Lagrangian Surplusection Phenomena

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Abstract. Suppose you have a family of Lagrangian submanifolds L_t and an auxiliary Lagrangian K. Suppose that K intersects some of the L_t more than the minimal number of times. Can you eliminate surplus intersection (surplusection) with all fibres by performing a Hamiltonian isotopy of K? Or will any Lagrangian isotopic to K surplusect some of the fibres? We argue that in several important situations, surplusection cannot be eliminated, and that a better understanding of surplusection phenomena (better bounds and a clearer understanding of how the surplusection is distributed in the family) would help to tackle some outstanding problems in different areas, including Oh's conjecture on the volume-minimising property of the Clifford torus and the concurrent normals conjecture in convex geometry. We pose many open questions.

Key words: symplectic geometry; Lagrangian intersections; Floer theory

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1 Introduction

In this paper, we draw attention to a class of Lagrangian intersection problems which we believe deserve further study. We will start by outlining the general class of problems, and then give some specific examples where we either know or suspect that the answer is interesting.

Definitions 1.1. Let (X, ω) be a symplectic manifold. Given Lagrangian submanifolds K and L in X, define their *geometric intersection number* to be the smallest number of (transverse) intersections that can be achieved between L and a Lagrangian Hamiltonian isotopic to K:

$$i(K, L) = \min\{\#(\phi(K) \pitchfork L) \mid \phi \in \mathrm{Ham}(X, \omega)\}.$$

We say that K and L *surplusect* if $\#(K \cap L) > i(K, L)$. Given a family $\mathcal{L} = \{L_t \mid t \in T\}$ of Lagrangian submanifolds (parametrised by a measure space T with measure μ) and an auxiliary Lagrangian K, we define the *surplusection locus* to be

$$\mathbb{S}_{\mathcal{L}}(K) = \{t \in T \mid \#(K \cap L_t) > i(K, L_t)\} \subseteq T$$

and the *mean surplusection* to be the integral

$$\frac{1}{\int_T \mathrm{d}\mu} \int_T (\#(K \cap L_t) - i(K, L_t))\mathrm{d}\mu,$$

assuming it is well defined.

In Figure 1, we see an example of a Lagrangian submanifold K which can be straightened by a Hamiltonian isotopy to remove all of its surplusection with the vertical Lagrangians. Our main thesis is that, in many situations of interest, you cannot achieve this, and surplusection is forced upon you no matter how you isotope K.
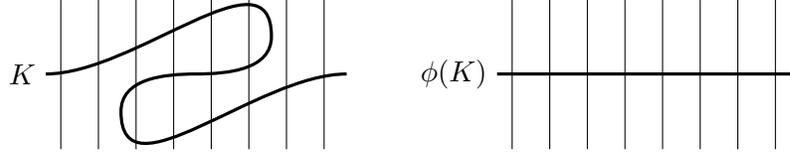
Figure 1. (1) The Lagrangian K surplusects some of the vertical fibres, (2) but this surplusection can be removed by an isotopy of K.

Questions 1.2.

(A) Is there a $\phi \in \mathrm{Ham}(X, \omega)$ for which $\mathbb{S}_{\mathcal{L}}(\phi(K))$ is empty? In other words, can we arrange for $\phi(K)$ to simultaneously intersect all L_t in the family minimally?

(B) If not, what is $\inf\{\mu(\mathbb{S}_{\mathcal{L}}(\phi(K))) \mid \phi \in \mathrm{Ham}(X, \omega)\}$? Or can we bound from below the mean surplusection?

In some cases, the quantity $i(K, L)$ might be hard to compute, in which case one could replace it by an Ersatz which bounds $i(K, L)$ from below (like the rank of Lagrangian Floer cohomology, assuming it is defined between K and L). In the rest of this note, we outline some of the situations in which these very general questions arise naturally.

2 Volume bounds

2.1 Crofton formula

Lê [19] proved the following remarkable formula[1] for Lagrangian submanifolds of $\mathbb{CP}^n$:

$$\mathrm{vol}(K) = \xi_n \int_{g \in \mathrm{PU}(n+1)} \#(K \cap gL) \mathrm{d}\mu, \tag{2.1}$$

where $\mathrm{vol}(K)$ is the unsigned Riemannian volume of K, ξ_n is a constant depending only on n, $\mathrm{PU}(n + 1)$ is the group of projective unitary isometries of $\mathbb{CP}^n$, L is the standard $\mathbb{RP}^n \subset \mathbb{CP}^n$, and μ is the Haar measure on $\mathrm{PU}(n+1)$. We can compute the number ξ_n by picking $K = \mathbb{RP}^n$: since[2] $\#(\mathbb{RP}^n \cap g\mathbb{RP}^n) = n+1$ for all but a measure-zero set of $g \in \mathrm{PU}(n + 1)$, we get

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[1]It is named the Crofton formula after an analogous result in integral geometry of Euclidean space. Oh [23, p. 503] mentions the existence of such a formula for Lagrangian submanifolds of $\mathbb{CP}^n$, attributing it to Kleiner. The proof is all the more remarkable for its simplicity: we sketch a proof following [17, Section 1.1]. Observe that both sides of the equation are additive under taking disjoint union, e.g., $\mathrm{vol}(K_1 \coprod K_2) = \mathrm{vol}(K_1) + \mathrm{vol}(K_2)$. Since one can approximate any Lagrangian arbitrarily closely by something which is piecewise linear, made of tiny patches of totally geodesic $\mathbb{RP}^n$s, it is sufficient to prove the formula for totally geodesic $\mathbb{RP}^n$s. Since these are all related by isometries, this just amounts to fixing the constant ξ_n.

