Symmetric Frameworks on Surfaces

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Abstract

We present a study of combinatorial constructions that are related to understanding the structure of bar-joint frameworks that are restricted to a subspace of $\mathbb{R}^d$. There are two such restrictions of $\mathbb{R}^d$ we approach.

We combine two recent extensions of the generic theory of rigid and flexible graphs by considering symmetric frameworks in $\mathbb{R}^3$ restricted to move on a surface. In Chapter 3 necessary combinatorial conditions are given for a symmetric framework on the sphere, cylinder, cone, elliptical cylinder and ellipsoid to be isostatic (i.e. minimally infinitesimally rigid) under any finite point group symmetry. In Chapter 4 we focus exclusively on the cylinder. In every case when the symmetry group is cyclic, which we prove restricts the group to being inversion, half-turn or reflection symmetry, these conditions are then shown to be sufficient under suitable genericity assumptions, giving precise combinatorial descriptions of symmetric isostatic graphs in these contexts.

Motivated by applications where boundary conditions play a significant role, one may generalise and consider linearly constrained frameworks where some vertices are constrained to move on fixed affine subspaces. Additional to Chapter 3, the necessary combinatorial conditions are given for a symmetric linearly constrained framework in $\mathbb{R}^d$ to be isostatic under a choice finite point group symmetries. In Chapter 5, we consider linearly constrained frameworks in the plane, and the case of rotation symmetry groups whose order is either 2 or odd. These conditions are then shown to be sufficient under suitable genericity assumptions, giving precise combinatorial descriptions of symmetric isostatic graphs in these contexts.

To conclude there is a short chapter in which we suggest ways for this research to be furthered.

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I'd like to thank my supervisors Anthony Nixon and Bernd Schulze. From the first interview I had for the studentship, they were both incredibly warm and welcoming. They designed an ideal first problem, considering the isostatic rigid frameworks on the cylinder with an inversion symmetry, which lends from both of their areas of expertise. This was in many ways an great introduction to wider surface and symmetry problems, and through that I gained much understanding and ability to tackle more technical problems visited in this thesis. Through the COVID-19 pandemic, they sought to provide the best supervision under difficult circumstances. In the later stages of my PhD, they assisted me in submitting two papers, helped me improve my writing, and without the time and effort they put into the papers, this would not have been possible. I am proud of the work we achieved together.

I'd like to thank Stephen Power, my HDC chair, for work in his role beyond what his duty bound him, taking a keen interest in my work, as well as his years of service to rigidity.

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I'd also like to thank my examiners, Sean Prendiville and Alex Fink. They were very welcoming during my viva and gave very detailed feedback which has improved the thesis throughout.

Finally, to my parents, David and Jayne. I'd like to thank them for having always nurtured my learning, encouraging me to push myself and take on challenges. They have given me both the independence to grow as a person, and support when I have been in need.

Declaration

I hereby declare that this thesis is my own work and none of its contents have previously been submitted for the award of a degree by any university.

The majority of this thesis is the product of research carried out in collaboration with my supervisors, Tony Nixon and Bernd Schulze. The work is based on joint work with Tony and Bernd, and closely follows these pre-prints, namely, work from [33] can be seen throughout Chapters 3 and 4, and work from [34] can be seen throughout Chapters 3 and 5.

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Chapter 1

Introduction

A (bar-joint) *framework* (G, p) is the combination of a finite simple graph $G = (V, E)$ and a map $p : V \to \mathbb{R}^d$ which assigns positions to the vertices, and hence lengths to the edges. With stiff bars for the edges and full rotational freedom for the joints representing the vertices, the topic of rigidity theory concerns whether the framework may be deformed without changing the graph structure or the bar lengths. While 'trivial' motions are always possible due to actions of the Euclidean isometry group, the framework is *flexible* if a non-trivial motion is possible and *rigid* if no non-trivial motion exists.

The problem of determining whether a given framework is rigid is computationally difficult for all $d \geq 2$ [1]. However, every graph has a typical behaviour in the sense that either all 'generic' (i.e. almost all) frameworks with the same underlying graph are rigid or all are flexible. So, generic rigidity depends only on the graph and is often studied using a linearisation known as infinitesimal rigidity, which is equivalent to rigidity for generic frameworks [3]. On the real line it is a simple folklore result that rigidity coincides with graph connectivity. In the plane a celebrated theorem due to Polaczek-Geiringer [36], often referred to as Laman's theorem due to a rediscovery in the 1970s [23], characterises the generically rigid graphs precisely in terms of graph sparsity counts, and these combinatorial conditions can be checked in polynomial time. Combinatorial characterisations of generically rigid graphs in dimension 3 or higher have not yet been found. Much of the recent work in rigidity has been motivated by this open problem in three dimensions. One natural approach to this setting is to consider restricting the framework, fully or partially, to some subspace of $\mathbb{R}^d$.

One such case is to replace $\mathbb{R}^d$ with a d-dimensional manifold (or d-fold for short). It seems unlikely that rigidity becomes easier on a d-fold when $d \geq 3$ and hence it is natural to consider rigidity for frameworks realised on 2-folds. Specifically, let S be a 2-fold embedded in $\mathbb{R}^3$ and let the framework (G, p) be such that $p : V \to S$, but the 'bars' are straight Euclidean bars (and not surface geodesics). Supposing S is smooth and an irreducible real algebraic set, and the subgroup of Euclidean isometries that preserve S has dimension at least 1, characterisations of generic rigidity were proved in [30, 31].

A linearly constrained framework is a bar-joint framework in which certain vertices are constrained to lie in given affine subspaces, in addition to the usual distance constraints between pairs of vertices. Linearly constrained frameworks arise naturally in practical applications where objects may be constrained to move, for example, on the ground, along a wall, or in a groove. In particular, slider joints are very common in mechanical engineering (see, e.g., [10]) and have been applied to modelling boundary conditions in biophysics [45].

Streinu and Theran [44] proved a characterisation of generic rigidity for linearly constrained frameworks in $\mathbb{R}^2$. The articles [8, 20] provide an analogous characterisation for generic rigidity of linearly constrained frameworks in $\mathbb{R}^d$ as long as the dimensions of the affine subspaces at each vertex are sufficiently small (compared to d), and [15] characterises the stronger notion of global rigidity of two dimensional linearly constrained frameworks.

Separately, the genericity hypothesis, while natural from an algebraic geometry viewpoint, does not apply in many practical applications of rigidity theory. In particular, structures in mechanical and structural engineering, computer-aided design, biophysics, and materials science often exhibit non-trivial symmetry. This has motivated multiple groups of researchers to study the rigidity of symmetric structures over the last two decades, which has led to an explosion of results in this area. We direct the reader to [6, 42] for details. Importantly, there are two quite different notions of symmetric rigidity that one may consider. Firstly, *forced symmetric* *rigidity* concerns frameworks that are symmetric and only motions that preserve the symmetry are allowed (that is, a framework may be flexible but if all the non-trivial motions destroy the symmetry then it is still 'forced symmetric rigid'). Secondly, *incidental symmetric rigidity* again concerns symmetric frameworks, but the question of whether they are rigid is the same as in the non-symmetric case.

It is incidental symmetry that we focus on in this thesis. More specifically, we are interested in describing, combinatorially, when a generic symmetric framework on a surface such as the infinite cylinder, or a linearly constrained framework is *isostatic*, i.e. minimally infinitesimally rigid in the sense that it is infinitesimally rigid but ceases to be so after deleting any edge. The corresponding question in the Euclidean plane has been studied in [38, 39]. In these papers, Laman-type results in the plane have been established for the groups generated by a reflection, the half-turn and a three-fold rotation, but these problems remain open for the other groups that allow isostatic frameworks.

