short, if it satisfies the following combinatorial conditions.

($\overline{\mathrm{PC}}$1) If there exists a sequence $\{(x^n_1, x^n_2)\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$ for some $p, q \in [m]$ such that $p \neq q$ and the sequences $\{x^n_1\}_n$ and $\{x^n_2\}_n$ are strictly increasing, then $|p^+, q^+| \in \mathcal{X}$.

($\overline{\mathrm{PC}}$2) If there exists a sequence $\{(x^n_1, x^n_2)\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$ for some $p, q \in [m]$ and the sequences $\{x^n_1\}_n$ and $\{x^n_2\}_n$ are respectively strictly decreasing and strictly increasing, then there exists a strictly decreasing sequence $\{y^n_1\}_n \subseteq C^{(p)}$ such that $\{|y^n_1, q^+|\}_n \subseteq \mathcal{X}$.

($\overline{\mathrm{PC}}2'$) If there exists a sequence $\{(x^n_1, x^n_2)\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$ for some $p, q \in [m]$ such that $p \neq q$, and the sequences $\{x^n_1\}_n$ and $\{x^n_2\}_n$ are respectively strictly increasing and strictly decreasing, then there exists a strictly decreasing sequence $\{y^n_2\}_n \subseteq C^{(q)}$ such that $\{(p^+, y^n_2)\}_n \subseteq \mathcal{X}$.

($\overline{\mathrm{PC}}$3) If there exists a sequence $\{(x_1, x^n_2)\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$ for some $p \in [m'] \cup [m]$ and $q \in [m]$ such that $p \neq q^+$ and the sequence $\{x^n_2\}_n$ is strictly increasing, then $|x_1, q^+| \in \mathcal{X}$.

($\overline{\mathrm{PC}}3'$) If there exists a sequence $\{(x^n_1, x_2)\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$ for some $p \in [m]$ and $q \in [m'] \cup [m]$ such that $p \neq q$, $q \neq p^+$, and the sequence $\{x^n_1\}_n$ is strictly increasing, then $(p^+, x_2) \in \mathcal{X}$.

Figure 4.7 illustrates some $\overline{\mathrm{PC}}$ conditions.

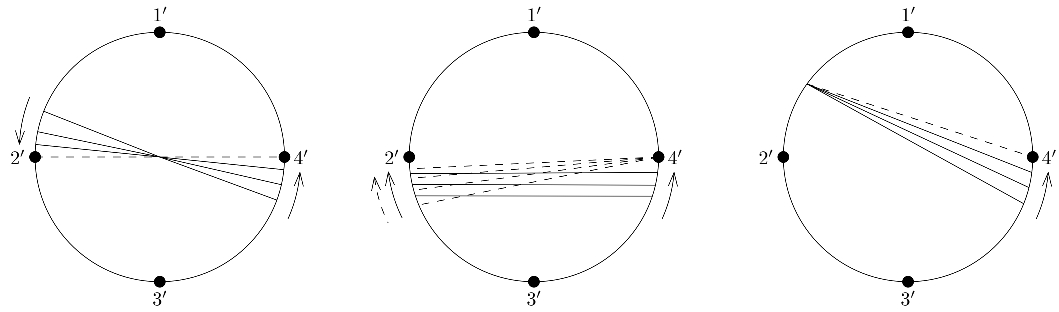


Figure 4.7: On the left ($\overline{\mathrm{PC}}$1), in the middle ($\overline{\mathrm{PC}}$2), and on the right ($\overline{\mathrm{PC}}$3).

The main result of this section is the following.

Theorem 4.4.3. *Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. The following statements are equivalent.*

1. *$\mathcal{X}$ is precovering in $\overline{\mathcal{C}}_m$.*
2. *$\pi^{-1}\mathcal{X} \cap \mathcal{A}$ is precovering in $\mathcal{C}_{2m}$.*
3. *$\mathcal{X}$ satisfies the $\overline{\mathrm{PC}}$ conditions.*

The following lemmas will be useful to prove the theorem above.

Lemma 4.4.4. *The following statements hold.*

1. *The category $\mathcal{A}$ is the aisle of a t-structure in $\mathcal{C}_{2m}$.*
2. *For each $x \in \operatorname{ind}\mathcal{C}_{2m}$ there exists an $\mathcal{A}$-cover $a \to x$ of x such that $\pi a \cong \pi x$ and a is indecomposable.*

Proof. Statement (1) follows from Theorem 3.6.3. As an alternative, from Proposition 2.3.27 it is enough to check that $\mathcal{A}$ is suspended and precovering. It is straightforward to check that $\mathcal{A}$ satisfies the PC conditions, therefore, by Theorem 3.3.2, $\mathcal{A}$ is precovering. For showing that $\Sigma\mathcal{A} \subseteq \mathcal{A}$, consider $a = (a_1, a_2) \in \operatorname{ind}\mathcal{A}$, then by Definition 4.4.1 $a_1-1, a_2-1 \in \bigcup_{p\in[m]}(p, z^0_{p+} - 1] \subseteq \bigcup_{p\in[m]}(p, z^0_{p+}]$. As a consequence, $\Sigma a = (a_1 - 1, a_2 - 1) \in \operatorname{ind}\mathcal{A}$. Now we prove that $\mathcal{A}$ is extension-closed. Let $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ be a triangle of $\mathcal{C}_{2m}$ with $a, b \in \mathcal{A}$. Then all the endpoints of the indecomposable summands of a and b belong to the set $\bigcup_{p\in[m]}(p, z^0_{p+}]$. Therefore, by [22, Lemma 3.4], all the endpoints of the indecomposable summands of e belong to the same set, and it follows that $e \in \mathcal{A}$.

Now we prove statement (2). Let $x = (x_1, x_2) \in \operatorname{ind}\mathcal{C}_{2m}$, by Lemma 2.3.21 x has an $\mathcal{A}$-cover $a \to x$. We show that $a \in \operatorname{ind}\mathcal{A}$ and that $\pi a \cong \pi x$. Let $p, q \in [m'] \cup [m]$ such that $x \in \mathbb{Z}^{(p,q)}$, and assume that $p \in [m']$ and $q \in [m]$, the other cases are analogous. If $x_1 \in (p, z^0_p]$ then $x \in \operatorname{ind}\mathcal{A}$ and $1_x\colon x \to x$ is an $\mathcal{A}$-cover of x. If $x_1 \notin (p, z^0_p]$ then let $a' = (z^0_p, x_2) \in \operatorname{ind}\mathcal{C}_{2m}$. We have that $a' \in \operatorname{ind}\mathcal{A}$. There exists a non-zero morphism $a' \to x$ and it is straightforward to check that it is an $\mathcal{A}$-precover, see Figure 4.8. Moreover $a' \to x$ is right-mimimal and therefore an $\mathcal{A}$-cover. Since covers are unique up to isomorphism, we

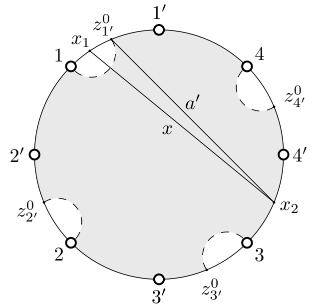


Figure 4.8: Illustration of the argument of statement (2) of Lemma 4.4.4.

have that $a \cong a'$. We also have that $\pi a \cong \pi a' = (\overline{z}^0_p, \overline{x}_2) = (\overline{x}_1, \overline{x}_2) = \pi x$. This concludes the argument. $\square$

Consider $a, b \in \operatorname{ind} \overline{\mathcal{C}}_m$ such that $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(b, a) \cong \mathbb{K}$. We recall, from Lemma 4.3.6, that we can fix $a' \in \operatorname{ind} \mathcal{C}_{2m}$ such that $\pi(a') \cong a$, and then there exists $b' \in \operatorname{ind} \mathcal{C}_{2m}$ (which depends on the choice of a') such that $\pi(b') \cong b$, $\operatorname{Hom}_{\mathcal{C}_{2m}}(b', a') \cong \mathbb{K}$, and any non-zero morphism $b' \to a'$ does not factor through $\mathcal{D}$.

Lemma 4.4.5. *Let $a, b \in \operatorname{ind} \overline{\mathcal{C}}_m$ be such that $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(b, a) \cong \mathbb{K}$, and let $a' \in \operatorname{ind} \mathcal{C}_{2m}$ be such that $\pi(a') \cong a$. Let $b' \in \operatorname{ind} \mathcal{C}_{2m}$ be such that $\pi(b') \cong b$, $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(b', a') \cong \mathbb{K}$, and any non-zero morphism $b' \to a'$ does not factor through $\mathcal{D}$. Let $b'' \to b'$ be the $\mathcal{A}$-cover of b'. Then $\operatorname{Hom}_{\mathcal{C}_{2m}}(b'', a') \cong \mathbb{K}$ and any non-zero morphism $b'' \to a'$ does not factor through $\mathcal{D}$.*

Proof. By Lemma 4.3.6 such b' exists. If $b' \in \mathcal{A}$ then $b'' \cong b'$ and we have the statement. Assume that $b' \notin \mathcal{A}$ and let $p, q \in [m'] \cup [m]$ be such that $b' = (b'_1, b'_2) \in \mathbb{Z}^{(p,q)}$. Consider the case $p \in [m']$ and $q \in [m]$, the other cases are analogous. Since $b' \notin \mathcal{A}$ we have that $b'_1 \notin (p, z^0_p]$, i.e. $b'_1 \in [z^0_p + 1, p^+)$. From the argument of Lemma 4.4.4, $b'' = (z^0_p, b'_2)$. We have that $a' \in H^+(b') \cup H^-(\Sigma^2 b')$, we show that $a' \in H^+(b'') \cup H^-(\Sigma^2 b'')$.

If $a' \in H^+(b')$ then $b'_1 \leq a'_1 \leq b'_2 - 2$ and $a'_2 \geq b'_2$. Since $b'_1 > z^0_p$, then $a' \in H^+(b'')$. Now, if $a' \in H^-(\Sigma^2 b')$ then $a'_1 \leq b'_1 - 2$ and $b'_1 \leq a'_2 \leq b'_2 - 2$. Assume that $a'_1 \nleq z^0_p - 2$, then $b'_1 - 2 \leq a'_1 \leq z^0_p - 1$. In particular, $a'_1 \in \mathbb{Z}^{(p)}$ and any non-zero morphism $b' \to a'$ factors through $\mathcal{D}$ giving a contradiction, see Figure 4.4. Therefore $a'_1 \leq z^0_p - 2$. Moreover, since $b'_1 > z^0_p$, from $b'_1 \leq a'_2 \leq b'_2 - 2$ we also have that $z^0_p \leq a'_2 \leq b'_2 - 2$ and obtain that $a' \in H^-(\Sigma^2 b'')$. We can conclude that $\operatorname{Hom}_{\mathcal{C}_{2m}}(b'', a') \cong \mathbb{K}$.

We show that any non-zero morphism $b'' \to a'$ does not factor through $\mathcal{D}$. If this is not the case, then $a' \in H^-(\Sigma^2 b'')$ and $a'_1 \in \mathbb{Z}^{(p)}$. As a consequence, $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(b, a) = 0$ giving a contradiction. We obtain that any non-zero morphism $b'' \to a'$ does not factor through $\mathcal{D}$, and this concludes the argument. $\square$

Lemma 4.4.6. *Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. The subcategory $\mathcal{X}$ satisfies the $\overline{\text{PC}}$ conditions if and only if $\pi^{-1}\mathcal{X} \cap \mathcal{A}$ satisfies the* PC *conditions.*

Proof. We show that $\mathcal{X}$ satisfies ($\overline{\text{PC}}$1) if and only if $\pi^{-1}\mathcal{X} \cap \mathcal{A}$ satisfies (PC 1), we refer to Figure 4.9 for an illustration. Assume that $\mathcal{X}$ satisfies ($\overline{\text{PC}}1$) and that there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathbb{Z}^{(p,q)} \cap (\pi^{-1}\mathcal{X} \cap \mathcal{A})$ for some $p, q \in [m'] \cup [m]$ such that $p \neq q$ with $\{x_1^n\}_n$ and $\{x_2^n\}_n$ strictly increasing sequences. Note that $p, q \notin [m']$, otherwise for n big enough we have $\{(x_1^n, x_2^n)\}_n \not\subseteq \mathcal{A}$. For each n we define $\pi x^n = y^n = (y_1^n, y_2^n)$, note that $\{y_1^n\}_n$ and $\{y_2^n\}_n$ are still strictly increasing sequences, and consider the sequence $\{y^n\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$. Since $\mathcal{X}$ satisfies ($\overline{\text{PC}}1$), then $|p^+, q^+| \in \mathcal{X}$ and $\pi^{-1}\mathcal{X}$ contains any arc of $\mathcal{C}_{2m}$ having one endpoint in $\mathbb{Z}^{(p^+)}$ and the other in $\mathbb{Z}^{(q^+)}$. In particular, there exist strictly decreasing sequences $\{z_1^n\}_n \subseteq \mathbb{Z}^{(p^+)}$ and $\{z_2^n\}_n \subseteq \mathbb{Z}^{(q^+)}$ such that $\{|z_1^n, z_2^n|\}_n \subseteq \pi^{-1}\mathcal{X} \cap \mathcal{A}$. This proves that $\pi^{-1}\mathcal{X} \cap \mathcal{A}$ satisfies (PC 1).

Now assume that $\pi^{-1}\mathcal{X} \cap \mathcal{A}$ satisfies (PC 1) and that there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$ for some $p, q \in [m]$ such that $p \neq q$, and $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are strictly increasing sequences. For each n there exists $y^n = (y_1^n, y_2^n) \in \operatorname{ind} \pi^{-1}\mathcal{X} \cap \mathcal{A}$ such that $\pi y^n \cong x^n$. Thus, there exists a sequence $\{|y_1^n, y_2^n|\}_n \subseteq (\pi^{-1}\mathcal{X} \cap \mathcal{A}) \cap C^{(p,q)}$ such that $\{y_1^n\}_n$ and $\{y_2^n\}_n$ are stricly increasing sequences. Since $\pi^{-1}\mathcal{X} \cap \mathcal{A}$ satisfies (PC 1), then there exist strictly decreasing sequences $\{z_1^n\}_n \subseteq \mathbb{Z}^{(p^+)}$ and $\{z_2^n\}_n \subseteq \mathbb{Z}^{(q^+)}$ such that $\{|z_1^n, z_2^n|\}_n \subseteq \pi^{-1}\mathcal{X} \cap \mathcal{A}$. As a consequence we have that $|p^+, q^+| \in \mathcal{X}$. This proves that $\mathcal{X}$ satisfies ($\overline{\text{PC}}1$).

It is straightforward to check that $\mathcal{X}$ satisfies ($\overline{\text{PC}}3$) and ($\overline{\text{PC}}3'$) if and only if $\pi^{-1}\mathcal{X} \cap \mathcal{A}$ satisfies (PC 3) and (PC 3$'$). Moreover, if $\mathcal{X}$ satisfies ($\overline{\text{PC}}2$), ($\overline{\text{PC}}2'$), ($\overline{\text{PC}}3$), and ($\overline{\text{PC}}3'$) then $\pi^{-1}\mathcal{X} \cap \mathcal{A}$ satisfies (PC 2) and (PC 2$'$). Finally, if $\pi^{-1}\mathcal{X} \cap \mathcal{A}$ satisfies (PC 2) and (PC 2$'$) then $\mathcal{X}$ satisfies ($\overline{\text{PC}}2$) and ($\overline{\text{PC}}2'$). We conclude that $\mathcal{X}$ satisfies the $\overline{\text{PC}}$ conditions if and only if $\pi^{-1}\mathcal{X} \cap \mathcal{A}$ satisfies the PC conditions. □

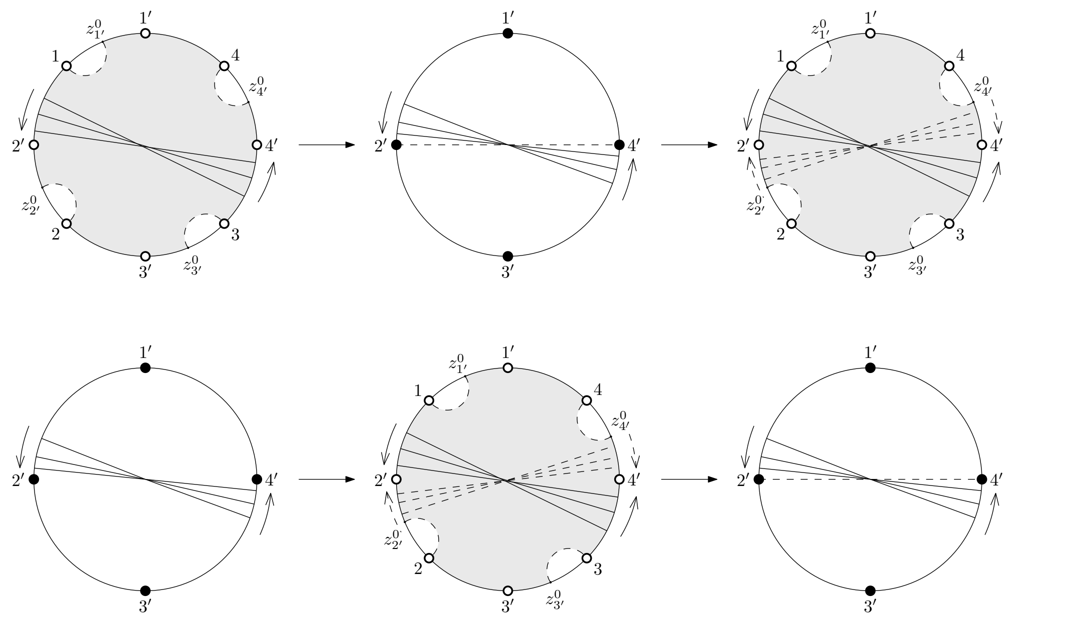


Figure 4.9: Illustration of the argument of Lemma 4.4.6.

Proposition 4.4.7. *Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. If $\pi^{-1}\mathcal{X} \cap \mathcal{A}$ is a precovering subcategory of $\mathcal{C}_{2m}$ then $\mathcal{X}$ is a precovering subcategory of $\overline{\mathcal{C}}_m$.*

Proof. From Remark 2.3.20 it is enough to check that $\mathcal{X}$ is precovering at the level of the indecomposable objects. Consider $a \in \operatorname{ind} \overline{\mathcal{C}}_m$, then there exists $a' \in \operatorname{ind} \mathcal{C}_{2m}$ such that $\pi a' \cong a$, and there exists $f\colon x \to a'$ a $\pi^{-1}\mathcal{X} \cap \mathcal{A}$-precover of a'. Consider $\pi f\colon \pi x \to a$, we show that πf satisfies the condition of Remark 2.3.20. First assume that f does not factor through $\mathcal{D}$. Consider $b \in \operatorname{ind} \mathcal{X}$ and $g\colon b \to a$ in $\overline{\mathcal{C}}_m$. Without loss of generality we can assume that $g \neq 0$. From Lemma 4.4.4 and Lemma 4.4.5, there exists $b' \in \operatorname{ind} \pi^{-1}\mathcal{X} \cap \mathcal{A}$ such that $\pi b' \cong b$ and there exists a non-zero morphism $g'\colon b' \to a'$ in $\mathcal{C}_{2m}$ which does not factor through $\mathcal{D}$. Since the Hom-spaces in $\overline{\mathcal{C}}_m$ are at most one dimensional, we have that $g = \lambda \pi g'$ for some $\lambda \in \mathbb{K}^*$. Moreover, since $f\colon x \to a'$ is a $\pi^{-1}\mathcal{X} \cap \mathcal{A}$-precover of a', there exists $h\colon b' \to x$ in $\mathcal{C}_{2m}$ such that $fh = g'$. We obtain that $\lambda\pi(f)\pi(h) = \pi(fh) = g$ in $\overline{\mathcal{C}}_m$. This proves that $\pi f\colon \pi x \to a$ is an $\mathcal{X}$-precover of a.

Now we consider the case when f factors through $\mathcal{D}$. We show that $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(b, a) = 0$ for all $b \in \operatorname{ind} \mathcal{X}$. Assume that there exists a non-zero morphism $g\colon b \to a$ in $\overline{\mathcal{C}}_m$ for some $b \in \operatorname{ind} \mathcal{X}$, then as above there exists $b' \in \operatorname{ind} \pi^{-1}\mathcal{X} \cap \mathcal{A}$ such that $\pi b' \cong b$ and there exists a non-zero morphism $g'\colon b' \to a'$ in $\mathcal{C}_{2m}$ which does not factor through $\mathcal{D}$. Since $f\colon x \to a'$ is a $\pi^{-1}\mathcal{X} \cap \mathcal{A}$-precover of a', there exists $h\colon b' \to x$ in $\mathcal{C}_{2m}$ such that $fh = g'$. Since f factors through $\mathcal{D}$, we have that g' factors through $\mathcal{D}$, giving a contradiction. We can conclude that if f factors through $\mathcal{D}$ then $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(b, a) = 0$ for all $b \in \operatorname{ind} \mathcal{X}$. As a consequence, $\pi f = 0$ is an $\mathcal{X}$-precover of a. □

The following proposition is the analogue of [21, Proposition 3.7] in $\overline{\mathcal{C}}_m$ and its proof is similar.

Proposition 4.4.8. *Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. If $\mathcal{X}$ is a precovering subcategory then it satisfies the $\overline{\mathrm{PC}}$ conditions.*

Proof. Assume that $\mathcal{X}$ is a precovering subcategory of $\overline{\mathcal{C}}_m$, we show that it satisfies ($\overline{\mathrm{PC}}1$). Assume that there is a sequence $\{x^n = (x_1^n, x_2^n)\}_n \subseteq \operatorname{ind} \mathcal{X} \cap C^{(p,q)}$ for some $p, q \in [m]$ with $p \neq q$ such that $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are strictly increasing sequences. We show that $a = |p^+, q^+| \in \operatorname{ind} \mathcal{X}$. Consider $\begin{pmatrix} f_1 & \cdots & f_k \end{pmatrix} : y_1 \oplus \cdots \oplus y_k \to a$ an $\mathcal{X}$-precover of a with $y_1, \ldots, y_k \in \operatorname{ind} \mathcal{X}$. Note that $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(x^n, a) \cong \mathbb{K}$ for each n. Fix n, and consider a non-zero morphism $g^n\colon x^n \to a$. Then there exists $h^n = \begin{pmatrix} h_1^n & \cdots & h_k^n \end{pmatrix}^T : x^n \to y_1 \oplus \cdots \oplus y_k$ such that $fh^n = g^n$. Then, for each n there exists $l \in \{1, \ldots, k\}$ such that g^n factors through f_l.

There exists $l \in \{1, \ldots, k\}$ such that g^n factors through f_l for infinitely many $n \in \mathbb{Z}$. Indeed, if for each l only finitely many of the g^n factor through f_l, then there are only finitely many g^n's and this contradicts the fact that the sequence $\{x^n\}_n$ is infinite. Now fix an l such that g^n factors through f_l for infinitely many $n \in \mathbb{Z}$. Without loss of generality we can assume that for each $n \in \mathbb{Z}$ the morphism g^n factors through f_l. Indeed, if this is not the case, we can extract an infinite subsequence of $\{x^n\}_n$ such that all $g^n\colon x^n \to a$ satisfy that property. From now on we denote the object y_l as y, the morphism $f_l\colon y_l \to a$ as $f\colon y \to a$, and we denote by $h^n\colon x^n \to y$ the morphism such that $fh^n = g^n$.

Since $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(y, a) \cong \mathbb{K}$ and $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(x^n, y) \cong \mathbb{K}$ for all $n \in \mathbb{Z}$, we have that

$$y \in \left(\bigcap_{n \in \mathbb{Z}} \overline{H}^+(x^n) \cup \overline{H}^-(\Sigma^2 x^n) \right) \cap \left(\overline{I}^-(a) \cup \overline{I}^+(\Sigma^{-2} a) \right).$$

We refer to Figure 4.10 for an illustration.

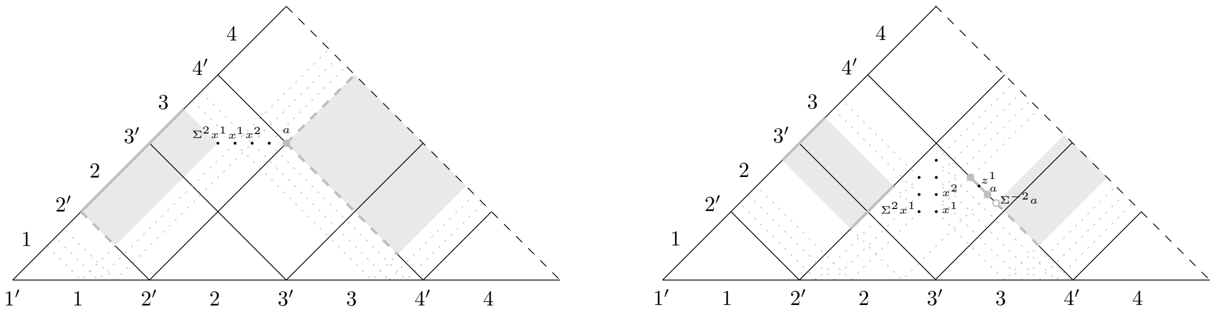


Figure 4.10: On the left the argument for ($\overline{\mathrm{PC}}1$), on the right for ($\overline{\mathrm{PC}}2$). The grey areas represent where the object y belongs.

We show that $y \cong a$. Assume that $y \ncong a$, from Proposition 4.3.7 there exists a non-zero morphism $f' \colon a \to y$ such that $h^n = f' g^n$ for each $n \in \mathbb{Z}$. Since $f h^n = g^n \neq 0$, then $f f' g^n \neq 0$ and $f f' \colon a \to a$ is non-zero. Thus, $f f' = \lambda 1_a$ for some $\lambda \in \mathbb{K}^*$ and $a \cong y$, which contradicts our assumption. We obtain that $a \cong y \in \mathcal{X}$, and we conclude that $\mathcal{X}$ satisfies ($\overline{\mathrm{PC}}1$).

Now we show that $\mathcal{X}$ satisfies ($\overline{\mathrm{PC}}2$). Assume that there is a sequence $\{(x_1^n, x_2^n)\} \subseteq \operatorname{ind} \mathcal{X} \cap C^{(p,q)}$ for some $p, q \in [m]$ such that $\{x_1^n\}_n$ is strictly decreasing and $\{x_2^n\}_n$ is strictly increasing. We show that there is a strictly decreasing sequence $\{y_1^n\}_n \subseteq C^{(p)}$ such that $\{|y_1^n, q^+|\}_n \subseteq \mathcal{X}$.

Consider an object $a = |a_1, q^+|$ with $a_1 \in \mathbb{Z}^{(p)}$ such that $x_1^1 < a_1 \leq x_2^1 - 2$. Then for each n there exists a non-zero morphism $g^n \colon x^n \to a$. Consider an $\mathcal{X}$-precover $(\, f_1 \cdots f_k \,) \colon y_1 \oplus \cdots \oplus y_k \to a$ of a. With the same argument as above there exists $l \in \{1, \ldots, k\}$ such that $g^n \colon x^n \to a$ factors through $f_l \colon y_l \to a$ for all n (up to taking subsequences). Let $y = y_l$, proceeding similarly as above we obtain that $y \in \left\{ |z, q^+| \middle| x_1^1 \leq z \leq a_1^1 \right\}$, see Figure 4.10. We define $z^1 = y$, which is the first element of our desired sequence. Now we consider $a' = |a_1', q^+|$ with $x_1^1 < a_1' \leq x_1^2$. By repeating the same argument there exists $z^2 \in \{|z, q^+| | a_1' \leq z \leq a_1\}$ which is an object of $\mathcal{X}$. With this procedure we obtain our desired sequence $\{z^n\}_n$. This proves that $\mathcal{X}$ satisfies ($\overline{\mathrm{PC}}2$).

The argument of ($\overline{\mathrm{PC}}2'$) is similar to the argument of ($\overline{\mathrm{PC}}2$), the arguments of ($\overline{\mathrm{PC}}3$) and ($\overline{\mathrm{PC}}3'$) are similar to the argument of ($\overline{\mathrm{PC}}1$). We can conclude that $\mathcal{X}$ satisfies the $\overline{\mathrm{PC}}$ conditions. $\square$

We now have our classification of the precovering subcategories of $\overline{\mathcal{C}}_m$.

Proof of Theorem 4.4.3. The claim follows directly from Theorem 3.3.2, Lemma 4.4.6, Proposition 4.4.7, and Proposition 4.4.8. $\square$

4.4.2 Preenveloping subcategories of $\overline{\mathcal{C}}_m$

Here we discuss a characterization of the preenveloping subcategories of $\overline{\mathcal{C}}_m$ dual to Theorem 4.4.3. First we define an auxiliary category $\mathcal{B}$, which is the dual version of $\mathcal{A}$. We recall that in Section 4.4.1 we fixed an integer $z^0 \in \mathbb{Z}$, now we define $w^0 = z^0 - 1$. For each $p \in [m']$ we denote by $w^0_p \in \mathcal{Z}_{2m}$ the copy of w^0 belonging to $\mathbb{Z}^{(p)}$.

Definition 4.4.9. We define the subcategory $\mathcal{B}$ of $\mathcal{C}_{2m}$ as

$$\mathcal{B} = \operatorname{add}\left\{ (b_1, b_2) \in \operatorname{ind}\mathcal{C}_{2m} \,\middle|\, b_1, b_2 \in \bigcup_{p \in [m']} [w^0_p, p^{++}) \right\}.$$

Figure 4.11 illustrates the subcategory $\mathcal{B}$.

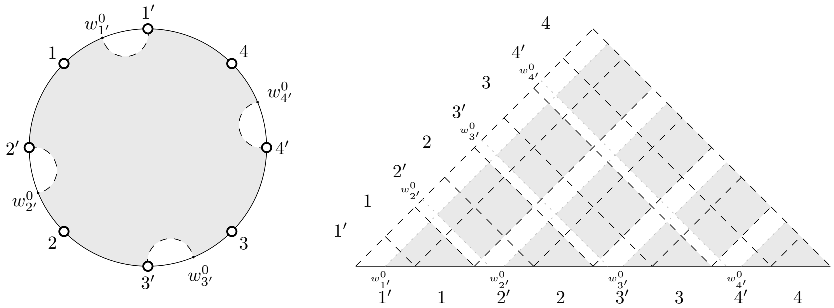


Figure 4.11: The category $\mathcal{B}$.

For convenience of the reader, we record the duals of Definition 4.4.2 and Theorem 4.4.3.

Definition 4.4.10. Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. We say that $\mathcal{X}$ satisfies the *completed preenveloping conditions*, $\overline{\text{PE}}$ for short, if it satisfies the following combinatorial conditions.

($\overline{\text{PE}}$1) If there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$ for some $p, q \in [m]$ such that $p \neq q$ and the sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are strictly decreasing, then $(p^-, q^-) \in \mathcal{X}$.

($\overline{\text{PE}}$2) If there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$ for some $p, q \in [m]$ such that $p \neq q$ and the sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are respectively strictly increasing and strictly decreasing, then there exists a strictly increasing sequence $\{y_1^n\}_n \subseteq C^{(p)}$ such that $\{(y_1^n, q^-)\}_n \subseteq \mathcal{X}$.

($\overline{\text{PE}}$2′) If there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$ for some $p, q \in [m]$ such that the sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are respectively strictly decreasing and strictly increasing, then there exists a strictly increasing sequence $\{y_2^n\}_n \subseteq C^{(q)}$ such that $\{(p^-, y_2^n)\}_n \subseteq \mathcal{X}$.

($\overline{\text{PE}}$3) If there exists a sequence $\{(x_1, x_2^n)\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$ for some $p \in [m'] \cup [m]$ and $q \in [m]$ such that $p \neq q$, $p \neq q^-$ and the sequence $\{x_2^n\}_n$ is strictly decreasing, then $(x_1, q^-) \in \mathcal{X}$.

($\overline{\text{PE}}$3′) If there exists a sequence $\{(x_1^n, x_2)\}_n \subseteq \mathcal{X} \cap C^{(p,q)}$ for some $p \in [m]$ and $q \in [m'] \cup [m]$ such that the sequence $\{x_1^n\}_n$ is strictly decreasing, then $(p^-, x_2) \in \mathcal{X}$.

Theorem 4.4.11. *Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. The following statements are equivalent.*

1. *$\mathcal{X}$ is preenveloping in $\overline{\mathcal{C}}_m$.*
2. *$\pi^{-1}\mathcal{X} \cap \mathcal{B}$ is preenveloping in $\mathcal{C}_{2m}$.*
3. *$\mathcal{X}$ satisfies $\overline{\mathrm{PE}}$ conditions.*

The following lemma will be useful in Section 4.7.2 for computing the heart of a t-structure.

Lemma 4.4.12. *The following statements hold.*

1. *The category $\mathcal{B}$ is the co-aisle of a t-structure in $\mathcal{C}_{2m}$.*
2. *For each $x \in \operatorname{ind}\overline{\mathcal{C}}_m$ there exists $x' \in \operatorname{ind}\mathcal{A} \cap \Sigma^{-1}\mathcal{B} \subseteq \operatorname{ind}\mathcal{A} \cap \Sigma\mathcal{B}$ such that $\pi x' \cong x$.*

Proof. Statement (1) is the dual of statement (1) of Lemma 4.4.4. For statement (2), consider the $\mathcal{A}$-cover $x' \to x$ of x as in statement (2) of Lemma 4.4.4. We have that $\pi(x') \cong x$, and it is straightforward to check that $x' \in \Sigma^{-1}\mathcal{B}$. Moreover, since $\Sigma^{-1}\mathcal{B} \subseteq \mathcal{B} \subseteq \Sigma\mathcal{B}$, we have that $x' \in \Sigma\mathcal{B}$. This concludes the proof. □

4.5 Extension-closed subcategories

In this section we classify the extension-closed subcategories of $\overline{\mathcal{C}}_m$. To do so, we use the fact that the extension-closed subcategories of $\mathcal{C}_m$ are precisely those closed under extensions having indecomposable outer terms, see Proposition 3.4.4. First we introduce the completed version of the PT condition, cf. Definition 3.4.1. We refer to Proposition 4.2.1 for the computation of the Hom-spaces of $\overline{\mathcal{C}}_m$.

Recall that we identify the indecomposable objects of $\overline{\mathcal{C}}_m$ with the arcs of $\overline{\mathcal{Z}}_m$.

Definition 4.5.1. Let $x, y \in \operatorname{ind}\overline{\mathcal{C}}_m$ be such that $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(x, \Sigma y) \cong \mathbb{K}$. The arcs of $\operatorname{ind}\overline{\mathcal{C}}_m \setminus \{x, y\}$ which connect the endpoints of x and y are called *Ptolemy arcs* of x and y. We say that a subcategory $\mathcal{X}$ of $\overline{\mathcal{C}}_m$ satisfies the *completed Ptolemy condition*, $\overline{\mathrm{PT}}$ condition for short, if it is closed under taking Ptolemy arcs.

Figure 4.12 provides an illustration of the Ptolemy arcs in $\overline{\mathcal{C}}_m$.

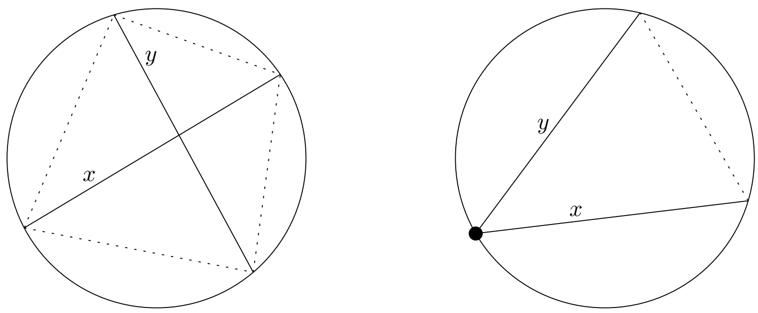


Figure 4.12: The dotted arcs are the Ptolemy arcs of x and y. On the left x and y cross, on the right they share one endpoint which is an accumulation point.

Now we prove the characterization of the extension-closed subcategories of $\overline{\mathcal{C}}_m$. The middle term of a non-split extension in $\overline{\mathcal{C}}_m$ having indecomposable outer terms was computed in [43, Section 3].

Proposition 4.5.2. *Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. The following statements are equivalent.*

1. *The subcategory $\mathcal{X}$ satisfies the $\overline{\mathrm{PT}}$ condition.*
2. *The subcategory $\mathcal{X}$ is closed under extensions of the form $x_1 \longrightarrow c \longrightarrow x_2 \longrightarrow \Sigma x_1$ with $x_1, x_2 \in \operatorname{ind} \overline{\mathcal{C}}_m$.*
3. *The subcategory $\mathcal{X}$ is closed under extensions.*

Proof. The proof of the equivalence of (1) and (2) is straightforward and follows from [43, Section 3]. The fact that (3) implies (2) is trivial. We prove that (2) implies (3). To this end, first we show that $\pi^{-1}\mathcal{X}$ is closed under extensions, and then that $\mathcal{X}$ is closed under extensions.

Assume that $\mathcal{X}$ is closed under extensions with indecomposable outer terms. Consider the preimage $\pi^{-1}\mathcal{X}$ in $\mathcal{C}_{2m}$. It is straightforward to check that $\pi^{-1}\mathcal{X}$ is an additive subcategory of $\mathcal{C}_{2m}$. We show that $\pi^{-1}\mathcal{X}$ is closed under extensions having indecomposable outer terms. Consider a triangle $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ in $\mathcal{C}_{2m}$ with $a, b \in \operatorname{ind} \pi^{-1}\mathcal{X}$. Then $\pi a \longrightarrow \pi e \longrightarrow \pi b \longrightarrow \pi \Sigma a$ is a triangle in $\overline{\mathcal{C}}_m$, see Section 2.3.7. Moreover, from [43, Proposition 3.10] it follows that πa and πb are either indecomposable objects or zero, and then $\pi a, \pi b \in \operatorname{ind} \mathcal{X}$. From (2) we obtain that $\pi e \in \mathcal{X}$, i.e. $e \in \pi^{-1}\mathcal{X}$. This proves that $\pi^{-1}\mathcal{X}$ is closed under extensions.

Now we show that $\mathcal{X}$ is closed under extensions. Consider a triangle $x_1 \longrightarrow c \longrightarrow x_2 \longrightarrow \Sigma x_1$ in $\overline{\mathcal{C}}_m$ with $x_1, x_2 \in \mathcal{X}$. Then there exists a triangle $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ in $\mathcal{C}_{2m}$ whose image after π is isomorhic in $\overline{\mathcal{C}}_m$ to $x_1 \longrightarrow c \longrightarrow x_2 \longrightarrow \Sigma x_1$. Thus $\pi a, \pi b \in \mathcal{X}$, i.e. $a, b \in \pi^{-1}\mathcal{X}$. Since $\pi^{-1}\mathcal{X}$ is closed under extensions, we have $e \in \pi^{-1}\mathcal{X}$ and then $c \cong \pi e \in \mathcal{X}$. This completes the proof. $\square$

4.6 Torsion pairs

We classify the torsion pairs in $\overline{\mathcal{C}}_m$. We recall from Proposition 2.3.27 that a torsion pair $(\mathcal{X}, \mathcal{Y})$ is uniquely determined by its torsion class $\mathcal{X}$, and therefore it is enough to classify the torsion classes.

Theorem 4.6.1. *Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. Then $\mathcal{X}$ is a torsion class in $\overline{\mathcal{C}}_m$ if and only if $\mathcal{X}$ satisfies the $\overline{\mathrm{PC}}$ conditions and the $\overline{\mathrm{PT}}$ condition. Moreover, there is a bijection.*

$$\left\{ \text{ Torsion-classes in } \overline{\mathcal{C}}_m \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Extension-closed subcategories } \mathcal{U} \subseteq \mathcal{C}_{2m} \\ \text{such that } \mathcal{D} \subseteq \mathcal{U} \text{ and } \mathcal{U} \cap \mathcal{A} \text{ is precovering} \end{array} \right\}$$

$$\mathcal{X} \longmapsto \pi^{-1}\mathcal{X}$$

$$\pi\mathcal{U} \longleftarrow \mathcal{U}$$

Proof. The first statement follows directly from Proposition 2.3.27, Theorem 4.4.3, and Proposition 4.5.2. The bijection follows from Proposition 2.3.38 and Theorem 4.4.3. □

We have the following corollaries which follow directly from Theorem 4.6.1 and will be useful in Section 4.7 and Section 4.8 to classify t-structures and co-t-structures.

Corollary 4.6.2. *Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. Then $\mathcal{X}$ is the aisle of a t-structure in $\overline{\mathcal{C}}_m$ if and only if $\mathcal{X}$ satisfies the $\overline{\mathrm{PC}}$ conditions, the $\overline{\mathrm{PT}}$ condition, and $\mathcal{X}$ is closed under clockwise rotations. Moreover, there is a bijection.*

$$\left\{ \text{ Aisles of t-structures in } \overline{\mathcal{C}}_m \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Suspended subcategories } \mathcal{U} \subseteq \mathcal{C}_{2m} \text{ such} \\ \text{that } \mathcal{D} \subseteq \mathcal{U} \text{ and } \mathcal{U} \cap \mathcal{A} \text{ is precovering} \end{array} \right\}$$
$$\mathcal{X} \longmapsto \pi^{-1}\mathcal{X}$$
$$\pi\mathcal{U} \longleftarrow\!\shortmid \mathcal{U}$$

Corollary 4.6.3. *Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. Then $\mathcal{X}$ is the aisle of a co-t-structure in $\overline{\mathcal{C}}_m$ if and only if $\mathcal{X}$ satisfies the $\overline{\mathrm{PC}}$ conditions, the $\overline{\mathrm{PT}}$ condition, and $\mathcal{X}$ is closed under anticlockwise rotations. Moreover, there is a bijection.*

$$\left\{ \text{ Aisles of co-t-structures in } \overline{\mathcal{C}}_m \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Co-suspended subcategories } \mathcal{U} \subseteq \mathcal{C}_{2m} \text{ such} \\ \text{that } \mathcal{D} \subseteq \mathcal{U} \text{ and } \mathcal{U} \cap \mathcal{A} \text{ is precovering} \end{array} \right\}$$
$$\mathcal{X} \longmapsto \pi^{-1}\mathcal{X}$$
$$\pi\mathcal{U} \longleftarrow\!\shortmid \mathcal{U}$$

4.7 T-structures

We classify the t-structures in $\overline{\mathcal{C}}_m$. We start by classifying the aisles of the t-structures, then we compute the co-aisles and the hearts. Finally, we classify the bounded and non-degenerate t-structures.

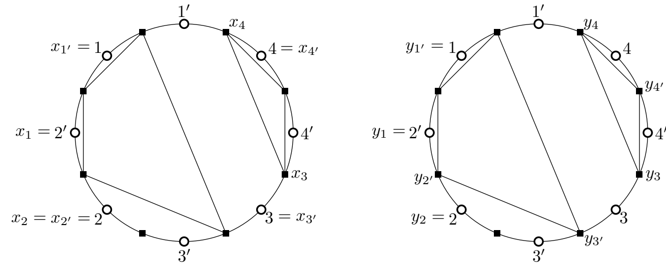
4.7.1 Aisles of t-structures

In Section 3.6 we discussed the classification of the aisles of t-structures of $\mathcal{C}_m$ in terms of decorated non-crossing partitions, see Definition 3.6.2. Here we introduce similar combinatorial objects which classify the aisles of t-structures in $\overline{\mathcal{C}}_m$.

Definition 4.7.1. A *half-decorated non-crossing partition* of $[m'] \cup [m]$ is a pair $(\mathcal{P}, X)$ given by a non-crossing partition $\mathcal{P}$ of $[m'] \cup [m]$ and a $2m$-tuple $X = (x_p)_{p \in [m'] \cup [m]}$ such that for each $p \in [m']$ we have that $x_p = p^+$, and for each $p \in [m]$

$$x_p \in \begin{cases} [p, p^+) & \text{if } \{p\} \in \mathcal{P}, \\ (p, p^+] & \text{if } p, p^+ \in B \text{ for some block } B \in \mathcal{P}, \\ (p, p^+) & \text{otherwise.} \end{cases}$$

Example 4.7.2. Figure 4.13 gives an example of non-crossing partition and half-decorated non-crossing partition.



$(\mathcal{P}, X)$ $(\mathcal{P}, Y)$

Figure 4.13: On the left $(\mathcal{P}, X)$ is a half-decorated non-crossing partition of $[4'] \cup [4]$, and on the right $(\mathcal{P}, Y)$ is a decorated non-crossing partition of $[4'] \cup [4]$, with $\mathcal{P} = \{\{1', 1, 2', 3'\}, \{2\}, \{3, 4', 4\}\}$.

Remark 4.7.3. Half-decorated non-crossing partitions and non-crossing partitions are closely related but distinct combinatorial objects. A decorated non-crossing partition of $[m'] \cup [m]$ may not be a half-decorated non-crossing partition of $[m'] \cup [m]$, and vice versa. For example, $(\mathcal{P}, X)$ of Figure 4.13 is not a decorated non-crossing partition, and $(\mathcal{P}, Y)$ is not a half-decorated non-crossing partition.

The main result of this section is the following analogue of Theorem 3.6.3. The notation employed in the statement will be defined in Definition 4.7.6 and Definition 4.7.9.

Theorem 4.7.4. *The following is a bijection.*

$$\left\{ \begin{array}{c} \textit{Half-decorated non-crossing} \\ \textit{partitions of } [m'] \cup [m] \end{array} \right\} \longleftrightarrow \{ \textit{ Aisles of t-structures in } \overline{\mathcal{C}}_m \ \}$$
$$(\mathcal{P}, X) \longmapsto \pi \mathcal{U}_{(\mathcal{P}, X)}$$
$$(\mathcal{P}_{\pi^{-1}\mathcal{X}}, X_{\pi^{-1}\mathcal{X}}) \longleftarrow\!\shortmid \mathcal{X}$$

To prove this result, we take an intermediate step through $\mathcal{C}_{2m}$. From Corollary 4.6.2 the aisles of t-structures in $\overline{\mathcal{C}}_m$ are in bijection with the suspended subcategories $\mathcal{U}$ of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering. These can be regarded as "almost aisles" of t-structures in $\mathcal{C}_{2m}$ and are classified in terms of half-decorated non-crossing partitions of $[m'] \cup [m]$, see in Proposition 4.7.5. The aisles of the t-structures in $\overline{\mathcal{C}}_m$ are then obtained by localising the "almost aisles" in $\mathcal{C}_{2m}$. Figure 4.14 illustrates this process.

The following proposition classifies the "almost aisles" of t-structures in $\mathcal{C}_{2m}$.

Proposition 4.7.5. *The following is a bijection.*

$$\left\{ \begin{array}{c} \textit{Half-decorated non-crossing} \\ \textit{partitions of } [m'] \cup [m] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Suspended subcategories } \mathcal{U} \subseteq \mathcal{C}_{2m} \textit{ such that} \\ \mathcal{D} \subseteq \mathcal{U} \textit{ and } \mathcal{U} \cap \mathcal{A} \textit{ is precovering} \end{array} \right\}$$
$$\alpha \colon (\mathcal{P}, X) \longmapsto \mathcal{U}_{(\mathcal{P}, X)}$$
$$(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}}) \longleftarrow\!\shortmid \mathcal{U} \colon \beta$$

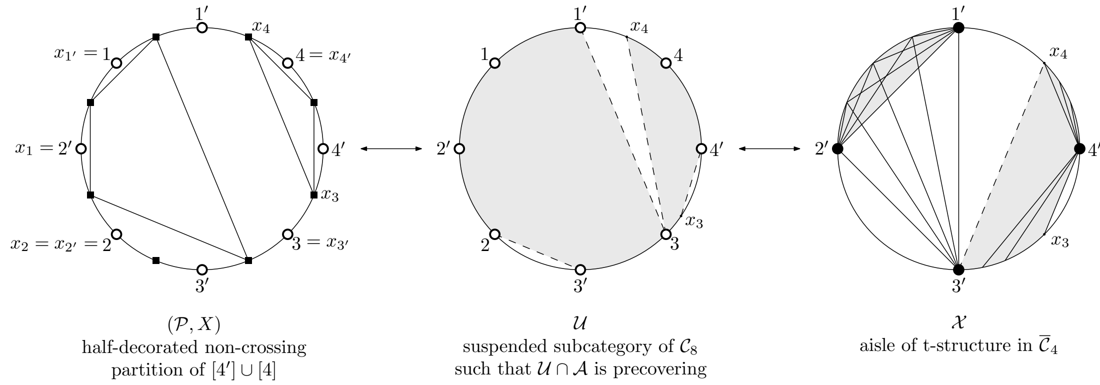


Figure 4.14: Illustration of how to obtain the aisle of a t-structure of $\overline{\mathcal{C}}_m$ from a half-decorated non-crossing partition of $[m'] \cup [m]$.

The rest of this section is devoted to prove Proposition 4.7.5. We start by defining the assignments of the maps α and β.

Definition 4.7.6. Let $(\mathcal{P}, X)$ be a half-decorated non-crossing partition of $[m'] \cup [m]$. We define

$$\mathcal{U}_{(\mathcal{P},X)} = \operatorname{add} \bigsqcup_{B \in \mathcal{P}} \left\{ (u_1, u_2) \in \operatorname{ind} \mathcal{C}_{2m} \,\middle|\, u_1, u_2 \in \bigcup_{p \in B} (p, x_p] \right\},$$

where we use the following convention: if $x_p = p$ then $(p, x_p] = \emptyset$, and if $x_p = p^+$ then $(p, x_p] = \mathbb{Z}^{(p)}$.

