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Enriched Koszul duality

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Abstract

We investigate the monoidal and enriched category properties of Koszul duality between the category of non-counital conilpotent dg-coalgebras and the category of non-unital dg-algebras. We find that the category of non-counital conilpotent dg-coalgebras has a non-unital monoidal structure compatible with its standard model structure. We then show that the category of non-unital dg-algebras carries a non-unital module category structure, over the category of non-counital conilpotent dg-coalgebras, compatible with its standard model structure. Furthermore we show that the Quillen equivalence between these two model categories extends to a non-unital module category Quillen equivalence. We also show the analogous results in the case of Koszul duality between the category of non-counital cocommutative conilpotent dg-coalgebras and the category of dg-Lie algebras. Thus we establish what we call an enriched form of Koszul duality.

We then proceed to show that the homotopy category of non-counital conilpotent dg-coalgebras and the category of non-unital dg-algebras inherit a semi-module structure over the homotopy category of reduced simplicial sets with the Quillen model structure. We also consider how our results can be used to possibly compute simplicial mapping spaces of non-unital dg-algebras and dg-Lie algebras and reach some partial results in this direction.

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Declaration

This thesis is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in section 1.1.

This thesis has not been submitted in substantially the same form for the award of any other degree or qualification.

This thesis does not exceed the permitted maximum of 80,000 words.

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CHAPTER 1

Introduction

Koszul duality appears in several areas of algebra, topology and geometry where its origin can be traced back to the inception of rational homotopy theory as developed by Quillen in [Qui69]. We will mainly be working in the context of *associative dg Koszul duality* which, in its modern formulation, can be expressed as Quillen equivalence between the category of non-unital differential graded algebras, DGA_0, and the category of conilpotent non-counital differential graded coalgebras, $\mathrm{coDGA}^{\mathrm{conil}}$, over some field k. This case was initially shown in [Lef03] and further developed in [Pos11]. A similar result is what we will refer to as *com-Lie dg Koszul duality*, previously obtained in [Hin01], this can be expressed as a Quillen equivalence between the category of differential graded Lie algebras, DGLA, and the category of conilpotent non-counital cocommutative differential graded coalgebras, $\mathrm{coCDGA}^{\mathrm{conil}}$, over some field k of characteristic zero. More generally, Koszul duality refers to similar correspondences at the level of e.g. (co)modules and (co)operads, however for our purposes *Koszul duality* will refer to the two Quillen equivalences detailed above. For a general survey of these results as well as further reading on Koszul duality we refer the reader to [Pos23].

On a somewhat different track, it has been shown in [AJ13] that the category DGA_0 is enriched, tensored, and cotensored over the closed symmetric monoidal category of non-counital differential graded coalgebras $(\mathrm{coDGA}_0, \otimes)$ equipped with the ordinary tensor product. This was further extended to the operadic setting in [LG19]. Our main aim is to provide a strengthening of Koszul duality that respects this enrichment of algebras over coalgebras. However, in the case of associative Koszul duality, we are dealing with the category of conilpotent coalgebras, $\mathrm{coDGA}^{\mathrm{conil}}$, which does not have a monoidal unit under the ordinary tensor product making it into a semi-monoidal category.

Motivated by the lack of a unit, we introduce the notion of semi-module categories over a semi-monoidal category. Taking this further in the homotopical direction, we introduce semi-monoidal model categories and semi-module model categories analogous to their unital counterparts. Remembering that a closed module category is precisely the same thing as a tensored and cotensored enriched category, over the same monoidal category, this also allows us to speak about what we will refer to as a semi-enrichment over a semi-monoidal category. While a priori a weaker concept than enriched category theory, it nevertheless puts structural limitations on the categories in question. Furthermore, while our main interest as well as initial motivation is that of Koszul duality, one quickly finds that semi-monoidal model categories that are not monoidal do appear naturally in homotopical algebra.

Having established the framework of semi-monoidal and semi-module categories as well as their model categorical analogues, we proceed to show that DGA_0 can be given a closed semi-module category structure over the semi-monoidal category $(\mathrm{coDGA}^{\mathrm{conil}}, \otimes)$ using the same procedure as in [AJ13]. Furthermore, we show that their semi-module structures are compatible with their respective model category structures as well as with the Quillen equivalence that is associative Koszul duality. Proceeding similarly in the case of com-Lie Koszul duality, one obtains the corresponding result in that setting. Specifically, our main result in the associative setting is

Theorem 1.0.1. *The category* $(\mathrm{coDGA}^{\mathrm{conil}}, \otimes, \underline{\mathrm{coDGA}}^{\mathrm{conil}})$ *is a semi-monoidal model category and* $(\mathrm{DGA}_0, \rhd, \overline{\mathrm{DGA}}_0, \{-,-\})$ *is a semi-module model category over* $\mathrm{coDGA}^{\mathrm{conil}}$*. Furthermore, the Quillen equivalence*

$$\mathrm{coDGA}^{\mathrm{conil}} \underset{B}{\overset{\Omega}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{DGA}_0$$

respects the $\mathrm{coDGA}^{\mathrm{conil}}$*-module structures making it a* $\mathrm{coDGA}^{\mathrm{conil}}$*-module Quillen equivalence.*

Here we used the notation $\underline{\mathrm{coDGA}}^{\mathrm{conil}}$ for the internal hom of $\mathrm{coDGA}^{\mathrm{conil}}$, while $\rhd$, $\overline{\mathrm{DGA}}_0$, and $\{-,-\}$ correspond to the tensoring, enrichment, and cotensoring functors of the closed semi-module structure of DGA_0 respectively. In particular the cotensoring $\{-,-\}$ is the convolution algebra functor.

The corresponding result in the com-Lie context is the following.

Theorem 1.0.2. *The category* $(\mathrm{coCDGA}^{\mathrm{conil}}, \otimes, \underline{\mathrm{coCDGA}}^{\mathrm{conil}})$ *is a semi-monoidal model category and* $(\mathrm{DGLA}, \rhd, \overline{\mathrm{DGLA}}, \{-,-\})$ *is a semi-module model category* *over* coCDGA$^{\text{conil}}$. *Furthermore the Quillen equivalence*

$$\text{coCDGA}^{\text{conil}} \underset{B}{\overset{\Omega}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \text{DGLA}$$

respects the coCDGA$^{\text{conil}}$*-module structures making it a* coCDGA$^{\text{conil}}$*-module Quillen equivalence.*

We have here used the notation $\underline{\text{coCDGA}}^{\text{conil}}$ for the internal hom functor of coCDGA$^{\text{conil}}$, while $\triangleright$, $\overline{\text{DGLA}}$, and $\{-,-\}$ correspond to the tensoring, enrichment, and cotensoring functors of the closed semi-module structure of DGLA respectively. In particular the cotensoring $\{-,-\}$ is the convolution algebra functor.

We note that similar results in the context of dg-categories have been obtained in [HL22], where they provide a homotopical enrichment of dg-categories over pointed coalgebras. Note that the category of pointed algebras is monoidal, as opposed to just semi-monoidal, and as such they provide the category of dg-categories with an enriched category structure.

Having established the above semi-module version of Koszul duality we set out with the initial goal of using this result to compute simplicial mapping spaces. To begin with we study the monoidal properties of the (co)bar construction. To do this we switch viewpoint and work mainly in the equivalent (co)augmented context. That is, we work with the category of augmented dg-algebras DGA_{aug} and view the category of conilpotent dg-coalgebras as being coaugmented which we denote as $\text{coDGA}^{\text{conil}}_{\text{coaug}}$. In this context the semi-monoidal structure considered earlier corresponds to the smash product $\wedge$. However, in addition to the smash product, we also consider the monoidal products given by the tensor product $\otimes$ of $\text{coDGA}^{\text{conil}}_{\text{coaug}}$ and DGA_{aug}. It turns out that the (co)bar construction is *not* a quasi-strong semi-monoidal functor with respect to the smash product $\wedge$. However it is quasi-strong with respect to the ordinary tensor product $\otimes$, which was shown in [HL22].

We next make a connection to the category of reduced simplicial sets qCat_0 given the Joyal model structure and to reduced simplicial sets sSet_0 given the Quillen model structure. To do this we show that the normalised chain coalgebra functor $C^N : \text{qCat}_0 \to \text{coDGA}^{\text{conil}}_{\text{coaug}}$ is quasi-strong semi-monoidal with respect to the smash product $\wedge$. As such the homotopy category of augmented dg-algebras $\text{Ho}(\text{DGA}_{\text{aug}})$ can be shown to carry an induced $\text{Ho}(\text{sSet}_0)$-module structure. Finally we consider if this structure can be used to compute simplicial mapping spaces of augmented dg-algebras. We find that not having a monoidal unit in our case impedes this and we are only able to achieve partial results.

1.1. Thesis outline and published work

Chapter 1 provides an introduction to the thesis and was partially written as an introduction for [Eur24].

In Chapter 2 we introduce some of the background material needed for the thesis. This includes introducing differential graded (dg) algebras, dg-coalgebras and the theory of model categories as well as brief coverage of monoidal and enriched categories. The chapter also introduces some of the terminology and notation we will use throughout the thesis. For the latter the reader may also choose to consult the summary of notation at the end of the thesis. Parts of Section 2.2 and Section 2.3 were originally written as part of the introductory material in [Eur24]. Parts of Section 2.4 were initially written as accompanying notes to a learning seminar given internally by the author.

Chapter 3 corresponds to [Eur24] and forms the main substance of the thesis. We here introduce the notion of semi-monoidal model categories and semi-module model categories. We then show that the category of conilpotent dg-coalgebras $\mathrm{coDGA}^{\mathrm{conil}}$ is a semi-monoidal model category and the category of non-unital dg-algebras DGA_0 is a semi-module model category over $\mathrm{coDGA}^{\mathrm{conil}}$. We then show the analogous results for the categories of cocommutative conilpotent non-counital dg-coalgebras $\mathrm{coCDGA}^{\mathrm{conil}}$ and the category of dg-Lie algebras DGLA. Finally we show that the obtained semi-module structure is compatible with Koszul duality and in both the associative and the com-Lie case is what we call semi-module Quillen equivalences.

In Chapter 4 we use the results obtained in Chapter 3 to give the homotopy categories of augmented dg-algebras $\mathrm{Ho}(\mathrm{DGA}_{\mathrm{aug}})$ closed $\mathrm{Ho}(\mathrm{sSet}_0)$-module category structures. To do this we begin by investigating the monoidal properties of the chain coalgebra functor C^N, showing it is quasi-strong semi-monoidal with respect to the smash product. We also investigate if the found semi-module structure can be used to calculate simplicial mapping spaces of augmented dg-algebras and find that the lack of a unit impedes this in general while still giving us some partial results.

CHAPTER 2

Preliminaries

2.1. Differential graded vector spaces and pointed objects

Throughout we will be working over some fixed field k. We denote by DGVec the category of differential graded (dg) vector spaces over k. We will throughout use the convention of homological grading, i.e. that the differential d is of degree -1.

The tensor product of two dg-vector spaces V and W is defined as

$$(V \otimes W)_n := \bigoplus_{i+j=n} V_i \otimes W_j,$$

with differential

$$d_{V\otimes W} = d_V \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes d_W.$$

We will here and throughout the thesis apply the Koszul sign rule, which states that when switching the order of symbols x and y of degrees $|x|$ and $|y|$ respectively, in a monoidal expression, we acquire a sign $(-1)^{|x||y|}$. In particular this means that the differential of the tensor product when acting on a pure tensor $v \otimes w$ is given by $d_{V\otimes W}(v \otimes w) := d_V v \otimes w + (-1)^{|v|} v \otimes d_W w$.

The category DGVec is closed symmetric monoidal under the tensor product. The braiding map $B_{V,W}$ is given by $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ and the internal hom functor is defined as

$$\underline{\mathrm{DGVec}}(V, W)_n := \prod_{i\in\mathbb{Z}} \mathrm{Vec}(V_i, W_{i+n}),$$

with differential

$$d_{\underline{\mathrm{DGVec}}(V,W)} := \mathrm{DGVec}(\mathrm{Id}_V, d_W) - \mathrm{DGVec}(d_V, \mathrm{Id}_W).$$

Explicitly by the Koszul sign this means that the differential acts as $d_{\underline{\mathrm{DGVec}}(V,W)} f = d_W \circ f - (-1)^{|f|} f \circ d_V$. As required the internal hom functor satisfies the hom tensor adjunction

$$\mathrm{DGVec} \underset{\underline{\mathrm{DGVec}}(V,-)}{\overset{V\otimes -}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{DGVec}\,.$$

Definition 2.1.1. A *pointed* dg-vector space is a dg-vector space V together with dg-linear maps $u : k \to V$ and $\epsilon : V \to k$, known as the *unit* and *counit* maps respectively, such that $\epsilon \circ u = \mathrm{Id}_k$.

Remark 2.1.2. This definition is chosen to agree with [AJ13] and has the benefit that when we in later sections introduce *augmented* dg-algebras and *coaugmented* dg-coalgebras their underlying dg-vector spaces will naturally be pointed. Note however that our definition differs from what would be a *pointed* object in DGVec, instead our definition would correspond to the notion of what could be called pointed copointed dg-vector spaces.

Definition 2.1.3. A morphism of pointed dg-vector spaces $f : V \to W$ is a morphism of dg-vector spaces commuting with the unit and counit maps, i.e. such that $u_W = f \circ u_V$ and $\epsilon_W \circ f = \epsilon_V$.

We denote the category of pointed dg-vector spaces by DGVec_* and note that a pointed vector space (V, u, ϵ) decomposes as $\overline{V} \oplus k$ where $\overline{V}$ is a dg-vector space given by $\overline{V} := \ker \epsilon$. As a consequence we have an equivalence of categories

$$\mathrm{DGVec} \cong \mathrm{DGVec}_* \,.$$

The coproduct in the category of pointed dg-vector spaces of V and W is the wedge product $V \vee W$ defined as the pushout

$$\begin{array}{ccc} k \oplus k & \xrightarrow{(u_V, u_W)} & V \oplus W \\ \downarrow & & \downarrow \\ k & \longrightarrow & V \vee W \end{array}$$

which we see may be computed as $V \vee W \cong \overline{V} \oplus \overline{W} \oplus k$. The closed symmetric monoidal structure of pointed vector spaces is the smash product $V \wedge W$ defined as the pushout

$$\begin{array}{ccc} V \oplus W & \xrightarrow{(v,w) \mapsto v \otimes w} & V \otimes W \\ {\scriptstyle \epsilon_V + \epsilon_W}\downarrow & & \downarrow \\ k & \longrightarrow & V \wedge W \end{array}$$

which may be computed as $V \wedge W \cong (\overline{V} \otimes \overline{W}) \oplus k$.

Correspondingly, the pointed hom functor is defined as the pullback

$$\begin{array}{ccc}
\underline{\mathrm{DGVec}}(V,k) \oplus \underline{\mathrm{DGVec}}(k,W) & \xleftarrow{(\epsilon_V, u_W) \leftarrow\!\shortmid 1} & k \\
\Big\uparrow {\scriptstyle \underline{\mathrm{DGVec}}(V,\epsilon_W) \oplus \underline{\mathrm{DGVec}}(\epsilon_V, W)} & & \Big\uparrow \\
\underline{\mathrm{DGVec}}(V,W) & \longleftarrow & \underline{\mathrm{DGVec}}_*(V,W)
\end{array}$$

which may be computed as $\underline{\mathrm{DGVec}}_*(V,W) \cong \underline{\mathrm{DGVec}}(\overline{V}, \overline{W}) \oplus k$.

The main importance of these constructions for us is that we will later need the analogous construction for augmented dg-algebras and coaugmented dg-coalgebras. The reader may also wish to consult [AJ13, Section 1.1.1] for a more detailed coverage.

2.2. Differential graded algebras and Lie algebras

Throughout this section we will be working over some field k. As before we denote by DGVec the category of differential graded vector spaces over k.

Definition 2.2.1. Let k be a field. A *non-unital dg-algebra* (A, m) consists of a dg-vector space A together with a dg-linear morphism

$$m : A \otimes A \to A,$$

known as the *multiplication* map, satisfying associativity i.e. that the diagram

$$\begin{array}{ccc} A & \xleftarrow{m} & A \otimes A \\ {\scriptstyle m}\uparrow & & \uparrow{\scriptstyle \mathrm{Id} \otimes m} \\ A \otimes A & \xleftarrow[m \otimes \mathrm{Id}]{} & A \otimes A \otimes A \end{array}$$

commutes.

Definition 2.2.2. A non-unital dg-algebra (A, m) is *commutative* if the multiplication map satisfies that $m = m \circ B_{A,A}$ where $B_{A,A} : a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$ is the braiding morphism of dg-vector spaces.

Definition 2.2.3. A *unital dg-algebra* (A, m, u) consists of a non-unital dg-algebra (A, m) together with a dg-linear morphism

$$u : k \to A,$$

known as the *unit* map, satisfying the left and right unit laws, i.e. that the diagram

$$\begin{array}{ccccc} A & & \xleftarrow{m} & & A \otimes A \\ & & & & \uparrow{\scriptstyle \mathrm{Id} \otimes u} \\ {\scriptstyle m}\uparrow & & \nwarrow{\scriptstyle \mathrm{Id}} & & A \otimes k \\ & & & & {\scriptstyle \cong}\uparrow{\scriptstyle \mathrm{Id} \otimes 1} \\ A \otimes A & \xleftarrow[u \otimes \mathrm{Id}]{} & k \otimes A & \xleftarrow[1 \otimes \mathrm{Id}]{\cong} & A \end{array}$$

commutes.

The requirement that the multiplication map m is dg-linear means it is a derivation, i.e. satisfies the Leibniz rule

$$d \circ m = m \circ (d \otimes \mathrm{Id} + \mathrm{Id} \otimes d).$$

Meanwhile the requirement that the unit u is dg-linear gives that $du = 0$.

Definition 2.2.4. An *augmented dg-algebra* (A, m, u, ϵ) is a unital dg-algebra (A, m, u) together with a dg-linear morphism $\epsilon : A \to k$ such that $\epsilon \circ u = \mathrm{Id}_k$.

Definition 2.2.5. Let (A, m) and (A', m') be non-unital dg-algebras. Then a non-unital dg-algebra morphism from A to A' is a dg-linear morphism $f : A \to A'$ such that

$$m' \circ (f \otimes f) = f \circ m.$$

Definition 2.2.6. Let (A, m, u) and (A', m', u') be unital dg-algebras. Then a dg-algebra morphism from A to A' is a non-unital dg-algebra morphism $f : A \to A'$ such that $f \circ u = u' \circ f$.

Definition 2.2.7. Let (A, m, u, ϵ) and (A', m', u', ϵ') be augmented dg-algebras. Then an augmented dg-algebra morphism from A to A' is a unital dg-algebra morphism $f : A \to A'$ such that $\epsilon' \circ f = \epsilon$.

For a fixed field k we will denote the category of unital dg-algebras by DGA, the category of non-unital dg-algebras by DGA_0, and the category of augmented dg-algebras by $\mathrm{DGA}_{\mathrm{aug}}$. We denote their commutative counterparts by cDGA, cDGA_0, and $\mathrm{cDGA}_{\mathrm{aug}}$ respectively.

We see that every augmented dg-algebra decomposes as

$$A \cong \bar{A} \oplus k,$$

where $\bar{A}$ is the non-unital dg-algebra given by the augmentation ideal $\ker \epsilon$. Hence we get the following.

PROPOSITION 2.2.8. *There exists an equivalence of categories*

$$\mathrm{DGA}_0 \cong \mathrm{DGA}_{\mathrm{aug}},$$

where the equivalence is given by mapping an augmented dg-algebra A to its augmentation ideal $\bar{A}$ and conversely a non-unital dg-algebra $\bar{A}$ to $\bar{A} \oplus k$, i.e. adding the unit k.

In addition to associative dg-algebras as defined above we will also be working with dg-Lie algebras. Again we fix some field k. However we caution the reader that later, in e.g. Chapter 3, when we consider the model structure on the category of dg-Lie algebras we will require k to be of characteristic zero.

Definition 2.2.9. A dg-Lie algebra $(\mathfrak{g}, [-,-])$ consists of a dg-vector space $\mathfrak{g}$ together with a dg-linear morphism

$$[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g},$$

known as the *Lie bracket*, satisfying the following:

i) The graded Jacobi identity

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0,$$

ii) Skew symmetry

$$[x,y] + (-1)^{|x||y|}[y,x] = 0,$$

for all $x, y, z \in \mathfrak{g}$.

For a fixed field k we will denote the category of dg-Lie algebras by DGLA.

The categories of DGLA and DGA are related by the adjunction

$$\mathrm{DGLA} \underset{\mathrm{Lie}}{\overset{\mathcal{U}}{\rightleftarrows}} \mathrm{DGA}$$

where $\mathcal{U}$: DGLA $\to$ DGA is the universal enveloping algebra functor and Lie : DGA $\to$ DGLA is the functor that assigns to an associative dg-algebra A the graded commutator as its Lie bracket. The existence of this adjunction goes back to the Poincaré–Birkhoff–Witt theorem. We refer the reader to [Bou06] for a proof.

2.2.1. Free algebra constructions. We will throughout the thesis make use of various free constructions for the categories of algebras we have defined.

We define the free non-unital algebra functor T_0 as the left adjoint to the forgetful functor to the category of differential graded vector spaces DGVec, i.e.

$$\mathrm{DGVec} \underset{U}{\overset{T_0}{\rightleftarrows}} \mathrm{DGA}_0 \,.$$

Explicitly, the free non-unital algebra of a dg-vector space V can be constructed as the non-unital tensor algebra $T_0V := \bigoplus_{n=1}^{\infty} V^{\otimes n}$.

Similarly, in the case of commutative non-unital dg-algebras, we have a free forgetful adjunction

$$\mathrm{DGVec} \underset{U}{\overset{S_0}{\rightleftarrows}} \mathrm{CDGA}_0,$$

where S_0 is the *symmetric tensor algebra* functor, consisting of graded symmetric tensors. In particular S_0 is the abelianisation of T_0, i.e. the coequaliser of

$$T_0 V \otimes T_0 V \underset{m \circ B}{\overset{m}{\rightrightarrows}} T_0 V$$

in the category of non-unital dg-algebras, and where B is the braiding morphism of dg-vector spaces and m is concatenation of tensors.

Similarly, when working in the unital or augmented case we will use the notation T and T_{aug} for the corresponding free functors. We will refer to dg-algebras of the form T_0V, TV, $T_{\text{aug}}V$ as non-unital free, free, and augmented free dg-algebras respectively. Note that that the only difference between the functors T and T_{aug} is if we regard the target as being augmented or not. Analogously in the commutative case we will use the notation S and S_{aug}. We say that dg-algebras of the form S_0V, SV, and $S_{\text{aug}}V$ are non-unital symmetric free, symmetric free, and augmented symmetric free dg-algebras respectively. Note that that the only difference between the functors S and S_{aug} is if we regard the target as being augmented or not.