This thesis begins with a chapter outlining the background material required when studying rigidity theory, and more specifically introduces frameworks restricted to lie on a surface, and linearly constrained frameworks, both with and without symmetry. We are then required to visit some basic material on representation theory.

This thesis contains the first analysis of incidental symmetric rigidity on 2-folds, given in [33] which focuses on the cylinder, and the first rigidity-theoretic analysis of symmetric linearly constrained frameworks, given in [34] which focuses on the plane. In Chapter 3, the representation-theoretic necessary conditions for isostaticity are given for all relevant symmetry groups of the cylinder, sphere, cone, elliptical cylinder and ellipsoid, as well as for linearly constrained frameworks for all relevant symmetry groups in the plane, three dimensions, and some symmetry groups of interest in d-dimensional space.

Typically it is much harder to prove sufficient conditions for rigidity. This is demonstrated by the following. Firstly, in [30] the necessary conditions for minimally rigid frameworks are established for all smooth two dimensional manifolds, but sufficiency has only been established for a few select surfaces. Secondly, in [7] the necessary conditions were established and conjectured to be sufficient for

2-dimensional symmetric isostatic frameworks. This was confirmed for frameworks subject to the symmetry constraints imposed by a reflection, a half-turn [39] or a three-fold rotation [38], but it remains open for the dihedral groups. Indeed, this difference is not restricted only to isostatic frameworks. In [41] the authors consider symmetric infinitesimally rigid frameworks on the plane, establishing necessary and sufficient conditions for symmetric frameworks for reflection, half-turn and three-fold rotation, and necessary conditions for any other cyclic group.

In order to prove sufficiency and hence give combinatorial characterisations we will develop detailed geometric and combinatorial tools. For this reason we restrict our scope to certain important cases. In the first such case we give combinatorial characterisations of frameworks on the cylinder. To see why the cylinder, first consider the 'simplest' 2-fold: the sphere. In this case, Laman-type theorems either follow from a projective transfer between infinitesimal rigidity in the plane and on the sphere [5, 10] or seem to be equally as challenging as the open problems in the plane. The cylinder provides the first case when the combinatorial sparsity counts change and hence lead to new classes of graphs and rigidity matroids to investigate. In Chapter 4, we give complete combinatorial characterisations of symmetry-generic isostatic frameworks on the cylinder for the groups generated by an inversion (Section 4.3), a half-turn (Section 4.4) and a reflection (Section 4.5). The proofs rely on symmetry-adapted Henneberg-type recursive construction moves described in Section 4.1. In the case of isostatic frameworks in $\mathbb{R}^2$ only the well known 0- and 1-extension operations are needed to prove Laman's theorem [36, 23]. For the cylinder several additional operations were needed with associated combinatorial and geometric difficulties [30]. The additional conditions isostatic frameworks under symmetry must satisfy differ for each group, necessitating group-by-group combinatorial (and hence geometric) analyses. Fortunately, in each of the cases we study in detail only moderate extensions of existing geometric arguments are needed and hence we present a number of those for an arbitrary symmetry group (Section 4.1). On the other hand there are significant additional combinatorial difficulties in the recursive construction proof technique which takes up the main technical parts of this chapter (Sections 4.3, 4.4 and 4.5).

In Chapter 5, we give the complete combinatorial characterisations of symmetry-generic isostatic frameworks for the groups generated by a rotation of order either 2 or odd. These results, Theorems 5.2.10 and 5.3.7, are proved in Sections 5.2 and 5.3. The proofs of these combinatorial characterisations rely on symmetry-adapted Henneberg-type recursive construction moves described in Section 5.1.

Chapter 2

Rigidity Theory

We will begin our study with the necessary background material for later chapters. We first introduce basic graph theory, and introduce symmetric graphs. We follow this with a short section on representation theory which will enable us to prove the results in Chapter 3.

2.1 Graph theory

A *multigraph* G is the triple (V, E, L), where V is a non-empty finite set, E is a multiset of unordered pairs of distinct elements in V, and L is a multiset of pairs of duplicate elements in V. A *looped simple graph* is a multigraph where E is restricted to be a set (L remains a multiset). A *simple graph* is a multigraph where E is restricted to be a set and $L = \emptyset$ (thus will be written $G = (V, E)$). We will simply say *graph*, to exclusively mean a simple graph or a looped simple graph, where the distinction should be clear from the context. It is non-standard to view a graph as containing loops. We take this approach as loops will be helpful for the purpose of defining the constraints at vertices in linearly constrained rigidity.

Given a graph $G = (V, E, L)$, we say $v \in V$ is a *vertex* of G, $\{u, v\} \in E$ is an *edge* of G, and $\{v, v\} \in L$ is a *loop* of G. Where two (or more) edges or loops share the notation above, we may indicate this with subscripts, for example $\{v, v\}_i$. For a graph H, we may write $V(H), E(H), L(H)$ for the *vertex set*, *edge set*, *loop set* of H respectively. We will often choose to denote the edge $\{u, v\}$ by uv or vu, and similarly the loop $\{v, v\}$ by vv. For $e = uv \in E$ and $l = ww \in L$, we say that u and v are *adjacent* in G, that u and v are *incident* to the edge e, that w is *incident* to the loop l, and that u and v are *endpoints* of the edge e.

Graphs $G_1 = (V_1, E_1, L_1)$ and $G_2 = (V_2, E_2, L_2)$ are said to be *isomorphic* if there exists a bijective function $f : V_1 \to V_2$, such that $uv \in E_1$ and $ww \in L_1$ if and only if $f(u)f(v) \in E_2$ and $f(w)f(w) \in L_2$. For isomorphic G and H, we write $G \cong H$.

A *subgraph* $H = (V', E', L')$ of G, denoted $H \leq G$, is a graph with $V' \subseteq V$, $E' \subseteq E$, $L' \subseteq L$. A subgraph is said to be *proper* unless $H \cong G$. Let $X \subset V$ be non-empty. The *induced subgraph* of X in G, written $G[X]$, has vertex set X, edge set $\{uv : uv \in E, u, v \in X\}$, and loop set $\{ww : ww \in L, w \in X\}$. In a simple graph, we will use $i_G(X)$ to denote the number of edges in the induced subgraph $G[X]$ and the set X will be called *k-critical*, for $k \in \mathbb{N}$, if $i_G(X) = 2|X| - k$. For looped simple graphs, we require further notation. Here we write $i_{E+L}(X)$, $i_E(X)$, $i_L(X)$ to denote the number of edges and loops, edges, loops in the induced subgraph $G[X]$ respectively, and the set X will be called *k-critical* and *k-edge-critical*, for $k \in \mathbb{N}$, if $i_{E+L}(X) = 2|X| - k$, and $i_E(X) = 2|X| - k$ respectively. We also write k_X and $\bar{k}_X$ to denote the critical and edge-critical values, that is $k_X = 2|X| - i_{E+L}(X)$ and $\bar{k}_X = 2|X| - i_E(X)$ for $X \subset V$.

Remark 2.1.1. The term *critical* has previously been used in rigidity theory (for example [27]) to refer to a vertex set where the induced subgraph has as many edges as may be permitted for such a graph to be minimally rigid. For symmetry reasons it will turn out that we need to deal with a range of densities of subgraphs, extending the complication involved to establish our results. This will be made evident in Chapters 4 and 5.