We check that the map α is well defined.

Proposition 4.7.7. *Let $(\mathcal{P}, X)$ be a half-decorated non-crossing partition of $[m'] \cup [m]$. Then $\mathcal{U}_{(\mathcal{P},X)}$ is a suspended subcategory of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}_{(\mathcal{P},X)}$ and $\mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A}$ is precovering.*

Proof. We show that $\mathcal{D} \subseteq \mathcal{U}$. Consider $d = (d_1, d_2) \in \operatorname{ind} \mathcal{D}$, then $d_1, d_2 \in \mathbb{Z}^{(p)}$ for some $p \in [m']$. Since $p \in B$ for some block $B \in \mathcal{P}$, then $\mathbb{Z}^{(p,p)} \subseteq \operatorname{ind} \mathcal{U}$, and then $d \in \mathcal{U}_{(\mathcal{P},X)}$. Morevoer, it is straightforward to check that $\Sigma \mathcal{U}_{(\mathcal{P},X)} \subseteq \mathcal{U}_{(\mathcal{P},X)}$. For showing that $\mathcal{U}_{(\mathcal{P},X)}$ is extension-closed, we can proceed as in the argument of [22, Proposition 4.8].

Now we show that $\mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A}$ is precovering. Let $\widetilde{X}$ be the vector $\widetilde{X} = (\widetilde{x}_p)_{p \in [m'] \cup [m]}$ where for each $p \in [m'] \cup [m]$

$$\widetilde{x}_p = \begin{cases} z_p^0 & \text{if } p \in [m'], \\ x_p & \text{if } p \in [m]. \end{cases}$$

Then $(\mathcal{P}, \widetilde{X})$ is a decorated non-crossing partition of $[m'] \cup [m]$ and we can associate to it the aisle of a t-structure $\mathcal{U}_{(\mathcal{P},\widetilde{X})}$, see Theorem 3.6.3. We recall that

$$\mathcal{U}_{(\mathcal{P},\widetilde{X})} = \operatorname{add} \bigsqcup_{B \in \mathcal{P}} \left\{ (u_1, u_2) \in \operatorname{ind} \mathcal{C}_{2m} \,\middle|\, u, u_2 \in \bigcup_{p \in B} (p, \widetilde{x}_p] \right\}.$$

It is straightforward to check that $\mathcal{U}_{(\mathcal{P},\widetilde{X})} = \mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A}$. Then $\mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A}$ is the aisle of a t-structure in $\mathcal{C}_{2m}$, and in particular it is precovering. This concludes the argument. $\square$

Now we define the assignment of the map β. To this end, given an "almost aisle" in $\mathcal{C}_{2m}$, we define an equivalence relation $\sim_{\mathcal{U}}$ on the set $[m'] \cup [m]$ in the same way as in Section 3.6. The same argument of [22, Lemma 4.10] shows that $\sim_{\mathcal{U}}$ is an equivalence relation.

Definition 4.7.8. Let $\mathcal{U}$ be a suspended subcategory of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering. The relation $\sim_{\mathcal{U}}$ on the set $[m'] \cup [m]$ is defined as follows: for any $p, q \in [m'] \cup [m]$ we have that $p \sim_{\mathcal{U}} q$ if and only if $p = q$ or there exists an arc of $\mathcal{U}$ with an endpoint in $\mathbb{Z}^{(p)}$ and the other in $\mathbb{Z}^{(q)}$.

Definition 4.7.9. Keeping the assumptions and notation of Definition 4.7.8, we define $\mathcal{P}_{\mathcal{U}}$ to be the partition of $[m'] \cup [m]$ given by the equivalence classes of $\sim_{\mathcal{U}}$. For each $p \in [m'] \cup [m]$ we define

$$x_p = \sup\{z \in \mathbb{Z}^{(p)} \mid \text{there exists an arc of } \mathcal{U} \text{ with an endpoint equal to } z\}.$$

We denote by $X_{\mathcal{U}}$ the $2m$-tuple $X_{\mathcal{U}} = (x_p)_{p \in [m'] \cup [m]}$.

With Proposition 4.7.13 we will show that $(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}})$ is a half-decorated non-crossing partition of $[m'] \cup [m]$. The following remark and lemmas are useful for that purpose.

Remark 4.7.10. Consider a suspended subcategory $\mathcal{U}$ of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering. We observe that $\mathcal{U} \cap \mathcal{A}$ is the aisle of a t-structure in $\mathcal{C}_{2m}$. We denote by $(\mathcal{P}_{\mathcal{U}\cap\mathcal{A}}, X_{\mathcal{U}\cap\mathcal{A}})$ the decorated non-crossing partition associated to $\mathcal{U} \cap \mathcal{A}$. We recall that $\mathcal{P}_{\mathcal{U}\cap\mathcal{A}}$ is defined as the set of equivalence classes of $[m'] \cup [m]$ under the equivalence relation $\sim_{\mathcal{U}\cap\mathcal{A}}$, see Section 3.6. The following lemmas relate $(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}})$ and $(\mathcal{P}_{\mathcal{U}\cap\mathcal{A}}, X_{\mathcal{U}\cap\mathcal{A}})$.

Lemma 4.7.11. *Let $\mathcal{U}$ be a suspended subcategory of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering. Let $(\mathcal{P}_{\mathcal{U}\cap\mathcal{A}}, X_{\mathcal{U}\cap\mathcal{A}})$ be the decorated non-crossing partition associated to $\mathcal{U}\cap\mathcal{A}$. Then $\mathcal{P}_{\mathcal{U}} = \mathcal{P}_{\mathcal{U}\cap\mathcal{A}}$.*

Proof. We show that for any $p, q \in [m'] \cup [m]$ we have that $p \sim_{\mathcal{U}} q$ if and only if $p \sim_{\mathcal{U}\cap\mathcal{A}} q$. It is straightforward to check that if $p \sim_{\mathcal{U}\cap\mathcal{A}} q$ then $p \sim_{\mathcal{U}} q$. Assume that $p \sim_{\mathcal{U}} q$. If $p = q$ then the claim is straightforward. If $p \neq q$ then there exists $u \in \operatorname{ind}\mathcal{U}$ having one endpoint in $\mathbb{Z}^{(p)}$ and the other endpoint in $\mathbb{Z}^{(q)}$. Note that there exists $n \geq 0$ such that $\Sigma^n u \in \mathcal{A}$ and then, since $\Sigma^n \mathcal{U} \subseteq \mathcal{U}$, we obtain that $u \in \mathcal{U} \cap \mathcal{A}$. Then we have that $p \sim_{\mathcal{U}\cap\mathcal{A}} q$. This concludes the argument. $\square$

Lemma 4.7.12. *Let $\mathcal{U}$ be a subcategory of $\mathcal{C}_{2m}$ as in Lemma 4.7.11, $X_{\mathcal{U}} = (x_p)_{p \in [m'] \cup [m]}$, and $X_{\mathcal{U}\cap\mathcal{A}} = (\widetilde{x}_p)_{p \in [m'] \cup [m]}$. Then for each $p \in [m'] \cup [m]$*

$$\widetilde{x}_p = \begin{cases} z_p^0 & \text{if } p \in [m'], \\ x_p & \text{if } p \in [m]. \end{cases}$$

Proof. Let $p \in [m'] \cup [m]$. We recall that by construction, see Section 3.6, we have that

$$\widetilde{x}_p = \sup\{z \in \mathbb{Z}^{(p)} \mid \text{there exists an arc of } \mathcal{U} \cap \mathcal{A} \text{ with an endpoint equal to } z\}.$$

If $p \in [m']$, there exists an arc of $\mathcal{U} \cap \mathcal{A}$ with an endpoint equal to z_p^0. Moreover, for any $z \in \mathbb{Z}^{(p)}$ such that $z > z_p^0$ there is no arc of $\mathcal{A}$, and then no arc of $\mathcal{U} \cap \mathcal{A}$, with an endpoint in z. Thus, $\widetilde{x}_p = z_p^0$. Now consider $p \in [m]$, we show that $\widetilde{x}_p = x_p$. We divide the argument into claims.

Claim 1. Let $z \in \mathbb{Z}^{(p)}$. If there exists an arc of $\mathcal{U}$ with an endpoint equal to z, then there exists an arc of $\mathcal{U} \cap \mathcal{A}$ with an endpoint equal to z.

Assume that there exists $u \in \operatorname{ind}\mathcal{U}$ having an endpoint equal to z. If $u \in \mathcal{U} \cap \mathcal{A}$, then we have the claim. Now assume that $u \in \mathcal{U}$ and $u \notin \mathcal{A}$. We denote $u = (u_1, u_2)$. We assume that $u_1 = z$, the other case is analogous. Since $u \notin \mathcal{A}$, we have that $u_2 \in \mathbb{Z}^{(q)}$ for some $q \in [m']$ and $u_2 > z_p^0$. Then we are in the situation of Figure 4.15.

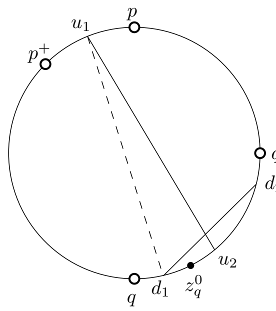


Figure 4.15: Illustration of the argument of Claim 1.

Consider $d = (d_1, d_2) \in \mathbb{Z}^{(q,q)}$ with $d_1 \leq z_p^0 < u_2 < d_2$. Since $d \in \mathcal{D} \subseteq \mathcal{U}$, u and d are crossing, and $\mathcal{U}$ is extension-closed, we obtain that $(z, d_1) = (u_1, d_1) \in \mathcal{U}$. Moreover, $(z, d_1) \in \mathcal{A}$. This concludes the argument of Claim 1.

Claim 2. If $\widetilde{x}_p = p$ then $x_p = p$.

Assume that $\widetilde{x}_p = p$, i.e. there is no $z \in \mathbb{Z}^{(p)}$ such that there is an arc of $\mathcal{U} \cap \mathcal{A}$ with an endpoint equal to z. By Claim 1 there is no $z \in \mathbb{Z}^{(p)}$ such that there is an arc of $\mathcal{U}$ with an endpoint equal to z, i.e. $x_p = p$. This concludes the argument of Claim 2.

Claim 3. If $\widetilde{x}_p = p^+$ then $x_p = p^+$.

The proof is straightforward.

Claim 4. $\widetilde{x}_p = x_p$.

If $\widetilde{x}_p = p$ or $\widetilde{x}_p = p^+$ then the claim follows from Claim 2 and Claim 3. Assume that there exists $z \in \mathbb{Z}^{(p)}$ such that $\widetilde{x}_p = z$. As a consequence, there is an arc of $\mathcal{U} \cap \mathcal{A}$ with an endpoint equal to z, and then there is an arc of $\mathcal{U}$ with an endpoint equal to z. Moreover, for any $z' \in \mathbb{Z}^{(p)}$ such that $z' > z$ there is no arc of $\mathcal{U} \cap \mathcal{A}$ with an endpoint equal to z'. Then, by Claim 1, for any $z' \in \mathbb{Z}^{(p)}$ such that $z' > z$ there is no arc of $\mathcal{U}$ with an endpoint equal to z'. Thus, $x_p = z$. This concludes the argument of Claim 4.

We can conclude that $\widetilde{x}_p = z_p^0$ for each $p \in [m]'$, and $\widetilde{x}_p = x_p$ for each $p \in [m]$. $\square$

Now we can prove that the map β of Proposition 4.7.5 is well defined.

Proposition 4.7.13. *Let $\mathcal{U}$ be a suspended subcategory of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering. Then $(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}})$ is a half-decorated non-crossing partition.*

Proof. We check that $(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}})$ satisfies the conditions of Definition 4.7.1. Consider $\mathcal{U} \cap \mathcal{A}$, which is the aisle of a t-structure, and its associated decorated non-crossing partition $(\mathcal{P}_{\mathcal{U}\cap\mathcal{A}}, X_{\mathcal{U}\cap\mathcal{A}})$. We recall that $\mathcal{P}_{\mathcal{U}}$ is a partition of $[m'] \cup [m]$ and that, from Lemma 4.7.11, $\mathcal{P}_{\mathcal{U}\cap\mathcal{A}} = \mathcal{P}_{\mathcal{U}}$. As a consequence, $\mathcal{P}_{\mathcal{U}}$ is a non-crossing partition of $[m'] \cup [m]$. Now, for the decorations we denote $X_{\mathcal{U}} = (x_p)_{p\in[m']\cup[m]}$ and $X_{\mathcal{U}\cap\mathcal{A}} = (\widetilde{x}_p)_{p\in[m']\cup[m]}$. From Lemma 4.7.12 we have that $x_p = \widetilde{x}_p$ for each $p \in [m]$. Moreover, for each $p \in [m']$, since $\mathbb{Z}^{(p,p)} \subseteq \operatorname{ind} \mathcal{D} \subseteq \operatorname{ind} \mathcal{U}$, we have that $x_p = p^+$. We can conclude that $(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}})$ is a half-decorated non-crossing partition of $[m'] \cup [m]$. $\square$

Given $\mathcal{U}$ a suspended subcategory of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering, the following lemma shows that any shift of $\mathcal{U}$ has the same properties of $\mathcal{U}$. This fact will be useful in the proof of Proposition 4.7.5.

Lemma 4.7.14. *Let $\mathcal{U}$ be a suspended subcategory of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering. Consider the associated half-decorated non-crossing partition $(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}})$ with $X_{\mathcal{U}} = (x_p)_{p\in[m']\cup[m]}$. The following statements hold.*

1. *For any $n \in \mathbb{Z}$ the subcategory $\Sigma^n\mathcal{U}$ of $\mathcal{C}_{2m}$ is suspended, $\mathcal{D} \subseteq \Sigma^n\mathcal{U}$, and $\Sigma^n\mathcal{U} \cap \mathcal{A}$ is precovering.*
2. *Consider $(\mathcal{P}_{\Sigma^n\mathcal{U}}, X_{\Sigma^n\mathcal{U}})$. Then $\mathcal{P}_{\Sigma^n\mathcal{U}} = \mathcal{P}_{\mathcal{U}}$ and $X_{\Sigma^n\mathcal{U}} = (x_p - n)_{p\in[m']\cup[m]}$.*

Proof. First we prove statement (1), statement (2) follows by construction, see Definition 4.7.6. It is straightforward to check that $\Sigma^n\mathcal{U}$ is extension-closed and contains $\mathcal{D}$, we show that $\Sigma^n\mathcal{U} \cap \mathcal{A}$ is precovering. By Proposition 2.3.38 we have that $\pi^{-1}\pi\mathcal{U} = \mathcal{U}$, and, since $\pi^{-1}\pi\mathcal{U} \cap \mathcal{A} = \mathcal{U} \cap \mathcal{A}$ is precovering, by Theorem 4.4.3 we have that $\pi\mathcal{U}$ is precovering in $\overline{\mathcal{C}}_m$. Now fix $n \in \mathbb{Z}$. Since $\pi\mathcal{U}$ is precovering, then $\Sigma^n\pi\mathcal{U}$ is precovering in $\overline{\mathcal{C}}_m$. As a consequence, from Theorem 4.4.3 we have that $\Sigma^n\mathcal{U} \cap \mathcal{A} = \pi^{-1}\pi\Sigma^n\mathcal{U} \cap \mathcal{A} = \pi^{-1}\Sigma^n\pi\mathcal{U} \cap \mathcal{A}$ is precovering. This concludes the proof. $\square$

Finally, we can prove Proposition 4.7.5.

Proof of Proposition 4.7.5. From Proposition 4.7.7 and Proposition 4.7.13 the maps α and β are well defined. We divide the proof into steps.

Step 1. The map β is injective.

Let $\mathcal{U}$ and $\mathcal{U}'$ be suspended subcategories of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$, $\mathcal{D} \subseteq \mathcal{U}'$, and such that $\mathcal{U} \cap \mathcal{A}$ and $\mathcal{U}' \cap \mathcal{A}$ are precovering. Assume that $(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}}) = (\mathcal{P}_{\mathcal{U}'}, X_{\mathcal{U}'})$, we show that $\mathcal{U} = \mathcal{U}'$. First we show that $\Sigma^n\mathcal{U} \cap \mathcal{A} = \Sigma^n\mathcal{U}' \cap \mathcal{A}$ for each $n \in \mathbb{Z}$. By Lemma 4.7.11, Lemma 4.7.12, and Lemma 4.7.14, $(\mathcal{P}_{\Sigma^n\mathcal{U}\cap\mathcal{A}}, X_{\Sigma^n\mathcal{U}\cap\mathcal{A}}) = (\mathcal{P}_{\Sigma^n\mathcal{U}'\cap\mathcal{A}}, X_{\Sigma^n\mathcal{U}'\cap\mathcal{A}})$. By Theorem 3.6.3 we obtain that $\Sigma^n\mathcal{U} \cap \mathcal{A} = \Sigma^n\mathcal{U}' \cap \mathcal{A}$.

Now we show that $\mathcal{U} \subseteq \mathcal{U}'$, the other inclusion can be obtained in the same way. Consider $u \in \operatorname{ind}\mathcal{U}$, we have that $\Sigma^n u \in \mathcal{A}$ for some $n \geq 0$. Then $\Sigma^n u \in \Sigma^n\mathcal{U} \cap \mathcal{A}$. Since $\Sigma^n\mathcal{U} \cap \mathcal{A} = \Sigma^n\mathcal{U}' \cap \mathcal{A}$, we have that $\Sigma^n u \in \Sigma^n\mathcal{U}' \cap \mathcal{A}$ and then $u \in \mathcal{U}'$. This concludes the argument of Step 1.

Step 2. We show that $\beta\alpha = \mathrm{id}$.

Let $(\mathcal{P}, X)$ be a half-decorated non-crossing partition of $[m'] \cup [m]$. Let $\mathcal{U}_{(\mathcal{P},X)}$ be the associated subcategory of $\mathcal{C}_{2m}$, which we denoted by $\mathcal{U}$. Let $(\mathcal{P}_\mathcal{U}, X_\mathcal{U})$ be the half-decorated non-crossing partition associated to $\mathcal{U}$. We show that $(\mathcal{P}, X) = (\mathcal{P}_\mathcal{U}, X_\mathcal{U})$.

Showing the equality $\mathcal{P} = \mathcal{P}_\mathcal{U}$ is equivalent to show that for any $p, q \in [m'] \cup [m]$ we have that $p \sim_\mathcal{U} q$ if and only if $p, q \in B$ for some block $B \in \mathcal{P}$. This follows directly from Definition 4.7.6 and Definition 4.7.8. Now we show that $X = X_\mathcal{U}$. We denote $X = (x_p)_{p\in[m']\cup[m]}$ and $X_\mathcal{U} = (y_p)_{p\in[m']\cup[m]}$. By construction, see Definition 4.7.6 and Definition 4.7.9, we have the following equalities.

$$y_p = \sup\{z \in \mathbb{Z}^{(p)} \mid \text{there exists an arc of } \mathcal{U} = \mathcal{U}_{(\mathcal{P},X)} \text{ with an endpoint equal to } z\} = x_p$$

Therefore we have that $(\mathcal{P}, X) = (\mathcal{P}_\mathcal{U}, X_\mathcal{U})$. This concludes the argument of Step 2. We can conclude that α and β are mutually inverse. □

4.7.2 Co-aisles of t-structures

From Theorem 4.7.4 we have a classification of the aisles of the t-structures in $\overline{\mathcal{C}}_m$, now we compute the corresponding co-aisles in terms of non-crossing partitions. As before, we take an intermediate step through $\mathcal{C}_{2m}$. Given a half-decorated non-crossing partition $(\mathcal{P}, X)$ of $[m'] \cup [m]$, we consider its complement $(\mathcal{P}, X)^c = (\mathcal{Q}, Y)$, where $\mathcal{Q} = \mathcal{P}^c$ is the Kreweras complement of $\mathcal{P}$, see Section 3.6. With a computation similar to Section 3.6, $(\mathcal{Q}, Y)$ corresponds to a subcategory $\mathcal{V}$ of $\mathcal{C}_{2m}$. This is a co-suspended subcategory of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{V}$ and $\mathcal{V} \cap \mathcal{B}$ is preenveloping, therefore $\mathcal{V}$ can be thought as an "almost co-aisle" in $\mathcal{C}_{2m}$. From such $\mathcal{V}$ we obtain the corresponding co-aisle in $\overline{\mathcal{C}}_m$ after localising. Figure 4.16 illustrates this process.

Definition 4.7.15. Let $(\mathcal{P}, X)$ be a half-decorated non-crossing partition of $[m']\cup[m]$ with $X = (x_p)_{p\in[m']\cup[m]}$. We define the *complement*, $(\mathcal{P}, X)^c$, of $(\mathcal{P}, X)$ to be the pair $(\mathcal{Q}, Y)$ where $\mathcal{Q} = \mathcal{P}^c$ is the Kreweras complement of $\mathcal{P}$, and Y is the $2m$-tuple $Y = (y_p)_{p\in[m']\cup[m]}$ with

$$y_p = \begin{cases} p & \text{if } p \in [m'], \\ x_p - 1 & \text{if } p \in [m]. \end{cases}$$

We describe how to obtain an "almost co-aisle" of t-structure in $\mathcal{C}_{2m}$ from the complement of a half-decorated non-crossing partition of $[m'] \cup [m]$.

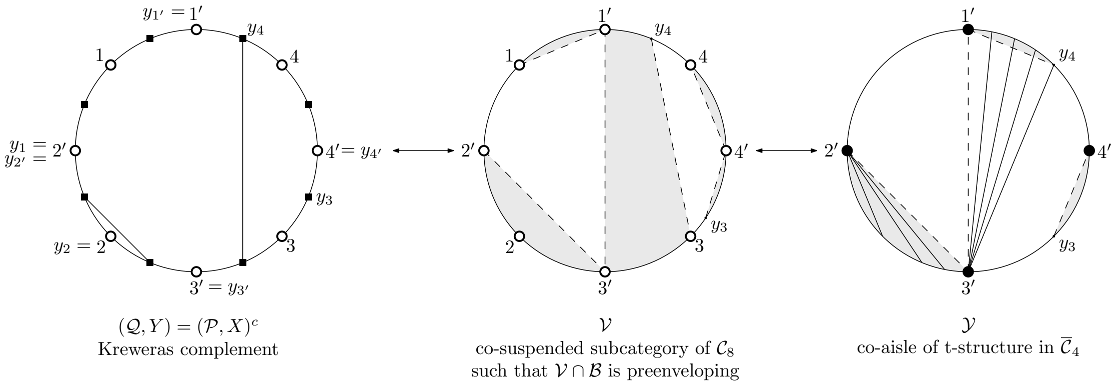


Figure 4.16: Illustration of how to obtain the co-aisle of the aisle of Figure 4.14.

Definition 4.7.16. Let $(\mathcal{P}, X)$ be a half-decorated non-crossing partition of $[m'] \cup [m]$ and let $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$. We define

$$\mathcal{V}_{(\mathcal{Q},Y)} = \operatorname{add} \bigsqcup_{B \in \mathcal{Q}} \left\{ (v_1, v_2) \in \operatorname{ind} \mathcal{C}_{2m} \;\middle|\; v_1, v_2 \in \bigcup_{p \in B} [y_p, p^+) \right\}.$$

Consider the complement $(\mathcal{Q}, Y)$ of a half-decorated non-crossing partition of $[m'] \cup [m]$. The following lemmas and remark establish some properties of the subcategory $\mathcal{V}_{(\mathcal{Q},Y)}$ of $\mathcal{C}_{2m}$. The first is analogous to Proposition 4.7.7.

Lemma 4.7.17. *Let $(\mathcal{P}, X)$ be a half-decorated non-crossing partition of $[m'] \cup [m]$ and let $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$. Then $\mathcal{V}_{(\mathcal{Q},Y)}$ is co-suspended and contains $\mathcal{D}$.*

Proof. The proof is analogous to the argument of Proposition 4.7.7. □

Lemma 4.7.18. *Let $(\mathcal{P}, X)$ be a half-decorated non-crossing partition of $[m'] \cup [m]$ and let $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$. Then $\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B} = \big(\mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A}\big)^{\perp}$.*

Proof. For each $p \in [m'] \cup [m]$ we define

$$\widetilde{y}_p = \begin{cases} y_p & \text{if } p \in [m], \\ w_p^0 & \text{if } p \in [m'] \end{cases}$$

where we recall from Section 4.4.2 that $w_p^0 = z_p^0 - 1$. It is straightforward to check that

$$\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B} = \operatorname{add} \bigsqcup_{B \in \mathcal{Q}} \left\{ (v_1, v_2) \in \operatorname{ind} \mathcal{C}_{2m} \;\middle|\; v_1, v_2 \in \bigcup_{p \in B} [\widetilde{y}_p, p^+) \right\}.$$

Moreover, by Corollary 3.6.4, the left hand side is equal to $\big(\mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A}\big)^{\perp}$. □

Remark 4.7.19. Let $\mathcal{U}_{(\mathcal{P},X)}$ and $\mathcal{V}_{(\mathcal{Q},Y)}$ be as in Lemma 4.7.18. Since $\mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A}$ is precovering and suspended, by Proposition 2.3.27 $\big(\mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A}, \mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B}\big)$ is a t-structure.

The following lemma and proposition show that an "almost co-aisle" in $\mathcal{C}_{2m}$ is, after localising, the co-aisle of a t-structure in $\overline{\mathcal{C}}_m$.

Lemma 4.7.20. *Let $\mathcal{X}$ be the aisle of a t-structure in $\overline{\mathcal{C}}_m$, let $(\mathcal{P}, X)$ be the half decorated non-crossing partition associated to $\mathcal{X}$, and let $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$. Then $\pi\mathcal{V}_{(\mathcal{Q},Y)} \subseteq \mathcal{X}^{\perp}$.*

Proof. Assume that there exist $x \in \operatorname{ind}\mathcal{X}$ and $y \in \operatorname{ind}\pi\mathcal{V}_{(\mathcal{Q},Y)}$ such that $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(x,y) \cong \mathbb{K}$. Note that there exists $y' \in \operatorname{ind}\mathcal{B}$ such that $\pi(y') \cong y$. Then $y' \in \pi^{-1}\pi\mathcal{V}_{(\mathcal{Q},Y)}$ and by Proposition 2.3.38 and Lemma 4.7.17, we have that $y' \in \operatorname{ind}\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B}$. We define $\mathcal{U} = \pi^{-1}\mathcal{X}$, we have that $\mathcal{U} = \mathcal{U}_{(\mathcal{P},X)}$. Now, by Lemma 4.4.4 and Lemma 4.4.5, there exists $x' \in \operatorname{ind}\mathcal{A}$ such that $\pi(x') \cong x$, and then $x' \in \operatorname{ind}\mathcal{U} \cap \mathcal{A}$, and $\operatorname{Hom}_{\mathcal{C}_{2m}}(x', y') \cong \mathbb{K}$. Since $x \in \operatorname{ind}\mathcal{U} \cap \mathcal{A}$ and $y' \in \operatorname{ind}\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B}$, this gives a contradiction with Lemma 4.7.18. Then we can conclude that $\operatorname{Hom}_{\overline{\mathcal{C}}_m}(\mathcal{X}, \pi\mathcal{V}_{(\mathcal{Q},Y)}) = 0$. ☐

Proposition 4.7.21. *Let $(\mathcal{X}, \mathcal{Y})$ be a t-structure in $\overline{\mathcal{C}}_m$, $\mathcal{U} = \pi^{-1}\mathcal{X}$, $(\mathcal{P}, X)$ be its associated half-decorated non-crossing partition, and $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$. Then the following equalities hold:*

$$\mathcal{Y} = \pi\mathcal{V}_{(\mathcal{Q},Y)} = \pi\left(\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B}\right) = \pi\left((\mathcal{U} \cap \mathcal{A})^{\perp}\right).$$

Proof. First we show that $\pi\mathcal{V}_{(\mathcal{Q},Y)} = \pi\left(\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B}\right)$. The inclusion $\pi\left(\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B}\right) \subseteq \pi\mathcal{V}_{(\mathcal{Q},Y)}$ is straightforward. We show the other inclusion. Consider $y \in \operatorname{ind}\pi\mathcal{V}_{(\mathcal{Q},Y)}$, then there exists $y' \in \operatorname{ind}\mathcal{B}$ such that $\pi(y') \cong y$. Since $y \in \pi\mathcal{V}_{(\mathcal{Q},Y)}$, we have that $y' \in \pi^{-1}\pi\mathcal{V}_{(\mathcal{Q},Y)}$. By Proposition 2.3.38 and Lemma 4.7.17 we have that $\pi^{-1}\pi\mathcal{V}_{(\mathcal{Q},Y)} = \mathcal{V}_{(\mathcal{Q},Y)}$. Thus, $y' \in \operatorname{ind}\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B}$ and then $y \cong \pi(y') \in \pi\left(\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B}\right)$. From Lemma 4.7.18 we also have the equality $\pi\left(\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B}\right) = \pi\left((\mathcal{U} \cap \mathcal{A})^{\perp}\right)$. It remains to show the equality $\mathcal{Y} = \pi\mathcal{V}_{(\mathcal{Q},Y)}$, to do so we check that $\left(\mathcal{X}, \pi\mathcal{V}_{(\mathcal{Q},Y)}\right)$ is a torsion pair.

Note that by Lemma 4.7.20 we have that $\pi\mathcal{V}_{(\mathcal{Q},Y)} \subseteq \mathcal{X}^{\perp}$, we show that $\mathcal{X} * \pi\mathcal{V}_{(\mathcal{Q},Y)} = \overline{\mathcal{C}}_m$. Since $\mathcal{X} = \pi(\mathcal{U} \cap \mathcal{A})$ and $\pi\mathcal{V}_{(\mathcal{Q},Y)} = \pi\left((\mathcal{U} \cap \mathcal{A})^{\perp}\right)$, it is equivalent to show that $\pi(\mathcal{U} \cap \mathcal{A}) * \pi\left((\mathcal{U} \cap \mathcal{A})^{\perp}\right) = \overline{\mathcal{C}}_m$.

Let $a \in \overline{\mathcal{C}}_m$, there exists $a' \in \mathcal{C}_{2m}$ such that $\pi(a') \cong a$. Since $(\mathcal{U} \cap \mathcal{A}) * (\mathcal{U} \cap \mathcal{A})^{\perp} = \mathcal{C}_{2m}$, there exists a triangle $u \longrightarrow a' \longrightarrow v \longrightarrow \Sigma a$ in $\mathcal{C}_{2m}$ with $u \in \mathcal{U} \cap \mathcal{A}$ and $v \in (\mathcal{U} \cap \mathcal{A})^{\perp}$. After localising we obtain the triangle $\pi(u) \longrightarrow a \longrightarrow \pi(v) \longrightarrow \Sigma\pi(u)$ in $\overline{\mathcal{C}}_m$. Note that $\pi(u) \in \pi(\mathcal{U} \cap \mathcal{A})$ and $\pi(v) \in \pi\left((\mathcal{U} \cap \mathcal{A})^{\perp}\right)$, thus we have that $a \in \pi(\mathcal{U} \cap \mathcal{A}) * \pi\left((\mathcal{U} \cap \mathcal{A})^{\perp}\right)$. We can conclude that $\left(\mathcal{X}, \pi\mathcal{V}_{(\mathcal{Q},Y)}\right)$ is a torsion pair, and as a consequence $\mathcal{Y} = \pi\mathcal{V}_{(\mathcal{Q},Y)}$. ☐

4.7.3 Hearts

With Theorem 4.7.4 we classified the aisles of t-structures in $\overline{\mathcal{C}}_m$, and with Proposition 4.7.21 we computed the corresponding co-aisle. Now we can compute the heart of a t-structure $(\mathcal{X}, \mathcal{Y})$ in $\overline{\mathcal{C}}_m$. We first consider the preimage of $(\mathcal{X}, \mathcal{Y})$ under π, which we denote by $(\mathcal{U}, \mathcal{V})$. Note that $(\mathcal{U}, \mathcal{V})$ is not a t-structure of $\mathcal{C}_{2m}$, but $(\mathcal{U} \cap \mathcal{A}, \mathcal{V} \cap \mathcal{B})$ is. We can compute the heart of $(\mathcal{U} \cap \mathcal{A}, \mathcal{V} \cap \mathcal{B})$ as in Proposition 3.6.5, and then obtain the heart of $(\mathcal{X}, \mathcal{Y})$ by localising. Figure 4.17 illustrates this process.

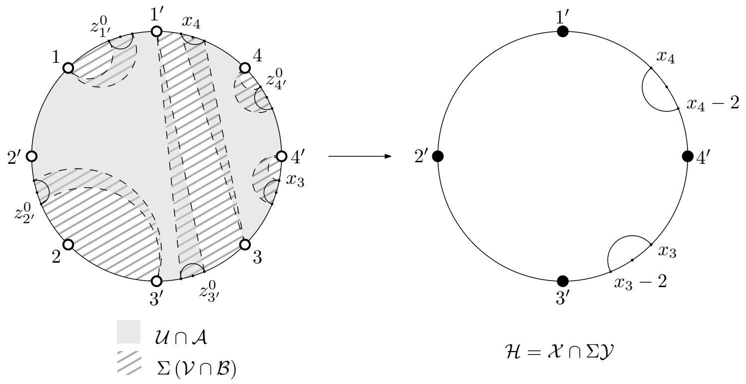


Figure 4.17: The heart of the t-structure $(\mathcal{X}, \mathcal{Y})$ of Figure 4.14 and Figure 4.16.

Corollary 4.7.22. *Let $(\mathcal{X}, \mathcal{Y})$ be a t-structure in $\overline{\mathcal{C}}_m$. Consider its associated decorated non-crossing partition $(\mathcal{P}, X)$ of $[m'] \cup [m]$, with $X = (x_p)_{p\in[m']\cup[m]}$. Then the heart $\mathcal{H} = \mathcal{X} \cap \Sigma\mathcal{Y}$ is given by*

$$\mathcal{H} = \operatorname{add}\{(x_p - 2, x_p) \mid p \in [m] \textit{ and } x_p \in \mathcal{Z}_m\}.$$

Proof. Let $\mathcal{U} = \pi^{-1}\mathcal{X}$, $\mathcal{V} = \pi^{-1}\mathcal{Y}$, $\mathcal{U}' = \mathcal{U}\cap\mathcal{A}$, and $\mathcal{V}' = \mathcal{V}\cap\mathcal{B}$. By Lemma 4.7.18, the pair $(\mathcal{U}', \mathcal{V}')$ is a t-structure in $\mathcal{C}_{2m}$. Consider the heart $\mathcal{H}' = \mathcal{U}' \cap \Sigma\mathcal{V}'$, we show that $\pi\mathcal{H}' = \mathcal{H}$. Then the claim follows directly from Proposition 3.6.5.

First we show the inclusion $\pi\mathcal{H}' \subseteq \mathcal{H}$. Consider $h' \in \operatorname{ind}\mathcal{H}'$. Since $h' \in \mathcal{U}' \subseteq \mathcal{U}$, we have that $\pi h' \in \pi\mathcal{U}$ and from Proposition 2.3.38 we have that $\pi\mathcal{U} = \mathcal{X}$. Similarly, since $h' \in \Sigma\mathcal{V}' \subseteq \Sigma\mathcal{V}$, we obtain that $\pi h' \in \pi\Sigma\mathcal{V} = \Sigma\pi\mathcal{V} = \Sigma\mathcal{Y}$. Thus, $\pi h' \in \mathcal{X}\cap\Sigma\mathcal{Y} = \mathcal{H}$.

Now we show the inclusion $\mathcal{H} \subseteq \pi\mathcal{H}'$. Let $h \in \operatorname{ind}\mathcal{H}$, from Lemma 4.4.12 there exists $h' \in \operatorname{ind}\mathcal{A}\cap\Sigma\mathcal{B}$ such that $\pi h' \cong h$. Since $h \in \mathcal{X}$, then $h' \in \pi^{-1}\mathcal{X} = \mathcal{U}$. Moreover, since $h \in \Sigma\mathcal{Y}$, then $h' \in \pi^{-1}\Sigma\mathcal{Y} = \Sigma\mathcal{V}$. Thus, $h' \in \mathcal{U}\cap\mathcal{A}$ and $h' \in \Sigma\mathcal{V}\cap\Sigma\mathcal{B}$. We obtain that $h' \in \mathcal{U}'\cap\Sigma\mathcal{V}' = \mathcal{H}'$, and then $h \cong \pi h' \in \pi\mathcal{H}'$. We can conclude that $\mathcal{H} = \pi\mathcal{H}'$. ☐

4.7.4 Boundedness

We classify the bounded t-structures in $\overline{\mathcal{C}}_m$, and we obtain that for each $m \geq 1$ there are no bounded t-structures in $\overline{\mathcal{C}}_m$. We refer to Proposition 3.6.5 for the classification of the bounded t-structures in $\mathcal{C}_m$.

Proposition 4.7.23. *Let $(\mathcal{X}, \mathcal{Y})$ be a t-structure in $\overline{\mathcal{C}}_m$, let $(\mathcal{P}, X)$ be its associated half-decorated non-crossing partition of $[m'] \cup [m]$, $\mathcal{U} = \pi^{-1}\mathcal{X}$ and $\mathcal{V} = \pi^{-1}\mathcal{Y}$. The following statements are equivalent.*

1. *The t-structure $(\mathcal{X}, \mathcal{Y})$ is left bounded in $\overline{\mathcal{C}}_m$.*
2. *The t-structure $(\mathcal{U}\cap\mathcal{A}, \mathcal{V}\cap\mathcal{B})$ is left bounded in $\mathcal{C}_{2m}$.*
3. *The non-crossing partition $\mathcal{P}$ has as unique block $\{1', 1, \ldots, m', m\}$.*

Proof. We prove the equivalence of statements (1) and (2), for the equivalence between

(2) and (3) we refer to Proposition 3.6.5. Assume that (1) holds, we check the inclusion $\mathcal{C}_{2m} \subseteq \bigcup_{n\in\mathbb{Z}} \Sigma^n(\mathcal{U} \cap \mathcal{A})$, the other inclusion is trivial. Consider $a \in \operatorname{ind}\mathcal{C}_{2m}$. Note that there exists $k \in \mathbb{Z}$ such that $a \in \Sigma^k\mathcal{A}$. Moreover, since $\pi(a) \in \overline{\mathcal{C}}_m$, there exists $l \in \mathbb{Z}$ such that $\pi(a) \in \Sigma^l\mathcal{X}$, and then $a \in \Sigma^l\pi^{-1}\mathcal{X} = \Sigma^l\mathcal{U}$. Thus, we have that $a \in \Sigma^l\mathcal{U} \cap \Sigma^k\mathcal{A}$. Let $n = \min\{k, l\}$, then $a \in \Sigma^n(\mathcal{U} \cap \mathcal{A})$.

Now we assume that (2) holds, we check the inclusion $\overline{\mathcal{C}}_m \subseteq \bigcup_{n\in\mathbb{Z}} \Sigma^n\mathcal{X}$, the other inclusion is trivial. Consider $a \in \operatorname{ind}\overline{\mathcal{C}}_m$, then there exists $a' \in \operatorname{ind}\mathcal{C}_{2m}$ such that $\pi(a') \cong a$. Then there exists $n \in \mathbb{Z}$ such that $a' \in \Sigma^n(\mathcal{U} \cap \mathcal{A}) \subseteq \Sigma^n\mathcal{U}$, and then $a \cong \pi(a') \in \pi\Sigma^n\mathcal{U} = \Sigma^n\mathcal{X}$. This concludes the proof. □

Dually, we have the following proposition.

Proposition 4.7.24. *Keeping the assumptions and notation of Proposition 4.7.23, the following statements are equivalent.*

1. *The t-structure $(\mathcal{X}, \mathcal{Y})$ is right bounded in $\overline{\mathcal{C}}_m$.*
2. *The t-structure $(\mathcal{U} \cap \mathcal{A}, \mathcal{V} \cap \mathcal{B})$ is right bounded in $\mathcal{C}_{2m}$.*
3. *The non-crossing partition $\mathcal{P}$ has as blocks $\{1'\}$, $\{1\}$, ..., $\{m'\}$, $\{m\}$.*

We have the following corollary of Proposition 4.7.23 and Proposition 4.7.24.

Corollary 4.7.25. *For each $m \geq 1$ there are no bounded t-structures in $\overline{\mathcal{C}}_m$.*

4.7.5 Non-degeneracy

We classify the non-degenerate t-structures in $\overline{\mathcal{C}}_m$. We refer to Proposition 3.6.5 for the classification of the non-degenerate t-structures in $\mathcal{C}_m$.

Proposition 4.7.26. *Let $(\mathcal{X}, \mathcal{Y})$ be a t-structure in $\overline{\mathcal{C}}_m$, let $(\mathcal{P}, X)$ be its associated half-decorated non-crossing partition with $X = (x_p)_{p\in[m']\cup[m]}$, and $\mathcal{U} = \pi^{-1}\mathcal{X}$. The following statements are equivalent.*

1. *The t-structure $(\mathcal{X}, \mathcal{Y})$ is left non-degenerate in $\overline{\mathcal{C}}_m$.*
2. *We have that $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U} = \mathcal{D}$.*
3. *For each $p \in [m]$ we have that $x_p \neq p^+$, and for each $p, q \in [m']$ if $p, q \in B$ for some block $B \in \mathcal{P}$, then $p = q$.*

Proof. First we show the equivalence between the statements (1) and (2) beginning with (1) implies (2). Assume that $(\mathcal{X}, \mathcal{Y})$ is left non-degenerate, i.e. $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{X} = 0$. The inclusion $\mathcal{D} \subseteq \bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U}$ is straightforward, we show the other inclusion. Consider $u \in \operatorname{ind}\mathcal{C}_{2m}$ such that $u \in \Sigma^n\mathcal{U}$ for all $n \in \mathbb{Z}$, then $\pi(u) \in \pi\Sigma^n\mathcal{U} = \Sigma^n\mathcal{X}$ for all $n \in \mathbb{Z}$. As a consequence, $\pi(u) = 0$ and then $u \in \mathcal{D}$.

Now to show that (2) implies (1), assume that $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U} = \mathcal{D}$, we show that $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{X} = 0$. Assume that there exists $x \in \operatorname{ind}\overline{\mathcal{C}}_m$ such that $x \in \Sigma^n\mathcal{X}$ for all $n \in \mathbb{Z}$. Then there exists $x' \in \operatorname{ind}\mathcal{C}_{2m}$ such that $\pi(x') \cong x$, and $x' \in \pi^{-1}(\Sigma^n\mathcal{X}) = \Sigma^n\mathcal{U}$ for all $n \in \mathbb{Z}$. Then $x' \in \mathcal{D}$

and $x \cong \pi(x') = 0$, contradicting the fact that $x \in \operatorname{ind}\overline{\mathcal{C}}_m$. This proves the equivalence between (1) and (2).

Now we prove the equivalence between statements (2) and (3). Assume that $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U} = \mathcal{D}$ and that there exists $p \in [m]$ such that $x_p = p^+$, then $\mathbb{Z}^{(p,p)} \subseteq \operatorname{ind}\mathcal{U}$. Let $u \in \mathbb{Z}^{(p,p)}$, then $x \in \Sigma^n\mathcal{U}$ for each $n \in \mathbb{Z}$. As a consequence $x \in \mathcal{D}$, and this contradicts the fact that $p \in [m]$. This proves that $x_p \neq p^+$. Now consider $p, q \in [m']$ such that $p, q \in B$ for some block $B \in \mathcal{P}$, then $\mathcal{U}$ contains all arcs having one endpoint in $\mathbb{Z}^{(p)}$ and the other in $\mathbb{Z}^{(q)}$. Consider such u, then $u \in \Sigma^n\mathcal{U}$ for each $n \in \mathbb{Z}$. As a consequence $u \in \mathcal{D}$, and then $p = q$. This proves that (2) implies (3).

Now assume that statement (3) holds, we show that $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U} = \mathcal{D}$. The inclusion $\mathcal{D} \subseteq \bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U}$ is straightforward, we show the other inclusion. Let $u \in \operatorname{ind}\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U}$, we show that $u \in \mathcal{D}$. Assume that u has an endpoint $z \in \mathbb{Z}^{(p)}$ for some $p \in [m]$. Since $u \in \operatorname{ind}\mathcal{U}$, then $z \in (p, x_p]$. Moreover, since $x_p \neq p^+$, there exists $n \in \mathbb{Z}$ such that $\Sigma^n u \notin \mathcal{U}$, and this contradicts the fact that $u \in \bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U}$. Thus, $u \in \mathbb{Z}^{(p,q)}$ for some $p, q \in [m']$. Then $p, q \in B$ for some $B \in \mathcal{P}$ and as a consequence $p = q$, i.e. $u \in \mathcal{D}$. This concludes the argument. $\square$

Dually, we have the following proposition.

Proposition 4.7.27. *Keeping the assumptions and notation of Proposition 4.7.26, let* $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$, *and let* $\mathcal{V} = \pi^{-1}\mathcal{Y}$. *The following statements are equivalent.*

1. *The t-structure* $(\mathcal{X}, \mathcal{Y})$ *is right non-degenerate.*
2. *We have that* $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{V} = \mathcal{D}$.
3. *For each* $p \in [m]$ *we have that* $x_p \neq p$, *and for each* $p, q \in [m']$ *if* $p, q \in C$ *for some block* $C \in \mathcal{Q}$, *then* $p = q$.

Combining Proposition 4.7.26 and Proposition 4.7.27 we obtain the following corollary.

Corollary 4.7.28. *Keeping the assumptions and notation of Proposition 4.7.26 and Proposition 4.7.27, the following statements are equivalent.*

1. *The t-structure* $(\mathcal{X}, \mathcal{Y})$ *is non-degenerate.*
2. *We have that* $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U} = \mathcal{D} = \bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{V}$.
3. *For each* $p \in [m]$ *we have that* $x_p \in \mathbb{Z}^{(p)}$, *and for each* $p, q \in [m']$ *if* $p, q \in B$ *for some block* $B \in \mathcal{P}$, *or* $p, q \in C$ *for some block* $C \in \mathcal{Q}$, *then* $p = q$.

With the following example we show that there exist half-decorated non-crossing partitions of $[m'] \cup [m]$ satisfying condition (3) of Corollary 4.7.28.

Example 4.7.29. Consider $\mathcal{P}$ the non-crossing partition $\mathcal{P} = \{\{1', 1\}, \{2', 2\}, \dots, \{m', m\}\}$ of $[m'] \cup [m]$, and $X = (x_p)_{p\in[m']\cup[m]}$ with $x_p \in \mathbb{Z}^{(p)}$ for each $p \in [m]$. Then $(\mathcal{P}, X)$ is a half-decorated non-crossing partition of $[m'] \cup [m]$, and $\mathcal{P}^c = \{\{1'\}, \{2'\}, \dots, \{m'\}, \{1, 2, \dots, m\}\}$, see Section 3.6. Note that $(\mathcal{P}, X)$ satisfies condition (3) of Corollary 4.7.28.

As a consequence, we have the following corollary.

Corollary 4.7.30. *Non-degenerate t-structures in $\overline{\mathcal{C}}_m$ exist for each $m \geq 1$.*

4.8 Co-t-structures

We know that in the category $\mathcal{C}_m$ the only co-t-structures are $(\mathcal{C}_m, 0)$ and $(0, \mathcal{C}_m)$, see Remark 3.6.6. In $\overline{\mathcal{C}}_m$ this is not the case, Figure 4.18 gives an example of a non-trivial co-t-structure in $\overline{\mathcal{C}}_m$. In this section we classify the aisles of the co-t-structures, we compute the co-aisles and co-hearts. We also classify the bounded and non-degenerate co-t-structures, and the co-t-structures having a left or right adjacent t-structure. Moreover, from the classification of the co-t-structures, we can easily obtain the classification of the recollements of $\overline{\mathcal{C}}_m$.

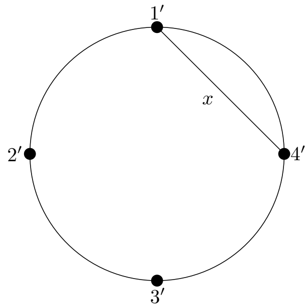


Figure 4.18: The subcategory add$\{x\}$ of $\overline{\mathcal{C}}_4$ is the aisle of a co-t-structure.

4.8.1 Aisles of co-t-structures

Here we classify the aisles of the co-t-structures in $\overline{\mathcal{C}}_m$ similarly to the classification of the aisles of the t-structures in Section 4.7.1. The definition below is similar to Definition 4.7.1.

Definition 4.8.1. A *half-decorated half-non-crossing partition* of $[m'] \cup [m]$ is a pair $(\mathcal{P}, X)$ given by a non-crossing partition $\mathcal{P}$ of $[m']$ and a $2m$-tuple $X = (x_p)_{p \in [m]}$ such that $x_p \in [p, p^+]$ for each $p \in [m]$.

The main result of this section is the following analogue of Theorem 4.7.4. The notation employed in the statement will be defined in Definition 4.8.4 and Definition 4.8.6.