There is also a free forgetful adjunction between dg-algebras and dg-Lie algebras,

$$\mathrm{DGVec} \underset{U}{\overset{T_{\mathrm{Lie}}}{\rightleftarrows}} \mathrm{DGLA},$$

where T_{Lie} denotes the free dg-Lie algebra functor. We will refer to dg-Lie algebras of the form $T_{\text{Lie}}V$ as free dg-Lie algebras. We refer the reader to [Bou06] for a proof.

2.3. Differential graded coalgebras

Definition 2.3.1. Let k be a field. A *non-counital dg-coalgebra* (C, Δ) consists of a dg-vector space C together with a dg-linear morphism

$$\Delta : C \to C \otimes C,$$

known as the *comultiplication* map satisfying, coassociativity i.e. that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{Id} \otimes \Delta} \\ C \otimes C & \xrightarrow[\Delta \otimes \mathrm{Id}]{} & C \otimes C \otimes C \end{array}$$

commutes.

Definition 2.3.2. A non-counital coalgebra (C, Δ) is said to be *cocommutative* if it satisfies that $\Delta = B_{C,C} \circ \Delta$ where $B_{C,C} : x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$ is the braiding morphism of dg-vector spaces.

Definition 2.3.3. A *counital dg-coalgebra* (C, Δ, ϵ) consists of a non-counital dg-coalgebra (C, Δ) together with a dg-linear morphism,

$$\epsilon : C \to k,$$

known as the *counit* morphism, satisfying the left and right counit laws, i.e. such that

$$\begin{array}{ccccc} C & & \xrightarrow{\Delta} & & C \otimes C \\ & \searrow^{\mathrm{Id}} & & & \downarrow{\scriptstyle \mathrm{Id} \otimes \epsilon} \\ {\scriptstyle\Delta}\downarrow & & \searrow & & C \otimes k \\ & & & \searrow & {\scriptstyle\cong}\downarrow{\scriptstyle \text{scalar mult.}} \\ C \otimes C & \xrightarrow[\epsilon \otimes \mathrm{Id}]{} & k \otimes C & \xrightarrow[\text{scalar mult.}]{\cong} & C \end{array}$$

commutes.

For a fixed field k we will denote by coDGA the category of counital dg-coalgebras, and by coDGA_0 the category of non-counital dg-coalgebras.

The requirement that the comultiplication Δ is dg-linear means that it is a *coderivation*, i.e. satisfies the co-Leibniz rule

$$\Delta \circ d = (\mathrm{Id} \otimes d + d \otimes \mathrm{Id}) \circ \Delta.$$

Meanwhile the requirement that the counit ϵ is dg-linear means that $\epsilon \circ d = 0$.

Definition 2.3.4. A *coaugmented* dg-coalgebra (C, Δ, ϵ, u) is a counital dg-coalgebra (C, Δ, ϵ) together with a dg-linear morphism $u : k \to C$ such that $\epsilon \circ u = \mathrm{Id}_k$.

Definition 2.3.5. Let (C, Δ) and (C', Δ') be non-counital dg-coalgebras. A non-counital dg-coalgebra morphism from C to C' is a dg-linear map $f : C \to C'$ satisfying

$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f.$$

Definition 2.3.6. Let (C, Δ, ϵ) and (C', Δ', ϵ') be counital dg-coalgebras. A dg-coalgebra morphism from C to C' is a non-counital dg-coalgebra morphism $f : C \to C'$ such that $\epsilon' \circ f = \epsilon$.

Definition 2.3.7. Let (C, Δ, ϵ, u) and $(C', \Delta', \epsilon', u')$ be coaugmented dg-coalgebras. A coaugmented dg-coalgebra morphism from C to C' is a dg-coalgebra morphism $f : C \to C'$ such that $u' \circ f = f \circ u$.

Definition 2.3.8. Let (C, Δ, ϵ) be a dg-coalgebra. A *coideal* I is then a dg-linear subspace $I \subset C$ such that $\Delta(I) \subset (I \otimes C) \oplus (C \otimes I)$ and $\epsilon(I) = 0$.

We denote the category of coaugmented dg-coalgebras by $\mathrm{coDGA}_{\mathrm{coaug}}$. We see that every coaugmented dg-coalgebra decomposes as

$$C \cong \bar{C} \oplus k,$$

where $\bar{C}$ is the non-counital dg-coalgebra given by the counit coideal $\ker \epsilon$. Hence we get the following.

Proposition 2.3.9. *There is an equivalence of categories*

$$\mathrm{coDGA}_0 \cong \mathrm{coDGA}_{\mathrm{aug}}\,.$$

A main feature of dg-coalgebras is that they satisfy the fundamental theorem of coalgebras.

Theorem 2.3.10. *Every (coaugmented) (non-)counital dg-coalgebra (C, Δ) is a colimit of its finite dimensional dg-subcoalgebras.*

Proof. See [Swe69] for the non dg case and e.g. [Gua+19, Theorem 5.3] for the extension to the dg case. □

Remark 2.3.11. Note that a similar theorem is false for algebras. For instance the only finite dimensional subalgebra of the polynomial algebra $k[x]$ is 0.

2.3.1. Conilpotent coalgebras. We will now introduce the concept of *conilpotent coalgebras*. We choose to work in the non-counital context in this section but everything translates straightforwardly to the coaugmented case. We will also from now take coalgebra to implicitly mean dg-coalgebra.

For convenience we will use the notation Δ^n to mean composition of the comultiplication n-times in the sense of

$$\Delta^n := (\Delta \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} + \cdots + \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \Delta) \circ \cdots \circ (\Delta \otimes \mathrm{Id} + \mathrm{Id} \otimes \Delta) \circ \Delta.$$

Definition 2.3.12. Let (C, Δ) be a non-counital coalgebra. We say that an element $x \in C$ is conilpotent if there exists some n such that $\Delta^n(x) = 0$. If all elements of C are conilpotent we say that C is conilpotent.

We will denote the category of conilpotent dg-coalgebras by $\mathrm{coDGA}^{\mathrm{conil}}$.

REMARK 2.3.13. In the alternative context of coaugmented coalgebras we say that a coaugmented coalgebra C is conilpotent if its corresponding non-counital coalgebra $\overline{C}$ is conilpotent. In the few instances when the distinction between the non-counital and the coaugmented context is of importance we will denote the category of conilpotent coaugmented dg-coalgebras by $\mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}}$.

Definition 2.3.14. Let (C, Δ, d) be a non-counital coalgebra. We say that an element $c \in C$ is an atom if $\Delta(c) = c \otimes c$ and $dc = 0$.

The set of atoms of a non-counital coalgebra C is in one-to-one correspondence with coalgebra morphisms $k \to C$ from the monoidal unit. We note that a conilpotent coalgebra C has exactly one atom $0 \in C$. In particular, if C is conilpotent, the zero morphism is the only morphism $k \to C$.

Example 2.3.15. An example of a non-counital dg-coalgebra that is *not* conilpotent is the coalgebra k, concentrated in degree 0, and given the comultiplication induced by $1 \mapsto 1 \otimes 1$. As both 0 and 1 are atoms of k it follows that k is not conilpotent.

A standard example of a conilpotent non-counital dg-coalgebra is the *cofree conilpotent coalgebra* or *tensor coalgebra* $T^{\mathrm{co}}V$ of a dg-vector space V given as a dg-vector space by

$$T^{\mathrm{co}}V = \bigoplus_{n=1}^{\infty} V^{\otimes n},$$

with the *cut comultiplication.* The *cut comultiplication* is defined by acting on pure tensor, i.e. a tensor of the form $v_0 \otimes \cdots \otimes v_n$ as

$$\Delta(v_1 \otimes \cdots \otimes v_n) := \sum_{i=0}^{n-1} (v_0 \otimes v_i) \otimes (v_{i+1} \otimes v_n),$$

We will later see that the tensor coalgebra is a cofree object in $\mathrm{coDGA}^{\mathrm{conil}}$.

PROPOSITION 2.3.16. *Let (C, Δ_C) be a conilpotent coalgebra and (D, Δ_D) an arbitrary non-counital coalgebra. Then $(C \otimes D, \Delta_{C\otimes D})$ is also conilpotent.*

PROOF. Let $c \otimes d$ be a pure tensor in $C \otimes D$. By assumption there exists some $n > 0$ such that $\Delta_C^n(c) = 0$. Then we have

$$\Delta^n_{C\otimes D}(c \otimes d) := (\Delta_C \otimes \Delta_D)^n(c \otimes d) = \Delta^n_C(c) \otimes \Delta^n_D(d) = 0,$$

where we have not done any reordering of the terms as it only affects the sign. □

Definition 2.3.17. Let $(C, d_C, \Delta_C) \in \mathrm{coDGA}^{\mathrm{conil}}$ be a conilpotent dg-coalgebra. An admissible filtration of C is an exhaustive increasing filtration F starting at $F_0 = 0$ compatible with the differential and comultiplication, meaning that

$$d(F_n) \subset F_n, \text{ and}$$
$$\Delta(F_n) \subset \bigoplus_{k=1}^{n-1} F_{n-k} \otimes F_k.$$

Example 2.3.18. For any conilpotent dg-coalgebra (C, Δ) there is a canonical admissible filtration, known as the coradical filtration, given by $F_n = \ker \Delta^n$.

Definition 2.3.19. We say that a morphism $f : C \to D$ in $\mathrm{coDGA}^{\mathrm{conil}}$ is a filtered quasi-isomorphism if there exist admissible filtrations $\mathcal{F}_C$ and $\mathcal{F}_D$ of C and D respectively such that the induced morphism of the associated graded complexes $\mathrm{gr}\, f : \mathrm{gr}(\mathcal{F}_C) \to \mathrm{gr}(\mathcal{F}_D)$ is a quasi-isomorphism in each degree.

2.3.2. Cofree constructions. Several of our proofs rely on the existence of cofree objects in the categories of coDGA_0 and $\mathrm{coDGA}^{\mathrm{conil}}$. The non-counital cofree functor $\check{T}_0$ is defined as the right adjoint to the forgetful functor to the category of differential graded vector spaces, DGVec, i.e.

$$\mathrm{coDGA}_0 \underset{\check{T}_0}{\overset{U}{\rightleftarrows}} \mathrm{DGVec}\,.$$

The existence of this adjunction was shown in [BL85]. We will say that a coalgebra of the form $\check{T}_0 V$ for some $V \in$ DGVec is cofree. Analogously, the conilpotent cofree functor T^{co} is defined to satisfy the adjunction

$$\mathrm{coDGA}^{\mathrm{conil}} \underset{T^{\mathrm{co}}}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{DGVec},$$

and we say that a conilpotent coalgebra of the form $T^{\mathrm{co}}V$ for some $V \in$ DGVec is conilpotent cofree. Recall that conilpotent cofree coalgebras are also known as tensor coalgebras as they are constructed analogously to the tensor algebra but instead given the cut comultiplication.

There is also an adjunction,

$$\mathrm{coDGA}^{\mathrm{conil}} \underset{R^{\mathrm{co}}}{\overset{\iota}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{coDGA}_0,$$

between the inclusion functor $\iota : \mathrm{coDGA}^{\mathrm{conil}} \to \mathrm{coDGA}_0$ and the conilpotent radical functor $R^{\mathrm{co}} : \mathrm{coDGA}_0 \to \mathrm{coDGA}^{\mathrm{conil}}$. The latter is defined by taking a coalgebra C to the subcoalgebra consisting of conilpotent elements of C. The adjunction follows from the fact that the image of a conilpotent element under a coalgebra morphism is also conilpotent. As a consequence of this adjunction that the conilpotent cofree coalgebra functor T^{co} is related to the non-counital cofree coalgebra functor $\check{T}_0$ by

$$T^{\mathrm{co}} \cong R^{\mathrm{co}}\check{T}_0.$$

When working with the category of cocommutative conilpotent non-counital dg-coalgebras $\mathrm{coCDGA}^{\mathrm{conil}}$ we will also make use of the adjunction

$$\mathrm{coCDGA}^{\mathrm{conil}} \underset{S^{\mathrm{co}}}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{DGVec}\,.$$

Here S^{co} denotes the cocommutative conilpotent cofree functor, which takes a dg-vector space V to the subcoalgebra of the tensor coalgebra $T^{\mathrm{co}}V$ consisting of graded-symmetric tensors. In particular, $S^{\mathrm{co}}V$ is the coabelianisation of $T^{\mathrm{co}}V$, i.e. the equaliser of

$$T^{\mathrm{co}}V \underset{B\circ\Delta}{\overset{\Delta}{\rightrightarrows}} T^{\mathrm{co}}V \otimes T^{\mathrm{co}}V,$$

in the category of non-counital dg-coalgebras, and where B is the braiding morphism of dg-vector spaces and Δ the cut comultiplication. Thus, the cocommutative conilpotent cofree coalgebra is the subcoalgebra of the tensor coalgebra consisting of graded-symmetric tensors. We say that a cocommutative conilpotent coalgebra of the form $S^{\mathrm{co}}V$ for $V \in \mathrm{DGVec}$ is cocommutative conilpotent cofree.

2.4. Monoidal and enriched categories

In this section we give a brief review of the definitions of monoidal and enriched categories. For further reading we refer the reader to [Kel74]. We will later in Section 3.2 introduce the non-unital analogues i.e. what we will refer to as *semi-monoidal* categories.

Definition 2.4.1. A monoidal category $(\mathcal{V}, \otimes, \mathbf{I}, a, l, r)$ consists of a *tensor product functor* $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and an *identity object* $\mathbf{I} \in \mathcal{C}$ together with three natural isomorphisms

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z),$$

$$l_X : \mathbf{I} \otimes X \to X,$$

$$r_X : X \otimes \mathbf{I} \to X,$$

known as the *associator*, *left unit*, and *right unit* respectively. Furthermore they are required to satisfy the coherence conditions that the diagrams

$$\begin{array}{ccc} (X \otimes \mathbf{I}) \otimes Y & \xrightarrow{a_{X,\mathbf{I},Y}} & X \otimes (\mathbf{1} \otimes Y) \\ & {\scriptstyle r_X \otimes \mathrm{Id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{Id}_X \otimes l_Y} & \\ & X \otimes Y & \end{array}$$

and

$$\begin{array}{ccccc} & & ((X \otimes Y) \otimes Z) \otimes W & & \\ & {\scriptstyle a_{X,Y,Z} \otimes \mathrm{Id}_W} \swarrow & & \searrow {\scriptstyle a_{X \otimes Y,Z,W}} & \\ (X \otimes (Y \otimes Z)) \otimes W & & & & (X \otimes Y) \otimes (Z \otimes W) \\ {\scriptstyle a_{X,Y \otimes Z,W}} \downarrow & & & & \downarrow {\scriptstyle a_{X,Y,Z \otimes W}} \\ X \otimes ((Y \otimes Z) \otimes W) & & \xrightarrow[\mathrm{Id}_X \otimes a_{Y,Z,W}]{} & & X \otimes (Y \otimes (Z \otimes W)) \end{array}$$

commute for all $X, Y, Z, W \in \mathcal{C}$. In the particular case when the associator and unit morphisms are identity morphisms we say that the monoidal category is *strict*.

Example 2.4.2. Examples of monoidal categories include the following:

i) The category of vector spaces Vec_k over a field k with the tensor product $\otimes$ and monoidal unit k.
ii) The category of abelian groups Ab with the tensor product $\otimes$ and monoidal unit $\mathbb{Z}$.
iii) The category of chain complexes Ch_R over a ring R with the tensor product $\otimes$ and unit R.

iv) The category of sets Set with the Cartesian product $\times$ and unit the set $*$ with one object.

v) The category of topological spaces Top with the Cartesian product $\times$ and unit the topological space $*$ consisting of one point.

Remark 2.4.3. We note that in Example 2.4.2 the categories Set and Top both have their monoidal product to be their categorical product. Such categories are known as *Cartesian monoidal* and the unit is by necessity the terminal object.

The provided examples in Example 2.4.2 are in fact all examples of what is known as a *symmetric monoidal* category where the monoidal product is commutative.

Definition 2.4.4. A symmetric monoidal category is a monoidal category $(\mathcal{C}, \otimes, \mathbf{I})$ together with a natural isomorphism

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying that

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{c_{X,Y}} & Y \otimes X \\ & \searrow^{\mathrm{Id}_{X\otimes Y}} & \downarrow{\scriptstyle c_{Y,X}} \\ & & X \otimes Y \end{array}$$

commutes for all $X, Y \in \mathcal{C}$ and furthermore satisfies the coherence axioms that the diagrams

$$\begin{array}{ccc} I \otimes X & \xrightarrow{\quad c_{I,X} \quad} & X \otimes I \\ & {\scriptstyle l_X}\searrow \quad \swarrow{\scriptstyle r_X} & \\ & X & \end{array}$$

and

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{c_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X \\ {\scriptstyle c_{X,Y}\otimes \mathrm{Id}_Z}\downarrow & & & & \downarrow{\scriptstyle a_{Y,Z,X}} \\ (Y \otimes X) \otimes Z & \xrightarrow[a_{Y,X,Z}]{} & Y \otimes (X \otimes Z) & \xrightarrow[\mathrm{Id}_Y \otimes c_{X,Z}]{} & Y \otimes (Z \otimes X). \end{array}$$

commutes for all $X, Y, Z \in \mathcal{C}$.

Definition 2.4.5. We say that a symmetric monoidal category $(\mathcal{V}, \otimes)$ is closed if there exists a functor

$$\underline{\mathcal{V}}(-,-) : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$$

such that for any $Y \in \mathcal{V}$ the functor $\underline{\mathcal{V}}(Y, -)$ is right adjoint to the tensor product functor $(-) \otimes Y$ i.e. in adjunction notation we have

$$\mathcal{V} \underset{\underline{\mathcal{V}}(Y,-)}{\overset{(-)\otimes Y}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{V}.$$

The functor $\underline{\mathcal{V}}$ is known as the *internal hom functor*.

Definition 2.4.6. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A lax monoidal functor (F, γ, e) from $\mathcal{C}$ to $\mathcal{D}$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural transformation

$$\gamma_{A,B} : FX \otimes FY \to F(X \otimes Y)$$

and a morphism

$$e : \mathbf{I}_{\mathcal{D}} \to F(\mathbf{I}_{\mathcal{C}})$$

satisfying that the diagrams

$$\begin{array}{ccc}
(FX \otimes FY) \otimes FZ & \xrightarrow{a_{FX,FY,FZ}} & FX \otimes (FY \otimes FZ) \\
\Big\downarrow{\scriptstyle \gamma_{X,Y} \otimes \mathrm{Id}} & & \Big\downarrow{\scriptstyle \mathrm{Id} \otimes \gamma_{Y,Z}} \\
F(X \otimes Y) \otimes FZ & & FA \otimes F(Y \otimes Z) \\
\Big\downarrow{\scriptstyle \gamma_{X \otimes Y, Z}} & & \Big\downarrow{\scriptstyle \gamma_{X, Y \otimes Z}} \\
F(X \otimes Y \otimes Z) & \xrightarrow[F\alpha_{X,Y,Z}]{} & F(X \otimes (Y \otimes Z)),
\end{array}$$

$$\begin{array}{ccc}
\mathbf{I} \otimes FX & \xrightarrow{e \otimes \mathrm{Id}} & F\mathbf{I} \otimes FX \\
\Big\downarrow{\scriptstyle l_{FX}} & & \Big\downarrow{\scriptstyle \gamma_{\mathbf{I},X}} \\
FX & \xleftarrow[Fl_X]{} & F(\mathbf{I} \otimes X),
\end{array}$$

and

$$\begin{array}{ccc}
FX \otimes \mathbf{I} & \xrightarrow{\mathrm{Id} \otimes e} & FX \otimes \mathbf{I} \\
\Big\downarrow{\scriptstyle r_{FX}} & & \Big\downarrow{\scriptstyle \gamma_{X,\mathbf{I}}} \\
FX & \xleftarrow[Fr_X]{} & F(X \otimes \mathbf{I})
\end{array}$$

commutes for all $X, Y, Z \in \mathcal{C}$.

Definition 2.4.7. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal categories. A lax symmetric monoidal functor (F, γ, e) between symmetric monoidal categories is a monoidal functor that additionally satisfies the diagram

$$\begin{array}{ccc}
FX \otimes FY & \xrightarrow{c_{FX \otimes FY}} & FY \otimes FX \\
\Big\downarrow{\scriptstyle \gamma_{X,Y}} & & \Big\downarrow{\scriptstyle \gamma_{Y,X}} \\
F(X \otimes Y) & \xrightarrow[Fc_{X,Y}]{} & F(Y \otimes X),
\end{array}$$

for all $X, Y \in \mathcal{C}$.

Definition 2.4.8. Let $(\mathcal{V}, \otimes, \mathbf{I})$ be a monoidal category. A $\mathcal{V}$-enriched category $\mathcal{C}$ consists of the following data:

i) A collection of objects $\mathrm{Ob}(\mathcal{C})$,
ii) For every pair of objects $X, Y \in \mathcal{C}$, an object $\overline{\mathcal{C}}(X, Y) \in \mathcal{V}$ called the *hom object*,
iii) For every triple of objects $X, Y, Z \in \mathcal{C}$, a morphism

$$\circ_{X,Y,Z} : \overline{\mathcal{C}}(Y, Z) \otimes \overline{\mathcal{C}}(X, Y) \to \overline{\mathcal{C}}(X, Z),$$

known as the *composition map*, satisfying the associativity condition that

$$\begin{array}{ccccc} & & \overline{\mathcal{C}}(Z,W) \otimes \overline{\mathcal{C}}(Y,Z) \otimes \overline{\mathcal{C}}(X,Y) & & \\ & \swarrow^{\mathrm{Id} \otimes \circ_{X,Y,Z}} & & \searrow^{\circ_{Y,Z,W} \otimes \mathrm{Id}} & \\ \overline{\mathcal{C}}(Z,W) \otimes \overline{\mathcal{C}}(X,Z) & & & & \overline{\mathcal{C}}(Y,W) \otimes \overline{\mathcal{C}}(X,Y) \\ & \searrow_{\circ_{X,Z,W}} & & \swarrow_{\circ_{X,Y,W}} & \\ & & \overline{\mathcal{C}}(X,W) & & \end{array}$$

commutes for all $X, Y, Z, W \in \mathcal{C}$.

iv) For every object $X \in \mathcal{C}$ a morphism

$$j_X : \mathbf{I} \to \overline{\mathcal{C}}(X, X),$$

known as the *unit* morphisms, satisfying that the diagram

$$\begin{array}{ccccc} \overline{\mathcal{C}}(X,Y) \otimes \mathbf{I} & \xrightarrow{r} & \overline{\mathcal{C}}(X,Y) & \xleftarrow{l} & \mathbf{I} \otimes \overline{\mathcal{C}}(X,Y) \\ \downarrow{\scriptstyle \mathrm{Id} \otimes j_X} & \nearrow_{\alpha_{X,X,Y}} & & \nwarrow_{\circ_{X,Y,Y}} & \downarrow{\scriptstyle j_Y \otimes \mathrm{Id}} \\ \overline{\mathcal{C}}(X,Y) \otimes \overline{\mathcal{C}}(X,X) & & & & \overline{\mathcal{C}}(Y,Y) \otimes \overline{\mathcal{C}}(X,Y) \end{array}$$

commutes for all $X, Y \in \mathcal{C}$.