For $X, Y \subset V$, $d_G(X, Y)$ will denote the number of edges of the form xy with $x \in X \setminus Y$ and $y \in Y \setminus X$. For a simple graph G, the *degree* of a vertex v in G, denoted $d_G(v)$, is the number of edges incident to v. In looped simple graphs, it is common to think of loops as degenerate edges, and the degree counts the number of edges coming ‘out’ from a vertex. Hence, for a looped simple graph G, the degree of v in G, still written $d_G(v)$, is the number of edges and twice the number of loops incident to v. With this notation, for $H \leq G$, $d_H(v)$ is the degree of v in the subgraph H. We will often suppress subscripts when the graph is clear from the context and use $d(v)$, $i(X)$ and $d(X,Y)$. In our work we will often be concerned with the *minimum degree* of a graph G. We denote this by $\delta(G)$. The *maximum degree* of a graph is denoted by $\Delta(G)$. The *open neighbourhood* of a vertex v, denoted $N(v)$, is the set of all vertices of G adjacent to v. The *closed neighbourhood* of v is the set $N[v] := N(v) \cup \{v\}$.

2.1.1 Graphs

In this short subsection we introduce graphs which we will regularly encounter during our studies. The *complete graph* (on n vertices), denoted K_n, is the simple graph with $uv \in E$ for every distinct pair $u, v \in V$. A *bipartite graph* has vertex set partitioned $V = A \cup B$ where $A \cap B = \emptyset$, and edge set so that $e \in E$ implies e has one endpoint in A and the other in B. We write $K_{n,m}$ as the bipartite graph where $|A| = n$ and $|B| = m$. The *empty graph* (on n vertices), denoted O_n, has $E, L = \emptyset$. A *tree* is a simple graph with $|V| = n$, $|E| = n-1$, such that all subgraphs $H = (V', E')$ satisfy $|E'| \leq |V'| - 1$. The *path* (on n vertices), denoted P_n, is a tree with $\Delta(G) = 2$. The *cycle*, denoted C_n, is the graph obtained from a path P_n by adding an edge between the two degree 1 vertices. Then, W_n denotes the *wheel* over a cycle on $n-1$ vertices ($n \geq 4$), where a new vertex v is added, with v adjacent to all $u \in V - v$. We write $Wd(n,k)$ to denote the windmill, which is k copies of K_n all joined at a single vertex. Examples of these graphs are depicted in Figure 2.1.

2.1.2 Graph operations

We will wish to construct new graphs through *graph operations*. The most basic of these operations are adding or deleting a single edge, loop or vertex. We write $G + x$ and $G - x$ for each of these operations, noting that when adding or removing a vertex, all of the edges or loops incident will be added or deleted. Our main operations of interest are the 0-extensions and 1-extensions.

For a simple graph G with $|V| \geq k$, a $(k,0)$-*extension* of G adds a new vertex of degree k, say v adjacent to $v_1, \dots, v_k$, to generate a new graph $G^+ = (V \cup \{v\}, E \cup$

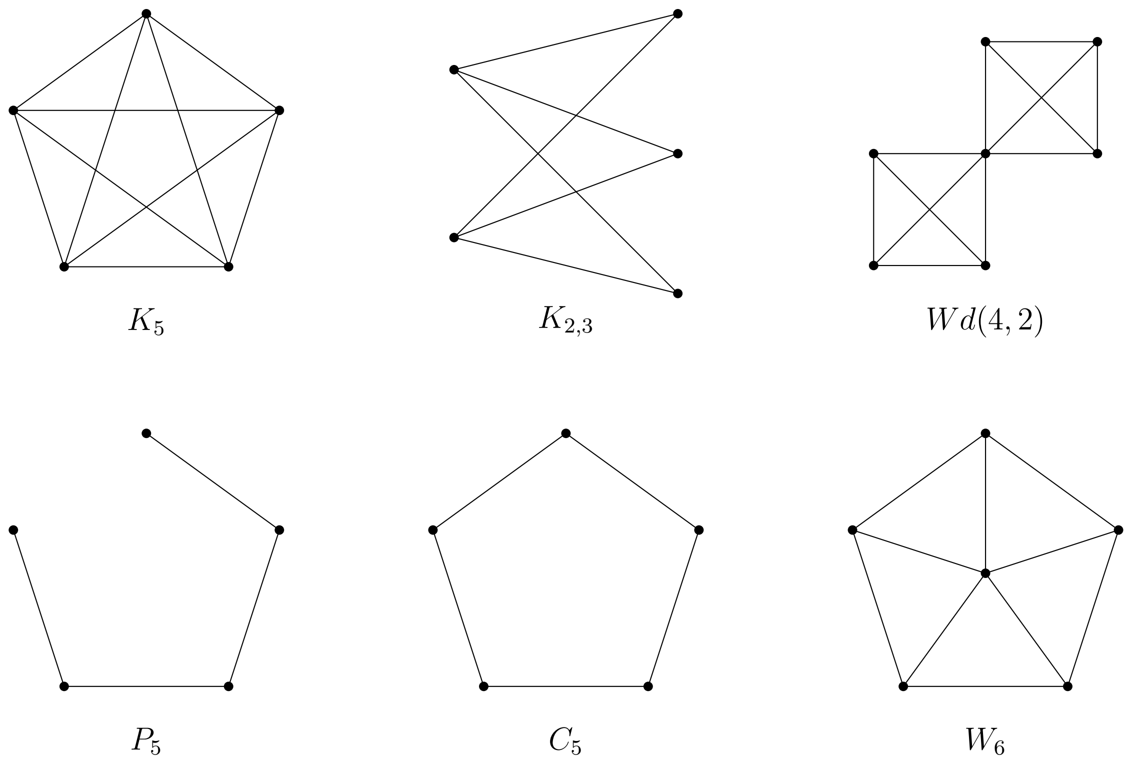

Figure 2.1: Illustrated graphs, $K_5, K_{2,3}, Wd(4,2), P_5, C_5, W_6$.

$\{vv_1, \ldots, vv_k\})$. Conversely, a $(k,0)$-*reduction* of G^+ at v deletes the vertex v and its incident edges, returning the graph G. Where the value of k is clear from the context, these operations will be known as a 0-extension and a 0-reduction.

For a simple graph G with $|V| \geq k+1$ and $|E| \geq 1$, a $(k,1)$-*extension* of G adds a new vertex of degree $k+1$, say v adjacent to $v_1, \ldots, v_{k+1}$, and deletes an edge $e \in E$, where $e = v_iv_j$ for some $1 \leq i, j \leq k+1$, to generate a new graph $G^+ = (V \cup \{v\}, E \cup \{vv_1, \ldots, vv_{k+1}\} \setminus \{e\})$. Conversely, a $(k,1)$-*reduction* of G^+ at v, deletes the vertex v and its incident edges, and adds an edge v_iv_j for $1 \leq i, j \leq k+1$ such that the resultant graph is simple. Where the value of k is clear from the context, these operations will be known as a 1-extension and a 1-reduction.

For a looped simple graph G with $|V| \geq k-1$, a *looped* $(k,0)$-*extension* of G adds a new vertex of degree $k+1$, say v adjacent to $v_1, \ldots, v_{k-1}$ and incident to a loop l, to generate a new graph $G^+ = (V \cup \{v\}, E \cup \{vv_1, \ldots, vv_{k-1}\}, L \cup \{l\})$. Conversely, a *looped* $(k,0)$-*reduction* of G^+ at v deletes the vertex v and its incident edges and loop, returning the graph G. Where the value of k is clear from the context, these operations will be known as a looped 0-extension and a looped 0-reduction.