Theorem 4.8.2. *The following is a bijection.*

$$\left\{ \begin{array}{c} \textit{Half-decorated half-non-crossing} \\ \textit{partitions of } [m'] \cup [m] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Aisles of co-t-structures in} \\ \overline{\mathcal{C}}_m \end{array} \right\}$$

$$(\mathcal{P}, X) \longmapsto \pi \mathcal{U}_{(\mathcal{P}, X)}$$

$$(\mathcal{P}_{\pi^{-1}\mathcal{X}}, X_{\pi^{-1}\mathcal{X}}) \longleftarrow \mathcal{X}$$

To prove this result, we proceed as in Section 4.7.1 by taking an intermediate step through $\mathcal{C}_{2m}$. From Corollary 4.6.3 the aisles of co-t-structures in $\overline{\mathcal{C}}_m$ are in bijection with certain subcategories of $\mathcal{C}_{2m}$, which can be regared as “almost aisles” of co-t-structures. These are co-suspended subcategories $\mathcal{U}$ of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering, and are classified with half-decorated half-non-crossing paritions of $[m'] \cup [m]$ in Proposition 4.8.3. The aisles of co-t-structures in $\overline{\mathcal{C}}_m$ are then obtained after localising the “almost aisles” of co-t-structures in $\mathcal{C}_{2m}$. Figure 4.19 illustrates this process.

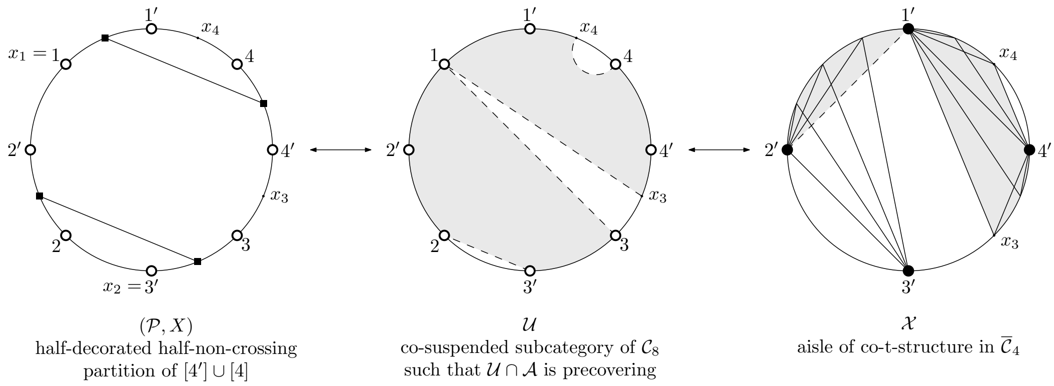


Figure 4.19: Illustration of how to obtain the aisle of a co-t-structure in $\overline{\mathcal{C}}_m$ from a half-decorated half-non-crossing partition of $[m'] \cup [m]$.

The rest of this section is devoted to prove the following proposition. The assignments of the maps α and β will be defined in Definition 4.8.4 and Definition 4.8.6.

Proposition 4.8.3. *The following is a bijection.*

$$\left\{ \begin{array}{c} \textit{Half-decorated half-non-crossing} \\ \textit{partitions of } [m'] \cup [m] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Co-suspended subcategories } \mathcal{U} \subseteq \mathcal{D} \textit{ such that} \\ \mathcal{D} \subseteq \mathcal{U} \textit{ and } \mathcal{U} \cap \mathcal{A} \textit{ is precovering} \end{array} \right\}$$

$$\alpha \colon (\mathcal{P}, X) \longmapsto \mathcal{U}_{(\mathcal{P},X)}$$
$$(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}}) \longleftarrow \mathcal{U} \colon \beta$$

The following definition and lemma define the map α and show that it is well defined.

Definition 4.8.4. Let $(\mathcal{P}, X)$ be a half-decorated half-non-crossing partition of $[m'] \cup [m]$. We define

$$\mathcal{U}_{(\mathcal{P},X)} = \operatorname{add} \bigsqcup_{B \in \mathcal{P}} \left\{ (u_1, u_2) \in \operatorname{ind} \mathcal{C}_{2m} \,\middle|\, u_1, u_2 \in \bigcup_{p \in B} [x_{p^-}, p^+) \right\},$$

where we use the following convention: if $x_{p^-} = p^-$ then $[x_{p^-}, p^+) = \mathbb{Z}^{(p^-)} \sqcup \mathbb{Z}^{(p)}$, and if $x_{p^-} = p$ then $[x_{p^-}, p^+) = \mathbb{Z}^{(p)}$.

Proposition 4.8.5. *Let $(\mathcal{P}, X)$ be a half-decorated half-non-crossing partition of $[m'] \cup [m]$. Then $\mathcal{U}_{(\mathcal{P},X)}$ is a co-suspended subcategory of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering.*

Proof. In order to show that $\mathcal{U}_{(\mathcal{P},X)}$ is extension-closed, contains $\mathcal{D}$, and is closed under Σ^{-1}, we can proceed similarly to the argument of Proposition 4.7.7. We show that $\mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A}$ is precovering. By Theorem 3.3.2 we know that this is equivalent to showing that $\mathcal{U}_{(\mathcal{P},X)}$

satisfies the PC conditions, see Definition 3.3.1.

We check that $\mathcal{U}_{(\mathcal{P},X)}$ satisfies (PC1), the other conditions are analogous. Assume that there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A} \cap \mathbb{Z}^{(p,q)}$ for some $p, q \in [m'] \cup [m]$ such that $p \neq q$ and the sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are strictly increasing. Then $p, q \in [m]$. By Definition 4.8.4 we have that $p^+, q^+ \in B$ for some block $B \in \mathcal{P}$. As a consequence, there exist strictly decreasing sequences $\{y_1^n\}_n \subseteq \mathbb{Z}^{(p^+)}$ and $\{y_2^n\}_n \subseteq \mathbb{Z}^{(q^+)}$ such that $\{|y_1^n, y_2^n|\}_n \subseteq \mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A}$. This proves that (PC1) holds, and concludes the argument. $\square$

With the following definition and proposition we define the map β of Proposition 4.8.3 and we check that it is well defined. Given a co-suspended subcategory $\mathcal{U}$ of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering, we define the equivalence relation $\sim_{\mathcal{U}}$ on the set $[m']$ as in Definition 4.7.8.

Definition 4.8.6. Let $\mathcal{U}$ be a co-suspended subcategory of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering. We define $\mathcal{P}_{\mathcal{U}}$ to be the partition of $[m']$ given by the equivalence classes of $\sim_{\mathcal{U}}$. For each $p \in [m]$ we define

$$x_p = \inf\{z \in \mathbb{Z}^{(p)} \mid \text{there exists } u \in \mathcal{U} \text{ with an endpoint equal to } z\}.$$

We denote by $X_{\mathcal{U}}$ the m-tuple $X_{\mathcal{U}} = (x_p)_{p \in [m]}$.

Proposition 4.8.7. *Keeping the notation of Definition 4.8.6, the pair $(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}})$ is a half-decorated half-non-crossing partition of $[m'] \cup [m]$.*

Proof. We already know that $\mathcal{P}_{\mathcal{U}}$ is a partition of $[m']$, we only need to check that $\mathcal{P}_{\mathcal{U}}$ non-crossing. To this end we can apply the same argument of [22, Lemma 4.12]. $\square$

The following lemma is useful for the argument of Proposition 4.8.3.

Lemma 4.8.8. *Let $\mathcal{U}$ be a co-suspended subcategory of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering. Consider the half-decorated half-non-crossing partition $(\mathcal{P}_{\mathcal{U}}, X_{\mathcal{U}})$ with $X_{\mathcal{U}} = (x_p)_{p \in [m'] \cup [m]}$. Let $p, q \in [m']$ be such that $p, q \in B$ for some block $B \in \mathcal{P}_{\mathcal{U}}$. Then any arc of $\mathcal{C}_{2m}$ having one endpoint in $[x_{p^-}, p^+)$ and the other in $[x_{q^-}, q^+)$ is an arc of $\mathcal{U}$.*

Proof. In order to simplify the notation, we assume that $q \neq 1'$. If $q = 1'$ we can proceed analogously. We denote by $[x_{p^-}, p^+) \times [x_{q^-}, q^+)$ the set of arcs $a = (a_1, a_2) \in \operatorname{ind} \mathcal{C}_{2m}$ such that $a_1 \in [x_{p^-}, p^+)$ and $a_2 \in [x_{q^-}, q^+)$. We show that $[x_{p^-}, p^+) \times [x_{q^-}, q^+) \subseteq \operatorname{ind} \mathcal{U}$. We have the equality

$$[x_{p^-}, p^+) \times [x_{q^-}, q^+) = \mathbb{Z}^{(p,q)} \sqcup \Big([x_{p^-}, p) \times \mathbb{Z}^{(q)}\Big) \sqcup \Big(\mathbb{Z}^{(p)} \times [x_{q^-}, q)\Big) \sqcup \big([x_{p^-}, p) \times [x_{q^-}, q)\big).$$

We assume that $x_{p^-} \neq p$ and $x_{q^-} \neq q$, the other cases are analogous. We divide the proof into steps.

Step 1. We show that $\mathbb{Z}^{(p,q)} \subseteq \operatorname{ind} \mathcal{U}$.

If $p = q$ we have the claim from the fact that $\mathcal{D} \subseteq \mathcal{U}$. Now assume that $p \neq q$. Since $p, q \in B$ for some block $B \in \mathcal{P}_{\mathcal{U}}$, there exists $u \in \operatorname{ind}\mathcal{U}$ such that $u \in \mathbb{Z}^{(p,q)}$. We show that $\Sigma^n u \in \operatorname{ind}\mathcal{U}$ for each $n \in \mathbb{Z}$, then, using the fact that $\mathcal{U}$ is extension-closed, it is straightforward to check that $\mathcal{U}$ contains any arc of $\mathbb{Z}^{(p,q)}$. Since $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$, we already know that $\Sigma^n u \in \mathcal{U}$ for each $n \leq 0$, it remains to check that $\Sigma^n u \in \mathcal{U}$ for each $n \geq 1$. Consider the arcs $a = (u_1 - 1, u_1 + 1) \in \mathbb{Z}^{(p,p)}$ and $b = (u_2 - 1, u_2 + 1) \in \mathbb{Z}^{(q,q)}$. The arcs u and a cross, and then, since $\mathcal{U}$ satisfies the PT condition, $u' = (u_1 - 1, u_2) \in \mathcal{U}$. Moreover, u' and b cross and $u'' = (u_1 - 1, u_2 - 1) = \Sigma u \in \mathcal{U}$. Using this argument we obtain that $\Sigma^n u \in \mathcal{U}$ for each $n \geq 1$. This concludes the argument of Step 1.

Step 2. We show that $[x_{p^-}, p) \times \mathbb{Z}^{(q)} \subseteq \operatorname{ind}\mathcal{U}$.

Let $a = (a_1, a_2) \in \operatorname{ind}\mathcal{C}_{2m}$ with $a_1 \in [x_{p^-}, p)$ and $a_2 \in \mathbb{Z}^{(q)}$, we show that $a \in \mathcal{U}$. First we show that there exists an arc of $\mathcal{U}$ with an endpoint equal to a_1. If there is not such arc, then there is no arc $u \in \operatorname{ind}\mathcal{U}$ with an endpoint in $[x_{p^-}, a_1]$, otherwise $\Sigma^n u \in \mathcal{U}$ has an endpoint equal to a_1 for some $n \leq 0$. Since

$$x_{p^-} = \inf\{z \in \mathbb{Z}^{(p^-)} \mid \text{there exists an arc of } \mathcal{U} \text{ with an endpoint equal to } z\}$$

this gives a contradiction, and therefore there exists an arc of $\mathcal{U}$ with an endpoint equal to a_1. Let u' be such arc, then $\Sigma^n u' \in \mathcal{U}$ for each $n \leq 0$. Moreover, since $\mathcal{U}$ satisfies the PT condition, we obtain that $(a_1, a_1 + 2), (a_1, a_1 + 3), \dots \in \mathcal{U}$. Note that these arcs are also in $\mathcal{A}$ because they belong to $\mathbb{Z}^{(p^-)}$. Therefore we have a sequence $\{(a_1, a_1 + 2 + n)\}_{n \geq 0} \subseteq \operatorname{ind}\mathcal{U} \cap \mathcal{A}$ such that $\{a_2 + 2 + n\}_{n \geq 0}$ is strictly increasing. Since $\mathcal{U} \cap \mathcal{A}$ is precovering and satisfies condition (PC 3), it follows that there exists an arc $v = (a_1, v_2) \in \mathbb{Z}^{(p^-,p)} \cap \mathcal{U}$ such that $v_2 \leq a_2$. Consider an arc of the form $z = (z_1, a_2) \in \mathbb{Z}^{(p,q)}$ with $p < z_1 < v_2$. Since the arcs v and z cross and $\mathbb{Z}^{(p,q)} \subseteq \mathcal{U}$, then $a = (a_1, a_2) \in \mathcal{U}$. This proves Step 2.

Step 3. Analogously as in Step 2 we have that $\mathbb{Z}^{(p)} \times [x_{q^-}, q) \subseteq \operatorname{ind}\mathcal{U}$.

Step 4. We show that $[x_{p^-}, p) \times [x_{q^-}, q) \subseteq \operatorname{ind}\mathcal{U}$.

Let $a = (a_1, a_2) \in [x_{p^-}, p) \times [x_{q^-}, q)$, we show that $a \in \mathcal{U}$. If $p = q$, consider the sequence $\{(a_1, a_1+2+n)\}_{n \geq 0} \subseteq \mathcal{U} \cap \mathcal{A}$ of Step 2. Since $(a_1, a_2) = (a_1, a_1+2+n) \in \mathcal{U}$ for some $n \geq 0$, we have that $a \in \mathcal{U}$. Now assume that $p \neq q$. We consider the arc $v = (a_1, v_2) \in \mathbb{Z}^{(p^-,p)}$ of Step 2, and an arc $z = (z_1, a_2) \in \mathbb{Z}^{(p,q^-)}$ with $p < z_1 < v_2$. Since the arcs v and z cross and $z \in \mathbb{Z}^{(p)} \times [x_{q^-}, q) \subseteq \mathcal{U}$, by Step 3 $a = (a_1, a_2) \in \mathcal{U}$. This concludes the argument of Step 4. We can conclude that $[x_{p^-}, p^+) \times [x_{q^-}, q^+) \subseteq \operatorname{ind}\mathcal{U}$. □

Finally, we can prove Proposition 4.8.3.

Proof of Proposition 4.8.3. From Definition 4.8.4 and Proposition 4.8.5 we have that the maps are well defined. We divide the proof into steps.

Step 1. The map α is injective.

Let $(\mathcal{P}, X)$ and $(\mathcal{Q}, Y)$ be two half-decorated half-non-crossing partitions of $[m'] \cup [m]$ such that $\mathcal{U}_{(\mathcal{P},X)} = \mathcal{U}_{(\mathcal{Q},Y)}$, we show that $(\mathcal{P}, X) = (\mathcal{Q}, Y)$. Assume that $\mathcal{P} \neq \mathcal{Q}$. Then there exist $p, q \in [m']$ with $p \neq q$ such that p and q belong to the same block of $\mathcal{P}$ and to distinct blocks of $\mathcal{Q}$, or vice versa p and q belong to the same block of $\mathcal{Q}$ and to distinct blocks of $\mathcal{P}$. In the first case, there exists an arc of $\mathcal{U}_{(\mathcal{P},X)}$ with an endpoint in $\mathbb{Z}^{(p)}$ and the other in $\mathbb{Z}^{(q)}$, while there is no such arc in $\mathcal{U}_{(\mathcal{Q},Y)}$. As a consequence $\mathcal{U}_{(\mathcal{P},X)} \neq \mathcal{U}_{(\mathcal{Q},Y)}$, giving a contradiction. In the second case the role of $\mathcal{P}$ and $\mathcal{Q}$ exchange and we obtain the same contradiction. Thus we have that $\mathcal{P} = \mathcal{Q}$.

Now we show that $X = Y$. We denote $X = (x_p)_{p\in[m]}$ and $Y = (y_p)_{p\in[m]}$, and we assume that $X \neq Y$. Let $p \in [m]$ be such that $x_p \neq y_p$, then either $x_p < y_p$ or $x_q < y_p$. Assume that $x_p < y_p$, the other case is analogous. Since $p \leq x_p < y_p$, there is an arc of $\mathcal{U}_{(\mathcal{Q},Y)}$ with an endpoint greater that x_p, while there is no such arc in $\mathcal{U}_{(\mathcal{P},X)}$. We obtain that $\mathcal{U}_{(\mathcal{P},X)} \neq \mathcal{U}_{(\mathcal{Q},Y)}$, giving a contradiction. This concludes the argument of Step 1.

Step 2. We show that $\alpha\beta = \mathrm{id}$.

Consider $\mathcal{U}$ a co-suspended subcategory of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{A}$ is precovering. We show that $\mathcal{U} = \mathcal{U}_{(\mathcal{P}_\mathcal{U},X_\mathcal{U})}$. First we show the inclusion $\mathcal{U}_{(\mathcal{P}_\mathcal{U},X_\mathcal{U})} \subseteq \mathcal{U}$. Consider $u = (u_1, u_2) \in \mathrm{ind}\,\mathcal{U}_{(\mathcal{P}_\mathcal{U},X_\mathcal{U})}$, then there exist a block $B \in \mathcal{P}_\mathcal{U}$ and $p, q \in B$ such that $u_1 \in [x_{p^-}, p^+)$ and $u_2 \in [x_{q^-}, q^+)$, where $X_\mathcal{U} = (x_p)_{p\in[m]}$. Then $u \in \mathrm{ind}\,\mathcal{U}$ by Lemma 4.8.8.

Now we show the inclusion $\mathcal{U} \subseteq \mathcal{U}_{(\mathcal{P}_\mathcal{U},X_\mathcal{U})}$. Consider $u = (u_1, u_2) \in \mathrm{ind}\,\mathcal{U}$, then there exist $p, q \in [m']$ such that $u_1 \in [x_{p^-}, p^+)$ and $u_2 \in [x_{q^-}, q^+)$ where

$$x_{p^-} = \inf\{z \in \mathbb{Z}^{(p^-)} \mid \text{there exists an arc of } \mathcal{U} \text{ with an endpoint equal to } z\}$$

and x_{q^-} is defined similarly. We show that $p, q \in B$ for some block $B \in \mathcal{P}_\mathcal{U}$, i.e. that there exists an arc of $\mathcal{U}$ with an endpoint in $\mathbb{Z}^{(p)}$ and the other in $\mathbb{Z}^{(q)}$. Then we can conclude that $u_1, u_2 \in \bigcup_{p\in B}[x_{p^-}, p^+)$, and then $u \in \mathcal{U}_{(\mathcal{P}_\mathcal{U},X_\mathcal{U})}$. If $p = q$ the claim is straightforward, we assume that $p \neq q$. We can write $[x_{p^-}, p^+) = [x_{p^-}, p) \sqcup \mathbb{Z}^{(p)}$ and $[x_{q^-}, q^+) = [x_{q^-}, q) \sqcup \mathbb{Z}^{(q)}$. If $u_1 \in \mathbb{Z}^{(p)}$ and $u_2 \in \mathbb{Z}^{(q)}$ then the claim follows directly. We assume that $u_1 \notin \mathbb{Z}^{(p)}$ or $u_2 \notin \mathbb{Z}^{(q)}$.

Assume that $u_1 \in [x_{p^-}, p)$ and $u_2 \in [x_{q^-}, q)$. We consider the sequence $\{\Sigma^{-n}u = (u_1 + n, u_2 + n)\}_{n\geq 0} \subseteq \mathrm{ind}\,\mathcal{U}$. This sequence is also in $\mathcal{A}$ because it is contained in $\mathbb{Z}^{(p^-,q^-)}$. Since $\mathcal{U} \cap \mathcal{A}$ is precovering and satisfies condition (PC 1), there exists an arc of $\mathcal{U}$ with an endpoint in $\mathbb{Z}^{(p)}$ and the other in $\mathbb{Z}^{(q)}$. This gives the claim.

Now assume that $u_1 \in \mathbb{Z}^{(p)}$ and $u_2 \in [x_{q^-}, q)$. We consider the sequence $\{\Sigma^{-n}u = (u_1+n, u_2+n)\}_{n\geq 0} \subseteq \mathrm{ind}\,\mathcal{U}$. The sequence $\{(u_2, u_2+2+n)\}_{n\geq 0} \subseteq \mathbb{Z}^{(q^-,q^-)}$ is obtained from the crossings of the sequence $\{\Sigma^{-n}u\}_{n\geq 0}$. We have that $\{(u_2, u_2 + 2 + n)\}_{n\geq 0} \subseteq \mathrm{ind}\,\mathcal{U} \cap \mathcal{A}$ because this sequence is contained in $\mathbb{Z}^{(q^-,q^-)}$ and $\mathcal{U}$ satisfies the PT condition. Since $\mathcal{U} \cap \mathcal{A}$ is precovering and satisfies condition (PC 3), there exists an arc $x \in \mathrm{ind}\,\mathcal{U}$ with an endpoint equal to u_2 and the other in $\mathbb{Z}^{(q)}$. The arcs $\Sigma^{-1}u$ and x cross, and from this crossing we obtain an arc $u' \in \mathrm{ind}\,\mathcal{U}$ with an endpoint in $\mathbb{Z}^{(p)}$ and the other endpoint in $\mathbb{Z}^{(q)}$. The case where $u_1 \in [x_{p^-}, p)$ and $u_2 \in \mathbb{Z}^{(q)}$ is analogous, therefore we have the claim.

This concludes the argument of Step 2. We can conclude that the two maps of the claim are mutually inverse. $\square$

4.8.2 Co-aisles of co-t-structures

We compute the co-aisles of co-t-structures in $\overline{\mathcal{C}}_m$ using a method similar to Section 4.7.2. From a half-decorated half-non-crossing partition $(\mathcal{P}, X)$ of $[m'] \cup [m]$, we consider its complement $(\mathcal{P}, X)^c$, obtained from the Kreweras complement $\mathcal{P}^c$ of $\mathcal{P}$, see Section 3.6. This corresponds to a subcategory $\mathcal{V}$ of $\mathcal{C}_{2m}$, which can be thought as an "almost co-aisle" of a co-t-structure in $\mathcal{C}_{2m}$. This is a suspended subcategory $\mathcal{V}$ of $\mathcal{C}_{2m}$ such that $\mathcal{D} \subseteq \mathcal{V}$ and $\mathcal{V} \cap \mathcal{B}$ is preenveloping. The subcategory $\mathcal{V}$ gives a co-aisle of a co-t-structure in $\overline{\mathcal{C}}_m$ after localising. Figure 4.20 illustrates this process.

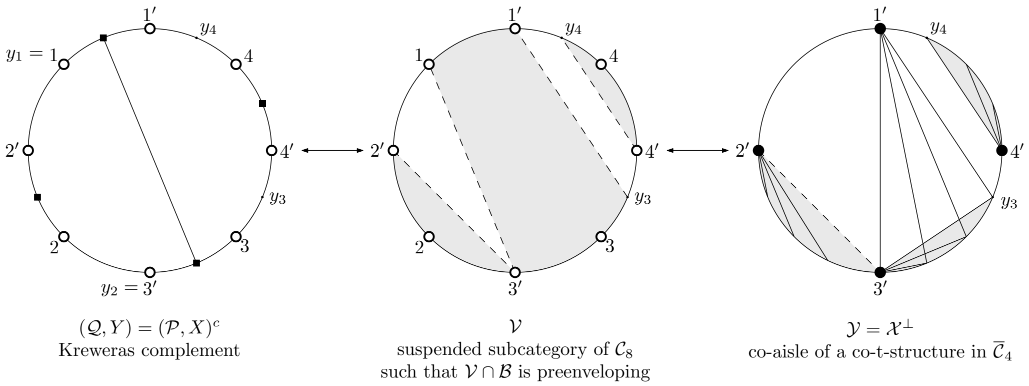


Figure 4.20: Illustration of how to obtain the co-aisle of the aisle of Figure 4.19

Definition 4.8.9. Let $(\mathcal{P}, X)$ be a half-decorated half-non-crossing partition of $[m'] \cup [m]$ with $X = (x_p)_{p \in [m'] \cup [m]}$. We define the *complement*, $(\mathcal{P}, X)^c$, of $(\mathcal{P}, X)$ to be the pair $(\mathcal{Q}, Y)$ where $\mathcal{Q} = \mathcal{P}^c$ is the Kreweras complement of $\mathcal{P}$, and Y is the m-tuple $Y = (y_p)_{p \in [m]}$ where $y_p = x_p - 1$ for each $p \in [m]$.

From the complement of a half-decorated half-non-crossing partition of $[m'] \cup [m]$ we obtain an "almost co-aisle" of co-t-structure in $\mathcal{C}_{2m}$. The following definition is similar to Definition 4.7.16.

Definition 4.8.10. Let $(\mathcal{P}, X)$ be a half-decorated half-non-crossing partition of $[m'] \cup [m]$ and let $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$. We define

$$\mathcal{V}_{(\mathcal{Q},Y)} = \operatorname{add} \bigsqcup_{B \in \mathcal{Q}} \left\{ (v_1, v_2) \in \operatorname{ind} \mathcal{C}_{2m} \,\middle|\, v_1, v_2 \in \bigcup_{p \in B} (p, y_{p^+}] \right\}.$$

Lemma 4.8.11. *Let $(\mathcal{P}, X)$ be a half-decorated half-non-crossing partition of $[m'] \cup [m]$ and let $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$. Then $\mathcal{V}_{(\mathcal{Q},Y)}$ contains $\mathcal{D}$ and is extension-closed.*

Proof. The proof is analogous to the argument of Proposition 4.8.5. $\square$

Consider the complement $(\mathcal{Q}, Y)$ of a half-decorated half-non-crossing partition of $[m'] \cup [m]$. The following lemmas and remark describe some properties of the subcategory $\mathcal{V}_{(\mathcal{Q},Y)}$.

Lemma 4.8.12. *Let $(\mathcal{P}, X)$ be a half-decorated half-non-crossing partition of $[m'] \cup [m]$ and let $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$. Then $\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B} = (\mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A})^{\perp}$.*

Proof. It is straightforward to check that

$$\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B} = \operatorname{add} \bigsqcup_{B \in \mathcal{Q}} \left\{ (v_1, v_2) \in \operatorname{ind} \mathcal{C}_{2m} \,\middle|\, v_1, v_2 \in \bigcup_{p \in B} [w_p^0, y_{p^+}] \right\}$$

where we recall from Section 4.4.1 that $w_p^0 = z_p^0 - 1$ for each $p \in [m']$. We denote the right hand side of the equality by $\mathcal{W}$. Proceeding analogously as in the argument of [22, Corollary 4.14], it is straightforward to check that $\Sigma^{-1}\mathcal{W}$ consists precisely of all the arcs of $\mathcal{C}_{2m}$ which do not cross $\mathcal{U} \cap \mathcal{A}$. As a consequence $\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B} = (\mathcal{U} \cap \mathcal{A})^{\perp}$. □

Remark 4.8.13. Let $\mathcal{U}_{(\mathcal{P},X)}$ and $\mathcal{V}_{(\mathcal{Q},Y)}$ be as in Lemma 4.8.12. Since $\mathcal{U}_{(\mathcal{P},X)} \cap \mathcal{A}$ is precovering and extension-closed, by Proposition 2.3.27, $\big(\mathcal{U} \cap \mathcal{A}, \mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B}\big)$ is a torsion pair. It is not a t-structure nor a co-t-structure because in general $\mathcal{U} \cap \mathcal{A}$ is not closed under Σ or Σ^{-1}, cf. Remark 4.7.19.

Let $(\mathcal{Q}, Y)$ be the complement of a half-decorated half-non-crossing partition of $[m'] \cup [m]$. With the following proposition we prove that by localising $\mathcal{V}_{(\mathcal{Q},Y)}$ we obtain the co-aisle of a co-t-structure in $\overline{\mathcal{C}}_m$. The argument is the same of Proposition 4.7.21.

Proposition 4.8.14. *Let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure in $\overline{\mathcal{C}}_m$, $\mathcal{U} = \pi^{-1}\mathcal{X}$, $(\mathcal{P}, X)$ be its associated half-decorated half-non-crossing partition, and $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$. Then the following equalities hold.*

$$\mathcal{Y} = \pi\mathcal{V}_{(\mathcal{Q},Y)} = \pi\left(\mathcal{V}_{(\mathcal{Q},Y)} \cap \mathcal{B}\right) = \pi\left((\mathcal{U} \cap \mathcal{A})^{\perp}\right)$$

4.8.3 Co-hearts

We classified the aisles of co-t-structures in $\overline{\mathcal{C}}_m$ in Theorem 4.8.2, and we computed the co-aisle of a co-t-structure in Proposition 4.8.14. Here we compute the co-heart of a co-t-structure in $\overline{\mathcal{C}}_m$.

First we introduce some notation. Let $(\mathcal{P}, X)$ be a half-decorated half-non-crossing partition of $[m'] \cup [m]$. Consider $p, q \in [m'] \cup [m]$, we write $q = p^{+_B}$ if

- $p, q \in [m']$, and
- $p, q \in B$ for some block $B \in \mathcal{P}$, and
- q is the next element of $[m'] \cap B$ we meet while moving from p along S^1 in the anticlockwise direction.

If $B = \{p\}$, then by convention $p^{+_B} = p$.

Now let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure in $\overline{\mathcal{C}}_m$. We consider the preimage $(\pi^{-1}\mathcal{X}, \pi^{-1}\mathcal{Y})$ of $(\mathcal{X}, \mathcal{Y})$, which we denote by $(\mathcal{U}, \mathcal{V})$. The pair $(\mathcal{U}, \mathcal{V})$ is not a torsion pair, but $(\mathcal{U} \cap \mathcal{A}, \mathcal{V} \cap \mathcal{B})$ is. Moreover, $(\mathcal{U} \cap \mathcal{A}, \mathcal{V} \cap \mathcal{B})$ is not a co-t-structure, but we can still compute $\mathcal{S}' = (\mathcal{U} \cap \mathcal{A}) \cap \Sigma^{-1}(\mathcal{V} \cap \mathcal{B})$ similarly. The co-heart of $(\mathcal{X}, \mathcal{Y})$ is obtained by localising $\mathcal{S}'$. Figure 4.21 illustrates this process.

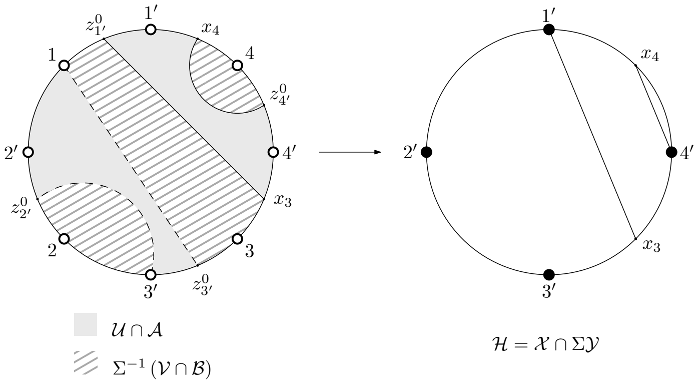


Figure 4.21: Illustration of how to obtain the co-heart of the co-t-structure of Figure 4.19 and Figure 4.20.

In the proposition below we recall that $|z^0_p, x_{q^-}|$ is equal to (z^0_p, x_{q^-}) if $p < q^-$ and is equal to (x_{q^-}, z^0_p) if $q^- < p$, see Section 3.1.

Proposition 4.8.15. *Let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure in $\overline{\mathcal{C}}_m$, $(\mathcal{P}, X)$ be its associated half-decorated half-non-crossing partition of $[m'] \cup [m]$ with $X = (x_p)_{p\in[m]}$, $\mathcal{U} = \pi^{-1}\mathcal{X}$, and $\mathcal{V} = \pi^{-1}\mathcal{Y}$. Then*

$$(\mathcal{U} \cap \mathcal{A}) \cap \Sigma^{-1}(\mathcal{V} \cap \mathcal{B}) = \operatorname{add}\left\{ |z^0_p, x_{q^-}| \middle| p, q \in [m'],\ q = p^{+_B} \text{ for some } B \in \mathcal{P}, x_{q^-} \in \mathbb{Z}^{(q^-)} \right\}.$$

Proof. First we show that arcs of the form $a = |z^0_p, x_{q^-}|$, where $q = p^{+_B}$ for some block $B \in \mathcal{P}$ and $x_{q^-} \in \mathbb{Z}^{(q^-)}$, belong to $(\mathcal{U} \cap \mathcal{A}) \cap \Sigma^{-1}(\mathcal{V} \cap \mathcal{B})$. By Definition 4.4.1 and Definition 4.8.4 we have that $a \in \operatorname{ind}\mathcal{U} \cap \mathcal{A}$, we check that $a \in \operatorname{ind}\Sigma^{-1}(\mathcal{V} \cap \mathcal{B})$. From Lemma 4.8.12 this is equivalent to check that a does not cross any arc $u \in \operatorname{ind}\mathcal{U} \cap \mathcal{A}$. Note that $z^0_p \in [z^0_p, x_{p^+}]$ and $x_{q^-} \in [z^0_{q^{--}}, x_{q^-}]$. Moreover, since $q = p^{+_B}$ for some block $B \in \mathcal{P}$, we have that $p, q^{--} \in C$ for some block $C \in \mathcal{P}^c$, see Section 3.6. From Definition 4.4.9 and Definition 4.8.10 this implies that $a = |z^0_p, x_{q^-}| \in \operatorname{ind}\Sigma^{-1}\mathcal{V} \cap \mathcal{B}$.

Now we show that any arc $a \in \operatorname{ind}(\mathcal{U} \cap \mathcal{A}) \cap \Sigma^{-1}(\mathcal{V} \cap \mathcal{B})$, provided that it exists, is of the form $a = |z^0_p, x_{q^-}|$ with $p, q \in [m']$ such that $q = p^{+_B}$ for some block $B \in \mathcal{P}$, and $x_{q^-} \in \mathbb{Z}^{(q^-)}$. We divide the argument into steps.

Step 1. Let z be an endpoint of a. We show that $z = z^p_0$ for some $p \in [m']$, or $z = x_{p^-}$ for some $p \in [m']$ such that $x_{p^-} \in \mathbb{Z}^{(p^-)}$.

Since $a \in \operatorname{ind}\mathcal{U} \cap \mathcal{A}$, then $z \in [x_{p^-}, z^0_p]$ for some $p \in [m']$, and since $a \in \operatorname{ind}\mathcal{V} \cap \mathcal{B}$, then $z \in [z^0_q, x_{q^+}]$ for some $q \in [m']$. If $z \in (p, z^0_p]$ then $q = p$ and $z = z^0_p$. Therefore we have the claim.

Step 2. Let $p, q \in [m']$ be such that one endpoint of a is of the form x_{p^-} or z^0_p, and the other endpoint is of the form x_{q^-} or z_q^0. We show that $a \ncong |z_p^0, z_q^0|$ and $a \ncong |x_{p^-}, x_{q^-}|$.

If $a \cong |z_p^0, z_q^0|$ then a and Σa are crossing and, since $\Sigma a = |z_p^0 - 1, z_q^0 - 1| \in \operatorname{ind}\mathcal{U} \cap \mathcal{A}$, this gives a contradiction. Similarly, if $a \cong |x_{p^-}, x_{q^-}|$ then a is crossed by $\Sigma^{-1}a = |x_{p^-} + 1, x_{q^-} + 1| \in \operatorname{ind}\mathcal{U} \cap \mathcal{A}$ and we obtain again a contradiction.

Step 3. We know that $a \cong |z_p^0, x_{q^-}|$ for some $p, q \in [m']$ such that $p, q \in B$ for some block $B \in \mathcal{P}$ and $x_{q^-} \in \mathbb{Z}^{(q^-)}$. We show that $q = p^{+_B}$.

Assume that $q \neq p^{+_B}$. Then $B \neq \{p\}$, otherwise $p = q$ and $q = p^{+_B}$. If $p = q$ consider $r \in B \setminus \{p\}$. Then there exists an arc in $\operatorname{ind}\mathcal{U} \cap \mathcal{A}$ with an endpoint in $(p, z_p^0]$ and the other endpoint in $(r, z_0^r]$ which crosses a, and this gives a contradiction. Now assume that $p \neq q$, then there exists $r \in B \setminus \{p, q\}$ such that p, r, q are in cyclic order. The arc $|z_0^r, z_0^q| \in \mathcal{U} \cap \mathcal{A}$ crosses a, and this gives a contradiction. This concludes the argument. $\square$

The following corollary can be proved with the same argument of Corollary 4.7.22.

Corollary 4.8.16. *Let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure in $\overline{\mathcal{C}}_m$. Consider $(\mathcal{P}, X)$ its half-decorated half-non-crossing partition of $[m'] \cup [m]$ with $X = (x_p)_{p \in [m]}$. Then the co-heart $\mathcal{S} = \mathcal{X} \cap \Sigma^{-1}\mathcal{Y}$ is given by*

$$\mathcal{S} = \operatorname{add}\left\{ |p, x_{q^-}| \,\middle|\, p, q \in [m'],\ q = p^{+_B} \text{ for some } B \in \mathcal{P}, \text{ and } x_{q^-} \in \mathbb{Z}^{(q^-)} \right\}.$$

4.8.4 Boundedness

We study the bounded co-t-structures in $\overline{\mathcal{C}}_m$. We find that for $m \geq 2$ there are no bounded co-t-structures.

Proposition 4.8.17. *Let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure in $\overline{\mathcal{C}}_m$, $(\mathcal{P}, X)$ be its associated half-decorated half-non-crossing partition with $X = (x_p)_{p \in [m]}$, and $\mathcal{U} = \pi^{-1}\mathcal{X}$. The following statements are equivalent.*

1. *The co-t-structure $(\mathcal{X}, \mathcal{Y})$ is left bounded in $\overline{\mathcal{C}}_m$.*
2. *We have that $\bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{U} = \mathcal{C}_{2m}$.*
3. *The half-non-crossing partition $\mathcal{P}$ has as unique block $\{1', \ldots, m'\}$ and for each $p \in [m]$ we have that $x_p \neq p^+$.*

Proof. The equivalence between the statements (1) and (2) is straightforward, we show the equivalence between (2) and (3). Assume that $\mathcal{P} = \{\{1', \ldots, m'\}\}$ and $x_p \neq p^+$ for each $p \in [m]$. Let $a = (a_1, a_2) \in \operatorname{ind}\mathcal{C}_{2m}$, we check that $a \in \Sigma^n \mathcal{U}$ for some $n \in \mathbb{Z}$. There exists $n \geq 0$ such that $a_1 + n \in [x_{p^-}, p^+)$ and $a_2 + n \in [x_{q^-}, q^+)$ for some $p, q \in [m']$. Since p and q belong to the same block of $\mathcal{P}$, we have that $\Sigma^{-n}a = (a_1 + n, a_2 + n) \in \mathcal{U}$, and then $a \in \Sigma^n \mathcal{U}$.

Now assume that $\bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{U} = \mathcal{C}_{2m}$, we check that (3) holds. Let $p, q \in [m']$, and consider $a \in \operatorname{ind}\mathcal{C}_{2m}$ with an endpoint in $\mathbb{Z}^{(p)}$ and the other in $\mathbb{Z}^{(q)}$. By assumption there exists $n \in \mathbb{Z}$ such that $a \in \Sigma^n \mathcal{U}$, and then $\Sigma^{-n}a \in \mathcal{U}$. Since the endpoints of $\Sigma^{-n}a$ still belong to $\mathbb{Z}^{(p)}$ and $\mathbb{Z}^{(q)}$, we have that $p, q \in B$ for some block $B \in \mathcal{P}$. This means that any two elements of $[m']$ belong to the same block of $\mathcal{P}$, i.e. $\mathcal{P} = \{\{1', \dots, m'\}\}$. Now, assume that $x_p = p^+$ for some $p \in [m]$. Consider an arc $a \in \mathbb{Z}^{(p,p)}$, we observe that $\Sigma^n a \notin \mathcal{U}$ for each $n \in \mathbb{Z}$, and this gives a contradiction. This concludes the argument. $\square$

Dually, we have the following proposition.

Proposition 4.8.18. *Let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure in $\overline{\mathcal{C}}_m$, let $(\mathcal{P}, X)$ be its associated half-decorated half-non-crossing partition with $X = (x_p)_{p \in [m]}$, and $\mathcal{V} = \pi^{-1}\mathcal{X}$. The following statements are equivalent.*

1. *The co-t-structure $(\mathcal{X}, \mathcal{Y})$ is right bounded in $\overline{\mathcal{C}}_m$.*
2. *We have that $\bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{V} = \mathcal{C}_{2m}$.*
3. *The half-non-crossing partition $\mathcal{P}$ has as blocks $\{1'\}$, …, $\{m'\}$, and $x_p \neq p$ for each $p \in [m]$.*

Corollary 4.8.19. *For each $m \geq 2$ there are no bounded co-t-structures in $\overline{\mathcal{C}}_m$.*

Proof. Assume that $m \geq 2$. If there exists a bounded co-t-structure in $\overline{\mathcal{C}}_m$, then, by Proposition 4.8.17 and Proposition 4.8.18, its associated half-decorated half-non-crossing partition $(\mathcal{P}, X)$ of $[m'] \cup [m]$ is such that $\mathcal{P} = \{1', \dots, m'\} = \{\{1'\}, \dots, \{m'\}\}$, giving a contradiction. Therefore, there are no bounded co-t-structures in $\overline{\mathcal{C}}_m$ if $m \geq 2$. $\square$

4.8.5 Non-degeneracy

We classify the non-degenerate co-t-structures in $\overline{\mathcal{C}}_m$. We find that for $m \geq 2$ there are no non-degenerate co-t-structures. In general it is straightforward to check that left or right bounded co-t-structures are also right or left non-degenerate respectively. We will see that also the converse holds in $\overline{\mathcal{C}}_m$.

Proposition 4.8.20. *Let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure in $\overline{\mathcal{C}}_m$, $(\mathcal{P}, X)$ be its associated half-decorated half-non-crossing partition with $X = (x_p)_{p \in [m'] \cup [m]}$, and $\mathcal{U} = \pi^{-1}\mathcal{X}$. The following statements are equivalent.*

1. *The co-t-structure $(\mathcal{X}, \mathcal{Y})$ is left non-degenerate.*
2. *We have that $\bigcap_{n \in \mathbb{Z}} \Sigma^n \mathcal{U} = \mathcal{D}$.*
3. *The half-non-crossing partition $\mathcal{P}$ has blocks $\{1'\}$, …, $\{m'\}$, and $x_p \neq p$ for each $p \in [m]$.*

Proof. For the equivalence between the statements (1) and (2) we can use the same argument of Proposition 4.7.26. We prove the equivalence between (2) and (3). Assume that $\bigcap_{n \in \mathbb{Z}} \Sigma^n \mathcal{U} = \mathcal{D}$ and that there exist $p, q \in [m']$ such that $p, q \in B$ for some $B \in \mathcal{P}$. Then $\mathcal{U}$ contains any arc having one endpoint in $\mathbb{Z}^{(p)}$ and the other endpoint in $\mathbb{Z}^{(q)}$. Consider such arc u, then $\Sigma^n u \in \mathcal{U}$ for each $n \in \mathbb{Z}$, i.e. $u \in \bigcap_{n \in \mathbb{Z}} \Sigma^n \mathcal{U}$. Then $u \in \mathcal{D}$ and $p = q$. Now assume that there exists $p \in [m]$ such that $x_p = p$, then $\mathcal{U}$ contains any arc $u \in \mathbb{Z}^{(p,p)}$.

Thus, $u \in \bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U} = \mathcal{D}$, and then $u \in \mathbb{Z}^{(q,q)}$ for some $q \in [m']$ and this contradicts the fact that $p \in [m]$. This proves that (3) holds.

Now we assume that statement (3) holds, we check that $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U} \subseteq \mathcal{D}$, the other inclusion is straightforward. Let $u \in \operatorname{ind}\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U}$, then $u \in \mathcal{U}$ and there exist $p, q \in [m']$ such that u has one endpoint in $[x_{p^-}, p^+)$ and the other endpoint in $[x_{q^-}, q^+)$. Then $p, q \in B$ for some block $B \in \mathcal{P}$, and as a consequence $p = q$ and u has both endpoints in $[x_{p^-}, p^+)$. Assume that u has an endpoint in $[x_{p^-}, p)$, then, since $x_p \neq p$, there exists $n \in \mathbb{Z}$ such that $\Sigma^n u \notin \mathcal{U}$, i.e. $u \notin \bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{U}$. Then $u \in \mathbb{Z}^{(p,p)}$, and as a consequence $u \in \mathcal{D}$. This concludes the argument. □

Dually, we have the following proposition.

Proposition 4.8.21. *Let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure of $\overline{\mathcal{C}}_m$, $(\mathcal{P}, X)$ be its associated half-decorated half-non-crossing partition with $X = (x_p)_{p\in[m']\cup[m]}$, and $\mathcal{V} = \pi^{-1}\mathcal{X}$. The following statements are equivalent.*

1. *The co-t-structure $(\mathcal{X}, \mathcal{Y})$ is right non-degenerate.*
2. *We have that $\bigcap_{n\in\mathbb{Z}} \Sigma^n\mathcal{V} = \mathcal{D}$.*
3. *The half-non-crossing partition $\mathcal{P}$ has as unique block $\{1', \ldots, m'\}$ and for each $p \in [m]$ we have that $x_p \neq p^+$.*

Corollary 4.8.22. *For each $m \geq 2$ there are no non-degenerate co-t-structures in $\overline{\mathcal{C}}_m$.*

We also have the following corollary, which combines these results with those in Section 4.8.4.

Corollary 4.8.23. *Let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure in $\overline{\mathcal{C}}_m$. Then $(\mathcal{X}, \mathcal{Y})$ is left bounded if and only if it is right non-degenerate, and $(\mathcal{X}, \mathcal{Y})$ is right-bounded if and only if it is left non-degenerate.*

4.8.6 Adjacent triples

We classify the co-t-structures in $\overline{\mathcal{C}}_m$ having a left adjacent or right adjacent t-structure.

Theorem 4.8.24. *Let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure in $\overline{\mathcal{C}}_m$ and $(\mathcal{P}, X)$ be its associated half-decorated half-non-crossing partition with $X = (x_p)_{p\in[m]}$. The following statements hold.*

1. *The co-t-structure $(\mathcal{X}, \mathcal{Y})$ has a right adjacent t-structure if and only if for each $p \in [m]$ if $x_p = p^+$ then $\{p^+\} \in \mathcal{P}$.*
2. *The co-t-structure $(\mathcal{X}, \mathcal{Y})$ has a left adjacent t-structure if and only if for each $p \in [m]$ if $x_p = p$ then $p^-, p^+ \in B$ for some block $B \in \mathcal{P}$.*

Proof. We prove statement (1), statement (2) is dual. Let $\mathcal{V} = \pi^{-1}\mathcal{Y}$ and $(\mathcal{Q}, Y) = (\mathcal{P}, X)^c$ with $Y = (y_p)_{p\in[m]}$. If $(\mathcal{X}, \mathcal{Y})$ has a right adjacent t-structure, then $\mathcal{Y}$ is precovering and $\mathcal{V} \cap \mathcal{A}$ satisfies the PC conditions, see Theorem 4.4.3. Let $p \in [m]$ be such that $x_p = p^+$, then $y_p = p^+$. We show that $p^-, p^+ \in C$ for some block $C \in \mathcal{Q}$. Since $y_p = p^+$, $\mathcal{V} \cap \mathcal{A}$

contains all the arcs having one endpoints in $(p^-, z^0_{p^-}]$ and the other in $\mathbb{Z}^{(p)}$, see Definition 4.4.1 and Definition 4.8.10. By (PC 3) or (PC 3′) there exists an arc of $\mathcal{V}$ with an endpoint in $\mathbb{Z}^{(p^-)}$ and the other in $\mathbb{Z}^{(p^+)}$. Thus, $p^-, p^+ \in C$ for some block $C \in \mathcal{Q}$. Since $\mathcal{Q} = \mathcal{P}^c$, this is equivalent to $\{p^+\} \in \mathcal{P}$.

Now assume that (2) holds, i.e. if $y_p = p^+$ then $p^-, p^+ \in C$ for some block $C \in \mathcal{Q}$. We show that $\mathcal{Y}$ is precovering, i.e. that $\mathcal{V} \cap \mathcal{A}$ is precovering. We check that $\mathcal{V} \cap \mathcal{A}$ satisfies (PC 1) the other conditions are analogous. Assume that there exists a sequence $\{(v^n_1, v^n_2)\}_n \subseteq \mathcal{V} \cap \mathcal{A} \cap \mathbb{Z}^{(p,q)}$ for some $p, q \in [m'] \cup [m]$ such that $p \neq q$ with $\{v^n_1\}_n$ and $\{v^n_2\}_n$ strictly increasing. Then $p, q \in [m]$ and, since there exist arcs of $\mathcal{V}$ in $\mathbb{Z}^{(p,q)}$, $p^-, q^- \in C$ for some $C \in \mathcal{Q}$. Moreover, $y_p = p^+$ and $y_q = q^+$, and then by assumption $p^-, p^+, q^-, q^+ \in C$. Then, $\mathcal{V} \cap \mathcal{A}$ contains any arc having one endpoint in $(p^+, z^0_{p^+}]$ and the other endpoint in $(q^+, z^0_{q^+}]$. In particular, there exist strictly decreasing sequences $\{w^n_1\}_n \subseteq \mathbb{Z}^{(p^+)}$ and $\{w^n_2\}_n \subseteq \mathbb{Z}^{(q^+)}$ such that $\{|w^n_1, w^n_2|\}_n \subseteq \mathcal{V} \cap \mathcal{A}$. This concludes the argument. $\square$

4.8.7 Recollements

We recall that in a triangulated category recollements are in bijection with TTF triples and functorially finite thick subcategories, see Section 2.3.5. Thick subcategories of $\mathcal{C}_m$ and $\overline{\mathcal{C}}_m$ were classified in [22] and [40] respectively. The category $\mathcal{C}_m$ has only trivial functorially finite thick subcategories, see Remark 3.6.6. This is no longer the case for $\overline{\mathcal{C}}_m$, where there exist non-trivial precovering or preenveloping thick subcategories, see Figure 4.18 for an example.