Definition 2.4.9. Let $\mathcal{C}, \mathcal{D}$ be two $\mathcal{V}$-categories. Then a $\mathcal{V}$-functor from $\mathcal{C}$ to $\mathcal{D}$ consists of a function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and morphisms

$$F_{X,Y} : \overline{\mathcal{C}}(X, Y) \to \overline{\mathcal{D}}(FX, FY)$$

satisfying that the diagrams

$$\begin{array}{ccc} & & \overline{\mathcal{C}}(X,X) \\ & \overset{j_X}{\nearrow} & \\ \mathbf{I} & & \downarrow F_{X,X} \\ & \underset{j_{FX}}{\searrow} & \\ & & \overline{\mathcal{D}}(FX,FX) \end{array}$$

and

$$\begin{array}{ccc} \overline{\mathcal{C}}(Y,Z) \otimes \overline{\mathcal{C}}(X,Y) & \xrightarrow{\circ_{X,Y,Z}} & \overline{\mathcal{C}}(X,Z) \\ {\scriptstyle F_{Y,Z} \otimes F_{X,Y}} \downarrow & & \downarrow {\scriptstyle F_{X,Z}} \\ \overline{\mathcal{D}}(FY,FZ) \otimes \overline{\mathcal{D}}(FX,FY) & \xrightarrow[\circ_{FX,FY,FZ}]{} & \overline{\mathcal{D}}(FX,FZ) \end{array}$$

commute for all $X, Y, Z \in \mathcal{C}$.

2.5. Model categories

Model categories were introduced by Quillen in [Qui69] as an axiomatisation of homotopy theory, as used in the study of topological spaces. The framework of model categories allows homotopy theoretic methods to be used in the study of a wide variety of different categories. An accessible introduction to model categories is [DS95] while more in depth coverage can be found in [Hov99],[Hir03], [GJ09], and [Bal21]. We here give a brief introduction to model categories that should be sufficient for our needs.

Definition 2.5.1. A *model category* $\mathcal{C}$ is a category $\mathcal{C}$ together with three classes of morphisms,

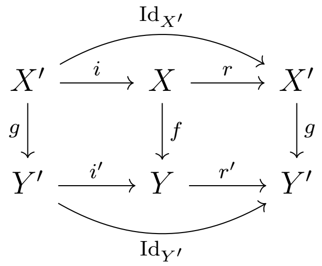
i) Weak equivalences $\xrightarrow{\sim}$,
ii) Fibrations $\twoheadrightarrow$,
iii) Cofibrations $\hookrightarrow$,

each closed under composition and containing the identity morphisms. A morphism that is both a weak equivalence and a fibration is known as an *acyclic fibration* while a morphism that is both a weak equivalence and a cofibration is known as an *acyclic cofibration*. The classes are furthermore required to satisfy the axioms:

(MC1) $\mathcal{C}$ is complete and cocomplete, i.e. has all small limits and colimits.

(MC2) Weak equivalences satisfy the two out of three property. That is if two out three of the morphisms $f : X \to Y$, $g : Y \to Z$ and $g \circ f : X \to Z$ are weak equivalences then so is the third.

(MC3) Weak equivalences, fibrations, and cofibrations are closed under retracts. That is for every commutative diagram



where f is a weak equivalence, fibration, or a cofibration it follows that g also belongs to the respective class.

(MC4) The commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

in $\mathcal{C}$, with f and g being arbitrary morphisms, admits a lift $h : B \to X$ in either the case of

i) i is an acyclic cofibration and p a fibration.

ii) i is a cofibration and p an acyclic fibration.

We say that i has the left lifting property (LLP) with respect to p and conversely that p has the right lifting property (RLP) with respect to i.

(MC5) Every morphism can functorially be factorised in the following two ways.

i) As an acyclic cofibration followed by a fibration.

ii) As a cofibration followed by an acyclic fibration.

It turns out that a model category $\mathcal{C}$ is uniquely determined by specifying the class of weak equivalences together with either the class of fibrations or the class of cofibrations. Specifically we have the following proposition.

PROPOSITION 2.5.2. *Let $\mathcal{C}$ be a model category. Then*

i) A morphism in $\mathcal{C}$ is a cofibration if and only if it has the LLP with respect to all acyclic fibrations.

ii) A morphism in $\mathcal{C}$ is an acyclic cofibration if and only if it has the LLP with respect to all fibrations.

iii) A morphism in $\mathcal{C}$ is a fibration if and only if it has the RLP with respect to all acyclic cofibrations.

iv) A morphism in $\mathcal{C}$ is an acyclic fibration if and only if it has the RLP with respect to all cofibrations.

PROOF. See [DS95, Proposition 3.13] □

As such it is common, when specifying a model category structure, to define either the fibrations to be those morphisms with the RLP with respect to acyclic cofibrations or the cofibrations as those morphisms with the LLP with respect to all acyclic fibrations.

Definition 2.5.3. Let $\mathcal{C}$ be a model category and X an object in $\mathcal{C}$. We say that X is *cofibrant* if the morphism $\emptyset \to X$ from the initial object is a cofibration.

Dually we say that X is *fibrant* if the morphism $X \to *$ to the terminal object is a fibration. We say that an object that is both fibrant and cofibrant is *bifibrant.*

In particular, MC5 tells us that for every object X in a model category $\mathcal{C}$ there are factorisations

$$\emptyset \hookrightarrow QX \overset{\sim}{\twoheadrightarrow} X$$

for some cofibrant object $QX \in \mathcal{C}$. Since the factorisation is functorial this defines a functor

$$Q : \mathcal{C} \to \mathcal{C}_c,$$

where $\mathcal{C}_c$ is the full subcategory of $\mathcal{C}$ consisting of cofibrant objects. We say that QX is a *cofibrant replacement* of X and Q is the *cofibrant replacement functor.*

Dually we have a factorisation

$$X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow *$$

for some fibrant object $RX \in \mathcal{C}$. This defines a functor

$$R : \mathcal{C} \to \mathcal{C}_f,$$

where $\mathcal{C}_f$ is the full subcategory of $\mathcal{C}$ consisting of fibrant object. We say that RX is a *fibrant replacement* for X and R is the *fibrant replacement functor.*

2.5.1. The homotopy category of a model category.

Definition 2.5.4. Let $\mathcal{C}$ be a model category. A *cylinder object* for $X \in \mathcal{C}$ is an object $X \times I$ together with factorisation

$$X \coprod X \underset{i}{\hookrightarrow} X \times I \underset{p}{\overset{\sim}{\longrightarrow}} X, \qquad p \circ i = \mathrm{Id}_X + \mathrm{Id}_X$$

where i is a cofibration and p a weak equivalence. If furthermore p is a fibration we say that $X \wedge I$ is a *good cylinder object* for X.

Definition 2.5.5. Let $\mathcal{C}$ be a model category and $f, g : X \to Y$ two morphisms in $\mathcal{C}$. A *left homotopy* from f to g is a morphism

$$h : X \times I \to Y$$

for some cylinder object $X \times I$ of X such that

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{f+g} & Y \\ {\scriptstyle i}\big\downarrow & \nearrow_{h} & \\ X \times I & & \end{array}$$

commutes, where i is the cofibration as in Definition 2.5.4. We say that f is *left homotopic* to g and denote this by $f \sim_l g$.

Definition 2.5.6. Let $\mathcal{C}$ be a model category. A *path object* for $Y \in \mathcal{C}$ is an object Y^I in $\mathcal{C}$ together with a factorisation

$$Y \xrightarrow[i]{\sim} Y^I \xrightarrow[p]{} Y \times Y, \qquad p \circ i = (\mathrm{Id}_Y, \mathrm{Id}_Y)$$

where i is a weak equivalence and p a fibration. If, furthermore, i is a cofibration we say that Y^I is a *good path object* for Y.

Definition 2.5.7. Let $\mathcal{C}$ be a model category and $f, g : X \to Y$ be two morphisms in $\mathcal{C}$. A *right homotopy* from f to g is a morphism

$$h : X \to Y^I$$

for some path object Y^I of Y such that

$$\begin{array}{ccc} & & Y^I \\ & \nearrow^{h} & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow[(f,g)]{} & Y \times Y \end{array}$$

commutes. We say that f is *right homotopic* to g and denote this by $f \sim_r g$.

PROPOSITION 2.5.8. *Let $\mathcal{C}$ be a model category, X a cofibrant object, and Y a fibrant object in $\mathcal{C}$. Then left and right homotopies coincide and form an equivalence relation on $\mathcal{C}(X, Y)$. We say that two morphisms $f, g : X \to Y$ satisfying this equivalence relation are* homotopic *and denote this by $f \sim g$.*

PROOF. See [DS95, Lemma 4.7] and [DS95, Lemma 4.21]. □

PROPOSITION 2.5.9. *Let X and Y be bifibrant objects in a model category $\mathcal{C}$. Then a morphism $f : X \to Y$ is a weak equivalence if and only if it has a homotopy inverse.*

PROOF. See [DS95, Lemma 4.24]. □

Definition 2.5.10. Let $\mathcal{C}$ be a model category. We define the homotopy category $\mathrm{Ho}(\mathcal{C})$ of $\mathcal{C}$ to be the category whose objects are bifibrant objects and whose morphisms are homotopy classes of morphisms in $\mathcal{C}$. We have a canonical functor

$$RQ : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$$

that maps an object X to its bifibrant replacement RQX and a morphism $f : X \to Y$ to the homotopy class represented by $RQf : RQX \to RQY$.

Remark 2.5.11. By Proposition 2.5.9 we see that $\mathrm{Ho}(\mathcal{C})$ is also equivalent to the category $\mathcal{C}/\sim$ whose objects are the same as $\mathcal{C}$ and whose morphisms are homotopy classes of morphisms of $\mathcal{C}$.

Theorem 2.5.12. *Let $\mathcal{C}$ be a model category with class of weak equivalences $\mathcal{W}$. Denote by $\mathcal{C}[\mathcal{W}^{-1}]$ the localisation of $\mathcal{C}$ with respect to $\mathcal{W}$. Then the inclusion functor* $\mathrm{Ho}(\mathcal{C}) \xrightarrow{\cong} \mathcal{C}[\mathcal{W}^{-1}]$ *is an equivalence of categories.*

Proof. See [DS95, Theorem 6.2]. □

Example 2.5.13. Standard examples of model categories include the following.

(1) The Quillen model structure on the category of topological spaces Top.
   - $\xrightarrow{\sim}$ Weak homotopy equivalences,
   - $\twoheadrightarrow$ Serre fibrations,
   - $\hookrightarrow$ LLP with respect to acyclic fibrations.

   Every object is fibrant in this model structure while cofibrant objects are retracts of relative cell complexes.

(2) The Quillen model structure on the category of simplicial sets sSet.
   - $\xrightarrow{\sim}$ Weak homotopy equivalences,
   - $\twoheadrightarrow$ Kan fibrations,
   - $\hookrightarrow$ Injections.

   Every object is cofibrant in this model structure while the fibrant objects are Kan complexes.

(3) The projective model structure on chain complexes Ch_R over a ring R.
   - $\xrightarrow{\sim}$ Quasi-isomorphisms,
   - $\twoheadrightarrow$ Degreewise surjections,
   - $\hookrightarrow$ LLP with respect to acyclic fibrations.

Every object is fibrant in this model structure. The class of cofibrant objects includes bounded below complexes of projective modules.

(4) The injective model structure on chain complexes Ch_R over a ring R.
- $\xrightarrow{\sim}$ Quasi-isomorphisms,
- $\hookrightarrow$ Degreewise injections,
- $\twoheadrightarrow$ RLP with respect to acyclic cofibrations.

Every object is cofibrant in this model structure. The class of fibrant objects include bounded above complexes of injective modules.

2.5.2. Quillen adjunctions and derived functors.

Definition 2.5.14. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. A *Quillen adjunction* is an adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{D}$$

such that the following equivalent properties are satisfied:

i) F preserves cofibrations and acyclic cofibrations,

ii) G preserves fibrations and acyclic fibrations,

iii) F preserves cofibrations and G preserves fibrations,

iv) F preserves acyclic cofibrations and G preserves acyclic fibrations.

We say that F is a *left Quillen* functor and G a *right Quillen* functor.

Proposition 2.5.15. *A Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ induces an adjunction on homotopy categories*

$$\mathrm{Ho}(\mathcal{C}) \underset{\mathbb{R}G}{\overset{\mathbb{L}F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Ho}(\mathcal{D}).$$

The functor $\mathbb{L}F$ is known as the left derived *functor of F. The functor $\mathbb{R}G$ is known as the* right derived *functor of G.*

Proof. See [DS95, Theorem 9.7]. □

Remark 2.5.16. By Remark 2.5.11 we know that the homotopy category is equivalent to the category $\mathcal{C}/\sim$. In this case the left derived functor $\mathbb{L}F$ would be defined as the composite $F \circ Q$ with the cofibrant replacement functor. Analogously, the right derived functor of G would be defined as the composite $G \circ R$, with the fibrant replacement functor. This allows us to speak of the left and right derived functors between arbitrary objects in $\mathcal{C}$.

Definition 2.5.17. A Quillen adjunction $F \rightleftarrows G$ that induces an equivalence of homotopy categories $\mathrm{Ho}(\mathcal{C}) \cong \mathrm{Ho}(\mathcal{D})$ is known as an *Quillen equivalence.*

Example 2.5.18. Examples of Quillen equivalences include the following.

i) The nerve realisation adjunction

$$\mathrm{Top} \underset{|\cdot|}{\overset{\mathrm{Sing}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{sSet},$$

between the Quillen model structure of topological spaces and the Quillen model structure on simplicial sets.

ii) The identity adjunction

$$\mathrm{Ch}_R^{\mathrm{proj}} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Ch}_R^{\mathrm{inj}},$$

between the projective model structure and the injective model structure on chain complexes over a ring R. This result is shown in [Hov99, Chapter 2.3].

2.5.3. Homotopy pushouts and pullbacks.

Definition 2.5.19. Let $\mathcal{C}$ be a model category. Denote by $\mathrm{Psh}(\mathcal{C})$ the category whose objects are diagrams

$$Y \leftarrow X \rightarrow Z,$$

and whose morphisms are triples (f_x, f_y, f_z) of morphisms in $\mathcal{C}$ such that

$$\begin{array}{ccccc} Y & \longleftarrow & X & \longrightarrow & Z \\ {\scriptstyle f_y}\downarrow & & {\scriptstyle f_x}\downarrow & & {\scriptstyle f_z}\downarrow \\ Y' & \longleftarrow & X' & \longrightarrow & Z' \end{array}$$

commutes.

Proposition 2.5.20. *The category* $\mathrm{Psh}(\mathcal{C})$ *admits the* projective model structure *where*

- $\xrightarrow{\sim}$ *If f_y, f_x, and f_z are weak equivalences in $\mathcal{C}$,*
- $\twoheadrightarrow$ *If f_y, f_x, and f_z are fibrations in $\mathcal{C}$,*
- $\hookrightarrow$ *LLP with respect to acyclic fibrations.*

Proof. See [DS95, Proposition 10.6]. □

Remark 2.5.21. An object

$$Y \xleftarrow{i} X \xrightarrow{j} Z$$

in Psh($\mathcal{C}$) is cofibrant if and only if X is cofibrant and i and j are cofibrations in $\mathcal{C}$.

Proposition 2.5.22. *The adjunction*

$$\mathrm{Psh}(\mathcal{C}) \underset{\Delta}{\overset{\varinjlim}{\rightleftarrows}} \mathcal{C},$$

between the colimit functor $\varinjlim$ *and the diagonal functor* Δ*, that takes an object to the constant diagram, is a Quillen adjunction.*

Proof. See [DS95, Proposition 10.7]. □

Definition 2.5.23. The homotopy pushout of a diagram in Psh($\mathcal{C}$) is defined as the image of the left derived functor

$$\mathbb{L}\varinjlim : \mathrm{Ho}(\mathrm{Psh}(\mathcal{C})) \to \mathrm{Ho}(\mathcal{C}).$$

We now turn our attention to the dual concept of homotopy pullbacks.

Definition 2.5.24. Let $\mathcal{C}$ be a model category. Denote by Pull($\mathcal{C}$) the category whose objects are diagrams

$$Y \to X \leftarrow Z$$

and whose morphisms are triples (f_x, f_y, f_z) of morphisms in $\mathcal{C}$ such that

$$\begin{array}{ccccc} Y & \longrightarrow & X & \longleftarrow & Z \\ {\scriptstyle f_y}\downarrow & & {\scriptstyle f_x}\downarrow & & {\scriptstyle f_z}\downarrow \\ Y' & \longrightarrow & X' & \longleftarrow & Z' \end{array}$$

commutes.

Proposition 2.5.25. *The category* Pull($\mathcal{C}$) *admits the* injective model structure*.*

- $\xrightarrow{\sim}$ *If f_y, f_x, and f_z are weak equivalences in $\mathcal{C}$*
- $\hookrightarrow$ *If f_y, f_x, and f_z are cofibrations in $\mathcal{C}$*
- $\twoheadrightarrow$ *RLP with respect to acyclic cofibrations.*

Remark 2.5.26. An object

$$Y \xrightarrow{i} X \xleftarrow{j} Z$$

in Pull($\mathcal{C}$) is fibrant if and only if X is fibrant and i and j are fibrations in $\mathcal{C}$.

PROPOSITION 2.5.27. *The adjunction*

$$\mathcal{C} \underset{\varprojlim}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Pull}(\mathcal{C}),$$

between the diagonal functor Δ *and the limit functor, is a Quillen adjunction.*

PROOF. See [DS95, Proposition 10.12]. $\square$

Definition 2.5.28. The homotopy pullback of a diagram in $\mathrm{Pull}(\mathcal{C})$ is defined as the image of the right derived functor

$$\mathbb{R}\varprojlim : \mathrm{Ho}(\mathrm{Pull}(\mathcal{C})) \to \mathrm{Ho}(\mathcal{C}).$$

2.5.3.1. *Proper model categories.* We will now introduce certain classes of model categories, where homotopy pushouts and homotopy pullbacks are particularly easy to compute.

Definition 2.5.29. We say that a model category $\mathcal{C}$ is *left proper* if weak equivalences are stable under pushout along cofibrations. That is, for every pushout diagram

$$\begin{array}{ccc} X & \xrightarrow[\sim]{j} & Y \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i'} \\ Z & \xrightarrow[j']{} & Y \sqcup_X Z \end{array}$$

in $\mathcal{C}$ where i is a cofibration and j a weak equivalence the morphism j' is also a weak equivalence.

Definition 2.5.30. We say that a model category $\mathcal{C}$ is *right proper* if weak equivalences are stable under pullbacks along fibrations. That is, for every pullback diagram

$$\begin{array}{ccc} X & \xleftarrow[\sim]{j} & Y \\ {\scriptstyle i}\big\uparrow & & \big\uparrow{\scriptstyle i'} \\ Z & \xleftarrow[j']{} & Y \prod_X Z \end{array}$$

in $\mathcal{C}$ where i is a fibration and j a weak equivalence the morphism j' is also a weak equivalence.

PROPOSITION 2.5.31. *A model category* $\mathcal{C}$ *where every object is cofibrant is left proper. Dually a model category* $\mathcal{C}$ *where every object is fibrant is right proper.*

PROOF. See [Lur09, Proposition A.2.4.2]. $\square$

PROPOSITION 2.5.32. *Let $\mathcal{C}$ be a left proper model category and let*

$$Y \overset{i}{\leftarrow} X \overset{j}{\hookrightarrow} Z$$

be a pushout diagram in $\mathcal{C}$ where j is a cofibration. Then the ordinary pushout of the diagram represents its homotopy pushout.

PROOF. See [Bar10, Proposition 1, 19]. □

Dually we have the following proposition.

PROPOSITION 2.5.33. *Let $\mathcal{C}$ be a right proper model category and*

$$Y \overset{i}{\rightarrow} X \overset{j}{\twoheadleftarrow} Z$$

a pullback diagram in $\mathcal{C}$ where j is a fibration. Then the ordinary pullback of the diagram represents its homotopy pullback.

2.5.4. Simplicial mapping spaces. The homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ can be given the structure of a tensored and cotensored Ho(sSet)-category. The construction of this enrichment is the topic of e.g. [Hov99, Chapter 5] and [Hir03, Chapter 17]. In particular we will denote the enrichment functor by $\mathrm{Map}(X, Y)$. This simplicial set is commonly known as the simplicial *mapping space* or simplicial *function complex*.

THEOREM 2.5.34. *There exists a natural isomorphism*

$$\pi_0 \mathrm{Map}(X, Y) \cong \mathrm{Ho}(\mathcal{C})(X, Y).$$

PROOF. See [Hir03, Theorem 17.7.2]. □

REMARK 2.5.35. Alternatively a simplicial set, satisfying the condition of Theorem 2.5.34, can be constructed by taking the Hammock localisation $\mathcal{C}[\mathcal{W}^{-1}]^H$ of the model category $\mathcal{C}$. We refer the reader to [DK80] for this approach.

CHAPTER 3

Enriched Koszul duality

We will in this chapter introduce the notion of semi-monoidal model categories and semi-module model categories. We then show that the category of conilpotent non-counital dg-coalgebras $\mathrm{coDGA}^{\mathrm{conil}}$ is a semi-monoidal model category with respect to the ordinary tensor product and that the category of non-unital dg-algebras DGA_0 is a semi-module model category over the category $\mathrm{coDGA}^{\mathrm{conil}}$. We furthermore show the analogous result in the com-Lie case of Koszul duality between the category of cocommutative conilpotent non-counital coalgebras $\mathrm{coCDGA}^{\mathrm{conil}}$ and the category of dg-Lie algebras DGLA.

Finally we show that Koszul duality, both for the associative and the com-Lie case, respects the semi-module category structures and hence provides a strengthened form of Koszul duality. Thus we will have shown Theorem 1.0.1 and Theorem 1.0.2.