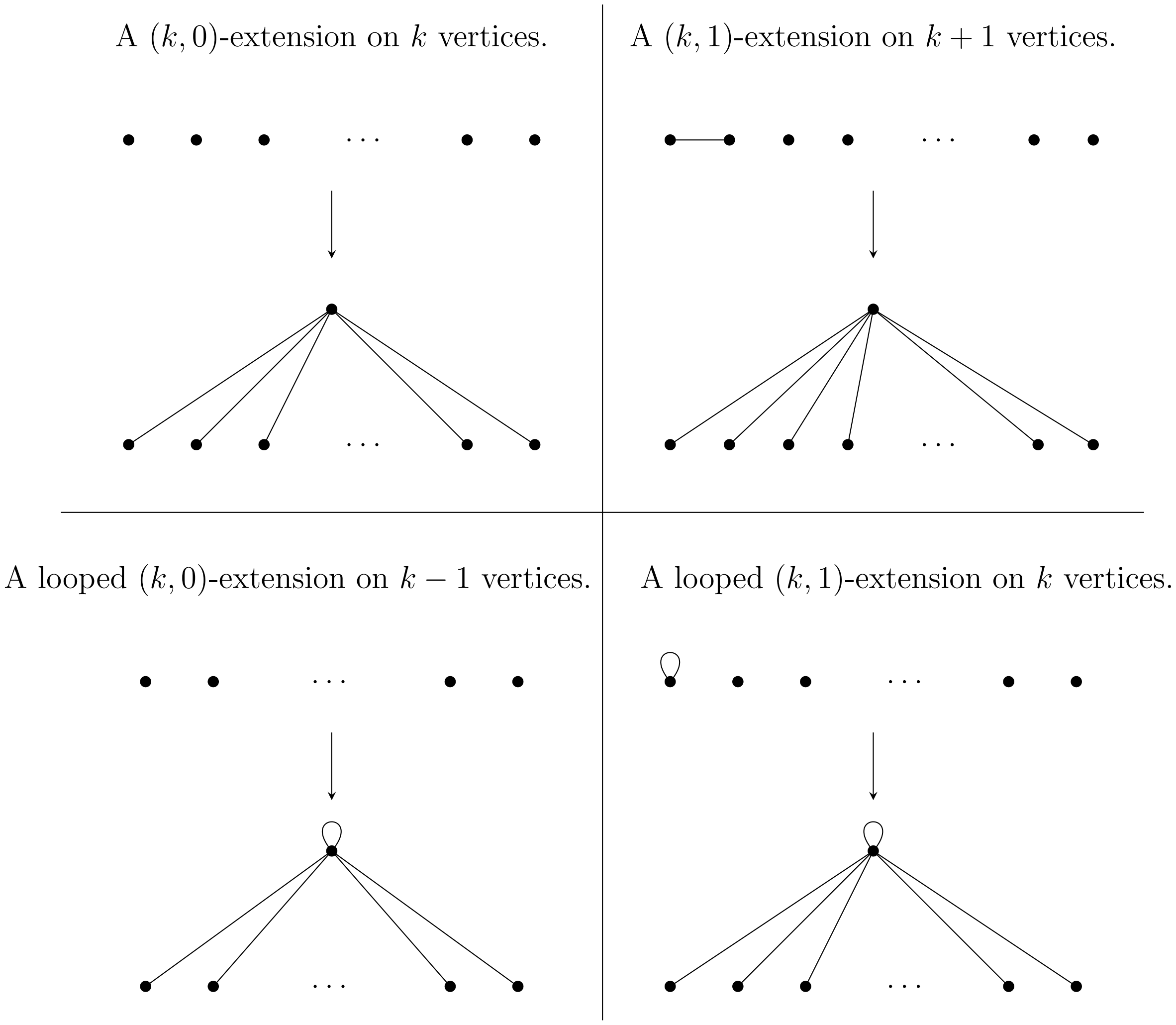

Figure 2.2: Four extension operations which will be visited throughout this thesis.

For a looped simple graph G with $|V| \geq k$ and $|E| \geq 1$, a *looped* $(k, 1)$*-extension* of G adds a new vertex of degree $k + 2$, say v adjacent to $v_1, \dots, v_k$ and incident to a loop l, deletes a loop $l^* \in E$, where $l^* = v_i v_i$ for some $1 \leq i \leq k$, to generate a new graph $G^+ = (V \cup \{v\}, E \cup \{vv_1, \dots, vv_k\}, L \cup \{l\} \setminus \{l^*\})$. Conversely, a $(k, 1)$*-reduction* of G^+ at v deletes the vertex v and its incident edges and loop, and adds a loop $v_i v_i$ for $1 \leq i, j \leq k + 1$. Unlike for a $(k, 1)$-reduction, this loop can cause multiplicity at that vertex since the resultant graph will still be looped simple.

We depict these operations in Figure 2.2. Where the value of k is clear from the context, these operations will be known as a looped 1-extension and a looped 1-reduction. We shall revisit the notion of graph operations throughout.

2.2 Rigidity and frameworks

Let $a, b \in \mathbb{R}^d$ and write $a = (a_1, \dots, a_d), b = (b_1, \dots, b_d)$. *Euclidean space* is the inner product space $(\mathbb{R}^d, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is given by $\langle a, b \rangle = \sqrt{a_1 b_1 + \cdots + a_d b_d}$. The *Euclidean norm* is the function $|| \cdot || : \mathbb{R}^d \to \mathbb{R}$ given by $||a|| = \langle a, a \rangle$. In the following subsection, we introduce rigidity and frameworks which lie in Euclidean space. Similar introductions can be found in [13, 14, 28] which provided me with background throughout my studies. We will also restrict ourselves to simple graphs, and introduce rigidity for looped simple graphs later.

A *configuration* $p : V \to \mathbb{R}^d$ of a graph G is a function that assigns positions to the vertices (and in doing so lengths to the edges) in d-dimensional Euclidean space. A *framework* is a pair (G, p) of a graph and its configuration.

The pair (G, p) as defined above is usually called a *bar-joint* framework. This is due to the view of the framework as having 'rigid' bars for edges, which cannot stretch, bend, break or compress, and perfect joints at the vertices, which allow full rotational freedom.

Two frameworks (G, p) and (G, q) are *equivalent* if $||p(u) - p(v)||^2 = ||q(u) - q(v)||^2$ for all $u, v \in V$ where $uv \in E$. The two frameworks are *congruent* if $||p(u) - p(v)||^2 = ||q(u) - q(v)||^2$ for all $u, v \in V$.

A framework (G, p) in $\mathbb{R}^d$ is *rigid* if there exists an $\epsilon > 0$ such that all frameworks (G, q) which are equivalent to (G, p) and satisfy $||p(v) - q(v)|| < \epsilon$ for all $v \in V$ are also congruent to (G, p). In the above, we say that (G, q) is *sufficiently close* to (G, p). A framework (G, p) in $\mathbb{R}^d$ is *globally rigid* if all equivalent frameworks are also congruent. An alternate but equivalent definition of rigidity comes from motions of the framework. A *continuous motion* of a framework (G, p) is a function $\lambda : I \times V \to \mathbb{R}^d$, where:

- $I \subseteq \mathbb{R}$ is an interval, and for each $v \in V$, $\lambda(t, v)$ is a *curve* (a continuous image of an interval) in $\mathbb{R}^d$;
- for each $t \in I$, λ_t is a configuration of G, with $\lambda_t(v) := \lambda(t, v)$;
- for some $t_0 \in I$, $\lambda_{t_0} = p$;

- for any $t \in I$, (G, λ_t) is equivalent to (G, p).

A framework is rigid if there exists an $\epsilon > 0$ for which all continuous motions of (G, p), say λ with $\lambda_{t_0} = p$, satisfy the condition that (G, λ_t) is congruent to (G, p) for all $|t - t_0| < \epsilon$. A continuous motion is called *trivial* if for any $t \in I$, (G, λ_t) is congruent to (G, p).

