Bubbles and Crashes: A Tale of Quantiles

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Abstract

Periodically collapsing bubbles, if they exist, induce asymmetric dynamics in asset prices. In this paper, I show that unit root quantile autoregressive models can approximate such dynamics by allowing the largest autoregressive root to take values below unity at low quantiles, which correspond to price crashes, and above unity at upper quantiles, that correspond to bubble expansions. On this basis, I employ two unit root tests based on quantile autoregressions to detect bubbles. Monte Carlo simulations suggest that the two tests have good size and power properties, and can outperform recursive least-squares based tests. The merits of the two tests are further illustrated in three empirical applications that examine Bitcoin, U.S. equity and U.S. housing markets. In the empirical applications, special attention is given to the issue of controlling for economic fundamentals. The estimation results indicate the presence of asymmetric dynamics that closely match those of the simulated bubble processes.

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1 Introduction

Over the last decades, and especially after the financial crisis of 2007–09, a large literature has emerged that deals with the development and application of econometric tests of speculative bubbles. Motivated by the theoretical predictions of rational expectation models, this literature has mainly concentrated on testing for an explosive root in asset prices. In this vein, early studies on bubble detection employed conventional integration tests based on ordinary least squares, such as the standard Augmented Dickey Fuller (ADF). A major shortcoming of such tests is that they frequently fail to detect explosive dynamics when bubbles periodically collapse (Evans, 1991; van Norden, 1996). One of the reasons for this failure is that market crashes generate extreme realizations in the left tail of the conditional asset price distribution. Because the least-squares estimator is highly sensitive to extreme values, conventional tests often indicate stationarity even though the series under examination is inherently nonstationary.

In recent years, new econometric methodologies have been proposed that attempt to deal with this shortcoming by allowing for time variation in persistence (Astill et al., 2023, 2018; Homm and Breitung, 2012). Two tests that have gained substantial popularity are the supremum ADF (SADF) of Phillips et al. (2011) and the generalized SADF (GSADF) derived by Phillips et al. (2015a,b). To deal with the effect of market crashes on the test's performance, the SADF and GSADF use a recursive least-squares algorithm that estimates ADF regressions on subsamples of data and retrieves the maximum ADF statistic. The SADF employs a forward expanding window, while its extension, the GSADF, tests for exuberance using all possible subsamples of a time series given a minimum window size. Between the two procedures, the GSADF is particularly attractive because it minimizes the impact of previous boom-bust episodes on estimation and thereby is consistent with multiple changes in regime. Since the seminal work of Phillips et al. (2011) and Phillips et al. (2015a,b), numerous studies have contributed to the literature on recursive right-tailed unit root testing (e.g., Harvey et al., 2015, 2020b, 2016; Phillips and Shi, 2018, 2019).[1]

The present paper proposes an alternative approach for bubble detection based on unit root quantile autoregressive models (Galvao, 2009; Koenker, 2017; Koenker and Xiao, 2004, 2006). Compared to ordinary least squares, quantile methods provide a more robust and efficient approach in the presence of outliers and/or non-Gaussian error distributions. The main advantage of unit root quantile autoregressions, however, is that they offer a mechanism for estimating the full range of conditional quantile functions rather than relying on a single measure of conditional central tendency. Thus, instead of examining variations in the degree of persistence over subsamples of data, like the SADF and GSADF, unit root tests based on quantile regressions look at the full sample

[1] As pointed out by a referee, recent studies on bubble detection relate to the broad literature that emerged following the influential paper of Perron (1989) on unit root testing in the presence of structural change. Similar to the findings for bubble processes that display episodes of explosive dynamics followed by crashes, this body of work has demonstrated that conventional integration tests cannot distinguish between a stationary time series with breaks and a unit root process. From this perspective, flexible-window tests that allow deviations from the null hypothesis to occur episodically offer natural solutions. In line with recent studies, the SADF and GSADF tests are chosen as benchmarks due to their popularity.

of data to explore the presence of heterogeneous dynamics across conditional quantiles. As shown in Sections 4 and 5, allowing for asymmetric dynamics is crucial for bubble detection. Intuitively, in the presence of periodically collapsing bubbles, autoregressive coefficient estimates increase with the quantile, taking values below unity at low quantiles, that correspond to market crashes, and above unity at upper quantiles, which correspond to bubble eruptions. Thus, researchers can test for speculative bubbles by examining the unit root property at high quantiles.

Another way to motivate the use of quantile autoregressive models in the present context is to note that they admit a conventional random-coefficient autoregressive representation. In their random-coefficient representation, autoregressive parameters vary over time as functions of a single random variable (Koenker and Xiao, 2006). Hence, even though unit root quantile autoregressive models are estimated over the entire sample, they allow the degree of persistence of asset prices to vary during boom-bust cycles, such as those generated by periodically collapsing bubbles.[2]

I employ two unit root tests based on quantile autoregressions for bubble detection. The first is the coefficient-based U_n test and the second is a right-sided version of the Kolmogorov-Smirnov type QKS_α test proposed by Koenker and Xiao (2004). Monte Carlo simulation experiments indicate that both tests have good size properties under different sample sizes, error distributions and lag length specifications. Moreover, the two tests outperform the SADF and GSADF in detecting Evans (1991) periodically collapsing bubbles by, in many cases, a substantial margin. The superior power properties of U_n and QKS_α can be attributed to the fact that, being full-sample tests, they use information from all bubble episodes collectively. Whereas, the SADF and GSADF test statistics are based on a single subsample of data which, in the presence of multiple relatively short-lived bubble episodes, may not provide a sufficiently strong signal to reject the null hypothesis (see, Phillips et al., 2011, Section 6.2).[3]

Although the primary focus of the paper is on rational, periodically collapsing bubbles, I also examine the power properties of the tests under the widely adopted bubble process proposed by Phillips et al. (2015a,b). This process alternates between a unit root, a mildly explosive, and a collapse regime at fixed points in time. In line with the simulation results for periodically collapsing bubbles, I find that the performance of QKS_α is comparable to those of recursive least-squares based tests when there is a single bubble episode and it is superior in the presence of multiple episodes, especially when these episodes are short-lived. Furthermore, I show that, unlikely for GSADF, the distance between bubble episodes does not have a substantial impact on the power of quantile-regression based tests. As an implication, QKS_α performs remarkably better when bubble episodes occur *close* to each other.

[2]Due to their flexibility, quantile autoregressive models have been used in various applications. For instance, Koenker and Xiao (2004) fit quantile autoregressions to short-term U.S. interest rates and find substantial differences in persistence during expansions and recessions. In related work, Koenker and Xiao (2006) show that the U.S. unemployment rate displays asymmetric dynamics across different parts of its conditional distribution, and Liu (2020) provides evidence that the degree of persistence of real GDP growth rates differs between contractions and expansions. Among other series, quantile autoregressions have also been applied to inflation rates, housing returns, and real exchange rates (Gaglianone et al., 2018; Galvao Jr et al., 2013; Nikolaou, 2008).

[3]The fact that U_n and QKS_α are full-sample tests also implies that they do not come with an accompanying date-stamping procedure. Section 8 provides a more detailed discussion of this issue.

In addition to the Monte Carlo experiments, I employ unit root quantile autoregressions to examine the presence of speculative dynamics in three distinct asset markets. The first empirical application deals with a leading cryptoasset known for its remarkably wild price fluctuations, Bitcoin. The second examines the S&P 500 index, which is a key benchmark for U.S. equity performance; and the final investigates U.S. housing, which constitutes a critical component of U.S. household wealth and the U.S. economy as a whole. Depending on data availability, I account for economic fundamental influences when testing for bubbles by adopting either a direct approach based on observed measures (i.e., price-dividend and price-rent ratios) or/and a recently proposed indirect approach based on futures prices (Pavlidis et al., 2017, 2018). The empirical applications provide novel insights about the persistence properties of the time series. In summary, the results for the unit root quantile autoregressive models suggest that there is substantial heterogeneity in persistence across quantiles for all three assets, with the observed pattern of autoregressive coefficient estimates closely resembling that for periodically collapsing bubbles. Not surprisingly, among the three assets, Bitcoin is found to be the most speculative.

The rest of the paper is structured as follows. Section 2 describes the theoretical asset pricing framework. Section 3 provides an outline of unit root tests based on quantile autoregressions. Sections 4 and 5 demonstrate the applicability of quantile autoregression models for bubble detection and investigate the finite-sample properties of unit root tests through Monte Carlo simulations. Section 6 outlines the indirect approach adopted to account for market fundamentals. Section 7 presents the three empirical applications. Section 8 provides a discussion of date-stamping episodes of exuberance, and the final section concludes.

2 Speculative Dynamics in Asset Prices

I consider rational expectation models in which the spot price of an asset, P_t, consists of an economic fundamentals component, x_t, and a speculative bubble component, B_t:

$$P_t = x_t + B_t. \tag{1}$$

There are many asset pricing models that take this general form. Among others, these include the dividend-discount model for stocks (Campbell and Shiller, 1988), Pindyck's (1993) model of rational commodity pricing, the present-value relation for housing prices (Glaeser and Nathanson, 2015; Meese and Wallace, 1994), and the monetary model of exchange rate determination (Devereux and Smith, 2021). In these theoretical formulations, the fundamental component of asset prices is a function of dividends, convenience yields, housing rents, and relative money supplies and relative income, respectively, and is typically assumed to follow a unit root process. On the contrary, in empirical applications, the intrinsic value of an asset as well as its time series properties are unknown. This gives rise to the well-known joint hypothesis problem, which states that econometric tests for bubbles actually examine a composite hypothesis of no bubbles and the correct model for fundamentals. For simplicity, I assume for now that x_t follows a random walk process:

$$x_t = x_{t-1} + \epsilon_t, \qquad \epsilon_t \sim iid(0, \sigma_\epsilon^2), \tag{2}$$

and return to the important issue of controlling for unobserved economic fundamentals in empirical applications in Section 6.