We will throughout this chapter work in the non-(co)unital context over some arbitrary field k. When we later also consider dg Lie algebras we will demand that the field k has characteristic 0.

3.1. Maurer-Cartan elements and Koszul duality

We begin by introducing Maurer-Cartan elements and Koszul duality. We will here and throughout the chapter adopt the convention of using homological grading and the Koszul sign rule convention.

Definition 3.1.1. Let (A, d_A) be a non-unital dg-algebra. We say an element $a \in A$ of degree -1 is a Maurer-Cartan element if it satisfies the Maurer-Cartan equation

$$d_A a + a^2 = 0.$$

We denote the set of Maurer-Cartan elements of A by $\mathrm{MC}(A)$.

We define the universal Maurer-Cartan algebra **mc** as the non-unital free graded algebra $T_0\langle x\rangle$, where $\langle x\rangle$ denotes the vector space generated by the variable

x of degree -1, and given the differential induced from $d : x \mapsto -x^2$. Note in particular the differential of $\mathbf{mc}$ is not the free one coming from T_0. Noting that any Maurer-Cartan element $a \in A$ corresponds to the unique morphism in $\mathrm{DGA}(\mathbf{mc}, A)$ generated by $x \mapsto a$ we get the following proposition.

PROPOSITION 3.1.2. *The functor* $\mathrm{MC} : \mathrm{DGA}_0 \to \mathrm{Set}$ *taking a non-unital dg-algebra to its set of Maurer-Cartan elements is representable by the universal Maurer-Cartan algebra* $\mathbf{mc}$.

Similarly in the case of dg-Lie algebras we have the following definition.

Definition 3.1.3. Let $(\mathfrak{g}, [-,-], d)$ be a dg-Lie algebra. We say an element $x \in \mathfrak{g}$ of degree -1 is a Maurer-Cartan element if it satisfies the Maurer-Cartan equation

$$dx + \frac{1}{2}[x, x] = 0.$$

We denote the set of Maurer-Cartan elements of $\mathfrak{g}$ by $\mathrm{MC}_{\mathrm{Lie}}(\mathfrak{g})$.

Similar to the associative case, we define the universal Maurer-Cartan Lie algebra $\mathbf{mc}_{\mathrm{Lie}}$ as the free graded Lie algebra $T_{\mathrm{Lie}}\langle x\rangle$, where $\langle x\rangle$ denotes the vector space generated by the variable x of degree -1, and given the differential induced from $d : x \mapsto -\frac{1}{2}[x, x]$. Note in particular the differential of $\mathbf{mc}_{\mathrm{Lie}}$ is not the free one coming from T_{Lie}. Noting that any Maurer-Cartan element $a \in \mathfrak{g}$ corresponds to the unique morphism in $\mathrm{DGLA}(\mathbf{mc}_{\mathrm{Lie}}, \mathfrak{g})$ generated by $x \mapsto a$ we get the following proposition.

PROPOSITION 3.1.4. *The functor* $\mathrm{MC}_{\mathrm{Lie}} : \mathrm{DGLA} \to \mathrm{Set}$ *taking a dg-Lie algebra to its set of Maurer-Cartan elements is representable by the universal Maurer-Cartan algebra* $\mathbf{mc}_{\mathrm{Lie}}$.

Next we remind ourselves of the convolution coalgebra construction.

Definition 3.1.5. Let (C, Δ_C, d_C) be a non-counital dg-coalgebra and (A, μ_A, d_A) a non-unital dg-algebra. Then the internal hom of dg-vector spaces, $\underline{\mathrm{DGVec}}(C, A)$ has the structure of a non-unital dg-algebra with multiplication defined as the convolution product

$$f * g := C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu_A} A,$$

and differential

$$df := d_A f - (-1)^{|f|} f d_C.$$

We will refer to this construction as the convolution algebra of C into A and denote it by $\{C, A\}$.

We caution the reader that our choice of notation for the convolution algebra conflicts with the notation used in [AJ13]. They use the notation $[-,-]$ for the convolution algebra while instead using $\{-,-\}$ to denoted the enrichment functor of dg-algebras in the category of dg-coalgebras, see e.g. [AJ13, Chapter 3.5] for this construction.

PROPOSITION 3.1.6. *Let C be a cocommutative non-unital coassociative dg-coalgebra and $\mathfrak{g}$ a dg-Lie algebra. Then the convolution algebra $\{C, \mathfrak{g}\}$, defined as above, takes the form of a dg-Lie algebra.*

PROOF. We check that $\{C, \mathfrak{g}\}$ is a dg-Lie algebra. As C is cocommutative we have that $\Delta = \Delta \circ B_{C,C}$, where $B_{C,C}$ is the braiding morphism of dg-vector spaces. Thus in Sweedler notation we have that $c^{(1)} \otimes c^{(2)} = (-1)^{|c^{(1)}||c^{(2)}|} c^{(2)} \otimes c^{(1)}$ for all $c \in C$. Combining this with the skew symmetry of $\mathfrak{g}$ we get

$$(f * g)(c) = [f, g] \circ \Delta_C(c) = [f(c^{(1)}), g(c^{(2)})] = -(-1)^{|f(c^{(1)})||g(c^{(2)})|}[g(c^{(2)}), f(c^{(1)})] =$$

$$-(-1)^{|f||g|}[g(c^{(1)}), f(c^{(2)})] = -(-1)^{|f||g|}[g, f] \circ \Delta_C(c) = -(-1)^{|f||g|}(g * f),$$

for all $f, g \in \{C, \mathfrak{g}\}$. Thus we have shown that the convolution algebra satisfies skew symmetry.

We also have the Jacobi identity as for all $f, g, h \in \{C, A\}$ we have that

$$(f * (g * h) + g * (h * f) + h * (f * g)) \, \Delta^2(c) = 0.$$

This follows by cocommutivity, where switching the order of terms in $\Delta^2(c) = c^{(1)} \otimes c^{(2)} \otimes c^{(3)}$ gives the appropriate sign corresponding to Definition 2.2.9. $\square$

We will from now on consider the convolution algebra functor restricted to the category of conilpotent coalgebras

$$\{-,-\} : \left(\mathrm{coDGA}^{\mathrm{conil}}\right)^{\mathrm{op}} \times \mathrm{DGA}_0 \to \mathrm{DGA}_0 .$$

Let us now briefly recall Koszul duality and the bar and cobar constructions. We refer the reader to [Pos11] for the proofs and further background. For an algebra $(A, m, d_A) \in \mathrm{DGA}_0$, we define the bar construction BA as a graded coalgebra to be the conilpotent cofree coalgebra $T^{\text{co}}\Sigma A$ where ΣA is the shifted complex with $(\Sigma A)_n := A_{n-1}$. The bar construction BA is then given the differential induced by the map $d_A + m : T^{\text{co}}\Sigma A \to \Sigma A$, by cofreely extending it to a coalgebra morphism into $T^{\text{co}}\Sigma A$.

Conversely, given a coalgebra (C, Δ, d_C) we define the cobar construction ΩC as a graded algebra to be the free non-unital algebra $T_0\Sigma^{-1}C$ with differential induced from $d_C + \Delta$ by freely extending it to $T_0\Sigma^{-1}C$. Note that the differentials of BA and ΩC differ from the ones coming from the (co)free constructions as they also have a non-linear term from the (co)multiplication.

The bar and cobar functors can be shown to be adjoint by

$$\mathrm{DGA}_0(\Omega C, A) \cong \mathrm{MC}(\{C, D\}) \cong \mathrm{coDGA}^{\mathrm{conil}}(C, BA).$$

Furthermore when considering categories DGA_0 and $\mathrm{coDGA}^{\mathrm{conil}}$ with their standard model structures the above adjunction is a Quillen equivalence. Explicitly the model structure on DGA_0 is given as follows. We say a morphism in DGA_0 is a

i) weak equivalence if it is a quasi-isomorphism,
ii) fibration if it is surjective,
iii) cofibration if it has the left lifting property with respect to all acyclic fibrations.

The category of $\mathrm{coDGA}^{\mathrm{conil}}$ admits the left transferred model structure over the above adjunction. We say that a morphism in $\mathrm{coDGA}^{\mathrm{conil}}$ is a

i) weak equivalence if it belongs to the minimal class of morphisms generated by filtered quasi-isomorphism under the 2 out of 3 property,
ii) cofibration if it is injective,
iii) fibration if it has the right lifting property with respect to all acyclic cofibrations.

The Quillen equivalence between these categories is what we refer to as *associative Koszul duality* and is the content of [Lef03, Theorem 1.3.1.1]. Alternatively the reader can find a proof of the existence of these model structures as well as the Quillen equivalence in [Pos11, Section 9].

The story in the com-Lie case of Koszul duality is very similar but we will in this case require that the ground field k has characteristic zero. This is needed for the existence of the model structure on DGLA and $\mathrm{coCDGA}^{\mathrm{conil}}$ as shown in [Hin01]. As in the associative case we say that a morphism in DGLA is a

i) weak equivalence if it is a quasi-isomorphism,
ii) fibration if it is surjective,
iii) cofibration if it has the left lifting property with respect to all acyclic fibrations.

Similarly we say that a morphism in $\mathrm{coCDGA}^{\mathrm{conil}}$ is a

i) weak equivalence if it belongs to the minimal class of morphisms generated by filtered quasi-isomorphism under the 2 out of 3 property,
ii) cofibration if it is injective,
iii) fibration if it has the right lifting property with respect to all acyclic cofibrations.

For a Lie-algebra $(\mathfrak{g}, [-,-], d_{\mathfrak{g}}) \in \mathrm{DGLA}$ we define the bar construction $B\mathfrak{g}$ to be $S^{co}(\Sigma\mathfrak{g})$ with differential induced from $d_{\mathfrak{g}} + [-,-]$. In the other direction for a cocommutative coalgebra $(C, \Delta, d_C) \in \mathrm{coCDGA}^{\mathrm{conil}}$ we define the cobar construction ΩC as the free Lie algebra $T_{\mathrm{Lie}}(\Sigma^{-1}C)$ with differential induced by $d_C + \Delta$. These functors are Quillen equivalent by the adjunction

$$\mathrm{DGLA}(\Omega C, \mathfrak{g}) \cong \mathrm{MC}_{\mathrm{Lie}}(\{C, \mathfrak{g}\}) \cong \mathrm{coCDGA}^{\mathrm{conil}}(C, B\mathfrak{g}),$$

as which is shown in [Hin01, Theorem 3.2]. Note in particular that $\{C, A\}$ has the structure of a dg-Lie algebra by Proposition 3.1.6. We will refer to this Quillen equivalene as *com-Lie Koszul duality.*

3.2. Semi-monoidal categories, semi-module categories, and semi-enrichments

Categories that are monoidal except missing a unit, known as semi-monoidal, semi-groupal or non-unital monoidal in the literature have previously been studied in e.g. [Koc08], [Abu13], and [LYH19]. We will introduce the notion of semi-module categories over a semi-monoidal category, fully analogous to the definition in the unital case. Working in the model category setting this will lead us to the definition of semi-monoidal model categories and semi-module model categories. For a brief coverage for the corresponding unital case the reader may want to consult Section 2.4.

We will take the definition of (symmetric) semi-monoidal categories, semi-monoidal functors, and semi-monoidal natural transformations to be fully analogous to the monoidal ones by dropping the unit and unit axioms at every step. Similarly, we take the definitions of semi-modules, semi-module functors and semi-module natural transformations to be those found in appendix B of [AJ13] or chapter 4 in [Hov99] by dropping the unit axioms. When working over a semi-monoidal category $\mathcal{V}$ we will commonly use the terminology $\mathcal{V}$-module to mean a semi-module over $\mathcal{V}$ etc.

Definition 3.2.1. A semi-monoidal category $(\mathcal{V}, \otimes, a)$ consists of a category $\mathcal{V}$ together with a functor

$$\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V},$$

and a natural isomorphism

$$a : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z),$$

satisfying the coherence diagram

$$\begin{array}{ccccc} & & ((X \otimes Y) \otimes Z) \otimes W & & \\ & \swarrow^{a_{X,Y,Z} \otimes \mathrm{Id}_W} & & \searrow^{a_{X \otimes Y, Z, W}} & \\ (X \otimes (Y \otimes Z)) \otimes W & & & & (X \otimes Y) \otimes (Z \otimes W) \\ \downarrow{\scriptstyle a_{X, Y \otimes Z, W}} & & & & \downarrow{\scriptstyle a_{X,Y,Z \otimes W}} \\ X \otimes ((Y \otimes Z) \otimes W) & & \xrightarrow[\mathrm{Id}_X \otimes a_{Y,Z,W}]{} & & X \otimes (Y \otimes (Z \otimes W)) \end{array}$$

for all $X, Y, Z, W \in \mathcal{V}$.

Definition 3.2.2. A symmetric semi-monoidal category $(\mathcal{V}, \otimes, a, c)$ is a semi-monoidal category $(\mathcal{V}, \otimes, a)$ together with a natural isomorphism $c : X \otimes Y \to Y \otimes X$ satisfying that

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{c_{X,Y}} & Y \otimes X \\ & \searrow_{\mathrm{Id}_{X \otimes Y}} & \downarrow{\scriptstyle c_{Y,X}} \\ & & X \otimes Y \end{array}$$

and

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{c_{X, Y \otimes Z}} & (Y \otimes Z) \otimes X \\ \downarrow{\scriptstyle c_{X,Y} \otimes \mathrm{Id}_Z} & & & & \downarrow{\scriptstyle a_{Y,Z,X}} \\ (Y \otimes X) \otimes Z & \xrightarrow[a_{Y,X,Z}]{} & Y \otimes (X \otimes Z) & \xrightarrow[\mathrm{Id}_Y \otimes c_{X,Z}]{} & Y \otimes (Z \otimes X). \end{array}$$

commute for all $X, Y, Z \in \mathcal{V}$.

Definition 3.2.3. We say that a semi-monoidal category $(\mathcal{V}, \otimes)$ is (left) closed if there exists a functor

$$\underline{\mathcal{V}} : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V},$$

such that $\underline{\mathcal{V}}(Y, -)$ is right adjoint to $- \otimes Y$. The functor $\underline{\mathcal{V}}$ is known as the internal hom functor.

Definition 3.2.4. Let $(\mathcal{V}, \otimes, a)$ and $(\mathcal{V}, \otimes', a')$ be two semi-monoidal categories. Then a (strong) semi-monoidal functor (F, m) from $\mathcal{V}$ to $\mathcal{V}'$ consists of a functor $F : \mathcal{V} \to \mathcal{V}'$ and a natural isomorphism

$$m : F(X) \otimes' F(Y) \to F(X \otimes Y),$$

satisfying the coherence diagram

$$\begin{array}{ccc}
(F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{a'} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\
{\scriptstyle m \otimes' \mathrm{Id}_{F(Z)}} \downarrow & & \downarrow {\scriptstyle \mathrm{Id}_{F(X)} \otimes' m} \\
F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\
{\scriptstyle m} \downarrow & & \downarrow {\scriptstyle m} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(a)} & F(X \otimes (Y \otimes Z))
\end{array}$$

for all $X, Y, Z \in \mathcal{V}$.

Definition 3.2.5. Let (F, m) and (F', m') be two semi-monoidal functors from $(\mathcal{V}, \otimes)$ to $(\mathcal{V}', \otimes')$. A semi-monoidal natural transformation $\eta : F \to F'$ is a natural transformation such that

$$\begin{array}{ccc}
F(X) \otimes' F(Y) & \xrightarrow{\eta \otimes' \eta} & F'(X) \otimes' F'(Y) \\
{\scriptstyle m} \downarrow & & \downarrow {\scriptstyle m'} \\
F(X \otimes Y) & \xrightarrow{\eta} & F'(X \otimes Y)
\end{array}$$

commutes for all $X, Y \in \mathcal{V}$.

We could at this point further proceed with the theory of semi-monoidal categories, by defining semi-monoidal adjunctions and semi-monoidal equivalences etc. As we will not explicitly need them, we instead proceed with the definition of semi-module categories.

Definition 3.2.6. Let $(\mathcal{V}, \otimes, a)$ be a symmetric semi-monoidal category. A (left) $\mathcal{V}$-module is a category $\mathcal{C}$ together with a functor

$$- \triangleright - : \mathcal{V} \times \mathcal{C} \to \mathcal{C},$$

known as the *tensoring functor*, and a natural isomorphism

$$\alpha : (X \otimes Y) \triangleright A \to X \triangleright (Y \triangleright A),$$

such that

$$\begin{array}{ccc} & ((X \otimes Y) \otimes Z) \rhd A & \\ {}^{a \rhd \mathrm{Id}_A}\swarrow & & \searrow^{\alpha} \\ (X \otimes (Y \otimes Z)) \rhd A & & (X \otimes Y) \rhd (Z \rhd A) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ X \rhd ((Y \otimes Z) \rhd A) & \xrightarrow[\mathrm{Id}_X \rhd \alpha]{} & X \rhd (Y \rhd (Z \rhd A)) \end{array}$$

commutes for all $X, Y, Z \in \mathcal{V}$ and $A \in \mathcal{C}$.

A module over the opposite symmetric semi-monoidal category $(\mathcal{V}^{op}, \otimes)$ is known as a $\mathcal{V}$-opmodule. We should at this point note that Mac Lane's coherence theorem holds in the case of semi-monoidal categories and the corresponding version for module categories similarly holds for semi-module categories.

Definition 3.2.7. We say that a $\mathcal{V}$-module $\mathcal{C}$ is right closed if there exists a functor

$$[-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

such that $[X, -]$ is right adjoint to $X \rhd -$ for all $X \in \mathcal{V}$. Similarly we say that $\mathcal{C}$ is left closed if there exists a functor

$$\overline{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$$

such that $\overline{\mathcal{C}}(A, -)$ is right adjoint to $- \rhd A$ for all $A \in \mathcal{C}$. If $\mathcal{C}$ is both left and right closed we say it is closed.

Particularly in the closed case, we will refer to the $[-,-]$ functor as the cotensoring functor and $\overline{\mathcal{C}}(-,-)$ as the enrichment functor. Note that the cotensoring functor $[-,-]$ makes $\mathcal{C}$ into a $\mathcal{V}$-*opmodule*. The motivation for referring to $\overline{\mathcal{C}}(-,-)$ as an enrichment is that in the monoidal case the notion of a closed $\mathcal{V}$-module is equivalent to a tensored and cotensored $\mathcal{C}$-category.

Definition 3.2.8. Let $(\mathcal{V}, \otimes)$ be a symmetric semi-monoidal category and $(\mathcal{C}, \rhd, \alpha)$ and $(\mathcal{D}, \rhd', \alpha')$ two $\mathcal{V}$-modules. A *lax* $\mathcal{V}$-module functor (F, m) from $\mathcal{C}$ to $\mathcal{D}$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural transformation

$$m : X \rhd' FA \to F(X \rhd A),$$

such that

$$\begin{array}{ccc} (X \otimes Y) \rhd' FA & \xrightarrow{m} & F((X \otimes Y) \rhd A) \\ {\scriptstyle \alpha'}\downarrow & & \\ X \rhd' (Y \rhd' FA) & & \downarrow {\scriptstyle F\alpha} \\ {\scriptstyle \mathrm{Id}_X \rhd' m}\downarrow & & \\ X \rhd' F(Y \rhd A) & \xrightarrow{m} & F(X \rhd (Y \rhd A)) \end{array}$$

commutes for all $X, Y \in \mathcal{V}$ and $A \in \mathcal{C}$. In case the natural transformation m is a natural isomorphism we say that (F, m) is a (strong) $\mathcal{V}$-module functor.

Proposition 3.2.9. *Let $(\mathcal{C}, \rhd)$ be a $\mathcal{V}$-module. Then for any $Y \in \mathcal{V}$ the tensoring functor $Y \rhd -$ is itself a $\mathcal{V}$-module functor.*

Proof. We take the natural isomorphism,

$$m : X \rhd (Y \rhd A) \to Y \rhd (X \rhd A),$$

to be the composition

$$X \rhd (Y \rhd A) \xrightarrow{\sim} (X \otimes Y) \rhd A \xrightarrow{\sim} (Y \otimes X) \rhd A \xrightarrow{\sim} Y \rhd (X \rhd A).$$

□

Definition 3.2.10. Let $\mathcal{V}$ be a symmetric semi-monoidal category and (F, m) and (F', m') be two $\mathcal{V}$-module functors from $(\mathcal{C}, \rhd)$ to $(\mathcal{D}, \rhd')$. A $\mathcal{V}$-module natural transformation is a natural transformation $\eta : F \to F'$ such that

$$\begin{array}{ccc} X \rhd' FA & \xrightarrow{m} & F(X \rhd A) \\ {\scriptstyle \mathrm{Id}_X \rhd' \eta_A}\downarrow & & \downarrow {\scriptstyle \eta_{X \rhd A}} \\ X \rhd' F'A & \xrightarrow{m'} & F'(X \rhd A) \end{array}$$

commutes for all $X \in \mathcal{V}$ and $A \in \mathcal{C}$.

We define the concept of semi-module adjunctions and equivalences as follows.

Definition 3.2.11. A $\mathcal{V}$-module adjunction (F, U, ϕ, m) is an adjunction (F, U, ϕ) such that (F, m) is a $\mathcal{V}$-module functor.

Definition 3.2.12. A $\mathcal{V}$-module equivalence (F, U, ϕ, m) is an equivalence of categories (F, U, ϕ) such that (F, m) is a $\mathcal{V}$-module functor.

Remark 3.2.13. Note that these are the notions of adjunction and equivalence in the 2-category of $\mathcal{V}$-module categories and *lax* $\mathcal{V}$-module functors. This comes from the fact that a $\mathcal{V}$-module adjunction, as defined above, induces a *lax* $\mathcal{V}$-module functor structure on its right adjoint. We refer the reader to [Kel74] for the general theory behind this.

Finally we would like to mention that we can define the concept of what we call semi-enriched categories by extracting the properties of the enrichment functor in a closed semi-module category. Proceeding similarly as in the monoidal case we get the definition of such a structure.

Definition 3.2.14. Let $(\mathcal{V}, \otimes, a)$ be a symmetric semi-monoidal category. A semi-enriched $\mathcal{V}$-category is a category $\mathcal{C}$ together with a functor

$$\overline{\mathcal{C}(-,-)} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$$

and a natural transformation

$$\mathbf{c} : \overline{\mathcal{C}}(B, C) \otimes \overline{\mathcal{C}}(A, B) \to \overline{\mathcal{C}}(A, C)$$

such that

$$\begin{array}{ccccc}
 & & (\overline{\mathcal{C}}(C,D) \otimes \overline{\mathcal{C}}(B,C)) \otimes \overline{\mathcal{C}}(A,B) & & \\
 & \swarrow^{a} & & \searrow^{\mathbf{c}\otimes \mathrm{Id}} & \\
\overline{\mathcal{C}}(C,D) \otimes (\overline{\mathcal{C}}(B,C) \otimes \overline{\mathcal{C}}(A,B)) & & & & \overline{\mathcal{C}}(B,D) \otimes \overline{\mathcal{C}}(A,B) \\
\downarrow{\scriptstyle \mathrm{Id}\otimes\mathbf{c}} & & & & \downarrow{\scriptstyle \mathbf{c}} \\
\overline{\mathcal{C}}(C,D) \otimes \overline{\mathcal{C}}(A,C) & & \xrightarrow{\quad\mathbf{c}\quad} & & \overline{\mathcal{C}}(A,D)
\end{array}$$

commutes.