[2]Oh [24] calls this property *global tightness*, and showed that the only globally tight Lagrangian submanifolds of $\mathbb{CP}^n$ are the standard $\mathbb{RP}^n$s.

$\xi_n = \mathrm{vol}(\mathbb{RP}^n)/(n+1)$. We will normalise the Fubini–Study metric by assuming that $\mathrm{vol}(\mathbb{RP}^n)$ is half the surface area of a unit Euclidean n-sphere, that is,

$$\mathrm{vol}(\mathbb{RP}^n) = \frac{\pi^{\frac{n+1}{2}}}{\Gamma((n+1)/2)}, \qquad \xi_n = \frac{\pi^{\frac{n+1}{2}}}{(n+1)\Gamma((n+1)/2)}, \tag{2.2}$$

where Γ is the Gamma function. So, for an arbitrary Lagrangian K for which $i(K,\mathbb{RP}^n)$ is known, the problem of bounding from below the mean surplusection of $\phi(K)$ and $\{g\mathbb{RP}^n \mid g \in \mathrm{PU}(n+1)\}$ is equivalent to finding a lower bound on $\mathrm{vol}(\phi(K))$ (assuming we know $i(K,L)$). Bounding the volume of $\phi(K)$ from below is a notoriously thorny problem in general: Oh [25, p. 192] conjectured that the monotone Clifford torus minimises volume amongst Lagrangians in its Hamiltonian isotopy class, but this conjecture remains open thirty years later.

Definition 2.1. Let $\mu\colon \mathbb{CP}^n \to \mathbb{R}^n$ be the moment map for the standard torus action, normalised so that its image is the simplex with its vertices at the origin and the standard basis vectors. The monotone Clifford torus is the Lagrangian torus $\mathbb{T}^n = \mu^{-1}\big(\frac{1}{n+1},\ldots,\frac{1}{n+1}\big)$. It is the unique nondisplaceable fibre of the moment map. We say that a Lagrangian torus is *Clifford-type* if it has the form $K = \phi(\mathbb{T}^n)$ for some Hamiltonian symplectomorphism ϕ.

2.2 The bound of Alston and Amorim

Alston [3] and Alston–Amorim [4] used the Crofton formula (2.1) to give lower bounds on the volume of a Clifford-type torus $\phi(\mathbb{T}^n)$ by showing that[3]

$$i(\mathbb{T}^n, \mathbb{RP}^n) \geq 2^{\lceil n/2 \rceil}. \tag{2.3}$$

The resulting volume bound is

$$\mathrm{vol}(\mathbb{T}^n) \geq \frac{\pi^{\frac{n+1}{2}} 2^{\lceil \frac{n}{2} \rceil}}{(n+1)\Gamma((n+1)/2)}. \tag{2.4}$$

If $n = 1$, then the lower bound from (2.4) is sharp because it equals $\mathrm{vol}\big(\mathbb{T}^1\big) = \pi$. But if $n \geq 2$, the bound is substantially lower than

$$\mathrm{vol}(\mathbb{T}^n) = \frac{(2\pi)^n}{(n+1)^{\frac{n+1}{2}}}, \tag{2.5}$$

as you can see from Table 1 below. Since $\mathbb{T}^n$ is conjectured to minimise volume in its Hamiltonian isotopy class, this suggests that there should be lots of surplusection.

Remark 2.2. In fact, as we will see in a moment, there exist much stronger bounds (due to Goldstein) on the volume of a Clifford-type torus (also shown in Table 1). However, the bound (2.3) on $i(K,\mathbb{RP}^n)$ is actually optimal: the lower bound of $2^{\lceil n/2 \rceil}$ intersections is realised by intersecting $\mathbb{T}^n$ with a specific $\mathbb{RP}^n$. Namely, let L be the fixed point set of the antisymplectic involution

$$[z_0 : z_1 : \cdots : z_{n-1} : z_n] \mapsto \begin{cases} [\bar{z}_1 : \bar{z}_0 : \cdots : \bar{z}_n : \bar{z}_{n-1}] & \text{if } n \text{ is odd,} \\ [\bar{z}_1 : \bar{z}_0 : \cdots : \bar{z}_{n-1} : \bar{z}_{n-2} : \bar{z}_n] & \text{if } n \text{ is even.} \end{cases}$$

This intersects $\mathbb{T}^n$ in a subtorus of dimension $\lceil n/2 \rceil$, so if $g \in \mathrm{PU}(n+1)$ is a generic element sufficiently close to the identity then $\mathbb{T}^n \cap gL$ consists of $2^{\lceil n/2 \rceil}$ transverse intersections.

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[3] If n is odd, then this bound is established by calculating (characteristic 2) Floer cohomology between $\mathbb{RP}^n$ and (a suitable local system on) K; if n is even, then all such Floer cohomology groups vanish, and the bound on geometric intersection is proved instead using the Abreu–Macarini trick [2], that is, by computing Floer cohomology between $\mathbb{RP}^n \times \mathbb{RP}^n$ and $K \times K$ in $\mathbb{CP}^n \times \mathbb{CP}^n$.

n	Alston–Amorim	Goldstein	$\mathrm{vol}(\mathbb{T}^n)$
1	$3.14159\ldots$	$3.14159\ldots$	$3.14159\ldots$
2	$4.18879\ldots$	$7.25519\ldots$	$7.59762\ldots$
3	$9.86960\ldots$	$13.95772\ldots$	$15.50313\ldots$
4	$10.52757\ldots$	$23.54038\ldots$	$27.88010\ldots$
5	$20.67085\ldots$	$35.80296\ldots$	$45.33624\ldots$
6	$18.89906\ldots$	$64.93939\ldots$	$67.80099\ldots$

Table 1. Lower bounds on $\mathrm{vol}(\phi(\mathbb{T}^n))$ for $\phi \in \mathrm{Ham}(\mathbb{CP}^n)$, due to Alston–Amorim and Goldstein, compared with the volume of the standard Clifford torus (final column).