With regard to the second component of the spot price, rational bubbles satisfy the condition (see, Diba and Grossman, 1988)

$$E_t(B_{t+1}) = (1+r)B_t, \tag{3}$$

where r is a positive constant derived from the structural model describing the economy and $E_t(\cdot)$ is the expectation operator. Aside from the linear AR(1), several nonlinear bubble processes have been proposed that meet (3). These nonlinear processes appear more realistic in that they allow the bubble to exhibit rich dynamics, growing exponentially in some periods and crashing in others (Blanchard and Watson, 1982; Evans, 1991; Homm and Breitung, 2012). I adopt the widely used periodically collapsing bubble process proposed by Evans (1991):

$$B_{t+1} = \begin{cases} (1+r)B_t\eta_{t+1}, & \text{if } \mathrm{B}_t \leq b, \\ \left[\lambda + \frac{1}{\pi}(1+r)\zeta_{t+1}\left(B_t - \frac{1}{(1+r)}\lambda\right)\right]\eta_{t+1}, & \text{if } B_t > b, \end{cases} \tag{4}$$

where λ and b are positive constants that satisfy $\lambda < (1+r)b$; $\eta_{t+1} = \exp(\kappa_{t+1} - \sigma_\kappa^2/2)$ with $\kappa_{t+1} \sim \mathcal{N}(0, \sigma_\kappa^2)$ is a lognormal variable scaled to have a mean of unity; and ζ_{t+1} is a Bernoulli process that takes the value of one with probability π and the value of zero with probability $1-\pi$. By taking expectations of both sides, it is easy to verify that the expected gross growth rate of the bubble is $1+r$ and thus the process satisfies the condition for a rational bubble. The bubble process has also the appealing property that, conditional on a positive initial value B_0, B_t remains positive for all future time periods $t > 0$. Consider, for example, a bubble that starts below the threshold b. The bubble initially grows exponentially at a constant expected rate of $1+r$. Eventually the bubble exceeds b and its expected growth rate, conditional on the bubble not collapsing, increases to $(1+r)/\pi$. When the bubble collapses, it drops to a positive expected value of λ, and the cycle begins again. Due to its rich dynamics and attractive properties, Equation (4) constitutes the most popular rational bubble process and serves as a benchmark for evaluating bubble detection tests (Chan and Santi, 2021; Morita et al., 2024; Otero et al., 2022; Phillips and Shi, forthcoming; Phillips et al., 2015a).

A direct implication of condition (3) and Equation (1) is that, if bubbles are present in asset markets, then the spot price will display explosive dynamics. Starting with the seminal paper of Diba and Grossman (1988), a large literature has developed that employs right-tailed unit root tests to examine the presence of speculative dynamics in asset markets on the basis of this implication. This literature has mainly focused on least-squares unit root tests, leaving tests based on quantile autoregressions unexplored.

3 Unit Root Quantile Autoregressions

Consider the Augmented Dickey Fuller (ADF) regression equation,

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \sum_{j=1}^{q} \alpha_{j+1} \Delta y_{t-j} + u_t. \tag{5}$$

Following Koenker and Xiao (2004, 2006), I adopt the corresponding unit root quantile autoregression (QAR) model given by[4]

$$Q_{y_t}(\tau | y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-q}) = \alpha_0(\tau) + \alpha_1(\tau) y_{t-1} + \sum_{j=1}^{q} \alpha_{j+1}(\tau) \Delta y_{t-j}. \tag{6}$$

In the above QAR model, the τth conditional quantile of y_t, $Q_{y_t}(\tau|y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-q})$, is a linear function of the lagged value of the series and q lagged first differences. Notice that regression parameter values are allowed to vary across quantiles. As an implication, the null hypothesis of a unit root,

$$H_0 : \alpha_1(\tau) = 1, \tag{7}$$

may be rejected in favour of the one-sided alternative of explosive dynamics in y_t,

$$H_1 : \alpha_1(\tau) > 1, \tag{8}$$

at some but not all quantiles.

Letting $\mathcal{F}_t$ denote the σ-field generated by $\{u_s, s \leq t\}$, and defining $x_t = (1, y_{t-1}, \Delta y_{t-1}, \ldots, \Delta y_{t-q})^\intercal$ and $\alpha(\tau) = (\alpha_0(\tau), \ldots, \alpha_{q+1}(\tau))^\intercal$, Equation (6) can be written more compactly as

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = x_t^\intercal \alpha(\tau). \tag{9}$$

For a selected value of τ, parameter estimation requires solving the minimization problem

$$\min_{\alpha \in \mathcal{R}} \sum_{t=1}^{n} \rho_\tau(y_t - x_t^\intercal \alpha), \tag{10}$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$ is the check loss function of Koenker and Bassett Jr (1978). The solution $\hat{\alpha}(\tau)$ for a given τ is called the τth autoregression quantile; while, viewed as a function of τ, $\hat{\alpha}(\tau)$ is referred to as the QAR process.

[4] For a detailed exposition of the general unit root quantile autoregression with deterministic terms and covariates see Galvao (2009). A limitation, not only of linear quantile autoregressions, but of all linear quantile regression models is that at some point/s there will be 'crossings' of the conditional quantile functions. For this reason, these models should be interpreted as useful local approximations to the true model over a specific region. Despite this shortcoming, linear quantile autoregression models are very valuable due to their simplicity, interpretability, and the important insights they can provide about the dynamics of economic series. Furthermore, as will be shown later, they have advantages over methods based on recursive least squares in bubble detection.

Koenker and Xiao (2004) propose several unit root tests based on (6). The two tests employed in this paper are the coefficient-based $U_n(\tau)$ (analogous to the coefficient-based ADF) and a right-sided version of their Kolmogorov-Smirnov type test, QKS_α. The latter examines the unit root property over a range of quantiles, $\tau \in \mathcal{T}$, by exploiting the QAR process. The two test statistics are given by

$$U_n(\tau) = n(\hat{\alpha}_1(\tau) - 1) \qquad \text{and} \qquad \mathrm{QKS}_\alpha = \sup_{\tau \in \mathcal{T}} U_n(\tau). \tag{11}$$

In practice, the $U_n(\tau)$ statistic can be computed over a grid of values, and the QKS_α statistic can be obtained by taking the maximum value.

Under the null, both test statistics have nonstandard limiting distributions that depend on nuisance parameters. Accurate finite-sample critical values can be computed, however, using the following bootstrap unit root procedure:

1. Let $\omega_t = \Delta y_t$. Impose the null of a unit root and estimate the restricted regression

$$\omega_t = \sum_{j=1}^{q} \xi_j \omega_{t-j} + v_t,$$

   by ordinary least squares to obtain the coefficient estimates $\hat{\xi}_1, \ldots, \hat{\xi}_q$ and the residuals $\hat{v}$.

2. Generate bootstrap residuals, v_t^b, by sampling with replacement draws from the centered residuals $\hat{v}_t - 1/(n-q)\sum_{j=q+1}^{n} \hat{v}_j$.

3. Use the bootstrap residuals and the estimated coefficients to recursively generate bootstrap samples for first differences,

$$\omega_t^b = \sum_{j=1}^{q} \hat{\xi}_j \omega_{t-j}^b + v_t^b,$$

   and for levels,

$$y_t^b = y_{t-1}^b + w_t^b,$$

   with $\omega_j^b = \Delta y_j$ for $j = 1, \ldots, q$ and $y_1^b = y_1$.

4. Estimate the quantile autoregression (6) and compute the coefficient-based and Kolmogorov-Smirnov statistics, $U_n^b(\tau)$ and QKS_α^b.

5. Repeat steps (2) to (4) N times to approximate the limiting distributions under the null, and compute the bootstrap p-value as the percentage of times the simulated statistic is as or more extreme than the original.

As discussed in Koenker and Xiao (2004), the validity of the above procedure derives from an invariance principle, which concerns the weak convergence of the bootstrap partial sum process to Brownian motion (Chang et al., 2006; Park, 2002).

With regard to computational costs, Online Appendix A discusses the complexity of the quantile regression and the recursive least-squares methods employed and evaluates their run-time performance by conducting a simulation experiment. The results suggest that the computational costs associated with estimating U_n and QKS_α are very low, which makes the quantile regression estimation method well-suited for use with bootstrapping techniques.

4 Periodically Collapsing Bubbles and Quantile Autoregressions

To illustrate the applicability of unit root quantile autoregressions for bubble detection, I first examine a single realization from the theoretical asset pricing model described by Equations (1), (2), and (4). The sample size n is set equal to 200 observations and the values of the structural parameters describing the economy are $\pi = 0.5$, $r = 0.015$, $\lambda = 0.5$, $b = 1$, $\sigma_k = 0.05$, $\sigma_\epsilon = 0.7$, $B_0 = 0.5$, and $x_0 = 30$. Following previous studies, I scale up the bubble process by a factor of 20.

Figure 1 displays the simulated price and fundamental series. What stands out is the large bubble eruption occurring before the middle of the sample period, with the asset price increasing from less than 40 to almost 80 units in a short period of time. At the peak of the bubble, the speculative component constitutes over half of the asset price. This remarkable market boom ends at $t = 93$ with a dramatic market collapse that erases all capital gains. A second bubble eruption takes place in the last part of the sample, but in this case the associated price rally is not substantial. This is due to the fact that, on the one hand, the bubble eruption is smaller in magnitude and, on the other, it coincides with a reduction in the value of fundamentals. Nonetheless, there is a sizeable market crash at t =169 when the bubble pops and the asset price implodes by 26 percent.

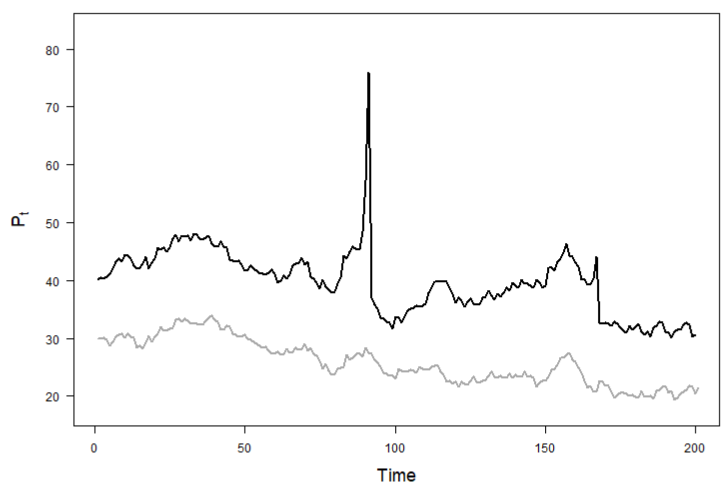

Figure 1: Simulated price (black line) and economic fundamental (grey line) series.

The market booms and crashes are also apparent in Figure 2a. This figure presents the scatter plot of the simulated price series against its lagged value. Superimposed on the plot are the fitted quantile regression lines corresponding to $\tau = \{0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95\}$ and the fitted least-squares line. To facilitate the analysis, the accompanying Figure 2b shows the QAR slope coefficient estimates against the same set of quantiles.