The following theorem follows directly from Theorem 4.8.2.

Theorem 4.8.25. *Let $(\mathcal{X}, \mathcal{Y})$ be a co-t-structure of $\overline{\mathcal{C}}_m$ and $(\mathcal{P}, X)$ be its associated half-decorated half-non-crossing partition. The following statements are equivalent.*

1. *$\mathcal{X}$ is a precovering thick subcategory.*
2. *$\mathcal{Y}$ is a preenveloping thick subcategory.*
3. *For each $p \in [m]$ either $x_p = p$ or $x_p = p^+$.*

The following corollary combines Theorem 4.8.24 and Theorem 4.8.25.

Corollary 4.8.26. *Let $\mathcal{X}$ be a subcategory of $\overline{\mathcal{C}}_m$. The following statements are equivalent.*

1. *$\mathcal{X}$ is a functorially finite thick subcategory.*
2. *The half-decorated half-non-crossing partition of $[m'] \cup [m]$ associated to the co-t-structure $(\mathcal{X}, \mathcal{X}^{\perp})$, which we denote by $(\mathcal{P}, X)$ with $X = (x_p)_{p \in [m]}$, satisfies the following condition: for each $p \in [m]$ either $x_p = p$ or $x_p = p^+$, and if $x_p = p$ then $p^-, p^+ \in B$ for some block $B \in \mathcal{P}$.*

4.8.8 Functorially finite co-t-structures

We compare the co-t-structures in $\overline{\mathcal{C}}_m$, which we classified in terms of half-decorated half-non-crossing partitions, with the t-structures in $\mathcal{C}_m$, which are classified in terms of decorated non-crossing partitions. Indeed, the similarity of these combinatorial objects suggests there may be a connection between them. We find the following result, which we prove using combinatorial arguments, although we expect there may be an (at least partial) homological argument.

We recall that a co-t-structure is functorially finite if both $\mathcal{X}$ and $\mathcal{Y}$ are functorially finite, i.e. $(\mathcal{X}, \mathcal{Y})$ admits both a left adjacent t-structure and a right adjacent t-structure.

Theorem 4.8.27. *There exist bijections between the following classes of torsion pairs.*

1. *The functorially finite co-t-structures in $\overline{\mathcal{C}}_m$ and the t-structures in $\mathcal{C}_m$.*

2. *The left bounded co-t-structures in $\overline{\mathcal{C}}_m$ and the right bounded t-structures in $\mathcal{C}_m$.*

3. *The right bounded co-t-structures in $\overline{\mathcal{C}}_m$ and the left bounded t-structures in $\mathcal{C}_m$.*

We start with the following observation about decorated non-crossing partitions. We refer to Definition 3.6.2.

Observation 4.8.28. Let $(\mathcal{P}, X)$ be a decorated non-crossing partition of $[m]$. Then $(\mathcal{P}, X)$ can be "embedded" in $[m'] \cup [m]$, i.e. $(\mathcal{P}, X)$ can be regarded as a pair where $\mathcal{P}$ is a non-crossing partition of $[m]$, and $X = (x_p)_{p \in [m]}$ is such that

- if $x_p = p$ then $\{p\} \in \mathcal{P}$, and

- if $x_p = p^+$ then $\{p, p^{++}\} \in B$ for some block $B \in \mathcal{P}$.

We introduce the following terminology.

Definition 4.8.29. A half-decorated half-non-crossing partition of $[m'] \cup [m]$ is called *functorially finite* if it corresponds to a functorially finite co-t-structure.

We have the following characterization of functorially finite half-decorated half-non-crossing partitions.

Remark 4.8.30. By Theorem 4.8.24, a half decorated half-non-crossing partition $(\mathcal{P}, X)$ of $[m'] \cup [m]$, with $X = (x_p)_{p \in [m]}$, is functorially finite if and only if, for each $p \in [m]$, we have that

- if $x_p = p$ then $p^-, p^+ \in B$ for some block $B \in \mathcal{P}$, and

- if $x_p = p^+$ then $\{p^+\} \in \mathcal{P}$.

We recall that in Section 3.5 we defined the notation $\mathcal{P}^c$, ${}^c\mathcal{P}$, $\mathcal{P}^-$, and $\mathcal{P}^+$ for a non-crossing partition $\mathcal{P}$.

Lemma 4.8.31. *The following is a bijection.*

$$\left\{ \begin{array}{c} \textit{Functorially finite half-decorated} \\ \textit{half-non-crossing partitions of} \\ [m'] \cup [m] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Decorated non-crossing} \\ \textit{partitions of } [m] \end{array} \right\}$$

$$\alpha\colon (\mathcal{P}, X) \longmapsto ((\mathcal{P}^c)^+, X)$$
$$((^c\mathcal{P})^-, X) \longleftarrow (\mathcal{P}, X)\colon \beta$$

Proof. We check that the map α is well defined, for β the proof is analogous. Since $\mathcal{P}$, and thus $\mathcal{P}^c$, is a non-crossing partition of $[m']$, by Observation 4.8.28, $(\mathcal{P}^c)^+$ is a non-crossing partition of $[m]$. Now let $p \in [m]$. If $x_p = p$ then $p^-, p^+ \in B$ for some block $B \in \mathcal{P}$, i.e. $\{p^-\} \in \mathcal{P}^c$ and as a consequence $\{p\} \in (\mathcal{P}^c)^+$. Similarly, if $x_p = p^+$ then $\{p^+\} \in \mathcal{P}$, i.e. $p^-, p^+ \in C$ for some $C \in \mathcal{P}^c$, and then $p, p^{++} \in C^+$ which is a block of $(\mathcal{P}^c)^+$. This proves that α is well defined. The fact that α and β are mutually inverse is straightforward to check and follows from the fact that $^c(\mathcal{P}^c) = (^c\mathcal{P})^c$, see Section 3.5. $\square$

Now we can prove our result.

Proof of Theorem 4.8.27. We prove statement (1). The functorially finite co-t-structures in $\overline{\mathcal{C}}_m$ are classified in terms of functorially finite half-decorated half-non-crossing partitions of $[m'] \cup [m]$, which, by Lemma 4.8.31, are in bijection with decorated non-crossing partitions of $[m]$. Since these are in bijection with the t-structures in $\mathcal{C}_m$, we obtain the claim.

Now we prove (2), (3) is dual. By Proposition 4.8.17, left bounded co-t-structures in $\overline{\mathcal{C}}_m$ are in bijection with half-decorated half-non-crossing partitions $(\mathcal{P}, X)$ of $[m'] \cup [m]$ of the form: $\mathcal{P} = \{1', \dots, m'\}$ and $X = (x_p)_{p\in[m]}$ is such that $x_p \neq p^+$ for each $p \in [m]$. Moroever, by Proposition 3.6.5, right bounded t-structures in $\mathcal{C}_m$ are in bijection with decorated non-crossing partitions $(\mathcal{P}, X)$ of $[m]$ of the form $\mathcal{P} = \{\{1\}, \dots, \{m\}\}$ and $X = (x_p)_{p\in[m]}$ is such that $x_p \neq p^+$ for each $p \in [m]$. It is straightforward to check that the maps α and β of Lemma 4.8.31 restrict to these subsets, and therefore we have a well-defined bijection between them. This concludes the proof. $\square$

Chapter 5

Higher-CY discrete cluster categories

We introduce the category $\mathcal{C}_{w,m}$, which is a candidate higher-CY version of $\mathcal{C}_m$. When $m = 1$, $\mathcal{C}_{w,1}$ is equivalent as an additive category to the Holm–Jørgensen category $\mathcal{T}_w$, studied in [27]. We define $\mathcal{C}_{w,m}$ as a subcategory of $\mathcal{C}_m$ which is not full: it consists of the *w-admissible* objects and morphisms of $\mathcal{C}_m$, where being w-admissible is a property defined combinatorially. Under the assumption that a certain conjecture holds, the restriction to $\mathcal{C}_{w,m}$ of the triangles and shift functor of $\mathcal{C}_m$ form a triangulated structure for $\mathcal{C}_{w,m}$. Assuming this conjecture, we prove that $\mathcal{C}_{w,m}$ is w-CY and we describe its AR quiver. We also classify the precovering subcategories, the (weakly) w-cluster tilting subcategories, and the torsion pairs in $\mathcal{C}_{w,m}$, generalising some existing results in [27] and [14] for the case $m = 1$, and in [21] for $w = 2$.

5.1 The category $\mathcal{C}_{w,m}$

For the rest of this chapter we denote $\mathcal{C}_m$ by $\mathcal{C}_{2,m}$. We fix an integer $w \geq 2$ and we introduce the category $\mathcal{C}_{w,m}$. We start by defining *w-admissible* objects and morphisms.

Definition 5.1.1 ([27, Definition 2.3])**.** An indecomposable object $a = (a_1, a_2) \in \operatorname{ind}\mathcal{C}_{2,m}$ is called *w-admissible* $a_2 - a_1 \equiv 1 \mod (w-1)$. An object of $\mathcal{C}_{2,m}$ is called *w-admissible* if it is the zero object, or if all its indecomposable direct summands are w-admissible.

We check that the property of being w-admissible is preserved under isomorphism.

Lemma 5.1.2. *Let $a, b \in \mathcal{C}_{2,m}$. If a is w-admissible and $a \cong b$, then b is w-admissible.*

Proof. Since $\mathcal{C}_{2,m}$ is Krull–Schmidt, we can further assume that a and b are indecomposable. The claim follows because, since $a \cong b$, a and b have the same coordinates. $\square$

We introduce the following concept before defining w-admissible morphisms.

Definition 5.1.3. Let $a = (a_1, a_2), b = (b_1, b_2) \in \operatorname{ind} \mathcal{C}_{2,m}$ be w-admissible. We say that b is *w-compatible* with a if $b \in H^+(a) \cup H^-(\Sigma^2 a)$, and

- if $b \in H^+(a)$, we have $b_1 - a_1 \equiv 0 \mod (w-1)$,
- if $b \in H^-(\Sigma^2 a)$, we have $a_1 - b_1 \equiv 1 \mod (w-1)$.

Definition 5.1.4. Let $a, b \in \operatorname{ind} \mathcal{C}_{2,m}$ and let $f \in \operatorname{Hom}_{\mathcal{C}_{2,m}}(a, b)$. We say that f is *w-admissible* if

- $f = 0$ and the objects a and b are w-admissible;
- $f \neq 0$, the objects a and b are w-admissible and b is w-compatible with a.

Let $a_1, \ldots, a_n, b_1, \ldots, b_m \in \operatorname{ind} C_{2,m}$. A morphism $f\colon \bigoplus_{j=1}^n a_j \to \bigoplus_{i=1}^m b_i$ of $C_{2,m}$ is called *w-admissible* if its entries $f_{i,j}\colon a_j \to b_i$ are w-admissible for each i and j.

We prove that the class of w-admissible morphisms is closed under composition.

Proposition 5.1.5. *The composition of two w-admissible morphisms is a w-admissible morphism.*

Proof. Let $a, b, c \in \mathcal{C}_{2,m}$ be w-admissible, let $f\colon a \to b$ and $g\colon b \to c$ be w-admissible. We show that $gf\colon a \to c$ is w-admissible. We first assume that $a = (a_1, a_2)$, $b = (b_1, b_2)$, and $c = (c_1, c_2)$ are indecomposable.

If $gf = 0$ then it is w-admissible. Now assume that $gf \neq 0$. We prove that c is w-compatible with a. Since $f \neq 0$ and $g \neq 0$, $b \in H^+(a) \cup H^-(\Sigma^2 a)$ and $c \in (H^+(b) \cup H^-(\Sigma^2 b)) \cap (H^+(a) \cup H^-(\Sigma^2 a))$. By Lemma 3.2.7 we have the following possibilities: $b \in H^+(a)$ and $c \in H^+(a) \cap H^+(b)$, $b \in H^+(a)$ and $c \in H^-(\Sigma^2 a) \cap H^+(\Sigma^2 b)$, and $b \in H^-(\Sigma^2 a)$ and $c \in H^-(\Sigma^2 a) \cap H^+(b)$. In the first case we have that $c_1 - a_1 = (c_1 - b_1) + (b_1 - a_1) \equiv 0 \mod (w-1)$, and then c is w-compatible with a, the other cases are analogous. Therefore, $gf\colon a \to c$ is w-admissible.

Now we remove the assumption that a, b, and c are indecomposable and we consider their decomposition into indecomposable w-admissible direct summands: $a = \bigoplus_{j=1}^n a_j$, $b = \bigoplus_{i=1}^l b_i$, $c = \bigoplus_{h=1}^k c_h$. Consider an entry $(gf)_{hj}\colon a_j \to c_h$ of gf and assume that $(gf)_{hj} \neq 0$. We show that c_h is w-compatible with a_j. Note that $0 \neq (gf)_{hj} = \sum_{i=1}^l g_{hi} f_{ij}$ implies that $g_{hi} f_{ij} \neq 0$ for some $1 \leq i \leq m$. As a consequence, since $0 \neq g_{hi} f_{ij}\colon a_j \to c_h$ is w-admissible, then c_h is w-compatible with a_j. We obtain that $(gf)_{hj}\colon a_j \to c_h$ is w-admissible. We can conclude that gf is w-admissible beacuse all its entries are w-admissible. $\square$

We introduce the category $\mathcal{C}_{w,m}$ as a subcategory of $\mathcal{C}_{2,m}$, though not a full one. Indeed, the Hom-sets of $\mathcal{C}_{w,m}$ are vector subspaces of the Hom-spaces of $\mathcal{C}_{2,m}$.

Definition 5.1.6. We define $\mathcal{C}_w$ as the subcategory of $C_{2,m}$ having as objects the w-admissible objects of $\mathcal{C}_{2,m}$ and as morphisms the w-admissible morphisms of $\mathcal{C}_{2,m}$.

Proposition 5.1.7. *Let $a, b \in \mathcal{C}_{w,m}$. Then $\operatorname{Hom}_{\mathcal{C}_{w,m}}(a, b)$ is a subspace of $\operatorname{Hom}_{\mathcal{C}_{2,m}}(a, b)$.*

Proof. Let $f, g \in \operatorname{Hom}_{\mathcal{C}_{w,m}}(a,b)$ and $\lambda \in K$. We have to show that $f+\lambda g \in \operatorname{Hom}_{\mathcal{C}_{w,m}}(a,b)$. We consider the decompositions of a and b into w-admissible indecomposable direct summands: $a = \bigoplus_{j=1}^{n} a_j$ and $b = \bigoplus_{i=1}^{l} b_i$. We show that all the entries $(f+\lambda g)_{i,j}$ of $f+\lambda g$ are w-admissible. We fix i and j. If $(f+\lambda g)_{i,j} = 0$ then it is w-admissible. If $f_{i,j} + \lambda g_{i,j} = (f+\lambda g)_{i,j} \neq 0$, then $f_{i,j} \neq 0$ or $g_{i,j} \neq 0$. Since $f_{i,j}, g_{i,j} \colon a_j \to b_i$ are w-admissible, it follows that b_i is w-compatible with a_j, and then $(f+\lambda g)_{i,j} \colon a_j \to b_i$ is w-admissible. □

Proposition 5.1.8. *Let $a, b \in \operatorname{ind} \mathcal{C}_{w,m}$. Then*

$$\operatorname{Hom}_{\mathcal{C}_w}(a,b) \cong \begin{cases} \mathbb{K} & \text{if } b \in H^+(a) \cup H^-(\Sigma^2 a) \text{ and } b \text{ is } w\text{-compatible with } a, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $b \notin H^+(a) \cup H^-(\Sigma^2 a)$ then, since $\operatorname{Hom}_{\mathcal{C}_{2,m}}(a,b) = 0$, we obtain $\operatorname{Hom}_{\mathcal{C}_{w,m}}(a,b) = 0$. Assume that $b \in H^+(a) \cup H^-(\Sigma^2 a)$, then $\operatorname{Hom}_{\mathcal{C}_{2,m}}(a,b) \cong \mathbb{K}$. If b is w-compatible with a, then any morphism of $\operatorname{Hom}_{\mathcal{C}_{2,m}}(a,b)$ is w-admissible, i.e. $\operatorname{Hom}_{\mathcal{C}_{w,m}}(a,b) = \operatorname{Hom}_{\mathcal{C}_{2,m}}(a,b) \cong \mathbb{K}$. If b is not w-compatible with a, then the only w-admissible morphism of $\operatorname{Hom}_{\mathcal{C}_{2,m}}(a,b)$ is zero, i.e. $\operatorname{Hom}_{\mathcal{C}_{w,m}}(a,b) = 0$. □

5.2 Triangulated structure

We observe that the category $\mathcal{C}_{w,m}$ is $\mathbb{K}$-linear, Hom-finite, and Krull–Schmidt, see Section 5.1. Now we want to prove that $\mathcal{C}_{w,m}$ has a triangulated structure. To do so, we first define the shift functor and the collection of triangles. The shift functor $\Sigma \colon \mathcal{C}_{w,m} \to \mathcal{C}_{w,m}$ is defined as the restriction of the shift functor $\Sigma \colon \mathcal{C}_{2,m} \to \mathcal{C}_{2,m}$ to $\mathcal{C}_{w,m}$. We have to check that this functor is well defined.

Lemma 5.2.1. *Let Σ be the restriction of the shift functor $\Sigma \colon \mathcal{C}_{2,m} \to \mathcal{C}_{2,m}$ to the subcategory $\mathcal{C}_{w,m}$. Then Σ sends objects of $\mathcal{C}_{w,m}$ to objects of $\mathcal{C}_{w,m}$, and morphisms of $\mathcal{C}_{w,m}$ to morphisms of $\mathcal{C}_{w,m}$.*

Proof. By the additivity of Σ it is enough to consider indecomposable objects and morphisms between indecomposable objects. Consider $a = (a_1, a_2)$ an indecomposable w-admissible object, it is straightforward to check that $(a_2 - 1) - (a_1 - 1) = a_2 - a_1 \equiv 1 \mod (w-1)$, i.e. Σa is still w-admissible. Now consider $a, b \in \operatorname{ind} \mathcal{C}_{w,m}$ and a w-admissible morphism $f \colon a \to b$. If $f = 0$ then $\Sigma f = 0$ is w-admissible. If $f \neq 0$ then b is w-compatible with a and it is straightforward to check that Σb is w-compatible with Σa. As a consequence, Σf is a w-admissible morphism. □

Now we define a triangulated structure of $\mathcal{C}_{w,m}$, from the triangles of $\mathcal{C}_{2,m}$ which are *w-admissible.*

Definition 5.2.2. A triangle $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ of $\mathcal{C}_{2,m}$ is called *w-admissible* if a, b and c are w-admissible objects and f and g are w-admissible morphisms.

We want to prove the w-admissible triangles and the shift functor give a triangulated structure of $\mathcal{C}_{w,m}$, see Theorem 5.2.17. This result will rely on some conjectures, see Section 5.2.2.

5.2.1 Making morphisms admissible

Before discussing Theorem 5.2.17, we need some technical lemmas about admissible morphisms. We start with the following.

Lemma 5.2.3. *Let $a, b, c \in \operatorname{ind}\mathcal{C}_{w,m}$, $f\colon a \to b$, and $g\colon b \to c$ be morphisms such that gf is w-admissible and non-zero. Then f is w-admissible if and only if g is w-admissible.*

Proof. We prove that if f is w-admissible then g is w-admissible, the other implication is analogous. Since $gf \neq 0$, by Lemma 3.2.7, we have the following possibilities: $b \in H^+(a)$ and $c \in H^+(a) \cap H^+(b)$, $b \in H^+(a)$ and $c \in H^-(\Sigma^2 b) \cap H^-(\Sigma^2 a)$, $b \in H^-(\Sigma^2 a)$ and $c \in H^+(b) \cap H^-(\Sigma^2 a)$. Assume that the first case holds, the other cases are analogous. We have that $c_1 - b_1 = (c_1 - a_1) - (b_1 - a_1) \equiv 0 \mod (w-1)$ because $c_1 - a_1 \equiv 0 \mod (w-1)$ and the same holds for $b_1 - a_1$. Thus, c is w-compatible with b and then g is w-admissible. $\square$

From a morphism of $\mathcal{C}_{2,m}$ between two w-admissible objects, we can obtain a w-admissible morphism by taking its *w-admissible part.*

Definition 5.2.4. Let $a_1, \ldots, a_n, b_1, \ldots, b_k \in \operatorname{ind}\mathcal{C}_{w,m}$ and $f = (f_{i,j})_{i,j}\colon \bigoplus_{i=1}^n a_j \to \bigoplus_{i=1}^k b_i$. We define $\overline{f}$, the *w-admissible part* of f, as

$$(\overline{f})_{i,j} = \begin{cases} f_{i,j} & \text{if } f_{i,j} \text{ is } w\text{-admissible,} \\ 0 & \text{otherwise.} \end{cases}$$

The following lemmas describe how the w-admissible part of a morphism behaves with respect to composition. Note that in the lemma below the objects a, b and c are not necessarily indecomposable.

Lemma 5.2.5. *Let $a, b, c \in \mathcal{C}_{w,m}$, $f\colon a \to b$, and $g\colon b \to c$ be morphisms in $\mathcal{C}_{2,m}$. The following statements hold.*

1. *If f is w-admissible and gf is w-admissible, then $gf = \overline{g}f$.*

2. *If g is w-admissible and gf is w-admissible, then $gf = g\overline{f}$.*

Proof. We prove statement (1), statement (2) is analogous. We write the decomposition of a, b and c into indecomposable w-admissible direct summands as: $a = \bigoplus_{j=1}^n a_j$, $b = \bigoplus_{i=1}^k b_i$, and $c = \bigoplus_{h=1}^k c_h$. We write the morphisms as: $f = (f_{i,j})_{i,j}$ and $g = (g_{h,i})_{h,i}$. We fix h and j, we prove that $(\overline{g}f)_{h,j} = (gf)_{h,j}$. We define the following sets of indices:

$$I_1 = \{1 \leq i \leq k \mid g_{h,i} \text{ is } w\text{-admissible and } g_{h,i}f_{i,j} \neq 0\}, \text{ and}$$
$$I_2 = \{1 \leq i \leq k \mid g_{h,i} \text{ is not } w\text{-admissible and } g_{h,i}f_{i,j} \neq 0\}.$$

Note that we can write $(gf)_{h,j}$ and $(\overline{g}f)_{h,j}$ as follows:

$$(gf)_{h,j} = \sum_{i=1}^{k} g_{h,i}f_{i,j} = \sum_{i\in I_1} g_{h,i}f_{i,j} + \sum_{i\in I_2} g_{h,i}f_{i,j}, \text{ and}$$
$$(\overline{g}f)_{h,j} = \sum_{i=1}^{k} \overline{g}_{h,i}f_{i,j} = \sum_{i\in I_1} g_{h,i}f_{i,j}.$$

We denote $S_1 = \sum_{i\in I_1} g_{h,i}f_{i,j}$ and $S_2 = \sum_{i\in I_2} g_{h,i}f_{i,j}$. We prove that $S_2 = 0$. Assume that $S_2 \neq 0$. If $(gf)_{h,j} \neq 0$ then c_h is w-compatible with a_j. Moreover, since $S_2 \neq 0$, there exists i such that $g_{h,i}$ is not w-admissible and $g_{h,i}f_{i,j} \neq 0$. Since c_h is w-compatible with a_j, $g_{h,i}f_{i,j}$ is w-admissible. By Lemma 5.2.3 we have that $g_{h,i}$ is w-admissible, giving a contradiction.

Now, if $(gf)_{h,j} = 0$, we have that $S_1 = -S_2 \neq 0$. As a consequence, since $S_1 \neq 0$, there exists i such that $g_{h,i}$ is w-admissible and $g_{h,i}f_{i,j} \neq 0$. Since both $g_{h,i}$ and $f_{i,j}$ are w-admissible, c_h is w-compatible with a_j. This contradicts the fact that $S_2 \neq 0$ as above. Therefore, we can conclude that $S_2 = 0$. This implies that $(gf)_{h,j} = (\overline{g}f)_{h,j}$. $\square$

Lemma 5.2.6. *Let $a, b, c \in \mathcal{C}_{w,m}$, $f\colon a \to b$ and $g\colon b \to c$ be morphisms. The following statements hold.*

1. *If f is w-admissible, then $\overline{gf} = \overline{g}f$.*
2. *If g is w-admissible, then $\overline{gf} = g\overline{f}$.*

Proof. We prove statement (1), statement (2) is analogous. Keeping the same notation of the argument of Lemma 5.2.5, for each pair of indices h and j we have that

$$(gf)_{h,j} = \sum_{i=1}^{k} g_{h,i}f_{i,j} = \sum_{i\in I_1} g_{h,i}f_{i,j} + \sum_{i\in I_2} g_{h,i}f_{i,j},$$
$$(\overline{gf})_{h,j} = \sum_{i=1}^{k} \overline{g_{h,i}f_{i,j}} = \sum_{i\in I_1} \overline{g_{h,i}f_{i,j}} + \sum_{i\in I_2} \overline{g_{h,i}f_{i,j}}, \text{ and}$$
$$(\overline{g}f)_{h,j} = \sum_{i=1}^{k} \overline{g}_{h,i}f_{i,j} = \sum_{i\in I_1} g_{h,i}f_{i,j}.$$

Note that for each $i \in I_1$ we have that both $g_{j,i}$ and $f_{i,j}$ are w-admissible, and then $\overline{g_{h,i}f_{i,j}} = g_{h,i}f_{i,j}$. Moreover, for each $i \in I_2$, by Lemma 5.2.3, we have that $g_{h,i}f_{i,j}$ is not w-admssible, and then $\overline{g_{h,i}f_{i,j}} = 0$. Thus, we obtain that $(\overline{gf})_{h,j} = (\overline{g}f)_{h,j}$. Therefore, we can conclude that $\overline{gf} = \overline{g}f$. $\square$

With the following lemmas we prove that the w-admissible part of an isomorphism between w-admissible objects is still an isomorphism.

Lemma 5.2.7. *Let n_1 and n_2 be positive integers, $a_1, a_2 \in \operatorname{ind}\mathcal{C}_{w,m}$ be such that $a_1 \ncong a_2$, $f\colon a_1^{n_1} \to a_2^{n_2}$ and $g\colon a_2^{n_2} \to a_1^{n_1}$ be morphisms. Then $gf = 0$ and $fg = 0$.*

Proof. We prove that $gf = 0$, the other claim is similar. We write $f = (f_{i,j})_{i,j}$ and $g = (g_{h,j})_{h,j}$. For each h and j we have that $(gf)_{h,j} = \sum_{i=1}^{n_2} g_{h,i}f_{i,j}$. Note that $g_{h,i}f_{i,j}\colon a_1 \to a_1$ is zero, otherwise $f_{i,j}\colon a_1 \to a_2$ is a split monomorphism, which contradicts the fact that $a_1 \ncong a_2$. Thus, all the entries $(gf)_{h,j}$ are equal to zero. □

Lemma 5.2.8. *Let $a \in \mathcal{C}_{w,m}$. If $\varphi\colon a \to a$ is an isomorphism, then $\overline{\varphi}$ is an isomorphism.*

Proof. After reordering the indecomposable direct summands of a, we can write $a = \bigoplus_{i=1}^{n} a_i^{k_i}$, where $a_1, \dots, a_n \in \operatorname{ind}\mathcal{C}_{w,m}$ are pairwise non-isomorphic. We write $\varphi = (\varphi_{i,j})_{i,j}$ where $\varphi_{i,j}\colon a_j^{k_j} \to a_i^{k_i}$. We proceed by induction on n.

If $n = 1$, it is straightforward to check that $\varphi\colon a_1^{k_1} \to a_1^{k_1}$ is w-admissible, and then $\overline{\varphi} = \varphi$ is an isomorphism. If $n \geq 2$, we define $a' = \bigoplus_{i=1}^{n-1} a_i^{k_i}$, and we write: $a = a' \oplus a_n^{k_n}$ and $\varphi = \left(\begin{smallmatrix} \varphi'_{1,1} & \varphi'_{1,2} \\ \varphi'_{2,1} & \varphi_{n,n} \end{smallmatrix}\right)$, where $\varphi'_{1,1}\colon a' \to a'$, $\varphi'_{1,2}\colon a_n^{k_n} \to a'$, and $\varphi'_{2,1}\colon a' \to a_n^{k_n}$. We prove that $\varphi'_{1,1}$ and $\varphi_{n,n}$ are isomorphisms.

Since φ is invertible, there exists φ^{-1}, which we denote by $\psi = \left(\begin{smallmatrix} \psi'_{1,1} & \psi'_{1,2} \\ \psi'_{2,1} & \psi_{n,n} \end{smallmatrix}\right)$. Thus, by Lemma 5.2.7, we have that $1_{a'} = \varphi'_{1,1}\psi'_{1,1} + \varphi'_{1,2}\psi'_{2,1} = \varphi'_{1,1}\psi'_{1,1}$ and $1_{a'} = \psi'_{1,1}\varphi'_{1,1} + \psi'_{1,2}\varphi'_{2,1} = \psi'_{1,1}\varphi'_{1,1}$. As a consequence $\varphi'_{1,1}$ is an isomorphism, and similarly we obtain that $\varphi_{n,n}$ is an isomorphism.

By the induction hypothesis, we have that $\overline{\varphi'}_{1,1}$ and $\overline{\varphi}_{n,n} = \varphi_{n,n}$ have inverses, which we denote by $\alpha_{1,1}$ and $\alpha_{n,n}$ respectively. Now, we define $\alpha_{1,2} = -\alpha_{1,1}\overline{\varphi'}_{1,2}\alpha_{n,n}$ and $\alpha_{2,1} = -\alpha_{2,2}\overline{\varphi'}_{2,1}\alpha_{n,n}$. Consider the morphism $\alpha = \left(\begin{smallmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{n,n} \end{smallmatrix}\right)$. It is straightforward to check that $\alpha\overline{\varphi} = \left(\begin{smallmatrix} 1_{a'} & 0 \\ 0 & 1_{a_n^{k_n}} \end{smallmatrix}\right) = \overline{\varphi}\alpha$. We can conclude that $\overline{\varphi}$ is an isomorphism. □

Lemma 5.2.9. *Let φ be a w-admissible isomorphism, then φ^{-1} is w-admissible.*

Proof. We denote $\psi = \varphi^{-1}$. Since $\varphi\psi = 1$ is w-admissible, then, by Lemma 5.2.5, $\varphi\overline{\psi} = \varphi\psi = 1$. Similarly, we also obtain that $\overline{\psi}\varphi = 1$ and therefore $\psi = \overline{\psi}$, i.e. ψ is w-admissible. □

5.2.2 The conjecture

In Theorem 5.2.17 we want to prove that the functor $\Sigma\colon \mathcal{C}_{w,m} \to \mathcal{C}_{w,m}$ and the w-admissible triangles, from Definition 5.2.2, form a triangulated structure for $\mathcal{C}_{w,m}$. We can prove this result under the assumption that the following conjecture holds.

Conjecture 5.2.10. Any w-admissible morphism $h\colon b \to \Sigma a$ can be extended to a w-admissible triangle.

We prove Conjecture 5.2.10 for some cases. We first consider the case of a triangle having indecomposable outer terms, then the triangles having only first term indecomposable.

5.2.3 Triangles having indecomposable outer terms

Lemma 5.2.11. *Let $a, b \in \operatorname{ind} \mathcal{C}_{w,m}$ be such that $b \ncong \Sigma a$, and $h\colon b \to \Sigma a$ be a non-zero w-admissible morphism. Then any triangle $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ is w-admissible.*

Proof. Consider a triangle $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$, we prove that e, f, and g are w-admissible. We recall that the middle term e is either indecomposable or has two indecomposable summands, see Section 3.4. Assume that e has two indecomposable summands, for the other case we can proceed similarly. We write $f = \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) : a \to e_1 \oplus e_2$ and $g = \left(\begin{smallmatrix} g_1 & g_2 \end{smallmatrix}\right) : e_1 \oplus e_2 \to b$. Since $a = (a_1, a_2)$ and $b = (b_1, b_2)$ cross, we have the following possibilities: $a_1 < b_1 < a_2 < b_2$, i.e. $\Sigma a \in H^-(\Sigma^2 b)$, or $b_1 < a_1 < b_2 < a_2$, i.e. $\Sigma a \in H^+(b)$, see Section 3.4. Assume that the first case holds, the other case is analogous. We have that $e \cong e_1 \oplus e_2$ with $e_1 = (a_1, b_2)$ and $e_2 = (b_1, a_2)$.

Note that e_1 and e_2 are w-admissible, indeed $b_2 - a_1 = (b_2 - b_1) + (b_1 - (a_1 - 1)) - 1 \equiv 1 \mod (w-1)$ and $a_2 - b_1 = (a_2 - a_1) - (b_1 - (a_1 - 1)) + 1 \equiv 1 \mod (w-1)$. Moreover, since $e_1, e_2 \in H^+(a)$ and $b \in H^+(e_1) \cap H^+(e_2)$, it is straightforward to check that e_1 and e_2 are w-compatible with a, and b is w-compatible with e_1 and e_2. We can conclude that f_1, f_2, g_1, and g_2 are w-admissible, i.e. f and g are w-admissible. Thus, $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ is a w-admissible triangle. □

Proposition 5.2.12. *Let $a, b \in \operatorname{ind} \mathcal{C}_{w,m}$ and $h\colon b \to \Sigma a$ be a w-admissible morphism. Then there exists a w-admissible triangle $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$.*

Proof. If $h = 0$ or h is an isomorphism, then $a \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} a \oplus b \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)} b \xrightarrow{0} \Sigma a$ or $a \longrightarrow 0 \longrightarrow b \xrightarrow{h} \Sigma a$ are, respectively, w-admissible triangles. Now assume that h is not an isomorphism and $h \neq 0$. Consider a triangle $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ in $\mathcal{C}_{2,m}$, by Lemma 5.2.11 we have that it is w-admissible. □

5.2.4 Triangles having one indecomposable outer term

Lemma 5.2.13. *Let $a, b_1, \dots, b_n \in \operatorname{ind} \mathcal{C}_{w,m}$ be such that the objects $b_1, \dots, b_n$ are pairwise* Hom*-orthogonal and $b_i \ncong \Sigma a$ for each i. Let $b = \bigoplus_{i=1}^n b_i$ and $h = \left(\begin{smallmatrix} h_1 & \dots & h_n \end{smallmatrix}\right) : b \to \Sigma a$ be a w-admissible morphism such that $h_i \neq 0$ for each i. Then any triangle $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$ is w-admissible.*

Proof. The proof is by induction on n. The case $n = 1$ is Lemma 5.2.11. Assume that $n \geq 2$. Consider a triangle $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$, we prove that it is w-admissible. Since $\Sigma a \in H^+(b_i) \cup H^-(\Sigma^2 b_i)$ for each i, we have that $b_1, \dots, b_n \in H^+(\tau^{-1} a) \cup H^-(\Sigma a)$. Assume that $b_1, \dots, b_n \in H^+(\tau^{-1} a)$, the other cases are analogous. After reordering the objects $b_1, \dots, b_n$, we can assume that their first coordinates are in increasing order, Figure 3.6 provides an illustration.

By [22, Lemma 4.16], the object e has either n indecomposable direct summands, when the first coordinate of b_n is equal to $a_2 - 1$, or $n+1$ indecomposable direct summands, otherwise. Assume that e has $n+1$ indecomposable direct summands $e \cong e_1 \oplus \cdots \oplus e_{n+1}$, the other case is analogous.

It is straightforward to check that $\mathrm{Hom}_{\mathcal{C}_{2,m}}(a, e_j) \cong \mathbb{K}$ for each j, and $\mathrm{Hom}_{\mathcal{C}_{2,m}}(e_j, b_i) \cong \mathbb{K}$ if and only if $i = j$ or $i = j-1$. Thus, in order to show that the triangle $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ is w-admissible, it is enough to check that $e_1, \ldots, e_{n+1}$ are w-admissible, $e_1, \ldots, e_{n+1}$ are w-compatible with a, and b_i is w-compatible with e_i and e_{i+1} for each i.

Let $h' = \begin{pmatrix} h_1 & \cdots & h_{n-1} \end{pmatrix}$ and $h'' = \begin{pmatrix} h_2 & \cdots & h_n \end{pmatrix}$. Consider the triangles

$$a \longrightarrow e_1 \oplus \cdots \oplus e_{n-1} \oplus e_n' \longrightarrow \bigoplus_{i=1}^{n-1} b_i \xrightarrow{h'} \Sigma a \quad \text{and}$$
$$a \longrightarrow e_2'' \oplus e_3 \oplus \cdots \oplus e_{n+1} \longrightarrow \bigoplus_{i=2}^{n} b_i \xrightarrow{h''} \Sigma a.$$

By the induction hypothesis, $e_1, \ldots, e_{n+1}$ are w-admissible. We denote $b_i = (b_{i,1}, b_{i,2})$ for each i. Thus, by [22, Lemma 4.16], we have that $e_1 = (a_1, b_{1,2})$, $e_i = (b_{i-1,1}, b_{i,2})$ for each $2 \leq i \leq n$, and $e_{n+1} = (b_{n-1}, a_2)$. Moreover, $e_i \in H^+(a)$ for each $1 \leq i \leq n+1$ and $b_i \in H^+(e_i) \cap H^+(e_{i+1})$ for each $1 \leq i \leq n$, see Figure 5.1.

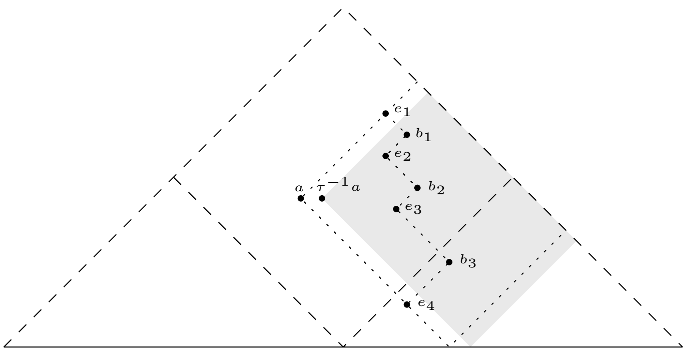


Figure 5.1: The middle term $e \cong e_1 \oplus e_2 \oplus e_3 \oplus e_4$ of the triangle $a \longrightarrow e \longrightarrow b_1 \oplus b_2 \oplus b_3 \longrightarrow \Sigma a$ in $\mathcal{C}_{2,2}$.

Therefore, by Definition 5.1.3, $e_1, \ldots, e_{n+1}$ are w-compatible with a and b_i is w-compatible with e_i and e_{i+1} for each $1 \leq i \leq n$. We can conclude that the triangle $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ is w-admissible. $\square$

Proposition 5.2.14. *Let $h \colon b \to \Sigma a$ be a w-admissible morphism such that a is indecomposable. Then there exists a w-admissible triangle $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$.*

Proof. Consider a triangle $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$. We write $b = \bigoplus_{i=1}^n b_i$ with $b_1, \ldots, b_n \in \operatorname{ind} \mathcal{C}_{w,m}$, and $h = \begin{pmatrix} h_1 & \cdots & h_n \end{pmatrix} : \bigoplus_{i=1}^n b_i \to \Sigma a$. We divide the proof into steps.

Step 1. We show that, without loss of generality, we can assume that $h_1, \ldots, h_n \neq 0$.

Up to reordering the summands of b, we can write $h = \begin{pmatrix} h_1 & \cdots & h_k & 0 & \cdots & 0 \end{pmatrix} : \left(\bigoplus_{i=1}^{k} b_i\right) \oplus \left(\bigoplus_{i=k+1}^{n} b_i\right) \to \Sigma a$. We denote $b' = \bigoplus_{i=1}^{k} b_i$, $b'' = \bigoplus_{i=k+1}^{n} b_i$, and $h' = \begin{pmatrix} h_1 & \cdots & h_k \end{pmatrix}$. Thus $h = \begin{pmatrix} h' & 0 \end{pmatrix}$. Note that, if h' extends to a w-admissible triangle, i.e. there exists a w-admissible triangle $a \xrightarrow{f'} e' \xrightarrow{g'} b' \xrightarrow{h'} \Sigma a$, then

$$a \xrightarrow{\left(\begin{smallmatrix} f' \\ 0 \end{smallmatrix}\right)} e' \oplus b'' \xrightarrow{\left(\begin{smallmatrix} g' & 0 \\ 0 & 1 \end{smallmatrix}\right)} b' \oplus b'' \xrightarrow{h} \Sigma a$$

is a w-admissible triangle, and we obtain the claim.

Step 2. We prove the statement assuming that $h_1, \ldots, h_k \neq 0$.

Now assume that all the entries of h are non-zero. We recall that, by Lemma 3.4.2, there exist $b' = \bigoplus_{i=1}^{k} b'_i$ a direct summand of b and a morphism $h' = \begin{pmatrix} h'_1 & \cdots & h'_k \end{pmatrix} : b' \to \Sigma a$ such that $b'_1, \ldots b'_k$ are pairwise Hom-orthogonal, $b'_i \ncong \Sigma a$ and $h'_i \neq 0$ for each i. Moreover, there is the following isomorphism of triangles.

$$\begin{tikzcd} a \arrow[r] \arrow[d, "1"] & e \arrow[r] \arrow[d, "\wr"] & b \arrow[r, "h"] \arrow[d, "\varphi"] & \Sigma a \arrow[d, "1"] \\ a \arrow[r, "\left(\begin{smallmatrix} f' \\ 0 \end{smallmatrix}\right)"'] & e' \oplus b'' \arrow[r, "\left(\begin{smallmatrix} g' & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & b' \oplus b'' \arrow[r, "\left(\begin{smallmatrix} h' & 0 \end{smallmatrix}\right)"'] & \Sigma a \end{tikzcd} \tag{5.1}$$

Since the second row is a triangle in $\mathcal{C}_{2,m}$, we have that $a \xrightarrow{f'} e' \xrightarrow{g'} b' \xrightarrow{h'} \Sigma a$ is a triangle because it is a direct summand of a triangle, see [29, Lemma A.1]. Note that b' is w-admissible because it is a direct summand of b, and h' is w-admissible because h is w-admissible and all its entries are non-zero. Since h' is w-admissible and the assumptions of Lemma 5.2.13 are satisfied, it follows that $a \xrightarrow{f'} e' \xrightarrow{g'} b' \xrightarrow{h'} \Sigma a$ is a w-admissible triangle, and as a consequence the second row of 5.1 is a w-admissible triangle.

By Lemma 5.2.5, Lemma 5.2.8, and Lemma 5.2.9, we have that: $\overline{\varphi}$ and $(\overline{\varphi})^{-1}$ are isomorphisms, $(\overline{\varphi})^{-1} \left(\begin{smallmatrix} g' & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is w-admissible, and $\begin{pmatrix} h' & 0 \end{pmatrix} \overline{\varphi} = h$. We denote $g'' = (\overline{\varphi})^{-1} \left(\begin{smallmatrix} g' & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Thus, the following diagram commutes.

$$\begin{tikzcd} a \arrow[r, "\left(\begin{smallmatrix} f' \\ 0 \end{smallmatrix}\right)"] \arrow[d, "1"] & e' \oplus b'' \arrow[r, "g''"] \arrow[d, "1"] & b \arrow[r, "h"] \arrow[d, "\overline{\varphi}"] & \Sigma a \arrow[d, "1"] \\ a \arrow[r, "\left(\begin{smallmatrix} f' \\ 0 \end{smallmatrix}\right)"'] & e' \oplus b'' \arrow[r, "\left(\begin{smallmatrix} g' & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & b' \oplus b'' \arrow[r, "\left(\begin{smallmatrix} h' & 0 \end{smallmatrix}\right)"'] & \Sigma a \end{tikzcd}$$

Since the second row of the diagram above is a triangle, we have that $a \xrightarrow{\left(\begin{smallmatrix} f' \\ 0 \end{smallmatrix}\right)} e' \oplus b'' \xrightarrow{g''} b \xrightarrow{h} \Sigma a$ is a w-admissible triangle. We conclude that h extends to a w-admissible triangle. □

Proposition 5.2.15. *Let $h \colon b \to \Sigma a$ be a w-admissible morphism and $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$ be a triangle in $\mathcal{C}_{2,m}$. Then $e \in \mathcal{C}_{w,m}$.*

Proof. We write $a = \bigoplus_{i=1}^{n} a_i$ with $a_1, \ldots, a_n \in \operatorname{ind} \mathcal{C}_{w,m}$. We proceed by induction on n.

If $n = 1$ then, by Proposition 5.2.14 we have that h extends to a w-admissible triangle $a \longrightarrow e' \longrightarrow b \xrightarrow{h} \Sigma a$. Thus, $e \cong e'$ and, by Lemma 5.1.2 we have that e is w-admissible.

Now assume that $n \geq 2$. Let $a' = \bigoplus_{i=1}^{n-1} a_i$, then $a = a' \oplus a_n$, $h = \left(\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right) : b \to \Sigma a' \oplus \Sigma a_n$, and $f = (\, f_1 \; f_2 \,)$. We consider the following Octahedral Axiom diagram

$$\begin{tikzcd}
 & a' \arrow[r, "1"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & a' \arrow[d, "f_1"] & \\
\Sigma^{-1}b \arrow[r, "-\Sigma^{-1}h"] \arrow[d, "1"] & a' \oplus a_n \arrow[r, "f"] \arrow[d, "(0\ 1)"] & e \arrow[r, "g"] \arrow[d, "u"] & \Sigma a \arrow[d, "1"] \\
\Sigma^{-1}b \arrow[r, "-\Sigma^{-1}h_2"] & a_n \arrow[r, "f'"] \arrow[d, "0"] & x \arrow[r, "g'"] \arrow[d, "v"] & b \arrow[d, "h"] \\
 & \Sigma a' \arrow[r, "1"] & \Sigma a' \arrow[r, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"] & \Sigma a' \oplus \Sigma a_n
\end{tikzcd}$$

where $a_n \xrightarrow{f'} x \xrightarrow{g'} b \xrightarrow{h_2} \Sigma a_n$ is a w-admissible triangle, which exists, by Proposition 5.2.14, because h_2 is w-admissible and a_n is indecomposable. Since $\left(\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right) = hg'$ is w-admissible, then v is w-admissible. Therefore, by induction hypothesis, e is w-admissible. $\square$

5.2.5 The axioms

We prove that if Conjecture 5.2.10 holds then the functor $\Sigma\colon \mathcal{C}_{w,m} \to \mathcal{C}_{w,m}$ and the w-admissible triangles form a triangulated structure for $\mathcal{C}_{w,m}$.

Proposition 5.2.16. *Assume that Conjecture 5.2.10 holds. Let $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ be a triangle in $\mathcal{C}_{2,m}$ with $a, e, b \in \mathcal{C}_{w,m}$ and g, h w-admissible morphisms. Then $a \xrightarrow{\overline{f}} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ is a w-admissible triangle.*

Proof. By Conjecture 5.2.10, there exists a w-admissible triangle $a \xrightarrow{f'} e \xrightarrow{g'} b \xrightarrow{h} \Sigma a$. Then there exists an isomorphism $\varphi\colon e \to e$ such that the diagram below commutes.

$$\begin{tikzcd}
a \arrow[r, "f'"] \arrow[d, "1"] & e \arrow[r, "g'"] \arrow[d, dashed, "\varphi"] & b \arrow[r, "h"] \arrow[d, "1"] & \Sigma a \arrow[d, "1"] \\
a \arrow[r, "f"] & e \arrow[r, "g"] & b \arrow[r, "h"] & \Sigma a
\end{tikzcd}$$

By Lemma 5.2.5 we have that $g\overline{\varphi} = g\varphi = g'$. Moreover, since f' is w-admissible, by Lemma 5.2.6 we have that $\overline{\varphi} f' = \overline{\varphi f'} = \overline{f}$. Now, by Lemma 5.2.8, $\overline{\varphi}$ is an isomorphism, and the diagram below commutes.

$$\begin{tikzcd}
a \arrow[r, "f'"] \arrow[d, "1"] & e \arrow[r, "g'"] \arrow[d, "\overline{\varphi}"] & b \arrow[r, "h"] \arrow[d, "1"] & \Sigma a \arrow[d, "1"] \\
a \arrow[r, "\overline{f}"] & e \arrow[r, "g"] & b \arrow[r, "h"] & \Sigma a
\end{tikzcd}$$

Therefore, $a \xrightarrow{\overline{f}} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ is a triangle in $\mathcal{C}_{2,m}$. Since it is also w-admissible, then we conclude that it is a w-admissible triangle. $\square$

Theorem 5.2.17. *Assume that Conjecture 5.2.10 holds. Then the w-admissible triangles and the functor $\Sigma\colon \mathcal{C}_{w,m} \to \mathcal{C}_{w,m}$ form a triangulated structure for $\mathcal{C}_{w,m}$.*

Proof. We check the axioms of Definition 2.3.1 individually.

(TR1) It is straightforward to check that the class if w-admissible triangles is closed under w-admissible isomorphism, and that $a \xrightarrow{1_a} a \longrightarrow 0 \longrightarrow \Sigma a$ is a w-admissible triangle for each $a \in \mathcal{C}_{w,m}$. Moreover, by Conjecture 5.2.10, each w-admissible morphism extends to a w-admissible triangle.