2.3 Goldstein's bound

In fact, Goldstein [12] gave a considerably better bound by exploiting the following identity from integral geometry:

$$\mathrm{vol}(K) \cdot \mathrm{vol}(L) = \zeta_n \int_{\mathrm{PU}(n+1)} \#(K \cap gL)\mathrm{d}g,$$

where K and L are Lagrangian submanifolds of $\mathbb{CP}^n$ and ζ_n is a constant (depending only on n) to be determined shortly. This identity is attributed to Howard [16].[4] To determine the constant, Goldstein applies this inequality in the case $K = L = \mathbb{RP}^n$

$$\mathrm{vol}(\mathbb{RP}^n)^2 = \zeta_n(n+1), \qquad \text{so} \qquad \zeta_n = \frac{\pi^{n+1}}{(n+1)\left(\Gamma((n+1)/2)\right)^2}.$$

Now he applies it when $K = L = \phi(\mathbb{T}^n)$ is a Clifford-type torus. Since $HF(\phi(\mathbb{T}^n), \phi(\mathbb{T}^n))$ has rank 2^n, this gives the bound

$$\mathrm{vol}(\phi(\mathbb{T}^n)) \geq \frac{1}{\Gamma((n+1)/2)} \sqrt{\frac{2^n \pi^{n+1}}{n+1}}$$

(tabulated for small n in Table 1).

Remark 2.3. The fact that the Goldstein bound is stronger than the Alston–Amorim bound means that *surplusection must occur* between $\mathbb{T}^n$ and $\{g\mathbb{RP}^n \mid g \in \mathrm{PSU}(n+1)\}$. But it seems like an interesting problem to determine how this surplusection is distributed over $\mathrm{PSU}(n+1)$: the volume bound only guarantees that the *mean* surplusection is large, but that could mean a very small surplusection locus with very high surplusection. We take a moment, therefore, to study the distribution of $\#\left(\mathbb{T}^2 \cap g\mathbb{RP}^2\right)$ for $g \in \mathrm{PSU}(3)$ where $\mathbb{T}^2$ is the Clifford torus in its standard (conjecturally minimal) position. This is the smallest interesting case, and the only one we have solved completely.

2.4 Crofton distribution for the Clifford torus

A point $(x, y, z) \in \mathbb{C}^3$ represents a point $[x : y : z]$ on the Clifford torus if and only if $|x|^2 = |y|^2 = |z|^2$. Given a matrix $g \in \mathrm{SU}(3)$, a point $(x, y, z) \in \mathbb{C}^3$ represents a point $[x : y : z] \in g\mathbb{RP}^2$

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[4]We can see how to prove it as follows: fix K and treat both sides as functionals of L, additive over concatenation. Just as with Crofton's formula, the identity then holds with a constant ζ_K potentially depending on K. But since both $\mathrm{vol}(K) \cdot \mathrm{vol}(L)$ and the integral are symmetric in K and L, we get the same constant $\zeta_K = \zeta_L$, and so the constant depends only on the choice of ambient space.

if and only if there exists a unit complex number u such that (ux, uy, uz) is contained in the real span of the columns of g. Therefore, the intersection $\mathbb{T}^2 \cap g\mathbb{RP}^2$ consists of points $[x : y : z]$ with (x, y, z) in the real span of the columns of g and which satisfy $|x|^2 = |y|^2 = |z|^2$. Write g_{ij} for the entries of g and let $a = (a_1, a_2, a_3) \in \mathbb{R}^3$. Let $(x_1, x_2, x_3) = ga^\mathsf{T}$ be the real linear combination of the columns of g with coefficients given by a_1, a_2, a_3. We have

$$|x_i|^2 = \sum_{j,k} g_{ij}\bar{g}_{ik}a_j a_k.$$

The intersection $\mathbb{T}^2 \cap g\mathbb{RP}^2$ is therefore in bijection with the subvariety of $\mathbb{RP}^2$ given by the intersection of the two conics

$$\sum_{j,k}(g_{2j}\bar{g}_{2k} - g_{1j}\bar{g}_{1k})a_j a_k = 0, \qquad \sum_{j,k}(g_{3j}\bar{g}_{3k} - g_{1j}\bar{g}_{1k})a_j a_k = 0.$$

In particular, this gives an upper bound of four intersection points (by Bézout's theorem). The lower bound of two intersections comes from Floer cohomology, and there must always be an even number of intersections if the intersection is transverse, so the only possibilities for $\#\left(\mathbb{T}^2 \cap g\mathbb{RP}^2\right)$ if the intersection is transverse are 2 and 4.

Corollary 2.4. *The probability that a randomly chosen $\mathbb{RP}^2$ intersects the standard Clifford torus at 2 (respectively 4) points is $2 - \frac{\pi}{\sqrt{3}} \approx 0.186$ (respectively $\frac{\pi}{\sqrt{3}} - 1 \approx 0.814$).*

Proof. The volume of the standard Clifford torus is $\frac{4\pi^2}{3\sqrt{3}}$ and the volume of $\mathbb{RP}^2$ is 2π. The expected number of intersections is given by the Crofton formula

$$\frac{4\pi^2}{3\sqrt{3}} = \frac{2\pi}{3}(2p_2 + 4p_4),$$

where p_i denotes the probability of i intersections. We also know that $p_2 + p_4 = 1$ because the probability of a non-transverse intersection is zero by Sard's theorem and the only possible numbers for transverse intersections are 2 or 4. This gives a pair of simultaneous equations for p_2 and p_4 which have the solutions as stated. ∎

2.5 Questions

The analogous problem in higher dimensions can also be recast as an intersection of real quadrics. Just knowing the volume and the fact that the probabilities sum to 1 is no longer enough to determine all the probabilities (for example, for $\mathbb{CP}^3$ we need to determine p_4, p_6 and p_8).