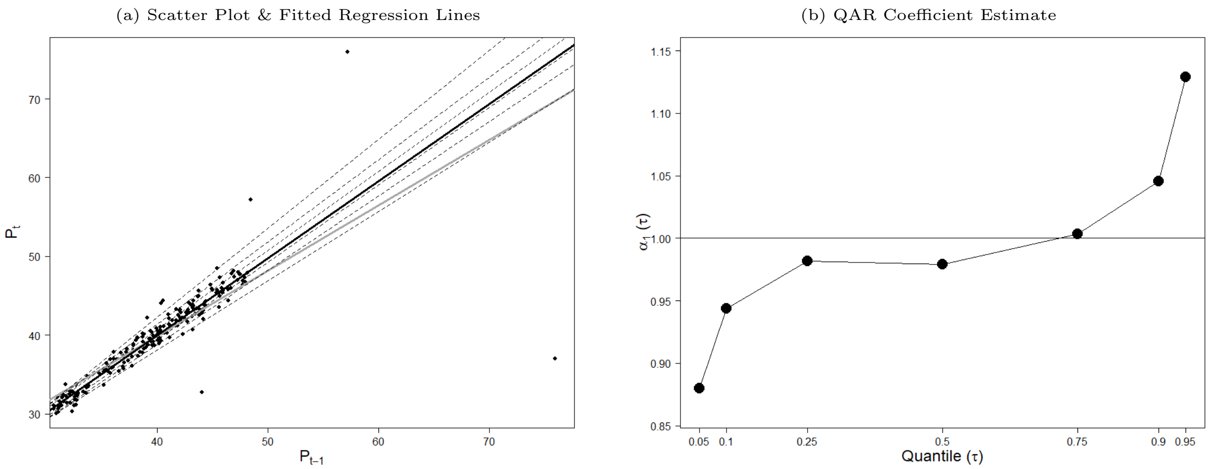

Figure 2: Scatter plot of simulated P_t against P_{t-1} with fitted quantile and least-squares regression lines (left). Median regression (solid black line), quantile regressions at $\tau = \{0.05, 0.1, 0.25, 0.75, 0.9, 0.95\}$ (dashed lines), and least-squares regression (solid grey line). QAR coefficient estimates against τ (right).

Starting with the least-squares results, it is evident that, between the extreme observations generated by market booms and market crashes, the latter have the largest influence on the regression line. Specifically, the abrupt price declines drive the least-squares estimate far below unity ($\hat{\alpha}_1 = 0.83$), falsely indicating a stationary process. This finding highlights the sensitivity of standard unit root tests to outliers and is in accord with previous studies that show that such tests have virtually no power to detect periodically collapsing bubbles (Evans, 1991; Franses and Haldrup, 1994; Harvey et al., 2001; Lucas, 1995; Phillips et al., 2011).

The quantile regression results, on the other hand, convey much richer information regarding the properties of the simulated asset price series. As can be seen from Figures 2a and 2b, the results suggest the presence of asymmetric dynamics, with the estimates for the autoregressive root increasing in a nonlinear fashion as we move from low to high quantiles. Specifically, the curve depicting the QAR process $\hat{\alpha}_1(\tau)$ is relative flat in the central region, while it displays a steep incline at the boundaries. At low quantiles, $\tau = \{0.05, 0.1\}$, the autoregressive coefficient estimates take small values, substantially below unity due to the market crashes. At $\tau = \{0.25, 0.5, 0.75\}$, the estimates are close to unity. While, at the upper quantiles $\tau = \{0.9, 0.95\}$ that correspond to bubble eruptions, they exceed unity, indicating that the time series exhibits explosive behaviour.

To generalize the above results, I now conduct a set of Monte Carlo experiments with 10,000 replications. For these experiments, the sample size n is set equal to 100, 200, 400, and 1,000 observations, and the probability π of the bubble not collapsing is set equal to 25, 50, 75 and 90 percent. The values for the remaining structural parameters are the same as in the previous exercise. Figure 3 and Table 1 report mean α_1 estimates for quantile and least-squares autoregressions of order one, respectively.

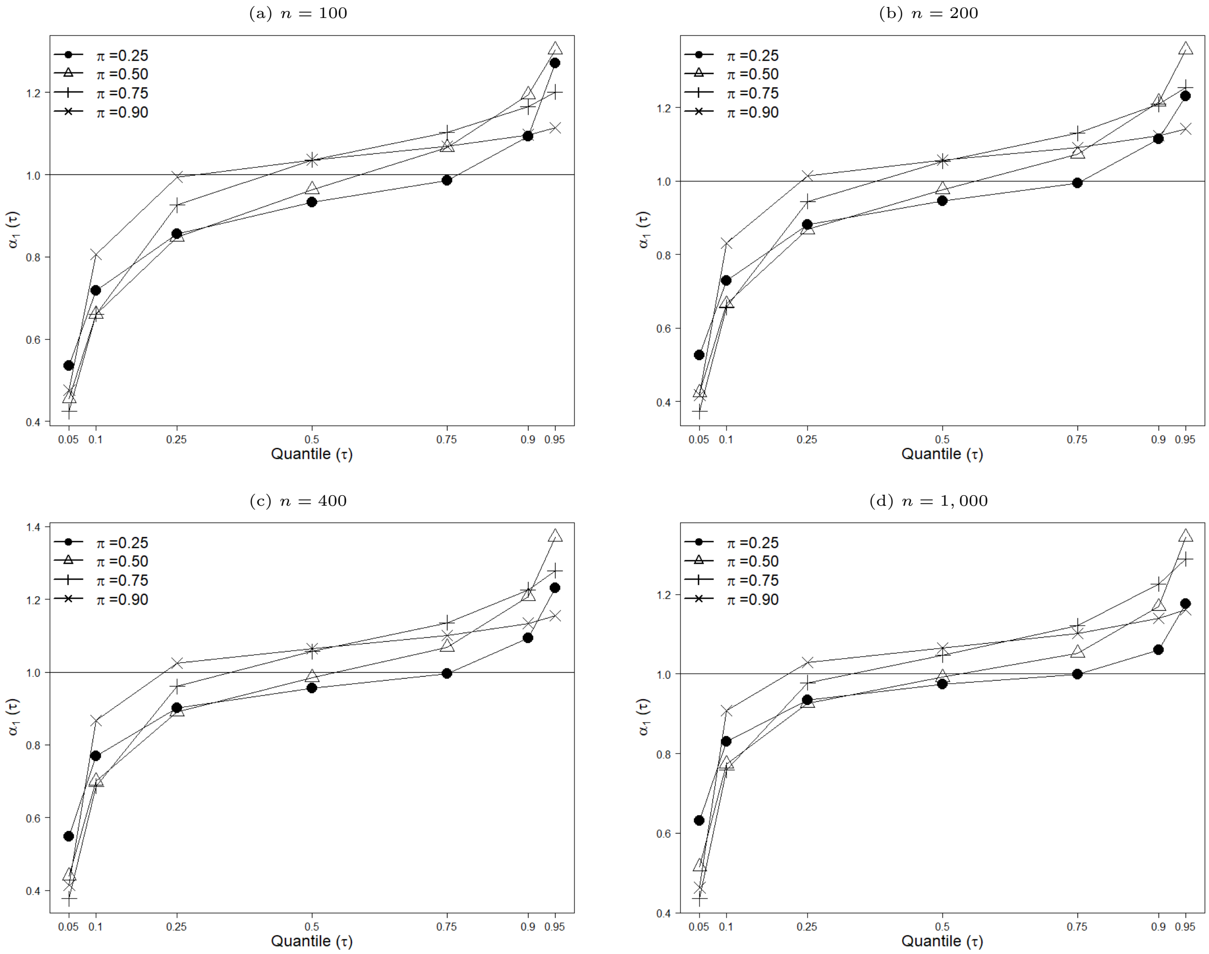

Figure 3: Mean quantile estimates of the QAR coefficient for different sample sizes n and probabilities of the bubble erupting π.

Several interesting conclusions emerge by visual inspection of Figure 3. The main conclusion is that, irrespective of the sample size and of the probability of collapse, the autoregressive coefficient estimates display a similar overall pattern: They monotonically increase with τ, always starting from values below unity at low quantiles and exceeding unity at higher quantiles. A closer look at the figure reveals that the range of mean α_1 estimates is wide, from less than 0.4 at the fifth quantile to almost 1.4 at the 95th quantile. Furthermore, it reveals that the quantile at which mean estimates exceed unity heavily depends on the probability of the bubble not collapsing. Not surprisingly, higher values of π are associated with lower quantiles. The curves corresponding to lower πs, however, exhibit a steeper incline at upper quantiles and thus take large values near the boundary ($\tau = 0.95$). This last observation can be explained by the fact that the number of sample periods characterized by a bubble erupting is positively related to π, while the intensity of the bubble, as measured by $(1 + r)/\pi$, negatively. That is, lower values of π result in fewer but more intense bubble eruptions, raising the autoregressive coefficient estimate more rapidly at upper quantiles. The main implication of the above observations is that researchers can test for speculative bubbles by examining the null hypothesis of a unit root against the alternative of explosive dynamics at sufficiently high quantiles.

Table 1: Mean least-squares α_1 estimates in the presence of periodically collapsing bubbles

	$n = 100$	$n = 200$	$n = 400$	$n = 1,000$
$\pi = 0.25$	0.745	0.702	0.672	0.652
$\pi = 0.50$	0.753	0.722	0.705	0.694
$\pi = 0.75$	0.814	0.802	0.797	0.794
$\pi = 0.90$	0.885	0.883	0.887	0.890

Notes: n and π denote the sample size and the probability of the bubble erupting, respectively.

It is also interesting to look at the results for median (least absolute deviations) and least-squares regressions. For median regressions, average $\alpha_1(0.5)$ estimates are positively related to the probability of the bubble erupting (see Figure 3). When this probability is high (i.e., $\pi = \{0.75, 0.9\}$), they take values above unity. In contrast, the average least-squares estimates reported in Table 1 are always below unity, with a minimum value of 0.652 (for $\pi = 0.25$, $n = 1,000$), and a maximum of 0.89 (for $\pi = 0.9$, $n = 1,000$). Hence, unit root tests based on least absolute deviations are expected to outperform their least-squares counterparts in detecting bubbles, but perform worse compared to tests focusing on upper quantiles. I examine the performance of the unit root tests next.

5 Monte Carlo Results: Empirical Size and Power

This section deals with the empirical size and power properties of $U_n(\tau)$ and QKS_α. For the implementation of the former test, I set $\tau = \{0.8, 0.85, 0.9, 0.95\}$ and, for the latter, I adopt the range $\mathcal{T} = [0.8, 0.95]$ and a step size of 0.01. This choice of quantiles strikes a balance between two opposing effects. On the one hand, the fact that the autoregressive coefficient increases with the quantile, exceeding unity at high quantiles, in the presence of bubbles; and, on the other, that the estimation of extreme quantiles can suffer from higher variance and instability issues due to data sparsity. In line with this latter point, simulation results reported in Online Appendix B suggest that the examination of more extreme quantiles, such as the 99th, can have an adverse impact on performance when the sample size is small.