Note this structure is different from that of semi-categories, i.e. categories without a unit morphism. Most notably, the underlying category is an essential part of the structure. This is necessary as if we had taken a definition as for semi-categories, i.e. enriched categories without units, as has been studied in [MBB02] and [Stu05] we would have no way of recovering an underlying category. In particular, going to back to our definition, the enriched hom objects don't necessarily provide information about the hom set of the underlying category.

3.3. Semi-monoidal model categories and semi-module model categories

We take the definitions of semi-monoidal model categories and semi-module model categories to be analogous to the corresponding definitions with units as found in [Hov99].

Before proceeding we begin by reminding the reader of the definition of two variable adjunctions and Quillen bifunctors.

Definition 3.3.1. Let $\mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ be categories. An adjunction of two variables consists of functors

$$\rhd : \mathcal{C} \times \mathcal{D} \to \mathcal{E},$$

$$\mathrm{Hom}_l : \mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{D},$$

and

$$\mathrm{Hom}_r : \mathcal{D}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{C},$$

together with natural isomorphisms

$$\mathcal{D}(D, \mathrm{Hom}_l(C, E)) \cong \mathcal{E}(C \rhd D, E) \cong \mathcal{C}(C, \mathrm{Hom}_r(D, E)),$$

for all $C \in \mathcal{C}$, $D \in \mathcal{D}$, and $E \in \mathcal{E}$.

Definition 3.3.2. Let $\mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ be model categories and $(\rhd, \mathrm{Hom}_l, \mathrm{Hom}_r)$ an adjunction of two variables between them. Then we say that

$$\rhd : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$$

is a Quillen bifunctor if for all cofibrations $i : C \to C'$ in $\mathcal{C}$ and cofibrations $j : D \to D'$ in $\mathcal{D}$ the induced pushout product map

$$i \wedge j : (C' \otimes D) \coprod_{C \otimes D} (C \otimes D') \to (C' \otimes D'),$$

is a cofibration, which is furthermore acyclic if either i or j is acyclic.

As a Quillen bifunctor is part of a two variable adjunction we have the following proposition which appears in [Hov99].

Proposition 3.3.3. *Let $\mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ be model categories and $(\rhd, \mathrm{Hom}_l, \mathrm{Hom}_r)$ an adjunction of two variables. Then the following are equivalent:*

i) The functor $\rhd : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a Quillen bifunctor.

ii) For $i : D \hookrightarrow D'$ a cofibration in $\mathcal{D}$ and $j : E \twoheadrightarrow E'$ a fibration in $\mathcal{E}$ the induced pullback map

$$\mathrm{Hom}_r(D', E) \to \mathrm{Hom}_r(D, E) \prod_{\mathrm{Hom}_r(D,E')} \mathrm{Hom}_r(D', E'),$$

is a fibration in $\mathcal{C}$ which furthermore is acyclic if either i or j is.

iii) For $i : C \hookrightarrow C'$ a cofibration in $\mathcal{C}$ and $j : E \twoheadrightarrow E'$ a fibration in $\mathcal{E}$ the induced pullback map

$$\mathrm{Hom}_l(C', E) \to \mathrm{Hom}_l(C, E) \prod_{\mathrm{Hom}_l(C,E')} \mathrm{Hom}_l(C', E'),$$

is a fibration in $\mathcal{D}$ which furthermore is acyclic if either i or j is.

Definition 3.3.4. Let $(\mathcal{V}, \otimes)$ be a closed symmetric semi-monoidal category. If furthermore $\mathcal{V}$ is a model category and the tensor functor,

$$\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V},$$

is a (left) Quillen bifunctor, then we say that $\mathcal{V}$ is a semi-monoidal model category.

Definition 3.3.5. Let $(\mathcal{V}, \otimes)$ be a semi-monoidal model category and $(\mathcal{C}, \triangleright)$ a closed $\mathcal{V}$-module. If furthermore $\mathcal{C}$ is a model category and the tensoring functor,

$$\triangleright : \mathcal{V} \times \mathcal{C} \to \mathcal{C},$$

is a (left) Quillen bifunctor, then we say that $\mathcal{C}$ is a $\mathcal{V}$-module model category.

Definition 3.3.6. Let $\mathcal{C}$ and $\mathcal{D}$ be semi-monoidal model categories. Then a semi-monoidal Quillen adjunction (F, U, ϕ, m) is a Quillen adjunction (F, U, ϕ) such that (F, m) is a semi-monoidal functor.

Definition 3.3.7. Let $\mathcal{V}$ be a semi-monoidal model category and $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-module model categories. A $\mathcal{V}$-module Quillen adjunction (F, U, ϕ, μ) is a Quillen adjunction such that (F, μ) is a $\mathcal{V}$-module functor.

Example 3.3.8 (Reduced simplicial sets)**.** Consider the category of reduced simplicial sets sSet_0, i.e. consisting of simplicial sets with a single vertex. The category sSet_0 together with the smash product $\wedge$ gives an example of a semi-monoidal model category that is not monoidal. To see this, consider the adjunction

$$\mathrm{sSet}_0 \underset{\mathscr{R}}{\overset{\iota}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{sSet}^{*/},$$

where ι is the inclusion into the category of pointed simplicial sets, $\mathrm{sSet}^{*/}$, and its adjoint $\mathscr{R}$ takes a pointed simplicial set to the subsimplicial set whose n-cells are those who have the marked point as 0-cells. Note that this adjunction is coreflective meaning that the inclusion functor of the full subcategory sSet_0 of sSet admits a right adjoint.

The category of reduced simplicial sets admits the left transferred model structure, from the classical model structure on $\mathrm{sSet}^{*/}$ by the above adjunction, as shown by Proposition 6.2 in [GJ09, Chapter V]. As the wedge product of pointed simplicial sets restricts to a functor

$$\wedge : \mathrm{sSet}_0 \times \mathrm{sSet}_0 \to \mathrm{sSet}_0,$$

we see that sSet_0 is closed semi-monoidal with internal hom functor

$$\underline{\mathrm{sSet}_0}(-,-) := \mathscr{R}\underline{\mathrm{sSet}}^{*/}(-,-).$$

Note that the unit, $* \sqcup *$, of $\mathrm{sSet}^{*/}$ is not reduced so sSet_0 is not monoidal.

As a consequence of sSet_0 being semi-monoidal together with the fact we have the coreflective Quillen adjunction to $\mathrm{sSet}^{*/}$ we get the following.

COROLLARY 3.3.9. *Any pointed simplicial model category* $\mathcal{M}$ *is canonically also a* sSet_0*-module model category.*

COROLLARY 3.3.10. *Let* $\mathcal{M}$ *be a pointed model category. Then its homotopy category can be given the structure of a closed* $\mathrm{Ho}(\mathrm{sSet}_0)$*-module.*

We will not further develop the theory here but only note that the standard proofs for monoidal- and module model categories also applies to the non-unital setting. In particular from Section 4.3 in [Hov99] we obtain the following statements.

THEOREM 3.3.11. *i) Let* $(\mathcal{V}, \otimes, \underline{\mathcal{V}})$ *be a symmetric semi-monoidal model category. Then its homotopy category* $\mathrm{Ho}(\mathcal{V})$ *has the structure of a semi-monoidal category* $(\mathrm{Ho}(\mathcal{V}), \otimes^L, R\underline{\mathcal{V}})$ *induced from* $\mathcal{V}$.

ii) Let $(\mathcal{C}, \overline{\mathcal{C}}, \rhd, [-,-])$ *be a* $\mathcal{V}$*-module model category. Then its homotopy category* $\mathrm{Ho}(\mathcal{C})$ *has the structure of a closed* $\mathrm{Ho}(\mathcal{V})$*-module* $(\mathrm{Ho}(\mathcal{C}), R\overline{\mathcal{C}}, \rhd^L, R[-,-])$ *induced from* $\mathcal{C}$.

iii) A semi-monoidal Quillen adjunction $(F, U, \varphi, m) : \mathcal{V} \to \mathcal{W}$ *between symmetric semi-monoidal model categories induces a symmetric semi-monoidal adjunction* $(LF, RU, R\varphi, m_{LF}) : \mathrm{Ho}(\mathcal{V}) \to \mathrm{Ho}(\mathcal{W})$ *of homotopy categories.*

iv) A $\mathcal{V}$*-module Quillen adjunction* $(F, U, \varphi, m) : \mathcal{C} \to \mathcal{D}$ *induces a* $\mathrm{Ho}(\mathcal{V})$*-module adjunction* $(LF, RU, R\varphi, m_{LF}) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ *of homotopy categories.*

3.4. The semi-module structure of DGA_0 over $\mathrm{coDGA}^{\mathrm{conil}}$

From [AJ13, Section 4.1] we know that DGA_0 admits a closed coDGA_0-module structure. To show that DGA_0 also admits a closed $\mathrm{coDGA}^{\mathrm{conil}}$-semi-module structure we will make use of the same procedure. Indeed, applying the conilpotent radical functor R^{co} to the internal hom functor and the enrichment functor in the coDGA_0 case would provide the results in the conilpotent case. Nevertheless, for the convenience of the reader we provide categorical proofs of the needed results.

3.4.1. The semi-monoidal structure of conilpotent dg-coalgebras. It is well known from e.g. [AJ13, Section 4.1] that the category of non-counital dg-coalgebras $(\mathrm{coDGA}_0, \otimes)$ is symmetric monoidal. We consider the full subcategory of conilpotent non-counital dg-coalgebras $\mathrm{coDGA}^{\mathrm{conil}}$. By Proposition 2.3.16 we know that the tensor product functor of conilpotent dg-coalgebras restricts to a functor

$$\otimes : \mathrm{coDGA}^{\mathrm{conil}} \times \mathrm{coDGA}^{\mathrm{conil}} \to \mathrm{coDGA}^{\mathrm{conil}},$$

thus providing a semi-monoidal structure on $\mathrm{coDGA}^{\mathrm{conil}}$. However $\mathrm{coDGA}^{\mathrm{conil}}$ does not quite form a monoidal category under the tensor product as the monoidal unit k of coDGA_0 is not conilpotent.

Theorem 3.4.1. *The category* $(\mathrm{coDGA}^{\mathrm{conil}}, \otimes)$ *is closed symmetric semi-monoidal.*

Proof. We need to construct an internal hom functor

$$\underline{\mathrm{coDGA}}^{\mathrm{conil}}(-,-) : \left(\mathrm{coDGA}^{\mathrm{conil}}\right)^{\mathrm{op}} \times \mathrm{coDGA}^{\mathrm{conil}} \to \mathrm{coDGA}^{\mathrm{conil}},$$

satisfying that $\underline{\mathrm{coDGA}}^{\mathrm{conil}}(C_1, -)$ is right adjoint to the tensor product functor $- \otimes C'$ for every $C' \in \mathrm{coDGA}^{\mathrm{conil}}$.

For the case of a conilpotent cofree dg-coalgebra $T^{\mathrm{co}}V$ we have the natural isomorphisms

$$\mathrm{coDGA}^{\mathrm{conil}}(C \otimes C', T_0^{\mathrm{co}}V) \cong \mathrm{DGVec}(C \otimes C', V) \cong$$
$$\mathrm{DGVec}(C, \underline{\mathrm{DGVec}}(C', V)) \cong \mathrm{coDGA}^{\mathrm{conil}}(C, T^{\mathrm{co}}\,\underline{\mathrm{DGVec}}(C', V)).$$

Hence we can define

$$\underline{\mathrm{coDGA}}^{\mathrm{conil}}(C, T^{\mathrm{co}}V) := T^{\mathrm{co}}\,\underline{\mathrm{DGVec}}(C, V).$$

In the case of an arbitrary conilpotent dg-coalgebra C'' we can write C'' as an equaliser

$$C \longrightarrow T^{\mathrm{co}}C' \rightrightarrows (T^{\mathrm{co}})^2C''.$$

We then define the internal hom $\underline{\mathrm{coDGA}}^{\mathrm{conil}}(C', C'')$ as the equaliser of

$$\begin{array}{ccc} \underline{\mathrm{coDGA}}^{\mathrm{conil}}(C', T^{co}C'') & \rightrightarrows & \underline{\mathrm{coDGA}}^{\mathrm{conil}}(C', (T^{co})^2C'') \\ \cong & & \cong \\ T^{co}\,\underline{\mathrm{DGVec}}(C', C'') & \rightrightarrows & T^{\mathrm{co}}\,\underline{\mathrm{DGVec}}(C', T^{co}C'') \end{array}$$

Note that we use the notation $(T^{\mathrm{op}})^2$ to mean composing the T^{co} functor twice acting on the underlying dg-vector space i.e. $(T^{\mathrm{op}})^2 := T^{\mathrm{op}}U(T^{\mathrm{op}})$, where U denotes the forgetful functor to dg-vector spaces.

Since the $\mathrm{coDGA}^{\mathrm{conil}}(C', -)$ functor preserves limits this indeed gives the desired natural bijection

$$\mathrm{coDGA}^{\mathrm{conil}}(C \otimes C', C'') \cong \mathrm{coDGA}^{\mathrm{conil}}(C, \underline{\mathrm{coDGA}}^{\mathrm{conil}}(C', C'')).$$

□

Note that the internal hom of $\mathrm{coDGA}^{\mathrm{conil}}$ is different from the internal hom of coDGA_0. The latter can be constructed analogously using the non-unital cofree functor T_0 in place of the conilpotent cofree functor T^{co}. In particular we have that $\underline{\mathrm{coDGA}}^{\mathrm{conil}}(C, D) \cong R^{\mathrm{co}}\underline{\mathrm{coDGA}_0}(C, D)$ for all conilpotent dg-coalgebras C and D.

Remark 3.4.2. As we have no unit in a semi-monoidal category we have no way of recovering the hom sets from the internal hom. Indeed in our case that information is lost as the only atom of $\underline{\mathrm{coDGA}}^{\mathrm{conil}}(C, D)$ is 0, corresponding to the zero morphism. This is also the reason we cannot consider some larger subcategory of coDGA_0 containing k as, even if such a category admits an internal hom, it will never be conilpotent.

3.4.2. The semi-module structure of non-unital DG-algebras. In addition to the internal hom adjunction for conilpotent coalgebras established in the previous section we will need to establish tensoring and cotensoring adjunctions.

Our starting point is to consider the convolution algebra functor restricted to conilpotent coalgebras

$$\{-,-\} : \left(\mathrm{coDGA}^{\mathrm{conil}}\right)^{\mathrm{op}} \times \mathrm{DGA}_0 \to \mathrm{DGA}_0 \, .$$

We will show that this will be the cotensoring functor of the closed semi-module structure of DGA_0. We construct the enriched hom functor as the left adjoint to the opposite convolution algebra functor.

Proposition 3.4.3. *There exists a functor*

$$\overline{\mathrm{DGA}}_0(-,-) : \mathrm{DGA}_0^{\mathrm{op}} \times \mathrm{DGA}_0 \to \mathrm{coDGA}^{\mathrm{conil}},$$

such that $\overline{\mathrm{DGA}}_0(-,B)$ *is right adjoint to the contravariant convolution algebra functor* $\{-,B\}^{\mathrm{op}}$ *for each algebra* $B \in \mathrm{DGA}_0$. *We will refer to* $\overline{\mathrm{DGA}}_0$ *as the enrichment functor.*

Proof. For a free algebra T_0V we have natural isomorphisms

$$\mathrm{DGA}_0(T_0V, \{C,B\}) \cong \mathrm{DGVec}(V, \underline{\mathrm{DGVec}}(C,B)) \cong \mathrm{DGVec}(V \otimes C, B) \cong$$
$$\mathrm{DGVec}(C, \underline{\mathrm{DGVec}}(V,B)) \cong \mathrm{coDGA}^{\mathrm{conil}}(C, T_0^{\mathrm{co}}\, \underline{\mathrm{DGVec}}(V,B)),$$

so we can define the enriched hom as

$$\overline{\mathrm{DGA}}_0(T_0V, B) := T_0^{\mathrm{co}}\, \underline{\mathrm{DGVec}}(V,B).$$

Given an arbitrary algebra A we can write it as a coequaliser

$$T_0^2 A \underset{T_0\epsilon}{\overset{\epsilon}{\rightrightarrows}} T_0 A \xrightarrow{\ \epsilon\ } A,$$

where ϵ is the counit of the free forgetful adjunction. We then define $\overline{\mathrm{DGA}}_0(A,B)$ as the equaliser of

$$\begin{array}{ccc} \overline{\mathrm{DGA}}_0(T_0A, B) & \rightrightarrows & \overline{\mathrm{DGA}}_0(T_0^2A, B) \\ \cong & & \cong \\ T^{\mathrm{co}}\, \underline{\mathrm{DGVec}}(A,B) & \rightrightarrows & T^{\mathrm{co}}\, \underline{\mathrm{DGVec}}(T_0A,B). \end{array}$$

Since $\mathrm{coDGA}^{\mathrm{conil}}(C,-)$ preserves limits we get the desired natural bijection

$$\mathrm{DGA}_0(A, \{C,B\}) \cong \mathrm{coDGA}^{\mathrm{conil}}(C, \overline{\mathrm{DGA}}_0(A,B)).$$

□

We construct what will be the tensoring functor similarly.

PROPOSITION 3.4.4. *There exists a functor*

$$(-) \rhd (-) : \mathrm{coDGA}^{\mathrm{conil}} \times \mathrm{DGA}_0 \to \mathrm{DGA}_0,$$

such that $C \rhd (-)$ is left adjoint to the convolution algebra functor $\{C, -\}$ for each coalgebra $C \in \mathrm{coDGA}^{\mathrm{conil}}$. We will refer to $\rhd$ as the tensoring functor.

PROOF. For a free algebra T_0V we have the natural isomorphisms

$$\mathrm{DGA}_0(T_0V, \{C, B\}) \cong \mathrm{DGVec}(V, \underline{\mathrm{DGVec}}(C, B)) \cong$$
$$\mathrm{DGVec}(C \otimes V, B) \cong \mathrm{DGA}_0(T_0(C \otimes V), B).$$

Hence we can define

$$C \rhd T_0V := T_0(C \otimes V).$$

Given an arbitrary algebra A we can write it as a coequaliser and define $C \rhd A$ as the coequaliser of

$$\begin{array}{ccc} C \rhd T_0^2A & \rightrightarrows & C \rhd T_0A \\ \cong & & \cong \\ T_0(C \otimes T_0A) & \rightrightarrows & T_0(C \otimes A). \end{array}$$

Since $\mathrm{DGA}_0(-, B)$ takes colimits to limits we get the desired natural bijection

$$\mathrm{DGA}_0(A, \{C, B\}) \cong \mathrm{DGA}_0(C \rhd A, B).$$

□

By combining Proposition 3.4.3 and Proposition 3.4.4 we also get a third adjunction between the tensoring and the enrichment functor.

COROLLARY 3.4.5. *The tensoring $(-) \rhd A$ is left adjoint to the enrichment functor $\overline{\mathrm{DGA}}_0(A, -)$ for each algebra $A \in \mathrm{DGA}_0$.*

In summary we have adjunctions,

$$\mathrm{coDGA}^{\mathrm{conil}} \underset{\overline{\mathrm{DGA}}_0(-,B)}{\overset{\{-,B\}^{\mathrm{op}}}{\rightleftarrows}} \mathrm{DGA}^{\mathrm{op}},$$

$$\mathrm{DGA}_0 \underset{\{C,-\}}{\overset{C \rhd (-)}{\rightleftarrows}} \mathrm{DGA}_0,$$

and

$$\mathrm{coDGA}^{\mathrm{conil}} \underset{\overline{\mathrm{DGA}}_0(A,-)}{\overset{(-) \rhd A}{\rightleftarrows}} \mathrm{DGA}_0\,.$$

These are analogous to those shown in [AJ13, Section 4.1] for the non-conilpotent case. In particular the tensoring and cotensoring functors are the same as in the non-conilpotent case while the enrichment functor differs.

3.4.2.1. *Measurings and coherence.* To give DGA_0 the structure of a module category we also need it to satisfy the coherence axiom. To show this we will use the concept of measurings developed in [Swe69]. We will briefly repeat the definition of measurings and some properties we will make use of, while referring the reader to [AJ13] for a more extensive coverage.

Definition 3.4.6. Let A, B be non-unital dg-algebras and C a non-counital dg-coalgebra. We say that a dg-linear morphism $f : C \otimes A \to B$ is a measuring if the adjoint morphism,

$$A \to \{C, B\},$$

is a morphism of non-unital dg-algebras. We denote the set of measurings from $C \otimes A$ to B by $\mathcal{M}(C, A, B)$.

As an immediate consequence of the definition we have the following.

PROPOSITION 3.4.7. *Let $v : C \otimes A \to B$ be a measuring, $f : A' \to A$ and $g : B \to B'$ be algebra maps and $h : C' \to C$ a coalgebra map. Then the composition*

$$g \circ v \circ (h \otimes f) : C' \otimes A' \to B'$$

is also a measuring.

LEMMA 3.4.8. *Let C, D be conilpotent dg-coalgebras and A a non-unital dg-algebra. Then the natural isomorphism of dg-vector spaces*

$$\phi : \{C \otimes D, A\} \to \{D, \{C, A\}\},$$

is a morphism of non-unital dg-algebras.

PROOF. Let $f : C \otimes D \to A$ and $g : C \otimes D \to A$ be morphisms of dg-vector spaces. The adjoint morphism $(f * g)^*$ of their convolution product is then given by the composition

$$D \xrightarrow{\Delta_D} D \otimes D \xrightarrow{(f \otimes g)^*} \{C \otimes C, A \otimes A\} \xrightarrow{\{\Delta_C, m_A\}} \{C, A\}.$$

By explicitly acting on an element $d \in D$ it can be seen that $(f * g)^* = f^* * g^*$. $\square$

PROPOSITION 3.4.9. *Let $f : D \otimes A \to A'$ and $g : C \otimes A' \to A''$ be measurings. Then the composition map*

$$(C \otimes D) \otimes A \cong C \otimes (D \otimes A) \xrightarrow{\mathrm{Id}_C \otimes f} C \otimes A' \xrightarrow{g} A'',$$

is also a measuring.