Question 2.5. Fix $n > 3$. Let p_{2i} be the probability that the standard Clifford torus $\mathbb{T}^n \subset \mathbb{CP}^n$ intersects $g\mathbb{RP}^n$ transversely in $2i$ points (where $g \in \mathrm{PSU}(n+1)$ and we use the Haar measure on $\mathrm{PSU}(n+1)$). Determine the probability distribution $p_{2\lceil n/2 \rceil}, \ldots, p_{2^n}$ for each n.

Remark 2.6. Equations (2.1), (2.2) and (2.5) give the mean of this probability distribution as

$$\sum_{m=2^{\lceil n/2 \rceil}}^{2^n} m p_m = \xi_n^{-1}\mathrm{vol}(\mathbb{T}^n) = \frac{(2\pi)^n}{(n+1)^{\frac{n+1}{2}}} \cdot \frac{(n+1)\Gamma((n+1)/2)}{\pi^{\frac{n+1}{2}}} = \frac{2^n \pi^{\frac{n-1}{2}}\Gamma((n+1)/2)}{(n+1)^{\frac{n-1}{2}}}.$$

As observed by Alston [3], using Stirling's approximation $\Gamma(x) \approx \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x$, the mean is approximately 2^{cn}, where $c = \frac{1}{2}\log_2(2\pi/e) \approx 0.604$.

Question 2.7. Can anything more be said about the asymptotics of this distribution as $n \to \infty$?

Question 2.8 (refined Oh conjecture). Let $\phi(\mathbb{T}^n)$ be a Clifford-type torus for $\phi \in \mathrm{Ham}(\mathbb{CP}^n)$, and let μ be the Haar measure on $\mathrm{PSU}(n+1)$. Let q_{2k} be the probability with respect to μ that $\#(\phi(\mathbb{T}^n) \cap g\mathbb{RP}^n) \geq 2k$. Is it true that $\sum_{k \geq \ell} p_{2k} \leq \sum_{k \geq \ell} q_{2k}$ for all ℓ? Even short of proving this (which would imply Oh's conjecture), any lower bound (independent of ϕ) on some specific sum $\sum_{k \geq \ell} q_{2k}$ would give more information about how the surplusection guaranteed by Goldman's bound is distributed.

Remark 2.9. We note that Viterbo [28, Proposition 3.13] established a lower bound for the volume of *any* Lagrangian submanifold $L \subset \mathbb{CP}^n$. The lower bound is a constant times $\tilde{d}(L)^{n/2}$ where $\tilde{d}(L)$ is the displacement energy of the preimage of the Lagrangian in $S^{2n-1} \subset \mathbb{C}^{n+1}$ under the symplectic reduction $S^{2n-1} \to \mathbb{CP}^n$. This indicates that surplusection is a ubiquitous phenomenon and should probably be closely related to action filtrations, spectral invariants, and persistence modules.

3   Chekanov-type tori

Having focused on the case of Clifford-type tori in all dimensions, and how they intersect all images of $\mathbb{RP}^n$ under Kähler isometries, we turn to a more specific problem: how *Chekanov-type* tori in $\mathbb{CP}^2$ intersect a particular *loop* of $\mathbb{RP}^2$s. In this case, a Chekanov-type torus can be displaced from $\mathbb{RP}^2$, so *any* intersection is surplus. We will be able to give better, in fact sharp, control on the measure of the surplusection locus.

Definition 3.1 (clean loops). For each $t \in [0,1]$, let L_t be the Lagrangian submanifold

$$L_t := \left\{ \left[(a_1 + \mathrm{i}a_2)\mathrm{e}^{\mathrm{i}\pi t/3} : (a_1 - \mathrm{i}a_2)\mathrm{e}^{\mathrm{i}\pi t/3} : a_3 \mathrm{e}^{-\mathrm{i}2\pi t/3} \right] \mid [a_1 : a_2 : a_3] \in \mathbb{RP}^2 \right\} \subset \mathbb{CP}^2.$$

As t varies this defines for us a loop of $\mathbb{RP}^2$s coming from a 1-parameter subgroup of $\mathrm{PU}(3)$. If $t_1 \neq t_2 \mod 1$, then the intersection $L_{t_1} \cap L_{t_2}$ consists of a single transverse intersection at $[0:0:1]$ and a circle of clean intersection $\{[a:b:0] \mid |a| = |b|\}$. We call this a *clean loop* of $\mathbb{RP}^2$s.

Consider the moment map

$$\mu \colon \ \mathbb{CP}^2 \to \mathbb{R}^2, \qquad \mu([z_1 : z_2 : z_3]) = \frac{1}{2\sum_{j=1}^3 |z_j|^2} \left(|z_1|^2, |z_2|^2 \right)$$

for the standard torus action. Let $\Delta \subset \mathbb{R}^2$ be the moment image and ℓ be the intersection of the diagonal $\{(x,x) \mid x \in \mathbb{R}\}$ with Δ. Similarly to Chekanov and Schlenk [8], consider the singular hypersurface $\mu^{-1}(\ell)$, which is preserved by the circle action $\left[\mathrm{e}^{\mathrm{i}\theta} z_1 : \mathrm{e}^{-\mathrm{i}\theta} z_2 : z_3 \right]$. The symplectic reduction of $\mu^{-1}(\ell)$ by this circle action is a singular sphere which we will call S; it has two singular points p and q living over the ends of ℓ. Here, p is over the endpoint where ℓ meets the slanted edge of Δ and q is over the endpoint where ℓ hits the lower-left corner of Δ. The preimage of p under reduction is a circle on which the circle action has stabiliser $\mathbb{Z}_2$; the preimage of q is a fixed point of the circle action. The submanifold $L_t \subset \mu^{-1}(\ell)$ projects to an arc $\alpha_t \subset S$ connecting p and q. See Figure 2.