In addition to $U_n(\tau)$ and QKS_α, I also present results for the standard ADF, the SADF of Phillips et al. (2011) and the GSADF test of Phillips et al. (2015a). This allows a comprehensive comparison of the full-sample quantile regression based tests against least-squares benchmarks with varying degrees of flexibility. Furthermore, the SADF test serves as a useful benchmark for evaluating the efficacy of the GSADF under different data generating processes. The least-squares tests are described in Online Appendix C.

For all experiments, the number of Monte Carlo simulations is set equal to 1,000, the number of bootstrap repetitions N is 2,000, and the nominal significance level is set equal to five percent.

Unless specified otherwise, the sample size n is equal to $100, 200, 300$ and 400.

5.1 Empirical Size

I conduct two sets of size experiments to examine the properties of the tests under different error distributions and lag lengths. The data generating process is a driftless random walk,

$$y_t = y_{t-1} + u_t, \tag{12}$$

where the error term, u_t, is an i.i.d. random variable. Following Koenker and Xiao (2004), in the first set of experiments, I consider the standard normal and two heavy-tailed distributions for u_t: the Student-t with three degrees of freedom, t_3, and the Student-t with two degrees of freedom, t_2. In the latter case, errors have infinite variance.

Table 2: Size of unit root tests with $\mathcal{N}(0,1)$, t_3 and t_2 errors

	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
$\mathcal{N}(0,1)$									
n =100	0.053	0.056	0.063	0.052	0.057	0.055	0.050	0.042	0.049
n =200	0.059	0.045	0.050	0.050	0.045	0.046	0.046	0.044	0.035
n =300	0.057	0.058	0.064	0.060	0.051	0.051	0.057	0.050	0.043
n =400	0.050	0.050	0.053	0.052	0.055	0.048	0.056	0.055	0.045
t_3									
n =100	0.055	0.047	0.048	0.057	0.050	0.044	0.056	0.064	0.085
n =200	0.056	0.058	0.050	0.046	0.037	0.043	0.040	0.060	0.089
n =300	0.061	0.064	0.060	0.063	0.051	0.043	0.055	0.052	0.094
n =400	0.064	0.057	0.068	0.061	0.051	0.044	0.052	0.053	0.116
t_2									
n =100	0.055	0.043	0.042	0.038	0.043	0.036	0.047	0.083	0.119
n =200	0.053	0.042	0.044	0.050	0.051	0.040	0.053	0.093	0.144
n =300	0.041	0.047	0.049	0.053	0.065	0.052	0.044	0.082	0.146
n =400	0.048	0.058	0.053	0.052	0.055	0.054	0.043	0.078	0.156

Notes: n denotes the sample size.

According to the results in Table 2, there are no substantial deviations of the empirical size from the nominal significance level for $U_n(\tau)$ and QKS_α. Thus, quantile unit root tests perform well for all sample sizes and distributions, even for the case of infinite-variance errors. These findings complement the Monte Carlo results of Koenker and Xiao who do not focus on high quantiles, but instead look at unit root tests based on median quantile autoregressions and examine a two-tailed Kolmogorov-Smirnov type test with $\mathcal{T} = [0.1, 0.9]$. Regarding the least-squares tests, the standard ADF does not exhibit any significant size distortions and the SADF appears to be only slightly oversized for t_2 errors. The more flexible GSADF test, however, is slightly oversized for t_3 and moderately oversized for t_2 errors, with the size distortions increasing with the sample size. The maximum rejection rate of the GSADF is 15.6 percent for $n = 400$ and t_2 errors. Hence, not surprisingly, quantile regression based tests perform better than recursive least-squares tests in the presence of extremely heavy-tailed data.

Table 3: Size of unit root tests for different lag lengths

	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
$n = 100$									
$q = 1$	0.029	0.046	0.043	0.050	0.055	0.053	0.063	0.054	0.095
$q = 3$	0.025	0.044	0.051	0.061	0.053	0.061	0.047	0.099	0.252
$q = 6$	0.023	0.033	0.046	0.048	0.054	0.054	0.066	0.187	0.670
$n = 200$									
$q = 1$	0.033	0.040	0.048	0.049	0.055	0.047	0.047	0.054	0.081
$q = 3$	0.033	0.032	0.037	0.037	0.050	0.042	0.051	0.084	0.190
$q = 6$	0.031	0.044	0.048	0.055	0.078	0.069	0.051	0.140	0.479
$n = 300$									
$q = 1$	0.033	0.043	0.046	0.052	0.076	0.070	0.052	0.061	0.093
$q = 3$	0.033	0.043	0.054	0.059	0.060	0.062	0.068	0.087	0.208
$q = 6$	0.030	0.045	0.043	0.046	0.047	0.061	0.044	0.095	0.383
$n = 400$									
$q = 1$	0.041	0.041	0.041	0.046	0.056	0.052	0.051	0.057	0.086
$q = 3$	0.036	0.041	0.045	0.040	0.051	0.057	0.048	0.060	0.172
$q = 6$	0.033	0.043	0.050	0.039	0.044	0.039	0.053	0.098	0.381

Notes: n and q denote the sample size and the lag length, respectively.

For the second set of size experiments, I focus on standard normal errors and consider lag lengths $q = 1, 3$ and 6. Table 3 reports the simulation results. The $U_n(\tau)$, QKS_α and ADF tests exhibit, again, good size properties with rejection rates close to the five percent level. The size properties of SADF and GSADF under different lag settings have already been examined in Phillips et al. (2015a). In line with the study of Phillips et al., the results in Table 3 indicate that the two tests exhibit size distortions that increase with the lag length and decrease with the sample size. The distortions are particularly severe for the GSADF test, reaching values as high as 67 percent. As argued by Phillips et al., the greater distortions for the GSADF test can be attributed to the smaller sample sizes in the flexible-window procedure for larger values of q. Due to this property, the authors recommend selecting a very small lag length for empirical use of the SADF and GSADF test procedures.

5.2 Empirical Power

Having examined the empirical size properties of the tests, I now evaluate their ability to detect periodically-collapsing rational bubbles using the data generating process proposed by Evans (1991). As an additional exercise, I consider the widely employed bubble process of Phillips et al. (2015a,b) in the second part of this section.

5.2.1 Empirical Power: Evans (1991) Periodically Collapsing Bubbles

The design of the first set of power experiments is the same as that of Section 2 with the exception that the structural parameter r in Evans (1991) data generating process takes values in

$\{0.01, 0.015, 0.02\}$. Thus, the experiments allow us to investigate the role of the following factors in the performance of the tests: the sample size n, the probability of the bubble erupting π, the structural parameter r, and, for $U_n(\tau)$, the quantile τ. Rejection rates are reported in Table 4.

Overall, the results for the unit root tests based on quantile autoregressions are very encouraging. Among these tests, the QKS_α and $U_n(0.95)$ are found to perform best, with the former displaying the same or slightly higher power than the latter. The two tests outperform the SADF and GSADF for all simulation settings by, in many cases, a substantial margin. The mean relative difference between the QKS and SADF (GSADF) is 39.7 (17.5) percent, and the mean absolute difference is 21 (9.5) percentage points. Interestingly, the largest differences in power between least-squares and quantile regression based tests occur when the probability of the bubble erupting is low and thus there are many bubble eruptions but of short duration. For the QKS_α-SADF pair, the maximum relative difference is 95.4 percent ($r = 0.015, \pi = 0.25$, $n = 300$) and, for the QKS_α-GSADF pair, it is 77.7 percent ($r = 0.015, \pi = 0.25$, $n = 100$). Thus, in the presence of many, short-lived bubble episodes, it is much more informative to examine the persistence of asset prices at high quantiles over the entire sample period, instead of evaluating the persistence in the mean over subsamples. For large values of n and π, power differences remain substantial for the QKS_α-SADF pair but they become negligible for the QKS_α-GSADF since both tests almost always detect bubbles.

Table 4: Power of unit root tests in the presence of periodically collapsing bubbles

	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
$r = 0.01$, $\pi = 0.25$									
$n = 100$	0.077	0.260	0.300	0.325	0.339	0.375	0.024	0.226	0.211
$n = 200$	0.079	0.349	0.429	0.489	0.513	0.583	0.011	0.301	0.348
$n = 300$	0.060	0.423	0.499	0.588	0.634	0.701	0.001	0.361	0.474
$n = 400$	0.064	0.441	0.514	0.610	0.664	0.727	0.000	0.407	0.570
$r = 0.01$, $\pi = 0.50$									
$n = 100$	0.101	0.386	0.408	0.449	0.459	0.480	0.033	0.355	0.368
$n = 200$	0.113	0.540	0.609	0.665	0.692	0.729	0.014	0.490	0.593
$n = 300$	0.108	0.632	0.696	0.759	0.796	0.831	0.004	0.551	0.719
$n = 400$	0.120	0.675	0.736	0.791	0.833	0.866	0.003	0.622	0.813
$r = 0.01$, $\pi = 0.75$									
$n = 100$	0.321	0.535	0.555	0.580	0.592	0.607	0.044	0.465	0.496
$n = 200$	0.452	0.743	0.779	0.816	0.831	0.844	0.013	0.654	0.751
$n = 300$	0.535	0.829	0.880	0.906	0.923	0.930	0.017	0.733	0.867
$n = 400$	0.592	0.890	0.915	0.939	0.962	0.969	0.007	0.787	0.937
$r = 0.01$, $\pi = 0.90$									
$n = 100$	0.435	0.583	0.587	0.588	0.587	0.600	0.089	0.477	0.491
$n = 200$	0.651	0.820	0.829	0.852	0.842	0.863	0.038	0.690	0.777
$n = 300$	0.770	0.908	0.933	0.944	0.941	0.951	0.022	0.794	0.899