It is computationally NP-hard to determine whether a specific framework is globally rigid in $\mathbb{R}^d$ [37, Lemma 4.4] and rigid in $\mathbb{R}^d$ for $d \geq 2$ [1, Theorem 1.2]. However, with an additional condition on the configuration p, rigidity of a framework relies only on the graph. A framework (and its configuration) is *generic* if the set of all positions of vertices is an algebraically independent set over $\mathbb{Q}$. That is to say, with the vertices of G labelled $v_1, \ldots, v_n$, there is no polynomial with rational coefficients not all zero, $\mathbf{P}(x_1, \ldots, x_n)$ with $x_i \in \mathbb{R}^d$ for all i, such that $\mathbf{P}(p(v_1), \ldots, p(v_n)) = 0$. This condition on p appears restrictive, and for many applications is too restrictive. Indeed, in both cases of our study, restricting the vertices to lie on a surface and for the framework to be symmetric, the configuration is not generic. In these cases, we will require slightly altered conditions on p, which will be introduced in Sections 2.3 and 2.4. For rigidity in $\mathbb{R}^d$, the set of generic configurations form an open and dense set $p(V) \in \mathbb{R}^{d|V|}$ [3], so almost all frameworks are generic. The primary motivation for studying generic frameworks is the following definition. A graph G is *rigid* (*globally rigid*) if there exist a generic framework (G, p) which is rigid (globally rigid respectively). It was proven in [3] that when one generic framework is rigid, all generic frameworks are rigid.

Example 2.2.1. *The graph K_4 is globally rigid as any pair of vertices is adjacent, so all equivalent frameworks must be congruent. If we remove any edge e, $K_4 - e$ being rigid in $\mathbb{R}^2$ is a classical result. Removing a further edge either results in $K_{1,3}$ or C_4, both of which are not rigid in $\mathbb{R}^2$.*

In Example 2.2.1 we see motivation for the following defintion. A graph G (and similarly its framework (G, p)) is *minimally rigid* or *isostatic* if it is rigid and removing any edge results in a graph (framework resp.) which is not rigid. [i]

[i]It is possible that a rigid graph has no spanning isostatic subgraph (we will see an example

Each edge gives a constraint on possible motions of the vertices, which provides a system of equations to classify if a framework is rigid. However, as presented, the system of equations are quadratic, and thus solutions difficult to acquire. Define the *rigidity map* of a framework (G, p) as the function $f_G : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$ by

$$f_G(p) = (\dots, ||p(u) - p(v)||^2, \dots) \text{ for all } uv \in E.$$

The reason we are interested in the square edge lengths is that we may take the Jacobian of this function to linearise the problem of rigidity. We say that $p \in \mathbb{R}^{d|V|}$ is a *regular point* of f_G if $\operatorname{rank} df_G(p) = \max\{\operatorname{rank} df_G(q) : q \in \mathbb{R}^{d|V|}\}$. Note that a generic configuration will also be a regular point of the rigidity map as any row dependencies would contradict the definition of generic configuration.

The *rigidity matrix* of a framework (G, p), denoted $R(G, p)$, is the $|E| \times d|V|$ matrix with $R(G, p) = \frac{1}{2} df_G|_p$. Each row of the matrix corresponds to one edge and each d columns to one vertex. The row corresponding to the squared length $||p(u) - p(v)||^2$ has d-tuple entry $p(u) - p(v)$ in the d columns for u and d-tuple entry $p(v) - p(u)$ in the d columns for v, with zeros otherwise.

An *infinitesimal motion* of a framework (G, p), denoted $\dot{p} : V \to \mathbb{R}^d$, is a continuous motion which assigns infinitesimal velocities to the vertices of G. Motivation of this study arises from infinitesimal motions which are a result of considering *differentiable motions* (a continuous motion which is differentiable) of (G, p), and studying the derivative at t_0. Since infinitesimal motions are linear, they satisfy the following equation:

$$(p(u) - p(v)) \cdot (\dot{p}(u) - \dot{p}(v)) = 0 \text{ for all } uv \in E.$$

This equation leads to the observation that for any infinitesimal motion $\dot{p}$ of (G, p), $R(G, p)\dot{p} = 0$. An infinitesimal motion is *trivial* if it satisfies $R(K_n, p)\dot{p} = 0$, where $|V| = n$. Any trivial infinitesimal motion belongs to the kernel of the rigidity matrix, and therefore we may apply the rank-nullity theorem to deduce the following lemma.

later in Figure 4.17). Hence, the study of isostatic frameworks which we undertake cannot always be extended to rigidity as a whole. In Section 4.6.2 we show to two symmetries where our results can be extended.

Lemma 2.2.2. *[4] Let (G, p) be a framework in $\mathbb{R}^d$ and $|V| = n$. Then*

$$rank(R(G,p)) \leq \begin{cases} \binom{n}{2} & \text{if } n \leq d; \\ dn - \binom{d+1}{2} & \text{if } n \geq d+1. \end{cases}$$

Furthermore, when $n \geq d+1$, (G, p) is infinitesimally rigid if and only if equality holds.

In the above lemma, we say that (G, p) is *infinitesimally rigid* if the only infinitesimal motions are the trivial solutions that arise from Euclidean congruences of $\mathbb{R}^d$.

It was shown in an unnumbered theorem of [4] that for p a regular point, (G, p) is rigid in $\mathbb{R}^d$ if and only if (G, p) is infinitesimally rigid in $\mathbb{R}^d$.

A graph $G = (V, E)$ is (k, l)*-sparse* if $|E'| \leq k|V'| - l$ for all subgraphs (V', E') of G with $|V'| \geq k+2$. G is (k, l)*-tight* if it is (k, l)-sparse and $|E| = k|V| - l$. A graph is *Laman* if it is $(2, 3)$-tight. Note from Lemma 2.2.2 it follows that for a framework to be minimally infinitesimally rigid in $\mathbb{R}^d$, the graph should be $(d, \binom{d+1}{2})$-tight.

Theorem 2.2.3. *[23, Theorems 6.3, 6.4] Every Laman graph can be constructed from K_2 by a sequence of $(2, 0)$-extensions and $(2, 1)$-extensions.*

Laman graphs and their construction form an important case of study in rigidity due to the following theorem. This was originally discovered by Polaczek-Geiringer [36] in 1927, with rediscovery by Laman in 1970, which we cite due to language of origin.

Theorem 2.2.4. *[23, Theorem 5.6] The generic framework (G, p) in $\mathbb{R}^2$ is isostatic if and only if G is a Laman graph.*

It is natural to hope that similar classifications exist in higher dimensions. Beginning first with three dimensions, analysis of the rank of the rigidity matrix gives that any isostatic framework must have a graph which is $(3, 6)$-tight. However, we can find examples of $(3, 6)$-tight graphs which are not rigid. One such example, named the double-banana, can be seen in Figure 2.3. This further leads to the condition that G must be 3-connected, that is, deleting any two vertices of G still

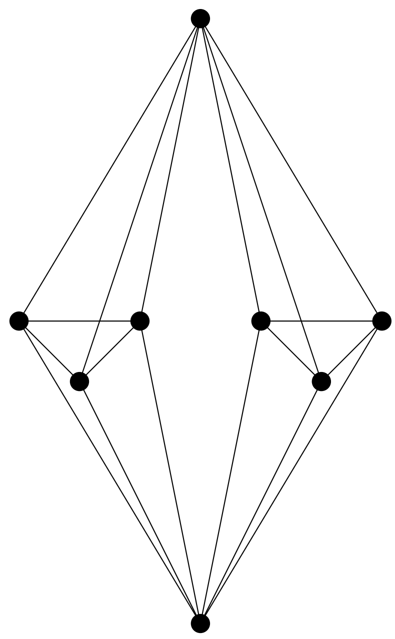

Figure 2.3: The double banana, an example of a $(3,6)$-tight graph which is not rigid in $\mathbb{R}^3$.

results in a connected graph. This provides an additional level of difficulty, and no characterisation has yet been found.