Definitions 3.2. For each loop $\gamma \subset S \setminus \{p, q\}$, there is a Lagrangian torus $T_\gamma \subset \mu^{-1}(\ell) \subset \mathbb{CP}^2$ which projects to γ along the symplectic reduction. Let P denote the component of $S \setminus \gamma$ containing p. If $\mathrm{area}(P) = \frac{1}{3}\mathrm{area}(S)$, then T_γ is monotone. There are two isotopy classes of loop $\gamma \subset S \setminus \{p, q\}$: loops of class A separate p from q; loops of class B do not. If γ has class A, then T_γ is a Clifford-type torus; if γ has class B, then T_γ is not Clifford-type. Pick your favourite γ of class B and call T_γ the standard Chekanov torus Ч. A *Chekanov-type torus* is any Lagrangian torus of the form $\phi(Ч)$ for $\phi \in \mathrm{Ham}(\mathbb{CP}^2)$, and if γ is of class B, then T_γ is Chekanov-type. See Figure 3.
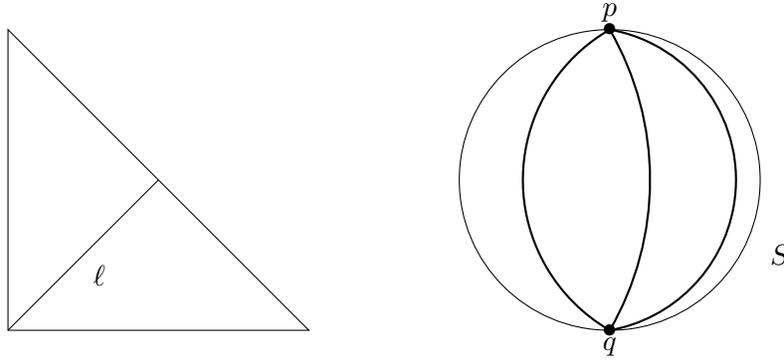
Figure 2. Left: The moment image of $\mathbb{CP}^2$ and the diagonal arc ℓ. Right: The symplectic reduction S of $\mu^{-1}(\ell)$ and several of the arcs α_t.
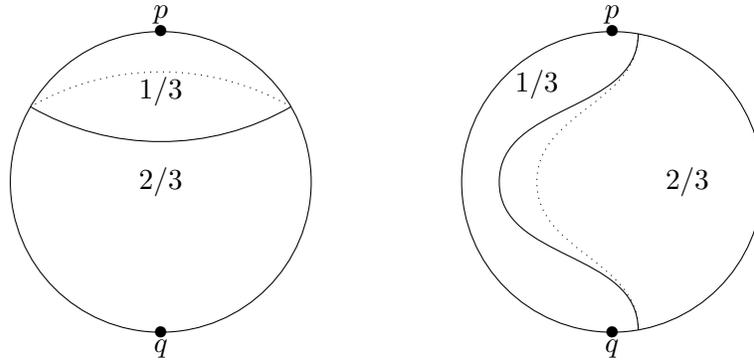
Figure 3. A loop of class A (left) and class B (right), yielding respectively a monotone Clifford-type and a Chekanov-type torus in $\mathbb{CP}^2$.

Lemma 3.3. *For any neighbourhood U of $L_t \cup L_{t+1/3}$, there is a monotone Chekanov-type torus Ч_U contained in U.*

Proof. Figure 4 shows a type B loop γ chosen in such a way that $\text{area}(P) = \frac{1}{3}\text{area}(S)$. This can be chosen to lie arbitrarily close to $\alpha_t \cup \alpha_{t+1/3}$. If it is close enough, then $T_\gamma \subset U$ and we can take $\text{Ч}_U = T_\gamma$. ∎

The fact that the Chekanov torus arises as a clean surgery of two $\mathbb{RP}^2$s in this way was observed by Abreu and Gadbled [1].

Corollary 3.4. *If $K = \phi(\text{Ч})$ is a Chekanov-type torus, then the intersection $K \cap (L_t \cup L_{t+1/3})$ is nonempty for any $t \in [0,1]$. Moreover, if K intersects both L_t and $L_{t+1/3}$ transversely and is disjoint from $L_t \cap L_{t+1/3}$, then the number of intersection points between K and $L_t \cup L_{t+1/3}$ is at least four.*

Proof. If K is disjoint from $L_t \cup L_{t+1/3}$, then it is disjoint from some neighbourhood U of this union, and hence from the torus Ч_U constructed in Lemma 3.3. But since both K and Ч_U are monotone Chekanov-type tori, the Floer cohomology group $HF(K, \text{Ч}_U)$ (with a suitable choice of local coefficient system) has rank four, so $i(K, \text{Ч}_U) \geq 4$, so this is impossible.