$n = 400$	0.848	0.961	0.966	0.969	0.978	0.978	0.016	0.838	0.948
$r = 0.015,\ \pi = 0.25$									
$n = 100$	0.091	0.350	0.392	0.432	0.479	0.535	0.007	0.335	0.351
$n = 200$	0.053	0.435	0.529	0.602	0.669	0.728	0.003	0.398	0.502
$n = 300$	0.057	0.475	0.569	0.673	0.762	0.813	0.000	0.447	0.617
$n = 400$	0.037	0.490	0.622	0.719	0.783	0.842	0.002	0.483	0.667
$r = 0.015,\ \pi = 0.50$									
$n = 100$	0.123	0.528	0.571	0.616	0.641	0.676	0.012	0.512	0.538
$n = 200$	0.101	0.693	0.761	0.805	0.858	0.871	0.005	0.637	0.760
$n = 300$	0.110	0.772	0.823	0.869	0.914	0.942	0.003	0.671	0.877
$n = 400$	0.090	0.796	0.869	0.916	0.951	0.963	0.002	0.677	0.922
$r = 0.015,\ \pi = 0.75$									
$n = 100$	0.441	0.702	0.735	0.767	0.795	0.800	0.029	0.643	0.704
$n = 200$	0.590	0.877	0.913	0.930	0.951	0.955	0.021	0.786	0.896
$n = 300$	0.709	0.954	0.963	0.981	0.986	0.989	0.011	0.813	0.964
$n = 400$	0.745	0.966	0.976	0.982	0.990	0.993	0.009	0.823	0.984
$r = 0.015,\ \pi = 0.90$									
$n = 100$	0.639	0.787	0.800	0.814	0.801	0.813	0.074	0.700	0.733
$n = 200$	0.828	0.940	0.953	0.963	0.959	0.969	0.034	0.841	0.927
$n = 300$	0.921	0.987	0.992	0.991	0.993	0.993	0.030	0.886	0.980
$n = 400$	0.956	0.989	0.991	0.994	0.994	0.994	0.019	0.894	0.988
$r = 0.02,\ \pi = 0.25$									
$n = 100$	0.077	0.390	0.459	0.527	0.564	0.630	0.005	0.405	0.435
$n = 200$	0.049	0.466	0.574	0.656	0.741	0.800	0.001	0.435	0.575
$n = 300$	0.039	0.497	0.624	0.730	0.796	0.858	0.000	0.439	0.643
$n = 400$	0.032	0.530	0.659	0.777	0.830	0.892	0.001	0.461	0.681
$r = 0.02,\ \pi = 0.50$									
$n = 100$	0.112	0.619	0.666	0.725	0.756	0.784	0.015	0.599	0.661
$n = 200$	0.091	0.768	0.835	0.864	0.910	0.925	0.006	0.662	0.844
$n = 300$	0.089	0.841	0.878	0.923	0.943	0.957	0.002	0.672	0.916
$n = 400$	0.080	0.865	0.914	0.952	0.979	0.988	0.000	0.667	0.934
$r = 0.02,\ \pi = 0.75$									
$n = 100$	0.544	0.817	0.837	0.872	0.889	0.900	0.025	0.753	0.831
$n = 200$	0.723	0.952	0.967	0.978	0.983	0.986	0.014	0.839	0.959
$n = 300$	0.811	0.984	0.993	0.992	0.993	0.996	0.008	0.857	0.983
$n = 400$	0.846	0.995	0.996	0.996	0.998	0.999	0.005	0.815	0.997
$r = 0.02,\ \pi = 0.90$									
$n = 100$	0.781	0.889	0.901	0.904	0.903	0.919	0.087	0.815	0.864
$n = 200$	0.930	0.978	0.985	0.988	0.989	0.993	0.032	0.892	0.975

$n = 300$	0.973	0.996	0.998	0.997	0.998	1.000	0.026	0.920	0.996
$n = 400$	0.986	1.000	1.000	1.000	1.000	1.000	0.016	0.917	0.999

Notes: n and π denote the sample size and the probability of the bubble erupting; r is a structural parameter that determines the growth rate of the bubble process.

In addition to the superior performance of QKS_α and $U_n(0.95)$ over their least-squares counterparts, several other conclusions emerge from Table 4. Focusing on the $U_n(\tau)$ tests with $\tau = \{0.80, 0.85, 0.9, 0.95\}$, rejection rates generally increase with the sample size n, the structural parameter r, the probability of the bubble erupting π, and the quantile τ. The positive relation between power and sample size is intuitively obvious. The same is true for the structural parameter r. This factor determines the value of the growth rate of the bubble process and, consequently, the degree of explosiveness of the asset price. When its value increases, the asset price becomes more explosive which raises power.

As far as the impact of the probability of the bubble erupting on the power of $U_n(\tau)$ is concerned, an increase in π has two effects that work in opposite directions. On the one hand, it lowers the growth rate of the ongoing bubble, which *ceteris paribus* leads to lower power of unit root tests. On the other, it increases the expected bubble duration which, in turn, raises power. This twofold impact reflects the complex dynamics at play in the model. The simulation results indicate that, at least for the simulation design used in this paper, the first effect outweighs the second so that higher values of π lead to higher rejections rates.

Regarding the last factor, the finding that the power of $U_n(\tau)$ generally increases with τ is in line with the analysis of the previous section which shows that higher quantiles are associated with larger autogressive root estimates. It should be noted, however, that the relationship between τ and the power of $U_n(\tau)$ depends on π and, for large values of π, it becomes flat. This observation can be attributed to the fact that, as the probability of the bubble regime increases, the estimated autoregressive roots exceed unity at lower quantiles (see Figure 3). Hence, the magnitude of differences in power across quantiles declines. Furthermore, as mentioned at the beginning of the section, the distribution of data is more sparse for upper quantiles, which may lead to less precise estimates. As a consequence, increasing τ beyond a point may actually lead to a drop in power.

Turning to $U_n(0.5)$, the results of the previous section indicate that, for median regressions, average α_1 estimates are positively related to π, only exceeding unity when π is higher than 50 percent. Consistent with this result, the power of $U_n(0.5)$ is very low when π is at or below the 50 percent threshold, and abruptly increases as π changes from 50 to 75 percent. For large n, r and π, $U_n(0.5)$ exhibits good power properties and, in a few cases, outperforms the SADF test, but not the GSADF. In contrast, as expected, the standard ADF has virtually no power to detect bubbles in all cases. Next, I consider the bubble process of Phillips et al. (2015a,b).

5.2.2 Empirical Power: Phillips et al. (2015a,b) Bubble Process

Phillips et al. (2015a,b) propose the following time-varying process that alternates between mildly explosive and martingale behaviour at fixed points in time,

$$
\begin{aligned}
y_t &= (y_{t-1}+u_t)1\{t\in T_0^N\}+(\delta_n y_{t-1}+u_t)1\{t\in T_i^B\} \\
&\quad +\sum_{i=1}^{K}\left(\sum_{l=t_{i,e}+1}^{t} u_l + y^{\star}_{t_{i,e}}\right)1\{t\in T_i^N\}.
\end{aligned} \tag{13}
$$

This process displays K bubble episodes that occur at $T_i^B=[t_{i,s},t_{i,e}]$ with $i=1,\ldots,K$. The rate of expansion of the process during these episodes is given by $\delta_n=1+cn^{-\alpha}$ with $c>0$ and $\alpha\in(0,1)$, and the termination of each episode is followed by a crash. When the series crashes, its value drops to the last pre-bubble observation plus a small perturbation, $y^{\star}_{t_{i,e}}=y_{t_{i,s}}+y_i^{\star}$ with $y_i^{\star}=O_p(1)$ for all i. During normal times when a bubble is neither erupting nor crashing (i.e., at $T_0^N=[1,t_{1,s})$, $T_i^N=(t_{i,e},t_{i+1,s})$ with $i=1,\ldots,K-1$, and $T_K^N=(t_{K,e},n])$, the process follows a pure random walk.[5]

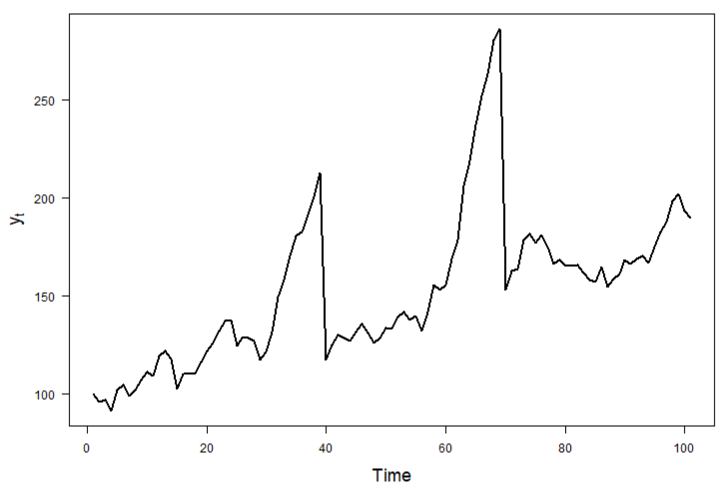

Figure 4: Simulated series with two bubble episodes from the data generating process of Phillips et al. (2015a,b).

Simulating from (13) allows investigation of the effect of several factors on the power properties of unit root tests. Among those, I consider the location, duration, and number of bubble episodes, as well as the distance between episodes. Like Phillips et al. (2015a,b), I set the sample size n equal to 100, the value of the initial observation y_0 to 100, the standard deviation of the error term σ_u to 6.79, the localizing parameter α to 0.6 and c to unity. The values of c, α and n correspond to an autoregressive coefficient δ_n of 1.06. Figure 4 shows a typical series with two bubble episodes simulated from (13) using these parameter values.

With respect to the effect of the location and duration of bubbles on power, Table 5 reports

[5] Equation (13) can be readily extended to allow for different values of the autoregressive coefficient δ_n during mildly explosive intervals and different dynamics for market collapses, see, e.g., Phillips and Shi (2018).

rejection rates when there is a single episode of mildly explosive dynamics that starts at $t_{1,s} = \{30, 60, 90\}$ and lasts for $d_r = \{5, 10\}$ observations. It is evident from the results that duration plays a crucial role in the ability of all tests to detect bubbles with the exception of the time-invariant ADF. Specifically, as the length of the bubble episode increases from five to 10 percent of the sample, rejection rates rise by a factor in excess of two.

Table 5: Phillips et al. (2015a,b) process with one bubble episode

(t_1^s, d_r)	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
(30,5)	0.067	0.167	0.182	0.198	0.202	0.217	0.018	0.191	0.175
(30,10)	0.334	0.539	0.557	0.561	0.480	0.572	0.004	0.593	0.577
(60,5)	0.070	0.161	0.176	0.220	0.230	0.233	0.011	0.128	0.162
(60,10)	0.421	0.592	0.591	0.605	0.545	0.624	0.001	0.501	0.605
(90,5)	0.087	0.232	0.234	0.255	0.236	0.241	0.005	0.115	0.221
(90,10)	0.403	0.555	0.581	0.578	0.514	0.591	0.001	0.428	0.598

Notes: t_1^s and d_r denote the start date and the duration of the bubble episode, respectively.

Turning to the impact of the location of the bubble episode, there is substantial heterogeneity across tests. For the GSADF and for the best performing quantile-regression based test, which is again the QKS_α, location does not appear to have a dramatic effect. On the contrary, changing the start date from the first part of the sample, $t_{1,s} = 30$, to the second part, $t_{1,s} = 90$, leads to a major reduction in the power of SADF that equals 40 (29) percent for $d_r = 5$ ($d_r = 10$). This finding is not surprising since the subsample of data used to obtain the SADF statistic runs from the first observation to the peak of the bubble episode. Hence, as the value of $t_{1,s}$ increases, a smaller fraction of the subsample displays mildly explosive dynamics and a larger fraction displays random walk behaviour, which causes a decline in power. The reason that GSADF does not exhibit such variations in power is that its estimation algorithm is more flexible, allowing both the start and end dates of the subsample to change.