PROOF. The adjoint morphism is given by the composition

$$A \xrightarrow{f^*} \{D, A'\} \xrightarrow{\{D, g^*\}} \{D, \{C, A''\}\} \cong \{C \otimes D, A''\},$$

which we see is a composition of morphisms of non-unital dg-algebras using Lemma 3.4.8. □

PROPOSITION 3.4.10. *Let C be a non-counital dg-coalgebra, A a non-unital dg-algebra and V a dg-vector space. Then a dg-linear map*

$$f : C \otimes V \to A,$$

extends uniquely to a measuring

$$f : C \otimes T_0 V \to A.$$

PROOF. By the tensor-hom adjunction for dg-vector spaces, the free forgetful adjunction for dg-algebras, and the definition of measurings, we have natural equivalences

$$\mathrm{DGVec}(C \otimes V, A) \cong \mathrm{DGVec}(V, \underline{\mathrm{DGVec}}(C, A)) \cong$$
$$\mathrm{DGA}_0(T_0 V, \{C, V\}) \cong \mathcal{M}(C, T_0 V, A).$$

□

By Proposition 3.4.7 the assignment $(C, A, B) \mapsto \mathcal{M}(C, A, B)$ of Definition 3.4.6 extends to a functor

$$\mathcal{M}(-, -; -) : (\mathrm{coDGA}_0)^{\mathrm{op}} \times \mathrm{DGA}_0^{\mathrm{op}} \times \mathrm{DGA}_0 \to \mathrm{Set},$$

which we will refer to as the measurement functor. We will consider the measurement functor restricted to conilpotent coalgebras,

$$\mathcal{M}(-, -; -) : \left(\mathrm{coDGA}^{\mathrm{conil}}\right)^{\mathrm{op}} \times \mathrm{DGA}_0^{\mathrm{op}} \times \mathrm{DGA}_0 \to \mathrm{Set},$$

which we will refer to as the restricted measurement functor. The measurement functor is representable in each variable, which is shown in Section 4.1 of [AJ13].

The argument to show that it is also representable in the restricted case is similar which we briefly repeat here.

PROPOSITION 3.4.11. *The restricted measurement functor* $\mathcal{M}$ *is represented in each variable.*

PROOF. By definition of measuring, the restricted measurement functor is represented in the second variable by

$$(\{C, B\}, \epsilon : C \otimes \{C, B\} \to B)\,,$$

where ϵ is the counit of the tensor-hom adjunction for dg-vector spaces. The representability in the remaining variables now follows from the tensored and cotensored adjunctions constructed in Proposition 3.4.3 and Proposition 3.4.4. That is we have isomorphisms,

$$\begin{aligned}&\mathcal{M}(C, A, B) \cong \\ &\mathrm{DGA}_0(A, \{C, B\}) \cong \mathrm{DGA}_0(C \rhd A, B) \cong \mathrm{coDGA}^{\mathrm{conil}}(C, \overline{\mathrm{DGA}}_0(A, B)),\end{aligned}$$

natural in each variable. ☐

REMARK 3.4.12. Explicitly we have that the restricted measuring functor is represented in the third variable by

$$(C \rhd A, u : C \otimes A \to C \rhd A)\,.$$

Here u comes from that the inclusion map

$$i : C \otimes A \to T_0(C \otimes A),$$

induces a measuring

$$i' : C \otimes T_0 A \to T_0(C \otimes A),$$

by Proposition 3.4.10. Taking the coequaliser as in the construction of $C \rhd A$, in Proposition 3.4.4, gives the measuring u.

Similarly, the restricted measuring functor is represented in the first variable by

$$(\overline{\mathrm{DGA}}_0(A, B), \mathbf{ev} : \overline{\mathrm{DGA}}_0(A, B) \otimes A \to B).$$

Here the evaluation map $\mathbf{ev}$ is the composition

$$\overline{\mathrm{DGA}}_0(A, B) \otimes A \xrightarrow{u} \overline{\mathrm{DGA}}_0(A, B) \rhd A \xrightarrow{\epsilon_{\rhd}} B,$$

with $\epsilon_{\triangleright}$ denoting the counit of the tensoring-enrichment functor adjunction of Corollary 3.4.5.

We can now proceed with showing the semi-module structure of $(\mathrm{DGA}_0, \triangleright)$.

Lemma 3.4.13. *There exists a unique natural isomorphism*

$$\alpha : (C \otimes D) \triangleright A \to C \triangleright (D \triangleright A),$$

such that

$$\begin{array}{ccc} (C \otimes D) \otimes A & \xrightarrow{a} & C \otimes (D \otimes A) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u \circ (\mathrm{Id} \otimes u)} \\ (C \otimes D) \triangleright A & \overset{\alpha}{\dashrightarrow} & C \triangleright (D \triangleright A) \end{array}$$

commutes.

Proof. We first note that the composition map

$$u \circ (\mathrm{Id} \otimes u) \circ a : (C \otimes D) \otimes A \to C \triangleright (D \triangleright A),$$

is a measuring by Proposition 3.4.9. We furthermore claim it is the universal element representing the restricted measurement functor $\mathcal{M}(C \otimes D, A, -)$.

This comes from natural isomorphisms

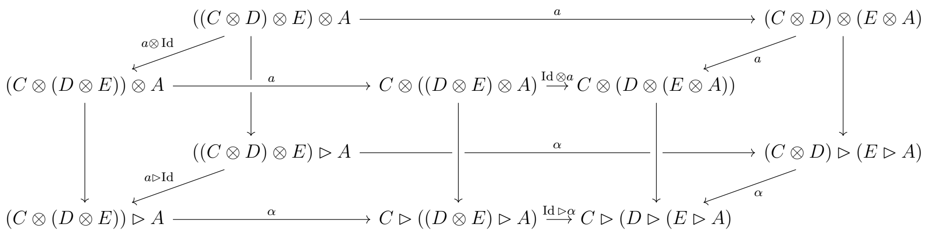
$$\mathcal{M}(C \otimes D, A, B) \cong \mathrm{DGA}_0(A, \{C \otimes D, B\}) \cong$$
$$\mathrm{DGA}_0(A, \{D, \{C, B\}\}) \cong \mathrm{DGA}_0(D \triangleright A, \{C, B\}) \cong \mathrm{DGA}_0(C \triangleright (D \triangleright A), B),$$

where we used Lemma 3.4.8 in the third step. Hence we have a natural bijection between measurings $(C \otimes D) \otimes A \to B$ and non-unital dg-algebra morphisms $C \triangleright (D \triangleright A) \to B$. It follows that the measuring $u \circ (\mathrm{Id} \otimes u) \circ a$ is a universal element. By the uniqueness of universal elements, α is then the unique natural isomorphism. □

We now have sufficient background to prove the main result of this section. The reader should however note that the following result also follows as a straightforward corollary from Theorem 4.1.18 in [AJ13].

Theorem 3.4.14. $(\mathrm{DGA}_0, \triangleright, \overline{\mathrm{DGA}}_0, \{-,-\})$ *is a closed* $\mathrm{coDGA}^{\mathrm{conil}}$*-module.*

Proof. It remains to show that the coherence axiom is satisfied for the associator α constructed in the Lemma 3.4.13. Consider the diagram



where the vertical arrows consist of the universal element u applied as demanded by the diagram. We conclude that every vertical face commutes by Lemma 3.4.13 and that the top face commutes by the monoidal structure of the tensor product $\otimes$ on vector spaces. Thus after precomposition with the morphism $u : ((C\otimes D)\otimes E)\otimes A \to ((C \otimes D) \otimes E) \rhd A$ the bottom face commutes. But by the universal property of u we have that u is right-cancellative on algebra morphisms. Hence the bottom face commutes. □

3.5. Semi-monoidal model structure on $\mathrm{coDGA}^{\mathrm{conil}}$

For showing that $\mathrm{coDGA}^{\mathrm{conil}}$ is a semi-monoidal model category, we will first establish that the tensor product functor of $\mathrm{coDGA}^{\mathrm{conil}}$ preserves (acyclic) cofibrations in each variable separately.

Lemma 3.5.1. *The tensor product functor*

$$\otimes : \mathrm{coDGA}^{\mathrm{conil}} \times \mathrm{coDGA}^{\mathrm{conil}} \to \mathrm{coDGA}^{\mathrm{conil}},$$

preserves cofibrations and weak equivalences in each variable separately.

Proof. We first note that the forgetful functor $U : \mathrm{coDGA}^{\mathrm{conil}} \to \mathrm{DGVec}$ commutes with the tensor product and both preserves and reflects cofibrations. The preservation of cofibrations under the tensor product of $\mathrm{coDGA}^{\mathrm{conil}}$ then follows from the DGVec case.

It remains to show the preservation of weak equivalences. Since the class of weak equivalences is the closure of the class of filtered quasi-isomorphisms under the 2 out of 3 property it suffices to show the preservation of filtered quasi-isomorphisms. Thus let $f : C \to D$ be a filtered quasi-isomorphism and let E be a conilpotent coalgebra. By assumption there exist admissible filtrations F_C and F_D, of C and D respectively, such that

$$\mathrm{gr}^i f : \mathrm{gr}^i_C F_C \to \mathrm{gr}^i_D F_D$$

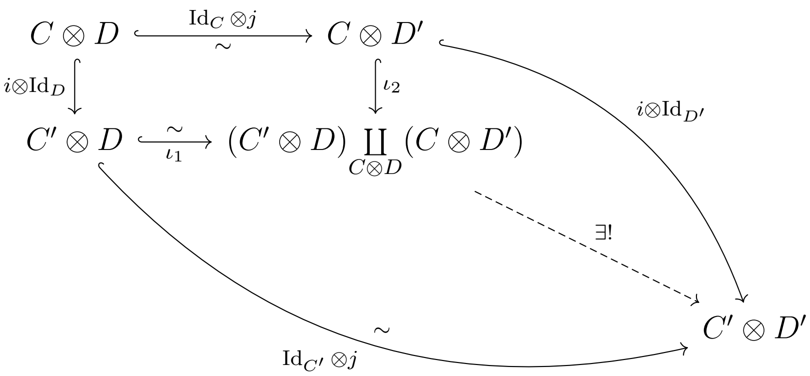
is a quasi-isomorphism in each degree. We define filtrations $F_{C\otimes E} := F_C \otimes E$ and $F_{D\otimes E} := F_D \otimes E$ and note that they are admissible. Further noting that $\mathrm{gr}_{C\otimes E} \cong \mathrm{gr}_C \otimes E$ we get an induced quasi-isomorphism,

$$\begin{array}{ccc} \mathrm{gr}^i F_{C\otimes E} & \xrightarrow{\sim} & \mathrm{gr}^i F_{D\otimes E} \\ \cong & & \cong \\ \mathrm{gr}^i F_C \otimes E & \xrightarrow[\mathrm{gr}^i f\otimes E]{\sim} & \mathrm{gr}^i F_D \otimes E, \end{array}$$

using that the tensor product of dg-vector spaces preserves quasi-isomorphisms. Thus $f \otimes E$ is a filtered quasi-isomorphism. □

THEOREM 3.5.2. $(\mathrm{coDGA}^{\mathrm{conil}}, \otimes)$ *is a symmetric semi-monoidal model category.*

PROOF. We have to show that $\otimes$ is a Quillen bifunctor. Let $i : C \to C'$ be a cofibration and $j : D \to D'$ an (acyclic) cofibration in $\mathrm{coDGA}^{\mathrm{conil}}$. The pushout diagram is



where we use that the (acyclic) cofibrations are closed under pushout, and Lemma 3.5.1.

We see that in the acyclic case we get that the pushout map is a weak equivalence by the 2 out of 3 property. That the pushout map is injective follows from the dg-vector space case as colimits and cofibrations are preserved by the forgetful functor to DGVec, which furthermore commutes with the tensor product. □

3.6. Homotopical enrichment of DGA_0

To show that DGA_0 is a model $\mathrm{coDGA}^{\mathrm{conil}}$ category we need to show that the tensoring functor $\triangleright$ is a Quillen bifunctor. However since cofibrations in $\mathrm{coDGA}^{\mathrm{conil}}$ and fibrations in DGA_0 are particularly easy to work with, we will make use of the equivalent condition for the cotensoring functor $\{-,-\}$. That is we will show that for every cofibration $i : C \to C'$ in $\mathrm{coDGA}^{\mathrm{conil}}$ and every fibration $j : A \to A'$ in

DGA_0 the induced map

$$\{C', A\} \to \{C, A\} \times_{\{C,A'\}} \{C', A'\},$$

is a cofibration, which furthermore is acyclic if either i or j is.

LEMMA 3.6.1. *The convolution algebra functor*

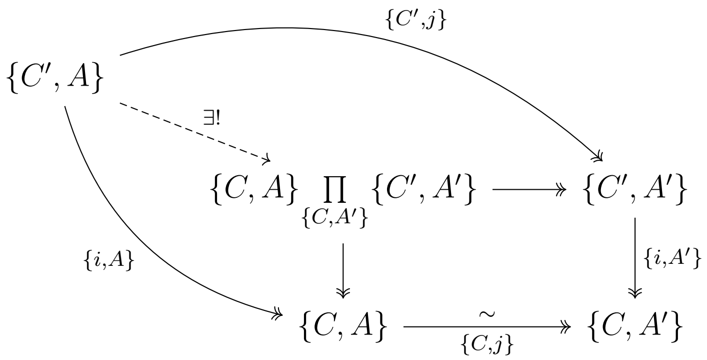
$$\{-,-\} : \left(\mathrm{coDGA}^{\mathrm{conil}}\right)^{\mathrm{op}} \times \mathrm{DGA}_0 \to \mathrm{DGA}_0,$$

takes (acyclic) cofibrations in the first variable and (acyclic) fibrations in the second variable to (acyclic) fibrations separately.

PROOF. That we get fibrations in either case is immediate. For the second part we show the stronger statement that the convolution algebra functor preserves quasi-isomorphisms. This is sufficient as every weak equivalence in $\mathrm{coDGA}^{\mathrm{conil}}$ by necessity is also a quasi-isomorphism. Next note that the forgetful functor to DGVec commutes with the cotensoring functor and both preserves and reflects quasi-isomorphisms. That is the convolution algebra functor is taken to the internal hom of dg-vector spaces. As the internal hom of dg-vector spaces is exact the preservation of quasi-isomorphisms follows. □

THEOREM 3.6.2. $(\mathrm{DGA}_0, \overline{\mathrm{DGA}}_0, \triangleright, \{-,-\})$ *is a* $\mathrm{coDGA}^{\mathrm{conil}}$*-model category.*

PROOF. Let $i : C \hookrightarrow C'$ be a cofibration in $\mathrm{coDGA}^{\mathrm{conil}}$ and $j : A \twoheadrightarrow A'$ a (acyclic) fibration in DGA_0. The pullback diagram corresponding to Proposition 3.3.3 ii) is



where we use that (acyclic) fibrations are closed on pullback, and Lemma 3.6.1. It follows from the 2 out of 3 property that the induced morphism is a weak equivalence. Similarly, had we instead started assuming that i was acyclic, we would've reached the same conclusion. That the pullback product map is surjective follows from the dg-vector space case as limits and fibrations are preserved by the forgetful functor to DGVec which furthermore commutes with the cotensoring. □

3.7. The com-Lie case

We have so far, in Sections 3.4 to 3.6, shown the homotopical enrichment corresponding to the case of associative dg Koszul duality. We will now similarly proceed with homotopical enrichment in the com-Lie case of Koszul duality. That is we will show that DGLA is a semi-module model category over $\mathrm{coCDGA}^{\mathrm{conil}}$. The procedure will be completely analogous to the associative case.

We first note that $(\mathrm{coCDGA}^{\mathrm{conil}}, \otimes)$ is symmetric semi-monoidal, which follows from the $\mathrm{coDGA}^{\mathrm{conil}}$ case. We have that it is also closed by the following.

THEOREM 3.7.1. *The category* $(\mathrm{coCDGA}^{\mathrm{conil}}, \otimes)$ *is closed symmetric semi-monoidal.*

PROOF. For a conilpotent cocommutative cofree coalgebra $S^{\mathrm{co}}V$ we define the internal hom $\underline{\mathrm{coCDGA}}^{\mathrm{conil}}(C, S^{\mathrm{co}}V)$ as

$$\underline{\mathrm{coCDGA}}^{\mathrm{conil}}(C, S^{\mathrm{co}}V) := S^{\mathrm{co}}\,\underline{\mathrm{DGVec}}(C, V).$$

For an arbitrary cocommutative coalgebra D we can write it as an equaliser

$$D \longrightarrow S^{\mathrm{co}}D \rightrightarrows (S^{\mathrm{co}})^2 D.$$

We then define the internal hom functor $\underline{\mathrm{coCDGA}}^{\mathrm{conil}}(C, D)$ as the equaliser of

$$\begin{array}{ccc} \underline{\mathrm{coCDGA}}^{\mathrm{conil}}(C, S^{\mathrm{co}}D) & \rightrightarrows & \underline{\mathrm{coCDGA}}^{\mathrm{conil}}(C, (S^{\mathrm{co}})^2 D) \\ \cong & & \cong \\ S^{\mathrm{co}}\,\underline{\mathrm{DGVec}}(C, D) & \rightrightarrows & S^{\mathrm{co}}\,\underline{\mathrm{DGVec}}(C, S^{\mathrm{co}}D). \end{array}$$

That this functor is right adjoint to the tensoring functor now follows by the same argument as in the proof of Theorem 3.4.1. □

We will next establish the $\mathrm{coCDGA}^{\mathrm{conil}}$-module structure of DGLA.

PROPOSITION 3.7.2. *There exists a functor*

$$\overline{\mathrm{DGLA}}(-,-) : \mathrm{DGLA}^{\mathrm{op}} \times \mathrm{DGLA} \to \mathrm{coCDGA}^{\mathrm{conil}},$$

such that $\overline{\mathrm{DGLA}}(-, \mathfrak{h})$ *is right adjoint to the opposite convolution algebra functor* $\{-, \mathfrak{h}\}^{\mathrm{op}}$ *for each Lie algebra* $\mathfrak{h} \in \mathrm{DGLA}$. *We will refer to* $\overline{\mathrm{DGLA}}$ *as the enrichment functor.*

PROOF. For a free Lie algebra $T_{\mathrm{Lie}}V$ we have the natural isomorphism

$$\mathrm{DGLA}(T_{\mathrm{Lie}}V, \{C, \mathfrak{h}\}) \cong \mathrm{DGVec}(V, \underline{\mathrm{DGVec}}(C, \mathfrak{h})) \cong \mathrm{DGVec}(V \otimes C, \mathfrak{h}) \cong$$

$$\mathrm{DGVec}(C, \underline{\mathrm{DGVec}}(V, \mathfrak{h})) \cong \mathrm{coCDGA}^{\mathrm{conil}}(C, S^{\mathrm{co}}(\underline{\mathrm{DGVec}}(V, \mathfrak{h})),$$

so we can define the enriched hom functor as

$$\overline{\mathrm{DGLA}}(T_{\mathrm{Lie}}V, \mathfrak{h}) := S^{\mathrm{co}}\,\underline{\mathrm{DGVec}}(V, \mathfrak{h}).$$

Given an arbitrary $\mathfrak{g} \in \mathrm{DGLA}$ we write it as a coequaliser

$$T^2_{\mathrm{Lie}}\mathfrak{g} \rightrightarrows T_{\mathrm{Lie}}\mathfrak{g} \longrightarrow \mathfrak{g}.$$

We then define $\overline{\mathrm{DGLA}}(\mathfrak{g}, \mathfrak{h})$ as the equaliser of

$$\begin{array}{ccc} \overline{\mathrm{DGLA}}(T^2_{\mathrm{Lie}}\mathfrak{g}, \mathfrak{h}) & \rightrightarrows & \overline{\mathrm{DGLA}}(T_{\mathrm{Lie}}\mathfrak{g}, \mathfrak{h}) \\ \cong & & \cong \\ S^{\mathrm{co}}\,\underline{\mathrm{DGVec}}(T_{\mathrm{Lie}}\mathfrak{g}, \mathfrak{h}) & \rightrightarrows & S^{\mathrm{co}}\,\underline{\mathrm{DGVec}}(\mathfrak{g}, \mathfrak{h}), \end{array}$$

where the bottom equaliser arrows corresponds to the top ones by pre- and post composition by the natural isomorphisms. Since $\mathrm{coCDGA}^{\mathrm{conil}}(C, -)$ preserves limits we get the desired natural bijection

$$\mathrm{DGLA}(\mathfrak{g}, \{C, \mathfrak{h}\}) \cong \mathrm{coCDGA}^{\mathrm{conil}}(C, \overline{\mathrm{DGLA}}(\mathfrak{g}, \mathfrak{h})).$$

□

We construct the tensoring functor similarly.

PROPOSITION 3.7.3. *There exists a functor*

$$(-) \rhd (-) : \mathrm{coCDGA}^{\mathrm{conil}} \times \mathrm{DGLA} \to \mathrm{DGLA},$$

such that $C \rhd (-)$ is left adjoint to the convolution algebra functor $\{C, -\}$ for each coalgebra $C \in \mathrm{coCDGA}^{\mathrm{conil}}$. We will refer to $\rhd$ as the tensoring functor.

PROOF. For a free dg-Lie algebra $T_{\mathrm{Lie}}V$ and $\mathfrak{h}$ a dg-Lie algebra we have the natural isomorphism

$$\mathrm{DGLA}(T_{\mathrm{Lie}}V, \{C, \mathfrak{h}\}) \cong \mathrm{DGVec}(V, \{C, \mathfrak{h}\})$$

$$\cong \mathrm{DGVec}(C \otimes V, \mathfrak{h}) \cong \mathrm{DGLA}(T_{\mathrm{Lie}}(C \otimes V), \mathfrak{h}).$$

Hence we define

$$C \rhd T_{\mathrm{Lie}}V := T_{\mathrm{Lie}}(C \otimes V).$$

Given an arbitrary Lie algebra $\mathfrak{g}$, we write it as a coequaliser and define $C \rhd \mathfrak{g}$ the coequaliser of

$$\begin{array}{ccc} C \rhd T^2_{\mathrm{Lie}}\mathfrak{g} & \rightrightarrows & C \rhd T_{\mathrm{Lie}}\mathfrak{g} \\ \cong & & \cong \\ T_{\mathrm{Lie}}(C \otimes_k T_{\mathrm{Lie}}\mathfrak{g}) & \rightrightarrows & T_{\mathrm{Lie}}(C \otimes_k \mathfrak{g}), \end{array}$$

where the bottom equaliser arrows corresponds to the top ones by pre- and post composition by the natural isomorphisms. Since $\mathrm{DGLA}(-, \mathfrak{h})$ takes colimits to limits we get the desired natural bijection

$$\mathrm{DGLA}(\mathfrak{g}, \{C, \mathfrak{h}\}) \cong \mathrm{DGLA}(C \rhd \mathfrak{g}, \mathfrak{h}).$$

□

By combining Proposition 3.7.2 and Proposition 3.7.3 we also get a third adjunction between the tensoring and the enrichement functor.