If the transversality conditions from the corollary hold, then there is a bijection between intersection points $K \cap (L_t \cup L_{t+1/3})$ and the intersection points $K \cap \text{Ч}_U$ for some sufficiently
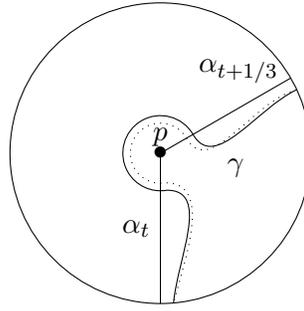
Figure 4. The arcs α_t and $\alpha_{t+1/3}$ and the loop γ, as viewed from the North pole (the dotted line indicates how it looks in the Southern hemisphere).

small neighbourhood U. The lower bound on the number of intersection points then follows from the fact that $i(K, Ч_U) \geq 4$. ∎

Corollary 3.5. *Take $\mathcal{L} = \{L_t \mid t \in [0,1]\}$ and let K be a monotone Chekanov-type torus. The surplusection locus $\mathbb{S}_{\mathcal{L}}(K) \subset [0,1]$ has Lebesgue measure at least $2/3$.*

Proof. Let $F = [0,1]\backslash\mathbb{S}_{\mathcal{L}}(K)$. If $t \in F$, then $t+1/3 \mod 1$ and $t-1/3 \mod 1$ belong to $\mathbb{S}_{\mathcal{L}}(K)$ by Corollary 3.4. This means that F, $F + 1/3 \mod 1$ and $F - 1/3 \mod 1$ are three disjoint subsets of $[0,1]$, each having the same measure, and hence the measure of each is at most $1/3$. Therefore, the measure of $\mathbb{S}_{\mathcal{L}}(K)$ is at least $2/3$. ∎

Remark 3.6. On a philosophical level, one may compare Corollary 3.4 with the recent works by Mak–Smith [21] and Polterovich–Shelukhin [27], which exhibit Lagrangian submanifolds which do not exhibit rigidity when taken alone but do exhibit rigidity when several copies are taken together.

Remark 3.7. One can use Corollary 3.5, together with standard formulas for integrating over Lie groups, to obtain volume bounds for the Chekanov torus from Crofton's formula,[5] however the resulting bound is only ≈ 3.27, which is significantly worse than Goldstein's bound. Goldstein's bound still applies to the Chekanov torus, and indeed any wide Lagrangian torus in $\mathbb{CP}^2$ (such as the Vianna tori), because the rank of Floer cohomology is still 4.

Remark 3.8. Despite the fact that this argument gives a weaker volume bound than Goldstein, it does give better control on the measure of the surplusection locus. In fact, it is sharp: one can (visibly, in the symplectic reduction) disjoin the Chekanov torus from $\bigcup_{t\in(\epsilon,1/3-\epsilon)} L_t$ for any ϵ. One can obtain still more information about the surplusection locus: Gathercole [11, Theorem 1.2] shows that $\phi(\mathbb{Y})$ cannot be disjoined from $L_s \cup L_{s+t}$ for any $s \in [0,1]$ and any $t \in [1/3, 2/3]$. This means that if $\phi(\mathbb{Y})$ misses (say) L_0 then the surplusection locus really contains the whole interval $[1/3, 2/3]$ (not just the endpoints).

The argument we used is very *ad hoc*, and does not easily generalise to tackle slight modifications of the problem. For example, the following.

Question 3.9. Can one see similar phenomena in higher dimensions?

Similar arguments appear to break down in $\mathbb{CP}^4$ and above. If we take a loop of diagonal matrices in $\mathrm{U}(n+1)$, where the first k entries coincide and the last $m = n+1-k$ entries coincide,

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[5]The interested reader can look at version 1 of this paper on arXiv for the details of how this bound is computed.

and apply this loop to $\mathbb{RP}^n$, then we get a clean loop of Lagrangian $\mathbb{RP}^n$s which intersect pairwise along a copy of $\mathbb{RP}^{k-1}$ and a copy of $\mathbb{RP}^{m-1}$. It appears (though we have not checked) that the clean surgery is the Oakley–Usher Lagrangian [22], which has vanishing Floer cohomology (it is actually displaceable) and our argument relies crucially on being able to use Floer cohomology with the result of surgery to get lower bounds on intersections.

Question 3.10. Let $\{g_t \mid t \in [0,1]\}$ be a *generic* 1-parameter subgroup in SU(3) isomorphic to U(1), so that $\mathbb{RP}^2 \cap g_t \mathbb{RP}^2$ consists of three points if $t \neq 0, 1$. Let $\mathcal{L}' = \{g_t \mathbb{RP}^2 \mid t \in [0,1]\}$ and let $K = \phi(\text{Ч})$ be a monotone Chekanov-type torus. Is it true that $\mathbb{S}_{\mathcal{L}'}(K)$ has Lebesgue measure at least $2/3$? The surgery of $\mathbb{RP}^2$ with $g_t \mathbb{RP}^2$ is now the non-orientable (Maslov 1) Lagrangian constructed by Abreu and Gadbled [1].

Question 3.11. Are we justified in harbouring an expectation that Chekanov tori should have bigger volume than Clifford tori? Or can one find Chekanov tori whose volumes get arbitrarily close to the volume of any Clifford torus? C. Evans[6] [10] showed that, under (volume-decreasing) Lagrangian mean curvature flow with surgeries, any Chekanov torus contained in the hypersurface $\mu^{-1}(\ell)$ satisfying a certain symmetry condition eventually flows with surgeries to the Clifford torus. However, this is a very special situation with lots of symmetry.