A comparison of the results in Table 5 for QKS_α with those for the two recursive least-squares tests reveals that, in the presence of a single bubble episode, QKS_α outperforms the SADF test in the vast majority of cases and displays very similar power with GSADF. The relative performance of QKS_α is particularly noteworthy given that the recursive least-squares procedures more closely align with the data generating process (13) with $K = 1$. This is so because the SADF and GSADF test statistics are obtained from the single subsample regression covering the most significant expansionary period. While, the quantile regression based tests exploit information from all bubble episodes and, thus, are expected to perform better for $K > 1$.

To illustrate this point, I now extend the analysis to two and three bubble episodes that occur at $(t_{1,s}, t_{2,s}) = (30, 60)$ and $(t_{1,s}, t_{2,s}, t_{3,s}) = (30, 60, 90)$. Table 6 reports the estimated rejection rates. To facilitate comparisons, the table also shows the results for a single bubble episode that starts at $t_{1,s} = 30$. The most striking result from this exercise is the sharp rise in the power of QKS_α compared to SADF and GSADF when K increases and the bubble duration is short. While all three tests exhibit similar power for $K = 1$ and $d_r = 5$, QKS_α becomes 84 (34) percent more powerful than SADF (GSADF) for $K = 2$, and 149 (50) percent more powerful for $K = 3$. The

Table 6: Phillips et al. (2015a,b) process with one, two and three bubble episodes

(K, d_r)	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS$_\alpha$	ADF	SADF	GSADF
(1,5)	0.067	0.167	0.182	0.198	0.202	0.217	0.018	0.191	0.175
(2,5)	0.068	0.279	0.327	0.349	0.340	0.391	0.003	0.212	0.292
(3,5)	0.158	0.478	0.493	0.504	0.475	0.557	0.002	0.224	0.371
(1,10)	0.334	0.539	0.557	0.561	0.480	0.572	0.004	0.593	0.577
(2,10)	0.650	0.798	0.795	0.781	0.736	0.824	0.000	0.662	0.769
(3,10)	0.785	0.855	0.851	0.823	0.770	0.867	0.000	0.654	0.835

Notes: K and d_r denote the number and duration of the bubble episode(s), respectively.

QKS$_\alpha$ test also outperforms the recursive least-squares tests for $d_r = 10$ but the differences are less striking. These results are in line with those for periodically-collapsing bubbles presented in Section 5.2.1 which suggest that the largest differences in power between least-squares and quantile regression based tests occur when the probability of the bubble erupting is low. That is, when there are many bubble eruptions but of short duration.

Table 7: Phillips et al. (2015a,b) process with two bubble episodes d_s periods apart

d_s	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS$_\alpha$	ADF	SADF	GSADF
5	0.457	0.604	0.632	0.619	0.572	0.664	0.001	0.327	0.345
10	0.500	0.636	0.645	0.650	0.596	0.673	0.002	0.385	0.400
15	0.458	0.644	0.649	0.652	0.605	0.686	0.000	0.381	0.596
20	0.484	0.632	0.641	0.615	0.591	0.671	0.000	0.390	0.613

Notes: d_s denotes the distance between the two bubble episodes. The duration of the first bubble equals five observations and of the second 10 observations.

As a final exercise, I evaluate the properties of the tests in the presence of two bubble episodes that occur *close* to each other. The first bubble episode starts at $t_{1,s} = 60$ and is short, lasting five observations. The second bubble episode originates $d_s = \{5, 10, 15, 20\}$ periods after the collapse of the first and is longer, having a duration of 10 observations. The main conclusion that emerges from looking at the results in Table 7 is that QKS$_\alpha$ is superior to all other tests. Specifically, it displays stable performance across different values of d_s and higher power. The performance of GSADF, on the other hand, crucially depends on the distance between bubbles. As d_s takes smaller values, its power deteriorates. This finding can be attributed to the fact that, as the distance between bubbles grows shorter, there are fewer or no subsamples corresponding to the second bubble episode that are not contaminated by the collapse of the first bubble. As a consequence, it becomes more likely for the GSADF test to fail to detect the second and longer lasting bubble. A comparison of the results for GSADF and QKS$_\alpha$ suggests that, for the shortest distance considered, the rejection rate of the recursive least-squares test is almost half of that of the quantile regression based test (34.5 percent versus 66.4 percent).

In summary, the simulation results indicate that the QKS$_\alpha$ test can perform comparably to recursive unit root tests in the case of a single bubble episode and displays superior performance in the presence of multiple bubbles, especially when their duration is short and/or they occur close to each other. Having examined the properties of the unit root tests, I now turn to the issue of controlling for economic fundamentals in empirical applications.

6 Controlling for Economic Fundamentals

Testing for speculative bubbles is confounded by the fact that the fundamental value of assets is unobserved. Early studies have attempted to address this issue by utilizing observed variables suggested by theory, such as dividends and rents. The main drawback of such direct approaches is that they crucially depend on the highly unrealistic assumption that the true model for fundamentals is known. Model misspecification or omitted variables can lead to false inference in favour of bubbles, rendering direct approaches invalid (Gürkaynak, 2008). The importance of appropriately accounting for fundamental factors is highlighted by, among others, the early studies of Flood and Garber (1980) and Hamilton and Whiteman (1985) in the context of testing for speculative inflationary bubbles during the interwar German hyperinflation. Likewise, Basse et al. (2021) raise concerns about the use of observed dividends for measuring the intrinsic value of a firm's shares; and Figuerola-Ferretti et al. (2015) show that the choice of fundamentals is crucial for assessing whether observed explosive dynamics in the price of non-ferrous metals since 2000 are indicative of bubbles.

To circumvent the above obstacle, recent studies have employed indirect approaches that exploit information about market fundamentals incorporated in derivative prices or survey data (Pavlidis et al., 2017, 2018). These studies show that, during the expansion phase of periodically collapsing bubbles, actual realizations of future spot prices and market expectations diverge. Under general conditions, this difference between actual prices and market expectations solely depends on the bubble process. As an implication, instead of using observed fundamental to proxy for intrinsic asset values, researchers can employ measures of market expectations.

To illustrate this most simply, consider the theoretical model of Section 2 and let F_t denote the market forecast for the price of the asset one period ahead

$$F_t = E_t(P_{t+1}) = E_t(x_{t+1}) + E_t(B_{t+1}). \tag{14}$$

It follows from (4) that, conditional on the bubble erupting, the market forecast for the speculative component of the asset price is biased

$$B_{t+1} - E_t(B_{t+1}) = (1+r)\left(\frac{1}{\pi}\eta_{t+1} - 1\right)B_t + \left(1 - \frac{1}{\pi}\right)\lambda\eta_{t+1}. \tag{15}$$

This bias arises because rational agents at time t correctly attach a nonzero probability to the bubble bursting at $t+1$. As a consequence, the expected growth rate of the bubble component $1+r$ is lower than the actual rate $(1+r)/\pi$. Being a function of B_t, the forecast error $B_{t+1} - E_t(B_{t+1})$ displays explosive dynamics. This property is propagated to the cumulative demeaned forecast errors for the asset price

$$P_t^f = \sum_{i=1}^{t} (\nu_i^f - \bar{\nu}^f), \tag{16}$$

where $\nu^f = P_{t+1} - E_t(P_{t+1})$ and $\bar{\nu}^f = 1/n \sum_{i=1}^{n} \nu^f$. Assuming that the forecast error for fundamentals is not explosive, researchers can test for bubbles by running unit root tests on P_t^f.[6] The main advantage of this approach is that, by exploiting the information incorporated in market expectations, it does not require the specification of market fundamentals and, thus, ameliorates the joint hypothesis problem. In practice, market expectations can be approximated by futures prices or survey data.

7 Empirical Applications

This section presents three empirical applications of the proposed bubble detection methods to Bitcoin, U.S. equity, and U.S. housing markets. All of these markets have a rich history of price run-ups and market crashes and, for this reason, they have attracted substantial attention by academics, policy makers and the media.

The Bitcoin Market. For the first application, I employ monthly data on Bitcoin spot and futures prices in U.S. dollars. The two series are downloaded from Bloomberg and span the period December 2017 to June 2023. The start date of the data set is chosen to coincide with the commencement of Bitcoin futures trading on the Chicago Mercantile Exchange.

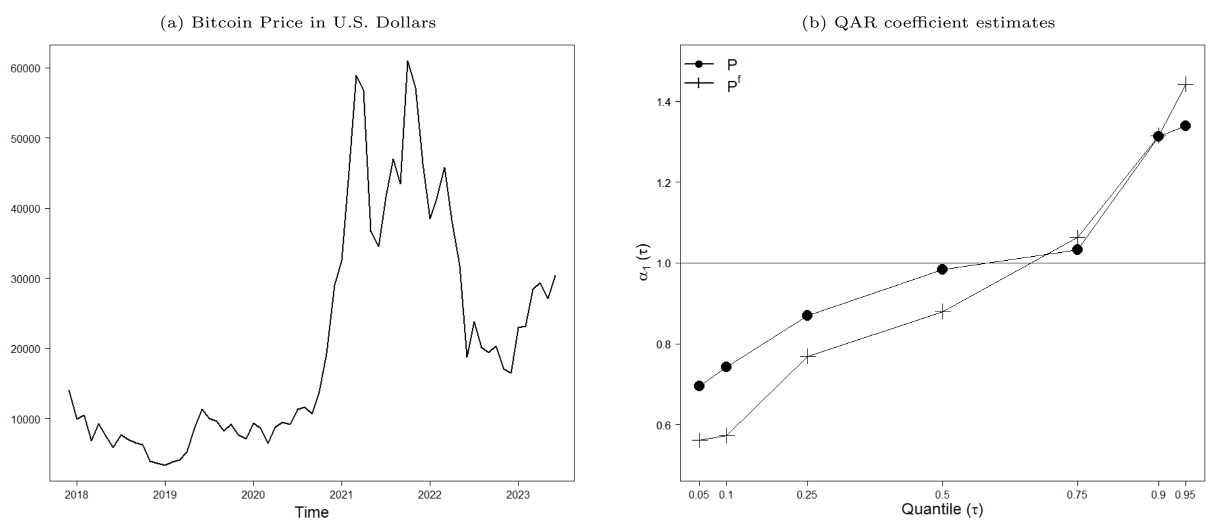

Figure 5: Bitcoin price in U.S. dollars (P_t) from December 2017 to June 2023 (left). Point estimates of the largest autoregressive root at different quantiles for P_t and P_t^f (right).