Corollary 3.7.4. *The tensoring* $(-) \rhd \mathfrak{h}$ *is left adjoint to the enrichment functor* $\overline{\mathrm{DGLA}}(\mathfrak{h}, -)$ *for each Lie algebra* $\mathfrak{h} \in \mathrm{DGLA}$*.*

In summary we have established adjunctions

$$\mathrm{coCDGA}^{\mathrm{conil}} \underset{\overline{\mathrm{DGLA}}(-,\mathfrak{h})}{\overset{\{-,\mathfrak{h}\}^{\mathrm{op}}}{\rightleftarrows}} \mathrm{DGLA}^{\mathrm{op}},$$

$$\mathrm{DGLA} \underset{\{C,-\}}{\overset{C \rhd (-)}{\rightleftarrows}} \mathrm{DGLA},$$

and

$$\mathrm{coCDGA}^{\mathrm{conil}} \underset{\overline{\mathrm{DGLA}}(\mathfrak{h},-)}{\overset{(-) \rhd A}{\rightleftarrows}} \mathrm{DGLA}\,.$$

As in the associative case we make use of the concept of measurings to show the coherence axiom for the module structure of DGLA. Adapted to the Lie algebra case the definition becomes the following.

Definition 3.7.5. Let A, B be dg-Lie algebras and C a cocommutative non-counital dg-coalgebra. We say that a dg-linear morphism $f : C \otimes A \to B$ is a measuring if the adjoint morphism

$$A \to \{C, B\},$$

is a morphism of Lie algebras.

As in the associative case we will denote by $\mathcal{M}(C, A, B)$ the set of measurings from $C \otimes A \to B$ and note this extends to a functor.

PROPOSITION 3.7.6. *The restricted measurement functor*

$$\mathcal{M}(-,-;-) : \left(\mathrm{coCDGA}^{\mathrm{conil}}\right)^{\mathrm{op}} \times \mathrm{DGLA}^{\mathrm{op}} \times \mathrm{DGLA} \to \mathrm{Set},$$

is represented in each variable.

PROOF. Same as the proof of Proposition 3.4.11. □

THEOREM 3.7.7. $(\mathrm{DGLA}, \triangleright, \overline{\mathrm{DGLA}}, \{-,-\})$ *is a closed* $\mathrm{coCDGA}^{\mathrm{conil}}$*-module.*

PROOF. Same as the proof of Lemma 3.4.13 and Theorem 3.4.14. □

We are now ready to proceed with the homotopical perspective, which also is shown fully analogously to the associative case.

PROPOSITION 3.7.8. $(\mathrm{coCDGA}^{\mathrm{conil}}, \otimes)$ *is a symmetric semi-monoidal model category.*

PROOF. Same as the proof of Theorem 3.5.2. □

LEMMA 3.7.9. *The convolution algebra functor*

$$\{-,-\} : \left(\mathrm{coCDGA}^{\mathrm{conil}}\right)^{\mathrm{op}} \times \mathrm{DGLA} \to \mathrm{DGLA},$$

takes (acyclic) cofibrations in the first variable and (acyclic) fibrations in the second variable to (acyclic) fibrations separately.

PROOF. Same as the proof of Lemma 3.6.1. □

THEOREM 3.7.10. $(\mathrm{DGLA}, \overline{\mathrm{DGLA}}(-,-), \triangleright, \{-,-\})$ *is a* $\mathrm{coCDGA}^{\mathrm{conil}}$*-module model category.*

PROOF. Using Lemma 3.7.9 we see that the same argument as in the proof of Theorem 3.6.2 goes through. □

3.8. Semi-enriched Koszul duality

Having established the semi-module category structures of DGA_0 and DGLA we will now return our attention to Koszul duality. Our main aim is to establish Theorem 1.0.1 and Theorem 1.0.2. That is we will show that the Quillen equivalences in both the associative and the com-Lie case of Koszul duality becomes semi-module Quillen equivalences.

As pointed out in [AJ13] the bar and cobar constructions are directly related to the constructed enrichment functor and the tensoring functor respectively. Specifically we have the following result.

Proposition 3.8.1. *There exist natural isomorphisms*

$$\Omega C \cong C \rhd \mathbf{mc} \quad \text{and} \quad BA \cong \overline{\mathrm{DGA}}_0(\mathbf{mc}, A),$$

where $\mathbf{mc}$ *is the universal Maurer-Cartan algebra.*

Proof. By the tensoring adjunctions and representability of the Maurer-Cartan functor MC we have natural isomorphisms

$$\begin{aligned}&\mathrm{DGA}_0(C \rhd \mathbf{mc}, A) \cong \mathrm{coDGA}^{\mathrm{conil}}(C, \overline{\mathrm{DGA}}_0(\mathbf{mc}, A))\\ &\cong \mathrm{DGA}_0(\mathbf{mc}, \{C, A\}) \cong \mathrm{MC}(\{C, A\}).\end{aligned}$$

Combining this with the bar-cobar adjunction

$$\mathrm{DGA}_0(\Omega C, A) \cong \mathrm{MC}(\{C, A\}) \cong \mathrm{coDGA}^{\mathrm{conil}}(C, BA),$$

we have natural isomorphisms

$$\begin{aligned}&\mathrm{DGA}_0(\Omega C, A) \cong \mathrm{DGA}_0(C \rhd \mathbf{mc}, A),\\ &\mathrm{coDGA}^{\mathrm{conil}}(C, BA) \cong \mathrm{coDGA}^{\mathrm{conil}}(C, \overline{\mathrm{DGA}}_0(\mathbf{mc}, A)),\end{aligned}$$

from which the statement follows by the Yoneda lemma. □

Similarly for the DGLA case we have the analogous result.

Proposition 3.8.2. *There exist natural isomorphisms*

$$\Omega C \cong C \rhd \mathbf{mc}_{\mathrm{Lie}} \quad \text{and} \quad BA \cong \overline{\mathrm{DGLA}}(\mathbf{mc}_{\mathrm{Lie}}, A),$$

where $\mathbf{mc}_{\mathrm{Lie}}$ *is the universal Maurer-Cartan Lie algebra.*

PROOF. By the tensoring adjunction and representability of the Maurer-Cartan functor $\mathrm{MC}_{\mathrm{Lie}}$ we have natural isomorphisms

$$\begin{aligned}\mathrm{DGLA}(C \triangleright \mathbf{mc}_{\mathrm{Lie}}, \mathfrak{g}) &\cong \mathrm{coCDGA}^{\mathrm{conil}}(C, \overline{\mathrm{DGLA}}(\mathbf{mc}_{\mathrm{Lie}}, \mathfrak{g}))\\ &\cong \mathrm{DGLA}(\mathbf{mc}_{\mathrm{Lie}}, \{C, \mathbf{g}\}) \cong \mathrm{MC}_{\mathrm{Lie}}(\{C, \mathbf{g}\}).\end{aligned}$$

Combining this with the bar-cobar adjunction

$$\mathrm{DGLA}(\Omega C, \mathfrak{g}) \cong \mathrm{MC}_{\mathrm{Lie}}(\{C, \mathfrak{g}\}) \cong \mathrm{coCDGA}^{\mathrm{conil}}(C, B\mathfrak{g}),$$

we have natural isomorphisms

$$\begin{aligned}&\mathrm{DGLA}(\Omega C, \mathfrak{g}) \cong \mathrm{DGLA}(C \triangleright \mathbf{mc}_{\mathrm{Lie}}, \mathfrak{g}),\\ &\mathrm{coCDGA}^{\mathrm{conil}}(C, B\mathfrak{g}) \cong \mathrm{coCDGA}^{\mathrm{conil}}(C, \overline{\mathrm{DGLA}}(\mathbf{mc}_{\mathrm{Lie}}, \mathfrak{g})),\end{aligned}$$

from which the statement follows by the Yoneda lemma. □

As a consequence of Proposition 3.8.1 we see that the bar-cobar adjunction, in the associative case, is a $\mathrm{coDGA}^{\mathrm{conil}}$-module Quillen equivalence. Similarly Proposition 3.8.2 implies that the bar-cobar adjunction, in the com-Lie case, upgrades to a $\mathrm{coCDGA}^{\mathrm{conil}}$-module Quillen equivalence. As a consequence, we have established Theorem 1.0.1 and Theorem 1.0.2.

REMARK 3.8.3. Note that it also follows that the bar construction B, in both the associative and the com-Lie case, is a quasi-strong semi-module functor. By this we mean that it has the structure of a lax semi-module functor $(B, m : C \otimes BA \to B(C \triangleright A))$ that induces a (strong) semi-module functor on homotopy categories. In particular this is the case when the natural transformation m is a weak equivalence.

Explicitly the weak equivalence in our case is given by

$$C \otimes BA \xrightarrow{\sim} B\Omega(C \otimes BA) \xrightarrow{\cong} B(C \triangleright \Omega BA) \xrightarrow{\sim} B(C \triangleright A),$$

here in the notation of the associative case.

REMARK 3.8.4. It may seem that our results should generalise to the operadic context of Koszul duality. This is however not the case as can be seen from considering a third case of Koszul duality between the category of conilpotent dg-Lie algebras, $\mathrm{coDGLA}^{\mathrm{conil}}$, and the category of commutative non-unital dg-algebras cDGA_0. This case was established in [LM15, Theorem 9.16] and shows that there is a Quillen equivalence

$$\mathrm{coDGLA}^{\mathrm{conil}} \underset{B}{\overset{\Omega}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{cDGA}_0\,.$$

The problem here, in regards to extending our results, is that $\mathrm{coDGLA}^{\mathrm{conil}}$ does not have the notion of a tensor product. Instead one could consider the monoidal structure given by the direct product, which indeed gives a closed monoidal structure on $\mathrm{coDGLA}^{\mathrm{conil}}$ albeit with a quite different internal hom functor from the other cases. However one quickly runs into trouble with defining a $\mathrm{coDGLA}^{\mathrm{conil}}$-module structure for cDGA_0 as we don't have the concept of a convolution algebra or the Sweedler theory adjunctions to rely on.

CHAPTER 4

Monoidal and module category properties of (co)algebras

In Chapter 3 we showed a version of Koszul duality that upgrades to a semi-module category equivalence. We will in this chapter proceed to show that this induces semi-module structures on the homotopy category of non-unital dg-algebras $\mathrm{Ho}(\mathrm{DGA}_0)$ and on the homotopy category of dg-Lie algebras Ho(DGLA), over the the homotopy category of reduced simplicial sets $\mathrm{Ho}(\mathrm{sSet}_0)$. We also consider if this structure can be used to compute simplicial mapping spaces of DGA_0 and DGLA, however the lack of a monoidal unit seems to impede this and it appears we can't recover this information directly from the semi-module structures on the respective categories.

4.1. Reduced simplicial sets and simplicial monoids

We will in this section provide a reminder on reduced simplicial sets and simplicial monoids and their respective model structures. Remember that a simplicial set X is *reduced* if it has a single vertex. We will denote the category of reduced simplicial sets by sSet_0. For background material on the homotopy theory of simplicial sets we refer the reader to [GJ09, Chapter I].

The category of reduced simplicial sets sSet_0 comes with two standard model structures. The first of these is the Quillen model structure given by the following.

- $\xrightarrow{\sim}$ Weak homotopy equivalences,
- $\hookrightarrow$ Monomorphisms,
- $\twoheadrightarrow$ RLP with respect to acyclic cofibrations.

This model structure appears in [GJ09, Proposition 6.2], and we will use sSet_0 to denote this model structure. The Quillen model structure is the right transferred model structure with respect to the Quillen adjunction

$$\mathrm{sSet} \underset{\iota}{\overset{\mathrm{Red}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{sSet}_0,$$

where ι is the inclusion and the functor Red identifies all vertices of a simplicial set. All objects are cofibrant in sSet_0 while the fibrant objects are reduced Kan complexes.

Definition 4.1.1. A reduced simplicial set X is a *Kan complex* if all horns of X have fillers i.e. there exists a lift in the diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

for all $0 \leq i \leq n$.

While a reduced Kan complex may deserve to go by the name "quasi-group", we avoid this term as it conflicts with its usage in algebra.

The other standard model structure is the Joyal model structure given by the following.

- $\xrightarrow{\sim}$ Weak categorical equivalences,
- $\hookrightarrow$ Monomorphisms,
- $\twoheadrightarrow$ RLP with respect to acyclic cofibrations.

This model structure was introduced in [CHL21] and further shown to be a simplicial model category in [Bur21]. Note that this differs from the Joyal model structure of all simplicial sets qCat, which is known not to be a simplicial model category. We will denote the Joyal model structure on reduced simplicial sets by qCat_0. This model structure is also referred to as the model structure for quasi-monoids.

The qCat_0 model category structure is the right transferred model structure along the Quillen adjunction

$$\mathrm{qCat} \underset{\iota}{\overset{\mathrm{Red}}{\rightleftarrows}} \mathrm{qCat}_0 .$$

All objects are cofibrant in qCat_0 while the fibrant objects are *reduced weak Kan complexes* or, as we will refer to them, *quasi-monoids*.

Definition 4.1.2. A reduced simplicial set X is a *quasi-monoid* if all inner horns of X have fillers i.e. there exists a lift in the diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

for all $0 < i < n$.

The Quillen model structure on reduced simplicial sets sSet_0 is a left Bousfield localization of the qCat_0 model structure, which is shown in [Bur21]. Thus we have a Quillen adjunction

$$\mathrm{qCat}_0 \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \mathrm{sSet}_0 \,.$$

We also have an adjunction between reduced Kan complexes $\mathrm{Fib}(\mathrm{sSet}_0)$ and quasi-monoids $\mathrm{Fib}(\mathrm{qCat}_0)$ given by

$$\mathrm{Fib}(\mathrm{sSet}_0) \underset{\mathrm{Core}}{\overset{\mathrm{Id}}{\rightleftarrows}} \mathrm{Fib}(\mathrm{qCat}_0),$$

which is the restriction to reduced simplicial sets of the usual inclusion core adjunction found in e.g. [Lur09, Proposition 1.2.5.3]. The Core functor here takes a quasi-monoid to its maximal Kan subcomplex.

The Quillen model structure on sSet_0 is also related to the category of pointed simplicial $\mathrm{sSet}^{*/}$ sets by the Quillen adjunction,

$$\mathrm{sSet}_0 \underset{\mathscr{R}}{\overset{\iota}{\rightleftarrows}} \mathrm{sSet}^{*/} \,.$$

Here ι is the inclusion while the functor $\mathscr{R}$ takes a simplicial set to the the subsimplicial set whose faces all have the marked point $*$ as their only vertex.

Similarly for the Joyal model structures qCat_0 and $\mathrm{qCat}^{*/}$ we have a Quillen adjunction

$$\mathrm{qCat}_0 \underset{\mathscr{R}}{\overset{\iota}{\rightleftarrows}} \mathrm{qCat}^{*/} \,.$$

Of particular interest for us is that this means that sSet_0 and qCat_0 have two semi-monoidal structures to consider, the Cartesian product $\times$ and the smash product $\wedge$. Both sSet_0 and qCat_0 are semi-monoidal model categories with either of these structures. Indeed in the case of the Cartesian product these categories are even monoidal model categories.

4.1.1. Simplicial groups and simplicial monoids. The sSet_0 and the qCat_0 model structures are Quillen equivalent to the standard model structures on simplicial groups sGrp and simplicial monoids sMon respectively. Recall that a simplicial group is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Grp}$ while a simplicial monoid is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Mon}$. Equivalently we may consider them to be simplicial sets with a group or monoid action respectively. Note also that a simplicial monoid is the same thing as a simplicial enriched category with one object.

The model structure on simplicial groups sGrp was shown in [Qui67, II.3 Theorem 2] and is given by the following.

- $\xrightarrow{\sim}$ Homotopy weak equivalences,
- $\twoheadrightarrow$ Kan fibrations,
- $\hookrightarrow$ LLP with respect to acyclic fibrations.

The Quillen model structure is related to the model structure of sGrp by the Quillen equivalence

$$\mathrm{sSet}_0 \underset{\overline{W}}{\overset{G}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{sGrp},$$

which is shown in [GJ09, Chapter V]. Here the functor G is the simplicial loop space functor and $\overline{W}$ the simplicial classifying space functor.

Similarly the category of simplicial monoids sMon has a model structure shown in [Qui69, II.4 Theorem 4]. The model structure is given by the following

- $\xrightarrow{\sim}$ Weak homotopy equivalences,
- $\twoheadrightarrow$ Kan fibrations,
- $\hookrightarrow$ LLP with respect to acyclic fibrations.

The Joyal model structure is related to the model structure of sMon by the Quillen equivalence,

$$\mathrm{qCat}_0 \underset{N^{\mathrm{hc}}}{\overset{\mathfrak{C}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{sMon},$$

which is shown in [Bur21, Chapter 3]. We borrow terminology from the non-reduced case and refer to $\mathfrak{C}$ as the *rigidification* functor or the *path category* functor and N^{hc} as the *homotopy coherent nerve* functor.

4.2. The normalised chain coalgebra functor

We next turn our attention to the connection between simplicial sets and dg-coalgebras. To do this we begin by reviewing the properties of the normalised chain coalgebra functor. A more extensive introduction to these topics can be found in [Nei10, Chapter 10.15] while we refer the reader to [Mac63, Chapter VIII] for the proofs.

Given a simplicial set X we can form its *unnormalised chains* complex $C(X)$ as the graded vector space over X with differential acting on a generator, i.e. an

n-simplex $\sigma \in X_n$, by

$$d(\sigma) = \sum_{i=0}^{n} (-1)^i d_i(\sigma),$$

where d_i are the face maps of X.

The unnormalised chains functor C has a lax monoidal structure given by the *Alexander-Whitney* map

$$\Delta : C(X \times Y) \to C(X) \otimes C(Y)$$

given on a generator $(x, y) \in X_n \times Y_n$ by

$$\Delta(x, y) = \sum_{i=0}^{n} {}_i x \otimes y_{n-i},$$

where ${}_i x := d_{i+1} \dots d_n x$ is the i:th front face of x and $y_{n-i} := d_0^i y$ the $(n-i)$:th back face of y.

The unnormalised chains functor C also has a colax monoidal structure given by the *Eilenberg-Zilber* map

$$\nabla : C(X) \otimes C(Y) \to C(X \times Y),$$

which is defined on generators $x \otimes y$ as

$$\nabla(x \otimes y) = \sum_{(\mu,\nu) \in \mathrm{Sh}(p,q)} \mathrm{sgn}(\mu, \nu) s_\nu x \times s_\mu y$$

where s_μ and s_ν are the iterated degeneracy maps given by

$$s_\mu := s_{\mu_p - 1} \circ \cdots \circ s_{\mu_1 - 1}.$$

The *Eilenberg-Zilber Theorem* states that the Alexander-Whitney map and the Eilenberg-Zilber map constitute chain homotopy equivalences, i.e. $\nabla \circ \Delta \simeq \mathrm{Id}_{C(X \times Y)}$ and $\Delta \circ \nabla \simeq \mathrm{Id}_{C(X) \otimes C(Y)}$. In particular both the Alexander-Whitney and the Eilenberg-Zilber maps are quasi-isomorphisms. We again refer to [Mac63, Chapter VIII] for a proof of this.

4.2.1. Normalised chains. Instead of working with the unnormalised chains complex it is often more convenient to work with its normalisation, the *normalised chains* complex. Let $D(X)$ be the subcomplex spanned by the degenerate simplices of X. The subcomplex $D(X)$ is acyclic which is shown in [Mac63, Chapter VIII.6]. The *normalised chains* complex $C^N(X)$ is defined as the quotient $C(X)/D(X)$. Since $D(X)$ is acyclic the projection $C(X) \to C^N(X)$ is a quasi-isomorphism.

Restricted to normalised chains, the composition $\Delta \circ \nabla : C^N(X) \otimes C^N(Y) \to C^N(X) \otimes C^N(Y)$ becomes the identity functor and hence ∇ is a deformation retract of Δ.

Letting X for convenience now be a pointed simplicial set, the Alexander-Whitney map induces a dg-coalgebra structure on $C^N(X)$ with comultiplication given by the composition

$$C^N(X) \xrightarrow{C(\Delta)} C^N(X \times X) \xrightarrow{\Delta} C^N(X) \otimes C^N(X),$$

where we have also used Δ to denote the diagonal map $X \to X \times X$. As we have demanded X to be pointed we have a canonical choice of counit

$$\epsilon : C^N(X) \to k[*] \cong k.$$

We thus view the normalised chains coalgebra C^N as a coaugmented counital dg-coalgebra. We refer to [Mac63, Chapter VIII] for a proof that the comultiplication is coassociative.

Working with reduced simplicial sets, the normalised chain coalgebra takes a particularly nice form.

Proposition 4.2.1. *Let X be a pointed simplicial set. Then the normalised chain coalgebra $C^N(X)$ of X is conilpotent if and only if X is reduced.*

Proof. Each element of $x \in C^N(X)_0$ is an atom i.e. having comultiplication $x \mapsto x \otimes x$ and thus not conilpotent. On the other hand if X is reduced we have that $\Delta^n(x) = 0$ for every $x \in C^N(X)_n$ as the Alexander-Whitney map reduces the degree. □

As a consequence, we see that the normalised chain coalgebra functor restricts to a functor

$$C^N : \mathrm{sSet}_0 \to \mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}},$$

which we call the *normalised chain coalgebra* functor. We have here used $\mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}}$ to denote that we here view the category of conilpotent dg-coalgebras as being coaugmented as opposed to non-counital.

Remark 4.2.2. By equivalence of categories we may also view C^N as functor into conilpotent non-counital dg-coalgebras $\mathrm{coDGA}^{\mathrm{conil}}$. We will freely switch between these perspectives.