(TR2) It is straightforward to check that rotations of w-admissible triangles are still w-admissible triangles.

(TR3) Let $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ and $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{r} \Sigma x$ be w-admissible triangles, and $\alpha\colon a \to x$, $\beta\colon e \to y$ be w-admissible morphisms such that $\beta f = u\alpha$. We show that there exists $\gamma\colon b \to z$ a w-admissible morphism such that the following diagram commutes.

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "\alpha"] & e \arrow[r, "g"] \arrow[d, "\beta"] & b \arrow[r, "h"] \arrow[d, dashed, "\gamma"] & \Sigma a \arrow[d, "\Sigma\alpha"] \\
x \arrow[r, "u"] & y \arrow[r, "v"] & z \arrow[r, "r"] & \Sigma x
\end{tikzcd}$$

Since the first and second rows of the diagram above are also triangles in $\mathcal{C}_{2,m}$, we know that there exists $\gamma\colon b \to z$ making the diagram above commutative. If γ is w-admissible, then we have the claim. If γ is not w-admissible, then we consider instead its w-admissible part $\overline{\gamma}$, see Definition 5.2.4. By Lemma 5.2.5, we obtain that $\overline{\gamma}g = \gamma g = v\beta$ and $r\overline{\gamma} = r\gamma = (\Sigma\alpha)h$, since $\gamma g = v\beta$ and $r\gamma = (\Sigma\alpha)h$ are w-admissible. Thus, there exists a w-admissible morphism making the diagram above commutative.

(TR4") Let $a \xrightarrow{f} e \xrightarrow{g} b \xrightarrow{h} \Sigma a$ and $a \xrightarrow{u} x \xrightarrow{v} y \xrightarrow{r} \Sigma a$ be w-admissible triangles, and let $\alpha\colon b \to y$ be a w-admissible morphism such that $r\alpha = h$. Since the two w-admissible triangles are also triangles in $\mathcal{C}_{2,m}$, there exists a morphism $\beta\colon e \to x$ such that the following diagram commutes

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "1_a"] & e \arrow[r, "g"] \arrow[d, dashed, "\beta"] & b \arrow[r, "h"] \arrow[d, "\alpha"] & \Sigma a \arrow[d, "1_{\Sigma a}"] \\
a \arrow[r, "u"] & x \arrow[r, "v"] & y \arrow[r, "r"] & \Sigma a
\end{tikzcd}$$

and $e \xrightarrow{\left(\begin{smallmatrix}\beta \\ -g\end{smallmatrix}\right)} x \oplus b \xrightarrow{(v\ \alpha)} y \xrightarrow{(\Sigma f)r} \Sigma e$ is a triangle in $\mathcal{C}_{2,m}$. As a consequence, by Lemma 5.2.5, if we replace β with $\overline{\beta}$, the diagram above still commutes. Moreover, by Proposition 5.2.16,

$$e \xrightarrow{\left(\begin{smallmatrix}\overline{\beta} \\ -g\end{smallmatrix}\right)} x \oplus b \xrightarrow{(v\ \alpha)} u \xrightarrow{(\Sigma f)r} \Sigma e$$

is a w-admissible triangle.

We conclude that $\mathcal{C}_{w,m}$ is triangulated. $\square$

For the rest of this chapter we assume that Conjecture 5.2.10 holds.

5.3 Calabi-Yau dimension

We prove that Σ^w is a Serre functor for $\mathcal{C}_{w,m}$, i.e. $\mathcal{C}_{w,m}$ has w-CY dimension. We start with the following lemmas.

Lemma 5.3.1. *Let $a \in \operatorname{ind}\mathcal{C}_{w,m}$. Then $\Sigma^w a \in H^-(\Sigma^2 a)$.*

Proof. We denote $a = (a_1, a_2)$. Note that, since $w \geq 2$, $a_1 - w \leq a_1 - 2$ and $a_2 - w \leq a_2 - 2$. In order to prove that $\Sigma^w a \in H^-(\Sigma^2 a)$, it remains to check that $a_2 - w \geq a_1$. This follows from the fact that $a_2 - a_1 \equiv 1 \mod (w-1)$, i.e. $a_2 = a_1 + 1 + k(w-1)$ for some $k \geq 1$. $\square$

Lemma 5.3.2. *Let $a, b \in \operatorname{ind}\mathcal{C}_{w,m}$ and assume that there exists $0 \neq f \in \operatorname{Hom}_{\mathcal{C}_{w,m}}(a, b)$. Then there exists $0 \neq g \in \operatorname{Hom}_{\mathcal{C}_{w,m}}(b, \Sigma^w a)$ such that $gf \neq 0$.*

Proof. Since $f \neq 0$, we have that $b \in H^+(a) \cup H^-(\Sigma^2 a)$. Assume that $b \in H^+(a)$, we show that $\Sigma^w a \in H^-(\Sigma^2 b)$. As $a = (a_1, a_2)$ is w-admissible, we can write $a_2 - a_1 = k(w-1) + 1$ for some $k \geq 1$, and as $b = (b_1, b_2)$ is w-compatible with a we can write $b_1 - a_1 = h(w-1)$ for some $h \geq 0$.

We have that $a_1 - w \leq b_1 - 2$ and $a_2 - w \leq b_2 - 2$. Indeed, since $b \in H^+(a)$, $a_1 \leq b_1 \leq b_1 + w - 2$ and also $a_2 \leq b_2 \leq b_2 + w - 2$. Thus, in order to prove that $\Sigma^w a \in H^-(\Sigma^2 b)$, it remains to check that $a_2 - w \geq b_1$. Since $b \in H^+(a)$, then $b_1 \leq a_2 - 2$ and $2 \leq a_2 - b_1 = a_2 - a_1 + a_1 - b_1 = (k-h)(w-1) + 1$. As a consequence, we have that $k - h \geq 1$, otherwise, if $k - h \leq 0$ we obtain that $2 \leq 1$ which is impossible. Then $a_2 - b_1 \geq (w-1) + 1 = w$, i.e. $a_2 - w \geq b_1$. This shows that $\Sigma^w a \in H^-(\Sigma^2 b)$. Moreover, $b_1 - (a_1 - w) \equiv 1 \mod (w-1)$ and then $\Sigma^w a$ is w-compatible with b. Thus, there exists a w-admissible non-zero morphism $g \colon b \to \Sigma^w a$. By Lemma 5.3.1 we have that $\Sigma^w a \in H^-(\Sigma^2 a)$. Since $b \in H^+(a)$ and $\Sigma^w a \in H^-(\Sigma^2 b) \cap H^-(\Sigma^2 a)$, then, by Proposition 4.3.7, $gf \neq 0$.

If $b \in H^-(\Sigma^2 a)$, then we can prove analogously that $\Sigma^w a \in H^+(b) \cap H^-(\Sigma^2 a)$ and that $\Sigma^w a$ is w-compatible with b. Thus, there exists a w-admissible non-zero morphism $g \colon b \to \Sigma^w a$ such that $gf \neq 0$. This concludes the proof. $\square$

The following result follows from the lemmas above. Moreover, by Theorem 2.1.5, it follows that Σ^w is a Serre functor.

Proposition 5.3.3. *Let $a, b \in \mathcal{C}_{w,m}$. The following is a non-degenerate bilinear form.*

$$\begin{aligned} \Phi_{a,b} \colon \operatorname{Hom}_{\mathcal{C}_{w,m}}(a, b) \times \operatorname{Hom}_{\mathcal{C}_{w,m}}(b, \Sigma^w a) &\longrightarrow \mathbb{K} \\ (f, g) &\longmapsto \operatorname{Tr}(gf) \end{aligned}$$

Proof. It is straightforward to check that $\Phi_{a,b}$ is bilinear, we prove that it is non-degenerate. Let $f\colon a \to b$ be non-zero, we show that there exists $g\colon b \to \Sigma^w a$ such that $\Phi_{a,b}(f,g) = \mathrm{Tr}(gf) \neq 0$. We denote $a = \bigoplus_{t=1}^n a_t$ and $b = \bigoplus_{s=1}^k b_s$ with $a_1, \dots, a_n, b_1, \dots, b_k \in \operatorname{ind} \mathcal{C}_{w,m}$, and $f = (f_{s,t})_{s,t}$ with $f_{s,t}\colon a_t \to b_s$. Since $f \neq 0$, there exist $1 \leq i \leq n$ and $1 \leq j \leq k$ such that $f_{i,j}\colon a_j \to b_i$ is non-zero. By Lemma 5.3.2, there exists $\gamma\colon b_i \to \Sigma^w a_j$ such that $\gamma f_{i,j} \neq 0$. Now, we define $g = (g_{s,t})_{s,t}\colon b \to \Sigma^w a$ as

$$g_{s,t} = \begin{cases} \gamma & \text{if } s = j \text{ and } t = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $(gf)_{t,t} = \gamma f_{i,j} \neq 0$ if $t = j$, and $(gf)_{t,t} = 0$ otherwise. As a consequence, $\mathrm{Tr}(gf) \neq 0$. Similarly, we can prove that given a non-zero morphism $g\colon b \to \Sigma^w a$, there exists $f\colon a \to b$ such that $\Phi_{a,b}(gf) = \mathrm{Tr}(gf) \neq 0$. We conclude that $\Phi_{a,b}$ is non-degenerate. $\square$

5.4 The AR quiver

We describe the AR quiver of $\mathcal{C}_{w,m}$. We introduce the Hom-hammocks and we study the factorization properties of the morphisms of $\mathcal{C}_{w,m}$. Then we discuss the irreducible morphisms and the almost split triangles.

5.4.1 The coordinate system

We introduce a coordinate system for $\mathcal{C}_{w,m}$, and in Section 5.4.3 we prove that it describes the AR quiver of $\mathcal{C}_{w,m}$. For each $p, q \in [m]$ and $0 \leq i \leq w-2$, we define the set of w-admissible arcs

$$\mathbb{Z}^{(p,q,i)} = \Big\{ a = (a_1, a_2) \in \operatorname{ind} \mathcal{C}_{w,m} \mid a \in \mathbb{Z}^{(p,q)} \text{ and } a_1 \equiv i \mod (w-1) \Big\}.$$

We can arrange the isoclasses of indecomposable objects of $\mathcal{C}_{w,m}$ into a coordinate system having

- $(w-1)m$ components of type $\mathbb{Z}A_\infty$, each corresponding to the sets of arcs $\mathbb{Z}^{(p,q,i)}$ for $p \in [m]$ and $0 \leq i \leq w-2$, and
- $(w-1)\binom{m}{2}$ components of type $\mathbb{Z}A_\infty^\infty$, each corresponding to the sets of arcs $\mathbb{Z}^{(p,q,i)}$ for $p, q \in [m]$, $p < q$, and $0 \leq i \leq w-2$.

Figure 5.2 illustrates the coordinate system of $\mathcal{C}_{w,m}$.

5.4.2 Hom-hammocks and factorization properties

We introduce the Hom-hammocks in $\mathcal{C}_{w,m}$. The following definition is the higher number of accumulation points version of [26, Definition 2.5] or [14, Section 2.2, p. 7].

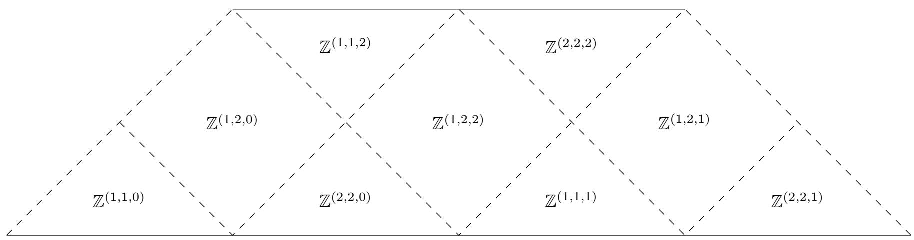


Figure 5.2: The coordinate system of $\mathcal{C}_{4,2}$.

Definition 5.4.1. Let $a = (a_1, a_2) \in \operatorname{ind}\mathcal{C}_{w,m}$. We define

$$H_w^+(a) = \{b = (b_1, b_2) \in \operatorname{ind}\mathcal{C}_{w,m} \mid b \in H^+(a) \text{ and } b_1 - a_1 \equiv 0 \mod (w-1)\} \quad \text{and}$$
$$H_w^-(a) = \{b = (b_1, b_2) \in \operatorname{ind}\mathcal{C}_{w,m} \mid b \in H^-(a) \text{ and } a_1 - b_1 \equiv 0 \mod (w-1)\}.$$

We observe that

$$H_w^+(a) = \{b = (b_1, b_2) \in \operatorname{ind}\mathcal{C}_{w,m} \mid a_1 \leq b_1 \leq a_2 - w \text{ and } b_2 \geq a_2\} \quad \text{and}$$
$$H_w^-(a) = \{b = (b_1, b_2) \in \operatorname{ind}\mathcal{C}_{w,m} \mid a_1 + w \leq b_2 \leq a_2 \text{ and } b_1 \geq a_1\}.$$

Figure 5.3 provides an illustration of the Hom-hammocks of $\mathcal{C}_{w,m}$. With Proposition 5.4.3 we will prove that the Hom-hammocks describe the Hom-spaces of $\mathcal{C}_{w,m}$. First, we need the following lemma. We recall that, by Lemma 5.3.1, $\Sigma^w a \in H^-(\Sigma^2 a)$ for each $a \in \operatorname{ind}\mathcal{C}_{w,m}$.

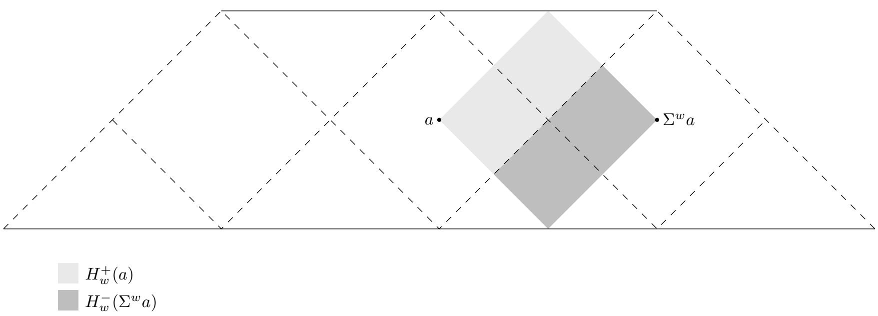


Figure 5.3: The Hom-hammocks $H_w^+(a)$ and $H_w^-(\Sigma^w a)$ for $a \in \operatorname{ind}\mathcal{C}_{4,2}$.

Lemma 5.4.2. *Let $a, b \in \operatorname{ind}\mathcal{C}_{w,m}$. The following statements hold.*

1. *$b \in H_w^+(a)$ if and only if $b \in H^+(a)$ and b is w-compatible with a.*
2. *$b \in H_w^-(\Sigma^w a)$ if and only if $b \in H^-(\Sigma^2 a)$ and b is w-compatible with a.*

Proof. Statement (1) is straightforward, we prove statement (2). We write $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Assume that $b \in H_w^-(\Sigma^w a)$, then $b \in H^-(\Sigma^w a)$ by Definition 5.4.1. We show that $b \in H^-(\Sigma^2 a)$. Note that $b_1 \leq a_1 - w \leq a_1 - 2$ and $b_2 \leq a_2 - w \leq a_2 - 2$, it remains to check that $b_2 \geq a_1$. Since b is w-admissible, $b_2 - b_1 = 1 + k(w-1)$ for some $k \geq 1$. Moreover, since $b \in H_w^-(\Sigma^w a)$, we have that $(a_1 - w) - b_1 \equiv 0 \mod (w-1)$, i.e.

$(a_1 - w) - b_1 = l(w-1)$ for some $l \geq 0$. Thus,

$$b_2 - a_1 = (b_2 - b_1) - (a_1 - b_1) = (b_2 - b_1) - ((a_1 - w) - b_1) - w = (k + l - 1)(w - 1).$$

We have that $k+l-1 \geq 0$. Indeed, since $b \in H^-(\Sigma^w a)$, $b_2 - a_1 \geq -w+2$, and this implies that $k + l - 1 \not\leq -1$, otherwise we have the contradiction $-w + 2 \leq b_2 - a_1 \leq -w + 1$. As a consequence, $b_2 - a_1 \geq 0$, and this proves that $b \in H^-(\Sigma^2 a)$. Moreover, $a_1 - b_1 = (a_1 - w) - b_1 + w \equiv 1 \mod (w-1)$, i.e. b is w-compatible with a.

Now assume that $b \in H^-(\Sigma^2 a)$ and b is w-compatible with a. We prove that $b \in H^-_w(\Sigma^w a)$. Note that $b_2 \geq a_1 \geq a_1 - w + 2$ and, since b is w-compatible with a, $a_1 - b_1 = 1 + l(w-1)$ for some $l \geq 1$, and as a consequence $a_1 - b_1 \geq w$. Therefore, it remains to check that $b_2 \leq a_2 - w$. Since a and b are w-admissible, $a_2 - a_1 = 1 + n(w-1)$ and $b_2 - b_1 = 1 + h(w+1)$ for some $n, h \geq 1$. Thus,

$$a_2 - b_2 = (a_2 - a_1) + (a_1 - b_1) - (b_2 - b_1) = 1 + (n + l - h)(w - 1).$$

Since $b_2 \leq a_2 - 2$, we obtain that $n + l - h \geq 1$ similarly as above. Then, $a_2 - b_2 \geq w$, i.e. $b_2 \leq a_2 - w$. This proves that $b \in H^-(\Sigma^w a)$. Moreover, $(a_1 - w) - b_1 = (a_1 - b_1) - w \equiv 0 \mod (w-1)$. This concludes the argument. $\square$

The following proposition follows directly from Definition 5.1.8 and Lemma 5.4.2.

Proposition 5.4.3. *Let $a, b \in \operatorname{ind} \mathcal{C}_{w,m}$. Then*

$$\operatorname{Hom}_{\mathcal{C}_{w,m}}(a,b) \cong \begin{cases} \mathbb{K} & \text{if } b \in H^+_w(a) \cup H^-_w(\Sigma^w a), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The factorization properties of the morphisms in $\mathcal{C}_{w,m}$ follow directly from Proposition 4.3.7 and Lemma 5.4.2. We refer to [14, Proposition 2.3] for the case $m = 1$ and to Lemma 3.2.5 for the case $w = 2$.

Proposition 5.4.4. *Let $a, b, c \in \operatorname{ind} \mathcal{C}_{w,m}$, $f \colon a \to b$, and $g \colon b \to c$ be non-zero w-admissible morphisms. Assume that one of the following conditions hold.*

1. *$b \in H^+_w(a)$ and $c \in H^+_w(a) \cap H^+_w(b)$.*
2. *$b \in H^+_w(a)$ and $c \in H^-_w(\Sigma^w a) \cap H^-_w(\Sigma^w b)$.*
3. *$b \in H^-_w(\Sigma^w a)$ and $c \in H^-_w(\Sigma^w a) \cap H^+_w(b)$.*

Then $gf \neq 0$.

5.4.3 Irreducible morphisms and almost-split sequences

With the following proposition we describe the irreducible morphisms of $\mathcal{C}_{w,m}$ and we prove that the coordinate system discussed in Section 5.4.1 yields the AR quiver of $\mathcal{C}_{w,m}$.

We recall that when we say that a morphism $f\colon a \to b$ factors through an object c, we mean that there exist $g\colon a \to c$ and $h\colon c \to b$ such that $f = hg$.

Proposition 5.4.5. *Let $a = (a_1, a_2), b = (b_1, b_2) \in \operatorname{ind} \mathcal{C}_{w,m}$. If $b = (a_1, a_2 + w - 1)$ or $b = (a_1 + w - 1, a_2)$, then any non-zero w-admissible morphism $a \to b$ is irreducible in $\mathcal{C}_{w,m}$. Moreover, there are no other w-admissible irreducible morphisms in $\mathcal{C}_{w,m}$ between indecomposable w-admissible objects.*

Proof. Assume that $b = (a_1, a_2 + w - 1)$, and consider a non-zero morphism $f\colon a \to b$. Moreover, assume that $f = hg$ for some w-admissible morphisms $g\colon a \to c$ and $h\colon c \to b$. We prove that g is a split monomorphism or g is a split epimorphism. Note that, by the one-dimensionality of the Hom-spaces, we can assume that c is indecomposable. We have that

$$c \in \left(H_w^+(a) \cup H_w^-(\Sigma^w a)\right) \cap \left(H_w^-(b) \cup H_w^+(\Sigma^{-w} b)\right) = \{a, b\}.$$

We refer to Figure 5.4 for an illustration. Thus, either $c \cong a$ or $c \cong b$, i.e. g is a split monomorphism or h is a split epimorphism. We conclude that f is irreducible. If $b = (a_1 + w - 1, a_2)$ then we can proceed with an analogous argument.

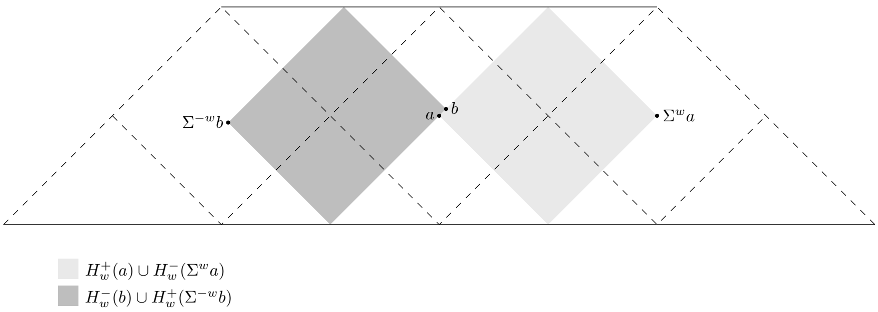


Figure 5.4: Illustration of the argument of Proposition 5.4.5.

Now consider an irreducible morphism $f\colon a \to b$ in $\mathcal{C}_{w,m}$ with $b \in \operatorname{ind} \mathcal{C}_{w,m}$. By Proposition 5.4.4, if $b_2 = a_2$ then f factors through $(a_1, a_2 + w - 1)$, and if $b_2 \neq a_2$ then f factors through $(a_1 + w - 1, a_2)$. Assume that the first case holds. There exist non-zero morphisms $g\colon a \to (a_1, a_2 + w - 1)$ and $h\colon (a_1, a_2 + w - 1) \to b$ such that $f = hg$. Since f is irreducible, g is a split monomorphism or h is a split epimorphism. Therefore, we obtain that $b \cong (a_1, a_2 + w - 1)$. If f factors through $(a_1, a_2 + w - 1)$, we obtain that $b \cong (a_1 + w - 1, a_2)$ for the same reason. This concludes the argument. $\square$

Since $\mathcal{C}_{w,m}$ has a Serre functor, see Proposition 5.3.3, then, by Proposition 2.3.6, it has almost split triangles. We denote the AR translate of $\mathcal{C}_{w,m}$ by τ_w.

Theorem 5.4.6. *Let $a = (a_1, a_2) \in \operatorname{ind} \mathcal{C}_{w,m}$. The following statements hold.*

1. *If $a_2 = a_1 + w$, then $(a_1 - w + 1, a_2 - w + 1) \longrightarrow (a_1 - w + 1, a_2) \longrightarrow (a_1, a_2) \longrightarrow \Sigma(a_1 - w + 1, a_2 - w + 1)$ is an almost split triangle.*

2. *If $a_2 \neq a_1 + w$, then $(a_1 - w + 1, a_2 - w + 1) \longrightarrow (a_1, a_2 - w + 1) \oplus (a_1 - w + 1, a_2) \longrightarrow (a_1, a_2) \longrightarrow \Sigma(a_1 - w + 1, a_2 - w + 1)$ is an almost split triangle.*

Thus, we have that $\tau_w a = (a_1 - w + 1, a_2 - w + 1)$.

Proof. It is straightforward to check that the triangles in the statement are w-admissible, we prove that they are almost split. Assume that $a_2 \neq a_1 + w$, the other case is analogous. We prove that the morphism $f\colon (a_1 - w + 1, a_2 - w + 1) \to (a_1, a_2 - w + 1) \oplus (a_1 - w + 1, a_2)$ is left almost-split. First, note that f is not a split monomorphism. Then, consider a w-admissible morphism $h\colon (a_1 - w + 1, a_2 - w + 1) \to b$ which is not a split monomorphism. Without loss of generality, we can further assume that b is indecomposable. By Proposition 5.4.4, h factors through $(a_1, a_2 - w + 1)$, if $b_2 = a_2$, or through $(a_1 - w + 1, a_2)$, otherwise. Thus, there exists $g\colon (a_1, a_2 - w + 1) \oplus (a_1 - w + 1, a_2) \to b$ such that $h = gf$. □

5.5 Precovering subcategories

We classify the precovering subcategories of $\mathcal{C}_{w,m}$ in terms of converging sequences of w-admissible arcs. We refer to Section 3.3 for the case $w = 2$, and to [14, Section 6] for the case $m = 1$. The following is the w-CY version of Definition 3.3.1. We recall that the accumulation points of $\mathcal{Z}_m$ are in cyclic order, i.e. $1 < 2 < \cdots < m < 1$, and, for each $p \in [m]$, p^+ denotes the next accumulation point of p with respect to the cyclic order. Moreover, in the definition below, if $q = m$ then $\mathbb{Z}^{(p,q^+,i)}$ means $\mathbb{Z}^{(1,p,i-1 \mod (w-1))}$. We refer to Section 4.1 for the notation $|x_1, x_2|$.

Definition 5.5.1. Let $\mathcal{X}$ be a subcategory of $\mathcal{C}_{w,m}$. We say that $\mathcal{X}$ satisfies the *w-precovering conditions*, w-PC for short, if it satisfies the following combinatorial conditions.

(w-PC 1) If there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathcal{X} \cap \mathbb{Z}^{(p,q,i)}$ for some $p, q \in [m]$ such that $p \neq q$ and the sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are strictly increasing, then there exist strictly decreasing sequences $\{y_1^n\}_n \subseteq \mathbb{Z}^{(p^+)}$ and $\{y_2^n\}_n \subseteq \mathbb{Z}^{(q^+)}$ such that $\{|y_1^n, y_2^n|\}_n \subseteq \mathcal{X} \cap \mathbb{Z}^{(p^+,q^+,i)}$.

(w-PC 2) If there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathcal{X} \cap \mathbb{Z}^{(p,q,i)}$ for some $p, q \in [m]$ such that $p \neq q^+$ and the sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are respectively strictly decreasing and strictly increasing, then there exist strictly decreasing sequences $\{y_1^n\}_n \subseteq \mathbb{Z}^{(p)}$ and $\{y_2^n\}_n \subseteq \mathbb{Z}^{(q^+)}$ such that $\{|y_1^n, y_2^n|\}_n \subseteq \mathcal{U} \cap \mathbb{Z}^{(p,q^+,i)}$.

(w-PC 2′) If there exists a sequence $\{(x_1^n, x_2^n)\}_n \subseteq \mathcal{U} \cap \mathbb{Z}^{(p,q,i)}$ for some $p, q \in [m]$ such that $q \neq p^+$, $p \neq q$, and the sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are respectively strictly increasing and strictly decreasing, then there exist strictly decreasing sequences $\{y_1^n\}_n \subseteq \mathbb{Z}^{(p^+)}$ and $\{y_2^n\}_n \subseteq \mathbb{Z}^{(q)}$ such that $\{(y_1^n, y_2^n)\}_n \subseteq \mathcal{X} \cap \mathbb{Z}^{(p^+,q,i)}$.

(w-PC 3) If there exists a sequence $\{(x_1, x_2^n)\}_n \subseteq \mathcal{X} \cap \mathbb{Z}^{(p,q,i)}$ for some $p, q \in [m]$ such that the sequence $\{x_2^n\}_n$ is strictly increasing, then there exists a strictly decreasing sequence $\{y_2^n\}_n \subseteq \mathbb{Z}^{(q^+)}$ such that $\{|x_1, y_2^n|\}_n \subseteq \mathcal{X} \cap \mathbb{Z}^{(p,q^+,i)}$.

(w-PC 3′) If there exists a sequence $\{(x_1^n, x_2)\}_n \subseteq \mathcal{U} \cap \mathbb{Z}^{(p,q,i)}$ for some $p, q \in [m]$ such that $p \neq q$ and the sequence $\{x_1^n\}_n$ is strictly increasing, then there exists a strictly decreasing sequence $\{y_1^n\}_n \subseteq \mathbb{Z}^{(p^+)}$ such that $\{(y_1^n, x_2)\}_n \subseteq \mathcal{X} \cap \mathbb{Z}^{(p^+,q,i)}$.

Figure 5.5 illustrates the w-PC conditions. The following theorem is analogous to Theorem 3.3.2, and we use an argument similar to those of [21, Theorem 3.1, Proposition 3.7]. Moreover, part of the argument of the theorem below is similar to Proposition 4.4.8. We write a full proof for the convenience of the reader.

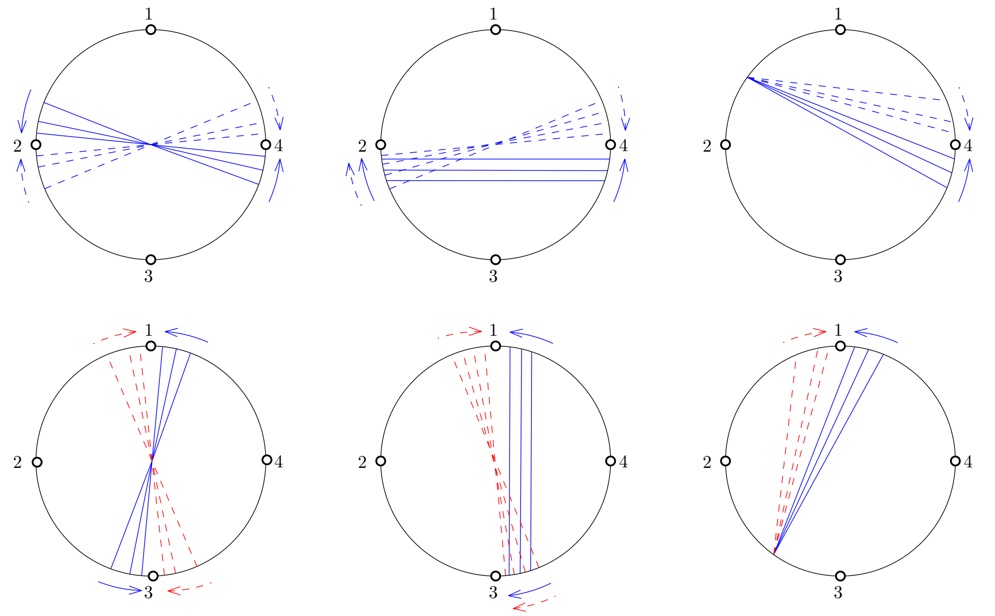


Figure 5.5: The first row illustrates (w-PC 1), on the left, (w-PC 2) in the middle, and (w-PC 3) on the right, when $q \neq m$. The second row illustrates the same PC conditions when $q = m$. The blue or red arcs are w-admissible. The blue arcs are of the form $x = (x_1, x_2)$ with $x_1 \equiv i \mod (w-1)$ and the red arcs are such that $x_1 \equiv i - 1 \mod (w-1)$ for some $0 \leq i \leq w-2$

.

Theorem 5.5.2. *A subcategory of $\mathcal{C}_{w,m}$ is precovering if and only if it satisfies the w-PC conditions.*

Proof. We divide the proof into steps.

Step 1. Assuming that $\mathcal{X}$ is a precovering subcategory of $\mathcal{C}_{w,m}$, we prove that $\mathcal{X}$ satisfies the w-PC conditions.

We prove that (w-PC 1) holds. Consider a sequence $\{x^n = (x_1^n, x_2^n)\}_n \subseteq \mathcal{X} \cap \mathbb{Z}^{(p,q,i)}$ such that $p \neq q$ and the sequences $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are strictly increasing. Then consider $a = (a_1, a_2) \in \operatorname{ind} \mathcal{C}_{w,m} \cap \mathbb{Z}^{(p^+,q^+,i)}$ such that $\operatorname{Hom}_{\mathcal{C}_{w,m}}(x^n, a) \cong \mathbb{K}$ for each $n \in \mathbb{Z}$, see Figure 5.6, and for each n consider a non-zero morphism $g_n \colon x^n \to a$. Since $\mathcal{X}$ is precovering, there exists $(\, f_1 \cdots f_k \,) : y_1 \oplus \cdots \oplus y_k \to a$, an $\mathcal{X}$-precover of a, with $y_1, \ldots, y_k \in \operatorname{ind} \mathcal{X}$. Note that each g_n factors through some of the morphisms $f_1, \ldots, f_k$. Therefore, we can extract a subsequence of $\{(x_1^n, x_2^n)\}_n$ for which each $g_n \colon (x_1^n, x_2^n) \to a$ factors through $f_l \colon y_l \to a$ for some $l \in \{1, \ldots, k\}$. For the rest of the argument, we denote y_l by y and f_l by f. We have that

$$y \in \left(\bigcap_{n \in \mathbb{Z}} H_w^+(x_n) \cup H_w^-(\Sigma^w x_n) \right) \cap \left(H_w^-(a) \cup H_w^+(\Sigma^{-w} a) \right).$$

Thus, $y = (y_1, y_2) \in \mathbb{Z}^{(p^+,q^+,i)}$, $y_1 < a_1$, and $y_2 < a_2$. We refer to Figure 5.6 for an illustration. Now consider $a' = (a'_1, a'_2) \in \mathbb{Z}^{(p^+,q^+,i)}$ such that $a'_1 < y_1$ and $a'_2 < y_2$. Then we repeat the same argument as above and we find $y' \in \mathcal{X} \cap \mathbb{Z}^{(p^+,q^+,i)}$ such that $y'_1 < a'_1$ and $y'_2 < a'_2$. Thus, we obtain the desired sequence of (w-PC 1).

Now, we prove that (w-PC 2) holds. Consider a sequence $\{x^n = (x_1^n, x_2^n)\}_n \subseteq \mathcal{X} \cap \mathbb{Z}^{(p,q,i)}$ such that $p \neq q^+$ and $\{x_1^n\}_n$ and $\{x_2^n\}_n$ are respectively strictly decreasing and strictly increasing. Let $a \in \mathbb{Z}^{(p,q^+,i)}$ be such that $\mathrm{Hom}_{\mathcal{C}_{w,m}}(x^n, a) \cong \mathbb{K}$ for each $n \in \mathbb{Z}$. By using the same argument as above, there exists $y = (y_1, y_2) \in \mathcal{X} \cap \mathbb{Z}^{(p^+,q,i)}$ such that $y_1 < a_1$ and $y_2 < a_2$. Then consider $a' = (a'_1, a'_2) \in \mathbb{Z}^{(p,q^+,i)}$ such that $a'_1 < y_1$, $a'_2 < y_2$, and $\mathrm{Hom}_{\mathcal{C}_{w,m}}(x^n, a) \cong \mathbb{K}$ for each $n \geq 2$. Then we can find $y' = (y'_1, y'_2) \in \mathcal{X} \cap \mathbb{Z}^{(p^+,q,i)}$ such that $y'_1 < a'_1$ and $y'_2 < a'_2$. Proceeding in this way we obtain the desired sequence of (w-PC 2). The arguments for (w-PC 2$'$), (w-PC 3), and (w-PC 3$'$) are similar.

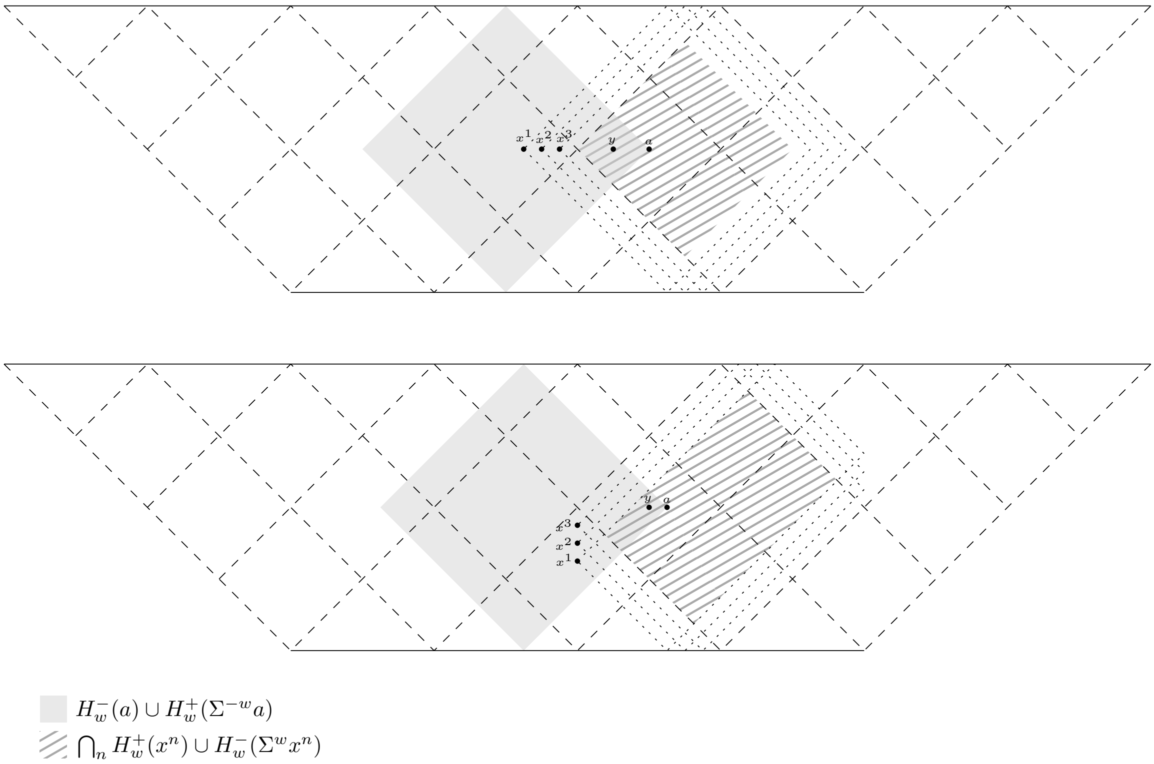


Figure 5.6: Illustration of the argument of Theorem 5.5.2 for proving that if $\mathcal{X}$ is precovering then it satisfies (w-PC 1) and (w-PC 2).

Step 2. Assume that $\mathcal{X}$ is a subcategory of $\mathcal{C}_{w,m}$ which satisfies the w-PC conditions, we prove that $\mathcal{X}$ is precovering.

Let $a = (a_1, a_2) \in \mathrm{ind}\,\mathcal{C}_{w,m}$, we show that there exists an $\mathcal{X}$-precover of a. Let $A^+ = H_w^+(\Sigma^{-w}a)$, $A^- = H_w^-(a)$ and $A = A^+ \cup A^-$. Note that if $A \cap \mathcal{X} = \varnothing$, then $0 \to a$ is an $\mathcal{X}$-precover of a. Now assume that $A \cap \mathcal{X} \neq \varnothing$. Then we construct a finite sequence of indecomposable objects $x^1, \ldots, x^n \in \mathcal{X}$ as follows. Let

$$\alpha = \begin{cases} \sup\{s \in \mathcal{Z}_m \mid \text{ there exists } (s,t) \in A^- \cap \mathcal{X}\} & \text{if } A^- \cap \mathcal{X} \neq \varnothing, \text{ or} \\ \sup\{s \in \mathcal{Z}_m \mid \text{ there exists } (t,s) \in A^+ \cap \mathcal{X}\} & \text{otherwise.} \end{cases}$$

From the fact that $\mathcal{X}$ satisfies the w-PC conditions, it follows that $\alpha \in \mathcal{Z}_m$. Indeed, if $\alpha \notin \mathcal{Z}_m$, then there exists a sequence $\{x^n = (x_1^n, x_2^n)\}_n \subseteq \mathcal{X}$ which belongs to A^- and is such that $\{x_1^n\}_n$ is strictly increasing, or belongs to A^+ and is such that $\{x_2^n\}_n$ is strictly increasing. Assume that the first case holds, the other case is analogous. As a consequence, up to extracting a subsequence of $\{x^n\}_n$, we have that x_2^n is constant, strictly increasing, or strictly decreasing. Thus, $\{x^n\}_n$ satisfies one of the (w-PC) conditions. As a consequence, there exists $y = (y_1, y_2) \in \mathcal{X} \cap A^-$ such that $y_1 > \alpha = \sup x_1^n$, giving a contradiction. Thus, $\alpha \in \mathcal{Z}_m$. Now we define

$$\beta = \begin{cases} \sup\{s \in \mathcal{Z}_m \mid (\alpha, s) \in A^-\} & \text{if } A^- \cap \mathcal{X} \neq \varnothing, \text{ or} \\ \sup\{s \in \mathcal{Z}_m \mid (s, \alpha) \in A^+\} & \text{otherwise.} \end{cases}$$

Since $\mathcal{X}$ satisfies the w-PC conditions, we have that $\beta \in \mathcal{Z}_m$ similarly as above. Then we denote $x^1 = (\alpha, \beta)$ if $A^- \cap \mathcal{X} \neq \varnothing$, and $x^1 = (\beta, \alpha)$ otherwise. We denote $H_1 = H_w^-(x^1) \cup H_w^+(\Sigma^{-w} x^1)$, and we consider $A_1^- = A^- \setminus H_1$, $A_1^+ = A^- \setminus H_1$, and $A_1 = A_1^- \cup A_1^+$. If $A_1 \cap \mathcal{X} = \varnothing$, then each $x \in \operatorname{ind} \mathcal{X}$ such that $\operatorname{Hom}_{\mathcal{C}_{w,m}}(x, a) \cong \mathbb{K}$ belongs to $A \cap H_1$. Moreover, it is straightforward to check that any non-zero morphism $x \to a$ factors through the non-zero morhism $x^1 \to a$. This implies that $x \to a$ is an $\mathcal{X}$-precover of a.

Now assume that $A_1 \cap \mathcal{X} \neq \varnothing$, then we find $x^2 \in A_1$ as above and define $A_2 = A \setminus (H_1 \cup H_2)$, where $H_2 = H_w^-(x^2) \cup H^+(\Sigma^{-w} x^2)$. By repeating this same procedure until we find k such that $A_k \cap \mathcal{X} = \varnothing$, we obtain a sequence $\{x^n = (x_1^n, x_2^n)\}_n \subseteq A \cap \mathcal{X}$. We prove that this sequence is finite. Assume that $\{x^n\}_n$ is infinite, then infinitely many of the objects x^n's belong to A^-, or infinitely many of the objects x^n's belong to A^+. Assume that the first case holds, the second case is analogous. Then, note that $x_1^{n+1} < x_1^n$ and $x_2^{n+1} > x_2^n$ for each n, see Figure 5.7. Thus, by (w-PC 2), there exists $y = (y_1, y_2) \in \mathcal{X} \cap A$ such that, since $x_1^1, x_1^2, \ldots$ are maximal, $y_1 = x_1^n$ and $y_2 > x_2^n$ for some n. This gives a contradiction with the maximality of x_2^n. Therefore, we have that the sequence $\{x^n\}_n = \{x^1, \ldots, x^k\}$ is finite. Figure 5.7 illustrates this sequence.

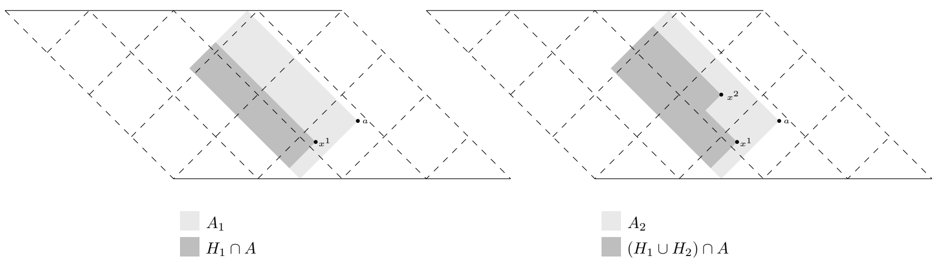


Figure 5.7: Illustration of the argument of Theorem 5.5.2 for proving that if $\mathcal{X}$ satisfies the w-PC conditions then it is precovering.

Note that, by construction, if $x \in \operatorname{ind} \mathcal{X}$ is such $\operatorname{Hom}_{\mathcal{C}_{w,m}}(x, a) \cong \mathbb{K}$, then $x \notin A_k$ because $A_k \cap \mathcal{X} = \varnothing$. Thus, $x \in A \setminus A_k = A \setminus (A \setminus (H_1 \cup \cdots \cup H_k)) = A \cap (H_1 \cup \cdots \cup H_k)$, see Figure 5.7, and then $x \in H_i$ for some $1 \leq i \leq k$. By Proposition 5.4.4, any non-zero morphism $x \to a$ factors through x^i. Therefore, we conclude that $\bigoplus_{n=1}^k x^n \to a$ is an $\mathcal{X}$-precover. $\square$

5.6 w-cluster tilting subcategories

In this section we classify the w-cluster tilting subcategories of $\mathcal{C}_{w,m}$ in terms of certain maximal collections of w-admissible arcs. We refer to [21] for the case $w = 2$ and to [27] for $m = 1$. We recall that a subcategory of $\mathcal{C}_{w,m}$ is w-cluster tilting if and only if it is weakly left w-cluster tilting and precovering, see Proposition 2.3.32. Moreover, since $\mathcal{C}_{w,m}$ is w-CY, being weakly left w-cluster tilting is equivalent to being weakly w-cluster tilting, see Remark 2.3.30.

In Section 5.5 we classified the precovering subcategories of $\mathcal{C}_{w,m}$, now we classify the weakly w-cluster tilting subcategories of $\mathcal{C}_{w,m}$. In [27] there is a classification of the weakly w-cluster tilting subcategories for the case $m = 1$. This classification can be extended to $m \geq 2$ in a straightforward way. The following proposition consists of the m accumulation points version of [27, Proposition 1.8]. Here we add some details to the argument of [27] for the convenience of the reader.

Proposition 5.6.1. *Let $x, y \in \operatorname{ind} \mathcal{C}_{w,m}$. Then x and y cross if and only if there exists $1 \leq i \leq w-1$ such that $\operatorname{Hom}_{\mathcal{C}_{w,m}}(x, \Sigma^i y) \cong \mathbb{K}$.*

Proof. We denote $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Note that y crosses x if and only if $y \in H^+(\tau^{-1}x)$, i.e. $x_1 + 1 \leq y_1 \leq x_2 - 1$ and $y_2 \geq x_2 + 1$, or $y \in H^-(\Sigma x)$, i.e. $y_1 \leq x_1 - 1$ and $x_1 + 1 \leq y_2 \leq x_2 - 1$. We divide the proof into steps.

Step 1. We have that $y \in H^+(\tau^{-1}x)$ if and only if there exists $1 \leq i \leq w-1$ such that $\Sigma^i y \in H^+_w(x)$.

Assume that $y \in H^+(\tau^{-1}x)$. Since x and y are w-admissible, $x_2 - x_1 = 1 + k(w-1)$ and $y_2 - y_1 = 1 + l(w-1)$ for some $k, l \geq 1$. Moreover, $y_1 - x_1 = i + n(w-1)$ for some $1 \leq i \leq w-1$ and $n \geq 0$. Note that $y_1 - i = x_1 + n(w-1) \geq x_1$ and, since $y_2 \geq x_2 + 1$,we have that $y_2 - i \geq x_2 + 1 - i \geq x_2$. In order to prove that $\Sigma^i y \in H^+_w(x)$, it remains to check that $y_1 - i \leq x_2 - w$. We have the following equality

$$y_1 - i = (y_1 - x_1 - i) - (x_2 - x_1) + x_2 = x_2 + (n-k)(w-1) + 1.$$

Since $y_1 \geq x_2 + 1$, then $n - k \ngeq 0$, otherwise $x_2 - 1 \geq y_1 \geq x_2 + 1 + i \geq x_2 + 2$, giving a contradiction. Thus, $n - k \leq -1$ and $y_1 - i \leq x_2 - w$. We obtain that $\Sigma^i y \in H^+_w(x)$.

Now assume that there exists $1 \leq i \leq w-1$ such that $\Sigma^i y \in H^+_w(x)$. Since $x_1 + 1 \leq x_1 + i \leq y_1 \leq x_2 - w + i \leq x_2 - 1$, and $y_2 \geq x_2 + i \geq x_2 + 1$, it follows that $y \in H^+(\tau^{-1}x)$.

Step 2. We have that $y \in H^-(\Sigma x)$ if and only if there exists $1 \leq i \leq w-1$ such that $\Sigma^i y \in H^+_w(\Sigma^w x)$.

We have that $y \in H^-(\Sigma x)$ if and only if $\Sigma x \in H^+(y)$. By Step 1, this is equivalent to $\Sigma^{j+1} x \in H^+_w(y)$ for some $0 \leq j \leq w-2$. This is equivalent to $\Sigma^{w-j-1} y \in H^-_w(\Sigma^w x)$ for some $1 \leq w-j-1 \leq w-1$ and concludes the proof. ☐

Given a subcategory $\mathcal{X}$ of $\mathcal{C}_{w,m}$, $\mathcal{X}$ is weakly left w-cluster tilting if and only if it is a maximal collection of pairwise non-crossing w-admissible arcs. In other words, $\mathcal{X}$ is weakly w-cluster tilting if and only if it consists of a $(w+1)$-angulation of $\mathcal{Z}_m$ made of w-admissile arcs.

The following result follows directly from Theorem 5.5.2 and Proposition 5.6.1.