[6]A wide range of processes for x_t satisfy the condition that the forecast error $x_{t+1} - E_t(x_{t+1})$ does not grow exponentially in-sample, including explosive linear AR models and nonlinear models, such as threshold and smooth transition autoregressive.

Figure 5a shows the evolution of the spot price over time. Evidently, the series exhibits wild fluctuations within the sample. Starting at 10,706 dollars in September 2020, the price surged by 450 percent in a period of six months, reaching 58,960 dollars in March 2021. During the price surge, on January 11, the UK Financial Conduct Authority issued a warning about the risks of investments advertising high returns based on cryptoassets, highlighting that "If consumers invest in these types of product, they should be prepared to lose all their money".[7] A few days earlier, on January 8, Michael Hartnett, chief investment strategist at Bank of America Securities, referred to Bitcoin as the "mother of all bubbles". Consistent with these concerns, the Bitcoin price plummeted by 40 percent between March and June 2021 to 34,585 dollars. The cryptoasset experienced a notable rebound in the subsequent months, hitting an all-time high of 60,975 dollars in October 2021. This peak was, again, short-lived and by June 2022 Bitcoin had lost almost 70 percent of its value, reaching a low of 18,731 dollars.

Aside from its remarkably turbulent behaviour, testing for speculative dynamics in the Bitcoin market is interesting because, unlike other assets, cryptocurrencies lack traditional fundamental valuation metrics. Indeed, a widely held view among academics and policy makers is that Bitcoin is a purely speculative asset with an intrinsic value of zero (Benigno and Rosa, 2023). The following quote from Fabio Panetta, member of the Executive Board of the European Central Bank, is indicative of the prevailing view:[8] "Unbacked cryptos lack any intrinsic value, too. They are speculative assets. Investors buy them with the sole objective of selling them on at a higher price. In fact, they are a gamble disguised as an investment asset." On the contrary, those in investment and entrepreneurial circles claim that the price of Bitcoin reflects fundamental factors, such as the underlying blockchain technology (Pagnotta, 2022). In the absence of conventional valuation metrics, accounting for possible fundamental influences using futures prices can offer valuable insights.

As a preliminary exercise, I fit the unit root quantile autoregressive model (6) to Bitcoin P_t and P_t^f. In this and the remaining two empirical applications, I use the Akaike Information Criterion (AIC) with a maximum lag order of six to choose the lag length q in non-recursive unit root autoregressive models, and set q equal to zero for the SADF and GSADF tests following the recommendation of Phillips et al. (2015a). Figure 5b displays the QAR process. The α_1 estimates indicate the presence of asymmetric dynamics in both series that closely resemble those of the simulated bubbles of Section 4. Specifically, the values of $\hat{\alpha}_1$ start far below unity at low quantiles, they increase with τ and eventually exceed unity at $\tau \geq 0.75$, indicating that the two series exhibit explosive behaviour at upper quantiles. The range of values of $\hat{\alpha}_1$ is remarkably wide. For P_t^f, the lowest α_1 estimate is below 0.6 at the fifth quantile and the highest above 1.4 at the 95th. For P_t, the range is somewhat narrower, from close to 0.7 to slightly below 1.4.

To formally test for the presence of bubbles, I investigate the integration properties of the two series. The unit root test results presented in Table 8 suggest that the null hypothesis can

[7] `https://www.fca.org.uk/news/news-stories/fca-warns-consumers-risks-investments-advertising-high-returns-based-cryptoassets`

[8] `https://www.ecb.europa.eu/press/blog/date/2023/html/ecb.blog230105~75d5aee900.es.html`

Table 8: Unit root test statistics for Bitcoin

	q	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
P	2	-1.084	10.695*	18.557*	19.998*	21.724*	22.337*	-1.378	6.589*	7.639*
P^f	0	-7.935	11.590*	20.182*	20.797*	29.186*	29.789*	-1.718	5.672*	6.321*

Notes: * denotes significance at the five percent level, q is the lag length selected by AIC.

be rejected in favour of the alternative of explosive dynamics at the five percent level by the all tests but two. Not surprisingly, the two tests that fail to reject the null are the standard ADF and $U_n(0.5)$, i.e., those with the lowest empirical power. The finding that Bitcoin prices display explosive dynamics is in accordance with the recent empirical literature on bubbles in cryptoassets (Harvey et al., 2020b).[9] Our study extends this literature by showing that P_t^f also displays explosive dynamics. By doing so, it provides novel evidence, based on an agnostic approach about market fundamentals, in favor of speculative bubbles.

The U.S. Equity Market. As a second empirical application, I examine the behavior of the Standard & Poor's 500 (S&P 500) index. This index goes back to the 19th century and constitutes a leading U.S. economic indicator as well as a benchmark for mutual and exchange-traded fund performance. Due to its importance and long history, it is one of the most commonly examined series for speculative dynamics.

I utilize two data sets that cover the recent period, following the launch of the E-mini S&P 500 futures contracts, from January 1998 to June 2023. Similar to Phillips et al. (2011) and Phillips et al. (2015a), the first data set consists of the real stock price index and an observed stock market fundamental series commonly used in the literature, namely real dividends. The second includes S&P 500 spot and E-mini futures prices.[10] The two data sets are downloaded from Robert Shiller's website and Bloomberg, respectively.

Figure 6a depicts the S&P 500 real price index. As can be seen from the figure, the period under examination is characterized by several market rallies and crashes. (For a historical perspective, see Shiller (2015).) The most notable include the dot-com bubble at the beginning of the sample period and the subsequent market crash; the severe market collapse associated with the 2007-09 financial crisis; the market crash at the onset of the Covid pandemic in February and March of 2020; and, finally, the price rally of 2020-21 and the abrupt price reversal of 2022.

Figure 6b displays the quantile autoregressive coefficient estimates for the S&P 500 real price index, P_t, the price-to-dividend ratio, P_t/D_t, and the price adjusted for fundamentals using information from the futures market, P_t^f. Like for Bitcoin, the results indicate the presence of asymmetric dynamics with $\hat{\alpha}_1$ increasing with the value of τ and exceeding unity at $\tau \geq 0.75$. The range of

[9] Harvey et al. (2020b) test for speculative bubbles in daily Bitcoin prices by running a novel sign-based recursive unit root test that is robust to deterministically time-varying volatility. Because time-varying volatility may be a feature of lower frequency data, as a robustness exercise, I run a wild-bootstrap version of the SADF and GSADF tests on P_t and P_t^f. The test statistics remain statistically significant at the five percent level.

[10] S&P 500 futures contracts expire four times a year in March, June, September, and December. Similarly to Chernenko et al. (2004) and Pavlidis et al. (2017), I obtain regularly spaced futures prices by linear interpolation of spot prices and near-term contracts.

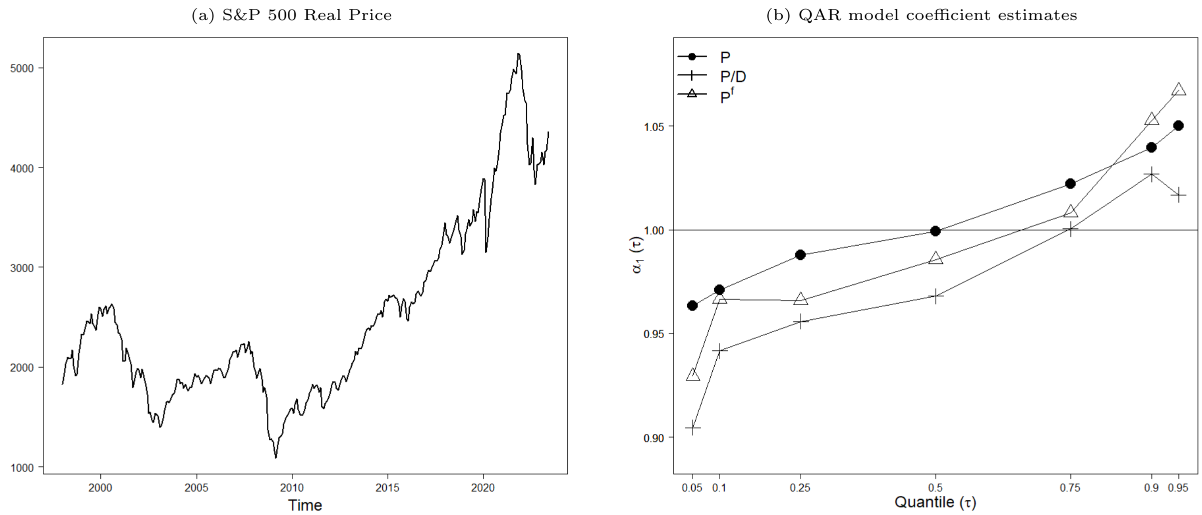

Figure 6: The S&P 500 real price index (P_t) from January 1998 to June 2023 (left). Point estimates of the largest autoregressive root at different quantiles for P_t, P_t/D_t and P_t^f (right).

estimates is, however, substantially narrower compared to Bitcoin. The minimum estimate is close to 0.9 at $\tau = 0.05$ and the maximum is slightly above 1.05 at $\tau = 0.95$. This finding reflects the highly speculative nature of the market for cryptocurrencies. It is also interesting to note that, like for Bitcoin, the P_t^f series displays a higher degree of asymmetry (as measured by the difference in the degree of persistence near the extremes) compared to P_t. While, a comparison between P_t and P_t/D_t provides mixed results. Specifically, the α_1 estimates for P_t/D_t always lie below those for P_t so that absolute deviations from unity are larger for the former series at low but not at high quantiles.

Table 9: Unit root test statistics for S&P 500

	q	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
P	1	-0.269	7.439⋆	11.570⋆	12.031⋆	15.205⋆	15.205⋆	-0.120	2.155⋆	2.397⋆
P/D	1	-9.742	0.715	8.386⋆	8.174⋆	5.121⋆	11.925⋆	-1.971	0.571	2.701⋆
P^f	2	-4.322	4.192⋆	6.126⋆	15.975⋆	20.362⋆	21.308⋆	-1.127	1.006	3.294⋆

Notes: ⋆ denotes significance at the five percent level, q is the lag length selected by AIC.

Table 9 presents the unit root test statistics for P_t, P_t/D_t, and P_t^f. The GSADF, QKS_α, and $U_n(\tau)$ tests with $\tau = \{0.85, 0.9, 0.95\}$ are consistent, indicating rejection of the null hypothesis for all series at the five percent level. On the contrary, the ADF and $U_n(0.5)$ statistics are not significant for any of the series, and the SADF statistic is only significant for P_t. Thus, in line with previous studies, the statistical evidence in this paper favours the presence of speculative dynamics in the S&P 500 (Homm and Breitung, 2012; Phillips et al., 2015a).