4 Concurrent normals

4.1 The original conjecture

We now move to another, completely different situation in which surplusection appears (and is conjectured to appear even more). Consider a not-too-eccentric ellipse in the plane and draw on all the lines which intersect it normally. Most points lie on two such normals, but there is a small star-shaped (astroid) region in the middle of the ellipse through every point of which four normals pass.
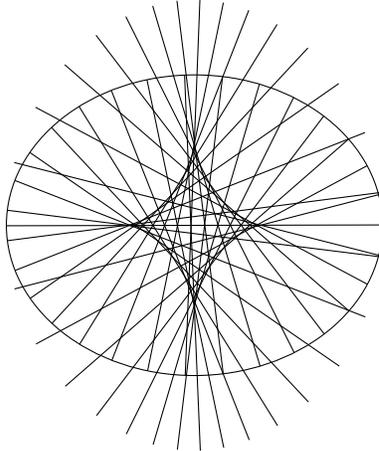
As the ellipse becomes more eccentric, the astroid starts to protrude out of the top and bottom; as the eccentricity goes to zero, the astroid shrinks to a point. In higher dimensions, the normals to ellipsoids $\sum \frac{x_i^2}{a_i^2} = 1$ have a similar behaviour: if the radii a_i are generic then the origin lies on $2n$ boundary normals (in the non-generic case it lies on infinitely many normals). There is a conjecture that any convex body has a caustic which is at least as complicated as this.

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[6]No relation.

Conjecture 4.1 ([9, Problem A3]). *If C is a convex body in $\mathbb{R}^n$ (with smooth boundary for the purposes of our discussion), then there exists a point $q \in C$ such that q lies on at least $2n$ inward normals to ∂C.*

Remark 4.2. This conjecture is known to hold in dimensions two (where it is easy), three (where it was proved by Heil [13, 14, 15]) and dimension four (where it was proved by Pardon [26]). It is also known when C is centrally symmetric, i.e., symmetric under the antipodal map, which was proved by Kuiper [18] as a corollary of Ljusternik–Schnirelmann theory [20]. The proofs in these cases translate it into a problem in Morse theory via the *support function*.

Definitions 4.3. The support function h_C of a convex body $C \subset \mathbb{R}^n$ is the function on the sphere $S^{n-1} = \{v \in \mathbb{R}^n \mid |v| = 1\}$ defined by setting $h_C(v) = \sup\{\langle u, v \rangle \mid u \in C\}$. If h_C is continuously differentiable (for example, if C has smooth boundary) and ∇h_C denotes its gradient with respect to the round metric, then we can recover ∂C as the image of the parametrisation

$$\varphi\colon\ S^{n-1} \to \mathbb{R}^n, \qquad \varphi(v) = h_C(v)v + \nabla h_C(v).$$

The inward normal vector to ∂C at $\varphi(v)$ is $-v$ and the boundary normals passing through the origin correspond to critical points of h_C. If we translate C so that a point $q \in C$ is the origin, then the support function becomes $h_{C-q}(v) = h_C(v) - \langle q, v \rangle$. This gives us a family of functions h_{C-q} parametrised by $q \in C$, which are Morse for generic $q \in C$. Those which are Morse are guaranteed to have at least 2 critical points, but the concurrent normals conjecture asserts that at least one of them has $2n$ critical points.

4.2   Reformulation in terms of surplusection

This problem can be viewed as a Lagrangian surplusection problem in the cotangent bundle T^*S^{n-1}. Namely, the space of oriented straight lines in $\mathbb{R}^n$ is naturally a symplectic manifold: it is the symplectic reduction (at height 1) of $(T^*\mathbb{R}^n, \sum \mathrm{d}p_i \wedge \mathrm{d}q_i)$ by the $\mathbb{R}$-action (cogeodesic flow) generated by the Hamiltonian $\frac{1}{2}\sum p_i^2$. Indeed, it is symplectomorphic to T^*S^{n-1}: intuitively, each oriented line has a given direction in S^{n-1} and there is a tangent hyperplane's worth of oriented lines pointing in a given direction. More formally, the unit cotangent bundle of $\mathbb{R}^n$ is strictly contactomorphic to the 1-jet bundle of S^{n-1} via Arnold's *hodograph transformation* [6, pp. 48–49], and the Reeb flow by which we symplectically reduce is the cogeodesic flow in one case and addition of a constant to the 1-jet in the other, so that in both cases symplectic reduction yields T^*S^{n-1}.

Given a submanifold $\Sigma \subset \mathbb{R}^n$, the space of lines orthogonal to it form a Lagrangian submanifold[7] L_Σ of T^*S^{n-1}. For example, if Σ is a point q then L_Σ is $\mathrm{graph}(\mathrm{d}(\langle q, - \rangle))$. If $\Sigma = \partial C$ is the (smooth) boundary of a convex body C then $L_\Sigma = \mathrm{graph}(\mathrm{d}h_C)$. Therefore, we have the following reformulation of the concurrent normals conjecture. Consider the family of Lagrangian submanifolds $L_q = \mathrm{graph}(\mathrm{d}\langle q, - \rangle) \subset T^*S^{n-1}$, with $q \in C$. Since L_q and $L_{\partial C}$ are Hamiltonian deformations of the zero-section in T^*S^{n-1}, the geometric intersection number $i(L_q, L_{\partial C})$ is equal to 2. However, we expect to find surplusection in this family.

Conjecture 4.4 (reformulation). *Let $C \subset \mathbb{R}^n$ be a convex body with smooth boundary. There is a point $q_* \in C$ with $\#(L_{q_*} \cap L_{\partial C}) = 2n$.*

Remark 4.5. It is not possible to bound the mean surplusection from below, because the locus of surplusection can be arbitrarily small (e.g., for a circle there is a single q which lies on *all* normals and every other point lies on precisely two). Note that the conjecture still seems to be difficult even without the restriction $q \in C$; see for example the discussion in [26, Section 5, Lemma 10].