Proposition 5.6.2. *Let $\mathcal{X}$ be a subcategory of $\mathcal{C}_{w,m}$. Then $\mathcal{X}$ is a w-cluster tilting subcategory of $\mathcal{C}_{w,m}$ if and only if it is a maximal collection of pairwise non crossing w-admissible arcs, and satisfies the w-PC conditions.*

Figure 5.8 provides an example of w-cluster tilting subcategory.

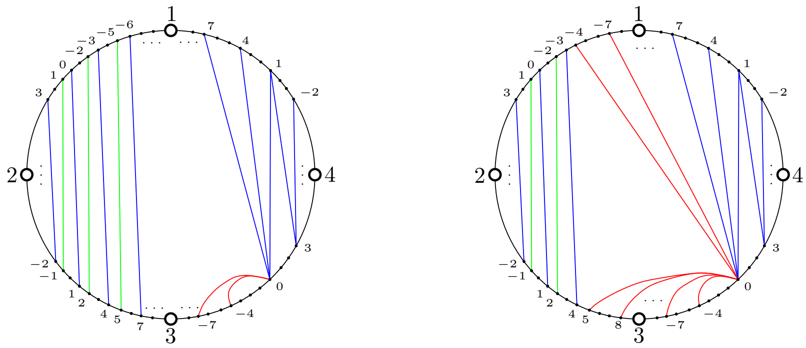


Figure 5.8: On the left: a weakly 4-cluster tilting subcategory of $\mathcal{C}_{4,4}$ which is not 4-cluster tilting. On the right: a 4-cluster tilting subcategory of $\mathcal{C}_{4,4}$. The coloured arcs are 4-admissible. The blue arcs are of the form (x_1, x_2) with $x_1 \equiv 0 \mod 3$, the green arcs are such that $x_1 \equiv 1 \mod 3$, and the red arcs are such that $x_1 \equiv 2 \mod 3$.

When $w = 2$, by [21, Theorem 5.7], a subcategory $\mathcal{X}$ of $\mathcal{C}_{2,m}$ is cluster-tilting if and only if it is a triangulation of $\mathcal{Z}_m$ such that at each accumulation point there exists a *fountain* or a *leapfrog* of $\mathcal{X}$ converging to it. We refer to [21, Definition 0.4] for the definitions of fountain and leapfrog.

We define a *sequence of parallel arcs* as a sequence of non-crossing arcs of $\mathcal{Z}_m$ which do not share any endpoint. If $w \neq 2$, each leapfrog of w-admissible arcs contains a subsequence of parallel w-admissible arcs, but there exist sequences of parallel w-admissible arcs which are not contained in a leapfrog of w-admissible arcs. We have the following conjecture, which is the $w \geq 2$ version of [21, Theorem 5.7], or the $m \geq 1$ version of [27, Theorem E].

Conjecture 5.6.3. Let $\mathcal{X}$ be a subcategory of $\mathcal{C}_{w,m}$. Then $\mathcal{X}$ is w-cluster tilting if and only if $\mathcal{X}$ is a $(w+1)$-angulation of $\mathcal{Z}_m$ such that each accumulation point of $\mathcal{Z}_m$ has either a fountain, a leapfrog, or a proper sequence of parallel arcs converging to it.

5.7 Torsion pairs

In this section we classify the torsion pairs in $\mathcal{C}_{w,m}$. These were classified in [14] for the case $m = 1$. We recall that, by Proposition 2.3.27, the torsion pairs in $\mathcal{C}_{w,m}$ are in bijection with the extension-closed precovering subcategories of $\mathcal{C}_{w,m}$. In Section 5.5 we classified the precovering subcategories in terms of the w-PC conditions.

In order to classify the extension-closed subcategories of $\mathcal{C}_{w,m}$, we introduce the w-Ptolemy condition. It consists of the w-CY version of Definition 3.4.1, or as the $m \geq 2$ version of [14, Definition 7.1]. We recall that, given two crossing arcs, their Ptolemy arcs are exactly the arcs obtained by joining their endpoints.

Definition 5.7.1. Let $\mathcal{X}$ be a subcategory of $\mathcal{C}_{w,m}$. We say that $\mathcal{X}$ satisfies *w-Ptolemy condition*, w-PT for short, if it is closed under taking w-admissible Ptolemy arcs.

We want to prove the following proposition.

Proposition 5.7.2. *Let $\mathcal{X}$ be a subcategory of $\mathcal{C}_{w,m}$. Then the following statements are equivalent.*

1. *The subcategory $\mathcal{X}$ satisfies the w-*PT *condition.*
2. *The subcategory $\mathcal{X}$ is closed under w-admissible extensions of the form $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ where $a, b \in \operatorname{ind} \mathcal{C}_{w,m}$.*
3. *The subcategory $\mathcal{X}$ is closed under w-admissible extensions.*

From the result above and Proposition 2.3.27, we obtain the following.

Theorem 5.7.3. *Let $(\mathcal{X}, \mathcal{Y})$ be a pair of subcategories of $\mathcal{C}_{w,m}$. Then $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{C}_{w,m}$ if and only if $\mathcal{X}$ satisfies the w-*PC *conditions, the w-*PT *condition, and $\mathcal{Y} = \mathcal{X}^{\perp_{\mathcal{C}_{w,m}}}$.*

In order to prove Proposition 5.7.2, we have the following w-admissible versions of Lemma 3.4.2 and Lemma 3.4.3, which can be proved using the same arguments.

Lemma 5.7.4. *Let $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$ be w-admissible with $a, b_1, \dots, b_n \in \operatorname{ind} \mathcal{C}_{w,m}$, $b = \bigoplus_{i=1}^n b_i$, and $h = (h_1, \dots, h_n)$. Then there exists $b' = \bigoplus_{i=1}^k b'_i$ a direct summand of b such that the objects $b'_1, \dots, b'_k \in \operatorname{ind} \mathcal{C}_{w,m}$ are pairwise* Hom*-orthogonal in $\mathcal{C}_{w,m}$, $b'_i \ncong \Sigma a$ for each i, all the entries of $h' = (h'_1, \dots, h'_k) \colon b' \to \Sigma a$ are non-zero, and there is the following w-admissible isomorphism of w-admissible triangles.*

$$\begin{tikzcd}
a \arrow[r] \arrow[d, "1_a"'] & e \arrow[r] \arrow[d, "\wr"'] & b \arrow[r, "h"] \arrow[d, "\wr"'] & \Sigma a \arrow[d, "1_{\Sigma a}"] \\
a \arrow[r] & e' \oplus b'' \arrow[r] & b' \oplus b'' \arrow[r, "{(h',0)}"] & \Sigma a
\end{tikzcd}$$

Lemma 5.7.5. *Let $\mathcal{X}$ be a subcategory of $\mathcal{C}_{w,m}$. If $\mathcal{X}$ is closed under w-admissible extensions of the form $a \longrightarrow e \longrightarrow b \longrightarrow \Sigma a$ with $a, b \in \operatorname{ind} \mathcal{C}_{w,m}$, then $\mathcal{X}$ is extension-closed in $\mathcal{C}_{w,m}$.*

The following lemma will also be useful for proving Proposition 5.7.2.

Lemma 5.7.6. *Let $a, b \in \operatorname{ind} \mathcal{C}_{w,m}$ be such that $b \ncong \Sigma a$, $h \colon b \to \Sigma a$ be a non-zero morphism of $\mathcal{C}_{2,m}$, and $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$ be a triangle in $\mathcal{C}_{2,m}$. Then the following statements are equivalent.*

1. *The morphism h is w-admissible.*

2. The object e is w-admissible.

3. There exists a w-admissible indecomposable direct summand of e.

Proof. First we prove the equivalence between statements (2) and (3). If (2) holds then (3) follows because all the indecomposable direct summands of e are w-admissible. Now assume that (3) holds. Since $h \neq 0$ is not an isomorphism, e is either indecomposable or has two indecomposable direct summands. In the first case, then the claim follows. Assume that the second case holds. Since $h \neq 0$, we have the following possibilities: $b \in H^+(\tau^{-1}a)$ or $b \in H^-(\Sigma a)$. Assume that $b \in H^+(\tau^{-1}a)$, then $e \cong (a_1, b_2) \oplus (b_1, a_2)$, see Section 3.4, the other case is analogous. If (a_1, b_2) is w-admissible, then $a_2 - b_1 = (a_2 - a_1) - (b_2 - a_1) + (b_2 - b_1) \equiv 1 \mod (w-1)$, i.e. (b_1, a_2) is w-admissible. Similarly, it is straightforward to check that if (b_1, a_2) is w-admissible, then (a_1, b_2) is w-admissible. This proves the equivalence between (2) and (3).

If statement (1) holds, then, by Lemma 5.2.11, $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$ is w-admissible and in particular e is w-admissible, i.e. (2) holds. Now assume that (2) holds, we prove (1). Since $h \neq 0$, we have that $b \in H^+(\tau^{-1}a)$ or $b \in H^-(\Sigma a)$. Assume that the first case holds, the second case is analogous. Then either $e = (a_1, b_2)$ is indecomposable, or $e \cong (a_1, b_2) \oplus (b_1, a_2)$. Since e is w-admissible, we have that (b_1, a_2) is w-admissible. Thus, $b_1 - (a_1 - 1) = -(b_2 - b_1) + (b_2 - a_1) + 1 \equiv 1 \mod (w-1)$, i.e. h is w-admissible. This concludes the argument. $\square$

Now we can prove Proposition 5.7.2.

Proof of Proposition 5.7.2. The equivalence between statements (2) and (3) can be proved with the same argument of Proposition 3.4.4. We prove that (1) implies (2). Consider a w-admissible triangle $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$ with $a, b \in \operatorname{ind} \mathcal{X}$. If $h = 0$ or $b \cong \Sigma a$, then $e \in \mathcal{X}$. Indeed, in the first case $e \cong a \oplus b$, and in the second case $e = 0$. If $h \neq 0$ and $b \ncong \Sigma a$, then $\operatorname{Hom}_{\mathcal{C}_{w,m}}(b, \Sigma a) \cong \mathbb{K}$ and, by Proposition 5.6.1, a and b cross. Moreover, the indecomposable direct summands of e are w-admissible Ptolemy arcs, see Section 3.4, and as a consequence $e \in \mathcal{X}$.

Now we prove that (2) implies (1). Let $a, b \in \operatorname{ind} \mathcal{X}$ be crossing arcs, we show that all their indecomposable Ptolemy arcs belong to $\mathcal{X}$. Since a and b cross, $a \ncong \Sigma b$ and there exist non-zero morphisms $h \colon b \to \Sigma a$ and $h' \colon a \to \Sigma b$ in $\mathcal{C}_{2,m}$. Thus, there exist triangles $a \longrightarrow e \longrightarrow b \xrightarrow{h} \Sigma a$ and $b \longrightarrow e' \longrightarrow a \xrightarrow{h'} \Sigma b$ in $\mathcal{C}_{2,m}$. The Ptolemy arcs of a and b are given by the indecomposable direct summands of e and e'. It is straightforward to check that if h is w-admissible, then h' is not w-admissible, and if h' is w-admissible then h is not w-admissible. Thus, we have the following possibilities: h and h' are both not w-admissible, h is w-admissible and h' is not, h' is w-admissible and h is not.

By Lemma 5.7.6, in the first case e and e' are not w-admissible, and as a consequence all the Ptolemy arcs of a and b are not w-admissible. In the second case, since h is w-admissible, $e \in \mathcal{X}$ is w-admissible, see Proposition 5.2.15, and e' is not w-admissible. Thus, $\mathcal{X}$ contains all the w-admissible Ptolemy arcs of a and b. The case when h is not w-admissible and h' is w-admissible is dual. Thus, we conclude that (1) and (2) are equivalent. $\square$

Chapter 6

(-1)-CY discrete cluster categories

We introduce a (-1)-CY version of Igusa–Todorov discrete cluster categories. To do so, we employ the results in [46] on continuous Nakayama representations for defining an infinite discrete version of symmetric Nakayama representations. We prove that these representations form an Krull–Schmidt abelian category, which is Frobenius, symmetric, and uniserial. After stabilising, we obtain a (-1)-CY triangulated category, we describe its AR quiver, and we observe that for m = 1 its geometric model coincides with the one for the Holm–Jørgensen category $\mathcal{T}_{-1}$.

6.1 Continuous representations

In this section we discuss the representations of $\mathbb{R}$, of S^1, and continuous Nakayama representations. These will be used in Section 6.2 for defining the infinite discrete version of symmetric Nakayama representations.

6.1.1 Representations of $\mathbb{R}$

We regard set of real numbers $\mathbb{R}$ as a category: the objects of $\mathbb{R}$ are the real numbers and for any $s, t \in \mathbb{R}$, if $s \leq t$, there is a unique morphism $f_{st} \colon s \to t$. For each $t \in \mathbb{R}$ the morphism f_{tt} coincides with the identity 1_t. For the rest of this chapter $\mathbb{K}$ will be a fixed field. We denote by $\operatorname{Vect}\mathbb{K}$ the category of vector spaces over $\mathbb{K}$.

Definition 6.1.1. A *representation* of $\mathbb{R}$ over $\mathbb{K}$ is a covariant functor $M \colon \mathbb{R} \to \operatorname{Vect}\mathbb{K}$. A *morphism* of representations is a natural transformation. We denote by $\operatorname{Rep}\mathbb{R}$ the category of representations of $\mathbb{R}$. A representation $M \in \operatorname{Rep}\mathbb{R}$ is *pointwise finite* if $\dim M(t) < \infty$ for each $t \in \mathbb{R}$. We denote by $\operatorname{Rep}^{\mathrm{pwf}}\mathbb{R}$ the category of pointwise finite representations of $\mathbb{R}$.

We fix some notation and terminology. Given $M, N \in \operatorname{Rep}\mathbb{R}$, by $\operatorname{Hom}_{\mathbb{R}}(M, N)$ we indicate, with an abuse of notation, the set of morphisms $M \to N$ in the category $\operatorname{Rep}\mathbb{R}$. For an interval $U \subseteq \mathbb{R}$, the *interval representation* M_U is given by

$$M_U(t) = \begin{cases} \mathbb{K} & \text{if } t \in U, \\ 0 & \text{otherwise} \end{cases}$$

with $M_U(f_{st})\colon M_U(s) \to M_U(t)$ equal to $1_{\mathbb{K}}$ if $s, t \in U$ and $s \leq t$, and equal to 0 otherwise. Now consider the intervals $U, V \subseteq \mathbb{R}$, the *left intersection* of V and U is defined as

$$V \cap_L U = \begin{cases} V \cap U & \text{if } v < u \text{ for any } (v, u) \in ((V \setminus U) \times U) \cup (V \times (U \setminus V)), \\ \varnothing & \text{otherwise.} \end{cases}$$

We refer to Figure 6.1 for an illustration. It is straightforward to check that $\mathrm{Hom}_{\mathbb{R}}(M_U, M_V) \cong \mathbb{K}$ if $V \cap_L U \neq \varnothing$, and $\mathrm{Hom}_{\mathbb{R}}(M_U, M_V) = 0$ otherwise.

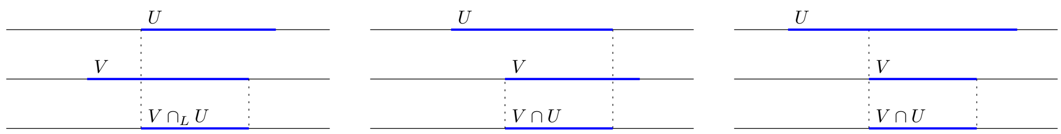


Figure 6.1: Illustration of left intersections of the intervals U and V. On the left, $V \cap_L U = V \cap U \neq \varnothing$. In the center and on the right, $V \cap U \neq \varnothing$ but $V \cap_L U = \varnothing$.

Definition 6.1.2. Let $U, V \subseteq \mathbb{R}$ be bounded intervals. We define the *standard morphism* $\varphi\colon M_U \to M_V$ as $\varphi(x) = 1_{\mathbb{K}}$ for each $x \in V \cap_L U$, and $\varphi(x) = 0$ for each $x \in \mathbb{R} \setminus (V \cap_L U)$.

Note that the composition of two standard morphisms is still a standard morphism. The following theorem gives the decomposition of the representations of $\mathbb{R}$.

Theorem 6.1.3 ([16, Theorem 1.1])**.** *Any pointwise finite representation of $\mathbb{R}$ decomposes uniquely, up to isomorphism and reordering the summands, as a direct sum of possibly infinitely many interval representations.*

6.1.2 Representations of S^1

Similarly as for $\mathbb{R}$, we also regard the circle S^1 as a category. Given $x, y \in S^1$ with $x \neq y$, there is a path $g_{xy}\colon x \to y$ moving from x to y along S^1 in the anticlockwise direction. By concatenating g_{xy} and g_{yx}, we obtain a path $\omega_x = g_{yx}g_{xy}\colon x \to x$ which consists of moving from x to x in the anticlockwise direction around S^1 exactly once. By convention, ω_x^0 and g_{xx} are the lazy paths $x \to x$, and we often denote them by 1_x. The objects of S^1, as a category, are the points of the circle, and for each pair of points $x, y \in S^1$ the morphisms $x \to y$ are the paths of the form $g_{xy}\omega_x^n = \omega_y^n g_{xy}$ with $n \in \mathbb{Z}$.

Analogously to Definition 6.1.1, the *representations* of S^1 over $\mathbb{K}$ are the covariant functors $M\colon S^1 \to \mathrm{Vect}\,\mathbb{K}$. We denote by $\mathrm{Rep}\,S^1$ the category of representations of S^1, and by $\mathrm{Rep}^{\mathrm{pwf}}\,S^1$ the category of pointwise finite representations. If $M, N \in \mathrm{Rep}\,S^1$, by $\mathrm{Hom}_{S^1}(M, N)$ we denote, with an abuse of notation, the set of morphisms $M \to N$ in the category $\mathrm{Rep}\,S^1$.

We introduce some notation from [46]. Consider a bounded interval $U \subseteq \mathbb{R}$ and the covering map $\gamma\colon \mathbb{R} \to S^1, t \mapsto t \mod 2\pi$. Let $x \in S^1$ and consider the set $\gamma^{-1}(x) \cap U$, we denote its cardinality by n_x and its elements by $b_{1,x} < \cdots < b_{n_x,x}$. Note that $n_x = 0$ if and only if $x \notin \gamma(U)$. Figure 6.2 provides an illustration. We now introduce some representations which are building blocks of pointwise finite representations.

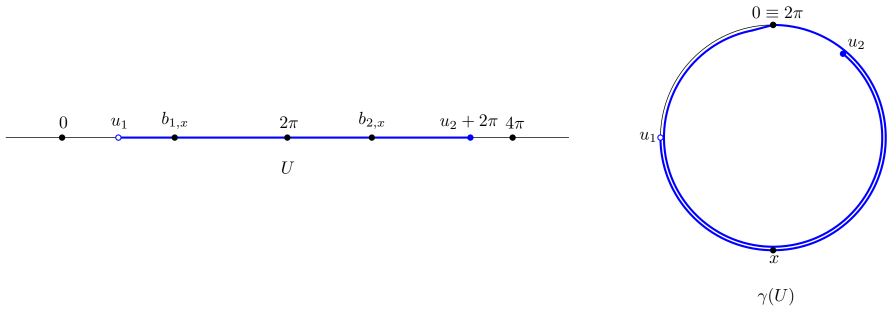


Figure 6.2: The bounded interval $U = (u_1, u_2 + 2\pi] \subseteq \mathbb{R}$ and the cyclically ordered set $\gamma(U) \subseteq S^1$. The empty blue circle indicates that the endpoints are excluded, while the full blue circle that the enpoints are included.

Definition 6.1.4 ([46, Section 2.2])**.** Let $U \subseteq \mathbb{R}$ be a bounded interval. Keeping the notation above, the representation $\overline{M}_U$ of S^1 defined as follows is called a *string*.

- For each $x \in S^1$ we have that $\overline{M}_U(x) = \mathbb{K}b_{1,x} \oplus \cdots \oplus \mathbb{K}b_{n_x,x}$.
- For each $x, y \in S^1$ we have that

$$\overline{M}_U(g_{xy})(b_{i,x}) = \begin{cases} b_{j,y} & \text{if there exists } b_{j,y} \text{ such that } 0 \leq b_{j,y} - b_{i,x} < 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 6.1.5. The representation $\overline{M}_U$ is pointwise finite. Moreover, $\overline{M}_U(\omega_x)(b_{i,x}) = b_{i+1,x}$ if $i \neq n_x$, and $\overline{M}_U(\omega_x)(b_{n_x,x}) = 0$.

Theorem 6.1.6 ([23, Theorem 3.8], [46, Corollary 2.8, 2.9])**.** *Let $U, V \subseteq \mathbb{R}$ be bounded intervals. The following statements hold.*

- *We have that $\overline{M}_U \cong \overline{M}_V$ if and only if $U = V + 2n\pi$ for some $n \in \mathbb{Z}$.*
- *The ring $\mathrm{End}_{S^1}(\overline{M}_U)$ is local, and therefore $\overline{M}_U$ is indecomposable.*
- *We have that $\mathrm{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \cong \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{R}}(M_U, M_{V+2n\pi})$.*

Note that, since the intervals U and V are bounded, the direct sum in the statement above is finite. The following theorem provides a decomposition of the pointwise finite representations of S^1. We refer to [23, Definition 3.5] for the definition of *band*.

Theorem 6.1.7 ([23, Theorem 5.6])**.** *Any pointwise finite representation of S^1 decomposes uniquely, up to isomorphism and reordering the summands, as a direct sum of possibly infinitely many strings and finitely many bands.*

Standard morphisms in $\operatorname{Rep}\mathbb{R}$ induce morphisms between strings in $\operatorname{Rep} S^1$. We introduce some notation which will be useful for Definition 6.1.9.

Notation 6.1.8. Let $U, V \subseteq \mathbb{R}$ be bounded intervals, $\varphi\colon M_U \to M_V$ be the standard morphism, and $x \in S^1$ be such that $\overline{M}_U(x) \neq 0$. We have that $\overline{M}_U(x) = \bigoplus_{i=1}^{n_x} \mathbb{K} b_{i,x}$ where $\{b_{i,x}\}_{i=1}^{n_x} = \gamma^{-1}(x) \cap U$. Note that if $b_{i,x} \in V \cap_L U$ for some i, then there exists a unique $c_{j,x} \in \gamma^{-1}(x) \cap V$ such that $b_{i,x} = c_{j,x}$. With a light abuse of notation, we denote $\varphi(b_{i,x}) = c_{j,x}$ if $b_{i,x} \in V \cap_L U$, and $\varphi(b_{i,x}) = 0$ otherwise.

Definition 6.1.9 ([46, p. 44])**.** Keeping U, V, and φ as in Notation 6.1.8, the morphism $\overline{\varphi}\colon \overline{M}_U \to \overline{M}_V$ is defined as $\overline{\varphi}(x)(b_{i,x}) = \varphi(b_{i,x})$ for each $x \in S^1$ such that $\overline{M}_U(x) \neq 0$.

The following lemma will be useful for Proposition 6.4.6 and Proposition 6.4.7.

Lemma 6.1.10. *Let $U, V, W \subseteq \mathbb{R}$ be bounded intervals. The following statements hold.*

1. *Let $\varphi\colon M_U \to M_V$ be a standard morphism. Then $\overline{\varphi} = 0$ if and only if $\varphi = 0$, or equivalently $V \cap_L U = \varnothing$.*
2. *If $\varphi\colon M_U \to M_V$ and $\psi\colon M_V \to M_W$ are standard morphisms, then $\overline{\psi\varphi} = \overline{\psi}\,\overline{\varphi}$.*
3. *Let $\varphi\colon M_U \to M_V$ and $\psi\colon M_{U+2n\pi} \to M_{V+2n\pi}$ be standard morphisms and $n \in \mathbb{Z}$. Then $\overline{\varphi} = \overline{\psi}$.*
4. *Assume that there exists a unique $n \in \mathbb{Z}$ such that $(V + 2n\pi) \cap_L U \neq \varnothing$. Then any non-zero morphism $f\colon \overline{M}_U \to \overline{M}_V$ is of the form $f = \lambda\overline{\varphi}$, where $\varphi\colon M_U \to M_{V+2n\pi}$ is a standard morphism and $\lambda \in \mathbb{K}^*$.*

Proof. We prove statement (1). If $\varphi = 0$ then $\overline{\varphi} = 0$ by Definition 6.1.9. If $\varphi \neq 0$ then there exists $t \in V \cap_L U \neq \varnothing$. Now consider $x = \gamma(t) \in S^1$, the sets $\gamma^{-1}(x) \cap U \neq \varnothing$ and $\gamma^{-1}(x) \cap V$ are non-empty because t belongs to both. Since $t \in \gamma^{-1}(x) \cap U = \{b_{i,x}\}_{i=1}^{n_x}$, we have that $t = b_{i,x}$ for some i. Therefore, $b_{i,x} \in V \cap_L U$, $\overline{\varphi}(x)(b_{i,x}) = \varphi(b_{i,x}) \neq 0$, and as a consequence $\overline{\varphi} \neq 0$.

Now we prove (2). If $\varphi = 0$ or $\psi = 0$, then $\psi\varphi = 0$ and $\overline{\psi\varphi} = 0$. Moreover, $\overline{\varphi} = 0$ or $\overline{\psi} = 0$ and then $\overline{\psi}\,\overline{\varphi} = 0 = \overline{\psi\varphi}$. Now assume that $\varphi \neq 0$ and $\psi \neq 0$, i.e. $V \cap_L U \neq \varnothing$ and $W \cap_L V \neq \varnothing$. Let $x \in S^1$ be such that $\overline{M}_U(x) \neq 0$, and consider $b_{i,x} \in \gamma^{-1}(x) \cap U$. We have the following possibilities: $W \cap_L U \neq \varnothing$ or $W \cap_L U = \varnothing$. Assume that the first case holds, the second case is similar. If $b_{i,x} \in U \cap_L W$, then $b_{i,x} \in V \cap_L U$, and the following equalities hold

$$\overline{\psi}(x)\overline{\varphi}(x)(b_{i,x}) = \overline{\psi}(x)(\varphi(b_{i,x})) = \overline{\psi}(x)(c_{j,x}) = \psi(c_{j,x}) = \psi\varphi(b_{i,x}) = \overline{\psi\varphi}(x)(b_{i,x})$$

where $c_{j,x} \in \gamma^{-1}(x) \cap V$ is such that $c_{j,x} = b_{i,x}$. Now assume that $b_{i,x} \notin U \cap_L W$, then $\overline{\psi\varphi}(x)(b_{i,x}) = \psi\varphi(b_{i,x}) = 0$. Moreover, if $\varphi(b_{i,x}) = 0$ then $\overline{\psi}(x)(\varphi(b_{i,x})) = 0$, and if $\varphi(b_{i,x}) \neq 0$, i.e. $b_{i,x} \in U \cap_L V$, then $\overline{\psi}(x)(\varphi(b_{i,x})) = \psi\varphi(b_{i,x}) = 0$ because $b_{i,x} \notin W \cap_L U$. In both cases $\overline{\psi}(x)\overline{\varphi}(x) = \overline{\psi}(x)(\varphi(b_{i,x})) = 0 = \overline{\psi\varphi}(x)(b_{i,x})$. Therefore, we obtain that $\overline{\psi}\,\overline{\varphi} = \overline{\psi\varphi}$.

Statement (3) is straightforward and follow from Definition 6.1.4. We prove statement (4). By Theorem 6.1.6 $\operatorname{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \cong \mathbb{K}$ and, since $(V + 2n\pi) \cap_L U \neq \varnothing$, we have that $\varphi \neq 0$ and then $\overline{\varphi} \neq 0$. Thus, $f = \lambda\overline{\varphi}$ for some $\lambda \in \mathbb{K}^*$. □

6.1.3 Continuous Nakayama representations

The continuous Nakayama representations of S^1 consist of pointwise finite representations of S^1 satisfying some conditions which are determined by a map called the *Kupisch function*. This map is the continuous analogue of the Kupisch series for Nakayama algebras and determines the length of the projective representations.

Definition 6.1.11 ([46, Definition 3.9])**.** A *Kupisch function* $\kappa \colon \mathbb{R} \to \mathbb{R}^{>0}$ is a map such that

- for each $t \in \mathbb{R}$ we have that $\kappa(t + 2\pi) = \kappa(t)$, and
- for each $t_1, t_2 \in \mathbb{R}$ if $t_1 \leq t_2$, then $t_1 + \kappa(t_1) \leq t_2 + \kappa(t_2)$.

Definition 6.1.12 ([46, Definition 3.9])**.** Let κ be a Kupisch function and $M \in \operatorname{Rep}^{\mathrm{pwf}} S^1$. We say that M is a *continuous Nakayama representation* of (S^1, κ) if each indecomposable direct summand of M is a string of the form $\overline{M}_U$, where $U \subseteq \mathbb{R}$ is a bounded interval such that $U \subseteq [\inf U, \inf U + \kappa(\inf U)]$. We denote by $\operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa)$ the category of continuous Nakayama representations of (S^1, κ).

Remark 6.1.13 ([46, Remark 3.10])**.** The category $\operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa)$ is an abelian subcategory of $\operatorname{Rep}(S^1)$, and any $M \in \operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa)$ decomposes uniquely, up to isomorphism and reordering the summands, as the direct sum of possibly infinitely many strings.

6.2 Discrete symmetric Nakayama representations

We want to study the representations of the infinite discrete versions of the symmetric Nakayama algebra $\mathbb{K}(C_n/\operatorname{rad}^{n+1})$, where C_n is the oriented cycle with n vertices. We first define a Kupisch function $\kappa_{\mathcal{Z}_m}$ from the ∞-gon $\mathcal{Z}_m$, and then we define the category of representations of $(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ as an abelian subcategory of $\operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa_{\mathcal{Z}_m})$.

We refer to Section 3.1 for the definition of the ∞-gon $\mathcal{Z}_m \subseteq S^1$. We often regard $\mathcal{Z}_m$ as a subset of the interval $(0, 2\pi) \subseteq \mathbb{R}$, where the accumulation point $1 \in [m]$ and the real number $0 \in \mathbb{R}$ are identified, see Figure 6.3. In this chapter, unlike the previous chapters, given $z \in \mathcal{Z}_m$ we denote its successor in $\mathcal{Z}_m$ by z^+, and its predecessor by z^-. By $z + 1$ we denote the the sum of the real numbers z and 1.

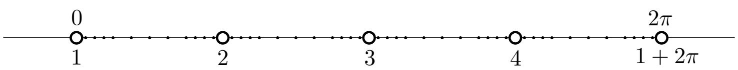


Figure 6.3: The ∞-gon $\mathcal{Z}_4$ regarded as a subset of $\mathbb{R}$.

Definition 6.2.1. Let $s, t \in \mathbb{R}$ be such that $t \equiv s \mod 2\pi$ with $s \in [0, 2\pi)$. We define

$$\kappa_{\mathcal{Z}_m}(t) = \begin{cases} 2\pi + z^+ - s & \text{if } s \in [z, z^+) \text{ for some } z \in \mathcal{Z}_m, \\ 2\pi & \text{if } s \in [m]. \end{cases}$$

Note that the map $\kappa_{\mathcal{Z}_m} : \mathbb{R} \to \mathbb{R}^{>0}$ is 2π-periodic.

Lemma 6.2.2. *The map $\kappa_{\mathcal{Z}_m}$ is a Kupisch function.*

Proof. Since κ is 2π-periodic by definition, we check that for any $t_1, t_2 \in \mathbb{R}$ if $t_1 \leq t_2$ then $t_1 + \kappa_{\mathcal{Z}_m}(t_1) \leq t_2 + \kappa_{\mathcal{Z}_m}(t_2)$. Since $\kappa_{\mathcal{Z}_m}$ is 2π-periodic, without loss of generality we can assume that $t_1 \in [0, 2\pi)$ and $t_2 \in [0, 2\pi) + 2h\pi$ for some $h \in \mathbb{Z}$ with $h \geq 0$.

If $h = 0$ then we can proceed with a case analysis where we distinguish when t_1 and t_2 are accumulation points of $\mathcal{Z}_m$ or not. We show the case where $t_1 \in [m]$ and $t_2 \in [z, z^+)$ for some $z \in \mathcal{Z}_m$. Since $t_1 < t_2 < z^+$, we have that $t_1 + \kappa_{\mathcal{Z}_m}(t_1) = t_1 + 2\pi < 2\pi + z^+ = t_2 + \kappa_{\mathcal{Z}_m}(t_2)$. The other cases are straightforward.

If $h = 1$, let $s = t_2 - 2\pi \in [0, 2\pi)$ and note that either $t_1 \leq s$ or $t_1 > s$. If $t_1 \leq s$ then, from the case $h = 0$ above, $t_1 + \kappa_{\mathcal{Z}_m}(t_1) \leq s + \kappa_{\mathcal{Z}_m}(s) \leq s + 2\pi + \kappa_{\mathcal{Z}_m}(s) = t_2 + \kappa_{\mathcal{Z}_m}(t_2)$. If $t_1 > s$, then we can divide the proof into cases where t_1 and s are or are not proper accumulation points of $\mathcal{Z}_m$. We show the case where $t_1 \in [z, z^+)$ for some $z \in \mathcal{Z}_m$ and that $s \in [m]$. Since $z^+ < 2\pi < s + 2\pi$, we have that $t_1 + \kappa_{\mathcal{Z}_m}(t_1) = 2\pi + z^+ < s + 4\pi = t_2 + 2\pi = t_2 + \kappa_{\mathcal{Z}_m}(t_2)$. The other cases are straightforward.

If $h = 2$, then $t_2 - 2\pi \in [0, 2\pi) + 2\pi$. From the case $h = 1$, we have that $t_1 + \kappa_{\mathcal{Z}_m}(t_1) \leq t_2 - 2\pi + \kappa_{\mathcal{Z}_m}(t_2 - 2\pi) = t_2 - 2\pi + \kappa_{\mathcal{Z}_m}(t_2) < t_2 + \kappa_{\mathcal{Z}_m}(t_2)$. If $h \geq 3$ we can proceed analogously. We can conclude that $t_1 + \kappa_{\mathcal{Z}_m}(t_1) \leq t_2 + \kappa_{\mathcal{Z}_m}(t_2)$. $\square$

We now define the category $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ of infinite discrete versions of symmetric Nakayama representations.

Definition 6.2.3. We define $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ as the full subcategory of $\operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa_{\mathcal{Z}_m})$ of representations M satisfying the following conditions.

1. If $x, y \in S^1$ are such that $z < x \leq y \leq z^+$ for some $z \in \mathcal{Z}_m$, then $M(g_{xy}) : M(x) \to M(y)$ is an isomorphism.

2. For each $p \in [m]$ there exist $a, b \in \mathcal{Z}_m$ such that $a < p < b$ are in cyclic order, and if $x, y \in S^1$ are such that $a < x \leq y \leq b$ are in cyclic order, then $M(g_{xy}) : M(x) \to M(y)$ is an isomorphism.

We prove that $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is a wide subcategory of $\operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa_{\mathcal{Z}_m})$, see Definition 2.2.5, and is therefore an abelian category.

Theorem 6.2.4. *The category $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is a wide subcategory of $\operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa_{\mathcal{Z}_m})$.*

Proof. We show that $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is an additive subcategory of $\operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa_{\mathcal{Z}_m})$ closed under extensions, kernels and cokernels. It is straightforward to check that $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ contains the zero object and that it is closed under isomorphisms and finite direct sums. We show that $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is closed under direct summands. Consider $M \in \operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ such that $M \cong L \oplus N$ with $L, N \in \operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa_{\mathcal{Z}_m})$, we check that $L, N \in \operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. Let $x, y \in S^1$ be such that $M(g_{xy})$ is an isomorphism, then so is $L(g_{xy}) \oplus N(g_{xy})\colon L(x) \oplus N(x) \to L(y) \oplus N(y)$, and as a consequence $L(g_{xy})$ and $N(g_{xy})$ are isomorphisms. Therefore, L and N satisfy Definition 6.2.3.

Now we show that $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is closed under extensions. Consider a short exact sequence $0 \longrightarrow L \xrightarrow{F} M \xrightarrow{G} N \longrightarrow 0$ in $\operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa_{\mathcal{Z}_m})$ with $L, N \in \operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, we show that $M \in \operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. Let $x, y \in S^1$ be such that $L(g_{xy})$ and $N(g_{xy})$ are isomorphisms, then the following is a commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & L(x) \arrow[r, "F(x)"] \arrow[d, "L(g_{xy})"'] & M(x) \arrow[r, "G(x)"] \arrow[d, "M(g_{xy})"'] & N(x) \arrow[r] \arrow[d, "N(g_{xy})"'] & 0 \\
0 \arrow[r] & L(y) \arrow[r, "F(y)"] & M(y) \arrow[r, "G(y)"] & N(y) \arrow[r] & 0
\end{tikzcd}$$

Since $L(g_{xy})$ and $N(g_{xy})$ are isomorphisms, by the Five Lemma, $M(g_{xy})$ is also an isomorphism. Therefore M satisfies the conditions of Definition 6.2.3.

Now we show that $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is closed under kernels. Consider $L, M \in \operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, a morphism $F\colon L \to M$, and its kernel $K = \operatorname{Ker} F \to L$ in $\operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa_{\mathcal{Z}_m})$. Let $x, y \in S^1$ be such that $L(g_{xy})$ and $M(g_{xy})$ are isomorphisms. We have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & K(x) \arrow[r] \arrow[d, "K(g_{xy})"'] & L(x) \arrow[r, "F(x)"] \arrow[d, "L(g_{xy})"'] & M(x) \arrow[d, "M(g_{xy})"'] \\
0 \arrow[r] & K(y) \arrow[r] & L(y) \arrow[r, "F(y)"] & M(y)
\end{tikzcd}$$

By the universal property of the kernel, $K(g_{xy})$ is an isomorphism. Thus, K satisfies the conditions of Definition 6.2.3. Dually, we can prove that $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is closed under cokernels. We can conclude that $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is a wide subcategory of $\operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa_{\mathcal{Z}_m})$. □

6.3 Indecomposable objects

We prove that the category $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is Krull–Schmidt and we describe its indecomposable objects. We arrange the indecomposable objects of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ into a coordinate system, which gives the AR quiver of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, as we will see in Section 6.8.

6.3.1 Krull–Schmidt property

This section is devoted to establish the following Krull–Schmidt decomposition theorem for the objects of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. First, we define the set of intervals

$$\mathcal{W} = \{(u_1, u_2 + 2h\pi] \subseteq \mathbb{R} \mid u_1, u_2 \in \mathcal{Z}_m, h \in \mathbb{Z}, \text{ and } u_1^+ \leq u_2 + 2h\pi \leq u_1^+ + 2\pi\}.$$

Note that either $h = 0$ or $h = 1$. The condition $u_1^+ \leq u_2 + 2h\pi \leq u_1^+ + 2\pi$ ensures that the "shortest" intervals of $\mathcal{W}$ are of the form $(u_1, u_1^+]$. Such intervals will correspond to the simple objects of $\operatorname{rep}(\mathcal{Z}_m\, \kappa_{\mathcal{Z}_m})$, see Section 6.7. The "longest" intervals in $\mathcal{W}$ are of the form $(u_1, u_1^+ + 2\pi]$, which correspond to indecomposable projective-injective objects of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, see Section 6.5.

Theorem 6.3.1. *The category* $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ *is Krull–Schmidt. Moreover, the indecomposable objects of* $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ *are exactly those of the form* $\overline{M}_U$ *with* $U \in \mathcal{W}$*, up to isomorphism.*

With Proposition 6.3.3 we prove that conditions (1) and (2) of Definition 6.2.3 imply that the isoclasses of indecoposable objects of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ are in bijection with the intervals in $\mathcal{W}$. We start with the following lemma.

Lemma 6.3.2. *Let* $U \in \mathcal{W}$*, then* $\overline{M}_U \in \operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$*.*

Proof. For each $U = (u_1, u_2 + 2h\pi] \in \mathcal{W}$ we have that $U \subseteq [u_1, u_1 + \kappa_{\mathcal{Z}_m}(u_1)]$, i.e. $\overline{M}_U \in \operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa_{\mathcal{Z}_m})$. Now we prove that condition (1) of Definition 6.2.3 holds. Let $x, y \in S^1$ be such that $z < x \leq y \leq z^+$ are in cyclic order for some $z \in \mathcal{Z}_m$. Since $U = (u_1, u_2 + 2h\pi]$ is bounded and $u_1, u_2 \in \mathcal{Z}_m$, the sets $\gamma^{-1}(x) \cap U = \{b_{i,x}\}_{i=1}^{n_x}$ and $\gamma^{-1}(y) \cap U = \{b_{i,y}\}_{i=1}^{n_y}$ have the same finite cardinality, possibly equal to 0, which we denote by n. Therefore, if $n = 0$, $\overline{M}_U(x) = \overline{M}_U(y) = 0$ and $\overline{M}_U(g_{xy})$ is an isomorphism. If $n \neq 0$, then $\overline{M}_U(x) = \bigoplus_{i=1}^n \mathbb{K}b_{i,x}$ and $\overline{M}_U(y) = \bigoplus_{i=1}^n \mathbb{K}b_{i,y}$. Moreover, $0 \leq b_{i,y} - b_{i,x} < 2\pi$, see Figure 6.4. Thus, $\overline{M}_U(g_{xy})(b_{i,x}) = b_{i,y}$ for each i, and as a consequence $\overline{M}_U(g_{xy})$ is an isomorphism.

Now we prove that condition (2) holds. Let $p \in [m]$, then there exist $a, b \in S^1$ such that $a < p < b$ are in cyclic order and, for each $x, y \in S^1$ such that $a < x \leq y < b$, the sets $\gamma^{-1}(x) \cap U = \{b_{i,x}\}_{i=1}^{n_x}$ and $\gamma^{-1}(y) \cap U = \{b_{i,y}\}_{i=1}^{n_y}$ have have the same finite cardinality, n, possibly equal to zero. Moreover, if $n \neq 0$, $0 \leq b_{i,y} - b_{i,x} < 2\pi$ for each i. We refer to Figure 6.4 for an illustration. As above, we obtain that $\overline{M}_U(g_{xy})$ is an isomorphism. □

Proposition 6.3.3. *Let* $M \in \operatorname{ind}\operatorname{Rep}^{\mathrm{pwf}}(S^1, \kappa_{\mathcal{Z}_m})$*. Then* $M \in \operatorname{ind}\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ *if and only if* $M \cong \overline{M}_U$ *for some* $U \in \mathcal{W}$*.*

Proof. By Lemma 6.3.2, the strings $\overline{M}_U$, with $U \in \mathcal{W}$, are indecomposable objects of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. Now consider $M \in \operatorname{ind}\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. By Definition 6.1.12 we know that $M \cong \overline{M}_U$ for some bounded interval $U \subseteq \mathbb{R}$ such that $U \subseteq [\inf U, \inf U + \kappa_{\mathcal{Z}_m}(\inf U)]$. Now let $u_1 = \inf U$ and $u_2 = \sup U$. We divide the proof into claims.

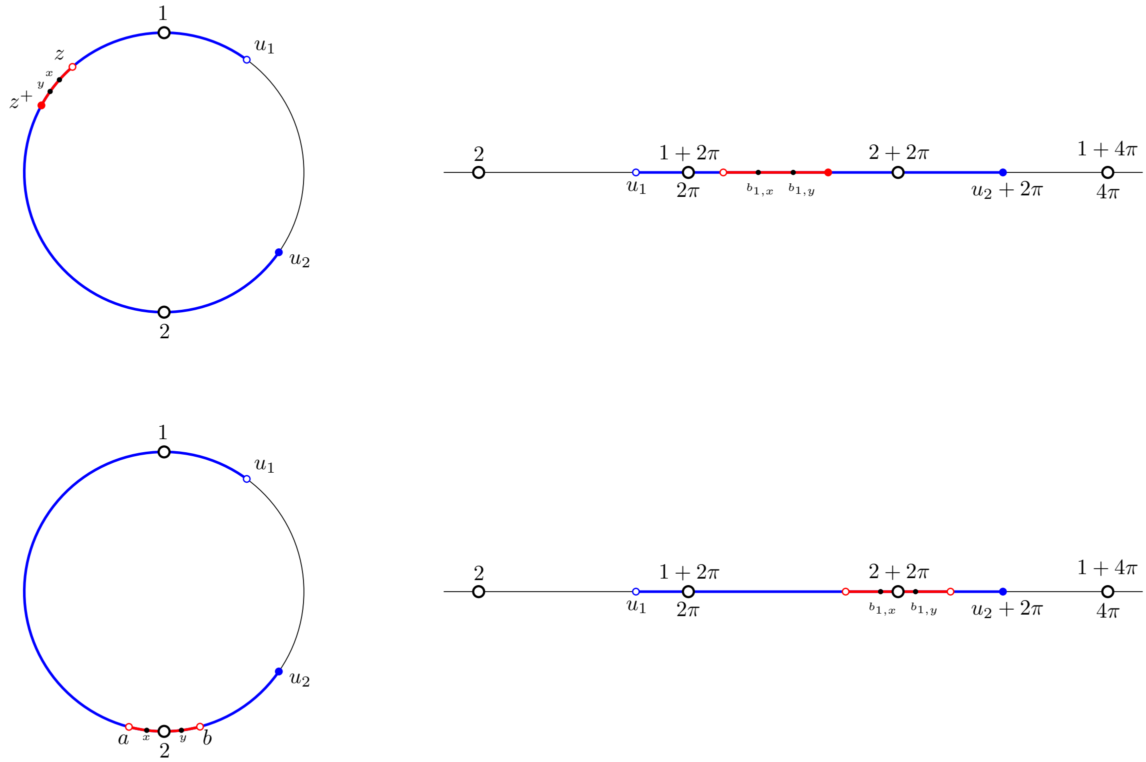


Figure 6.4: Illustration of the argument of Lemma 6.3.2 in $\mathcal{Z}_2$.

Claim 1. We have that $u_1, u_2 \notin [m] + 2\pi\mathbb{Z}$.

Assume that $u_1 = p + 2k\pi$ for some $p \in [m]$. By condition (2) of Definition 6.2.3, there exist $a, b \in \mathcal{Z}_m$ such that $a < p < b$ are in cyclic order and $\overline{M}_U(g_{xy})$ is an isomorphism for all $x, y \in S^1$ such that $a < x \leq y \leq b$ are in cyclic order. Since $U \subseteq [u_1, u_1 + \kappa_{\mathcal{Z}_m}(u_1)] = [p, p + 2\pi] + 2k\pi$, then $u_2 \leq u_1 + 2\pi$. If $u_2 < u_1 + 2\pi$ then there exist $x, y \in S^1$ such that $a < x < p < y < b$ are in cyclic order, $\overline{M}_U(x) = 0$, and $\overline{M}_U(y) \cong \mathbb{K}$. Then $\overline{M}_U(g_{xy})$ is not an isomorphism, and this gives a contradiction. If $u_2 = u_1 + 2\pi$, then consider $x, y \in S^1$ such that $a < x < p < y < b$ are in cyclic order. We have that $\overline{M}_U(x) \cong \mathbb{K}$ and $\overline{M}_U(y) \cong \mathbb{K}$, but $\overline{M}_U(g_{xy}) = 0$, see Definition 6.1.4, and this again gives a contradiction. Therefore, $u_1 \notin [m] + 2\pi\mathbb{Z}$. With an analogous argument we can prove that $u_2 \notin [m] + 2\pi\mathbb{Z}$.

Claim 2. We have that $u_1, u_2 \in \mathcal{Z}_m + 2\pi\mathbb{Z}$.

Assume that $u_1 \notin \mathcal{Z}_m + 2\pi\mathbb{Z}$, then there exist $z_1 \in \mathcal{Z}_m$ and $k \in \mathbb{Z}$ such that $z_1 + k\pi < u_1 < z_1^+ + 2k\pi$. We have the following possibilites: $u_2 < u_1 + 2\pi$, $u_2 = u_1 + 2\pi$, or $u_2 > u_1 + 2\pi$. If $u_2 < u_1 + 2\pi$, then there exist $x, y \in S^1$ such that $z_1 < x < \gamma(u_1) < y < z_1^+$ are in cyclic order, $\overline{M}_U(x) = 0$, and $\overline{M}_U(y) \cong \mathbb{K}$. If $u_2 = u_1 + 2\pi$, consider $x, y \in S^1$ such that $z_1 < x < \gamma(u_1) < y < z_1^+$ are in cyclic order. We have that $\overline{M}_U(x) \cong \mathbb{K}$ and $\overline{M}_U(y) \cong \mathbb{K}$, but $\overline{M}_U(g_{xy}) = 0$, see Definition 6.1.4. If $u_2 \geq u_1 + 2\pi$ then, since $U \subseteq [u_1, u_1 + \kappa_{\mathcal{Z}_m}(u_1)]$, we have that $u_2 \leq z_1^+ + 2(k+1)\pi$. Therefore, there exist $x, y \in S^1$ such that $z_1 < x < \gamma(u_1) < y < z_1^+$ are in cyclic order where $\overline{M}_U(x) = \mathbb{K}$ and $\overline{M}_U(y) \cong \mathbb{K}^2$. In each case we obtain that $\overline{M}_U(g_{xy})$ is not an isomorphism, which gives a contradiction with statement (1) of Definition 6.2.3. We conclude that $u_1 \in \mathcal{Z}_m + 2\pi\mathbb{Z}$. Using an analogous argument, we obtain that $u_2 \in \mathcal{Z}_m + 2\pi\mathbb{Z}$.