2.3 Frameworks on surfaces

Let S denote a 2-dimensional manifold embedded in $\mathbb{R}^3$. We will refer to S as a surface. A framework (G,p) on S is the combination of $G=(V,E)$ and a map $p: V \to \mathbb{R}^3$ such that $p(v) \in S$ for all $v \in V$ and $p(u) \neq p(v)$ for all $uv \in E$. We also say that (G,p) is a *realisation* of the graph G on S. (G,p) is *rigid on S* if every framework (G,q) on S that is sufficiently close to (G,p) arises from an isometry of S.

While much of this section remains true for a wider selection of surfaces, in the subsection that follows we will focus on the important case when S is a cylinder. Throughout, $\mathcal{Y}$ denotes the infinite circular cylinder; that is the real algebraic subvariety of $\mathbb{R}^3$ defined by the irreducible polynomial $x^2+y^2=1$.

As in the Euclidean case, it is a computationally challenging problem to determine if a given framework (G,p) is rigid on $\mathcal{Y}$. Hence we follow the standard path of linearising and considering infinitesimal motions as follows.

Given a framework $(G, \hat{p})$ on $\mathcal{Y}$, we are interested in the set of frameworks (G, p) on $\mathcal{Y}$ which are equivalent to $(G, \hat{p})$ where $\hat{p}(v_i) = (\hat{x}_i, \hat{y}_i, \hat{z}_i)$ and $p(v_i) = (x_i, y_i, z_i)$. The set of all frameworks on $\mathcal{Y}$ that are equivalent to $(G, \hat{p})$ is given by the set of solutions to the following system of equations:

$$\|p(v_i) - p(v_j)\|^2 = c_{ij} \qquad (v_i v_j \in E) \qquad (2.3.1)$$

$$x_i^2 + y_i^2 = 1 \qquad (v_i \in V) \qquad (2.3.2)$$

where $c_{ij} = \|\hat{p}(v_i) - \hat{p}(v_j)\|^2$. We can differentiate these equations to obtain the following linear system for the unknowns $\dot{p}(v_i)$, $v_i \in V$:

$$(p(v_i) - p(v_j)) \cdot (\dot{p}(v_i) - \dot{p}(v_j)) = 0 \qquad (v_i v_j \in E) \qquad (2.3.3)$$

$$x_i \dot{x}_i + y_i \dot{y}_i = 0 \qquad (v_i \in V). \qquad (2.3.4)$$

Solutions to this linear system are *infinitesimal motions*. We say that $(G, \hat{p})$ is *infinitesimally rigid* if the only infinitesimal motions are the trivial solutions that arise from Euclidean congruences of $\mathbb{R}^3$ that preserve $\mathcal{Y}$ (that is, translations in the z-direction and rotations about the z-axis, or combinations thereof). If (G, p) is not infinitesimally rigid it is called *infinitesimally flexible*. The trivial solutions may be referred to as the *trivial* infinitesimal motions, or simply *trivial motions*. Equivalently, $(G, \hat{p})$ is *infinitesimally rigid* if the rank of the matrix of coefficients of the system is $3|V| - 2$ [30]. This matrix, the *rigidity matrix of* (G, p) *on* $\mathcal{Y}$, denoted $R_{\mathcal{Y}}(G, p)$ has $3|V|$ columns and $|E| + |V|$ rows. The rows corresponding to (2.3.3) have the form

$$\begin{pmatrix} \dots & 0 & p(v_i) - p(v_j) & 0 & \dots & 0 & p(v_j) - p(v_i) & 0 & \dots \end{pmatrix}$$

and the rows corresponding to (2.3.4) have the form

$$\begin{pmatrix} \dots & 0 & (x_i, y_i, 0) & 0 & \dots \end{pmatrix}.$$

A framework (G, p) is called *isostatic* if it is infinitesimally rigid and *independent* in the sense that the rigidity matrix of (G, p) on $\mathcal{Y}$ has no non-trivial row dependence. Equivalently, (G, p) is isostatic if it is infinitesimally rigid and deleting any single edge results in a framework that is not infinitesimally rigid. A framework (G, p) on $\mathcal{Y}$ is *completely regular* if the rigidity matrix $R_{\mathcal{Y}}(K_{|V|}, p)$ of the complete graph on V and every square submatrix has maximum rank among all realisations of $K_{|V|}$ on $\mathcal{Y}$. In the completely regular case, rigidity and infinitesimal rigidity on a "smooth" surface (such as any considered in this thesis) coincide [30, Theorem 3.8]. Note that the set of all completely regular realisations of G on $\mathcal{Y}$ is an open dense subset of the set of all realisations of G on $\mathcal{Y}$. Thus, we may define a graph G to be *isostatic (independent, rigid) on* $\mathcal{Y}$ if there exists a framework (G, p) on $\mathcal{Y}$ that is isostatic (independent, infinitesimally rigid) on $\mathcal{Y}$.

It follows from the definitions that the smallest (non-trivial) rigid (or isostatic) graph on $\mathcal{Y}$ is the complete graph K_4. In [30] exactly which graphs are rigid on $\mathcal{Y}$ was characterised. The characterisation uses the following definition which will be one of the fundamental objects of study in this thesis. Recall a graph $G = (V, E)$ is $(2, 2)$-*sparse* if $|E'| \leq 2|V'| - 2$ for all subgraphs (V', E') of G. G is $(2, 2)$-*tight* if it is $(2, 2)$-sparse and $|E| = 2|V| - 2$.

Theorem 2.3.1. *[30, Theorem 5.4] A graph G is isostatic on $\mathcal{Y}$ if and only if G is $(2, 2)$-tight.*

Interestingly, the set of $(2, 2)$-tight graphs is exactly the set of simple graphs obtained as the edge-disjoint union of two spanning trees [26, Theorem 1]. We derive symmetry adapted results in Section 4.6.

While the theorem gives a complete answer in the generic case, this thesis will improve this answer to apply under the presence of non-trivial symmetry. To see the potential complications that can arise when the genericity hypothesis is weakened one might consider the results of [17] which apply to frameworks on $\mathcal{Y}$ that are generic except for one simple failure: two vertices are located in the same place.

2.4 Linearly constrained frameworks

Following [8], we define a *linearly constrained framework in* $\mathbb{R}^d$ to be a triple (G, p, q) where $G = (V, E, L)$ is a looped simple graph, $p : V \to \mathbb{R}^d$ is injective and $q : L \to \mathbb{R}^d$. For $v_i \in V$ and $\ell_j \in L$ we put $p(v_i) = p_i$ and $q(\ell_j) = q_j$.

An *infinitesimal motion* of (G, p, q) is a map $\dot{p} : V \to \mathbb{R}^d$ satisfying the system of linear equations:

$$(p_i - p_j) \cdot (\dot{p}_i - \dot{p}_j) = 0 \quad \text{for all } v_i v_j \in E \tag{2.4.1}$$

$$q_j \cdot \dot{p}_i = 0 \quad \text{for all incident pairs } v_i \in V \text{ and } \ell_j \in L. \tag{2.4.2}$$

The second constraint implies that the infinitesimal velocity of each $v_i \in V$ is constrained to lie on the hyperplane through p_i with normal vector q_j for each loop ℓ_j incident to v_i.

The *rigidity matrix* $R(G, p, q)$ of the linearly constrained framework (G, p, q) is the matrix of coefficients of this system of equations for the unknowns $\dot{p}$. Thus $R(G, p, q)$ is a $(|E| + |L|) \times d|V|$ matrix, in which: the row indexed by an edge $v_i v_j \in E$ has $p_i - p_j$ and $p_j - p_i$ in the d columns indexed by v_i and v_j, respectively and zeros elsewhere; and the row indexed by a loop $\ell_j = v_i v_i \in L$ has q_j in the d columns indexed by v_i and zeros elsewhere. The $|E| \times d|V|$ sub-matrix consisting of the rows indexed by E is the *bar-joint rigidity matrix* $R(G - L, p)$ of the bar-joint framework $(G - L, p)$.