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[7]See, for example, Arnold's survey on ray systems [5], in particular, Section 3.A, Example 4. To see why it is Lagrangian, if we think of T^*S^{n-1} as a symplectic reduction of $T^*\mathbb{R}^n$ via geodesic flow, L_Σ is the reduction of the (Lagrangian) conormal bundle of Σ.

The concurrent normals conjecture is therefore another motivation to study surplusection phenomena. Given that we know it is true for $n = 2, 3, 4$, we feel it is not unreasonable to ask a "multi-valued" analogue[8] of the concurrent normals conjecture; note that the condition $q \in C$ no longer makes sense in this generality.

Question 4.6. Does the surplusection associated with the concurrent normals conjecture continue to hold for Lagrangian spheres which are not graphical? More precisely, if $L \subset T^*S^{n-1}$ is a Lagrangian sphere which is Hamiltonian isotopic to the zero-section, does there exist a $q \in \mathbb{R}^n$ such that $\#(L \cap L_q) \geq 2n$?

Remark 4.7. The answer is clearly yes when $n = 2$: for graphical Lagrangians graph$(\mathrm{d}f)$, it follows from the concurrent normals conjecture to a convex body whose support function is f; for a non-graphical Lagrangian L, there is a cotangent fibre F which intersects L non-transversely but with total multiplicity 1. Now by tilting F slightly and approximating it by a suitable Lagrangian L_q as in Figure 5, we obtain four intersections.

For those wishing to learn about some failed attempts to tackle the concurrent normals conjecture (to help you avoid the same pitfalls), see the blog of the second author.[9]
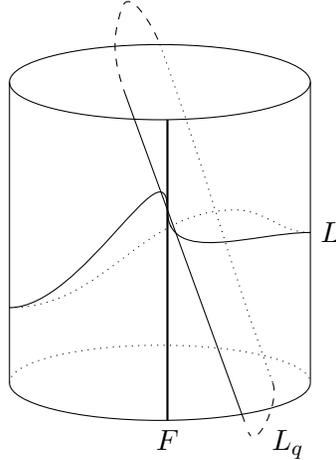
Figure 5. Question 4.6 has a positive answer for non-graphical Lagrangians when $n = 2$. Here, L is the non-graphical Lagrangian, F is a cotangent fibre meeting it non-transversely, and L_q is a circle from our family which approximates a tilt of F. The parts of L_q which go very far out of the picture are drawn dashed and anything at the back of the cylinder is drawn dotted.

5   Reformulation in T^*G

In this final section, we will show that a large class of surplusection problems can be recast in terms of a different surplusection problem in a cotangent bundle.

Definitions 5.1. Suppose that X admits a Hamiltonian action of a group G with equivariant moment map $\mu \colon X \to \mathfrak{g}^*$. Then there is a Lagrangian *moment correspondence* [29, p. 21]:

$$C_\mu \subset (T^*G)^- \times X^- \times X, \qquad \{(g, \mu(gx), x, gx) \mid g \in G, x \in X\}.$$

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[8]Following Arnold [7], we think of non-graphical Lagrangians in cotangent bundles as differentials of multi-valued functions.

[9]https://jde27.uk/blog/concurrent_normals.html.

Here, we identify T^*G with $G \times \mathfrak{g}^*$ by choosing a basis of left-invariant 1-forms and write a superscript minus sign to indicate that the sign of the symplectic form has been reversed. Composition with this correspondence gives a way of turning Lagrangian submanifolds in $X^- \times X$ into Lagrangian submanifolds in T^*G. For example, a product Lagrangian $K \times L$ turns into

$$C_\mu \circ (K \times L) = \left\{ (g, \mu(gx)) \mid g \in G, \, x \in K \cap g^{-1}L \right\}.$$

Suppose that we are interested in studying surplusection between K and the family $\{gL \mid g \in G\}$. Suppose moreover that μ is an embedding, for example, if X is a coadjoint orbit of G. For example, if $X = \mathbb{CP}^n$, $G = \mathrm{PU}(n+1)$, $L = \mathbb{RP}^n$, then this is precisely the situation we studied before.

Lemma 5.2. *The composite Lagrangian $M := C_\mu \circ (K \times L) \subset T^*G$ is embedded and has the following property. The intersection $M \cap T^*_g G$ is in bijection with $K \cap g^{-1}L$.*

Proof. Note that C_μ is the graph of the fibred coisotropic submanifold $G \times X \subset G \times \mathfrak{g}^* = T^*G$. A fibred coisotropic admits a symplectic reduction whose fibres are the leaves of the characteristic foliation; in this case, the symplectic reduction is $X^- \times X$, so M is simply the preimage of $K \times L$ under the projection $G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$. This shows that M is embedded.

To see the intersection, note that since μ is an embedding, $M \cap T^*_g G$ is in bijection with

$$I = \{(g, \mu(gx), x, gx) \mid x \in K, \, gx \in L\} \subset T^*_g G \times X \times X.$$

This set I is, in turn, in bijection with $\{x \in X \mid x \in K, \, gx \in L\} = K \cap g^{-1}L$. ∎

Remark 5.3. Let $\pi \colon T^*G \to G$ be the projection. The surplusection locus for the original problem is then the locus in G where $\pi|_M$ is more than $i(K, L)$-to-one, i.e., where M is "more folded than it needs to be" with respect to the fibration by cotangent fibres. If we think about what could be responsible for this folding, note that M is vertically "cramped": it is forced to live in the compact set $G \times X \subset T^*G$.

Remark 5.4. One can use this correspondence even if μ fails to be an embedding, but then the conclusions of Lemma 5.2 fail. However, it could still be useful: for example, if $i(K, L) = 0$, then the projection of M to the zero-section G still coincides with the surplusection locus.

Acknowledgements

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