Claim 3. By Claim 2, there exist $z_1, z_2 \in \mathcal{Z}_m$ and $k, l \in \mathbb{Z}$ such that $u_1 = z_1 + 2k\pi$, $u_2 = z_2 + 2l\pi$. Moreover, since $U \subseteq [u_1, u_1 + \kappa_{\mathcal{Z}_m}(u_1)] = [z_1, z_1^+ + 2\pi] + 2k\pi$, we have that $z_1^+ + 2k\pi \le u_2 \le z_1^+ + 2(k+1)\pi$.

Claim 4. We have that $U = (u_1, u_2]$.

Let $z_1, z_2 \in \mathcal{Z}_m$ and $k, l \in \mathbb{Z}$ be as above. We prove that $u_1 \notin U$ and $u_2 \in U$. Assume that $u_1 \in U$, and consider $x \in S^1$ such that $z_1^- < x < z_1$ are in cyclic order. Since $u_1 \in U$ and $\overline{M}_U(g_{xz_1})$ is an isomorphism, we have that $\dim \overline{M}_U(x) = \dim \overline{M}_U(z_1) \neq 0$. Moreover, since $U \subseteq [u_1, u_1 + \kappa_{\mathcal{Z}_m}(u_1)] = [z_1, z_1^+ + 2\pi] + 2k\pi$, either $u_2 = z_1 + 2\pi$ or $u_2 = z_1^+ + 2\pi$, otherwise we have that $\dim \overline{M}_U(x) = 0$. Therefore, $\dim \overline{M}_U(x) = \dim \overline{M}_U(z_1) = 1$ and then $\gamma^{-1}(x) \cap U = \{b_{1,x}\}$ and $\gamma^{-1}(z_1) \cap U = \{b_{1,z_1}\}$. Then, since $x + 2(k+1)\pi \in U$ and $u_1 = z_1 + 2k\pi \in U$, we obtain that $b_{1,x} = x + 2(k+1)\pi$ and $b_{1,z_1} = z_1 + 2k\pi$. We have that $\overline{M}_U(g_{xz_1}) = 0$ because $b_{1,x} > b_{1,z_1}$, contradicting the fact that $\overline{M}_U(g_{xz_1})$ is an isomorphism. Thus, $u_1 \notin U$.

Now assume that $u_2 \notin U$, and consider $x \in S^1$ such that $z_2^- < x < z_2$ are in cyclic order. Since $u_1 = z_1 + 2k\pi$ and then $u_1 < x + 2l$, we have that $x + 2l\pi \in U$ and as a consequence $\dim \overline{M}_U(x) \neq 0$. We also have that $\overline{M}_U(z_2) \neq 0$ because $\overline{M}_U(g_{xz_2})$ is an isomorphism. Moreover, $u_1 = z_2^- + 2(l-1)\pi$, otherwise $\dim \overline{M}_U(z_2) = 0$, and then $\dim \overline{M}_U(x) = 2$, while $\dim \overline{M}_U(z_2) = 1$. This contradicts the fact that $\overline{M}_U(g_{xz_2})$ is an isomorphism. Thus, $u_2 \in U$. Therefore $U = (u_1, u_2]$. $\square$

We recall from Theorem 6.1.7 that the objects of $\mathrm{Rep}(S^1, \kappa_{\mathcal{Z}_m})$ decompose as possibly infinite direct sums of indecomposable objects. With Proposition 6.3.4 we prove that condition (3) of Definition 6.2.3 implies that the objects of $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ have only finitely many indecomposable direct summands.

Proposition 6.3.4. *Let $M \in \mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. Then M has only finitely many indecomposable direct summands.*

Proof. By Theorem 6.1.7, M decomposes uniquely, up to isomorphism and reordering the summands, as a possibly infinite direct sum of strings. Since $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is closed under direct summands, see Theorem 6.2.4, by Proposition 6.3.3, the indecomposable direct summands of M are of the form $\overline{M}_U$ with $U = (u_1, u_2 + 2h\pi] \in \mathcal{W}$. Now consider an indecomposable direct summand $\overline{M}_U$ of M. For each $p \in [m]$ we fix $a_p < p < b_p$ as in condition (2) of Definition 6.2.3. We divide the proof into claims.

Claim 1. Let $p \in [m]$ and $x, y \in S^1$ be such that $a_p < x \le y < b_p$ are in cyclic order. Then $\overline{M}_U(g_{xy})$ is an isomorphism.

We have that $M \cong \overline{M}_U \oplus M'$ for some $M' \in \mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. Since $\begin{pmatrix} \overline{M}_U(g_{xy}) & 0 \\ 0 & M'(g_{xy}) \end{pmatrix}$ is an isomorphism, then $\overline{M}_U(g_{xy})$ is an isomorphism.

Now we denote $\mathcal{Z}'_m = \mathcal{Z}_m \setminus \bigcup_{p \in [m]} \{x \in S^1 \mid a_p < x < b_p \text{ are in cyclic order}\}$. Note that $\mathcal{Z}'_m$ is a finite set.

Claim 2. We have that $u_2 \in \mathcal{Z}'_m$.

If $u_2 \notin \mathcal{Z}'_m$ then there exist $p \in [m]$ and $x, y \in S^1$ such that $a_p < x \leq y < b_p$ are in cyclic order and $\overline{M}_U(g_{xy})$ is not an isomorphism, giving a contradiction with Claim 1. Indeed, let $x = u_2$ and consider y such that $u_2 < y < b_p$ are in cyclic order. Then $\overline{M}_U(g_{xy})$ is not an isomorphism.

Claim 3. There are only finitely many indecomposable direct summands of M.

For each $x \in \mathcal{Z}'_m$ there are only finitely many direct summands of M of the form $\overline{M}_U$ with $U = (u_1, u_2 + 2h\pi]$ and $u_2 = x$, otherwise $\dim M(x) = \infty$. Since $\mathcal{Z}'_m$ is a finite set, we can conclude that M has only finitely many indecomposable direct summands. $\square$

6.3.2 The coordinate system

For each $p, q \in [m]$ and $h \in \{0, 1\}$ we introduce the set of intervals

$$I_h^{(p,q)} = \left\{(u_1, u_2 + 2h\pi] \in \mathcal{W} \mid u_1 \in \mathbb{Z}^{(p)} \text{ and } u_2 \in \mathbb{Z}^{(q)}\right\}.$$

We can arrange the intervals of $\mathcal{W}$, or equivalently the isoclasses of indecomposable objects of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, into a coordinate system having

- $2m$ components of type $\mathbb{Z}A_\infty$, each corresponding to the sets $I_h^{(p,p)}$ for $p \in [m]$ and $h \in \{0, 1\}$, and
- $2\binom{m}{2}$ components of type $\mathbb{Z}A_\infty^\infty$, each corresponding to the sets $I_h^{(p,q)}$ for $p, q \in [m]$, $p \neq q$, and $h \in \{0, 1\}$.

Figure 6.5 illustrates the coordinate system of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, in Section 6.8 we will prove that this gives the AR quiver of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$.

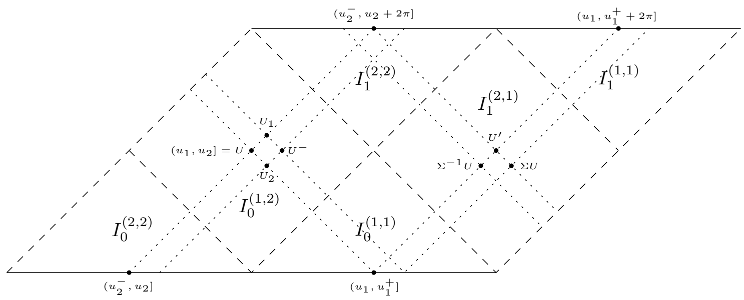


Figure 6.5: Illustration of the coordinate system of $\operatorname{rep}(\mathcal{Z}_2, \kappa_{\mathcal{Z}_2})$.

We now introduce some more intervals which play an important role in $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, see Figure 6.5 for an illustration.

Definition 6.3.5. Let $U = (u_1, u_2 + 2h\pi] \in \mathcal{W}$, we define the following intervals.

$$U_1 = (u_1^-, u_2 + 2h\pi] \qquad \Sigma^{-1}U = (u_2, u_1^+ + 2(1-h)\pi]$$
$$U_2 = (u_1, u_2^- + 2h\pi] \qquad \Sigma U = (u_2^-, u_1 + 2(1-h)\pi]$$
$$U^- = (u_1^-, u_2^- + 2h\pi] \qquad U' = (u_2^-, u_1^+ + 2(1-h)\pi]$$

With Proposition 6.5.4 we will prove that the interval U' determines the set of intervals $V \in \mathcal{W}$ such that any non-zero morphism $\overline{M}_U \to \overline{M}_V$ factors through a projective-injective object of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$.

The intervals U_1, U_2, and U^- determine the almost split sequence starting at U. Indeed, by Proposition 6.8.2 the almost split sequences in $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ are of the form $0 \longrightarrow \overline{M}_U \longrightarrow \overline{M}_{U_1} \oplus \overline{M}_{U_2} \longrightarrow \overline{M}_{U^-} \longrightarrow 0$, thus $\tau^{-1}\overline{M}_U \cong \overline{M}_{U^-}$.

Moreover, by Remark 6.6.5 and its dual version, we will obtain that $\Sigma^{-1}\overline{M}_U \cong \overline{M}_{\Sigma^{-1}U}$ and $\Sigma\overline{M}_U \cong \overline{M}_{\Sigma U}$, i.e., after stabilising $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, ΣU and $\Sigma^{-1}U$ determine the action of the shift functor on objects.

Observation 6.3.6. Let $U \in \mathcal{W}$. From Figure 6.5, it is straightforward to see that the following statements hold.

- $U_1 \in \mathcal{W}$ if and only if $u_1^+ \leq u_2 + 2h\pi \leq u_1 + 2\pi$, or equivalently $U \neq (u_1, u_1^+ + 2\pi]$.
- $U_2 \in \mathcal{W}$ if and only if $U_2 \neq \varnothing$, or equivalently $U \neq (u_1, u_1^+]$.
- $\Sigma^{-1}U \in \mathcal{W}$ if and only if $U \neq (u_1, u_1^+ + 2\pi]$. The same holds for ΣU.
- $U^- \in \mathcal{W}$. Moreover, $U^- = (u_2^-, u_2]$ if and only if $U = (u_1, u_1^+]$, and $U^- = (u_2^-, u_2 + 2\pi]$ if and only if $U = (u_1, u_1^+ + 2\pi]$.
- $U' \in \mathcal{W}$. Moreover, $U' = (u_2^-, u_2]$ if and only if $U = (u_1, u_1^+ + 2\pi]$, and $U' = (u_2^-, u_2 + 2\pi]$ if and only if $U = (u_1, u_1^+ + 2\pi]$.

6.4 Morphisms

In this section we describe the Hom-spaces of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ and the factorization properties of the morphisms.

6.4.1 Hom-hammocks

We define the Hom-hammocks and prove the following result.

Proposition 6.4.1. *Let $U = (u_1, u_2 + 2h\pi], V = (v_1, v_2 + 2k\pi] \in \mathcal{W}$. Then*

$$\operatorname{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \cong \begin{cases} \mathbb{K}^2 & \text{if } U = V = (u_1, u_1^+ + 2\pi], \\ \mathbb{K} & \text{if either } V \cap_L U \neq \varnothing \text{ or } (V - 2\pi) \cap_L U \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

We need the following lemma, which relates the Hom-spaces of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ to the intersections of intervals of $\mathcal{W}$.

Lemma 6.4.2. *Let $U = (u_1, u_2 + 2h\pi], V = (v_1, v_2 + 2k\pi] \in \mathcal{W}$. The following statements hold.*

1. *If $(V+2n\pi)\cap_L U \neq \varnothing$ for some $n \in \mathbb{Z}$, then $n \in \{0,-1\}$. Therefore $\operatorname{Hom}_{S^1}(\overline{M}_U, \overline{M}_V)$ is at most two dimensional.*

2. *We have that $V \cap_L U \neq \varnothing$ if and only if $v_1 \leq u_1$ and $u_1^+ \leq v_2 + 2k\pi \leq u_2 + 2h\pi$.*

3. *We have that $(V-2\pi) \cap_L U \neq \varnothing$ if and only if $k=1$ and $u_1^+ \leq v_2 \leq u_2 + 2h\pi$.*

4. *We have that $V \cap_L U \neq \varnothing$ and $(V-2\pi)\cap_L U \neq \varnothing$ if and only if $U = V = (u_1, u_1^+ + 2\pi]$.*

Proof. We prove statement (1). Assume that $(V+2n\pi)\cap_L U \neq \varnothing$ for some $n \in \mathbb{Z}$. If $n=1$ then, since $0 < u_1 < 2\pi$ and $2\pi < v_1 + 2\pi < 4\pi$, we have that $(V+2\pi)\cap_L U = \varnothing$. Moreover, if $n \geq 2$ or $n \leq -2$, then $(V+2n\pi)\cap U = \varnothing$ and as a consequence $(V+2n\pi)\cap_L U = \varnothing$. Thus, $n \in \{-1, 0\}$.

Statements (2) and (3) are straightforward, we prove statement (4). It is straightforward to check that if $U = V = (u_1, u_1^+ + 2\pi]$ then $V \cap_L U \neq \varnothing$ and $(V-2\pi)\cap_L U \neq \varnothing$. Now, if $V \cap_L U \neq \varnothing$ and $(V-2\pi)\cap_L U \neq \varnothing$, then in particular $k=1$, $u_1^+ \leq v_2 \leq u_2 + 2h\pi$, and $u_1^+ \leq v_2 + 2\pi \leq u_2 + 2h\pi$. Thus, $h=1$ and then $v_2 \leq u_2$. Since $u_2 + 2\pi \leq u_1^+ + 2\pi$, we have that $u_1^+ \leq v_2 \leq u_2 \leq u_1^+$, i.e. $v_2 = u_1^+$. Note that $v_1 \geq v_2^- + 2k\pi - 2\pi = u_1$ and, since $V \cap_L V \neq \varnothing$, also $v_1 \leq u_1$. Therefore, $v_1 = u_1$. Moreover, we have that $u_2 + 2h\pi = u_2 + 2\pi \leq u_1^+ + 2\pi$, i.e. $u_2 \leq u_1^+$. Since $u_2 + 2\pi \leq u_1^+ + 2\pi = v_2 + 2\pi \leq u_2 + 2\pi$, we obtain that $u_2 = u_1^+$. We can conclude that $U = V = (u_1, u_1^+ + 2\pi]$. $\square$

Now we can prove Proposition 6.4.1.

Proof of Proposition 6.4.1. By Theorem 6.1.6 we have that $\operatorname{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \cong \mathbb{K}^n$ with $n = |\{l \in \mathbb{Z} \mid (V+2l\pi)\cap_L U \neq \varnothing\}|$. Then the claim follows from Lemma 6.4.2. $\square$

We define the Hom-hammocks in $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ and then we prove that they describe exactly the Hom-spaces with Proposition 6.4.4. Figure 6.6 provides an illustration.

Definition 6.4.3. Let $U = (u_1, u_2 + 2h\pi] \in \mathcal{W}$. We define the following sets.

$$H^+(U) = \begin{cases} \{(v_1, v_2] \in \mathcal{W} \mid v_1 \leq u_1 \text{ and } u_1^+ \leq v_2 \leq u_2\} & \text{if } h = 0, \\ \{(v_1, v_2 + 2\pi] \in \mathcal{W} \mid u_2 \leq v_1 \leq u_1 \text{ and } v_2 \leq u_2\} & \text{if } h = 1. \end{cases}$$

$$H^-(U) = \begin{cases} \{(v_1, v_2] \in \mathcal{W} \mid u_1 \leq v_1 \leq u_2^- \text{ and } v_2 \leq u_2\} & \text{if } h = 0, \\ \{(v_1, v_2 + 2\pi] \in \mathcal{W} \mid v_1 \geq u_1 \text{ and } u_2 \leq v_2 \leq u_1\} & \text{if } h = 1. \end{cases}$$

$$P(U) = \begin{cases} \left\{ (v_1, v_2 + 2k\pi] \in \mathcal{W} \,\middle|\, \begin{array}{c} v_1 - 2k\pi \leq u_1 \text{ and } v_2 \geq u_2, \text{ or} \\ v_1 \leq u_1 \text{ and } v_2 + 2k\pi \geq u_2 \end{array} \right\} & \text{if } h = 0, \\ \{(v_1, v_2 + 2\pi] \in \mathcal{W} \mid v_1 \leq u_1 \text{ and } v_2 \geq u_2\} & \text{if } h = 1. \end{cases}$$

We extend those definitions to $U = \varnothing$ by imposing $H^+(\varnothing) = H^-(\varnothing) = P(\varnothing) = \varnothing$.

We want to prove the following result.

Proposition 6.4.4. *Let $U, V \in \mathcal{W}$, then $\mathrm{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \neq 0$ if and only if $V \in H^+(U) \sqcup H^-(\Sigma^{-1}U) \sqcup P(U')$.*

With Proposition 6.5.4 we will prove that $P(U')$ determines the set of intervals $V \in \mathcal{W}$ such that $\mathrm{Proj}(\overline{M}_U, \overline{M}_V) \neq 0$.

Observation 6.4.5. Let $U = (u_1, u_2 + 2h\pi], V = (v_1, v_2 + 2k\pi] \in \mathcal{W}$ be such that $u_2 + 2h\pi \neq u_1^+ + 2\pi$. The following statements hold, see Figure 6.6 for an illustration.

- The sets $H^+(U)$, $H^-(\Sigma^{-1}U)$, and $P(U')$ are pairwise disjoint.
- $V \in H^+(U)$ if and only if $V \notin P(U')$ and $k = h$. Moreover, if $V \in H^+(U)$ then $V \cap_L U \neq \varnothing$.
- $V \in H^-(\Sigma^{-1}U)$ if and only if $V \notin P(U')$ and $k = 1-h$. Moreover, if $V \in H^-(\Sigma^{-1}U)$ then $(V - 2k\pi) \cap_L U \neq \varnothing$.
- If $U = (u_1, u_1^+]$, then $P(U') = \{(u_1, u_1^+ + 2\pi]\}$.
- If $U = (u_1, u_1^+ + 2\pi]$, then $H^+(U) = H^-(\Sigma^{-1}U) = \varnothing$.

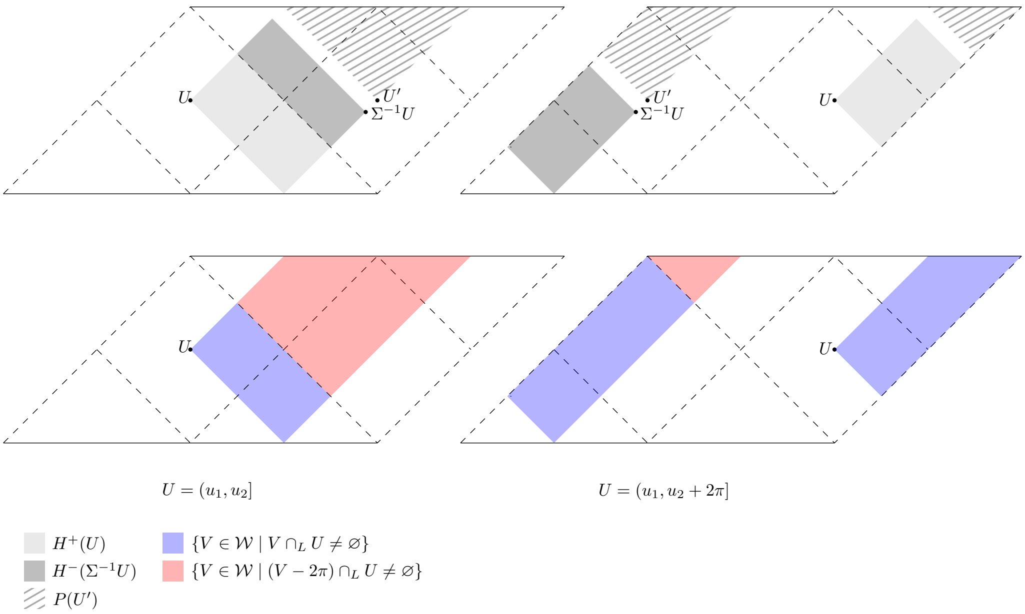


Figure 6.6: The interval $U \in \mathcal{W}$ and its Hom-hammocks when $m = 2$. In grey: $H^+(U)$, $H^-(\Sigma^{-1}U)$, and $P(U')$. In blue and red: how the intervals in $\mathcal{W}$ intersect U.

Now we can prove Proposition 6.4.4.

Proof of 6.4.4. We denote $U = (u_1, u_2 + 2h\pi]$ and we assume that $h = 0$, if $h = 1$ the proof is analogous. It is straightforward to check that the following equality holds.

$$\begin{aligned} H^+(U) \sqcup H^-(\Sigma^{-1}U) \sqcup P(U') = & \{(v_1, v_2] \in \mathcal{W} \mid u_1^+ \leq v_2 \leq u_2 \text{ and } v_1 \leq u_1\} \sqcup \\ & \{(v_1, v_2 + 2\pi] \in \mathcal{W} \mid u_1^+ \leq v_2 \leq u_2\} \end{aligned}$$

Let $V \in \mathcal{W}$. By Lemma 6.4.2 and Proposition 6.4.1, we know that $\mathrm{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \neq 0$

if and only if $V \cap_L U \neq \varnothing$ or $(V-2\pi) \cap_L U \neq \varnothing$. Thus, it is straightforward to check that

$$\{V \in \mathcal{W} \mid \operatorname{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \neq \varnothing\} = \{(v_1, v_2] \in \mathcal{W} \mid v_1 \leq u_1 \text{ and } u_1^+ \leq v_2 \leq u_2\} \sqcup \{(v_1, v_2 + 2\pi] \in \mathcal{W} \mid u_1^+ \leq v_2 \leq u_2\}.$$

We conclude that $V \in \mathcal{W}$ is such that $\operatorname{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \neq \varnothing$ if and only if $V \in H^+(U) \sqcup H^-(\Sigma^{-1}U) \sqcup P(U')$. $\square$

6.4.2 Factorization properties

We now prove the factorization properties of the morphisms of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. After stabilising, the proposition below can be reformulated as in Proposition 6.9.3.

Proposition 6.4.6. *Let $U, V, W \in \mathcal{W}$ be such that $\overline{M}_U$, $\overline{M}_V$, and $\overline{M}_W$ are non-isomorphic. Assume that there exist non-zero morphisms $f\colon \overline{M}_U \to \overline{M}_V$ and $g\colon \overline{M}_V \to \overline{M}_W$. Assume that one of the following conditions holds.*

1. *$V \cap_L U \neq \varnothing$, $W \cap_L V \neq \varnothing$, and $W \cap_L U \neq \varnothing$.*
2. *$V \cap_L U \neq \varnothing$, $(W-2\pi) \cap_L V \neq \varnothing$, and $(W-2\pi) \cap_L U \neq \varnothing$.*
3. *$(V-2\pi) \cap_L U \neq \varnothing$, $W \cap_L V \neq \varnothing$, and $(W-2\pi) \cap_L U \neq \varnothing$.*

Then $gf \neq 0$.

Proof. By assumption, we have that $(V+2n\pi) \cap_L U \neq \varnothing$, $(W+2l\pi) \cap_L (V+2n\pi) \neq \varnothing$, and $(W+2l\pi) \cap_L U \neq \varnothing$ for some $l, n \in \mathbb{Z}$. Since U, V, and W are pairwise non-isomorphic, by Proposition 6.4.1 we have that such l and n are unique. Let $\varphi\colon M_U \to M_{V+2n\pi}$ and $\psi\colon M_{V+2n\pi} \to M_{W+2l\pi}$ be standard morphisms, see Definition 6.1.2. Since $(W-2l\pi) \cap_L U \neq \varnothing$, $\psi\varphi \neq 0$, and then $\overline{\psi}\overline{\varphi} = \overline{\psi\varphi} \neq 0$, see Lemma 6.1.10. Moreover, $f = \lambda\overline{\varphi}$ and $g = \mu\overline{\psi}$ for some $\lambda, \mu \in \mathbb{K}^*$, and as a consequence $gf = \lambda\mu\overline{\psi}\overline{\varphi} \neq 0$. We conclude that $gf \neq 0$. $\square$

Proposition 6.4.7. *Let U, V, W, f, g be as in Proposition 6.4.6. Assume that one of the following conditions holds.*

1. *$(V-2\pi) \cap_L U \neq \varnothing$ and $(W-2\pi) \cap_L V \neq \varnothing$.*
2. *$V \cap_L U \neq \varnothing$, $(W-2\pi) \cap_L V \neq \varnothing$, and $W \cap_L U \neq \varnothing$.*
3. *$(V-2\pi) \cap_L U \neq \varnothing$, $W \cap_L V \neq \varnothing$, and $W \cap_L U \neq \varnothing$.*

Then $gf = 0$.

Proof. We proceeding similarly as in the argument of Proposition 6.4.6. By assumption we have that $(V+2n\pi) \cap_L U \neq \varnothing$ and $(W+2l\pi) \cap_L (V+2n\pi) \neq \varnothing$ for some $l, n \in \mathbb{Z}$. Let $\varphi\colon M_U \to M_{V+2n\pi}$ and $\psi\colon M_{V+2n\pi} \to M_{W+2l\pi}$ be standard morphisms, we obtain that $gf = \lambda\overline{\psi\varphi}$ for some $\lambda \in \mathbb{K}^*$. Note that $(W-2l\pi) \cap_L U = \varnothing$. Indeed, if condition (1) holds, we have that $l=-2$ and then $(W+2l\pi) \cap_L U = (W-4\pi) \cap_L U = \varnothing$. If condition

(2) or (3) holds, then $l = -1$ and, since $W \cap_L U \neq \varnothing$, we have that $(W - 2l\pi) \cap_L U = (W - 2\pi) \cap_L U = \varnothing$. Indeed, if $(W - 2\pi) \cap_L U \neq \varnothing$ then, since $W \cap_L U \neq \varnothing$, by Lemma 6.4.2, $U = W = (u_1, u_1^+ + 2\pi]$ and then $\overline{M}_U \cong \overline{M}_V$, giving a contradiction. As a consequence, $\psi\varphi = 0$ and we can conclude that $gf = 0$. $\square$

Lemma 6.4.8. *Let $U = (u_1, u_2 + 2h\pi], V = (v_1, v_2 + 2k\pi] \in \mathcal{W}$. The following statements hold.*

1. *Assume that $U \neq (u_1, u_1^+ + 2\pi]$. Then there exists a monomorphism $\overline{M}_U \to \overline{M}_V$ if and only if there exists $l \in \{0, -1\}$ such that $(V + 2l\pi) \cap_L U \neq \varnothing$ and $u_2 + 2h\pi = v_2 + 2(k + l)\pi$.*

2. *Assume that $V \neq (v_1, v_1^+ + 2\pi]$. Then there exists an epimorphism $\overline{M}_U \to \overline{M}_V$ if and only if $V \cap_L U \neq \varnothing$ and $u_1 = v_1$.*

Proof. We prove statement (1). Assume that $(V + 2l\pi) \cap_L U \neq \varnothing$ and $u_2 + 2h\pi = v_2 + 2(k+l)\pi$ for a unique $l \in \{0, -1\}$, then $\mathrm{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \cong \mathbb{K}$. Let $f\colon \overline{M}_U \to \overline{M}_V$ be non-zero, we prove that f is a monomorphism. Consider a non-zero morphism $g\colon \overline{M}_W \to \overline{M}_U$ with $W \in \mathcal{W}$, we check that $fg \neq 0$. Since $\mathrm{Hom}(\overline{M}_W, \overline{M}_U) \cong \mathbb{K}$, we have that either $U \cap_L W \neq \varnothing$ or $(U - 2\pi) \cap_L W \neq \varnothing$. If $(U - 2\pi) \cap_L W \neq \varnothing$ and $l = -1$, then $h = 1$ and we obtain that $u_2 + 2\pi = u_2 + 2h\pi = v_2 + 2(k + l)\pi = v_2 + 2k\pi - 2\pi$, i.e. $u_2 + 4\pi = v_2 + 2k\pi$, which is impossible because $k \in \{0, 1\}$.

Thus, we have the following possibilities: $U \cap_L W \neq \varnothing$ and $l = 0$, $U \cap_L W \neq \varnothing$ and $l = -1$, or $(U - 2\pi) \cap_L W \neq \varnothing$ and $l = 0$. Since $u_2 + 2h\pi = v_2 + 2(k + l)\pi$, in the first case we obtain that $V \cap_L W \neq \varnothing$, and for the remaining cases $(V - 2\pi) \cap_L W \neq \varnothing$, see Figure 6.7. By Proposition 6.4.6 we obtain that $fg \neq 0$. This proves that f is a monomorphism.

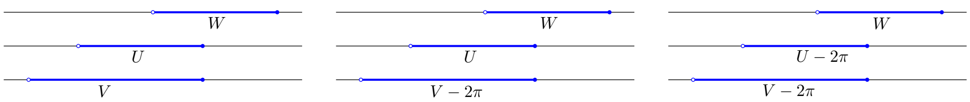


Figure 6.7: The intersection of intervals of the argument of Lemma 6.4.8.

Now assume that there exists a monomorphism $f\colon \overline{M}_U \to \overline{M}_V$. Since $f \neq 0$, there exists a unique $l \in \{0, -1\}$ such that $(V + 2l\pi) \cap_L U \neq \varnothing$, we prove that $u_2 + 2h\pi = v_2 + 2(k + l)\pi$. There exists a monomorphism $g\colon \overline{M}_{(u_2^-, u_2]} \to \overline{M}_U$. Indeed, it is straightforward to check that $(U - 2h\pi) \cap_L (u_2^-, u_2] \neq \varnothing$ and $u_2 = u_2 + 2(h - h)\pi$. Thus, $fg\colon \overline{M}_{(u_2^-, u_2]} \to \overline{M}_V$ is a monomorphism, and as a consequence $\mathrm{Hom}_{S^1}(\overline{M}_{(u_2^-, u_2]}, \overline{M}_V) \cong \mathbb{K}$. This implies that $v_2 = u_2$, see Lemma 6.4.2. Moreover, since $(V + 2l\pi) \cap_L U \neq \varnothing$, we have that $u_1^+ \leq v_2 + 2(k + l)\pi \leq u_2 + 2h\pi$. Thus, $v_2 + 2(k + l)\pi = u_2 + 2(k + l)\pi \in \gamma^{-1}(u_2) \cap U$. Since $U \neq (u_1, u_1^+ + 2\pi]$, $\gamma^{-1}(u_2) \cap U$ has only one element, namely $u_2 + 2h\pi$. Therefore, $v_2 + 2(k + l)\pi = u_2 + 2h\pi$. This concludes the argument of (1).

Now we prove statement (2). Similarly as above, we can prove that if $V \cap_L U \neq \varnothing$ and $v_1 = u_1$, then $\mathrm{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \cong \mathbb{K}$ and any non-zero morphism $\overline{M}_U \to \overline{M}_V$ is an epimorphism. Now assume that there exists an epimorphism $f\colon \overline{M}_U \to \overline{M}_V$, we prove that $V \cap_L U \neq \varnothing$ and $v_1 = u_1$. Since $(v_1, v_1^+] \cap_L V \neq \varnothing$, there is an epimorphism $g\colon \overline{M}_V \to \overline{M}_{(v_1,v_1^+]}$. Thus, $gf\colon \overline{M}_U \to \overline{M}_{(v_1,v_1^+]}$ is an epimorphism, and in particular $gf \neq 0$. As a consequence, either $(v_1, v_1^+] \cap_L U \neq \varnothing$ or $((v_1, v_1^+] - 2\pi) \cap_L U \neq \varnothing$. Note that the second case is impossible because $(v_1, v_1^+] \subseteq (0, 2\pi)$. We obtain that $V \cap_L U \neq \varnothing$, indeed, by Proposition 6.4.7, if $(V - 2\pi) \cap_L U \neq \varnothing$ then $gf = 0$, giving a contradiction. Moreover, since $(v_1, v_1^+] \cap_L U \neq \varnothing$, we have that $v_1 \leq u_1$ and $v_1^+ \geq u_1^+$, i.e. $v_1 = u_1$. This concludes the argument of (2). □

There is another formulation of statement (1) which is dual to (2). Indeed, if instead of our convention we consider intervals of the form $U = (u_1 - 2h\pi, u_2]$ with $u_1, u_2 \in \mathcal{Z}_m + 2\pi$ and $h \in \{0, 1\}$, then (1) is equivalent to: there exists a monomorphism $\overline{M}_U \to \overline{M}_V$ if and only if $V \cap_L U \neq \varnothing$ and $u_2 = v_2$.

6.5 Projective-injective objects

The category $\operatorname{rep}(\mathcal{Z}, \kappa_{\mathcal{Z}_m})$ is a Krull–Schmidt exact subcategory of $\operatorname{Rep}(S^1, \kappa_{\mathcal{Z}_m})$, see Theorem 6.2.4 and Theorem 6.3.1. In this section we prove that $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is a Frobenius category and that for each $z \in S^1$ we have that $P_z = I_z$, where P_z and I_z are respectively the indecomposable projective and indecomposable injective representation at z. Given $z \in \mathcal{Z}_m$, we denote $\overline{M}_{(z,z^+ + 2\pi]}$ by P_z or I_z. We want to prove the following result, which follows directly from Proposition 6.5.2 and Proposition 6.5.3.

Theorem 6.5.1. *The category* $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ *is Frobenius and its indecomposable projective-injective objects are exactly those isomorphic to* $P_z = I_z$ *for some* $z \in \mathcal{Z}_m$.

We start with the proposition below.

Proposition 6.5.2. *For each* $z \in \mathcal{Z}_m$ *the object* $P_z = I_z$ *is projective and injective in* $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$.

Proof. Let $z \in \mathcal{Z}_m$, we prove that P_z is projective. The proof that I_z is injective is dual. Consider a short exact sequence $0 \longrightarrow L \xrightarrow{F} M \xrightarrow{G} P_z \longrightarrow 0$, we show that this sequence splits, i.e. that G is a split epimorphism. We divide the proof into claims.

Claim 1. There exists a direct summand of M isomorphic to P_z.

Consider the following commutative diagram of vector spaces.

$$\begin{tikzcd}
0 \arrow[r] & L(z^+) \arrow[r, "F(z^+)"] \arrow[d, "L(\omega_{z^+})"] & M(z^+) \arrow[r, "G(z^+)"] \arrow[d, "M(\omega_{z^+})"] & P_z(z^+) \arrow[r] \arrow[d, "P_z(\omega_{z^+})"] & 0 \\
0 \arrow[r] & L(z^+) \arrow[r, "F(z^+)"] & M(z^+) \arrow[r, "G(z^+)"] & P_z(z^+) \arrow[r] & 0
\end{tikzcd}$$

We recall that, by Definition 6.1.4, $P_z(z^+) = \mathbb{K}b_{1,z^+} \oplus \mathbb{K}b_{2,z^+}$, $P_z(\omega_{z^+})(b_{1,z^+}) = b_{2,z^+}$, and $P_z(\omega_{z^+})(b_{2,z^+}) = 0$. Assume that $M(\omega_{z^+}) = 0$. Then $P_z(\omega_{z^+})G(z^+) = 0$, and this contradicts the fact that $G(z^+)$ is a split epimorphism and that $P_z(\omega_{z^+}) \neq 0$. Therefore $M(\omega_{z^+}) \neq 0$. Now consider the decomposition of M into indecomposable direct summands

$M = \bigoplus_{i=1}^{n} M_i$, we can write $M(\omega_{z^+}) = \bigoplus_{i=1}^{n} M_i(\omega_{z^+})$. Since $M(\omega_{z^+}) \neq 0$, there exists $i \in \{1, \ldots, n\}$ such that $M_i(\omega_{z^+}) \neq 0$, i.e. $M_i \cong P_z$.

Now we introduce some notation. After reordering the summands of M, we can write $M \cong \bigoplus_{i=1}^{k} M_i \oplus \bigoplus_{j=k+1}^{n} M_j$ where $M_i \cong P_z$ for each $i \in \{1, \ldots, k\}$ and $M_j \ncong P_z$ for each $j \in \{k+1, \ldots, n\}$. We also write $G\colon M \to P_z$ as $G = \begin{pmatrix} G_1 & \cdots & G_k & G_{k+1} & \cdots & G_n \end{pmatrix}$. We show that there exists $i \in \{1, \ldots, k\}$ such that $G_i\colon M_i \to P_z$ is an isomorphism.

Claim 2. For each $i \in \{1, \ldots, k\}$ and $j \in \{k+1, \ldots, n\}$, we have that $G_i(z^+) = \begin{pmatrix} \alpha_i & 0 \\ \beta_i & \alpha_i \end{pmatrix}$ and $G_j(z^+) = \begin{pmatrix} 0 & \gamma_j \end{pmatrix}^T$ for some $\alpha_i, \beta_i, \gamma_j \in \mathbb{K}$.

We recall that for each $i \in \{1, \ldots, k\}$ and $j \in \{k+1, \ldots, n\}$ the following diagrams commute.

$$\begin{tikzcd} M_i(z^+) \arrow[r, "G_i(z^+)"] \arrow[d, "M_i(\omega_{z^+})"] & P_z(z^+) \arrow[d, "P_z(\omega_{z^+})"] \\ M_i(z^+) \arrow[r, "G_i(z^+)"] & P_z(z^+) \end{tikzcd} \qquad \begin{tikzcd} M_j(z^+) \arrow[r, "G_j(z^+)"] \arrow[d, "M_j(\omega_{z^+})"] & P_z(z^+) \arrow[d, "P_z(\omega_{z^+})"] \\ M_j(z^+) \arrow[r, "G_j(z^+)"] & P_z(z^+) \end{tikzcd}$$

Then the linear map $G_i(z^+)\colon M_i(z^+) = \mathbb{K}b_{1,z^+} \oplus \mathbb{K}b_{2,z^+} \to P_z(z^+) = \mathbb{K}b_{1,z^+} \oplus \mathbb{K}b_{2,z^+}$ is of the form $G_i(z^+) = \begin{pmatrix} \alpha_i & 0 \\ \beta_i & \alpha_i \end{pmatrix}$ for some $\alpha_i, \beta_i \in \mathbb{K}$. Moreover, since $M_j \ncong P_z$, we have that $M_j(z^+) = 0$ or $M_j(z^+) \cong \mathbb{K}$ and in both cases $M_j(\omega_{z^+}) = 0$. Therefore, $G_j(z^+) = \begin{pmatrix} 0 & \gamma \end{pmatrix}^T$ for some $\gamma \in \mathbb{K}$.

Claim 3. There exists $i \in \{1, \ldots, k\}$ such that $G_i(z^+)\colon M_i(z^+) \to P_z(z^+)$ is an isomorphism.

We show that $\alpha_i \neq 0$ for some $i \in \{1, \ldots, k\}$. Indeed, if $\alpha_i = 0$ for all $i \in \{1, \ldots, k\}$, then

$$G(z^+) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \beta_1 & 0 & \cdots & \beta_k & 0 & \gamma_{k+1} & \cdots & \gamma_n \end{pmatrix}.$$

and this contradicts the fact that $G(z^+)$ is a split epimorphism. As a consequence, there exists $i \in \{1, \ldots, k\}$ such that $\alpha_i \neq 0$, and therefore $G_i(z^+)$ is an isomorphism.

Claim 4. Let $i \in \{1, \ldots, k\}$ be such that $G_i(z^+)$ is an isomorphism. We have that $G_i\colon M_i \to P_z$ is an isomorphism.

We prove that $G_i(x)\colon M_i(x) \to P_z(x)$ is an isomorphism for each $x \in S^1$. Let $x \in S^1$ and consider the following commutative diagram.

$$\begin{tikzcd} M_i(x) \arrow[r, "G_i(x)"] \arrow[d, "M_i(g_{xz^+})"] & P_z(x) \arrow[d, "P_z(g_{xz^+})"] \\ M_i(z^+) \arrow[r, "G_i(z^+)"] & P_z(z^+) \end{tikzcd} \tag{6.1}$$

If $z < x \leq z^+$ are in cyclic order then, since $M_i(g_{xz^+})$ and $P_z(g_{xz^+})$ are isomorphisms by Definition 6.1.4, and since $G_i(z^+)$ is an isomorphism by Claim 2, we obtain that $G_i(x)$ is an isomorphism. Now assume that $z^+ < x \leq z$ are in cyclic order. We recall that $M_i(x) = \mathbb{K}b_{1,x} = P_z(x)$ and $M_i(g_{xz^+})(b_{1,x}) = b_{1,z^+} = P_z(g_{xz^+})$. Since diagram (6.1) commutes, we obtain that $G_i(x) \neq 0$, and therefore $G_i(x)$ is an isomorphism. Thus, $G_i(x)$ is an isomorphism for each $x \in S^1$.

Claim 5. The morphism $G\colon M \to P_z$ is a split epimorphism.

From Claim 3 we know that $G_i\colon M_i \to P_z$ is an isomorphism of representations. Then we can conclude that $G\colon M \to P_z$ is a split epimorphism, i.e. the short exact sequence $0 \longrightarrow L \xrightarrow{F} M \xrightarrow{G} P_z \longrightarrow 0$ splits. □

The following are important properties for projective and injective objects of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$.

Proposition 6.5.3. *The category* $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ *has enough projectives and enough injectives. Moreover, each indecomposable projective or injective object of* $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ *is isomorphic to* $P_z = I_z$ *for some* $z \in \mathcal{Z}_m$.

Proof. By Theorem 6.3.1, each object of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ decomposes as a finite direct sum of indecomposable objects, thus it is enough to prove that each indecomposable object of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ has a projective cover and an injective envelope. Let $U = (u_1, u_2 + 2h\pi] \in \mathcal{W}$, by Lemma 6.4.8 there exist an epimorphism $P_{u_1} \to \overline{M}_U$ and a monomorphism $I_{u_2^-} \to \overline{M}_U$.

Now we prove that the indecomposable projective or injective objects are all of the form $P_z = I_z$ for some $z \in \mathcal{Z}_m$. Let $U = (u_1, u_2 + 2h\pi] \in \mathcal{W}$ and $\overline{M}_U$ be projective in $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, then there exists an epimorphism $P_{u_1} \to \overline{M}_U \to 0$. Since $\overline{M}_U$ is projective, this epimorphism splits and then $\overline{M}_U$ is a direct summand of P_{u_1}. Therefore, $\overline{M}_U \cong P_{u_1}$. If $\overline{M}_U$ is injective, we can proceed dually. □

Given $U, V \in \mathcal{W}$, we recall that $\operatorname{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \cong \mathbb{K}$ if and only if $V \in H^+(U) \sqcup H^-(\Sigma^{-1}U) \sqcup P(U')$, see Proposition 6.4.4. Now we characterise the intervals $V \in \mathcal{W}$ such that $\operatorname{Proj}(\overline{M}_U, \overline{M}_V) \neq 0$.

Proposition 6.5.4. *Let* $U, V \in \mathcal{W}$. *We have that* $\operatorname{Proj}(\overline{M}_U, \overline{M}_V) \neq 0$ *if and only if* $V \in P(U')$.

Proof. We denote $U = (u_1, u_2 + 2h\pi]$ and $V = (v_1, v_2 + k\pi]$. If $\overline{M}_U$ is projective, then $\operatorname{Proj}(\overline{M}_U, \overline{M}_V) = \operatorname{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \neq 0$ if and only if $V \in P(U')$, see Observation 6.4.5. Now assume that $\overline{M}_U$ is not projective. Let $f\colon \overline{M}_U \to \overline{M}_V$ be a non-zero morphism such that $f = \beta\alpha$ for some $\alpha\colon \overline{M}_U \to P$ and $\beta\colon P \to \overline{M}_V$, where P is a projective object of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. Since $\overline{M}_U$ is not projective, by Proposition 6.4.1, $\operatorname{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \cong \mathbb{K}$. Therefore, there exists an indecomposable direct summand P_z of P such that $f = \beta\alpha$ for some non-zero morphisms $\alpha\colon \overline{M}_U \to P_z$ and $\beta\colon P_z \to \overline{M}_U$.

If $h = 0$, since $(z, z^+ + 2\pi] \in P(U')$, we have that $u_1^+ \leq z^+ \leq u_2$ and $((z, z^+ + 2\pi] - 2\pi) \cap_L U \neq \varnothing$. We have that $V \cap_L (z, z^+ + 2\pi] \neq \varnothing$ and $(V - 2\pi) \cap_L U \neq \varnothing$, otherwise, by Proposition 6.4.7, $\beta\alpha = 0$. Thus, $v_1 \leq z \leq u_2^-$ and $v_2 \geq u_1^+$, i.e. $V \in P(U')$.

If $h = 1$, we have the following possibilites: either $(V - 2\pi) \cap_L U \neq \varnothing$ or $V \cap_L U \neq \varnothing$. In the first case, $v_2 \geq u_1^+$ and, since $k = 1$, $v_1 - 2k\pi = v_1 - 2\pi \leq u_2^-$. In the second case, $V \cap (z, z^+ + 2\pi] \neq \varnothing$ and $V \cap_L U \neq \varnothing$, otherwise $\beta\alpha = 0$ by Proposition 6.4.7. Therefore, $v_2 + 2k\pi \geq u_1^+$ and $v_1 \leq z \leq u_2^-$, and then $V \in P(U')$. This proves that if $\mathrm{Proj}(\overline{M}_U, \overline{M}_U) \neq 0$ then $V \in P(U')$.

Now we prove that if $V \in P(U')$ then $\mathrm{Proj}(\overline{M}_U, \overline{M}_V) \neq 0$. If $\overline{M}_U$ is projective, then $\mathrm{Proj}(\overline{M}_U, \overline{M}_V) = \mathrm{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \neq 0$. Otherwise, we have that $\mathrm{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \cong \mathbb{K}$ and $\mathrm{Proj}(\overline{M}_U, \overline{M}_V)$ is at most one dimensional. Let $f\colon \overline{M}_U \to \overline{M}_V$ be a non-zero morphism, we show that f factors through a projective object. Assume that $h = 0$. Note that $\big((u_2^-, u_2 + 2\pi] - 2\pi\big) \cap_L U \neq \varnothing$. Since $V \in P(U')$, we have that $v_1 \leq u_2^-$ and $v_2 \geq u_1^+$, and then $V \cap (u_2^-, u_2 + 2\pi]$ and $(V - 2\pi) \cap_L U \neq \varnothing$. Thus, by Proposition 6.4.6, f factors through $P_{u_2^-}$.

Now assume that $h = 1$. We have that $(u_2^-, u_2 + 2\pi] \cap U \neq \varnothing$. Since $V \in P(U')$, we have the following possibilities: $v_1 \leq u_2^-$ and $v_2 + 2k\pi \geq u_1^+$, or $v_1 - 2k\pi \leq u_2^-$ and $v_2 \geq u_1^+$. Thus, we have that $V \cap_L (u_2^-, u_2 + 2\pi] \neq \varnothing$ and $V \cap_L U \neq \varnothing$, or $(V - 2\pi) \cap_L (u_2^-, u_2 + 2\pi] \neq \varnothing$ and $(V - 2\pi) \cap_L U \neq \varnothing$. In both cases f factors through $P_{u_2^-}$. We conclude that any non-zero morphism of $\mathrm{Hom}_{S^1}(\overline{M}_U, \overline{M}_V)$ factors through a projective object, i.e. $\mathrm{Proj}(\overline{M}_U, \overline{M}_V) \cong \mathbb{K}$. □

6.6 Exact sequences

In this section we compute the middle terms of certain exact sequences in $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ of the form $0 \longrightarrow \overline{M}_U \longrightarrow M \longrightarrow \overline{M}_V \longrightarrow 0$. This computation will be useful in Section 6.7 for proving that $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is uniserial, and in Section 6.8 for describing the almost split sequences. Given $U \in \mathcal{W}$, we refer to Definition 6.3.5 for the notation U^-.

Setup 6.6.1. Let $U = (u_1, u_2 + 2h\pi], V = (v_1, v_2 + 2k\pi] \in \mathcal{W}$ be such that $\overline{M}_U$ is not projective, $\mathrm{Hom}_{S^1}(\overline{M}_{U^-}, \overline{M}_V) \cong \mathbb{K}$, and $\mathrm{Proj}(\overline{M}_{U^-}, \overline{M}_V) = 0$. Since $\mathrm{Hom}_{S^1}(\overline{M}_{U^-}, \overline{M}_V) \cong \mathbb{K}$, there exists a unique $l \in \{0, -1\}$ such that $(V + 2l\pi) \cap_L U^- \neq \varnothing$. We denote $I = (v_1, u_2 + 2(h - l)\pi]$ and $J = (u_1, v_2 + 2(k + l)\pi]$.

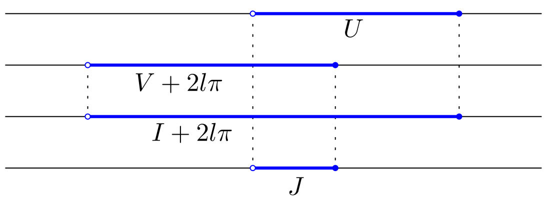


Figure 6.8: The intervals I and J of Setup 6.6.1.

The following lemmas will be useful to prove Proposition 6.6.4.