The framework (G, p, q) is *infinitesimally rigid* if its only infinitesimal motion is $\dot{p} = 0$, or equivalently if $\operatorname{rank} R(G, p, q) = d|V|$. A framework (G, p, q) is called *isostatic* if it is infinitesimally rigid and *independent* in the sense that the rigidity matrix of (G, p, q) has no non-trivial row dependence. Equivalently, (G, p, q) is isostatic if it is infinitesimally rigid and deleting any single edge results in a framework that is not infinitesimally rigid.

Example 2.4.1. *Suppose that G consists of a single vertex with one loop and (G, p, q) is a framework in $\mathbb{R}^2$. Then G is not infinitesimally rigid since the translation along the line corresponding to the loop is an infinitesimal motion (and any* *infinitesimal motion is considered non-trivial in our context).*

Similarly a complete graph, realised generically, is infinitesimally rigid in any dimension as a bar-joint framework. However as a linearly constrained framework it is not infinitesimally rigid since the translations and rotations are infinitesimal motions.

In this thesis, our attention is solely on infinitesimal motions, with all larger motions (those over an interval of positive length) having been discussed for background.

A linearly constrained framework (G, p, q) in $\mathbb{R}^d$ is *generic* if rank $R(G, p, q) \geq$ rank $R(G, p', q')$ for all frameworks (G, p', q') in $\mathbb{R}^d$.

We say that the looped simple graph G is *rigid* in $\mathbb{R}^d$ if rank $R(G, p, q) = d|V|$ for some realisation (G, p, q) in $\mathbb{R}^d$, or equivalently if rank $R(G, p, q) = d|V|$ for all *generic* realisations (G, p, q). Similarly, we define G to be *isostatic (independent)* if there exists a framework (G, p, q) that is isostatic (independent).

Streinu and Theran gave the following characterisation of looped simple graphs which are rigid in $\mathbb{R}^2$. We will say that $G = (V, E, L)$ is: *sparse* if $|E'| + |L'| \leq 2|V'|$ for all subgraphs (V', E') of G and $|E'| \leq 2|V'| - 3$ for all simple subgraphs with $|E'| > 0$; and *tight* if it is sparse and $|E| + |L| = 2|V|$.

Theorem 2.4.2. *[44, Theorem B] A generic linearly constrained framework* (G, p, q) *in* $\mathbb{R}^2$ *is isostatic if and only if* G *is tight.*

While the theorem gives a complete answer in the generic case, we will extend this to apply under the presence of non-trivial symmetries.

While for bar-joint frameworks little is known when $d \geq 3$, in the linearly constrained case characterisations are known when suitable assumptions are made on the affine subspaces defined by the linear constraints. Jackson, Nixon, and Tanigawa [20] extended the results of Cruickshank, Guler, Jackson, and Nixon [8] to give the following characterisation of looped simple graphs with each vertex incident to sufficiently many loops, which are rigid in $\mathbb{R}^d$. We will say G is *k-sparse* for some integer $k \geq 1$ if $i_{E+L}(X) \leq k|X|$ for all $X \subseteq V$, and that G is *k-tight* if it is k-sparse and $|E \cup L| = k|V|$. Recall $i_{E+L}(X)$ is the number of edges and loops in the induced subgraph $G[X]$.

Theorem 2.4.3. *[20, Theorem 3.2] Suppose $d \geq 2$ is an integer and G is a looped simple graph with the property that every vertex of G is incident with at least $\lfloor \frac{d}{2} \rfloor$ loops. Then G is isostatic in $\mathbb{R}^d$ if and only if G is d-sparse and K_{d+2}-free.*

It would be an interesting future project to extend our analysis to higher dimensions. In Sections 3.9 and 3.10 we begin such analysis by considering the fixed element counts for symmetric graphs.

2.5 Symmetric frameworks

2.5.1 Symmetric frameworks on surfaces

Let $G = (V, E)$ be a graph and Γ be a finite group. Then the pair (G, ϕ) is called Γ-symmetric if $\phi : \Gamma \to \text{Aut}(G)$ is a homomorphism, where $\text{Aut}(G)$ denotes the automorphism group of G. If ϕ is clear from the context we often also simply write G instead of (G, ϕ).

Let (G, ϕ) be a Γ-symmetric graph. Then, for a homomorphism $\tau : \Gamma \to O(\mathbb{R}^3)$ and an embedded surface S of $\mathbb{R}^3$, we say that a framework (G, p) is Γ-*symmetric on S (with respect to ϕ and τ)*, or simply $\tau(\Gamma)$-*symmetric*, if $\tau(\gamma)p_i = p_{\phi(\gamma)i}$ for all $i \in V$ and all $\gamma \in \Gamma$ and $p : V \to \mathbb{R}^3$ is such that $p(v) \in S$ for all $v \in V$. We will refer to $\tau(\Gamma)$ as a *symmetry group* and to elements of $\tau(\Gamma)$ as *symmetry operations* or simply *symmetries* of (G, p). We will often need to work with symmetric subgraphs and their frameworks. So for a Γ-symmetric graph (G, ϕ) we often consider a Γ-symmetric subgraph (H, ϕ'), where $\phi'(\gamma) = \phi(\gamma)|_{V(H)}$. In that case we often slightly abuse notation and write (H, ϕ) (or even just H) instead of (H, ϕ'). We also say that a subset X of V is Γ-*symmetric* if $(G[X], \phi)$ is a Γ-symmetric subgraph of the Γ-symmetric graph (G, ϕ).

A Γ-symmetric framework (G, p) on S (with respect to τ and ϕ) is *completely Γ-regular (with respect to τ and ϕ)* if the rigidity matrix $R_S(K_{|V|}, p)$ of the complete graph on V and every square submatrix has maximum rank among all Γ-symmetric realisations of $K_{|V|}$ on S (with respect to τ and ϕ). The set of all completely Γ-

regular realisations of G on S (with respect to τ and ϕ) is an open dense subset of the set of all Γ-symmetric realisations of G on S (with respect to τ and ϕ). Thus, we may say that a graph G is $\tau(\Gamma)$-*isostatic (independent, infinitesimally rigid, rigid)* on S if there exists a Γ-symmetric framework (G, p) on S (with respect to τ and ϕ) which is isostatic (independent, infinitesimally rigid, rigid). Later we will often remove ϕ from this notation and simply refer to a $\tau(\Gamma)$-isostatic (independent, infinitesimally rigid, rigid) graph on S (where ϕ is clear from the context).

An isometry of $\mathbb{R}^3$ that maps S onto itself is called a *surface-preserving isometry*. A symmetry group of a framework on S consisting of surface-preserving isometries is called a *surface-preserving symmetry group*.

2.5.2 Symmetric linearly constrained frameworks

Let $G = (V, E, L)$ be a looped simple graph and Γ be a finite group. Then the pair (G, ϕ) is called Γ-symmetric if $\phi : \Gamma \to \text{Aut}(G)$ is a homomorphism, where $\text{Aut}(G)$ denotes the automorphism group of G. Note that an automorphism $\phi(\gamma)$ of G consists of a permutation of the vertices, $\phi_1(\gamma)$, and a permutation of the loops, $\phi_2(\gamma)$, so that $v_iv_j \in E$ if and only if $\phi_1(v_i)\phi_1(v_j) \in E$, and v_i is incident to the loop l_j if and only if $\phi_1(v_i)$ is incident to $\phi_2(l_j)$. The permutation $\phi_1(\gamma)$ clearly induces a permutation of the edges in E. Moreover, $\phi_1(\gamma)$ must map a vertex with n loops to another vertex with n loops, and a loop can only be fixed by $\phi_2(\gamma)$ (i.e. $\phi_2(\gamma)(l_j) = l_j$) if it is incident to a vertex v_i that is fixed by $\phi_1(\gamma)$ (i.e. $\phi_1(\gamma)(v_i) = v_i$). In our context, a loop always represents a linear constraint in the plane. Thus, we will assume that a loop at v_i can only be fixed by $\phi_2(\gamma)$ if γ is the identity or an element of order 2 (since a line in the plane cannot be unshifted by an isometry of order greater than 2).