The U.S. Housing Market. The boom-bust episode in international housing markets from the late 1990s to the late 2000s generated a vast interest in the dynamics of house prices. A view shared by many academics and policy makers is that this episode was associated with house prices departing from their fundamental values, distorting investment decisions and leading to the 2008-09 global recession. Since then, many studies have provided evidence supporting this conjecture (Engsted et al., 2016; Greenaway-McGrevy and Phillips, 2016; Pavlidis et al., 2016; Shi, 2017; Shi et al., 2016; Shi and Phillips, 2023).

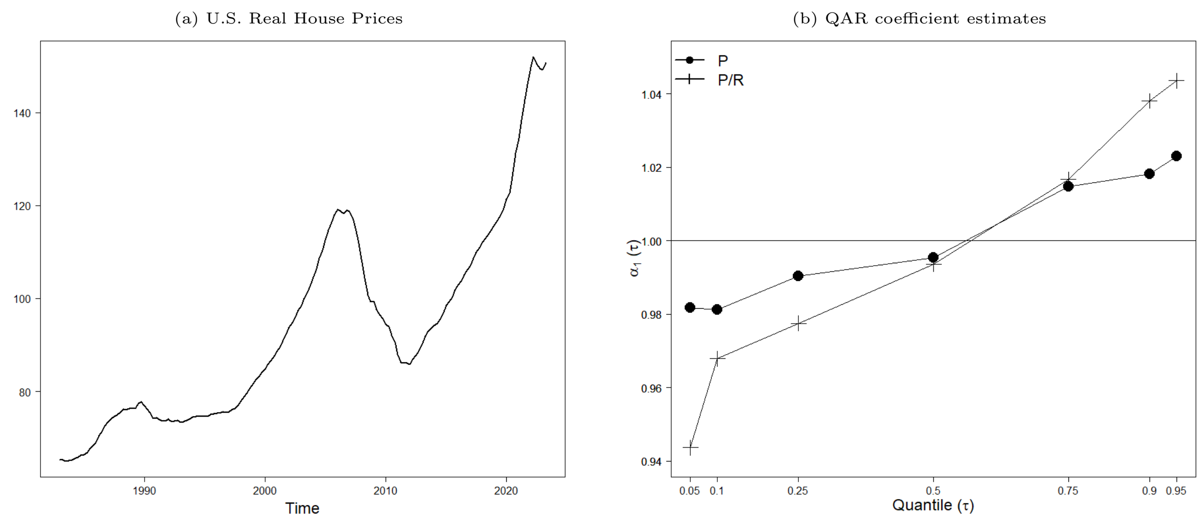

Figure 7: The U.S. real house price index (P_t) from 1983:Q1 to 2023:Q2 (left). Point estimates of the largest autoregressive root at different quantiles for P_t and P_t/R_t (right).

As a final application, I re-examine the integration properties of U.S. real house prices, P_t, and the U.S. house-price-to-rent ratio, P_t/R_t, using the quantile autoregressive model. The underlying series are downloaded from the OECD database and span the period from the 1st quarter of 1983 to the 2nd quarter of 2023. Regarding derivative prices, the Chicago Mercantile Exchange began trading housing futures contracts in May 2006 to allow investors to hedge against real estate risks. Unfortunately, since their launch, these securities have not attracted substantial attention by investors. Because the absence of sufficient liquidity can have a detrimental effect on the price discovery process, I omit futures prices from the analysis.

Figure 7a shows the evolution of the real house price index over time. Aside from the episode of the late 1990s/2000s, the U.S. housing market experienced a shorter and less intense boom-bust episode in the late 1980s/early 1990s; and displayed an impressive price surge after the Covid-19 recession. The latter rally in house prices raised issues of affordability and concerns about another U.S. housing bubble brewing, with the Federal Reserve Chairman Jerome Powell explicitly referring to a "housing bubble" and "housing prices going up at very unsustainable levels and overheating".[11]

Quantile autoregressive coefficient estimates and unit root test statistics are reported in Figure 7b and Table 10, respectively. Like for Bitcoin and S&P 500, there is a positive relationship between $\hat{\alpha}_1$ and τ. For house prices, the value of $\hat{\alpha}_1$ starts close to 0.98 at $\tau = 0.05$ and marginally exceeds

[11] Speech at the Brookings Institute Hutchins Center on Fiscal and Monetary Policy on 30th of November, 2022 https://www.brookings.edu/events/federal-reserve-chair-jerome-powell-the-economic-outlook-and-the-labor-market/

Table 10: Unit root test statistics for U.S. house prices

	q	$U_n(0.5)$	$U_n(0.8)$	$U_n(0.85)$	$U_n(0.9)$	$U_n(0.95)$	QKS_α	ADF	SADF	GSADF
P	4	-0.732	2.265*	2.459*	2.841*	3.618*	3.726*	0.123*	9.903*	17.102*
P/R	4	-1.011	2.966*	4.161*	5.975*	6.835*	7.553*	-0.809	10.755*	19.662*

Notes: * denotes significance at the five percent level, q is the lag length selected by AIC.

1.02 at $\tau = 0.95$. While, for the house-price-to-rent ratio, the coefficient curve is steeper, with a value of $\hat{\alpha}_1$ slightly above 0.94 at $\tau = 0.05$ and a value close to 1.04 at $\tau = 0.95$. Turning to the unit root test results, all statistics exceed their five percent critical value with the exception of $U_n(0.5)$, that fails to reject the null for both series, and the ADF, that only rejects the null for P_t. Thus, both recursive unit root tests and tests based on unit root quantile autoregressions detect explosive dynamics in the U.S. housing market.

Taken together, the findings for the Bitcoin, U.S. equity and U.S. housing markets contribute to a wider discussion and a long-standing debate on the existence of speculative bubbles.[12] While many prominent academics, investors, and policy makers believe that speculative bubbles are an inherent characteristic of asset markets, others hold a diametrically opposed view (Barlevy et al., 2018; Shiller, 2014). Most notably among the latter, Eugene F. Fama dismisses the notion that bubbles constitute a main feature of asset price movements and questions the evidence provided in the literature. Specifically, he argues that:[13] "For bubbles, I want a systematic way of identifying them ... Statistically, people have not come up with ways of identifying bubbles." The empirical approach adopted in this paper to account for fundamental factors provides a systematic attempt to address this criticism. Furthermore, it provides new insights on the properties of asset prices by showing that, consistent with speculative bubbles, they exhibit heterogeneous dynamics across conditional quantiles. Before concluding, some remarks are in order regarding date-stamping periods of market exuberance.

8 Remarks on Date-Stamping Episodes of Exuberance

The increased popularity of econometric tests for bubble detection has been accompanied by a growing interest in identifying periods of explosive behaviour. An attractive feature of recent least-squares procedures, such as the GSADF, is that their recursive design inherently enables date-stamping. As a result, these procedures can consistently estimate bubble origination and collapse dates under various scenarios and work well in practice (Phillips et al., 2015b). From an empirical perspective, recursive procedures shed light on the functioning of asset markets by providing information on the timing of regime shifts and can be used to enhance market surveillance.[14]

[12]This debate concerns both rational and irrational bubbles.

[13]Chicago Booth Review, June 30, 2016, `http://review.chicagobooth.edu/economics/2016/video/are-markets-efficient`

[14]For instance, following the 2007-09 financial crisis, the Federal Reserve Bank of Dallas has been providing quarterly releases of exuberance indicators for international housing markets. These indicators are estimated using the date-

Analogous to recursive least-squares methodologies, one might consider a flexible-window unit root quantile approach. However, such an approach would not yield a consistent dating strategy. As shown in Sections 4 and 5, quantile methods have the distinctive advantage of detecting bubbles even when market crashes occur within the sample period. This is so because bubble expansions drive the value of the autoregressive coefficient above unity at high quantiles, while bubble collapses drive it below unity at low quantiles. Therefore, given a substantial bubble expansion, a subsequent market crash will not cause the autoregressive coefficient to fall below unity at upper quantiles. Consequently, recursive quantile unit root tests will (correctly) continue to reject the null hypothesis of no bubbles after the collapse date. Using the same rationale, recursive quantile methods will also not identify subsequent bubble episodes.

A potential solution for researchers that are specifically interested in identifying periods of exuberance is to adopt a two-stage procedure that exploits both the efficacy of quantile methods in detecting bubbles and the effectiveness of recursive least-squares procedures in date-stamping. In particular, one could test for overall exuberance using a full-sample quantile unit root test and, conditional on rejection of the null hypothesis, utilize an off-the-shelf recursive least-squares algorithm to date the exact periods of explosive dynamics (see, e.g., Harvey et al., 2020a; Shi and Phillips, 2023).

9 Conclusion

This paper introduced a novel approach for testing for periodically collapsing bubbles using unit root quantile autoregressions. The underlying idea is that, within a conditional quantile regression framework, bubble expansions drive the largest autoregressive root of asset price series above unity at upper quantiles. From the different quantile autoregression unit root tests proposed in the literature, I employed the coefficient-based U_n and a modified version of the Kolmogorov-Smirnov QKS_α of Koenker and Xiao (2004) to examine this hypothesis. Monte Carlo experiments demonstrated that the two tests have good finite-sample size and power properties, and can outperform the popular recursive least-squares SADF and GSADF.

An application to Bitcoin, the S&P 500 index, and the U.S. housing market, showed that the null of a unit root can be rejected at upper quantiles for all markets. Furthermore, it revealed significant heterogeneity in persistence across quantiles, closely resembling the asymmetric dynamics implied by periodically collapsing bubbles. This finding is robust to accounting for economic fundamentals by using price-to-fundamental ratios and/or an indirect approach based on futures prices. Overall, the Monte Carlo experiments and the empirical applications suggest that unit root quantile autoregressions can provide new insights into the speculative dynamics characterizing asset markets and, thus, are a useful addition to the bubble detection toolkit of applied researchers.

Future research could extend the analysis to a multivariate setting. One promising avenue is to examine the relationship between asset prices and fundamentals under speculative bubbles

stamping algorithms of Phillips et al. (2011) and Phillips et al. (2015a,b). See `https://www.dallasfed.org/research/international/houseprice#tab2` and Pavlidis et al. (2016).

at different quantiles. Analogous to the least-squares PSY-IVX approach suggested by Shi and Phillips (2023), which allows for different degrees of persistence in fundamentals, one could adopt the IVX quantile regression method of Lee (2016). Another direction is to utilize spot and futures prices or survey data and employ Fama-type IVX quantile regressions to test the null hypothesis of no bubbles similar to Pavlidis et al. (2017, 2018). More broadly, a final direction for future work could involve developing quantile based methodologies for testing Granger causality. The work by Shi et al. (2018) and Shi et al. (2020) introduces innovative least-squares techniques that allow for episodic Granger causality between time series. Adopting a quantile approach could offer greater robustness to distributional assumptions and uncover new causal relationships.

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