Lemma 6.6.2. *Keeping Setup 6.6.1, the following statements hold.*

1. *We have that $I = (U - 2l\pi) \cup V$ and $J = U \cap (V + 2l\pi)$.*

2. *We have that $I \in \mathcal{W}$. Moreover, $\overline{M}_I$ is projective if and only if $u_2 + 2(h - l)\pi = v_1^+ + 2\pi$.*

3. *We have that $J \in \mathcal{W}$ if and only if $J \neq \varnothing$, or equivalently $v_2 + 2(k + l)\pi \neq u_1$.*

4. *We have that* $V = \Sigma^{-1}(U^-)$ *if and only if* $J = \varnothing$ *and* $\overline{M}_I$ *is projective.*

5. *There exist a monomorphism* $\overline{M}_U \to \overline{M}_I$ *and an epimorphism* $\overline{M}_I \to \overline{M}_V$*. Moreover, if* $J \neq \varnothing$*, there exist an epimorphism* $\overline{M}_U \to \overline{M}_J$ *and a monomorphism* $\overline{M}_J \to \overline{M}_V$*.*

Proof. Statement (1) is straightforward from Figure 6.8. We prove statement (2). Since $V \subseteq I$, we have that $I \neq \varnothing$. We check that $u_2 + 2(h-l)\pi \leq v_1^+ + 2\pi$. First we show that $h - l \in \{0, 1\}$. Since $h, -l \geq 0$, we have that $h - l \geq 0$. Moreover, if $l = 0$ then $h - l = h \leq 1$, and if $l = -1$ then $h = 0$, otherwise $\mathrm{Proj}(U^-, V) \neq 0$, see Figure 6.6, giving a contradiction.

Now, if $h - l = 0$, then $u_2 + 2(h-l)\pi = u_2 \leq v_1^+ + 2\pi$. If $h - l = 1$, we have the following possibilities: $h = 0$ and $l = -1$, or $h = 1$ and $l = 0$. We prove that in both cases $u_2 \leq v_1^+$. If $h = 0$ and $l = -1$, we also have $k = 1$, and, since $(V + 2l\pi) \cap_L U^- \neq \varnothing$, this implies that $v_2 = v_2 + 2(k+l)\pi \geq u_1$. If $h = 1$ and $l = 0$, we have that $v_2 + 2k\pi \geq u_1$. Since $\mathrm{Proj}(U^-, V) = 0$, i.e. $V \notin P((U^-)')$ where $(U^-)' = (u_2^{--}, u_1 + 2(1-h)\pi]$, we have that $v_1 \not\leq u_2^{--}$, see Definition 6.4.3. Thus, $v_1 \geq u_2^-$, i.e. $u_2 \leq v_1^+$ and then $u_2 + 2(h-l)\pi = u_2 + 2\pi \leq v_1^+ + 2\pi$. We obtain that $I \in \mathcal{W}$ and $\overline{M}_I$ is projective if and only if $u_2 + 2(h-l)\pi = v_1^+ + 2\pi$.

Now we prove statement (3). Since $(V + 2l\pi) \cap_L U^- \neq \varnothing$, we have that $v_2 + 2(k+l)\pi \leq u_2^- + 2h\pi \leq u_2 + 2h\pi \leq u_1^+ + 2\pi$, we have that $v_2 + 2(k+l)\pi \leq u_1^+ + 2\pi$. Thus, either $J = \varnothing$, i.e. $v_2 + 2(k+l)\pi = u_1$, or $J \neq \varnothing$ and then $J \in \mathcal{W}$.

We prove statement (4). Assume that $V = (v_1, v_2 + 2k\pi] = \Sigma^{-1}(U^-) = (u_2^-, u_1 + 2(1-h)\pi]$. Then $k = 1 - h = -l$, $u_2^- = v_1$, i.e. $u_2 = v_1^+$, and $v_2 = u_1$. Therefore, $u_2 + 2(h - l)\pi = v_1^+ + 2\pi$ and $v_2 + 2(k+l)\pi = u_1$. By statements (2) and (3) we have that $\overline{M}_I$ is projective and $J = \varnothing$. Now assume that the converse holds, i.e. $u_1 = v_2 + 2(k+l)\pi$ and $u_2 + 2(h-l)\pi = v_1^+ + 2\pi$. Then $v_1 = u_2^-$, $v_2 = u_1$, $k + l = 0$, and $h - l = 1$. As a consequence, $v_2 + 2k\pi = v_2 + 2(1-h)\pi = u_1 + 2(1-h)\pi$. We can conclude that $V = \Sigma^{-1}(U^-)$.

Finally, we prove statement (5). We check that U, V, I and J satisfy the conditions of Lemma 6.4.8. By Figure 6.8 it is straightforward to see that $(I + 2l\pi) \cap_L U \neq \varnothing$ and $(V + 2l\pi) \cap_L (I + 2l\pi) \neq \varnothing$, i.e. $V \cap_L I \neq \varnothing$, and, if $J \neq \varnothing$, $J \cap_L U \neq \varnothing$ and $(V + 2l\pi) \cap_L J \neq \varnothing$. Moreover, it is straightforward to check that the endpoints of U, V, I and J satisfy the remaining conditions of Lemma 6.4.8. Thus, there exist monomorphisms $\overline{M}_U \to \overline{M}_I$ and $\overline{M}_J \to \overline{M}_V$, and epimorphisms $\overline{M}_U \to \overline{M}_J$ and $\overline{M}_I \to \overline{M}_V$. □

Lemma 6.6.3. *Keeping Setup 6.6.1, the following statements hold.*

1. *If* $J \neq \varnothing$*, then* $\dim \overline{M}_U(x) + \dim \overline{M}_V(x) = \dim \overline{M}_I(x) + \dim \overline{M}_J(x)$ *for each* $x \in S^1$*.*

2. *If* $J = \varnothing$*, then* $\dim \overline{M}_U(x) + \dim \overline{M}_V(x) = \dim \overline{M}_I(x)$ *for each* $x \in S^1$*.*

Proof. We prove statement (1). Let $x \in S^1$. Since $\overline{M}_U$ and $\overline{M}_V$ are not projective, we have that $\dim \overline{M}_U(x), \dim \overline{M}_V(x) \in \{0,1\}$. We divide the argument into a case analysis.

Case 1. $\dim \overline{M}_U(x) = 0 = \dim \overline{M}_V(x)$.

By Definition 6.1.4, we have that $\gamma^{-1}(x) \cap U = \varnothing = \gamma^{-1}(x) \cap V$. Therefore, by Lemma 6.6.2, $\gamma^{-1}(x) \cap I = \gamma^{-1}(x) \cap ((U - 2l\pi) \cup V) = ((\gamma^{-1}(x) \cap U) - 2l\pi)) \cup (\gamma^{-1}(x) \cap V) = \varnothing$ and $\gamma^{-1}(x) \cap J = \gamma^{-1}(x) \cap U \cap (V + 2l\pi) = \varnothing$. Thus, $\dim \overline{M}_I(x) = 0 = \dim \overline{M}_J(x)$.

Case 2. $\dim \overline{M}_U(x) = 1$ and $\dim \overline{M}_V(x) = 0$.

We have that $\gamma^{-1}(x) \cap U = \{s\}$ for some $s \in \mathbb{R}$, and $\gamma^{-1}(x) \cap V = \varnothing$. Therefore, $\gamma^{-1}(x) \cap I = ((\gamma^{-1}(x) \cap U) - 2l\pi)) \cup (\gamma^{-1}(x) \cap U) = \{s - 2l\pi\}$ and $\gamma^{-1}(x) \cap J = U \cap ((\gamma^{-1}(x) \cap V) + 2l\pi) = \varnothing$. We obtain that $\dim \overline{M}_I(x) = 1$ and $\dim \overline{M}_J(x) = 0$.

Case 3. $\dim \overline{M}_U(x) = 0$ and $\dim \overline{M}_V(x) = 1$.

The proof is similar to Case 2.

Case 4. $\dim \overline{M}_U(x) = 1 = \dim \overline{M}_V(x)$.

In this case, $\gamma^{-1}(x) \cap U = \{s\}$ and $\gamma^{-1}(x) \cap V = \{t\}$ for some $s, t \in \mathbb{R}$. Thus, $\gamma^{-1}(x) \cap I = ((\gamma^{-1}(x) \cap U) - 2l\pi)) \cup (\gamma^{-1}(x) \cap V) = \{s - 2l\pi, t\}$ and $\gamma^{-1}(x) \cap J = (\gamma^{-1}(x) \cap U) \cap ((\gamma^{-1}(x) \cap V) + 2l\pi) = \{s\} \cap \{t + 2l\pi\}$. If $s = t + 2l\pi$, then $\gamma^{-1}(x) \cap I = \{t\}$ and $\gamma^{-1}(x) \cap J = \{s\}$, i.e. $\dim \overline{M}_I(x) = 1 = \dim \overline{M}_J(x)$. Moreover, if $s \neq t + 2l\pi$, then $\dim \overline{M}_I(x) = 2$ and $\dim \overline{M}_J(x) = 0$. This concludes the argument of statement (1).

Now we prove statement (2). We check that the claim holds when $\dim \overline{M}_U(x) = 1 = \dim \overline{M}_V(x)$, the other cases are analogous to the above. Let $s, t \in \mathbb{R}$ be such that $\gamma^{-1}(x) \cap U = \{s\}$ and $\gamma^{-1}(x) \cap V = \{t\}$. If $s = t + 2l\pi$, then $s \in (\gamma^{-1}(x) \cap U) \cap (\gamma^{-1} \cap (V + 2l\pi)) = \gamma^{-1} \cap (U \cap (V + 2l\pi)) = \gamma^{-1}(x) \cap J$. As a consequence, $J \neq \varnothing$, and this gives a contradiction. Thus, $s \neq t + 2l\pi$, and we obtain that $\dim \overline{M}_I(x) = 2$ similarly to the above. $\square$

Now we can prove that certain sequences of objects and morphisms in $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ are short exact sequences.

Proposition 6.6.4. *Keeping Setup 6.6.1, the following statements hold.*

1. *If $J \neq \varnothing$, the sequence $0 \longrightarrow \overline{M}_U \xrightarrow{\binom{f_1}{f_2}} \overline{M}_I \oplus \overline{M}_J \xrightarrow{(g_1\ g_2)} \overline{M}_V \longrightarrow 0$ with $f_1, f_2, g_1, g_2 \neq 0$ such that $g_1 f_1 + g_2 f_2 = 0$, is short exact.*
2. *If $J = \varnothing$, the sequence $0 \longrightarrow \overline{M}_U \xrightarrow{f_1} \overline{M}_I \xrightarrow{g_1} \overline{M}_V \longrightarrow 0$ with $f_1, g_1 \neq 0$ such that $g_1 f_1 = 0$, is short exact.*

Proof. We prove statement (1), statement (2) is analogous. It is enough to show that

$$0 \longrightarrow \overline{M}_U(x) \xrightarrow{\binom{f_1(x)}{f_2(x)}} \overline{M}_I(x) \oplus \overline{M}_J(x) \xrightarrow{(g_1(x)\ g_2(x))} \overline{M}_V(x) \longrightarrow 0 \tag{E}$$

is a short exact sequence of vector spaces for each $x \in S^1$. Let $x \in S^1$, since $\overline{M}_U$ and $\overline{M}_V$ are not projective, $\overline{M}_U(x)$ and $\overline{M}_V(x)$ are either equal to 0 or isomorphic to $\mathbb{K}$. We assume that $\overline{M}_U(x) \cong \mathbb{K}$ and $\overline{M}_V(x) \cong \mathbb{K}$, for the remaining cases the proof is analogous. By Lemma 6.6.3, we have that either $\overline{M}_I(x) \cong \mathbb{K}$ and $\overline{M}_J(x) \cong \mathbb{K}$, or $\overline{M}_I(x) \cong \mathbb{K}^2$ and $\overline{M}_J(x) = 0$, we proceed with a case analysis.

Case 1. $\overline{M}_I(x) \cong \mathbb{K}$ and $\overline{M}_J(x) \cong \mathbb{K}$.

By Lemma 6.4.8 , f_1, f_2, g_1 and g_2 are either monomorphisms or epimorphisms. Thus, $f_1(x)$, $f_2(x)$, $g_1(x)$, and $g_2(x)$ are either injective or surjective linear maps, in particular they are non-zero. Then, (E) is isomorphic to the short exact sequence

$$0 \longrightarrow \mathbb{K} \xrightarrow{\binom{1}{1}} \mathbb{K} \oplus \mathbb{K} \xrightarrow{(-1\ 1)} \mathbb{K} \longrightarrow 0$$

and is therefore is a short exact sequence.

Case 2. $\overline{M}_I(x) \cong \mathbb{K}^2$ and $\overline{M}_J(x) = 0$.

We denote $f_1(x) = (\lambda_1\ \lambda_2)^T$ and $g_1(x) = (\mu_1\ \mu_2)$ for some $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{K}$. We recall that, since $\overline{M}_I(x) \cong \mathbb{K}^2$, $\overline{M}_I$ is projective and $\overline{M}_I(\omega_x)\colon \overline{M}_I(x) \to \overline{M}_I(x)$ is of the form $\overline{M}_I(x) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, see Definition 6.1.4. Since $\overline{M}_I(\omega_x) f_1(x) = f_1(x)\overline{M}_U(\omega_x) = 0$, we obtain that $f_1(x) = (0\ \lambda_2)^T$. Moreover, since $f_1(x) \neq 0$ as in Case 1, we have that $\lambda_2 \neq 0$. Similarly, we obtain that $g_1(x) = (\mu_1\ 0)$ with $\mu_1 \neq 0$. Thus, (E) is isomorphic to the short exact sequence

$$0 \longrightarrow \mathbb{K} \xrightarrow{\binom{0}{1}} \mathbb{K} \oplus \mathbb{K} \xrightarrow{(1\ 0)} \mathbb{K} \longrightarrow 0.$$

We obtain that for each $x \in S^1$ (E) is a short exact sequence. $\square$

The following remark will be useful in Section 6.9 for computing the shift functor of the stable category of $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$.

Remark 6.6.5. Let $U = (u_1, u_2 + 2h\pi] \in \mathcal{W}$ be such that $\overline{M}_U$ is not injective. Since $U \neq (u_1, u_1^+ + 2\pi]$, we have that $\Sigma U \in \mathcal{W}$, see Observation 6.3.6. By Proposition 6.6.4 there is a short exact sequence $0 \longrightarrow \overline{M}_U \longrightarrow I_{u_2^-} \longrightarrow \overline{M}_{\Sigma U} \longrightarrow 0$. Thus, $\Sigma\overline{M}_U \cong \overline{M}_{\Sigma U}$, see Section 2.3.2.

6.7 Simple objects and uniseriality

With the following proposition we describe the simple objects of $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. For each $z \in \mathcal{Z}_m$ we denote $S_z = \overline{M}_{(z,z^+]}$.

Proposition 6.7.1. *The simple objects of* $\mathrm{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ *are exactly those isomorphic to* S_z *for some* $z \in \mathcal{Z}_m$.

Proof. Let $z \in \mathcal{Z}_m$, we show that $S_z = \overline{M}_{(z,z^+]}$ is simple. Let $V = (v_1, v_2 + 2k\pi] \in \mathcal{W}$ be such that there exists a monomorphism $\overline{M}_V \to S_z$. By Lemma 6.4.8 we have that $((z, z^+] + 2l\pi) \cap_L V \neq \varnothing$ and $z^+ = v_2 + 2(k + l)\pi$ for a unique $l \in \{0, -1\}$. Since $(z, z^+] \subseteq$

$(0, 2\pi)$, then $l = 0$ and as a consequence $k = 0$ and $v_2 = z^+$. Since $(z, z^+] \cap_L V \neq \varnothing$, we obtain that $V = (z, z^+]$. This proves that S_z is simple.

Now let $U = (u_1, u_2 + 2h\pi] \in \mathcal{W}$ and assume that $\overline{M}_U$ is simple. Note that, by Lemma 6.4.8, there exists a monomorphism $S_{u_1} \to \overline{M}_U$. Thus, $S_{u_1} \cong \overline{M}_U$, i.e. $U = (u_1, u_1^+]$. This concludes the argument. ☐

Now consider $U = (u_1, u_2 + 2h\pi] \in \mathcal{W}$. We prove that $\overline{M}_U$ admits the following, possibly infinite, series of inclusions

$$0 \subseteq \overline{M}_{(u_2^-, u_2]} \subseteq \overline{M}_{(u_2^{--}, u_2]} \subseteq \cdots \subseteq \overline{M}_{(z^-, u_2+2h\pi]} \subseteq \overline{M}_{(z, u_2+2h\pi]} \subseteq \overline{M}_{(z^+, u_2+2h\pi]} \subseteq \cdots$$
$$\subseteq \overline{M}_{(u_1^+, u_2+2h\pi]} \subseteq \overline{M}_{(u_1, u_2+2h\pi]} = \overline{M}_U$$

where $z \in \mathcal{Z}_m \cup (\mathcal{Z}_m + 2h\pi)$ is such that $u_1 \leq z \leq u_2^- + 2h\pi$. We prove this is a *composition series* of $\overline{M}_U$, i.e. for each z the cokernel of the inclusion $\overline{M}_{(z^+, u_2+2h\pi]} \subseteq \overline{M}_{(z, u_2+2h\pi]}$ is a simple object. Moreover, we show that composition series of indecomposable objects of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ are unique up to isomorphism, i.e. $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is *uniserial*.

Theorem 6.7.2. *For each $M \in \operatorname{ind}\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ there exists a unique monomorphism $f\colon L \to M$, up to isomorphism, such that* Coker f *is simple.*

Proof. Assume that M is simple, then $0 \to M$ is the only monomorphism whose cokernel is simple. Now, assume that M is not simple, and let $U = (u_1, u_2 + 2h\pi] \in \mathcal{W}$ be such that $M \cong \overline{M}_U$. Let $V = (u_1^+, u_2 + 2h\pi]$, note that $V \in \mathcal{W}$ and, by Lemma 6.4.8, there exists a monomorphism $f\colon \overline{M}_V \to \overline{M}_U$. Moreover, by Proposition 6.6.4 , $0 \longrightarrow \overline{M}_V \xrightarrow{f} \overline{M}_U \longrightarrow S_{u_1} \longrightarrow 0$ is a short exact sequence, i.e. Coker f is simple.

Now we prove the uniqueness. Assume that there exists $V = (v_1, v_2 + 2k\pi] \in \mathcal{W}$ and a monomorphism $f\colon \overline{M}_V \to \overline{M}_U$ such that Coker f is simple. Assume that $h = 0$, the other case is similar. By Lemma 6.4.8, $k = 0$ and $v_2 = u_2$. Moreover, by Proposition 6.6.4, it is straightforward to check that Coker $f \cong \overline{M}_{(u_1, v_1]}$. Thus, $v_1 = u_1^+$ and $V = (v_1, v_2] = (u_1^+, u_2]$. This concludes the proof. ☐

6.8 Irreducible morphisms and almost split sequences

In this section we describe the irreducible morphisms and the almost split sequences in $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. Given $U \in \mathcal{W}$, we refer to Definition 6.3.5 for the notation U_1, U_2, and U^-.

Proposition 6.8.1. *Let $U \in \mathcal{W}$. If $\overline{M}_U$ is not projective, let $f_1\colon \overline{M}_U \to \overline{M}_{U_1}$ be a non-zero morphism, and if $\overline{M}_U$ is not simple, let $f_2\colon \overline{M}_U \to \overline{M}_{U_2}$ be a non-zero morphism. The following statements hold.*

1. *Let $V \in \mathcal{W} \setminus \{U\}$ and $f\colon \overline{M}_U \to \overline{M}_V$. Then f factors through f_1 or f_2.*
2. *If $\overline{M}_U$ is not simple, then f_2 is irreducible.*
3. *If $\overline{M}_U$ is not projective, then f_1 is irreducible.*

4. *Let $V \in \mathcal{W}$ be such that that there exists an irreducible morphism $\overline{M}_U \to \overline{M}_V$. Then $\overline{M}_V \cong \overline{M}_{U_1}$ or $\overline{M}_V \cong \overline{M}_{U_2}$.*

Proof. We prove statement (1). Assume that $\overline{M}_U$ is not projective and not simple, for the other cases the proof is analogous. By Observation 6.3.6, $U_1, U_2 \in \mathcal{W}$ and it is straightforward to check that $U_1 \cap_L U \neq \varnothing$ and $U_2 \cap_L U \neq \varnothing$. Let $V = (v_1, v_2 + 2k\pi] \in \mathcal{W}$ and consider a morphism $g\colon \overline{M}_U \to \overline{M}_V$. If $g = 0$ then clearly f factors through f_1 and f_2, thus we assume that $g \neq 0$. If $v_1 = u_1$, then $V \cap_L U \neq \varnothing$, $V \cap_L U_2 \neq \varnothing$ and, by Proposition 6.4.6, h factors through f_2. If $v_1 \neq u_1$, then we have the following possibilities: either $V \cap_L U \neq \varnothing$, and then $V \cap_L U_1 \neq \varnothing$, or $(V - 2\pi) \cap_L U \neq \varnothing$, and then $(V - 2\pi) \cap_L U_1 \neq \varnothing$. In both cases we have that g factors through f_1. Therefore, we conclude that g factors through f_1 or f_2.

Now we prove statement (2), the proof of statement (3) is analogous. Since $\overline{M}_U$ is not simple we have that $U_2 \in \mathcal{W}$. Assume that $f_2\colon \overline{M}_U \to \overline{M}_{U_2}$ factors as $f_2 = \beta\alpha$ for some $\alpha\colon \overline{M}_U \to M$ and $\beta\colon M \to \overline{M}_{U_2}$ where $M \in \operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. We show that α is a split monomorphism or β is a split epimorphism. Since $\operatorname{Hom}_{S^1}(\overline{M}_U, \overline{M}_{U_2}) \cong \mathbb{K}$, without loss of generality we can assume that M is indecomposable, i.e. $M \cong \overline{M}_V$ for some $V \in \mathcal{W}$. Since $\beta\alpha = f_2$ and $U_2 \cap_L U \neq \varnothing$, we have that $V \cap_L U \neq \varnothing$ and $U_2 \cap_L V \neq \varnothing$. Indeed, in all the other cases, by Proposition 6.4.7 we have that $\beta\alpha = 0$, giving a contradiction. Thus, $V = U$ or $V = U_2$, i.e. α is a split monomorphism or β is a split epimorphism. We conclude that f_2 is irreducible.

We prove statement (4). Consider an irreducible morphism $f\colon \overline{M}_U \to \overline{M}_V$, by statement (1) f factors through f_1 or f_2. Assume that f factors through f_1, the other case is analogous. Thus, there exists $g\colon \overline{M}_{U_1} \to \overline{M}_V$ such that $f = gf_1$. Since f is irreducible and f_1 is not a split monomorphism, g is a split epimorphism and therefore $\overline{M}_{U_1} \cong \overline{M}_V$. $\square$

We want to prove the following proposition. We refer to Definition 2.2.10 for the definition of almost split sequences.

Proposition 6.8.2. *Let $U \in \mathcal{W}$ be such that $\overline{M}_U$ is not projective. The following statements hold.*

1. *Assume that $\overline{M}_U$ is not simple. Then $0 \longrightarrow \overline{M}_U \xrightarrow{\left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right)} \overline{M}_{U_1} \oplus \overline{M}_{U_2} \xrightarrow{(\,g_1\; g_2\,)} \overline{M}_{U^-} \longrightarrow 0$, where $f_1, f_2, g_1, g_2 \neq 0$ are such that $g_1 f_1 + g_2 f_2 = 0$, is an almost split sequence.*
2. *Assume that $\overline{M}_U$ is simple. Then $0 \longrightarrow \overline{M}_U \xrightarrow{f_1} \overline{M}_{U_1} \xrightarrow{g_1} \overline{M}_{U^-} \longrightarrow 0$, where $f_1, g_1 \neq 0$, is an almost split sequence.*

Thus, we have that $\tau \overline{M}_{U^-} \cong \overline{M}_U$.

Proof. We prove statement (1), statement (2) is analogous. By Lemma 6.6.4, the sequence in the statement is short exact. Since $\overline{M}_U$ and $\overline{M}_{U^-}$ are indecomposable, it remains to check that $\left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right)$ is left almost split. Consider a morphism $\alpha\colon \overline{M}_U \to M$ with $M \in \operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, and assume that α is not a split monomorphism, we show that h factors through $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. Without loss of generality we can assume that M is indecomposable. By Proposition 6.8.1, α factors through f_1 or f_2. Assume that $\alpha = \alpha_1 f_1$ for some $\alpha_1 \colon \overline{M}_{U_1} \to M$, the other case is analogous. Consider the morphism $\begin{pmatrix} \alpha_1 & 0 \end{pmatrix} : \overline{M}_{U_1} \oplus \overline{M}_{U_2} \to M$, we obtain that $\begin{pmatrix} \alpha_1 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \alpha$. Thus, we obtain that $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is left almost split. $\square$

6.9 The category $\mathcal{C}_{-1,m}$

We define the category $\mathcal{C}_{-1,m}$ as the stable category of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. The objects of $\mathcal{C}_{-1,m}$ are exactly those of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, and $\operatorname{Hom}_{\mathcal{C}_{-1,m}}(M, N) = \operatorname{Hom}_{S^1}(M, N)/\operatorname{Proj}(M, N)$ for each $M, N \in \operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$. We refer to Section 2.3.2 for some background on stable Frobenius categories. Here we list some important properties of $\mathcal{C}_{-1,m}$.

- Since $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is Hom-finite, $\mathbb{K}$-linear, and Krull–Schmidt, then so is $\mathcal{C}_{-1,m}$, see Lemma 2.3.10. The indecomposable objects of $\mathcal{C}_{-1,m}$ are exactly, up to isomorphism, the non-projective indecomposable objects of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, i.e. the objects of the form $\overline{M}_U$ with $U = (u_1, u_2 + 2h\pi] \in \mathcal{W}$ such that $u_1^+ \leq u_2 + 2h\pi \leq u_1 + 2\pi$. Moreover, the Hom-spaces of $\mathcal{C}_{-1,m}$ are at most one-dimensional because the Hom-spaces of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ between non-projective objects are.

- Since $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$ is Frobenius, the category $\mathcal{C}_{-1,m}$ is triangulated. Given an indecomposable non-projective object $\overline{M}_U \in \operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, the shift functor Σ acts as $\Sigma\overline{M}_U = \overline{M}_{\Sigma U}$, see Remark 6.6.5. The triangles of $\mathcal{C}_{-1,m}$ are obtained by stabilising the short exact sequences of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, see Proposition 2.3.8.

- The category $\mathcal{C}_{-1,m}$ has almost split triangles, which are obtained by stabilising the almost split sequences of $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, see Corollary 2.3.14. Therefore, $\mathcal{C}_{-1,m}$ has a Serre functor, see Proposition 2.3.6.

For the rest of this section, we denote the objects of $\mathcal{C}_{-1,m}$ by lower case letters.

6.9.1 The geometric model

We describe the geometric model of $\mathcal{C}_{-1,m}$ in terms of the ∞-gon $\mathcal{Z}_m$. We refer to Section 3.1 for the difference between elements of $\mathcal{Z}_m$. The following is a negative version of the definition of *w-admissible arc*, defined in [27, Definition 2.3] for $w \geq 2$.

Definition 6.9.1. Let $a_1, a_2 \in \mathcal{Z}_m$. We say that (a_1, a_2) is a (-1)-*admissible arc* if $a_1 \geq a_2 + 1$ and $a_1 - a_2 \equiv 1 \mod 2$.

Now we prove that the (-1)-admissible arcs are in bijection with the intervals $U \in \mathcal{W}$ such that $\overline{M}_U$ is not projective.

Proposition 6.9.2. *The following is a bijection.*

$$\varphi\colon \left\{ \begin{array}{c} U=(u_1,u_2+2h\pi]\in\mathcal{W} \text{ such that} \\ u_1^+ \le u_2+2h\pi \le u_1+2\pi \end{array} \right\} \longrightarrow \{\, (-1)\text{-admissible arcs of } \mathcal{Z}_m \,\}$$

$$(u_1,u_2+2h\pi] \longmapsto \begin{cases} (2u_2-1,2u_1) & \text{if } h=0, \\ (2u_1,2u_2-1) & \text{if } h=1. \end{cases}$$

Proof. It is straightforward to check that φ is well defined and injective. We show that φ is surjective. Let (x_1,x_2) be a (-1)-admissible arc and let $U=\left(\frac{x_1}{2},\frac{x_2+1}{2}+2\pi\right]$ if x_1 is even, and $U=\left(\frac{x_2}{2},\frac{x_1+1}{2}\right]$ if x_1 is odd. Then U belongs to the domain of φ and $\varphi(U)=(x_1,x_2)$. This proves that φ is a bijection. $\square$

Note that, given $a=(a_1,a_2)\in\operatorname{ind}\mathcal{C}_{-1,m}$, by Proposition 6.9.2 we have that $\Sigma a=(a_1-1,a_2-1)$ and $\Sigma^{-1}a=(a_1+1,a_2+1)$.

For each $p,q\in[m]$ and $i\in\{0,1\}$ we define the set of (-1)-admissible arcs

$$\mathbb{Z}^{(p,q,i)}=\Big\{a=(a_1,a_2)\in\operatorname{ind}\mathcal{C}_{-1,m}\mid a\in\mathbb{Z}^{(p,q)} \text{ and } a_1\equiv i \mod 2\Big\}.$$

We can arrange the isoclasses of indecomposable objects of $\mathcal{C}_{-1,m}$ into a coordinate system having

- $2m$ components of type $\mathbb{Z}A_\infty$, each corresponding to the sets of arcs $\mathbb{Z}^{(p,p,i)}$ for $p\in[m]$ and $i\in\{0,1\}$,
- $2\binom{m}{2}$ components of type $\mathbb{Z}A_\infty^\infty$, each corresponding to the sets of arcs $\mathbb{Z}^{(p,q,i)}$ for $p>q$ and $i\in\{0,1\}$.

Figure 6.9 provides an illustration of the coordinate system, with Proposition 6.9.4 we will prove that this yields the AR quiver of $\mathcal{C}_{-1,m}$.

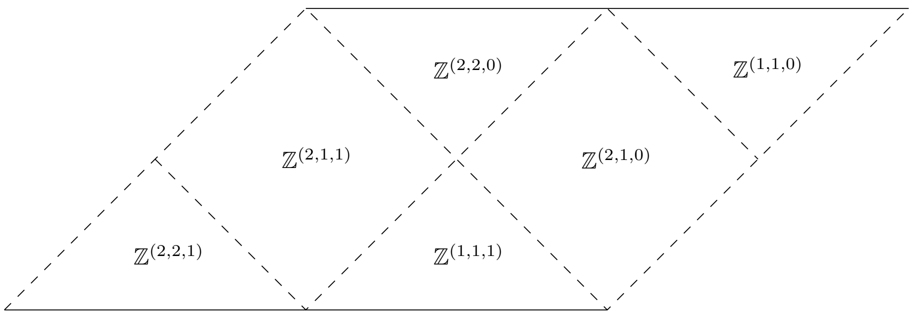


Figure 6.9: The coordinate system of $\mathcal{C}_{-1,2}$.

6.9.2 Factorization properties

Now we discuss the factorization properties of the morphisms of $\mathcal{C}_{-1,m}$. We start by discussing the Hom-hammocks and the Hom-spaces in $\mathcal{C}_{-1,m}$.

By Proposition 6.9.2, for an object $a=(a_1,a_2)\in\operatorname{ind}\mathcal{C}_{-1,m}$ we can re-write the Hom-hammocks of Definition 6.4.3 as

$$H^+(a) = \{(x_1, x_2) \in \operatorname{ind}\mathcal{C}_{-1,m} \mid a_2 + 1 \le x_1 \le a_1 \text{ and } x_2 \le a_2\}, \text{ and}$$
$$H^-(a) = \{(x_1, x_2) \in \operatorname{ind}\mathcal{C}_{-1,m} \mid x_1 \ge a_1 \text{ and } a_2 \le x_2 \le a_1 - 1\}$$

Given $b = (b_1, b_2) \in \operatorname{ind}\mathcal{C}_{-1,m}$, it is straightforward to check that $b \in H^+(a)$ if and only if $a \in H^-(b)$. Figure 6.10 provides an illustration of the Hom-hammocks. Moreover, by Proposition 6.4.1 we have that

$$\operatorname{Hom}_{\mathcal{C}_{-1,m}}(a,b) \cong \begin{cases} \mathbb{K} & \text{if } b \in H^+(a) \cup H^-(\Sigma^{-1}a), \\ 0 & \text{otherwise.} \end{cases}$$

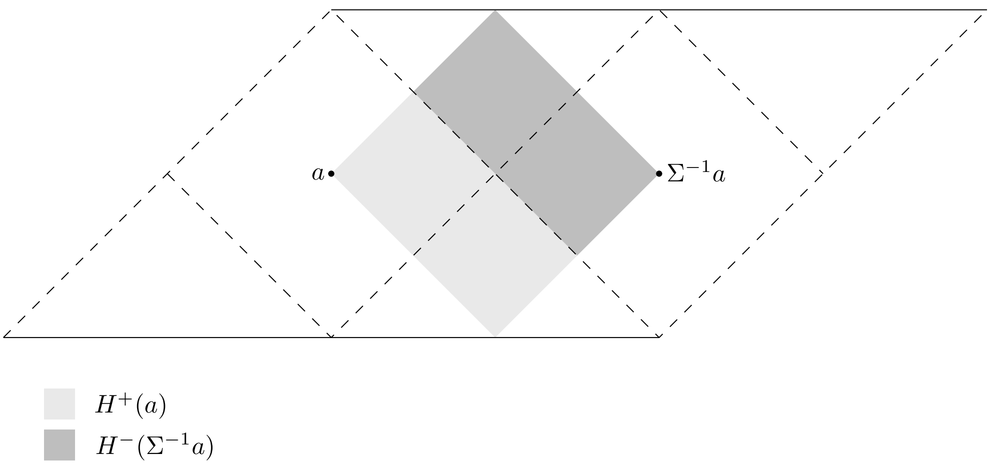


Figure 6.10: The Hom-hammocks $H^+(a)$ and $H^-(\Sigma^{-1}a)$ for $a \in \operatorname{ind}\mathcal{C}_{-1,2}$.

Now we prove the factorization properties of the morphisms in $\mathcal{C}_{-1,m}$. These are obtained from the factorization properties in $\operatorname{rep}(\mathcal{Z}_m, \kappa_{\mathcal{Z}_m})$, see Proposition 6.4.6.

Proposition 6.9.3. *Let $a, b, c \in \operatorname{ind}\mathcal{C}_{-1,m}$, $f\colon a \to b$, and $g\colon b \to c$ be non-zero morphisms. Assume that one of the following conditions hold.*

1. *$b \in H^+(a)$ and $c \in H^+(a) \cap H^+(b)$.*
2. *$b \in H^+(a)$ and $c \in H^-(\Sigma^{-1}a) \cap H^-(\Sigma^{-1}b)$.*
3. *$b \in H^-(\Sigma^{-1}a)$ and $c \in H^-(\Sigma^{-1}a) \cap H^+(b)$.*

Then $gf \neq 0$.

Proof. Let $U = (u_1, u_2 + 2h\pi], V = (v_1, v_2 + 2k\pi], W = (w_1, w_2 + 2l\pi] \in \mathcal{W}$ be intervals such that $\varphi(U) = a$, $\varphi(V) = b$, and $\varphi(W) = c$. Assume that (1) holds, i.e. $V \in H^+(U)$ and $W \in H^+(U) \cap H^+(V)$, then $V \cap_L U \neq \varnothing$, $W \cap_L V \neq \varnothing$, and $W \cap_L U \neq \varnothing$, see Observation 6.4.5. Thus, by Proposition 6.4.6, $gf \neq 0$.

Now assume that (2) holds, i.e. $V \in H^+(U)$ and $W \in H^-(\Sigma^{-1}U) \cap H^-(\Sigma^{-1}V)$. By Observation 6.4.5, if $h = 0$ then $k = h = 0$, $l = 1 - k = 1$, $V \cap_L U \neq \varnothing$, $(W - 2\pi) \cap_L V \neq \varnothing$, and $(W-2\pi)\cap_L U \neq \varnothing$. Thus, $gf \neq 0$. If $h=1$, then $k=h=1$, $l=1-k=0$, $V\cap_L U \neq \varnothing$, $W\cap_L V \neq \varnothing$, and $W\cap_L U \neq \varnothing$. As a consequence, $gf \neq 0$.

Finally, assume that (3) holds. If $h=0$, then $k=1-h=1$, $l=k=1$, $(V-2\pi)\cap_L U \neq \varnothing$, $W\cap_L V \neq \varnothing$, and $(W-2\pi)\cap_L U \neq \varnothing$. If $h=1$, then $k=1-h=0$, $l=k=0$, $V\cap_L U \neq \varnothing$, $W\cap_L V \neq \varnothing$, and $W\cap_L U \neq \varnothing$. In both cases we have that $gf \neq 0$. □

6.9.3 Irreducible morphisms and almost split sequences

We describe the irreducible morphisms and the almost split sequences in $\mathcal{C}_{-1,m}$. The following result follows directly from Proposition 2.3.13 and Proposition 6.8.1.

Proposition 6.9.4. *Let $a=(a_1,a_2), b=(b_1,b_2) \in \operatorname{ind}\overline{\mathcal{C}}_{-1,m}$. If $b=(a_1,a_2-2)$ or $b=(a_1-2,a_2)$, then any non-zero morphism $a \to b$ of $\mathcal{C}_{-1,m}$ is irreducible. Moreover, there are no other irreducible morphisms in $\overline{\mathcal{C}}_{-1,m}$ between indecomposable objects.*

The following result follows directly from 2.3.14 and Proposition 6.8.2.

Proposition 6.9.5. *Let $a=(a_1,a_2) \in \operatorname{ind}\mathcal{C}_{-1,m}$. The following statements hold.*

1. *If $a_1=a_2+1$, then $(a_1+2,a_2+2) \longrightarrow (a_1,a_2+2) \longrightarrow (a_1,a_2) \longrightarrow \Sigma(a_1+2,a_2+2)$ is an almost split triangle.*

2. *If $a_1 \neq a_2+1$, then $(a_1+2,a_2+2) \longrightarrow (a_1,a_2+2)\oplus(a_1+2,a_2) \longrightarrow (a_1,a_2) \longrightarrow \Sigma(a_1+2,a_2+2)$ is an almost split triangle.*

Thus, we have that $\tau a = (a_1+2,a_2+2)$.

6.9.4 The Calabi–Yau property

By Proposition 6.9.5, $\mathcal{C}_{-1,m}$ has almost split triangles, and as a consequence it has a Serre functor, by Proposition 2.3.6. In Proposition 6.9.7 we prove that Σ^{-1} is a Serre functor, i.e. $\mathcal{C}_{-1,m}$ is (-1)-CY. We refer to Section 2.1 for some background about Serre functors. We start with the following lemma.

Lemma 6.9.6. *Let $a,b \in \operatorname{ind}\mathcal{C}_{-1,m}$ and $f\colon a \to b$ be non-zero. Then there exists $g\colon b \to \Sigma^{-1}a$ such that $gf \neq 0$.*

Proof. Since $f \neq 0$, $b \in H^+(a)\cup H^-(\Sigma^{-1}a)$. If $b \in H^+(a)$ then $a \in H^-(b)$, i.e. $\Sigma^{-1}a \in H^-(\Sigma^{-1}b)$, and if $b \in H^-(\Sigma^{-1}a)$ then $\Sigma^{-1}a \in H^+(b)$. Thus, there exists a non-zero morphism $g\colon b \to \Sigma^{-1}a$. We have the following possibilities: $b \in H^+(a)$ and $\Sigma^{-1}a \in H^-(\Sigma^{-1}a)\cap H^-(\Sigma^{-1}b)$, or $b \in H^-(\Sigma^{-1}a)$ and $\Sigma^{-1}a \in H^-(\Sigma^{-1}a)\cap H^+(b)$. In both cases, by Proposition 6.9.3, we obtain that $gf \neq 0$. □

By Theorem 2.1.5, in order to prove that Σ^{-1} is a Serre functor, it is enough to check the existence of certain non-degenerate pairings. Thus, the following result implies that $\mathcal{C}_{-1,m}$ is (-1)-CY. We can use the same argument of Proposition 5.3.3.

Proposition 6.9.7. *Let $a, b \in \mathcal{C}_{-1,m}$. The following is a non-degenerate pairing.*

$$\begin{aligned} \Phi_{a,b} \colon \operatorname{Hom}_{\mathcal{C}_{-1,m}}(a,b) \times \operatorname{Hom}_{\mathcal{C}_{-1,m}}(b, \Sigma^{-1}a) &\longrightarrow \mathbb{K} \\ (f,g) &\longmapsto \operatorname{Tr}(gf) \end{aligned}$$

The category $\mathcal{C}_{-1,m}$ and the Holm-Jørgensen category $\mathcal{T}_{-1}$ are both $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt, algebraic triangulated, and (-1)-CY. We observe that $\mathcal{C}_{-1,1}$ and $\mathcal{T}_{-1}$ have the same geometric model, and therefore they are equivalent as additive categories. We refer for instance to [14, Section 2, Section 5] for the description of the properties and combinatorial model of $\mathcal{T}_{-1}$. We expect that $\mathcal{C}_{-1,1}$ and $\mathcal{T}_{-1}$ are triangle equivalent.

Conjecture 6.9.8. The categories $\mathcal{C}_{-1,1}$ and $\mathcal{T}_{-1}$ are triangle equivalent.

The idea of the proof is as follows. First we need to prove that the object $a = (a_1, a_1 - 2) \in \operatorname{ind} \mathcal{C}_{-1,1}$ is (-1)-*spherical*, see for instance [14, Section 2, p. 5] for the definition of spherical object. Then we should check that $\mathcal{C}_{-1,1}$ is *generated* by a, i.e. $\mathcal{C}_{-1,1}$ concides with its smallest thick subcategory containing a. By [35, Theorem 2.1], the category $\mathcal{T}_{-1}$ is the unique, up to triangle equivalence, algebraic triangulated category generated by a (-1)-spherical object, and therefore $\mathcal{C}_{-1,1}$ and $\mathcal{T}_{-1}$ are triangle equivalent.

Bibliography

[1] L. Angeleri Hügel, S. Koenig, Q. Liu, *Jordan Hölder theorems for derived module categories of piecewise hereditary algebras*, J. Algebra **352** (2021), 361-381.

[2] I. Assem, D. Simson, A. Skovroński, *Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory*, London Math. Soc. Stud. Texts, 65, Cambridge University Press, Cambridge, 2006.

[3] J. August, M. W. Cheung, E. Faber, S. Gratz, S. Schroll, *Cluster structures for the A_∞ singularity* J. Lond. Math. Soc. (2) **107** (2023), no. 6, 2121–2149.

[4] M. Auslander, S. O. Smalø, *Preprojective modules over Artin algebras*, J. Algebra **66** (1980), no. 1, 61–122.

[5] A. Beĭlinson, J. Bernstein, P. Deligne, O. Gabber, *Faisceaux pervers*, Société mathematique de France, 2018.

[6] M. V. Bondarko, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, J. K-Theory **6** (2010), no. 3, 387–504.

[7] A. B. Buan, B. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572-618.

[8] T. Bülher. *Exact categories*, Expo. Math. **28** (2010), no.1, 1–69.

[9] P. Caldero, F. Chapoton, R. Schiffler, *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1347-1364.

[10] I. Çanakçi, M. Kalck, M. Pressland, *Cluster categories for completed infinity-gons I: Categorifying triangulations*, arXiv:2401.08378.

[11] H. Chang, Y. Zhou, B. Zhu, *Cotorsion pairs in cluster categories of type A_∞^∞*, J. Combin. Theory Ser. A **156** (2018), 119-141.

[12] R. Coelho Simões, *Hom-configurations in triangulated categories generated by spherical objects*, J. Pure Appl. Algebra **219** (2015), no. 8, 3322–3336.

[13] R. Coelho Simões, *Mutations of simple-minded systems in Calabi-Yau categories generated by a spherical object*, Forum Math. **29** (2017), no. 5, 1065–1081.

[14] R. Coelho Simões, D. Pauksztello, *Torsion pairs in a triangulated category generated by a spherical object*, J. Algebra **448** (2016), 1-47.

[15] R. Coelho Simões, D. Pauksztello, *Simple-minded systems and reduction for negative Calabi–Yau triangulated categories*, Trans. Amer. Math. Soc. **373** (2020), no. 4, 2463–2498.

[16] W. Crawley-Boevey, *Decomposition of pointwise finite-dimensional persistence modules*, J. Algebra Appl. **14** (2015), no. 5, 1550066, 8 pp.

[17] C. Cummings, S. Gratz, *Metric completions of discrete cluster categories*, arXiv:2407.17369.

[18] A. Dugas, *Torsion pairs and simple-minded systems in triangulated categories*, Appl. Categ. Structures **23** (2015), no. 3, 507–526.

[19] T. Fisher, *On the enlargement by Prüfer objects of the cluster category of type A_∞*, arXiv:1411.4856.

[20] S. Franchini, *Torsion pairs, t-structures, and co-t-structures for completions of discrete cluster categories*, arXiv:2403.08735.

[21] S. Gratz, T. Holm, P. Jørgensen, *Cluster tilting subcategories and torsion pairs in Igusa–Todorov cluster categories of Dynkin type A_∞*, Math. Z. **292** (2019), no. 1-2, 33-56.

[22] S. Gratz, A. Zvonareva, *Lattices of t-structures and thick subcategories for discrete cluster categories*, J. Lond. Math. Soc. (2) **107** (2023), no. 3, 973-1001.

[23] E. Hanson, J. D. Rock, *Decomposition of pointwise finite-dimensional $\mathbb{S}^1$ persistence modules*, J. Algebra Appl. **23** (2024), no. 3, Paper No. 2450054, 24 pp.

[24] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Math. Soc. Lecture Note Ser., 119, Cambridge University Press, Cambridge, 1988.

[25] T. Holm, P. Jørgensen, *Triangulated categories: definitions, properties and examples*, Triangulated categories, 1–51, London Math. Soc. Lecture Note Ser., 375, Cambridge University Press, Cambridge, 2010.

[26] T. Holm, P. Jørgensen, *On a cluster category of infinite Dynkin type, and the relation to triangulations of the infinity-gon*, Math. Z. **270** (2012), no. 1-2, 277-295.

[27] T. Holm, P. Jørgensen, *Cluster tilting vs. weak cluster tilting in Dynkin type A infinity*, Forum Math. **27** (2015), no. 2, 1117–1137.

[28] T. Holm, P. Jørgensen, D. Yang, *Sparseness of t-structures and negative Calabi-Yau dimension in triangulated categories generated by a spherical object*, Bull. Lond. Math. Soc. **45** (2013), no. 1, 120–130.

[29] A. Hubery, *Characterising the bounded derived category of an hereditary abelian category*, arXiv:1612.06674.

[30] K. Igusa, G. Todorov, *Cluster categories coming from cyclic posets*, Comm. Algebra **43** (2015), no. 10, 4367-4402.

[31] O. Iyama, Y. Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. **172** (2008), no. 1, 117-168.

[32] P. Jørgensen, *Auslander–Reiten theory over topological spaces*, Comment. Math. Helv. **79** (2004), no. 1, 160–182.

[33] P. Jørgensen, *Auslander-Reiten triangles in subcategories*, J. K-Theory **3** (2009), no. 3, 583-601.

[34] B. Keller, *On triangulated orbit categories*, Doc. Math. **10** (2005), 551–581.

[35] B. Keller, D. Yang, G. Zhou, *The Hall algebra of a spherical object*, J. Lond. Math. Soc. (2) **80** (2009), no. 3, 771–784.

[36] H. Krause, *Localization theory for triangulated categories*, London Math. Soc. Lecture Note Ser., 375, Cambridge University Press, Cambridge, 2010.

[37] H. Krause, *Krull-Schmidt categories and projective covers*, Expo. Math. **33** (2015), no. 4, 535–549.

[38] G. Kreweras, *Sur les partitions non-croisées d'un cycle*, Discrete Math. **1** (1972), no. 4, 333–350.

[39] S. Liu, C. Paquette, *Cluster categories of type* $\mathbb{A}_\infty^\infty$ *and triangulations of the infinite strip*, Math. Z. **286** (2017), no. 1-2, 197-222.

[40] D. Murphy, *Bounding the Orlov spectrum for a completion of discrete cluster categories*, arXiv:2308.01767.

[41] P. Ng, *A characterization of torsion theories in the cluster category of Dynkin type* A_∞, arXiv:1005.4364.

[42] P. Nicolás, M. Saorín, *Lifting and restricting recollement data*, Appl. Categ. Structures **19** (2011), no. 3, 557–596.

[43] C. Paquette, E. Yıldırım, *Completions of discrete cluster categories of type* $\mathbb{A}$, Trans. London Math. Soc. **8** (2021), no. 1, 35-64.

[44] D. Pauksztello, *Compact corigid objects in triangulated categories and co-t-structures*, Cent. Eur. J. Math. **6** (2008), no. 1, 25–42.

[45] I. Reiten, M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), no. 2, 295–366.

[46] J. D. Rock, S. Zhu, *Continuous Nakayama representations*, Appl. Categ. Structures **31** (2023), no. 5, Paper No. 44, 25 pp.

[47] M. Saorín, A. Zvonareva, *Lifting of recollements and gluing of partial silting sets*, Proc. Roy. Soc. Edinburgh Sect. A **152** (2022), no. 1, 209–257.

[48] J. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque, tome 239 (1996).

[49] Y. Zhou, B. Zhu, *T-structures and torsion pairs in a 2-Calabi-Yau triangulated category*, J. Lond. Math. Soc. (2) **89** (2014), no. 1, 213-234.

[50] A. Zimmermann, *Representation theory. A homological algebra point of view*, Algebr. Appl., 19, Springer, Cham, 2014.