There is a Quillen adjunction

$$\mathrm{qCat}_0 \underset{N}{\overset{C^N}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{coDGA}^{\mathrm{conil}}$$

between the category of conilpotent dg-coalgebras $\mathrm{coDGA}^{\mathrm{conil}}$ and qCat_0, which was shown in [CHL21, Lemma 3.4]. The nerve functor N here is given as $\mathrm{coDGA}^{\mathrm{conil}}(C^N(\Delta^*), -)$ where $\Delta^* : \Delta \to \mathrm{sSet}$ is the standard cosimplicial simplicial set. Specifically the Quillen adjunction is the nerve realisation adjunction with respect to the cosimplicial conilpotent dg-coalgebra $C^N(\Delta^*) : \Delta \to \mathrm{coDGA}^{\mathrm{conil}}$ given as the composition

$$\Delta \xrightarrow{\Delta^*} \mathrm{sSet} \xrightarrow{C^N} \mathrm{coDGA}^{\mathrm{conil}} .$$

4.3. Monoidal properties of the (co)bar construction

We have in Chapter 3 worked in the non-(co)unital context. We will now switch to working in the equivalent (co)augmented context. In this context the ordinary tensor product of non-counital dg-coalgebras that we have used in Chapter 3 corresponds to the smash product of coaugmented dg-coalgebras. The smash product $C \wedge D$ of two coaugmented dg-coalgebras C and D is defined as the pushout

$$\begin{array}{ccc} C \oplus D & \longrightarrow & C \otimes C' \\ \downarrow & & \downarrow \\ k & \longrightarrow & C \wedge C' \end{array}$$

in the category of counital dg-coalgebras. Explicitly we may compute the smash product as $C \wedge D \cong (\overline{C} \otimes \overline{D}) \oplus k$ where $\overline{C}$ and $\overline{D}$ are the non-counital dg-coalgebras corresponding to C and D respectively.

We also have the pointed internal hom functor $\underline{\mathrm{coDGA}}^{\mathrm{conil}}_{\mathrm{coaug}}(C, D)$, from C to D, defined as the pullback

$$\begin{array}{ccc} \underline{\mathrm{coDGA}}(k, D) & \longleftarrow & k \\ \uparrow & & \uparrow \\ \underline{\mathrm{coDGA}}(C, D) & \longleftarrow & \underline{\mathrm{coDGA}}^{\mathrm{conil}}_{\mathrm{coaug}}(C, D) \end{array}$$

in the category of counital dg-coalgebras. Explicitly the pointed internal hom functor is computed as $\underline{\text{coDGA}}^{\text{conil}}_{\text{coaug}}(C, D) \cong \underline{\text{coDGA}}^{\text{conil}}(\overline{C}, \overline{D}) \oplus k$.

The reason we have worked with the smash product $\wedge$ in Chapter 3 is that it does admit a tensor hom adjunction albeit with the downside that we are missing a monoidal unit. However the category of $\text{coDGA}^{\text{conil}}_{\text{coaug}}$ also has a monoidal structure given by the tensor product $\otimes$, which does have a unit, namely, k. The downside here is that the tensor product is not closed semi-monoidal in $\text{coDGA}^{\text{conil}}_{\text{coaug}}$ and hence do not give rise to semi-monoidal model categories. Working in the (co)augmented context allows us to more easily consider both of these structures simultaneously.

Similarly the category of augmented dg-algebras DGA_{aug} has two monoidal structures given by the smash product $\wedge$ and the tensor product $\otimes$ with unit $k \oplus k$ and k respectively. Neither of these are closed semi-monoidal however so does not give rise a semi-monoidal model category. We may however still ask the question if the bar and cobar functors are quasi-strong semi-monoidal functors with respect to either the smash product or the tensor product. As we will see the (co)bar constructions are quasi-strong semi-monoidal with respect to the tensor product but not with respect to the smash product.

Remark 4.3.1. One way to see that the tensor product of conilpotent coaugmented dg-coalgebras is not closed is the following. As the monoidal unit k is also the zero object of coaugmented dg-coalgebras we see that the tensor product functor does not preserve initial objects and hence does not preserve colimits. Thus it does not admit a right adjoint.

A similar argument shows that DGA_{aug} is not closed monoidal under the tensor product.

To show that the cobar construction is not quasi-strong semi-monoidal with respect to the smash product, consider the following counterexample, here in the language of non-counital dg-coalgebras.

Example 4.3.2. Consider the conilpotent non-counital coalgebra k_0 defined to be k concentrated in degree 0 and given the zero comultiplication. We then have that Ωk_0 is given by

$$(\Omega k_0) = \begin{cases} k, & n \leq -1 \\ 0, & \text{otherwise,} \end{cases}$$

with zero differential. Next note that $\Omega k_0 \otimes \Omega k_0$ has zero differential and has $k \oplus k$ in degree -3. Thus we see that $\Omega k_0 \otimes \Omega k_0$ is not quasi-isomorphic to $\Omega(k_0 \otimes k_0) \cong \Omega k_0$.

Remark 4.3.3. It may seem that a similar counterexample would hold in the case of the ordinary tensor product. This however is not the case as the conilpotent coaugmented dg-coalgebra $(k_0 \oplus k) \otimes (k_0 \oplus k)$ has non-trivial comultiplication which then induces a differential on the cobar construction.

The cobar construction is however quasi-strong monoidal with respect to the ordinary tensor product. This was shown in [HL22] in the more general context of dg-categories from which our case follows. This corresponds to the following proposition.

Proposition 4.3.4. *The cobar construction* $\Omega : \mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}} \to \mathrm{DGA}_{\mathrm{aug}}$ *is quasi-strong monoidal with respect to the ordinary tensor product of coaugmented dg-coalgebras and augmented dg-algebras. In particular the inclusion map*

$$\Omega(C \otimes C') \to \Omega C \otimes \Omega C',$$

is a quasi-isomorphism.

Note that the cobar construction also respects the unit strongly $\Omega k \cong k$.

Corollary 4.3.5. *The bar construction* $B : \mathrm{DGA}_{\mathrm{aug}} \to \mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}}$ *is quasi-strong monoidal with respect to the ordinary tensor product.*

Proof. By Proposition 4.3.4, Koszul duality, and that the tensor product preserves quasi-isomorphisms of dg-vector spaces, we have weak equivalences

$$\Omega(BA \otimes BA') \xrightarrow{\sim} \Omega BA \otimes \Omega BA' \xrightarrow{\sim} A \otimes A'.$$

The adjoint map

$$BA \otimes BA' \xrightarrow{\sim} B(A \otimes A')$$

is thus a weak equivalence of $\mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}}$ by Koszul duality. □

4.4. A semi-module structure on $\mathrm{Ho}(\mathrm{coDGA}^{\mathrm{conil}})$ and $\mathrm{Ho}(\mathrm{DGA}_0)$

We will now turn our attention to the homotopy categories of conilpotent dg-coalgebras and non-unital dg-algebras and in particular their semi-module structures as defined in Chapter 3. We will see that these structures induce semi-module structures over the homotopy category of reduced simplicial sets $\mathrm{Ho}(\mathrm{sSet}_0)$.

We begin by spelling out some of the consequences of Chapter 3. By Proposition 3.8.1 we have natural isomorphisms

$$\Omega(C \otimes C') \cong (C \otimes C') \rhd \mathbf{mc} \cong C \rhd (C' \rhd \mathbf{mc}) \cong C \rhd \Omega C',$$

and

$$\overline{\mathrm{DGA}}_0(\Omega C, A) \cong \overline{\mathrm{DGA}}_0(C \rhd \mathbf{mc}, A) \cong \overline{\mathrm{DGA}}_0(\mathbf{mc}, \{C, A\}) \cong B\{C, A\}.$$

Using these natural isomorphisms and Theorem 1.0.1 we have the following corollaries.

Corollary 4.4.1. *The homotopy category of conilpotent dg-coalgebras* $\mathrm{Ho}(\mathrm{coDGA}^{\mathrm{conil}})$ *is closed semi-monoidal with tensor product*

$$C \otimes_{\mathbb{L}} C' \cong C \otimes C',$$

and internal hom

$$\mathbb{R}\, \underline{\mathrm{coDGA}}^{\mathrm{conil}}(C, C') \cong \underline{\mathrm{coDGA}}^{\mathrm{conil}}(C, B\Omega C') \cong B\{C, \Omega C'\}.$$

Corollary 4.4.2. *The homotopy category of non-unital dg-algebras* $\mathrm{Ho}(\mathrm{DGA}_0)$ *has the structure of a closed* $\mathrm{Ho}(\mathrm{coDGA}^{\mathrm{conil}})$*-module category with tensoring*

$$C \rhd_{\mathbb{L}} A \cong C \rhd \Omega BA \cong \Omega(C \otimes BA),$$

cotensoring

$$\mathbb{R}\{C, A\} \cong \{C, A\},$$

and enrichment functor

$$\mathbb{R}\, \overline{\mathrm{DGA}}_0(A, A') \cong \overline{\mathrm{DGA}}_0(\Omega BA, A') \cong B\{BA, A'\}.$$

Remark 4.4.3. Alternatively, we can view $\mathrm{Ho}(\mathrm{DGA}_0)$ as a closed semi-monoidal category with tensoring given by

$$BA \rhd A' \cong \Omega(BA \otimes BA'),$$

and internal hom given by

$$B\{BA, A'\}.$$

4.4.1. A $\mathrm{Ho}(\mathrm{sSet}_0)$-module structure for $\mathrm{Ho}(\mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}})$ and $\mathrm{Ho}(\mathrm{DGA}_{\mathrm{aug}})$. We will now proceed with giving $\mathrm{Ho}(\mathrm{sSet}_0)$ semi-module structures on the categories of $\mathrm{Ho}(\mathrm{coDGA}^{\mathrm{conil}})$ and $\mathrm{Ho}(\mathrm{DGA}_0)$ induced by the normalised chain coalgebra functor C^N.

PROPOSITION 4.4.4. *Let $C^N : \mathrm{qCat}_0 \to \mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}}$ be the normalised chain coalgebra functor. Then the Eilenberg-Zilber map*

$$\nabla_{X,Y} : C^N(X) \otimes C^N(Y) \xrightarrow{\sim} C^N(X \times Y),$$

is a weak equivalence of $\mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}}$.

PROOF. We need to show that the induced map

$$\Omega\nabla_{X,Y} : \Omega(C^N(X) \otimes C^N(Y)) \xrightarrow{\sim} \Omega C^N(X \times Y),$$

is a quasi-isomorphism. Consider the diagram

$$\begin{array}{ccc} \mathrm{qCat}_0 & \underset{N^{\mathrm{hc}}}{\overset{\mathfrak{C}}{\rightleftarrows}}\ (\perp) & \mathrm{sMon} \\ {\scriptstyle C^N}\downarrow & & \downarrow{\scriptstyle \mathcal{N}} \\ \mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}} & \underset{B}{\overset{\Omega}{\rightleftarrows}}\ (\perp) & \mathrm{DGA}_{\mathrm{aug}}, \end{array}$$

where $\mathcal{N}$ denotes the Moore complex functor. This diagram commutes up to homotopy by [RZ18, Proposition 7.3]. By Proposition 4.3.4 we know that the cobar construction Ω is quasi-strong monoidal while the rigidification functor $\mathfrak{C}$ is known to also be quasi-strong monoidal by e.g. [Lur09, Corollary 2.2.5.6]. The lax monoidal structure of the Moore complex $\mathcal{N}$ is the Eilenberg-Zilber map

$$\nabla_{X,Y} : N(X) \otimes N(Y) \to N(X \times Y),$$

which is known to be a quasi-isomorphism by the Eilenberg-Zilber theorem. See e.g. [Mac63, Theorem 8.1] for the standard argument. It follows that the Eilenberg-Zilber map of chain coalgebras is a weak equivalence. □

PROPOSITION 4.4.5. *The normalised chain coalgebra functor $C^N : \mathrm{qCat}_0 \to \mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}}$ is quasi-strong semi-monoidal with respect to the smash product.*

PROOF. We show that there is a natural weak equivalence

$$\eta_{X,Y} : C^N(X) \wedge C^N(Y) \xrightarrow{\sim} C^N(X \wedge Y),$$

induced by the Eilenberg-Zilber map.

Consider the commutative diagram

$$\begin{array}{ccccc} k & \longleftarrow & C^N(X)\oplus C^N(Y) & \longrightarrow & C^N(X)\otimes C^N(Y) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\sim \\ k & \longleftarrow & C^N(X\sqcup Y) & \longrightarrow & C^N(X\times Y), \end{array}$$

in the category of counital dg-coalgebras. The two first vertical maps are isomorphisms and the last one is a weak equivalence by Proposition 4.4.4. As $\mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}}$ is left proper and the two right maps are cofibrations, it follows that the ordinary pushout represents the homotopy pushout. Hence the pushouts induce a weak equivalence $C^N(X)\wedge C^N(Y)\xrightarrow{\sim} C^N(X\wedge Y)$. □

As we have shown that C^N is quasi-strong monoidal with respect to the smash product it hence induces a semi-monoidal functor $C^N : \mathrm{Ho}(\mathrm{qCat}_0)\to \mathrm{Ho}(\mathrm{coDGA}^{\mathrm{conil}})$ which allows us to transfer the semi-module structure to one over qCat_0. Note we have not needed to do any cofibrant replacement here as all objects in qCat_0 are cofibrant. As a consequence we get the following results.

COROLLARY 4.4.6. *The homotopy category of the category of conilpotent dg-coalgebras* $\mathrm{Ho}(\mathrm{coDGA}^{\mathrm{conil}})$ *has the structure of a closed* $\mathrm{Ho}(\mathrm{qCat}_0,\wedge)$*-module category with tensoring*

$$C^N(-)\otimes -,$$

cotensoring

$$B\{(C^N)^{\mathrm{op}}(-),\Omega -\},$$

and enrichment functor

$$NB\{-,\Omega -\}.$$

COROLLARY 4.4.7. *The homotopy category of the category of augmented dg-algebras* $\mathrm{Ho}(\mathrm{DGA}_{\mathrm{aug}})$ *has the structure of a closed* $\mathrm{Ho}(\mathrm{qCat}_0,\wedge)$*-module category with tensoring*

$$C^N(-)\rhd -,$$

cotensoring

$$\{(C^N)^{\mathrm{op}}(-),-\},$$

and enrichment functor

$$NB\{B-,-\}.$$

Using the inclusion core adjunction we can further transfer the semi-module structures over the homotopy category of reduced simplicial sets $\mathrm{Ho}(\mathrm{sSet}_0)$, giving us the following.

Corollary 4.4.8. *The homotopy category of the category of conilpotent dg-coalgebras* $\mathrm{Ho}(\mathrm{coDGA}^{\mathrm{conil}})$ *has the structure of a closed* $\mathrm{Ho}(\mathrm{sSet}_0)$*-module category with tensoring*

$$C^N(-) \otimes -,$$

cotensoring

$$B\{C^N(-), \Omega -\}$$

and enrichment functor

$$\mathrm{Core}\, N\{-, \Omega -\}.$$

Corollary 4.4.9. *The homotopy category of the category of augmented dg-algebras* $\mathrm{Ho}(\mathrm{DGA}_{\mathrm{aug}})$ *has the structure of a closed* $\mathrm{Ho}(\mathrm{sSet}_0, \wedge)$*-module category with tensoring*

$$C^N(-) \triangleright -,$$

cotensoring

$$\{C^N(-), -\},$$

and enrichment functor

$$\mathrm{Core}\, NB\{B-, -\}).$$

Remark 4.4.10. Note that we don't currently have the tools to state a similar result in the com-Lie case of Koszul duality. For instance, while composing our result with the coabelisation functor would give an enrichment functor it wouldn't satisfy the adjunction properties.

4.4.2. A remark on simplicial mapping spaces of (co)algebras. In this section we make an attempt to use the the semi-module structures of $\mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}}$, $\mathrm{coCDGA}^{\mathrm{conil}}$, $\mathrm{DGA}_{\mathrm{aug}}$, and DGLA to compute their simplicial mapping spaces in terms of the constructed enrichment functors. However, the lack of monoidal unit provides an obstacle and we are only able to obtain some special cases while leaving the general case as proposed further work. Our work is motivated by the similar case of dg-categories in [HL22], albeit with the difference that they do have a monoidal unit.

We observe that the semi-module adjunction we have shown in Chapter 3 holds at the level of mapping spaces. That is we have the following.

PROPOSITION 4.4.11. *Let C, C', C'' be conilpotent dg-coalgebras and A, A' augmented dg-algebras. Then there exist weak homotopy equivalences*

$$\mathrm{Map}(C \wedge C', C') \cong \mathrm{Map}(C, \underline{\mathrm{coDGA}}^{\mathrm{conil}}_{\mathrm{coaug}}(C', C')),$$

and

$$\mathrm{Map}(C \rhd A, A') \cong \mathrm{Map}(C, \underline{\mathrm{DGA}}_{\mathrm{aug}}(A, A') \cong \mathrm{Map}(A, \{C, A\}).$$

PROOF. We show the first statement; the others are similar. Let $C^\bullet$ be any Reedy cofibrant resolution of C. Then, as the smash product is left Quillen, it preserves colimits and cofibrant objects and hence Reedy cofibrant objects. It follows that $C^\bullet \wedge C'$ is a Reedy cofibrant resolution of $C \wedge C'$. But then by definition of mapping spaces we have

$$\mathrm{Map}(C \wedge C', C'') \cong \mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}}(C^\bullet \wedge C', C'') \cong$$
$$\mathrm{coDGA}^{\mathrm{conil}}(C^\bullet, \underline{\mathrm{coDGA}}^{\mathrm{conil}}_{\mathrm{coaug}}(C', C')) \cong \mathrm{Map}(C, \underline{\mathrm{coDGA}}^{\mathrm{conil}}_{\mathrm{coaug}}(C', C'')).$$

□

Similarly for the com-Lie case we have the following.

PROPOSITION 4.4.12. *Let C, C', C'' be cocommutative conilpotent dg-coalgebras and $\mathfrak{g}$, $\mathfrak{h}$ dg-Lie algebras. Then there exist weak homotopy equivalences*

$$\mathrm{Map}(C \wedge C', C') \cong \mathrm{Map}(C, \underline{\mathrm{coCDGA}}^{\mathrm{conil}}_{\mathrm{coaug}}(C', C')),$$

and

$$\mathrm{Map}(C \rhd \mathfrak{g}, \mathfrak{h}) \cong \mathrm{Map}(C, \underline{\mathrm{DGLA}}(\mathfrak{g}, \mathfrak{h}) \cong \mathrm{Map}(\mathfrak{g}, \{C, \mathfrak{h}\}).$$

We observe the following.

LEMMA 4.4.13. *Let C be a conilpotent dg-coalgebra, X a reduced simplicial set and $X^\bullet$ a Reedy cosimplicial resolution of X. Then $C \wedge C^N(X^\bullet)$ is a Reedy cosimplicial resolution of $C \wedge C^N(X)$.*

PROOF. This follows from that left Quillen functors preserve colimits and cofibrations and hence Reedy cofibrant objects. □

Using the fact that the nerve functor N is left Quillen together with Lemma 4.4.13 we get the following corollaries from Proposition 4.4.11 and Proposition 4.4.12.

COROLLARY 4.4.14. *Let X be a reduced simplicial set and C and D conilpotent dg-coalgebras. Then there are weak homotopy equivalences*

$$\mathrm{Map}(C^N(X) \wedge C, D) \cong \mathrm{Map}(X, N\, \underline{\mathrm{coDGA}}^{\mathrm{conil}}_{\mathrm{coaug}}(C, D)).$$

In particular we have that

$$\mathrm{Map}(k, D) \cong \mathrm{Map}(*, N\, \underline{\mathrm{coDGA}}^{\mathrm{conil}}_{\mathrm{coaug}}(C, D)) \cong *,$$

as expected.

COROLLARY 4.4.15. *Let X be a reduced simplicial set and A and A' augmented dg-algebras. Then there exist weak homotopy equivalences*

$$\mathrm{Map}(C^N(X) \rhd A, A') \cong \mathrm{Map}(X, N\{BA, A'\})).$$

In particular we have that

$$\mathrm{Map}(k, A) \cong \mathrm{Map}(*, N\{BA, A'\}) \cong *,$$

as expected.

COROLLARY 4.4.16. *Let X be a reduced simplicial set and C and D conilpotent cocommutative dg-coalgebras. Then there are weak homotopy equivalences*

$$\mathrm{Map}(C^N(X) \wedge C, D) \cong \mathrm{Map}(X, N\, \underline{\mathrm{coCDGA}}^{\mathrm{conil}}_{\mathrm{coaug}}(C, D)).$$

In particular we have that

$$\mathrm{Map}(k, D) \cong \mathrm{Map}(*, N\, \underline{\mathrm{coCDGA}}^{\mathrm{conil}}_{\mathrm{coaug}}(C, D)) \cong *,$$

as expected.

COROLLARY 4.4.17. *Let X be a reduced simplicial set and $\mathfrak{g}$ and $\mathfrak{h}$ dg-Lie algebras. Then there exist weak homotopy equivalences*

$$\mathrm{Map}(C^N(X) \rhd \mathfrak{g}, \mathfrak{h}') \cong \mathrm{Map}(X, N\{B\mathfrak{g}, \mathfrak{h}\})).$$

In particular we have that

$$\mathrm{Map}(k, A) \cong \mathrm{Map}(*, N\{B\mathfrak{g}, \mathfrak{h}\}) \cong *,$$

as expected.

Notation

We here provide a brief summary of the notation, for various categories and functors, occurring throughout the thesis.

dg	differential graded	Section 2.1
sSet	simplicial sets given the Quillen model structure	Section 2.5.1
sSet_0	reduced simplicial sets given the Quillen model structure	Section 4.1
$\mathrm{sSet}^{*/}$	pointed simplicial sets given the Quillen model structure	Section 4.1
qCat	simplicial sets given the Joyal the model structure	Section 4.1
qCat_0	reduced simplicial set given the Joyal model structure	Section 4.1
$\mathrm{qCat}^{*/}$	pointed simplicial set given the Joyal model structure	Section 4.1
DGVec	dg-vector spaces	Section 2.1
DGA	unital dg-algebras	Section 2.2
DGA_0	non-unital dg-algebras	Section 2.2
$\mathrm{DGA}_{\mathrm{aug}}$	augmented dg-algebras	Section 2.2
DGLA	dg-Lie algebras	Section 2.2
coDGA	dg-coalgebras	Section 2.3
coCDGA	cocommutative dg-coalgebras	Section 2.3
$\mathrm{coDGA}^{\mathrm{conil}}$	conilpotent dg-coalgebras	Section 2.3.1
$\mathrm{coDGA}^{\mathrm{conil}}_{\mathrm{coaug}}$	conilpotent coaugmented dg-coalgebras	Section 2.3.1
$\mathrm{coCDGA}^{\mathrm{conil}}$	conilpotent cocommutatative dg-coalgebras	Section 2.3.1
sMon	simplicial monoids given their standard model structure	Section 4.1
sGrp	simplicial groups given their standard model structure	Section 4.1
T_0	the free non-unital dg-algebra functor	Section 2.2.1
T_{Lie}	the free dg-Lie algebra functor	Section 2.2.1
T^{co}	the conilpotent cofree dg-coalgebra functor	Section 2.3.1
S^{co}	the conilpotent cocommutative cofree dg-coalgebra functor	Section 2.3.1

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