If Γ is clear from the context, then a Γ-symmetric graph will often simply be called *symmetric*. Similarly, if ϕ is clear from the context, we may simply refer to the Γ-symmetric graph (G, ϕ) by G. We also say that a subset X of V is $\tau(\Gamma)$-*symmetric* (*sparse*) if $G[X]$ is $\tau(\Gamma)$-symmetric (sparse).

Let (G, ϕ) be a Γ-symmetric looped simple graph. Then, for a homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$, we say that a linearly constrained framework (G, p, q) is Γ-*symmetric* *(with respect to ϕ and τ)*, or simply $\tau(\Gamma)$-*symmetric*, if

- $\tau(\gamma)p_i = p_{\phi(\gamma)i}$ for all $v_i \in V$ and all $\gamma \in \Gamma$;
- $\tau(\gamma)q_j = q_{\phi(\gamma)j}$ for all $l_j \in L$ and all $\gamma \in \Gamma$ whose order is not 2;
- $\tau(\gamma)q_j = -q_{\phi(\gamma)j}$ if $\tau(\gamma)$ is the half-turn and the loop l_j is fixed by $\gamma \in \Gamma$;
- $\tau(\gamma)q_j = \pm q_{\phi(\gamma)j}$ if $\tau(\gamma)$ is a reflection and the loop l_j is fixed by $\gamma \in \Gamma$.

We will refer to $\tau(\Gamma)$ as a *symmetry group* and to elements of $\tau(\Gamma)$ as *symmetry operations* or simply *symmetries* of (G, p, q).

A Γ-symmetric linearly constrained framework (G, p, q) is Γ-*generic (with respect to τ and ϕ)* if rank $R(G, p, q) \geq$ rank $R(G, p', q')$ for all linearly constrained frameworks (G, p', q') that are Γ-symmetric with respect to τ and ϕ. The set of all Γ-generic realisations of G (with respect to τ and ϕ) is an open dense subset of the set of all Γ-symmetric realisations of G (with respect to τ and ϕ). Thus, we may say that a graph G is $\tau(\Gamma)$-*isostatic (independent, infinitesimally rigid, rigid)* if there exists a Γ-symmetric framework (G, p, q) (with respect to τ and ϕ) which is isostatic (independent, infinitesimally rigid, rigid). Later we will often remove ϕ from this notation and simply refer to a $\tau(\Gamma)$-isostatic (independent, infinitesimally rigid, rigid) graph (where ϕ is clear from the context).

2.5.3 Symmetry operations and groups

Throughout this thesis, we will use a version of the Schoenflies notation for symmetry operations and groups of frameworks. This notation is primarily used to describe symmetries in three dimensions. However, we can easily restrict these operations to describe symmetries in two dimensional space, and in Section 3.10 we will describe an extension to d dimensions. The symmetry operations are the identity, denoted by id; rotations by $\frac{2\pi}{n}$, $n \in \mathbb{N}$, $n \geq 2$, denoted by c_n, which we will refer to as an *n-fold* rotation for $n \geq 2$, and more commonly *half-turn* when $n = 2$; reflections, denoted by σ; and improper rotations (i.e. rotations c_n followed by a reflection in the plane through the origin that is perpendicular to the c_n axis), $n \in \mathbb{N}$, $n \geq 2$, denoted by s_n; the inversion in the origin, denoted by φ. In two dimensions, the operation s_n is not defined, and for three dimensions $s_2 = \varphi$. There are occasions where we will further wish to specify an axis or plane for a rotation or reflection, and we will adjust the notation to make this axis clear. This will be dealt with case by case.

The symmetry groups these generate describe what are known in the literature [2] as *point groups* in three dimensions (although again we may make adjustments to other dimensions). These groups fall into the following families. Let $n \geq 2$ be an integer. The *reflection group* C_s is the order two group generated by σ. There are three *cyclic groups* [ii]. Firstly, C_n, is generated by a rotation c_n. The axis of rotation for c_n is known as the *primary axis* of the group whenever $n \geq 3$. This leads to two further classes of groups: C_{nv} is generated by C_n with a reflection plane which contains the primary axis; C_{nh} is generated by C_n with a reflection plane perpendicular to the primary axis. The *roto-reflection group* S_n is generated by the improper reflection s_n. We note that $S_n = C_{nh}$ for odd n. There are three classes of *dihedral groups*, D_n, D_{nh}, D_{nd}. D_n is generated by a rotation c_n and a half-turn c_2 whose axis is perpendicular to the primary axis. D_{nh} is generated by D_n with a reflection plane perpendicular to the primary axis. This reflection with the half-turn rotation composes to a reflection which contains the primary axis, so this group can be viewed as D_{nv}. D_{nd} is D_{nh} with a reflection that takes the primary axis to (one of) the half-turn axes. There are two *tetrahedral groups* T, T_d which are isometries of a tetrahedron. Firstly, T has the rotations which preserve the tetrahedron, that is four three-fold rotations (with axis through a vertex and its opposite face), and three half-turn rotations (with axis through the midpoints of opposite edges). T_d has 6 additional reflections, each containing an edge of the tetrahedron. This group contains all the symmetries which preserve the tetrahedron. T_h, does not preserve the surface of the tetrahedron, but is the group T with the inversion (and generates further reflections). There are two *octahedral groups*, O and O_h. O is the rotations which preserve the cube, namely three 4-fold axes between the centres of faces of the

[ii] We use the term cyclic group to match that of the literature. However, this is in conflict with that typically used in abstract alegbra, where a cyclic group is one which can be generated by a single element of the group. In this definition, the reflection groups and inversion groups are also cyclic.

cube, four 3-fold axes between the centres of edges of the cube, and six 2-fold axes between the opposite vertices of the cube. O_h is generated by O with a reflection perpendicular to a 4-fold rotation. This group also contains further reflections, improper rotations and the inversion. Finally, we have two *icosahedral* groups,I and I_h, being the rotations and symmetries of a icosahedron, respectively. I is formed by multiple rotations of orders 2 3 and 5, while I_h them rotations as well as reflections, the inversion, and roto-reflections. In total, I_h is a order 120 group, so we do not describe these symmetries further. This is indeed all of the point groups, which we shall use throughout Chapter 3.

Our studies in Chapter 3 will take interest in when the diagonal entries of the matrix representation of these symmetries takes integer values. Therefore we will require the following theorem.

Theorem 2.5.1. *[25, Theorem 1] Let n be a natural number. The value of $\cos(\frac{2\pi}{n})$ is rational if and only if $n = 1, 2, 3, 4, 6$.*

2.6 Representation theory

In the following, we give basic definitions that will be required in our work to establish necessary conditions for rigidity.

A *representation* of a group Γ on a d-dimensional vector space V over a field F is a homomorphism $\rho : \Gamma \to GL(V)$. Write (V, ρ) for this representation of Γ. The dimension d of V is called the *degree* of ρ. We say (V, ρ) is a *unitary* representation if ρ maps the identity element of Γ to the identity of $GL(V)$.

If $(V, \rho), (W, \sigma)$ are representations of Γ, a *homomorphism* of representations is a linear map $\varphi : V \to W$ so that $\varphi(\rho(\gamma)v) = \sigma(\gamma)\varphi(v)$ for all $\gamma \in \Gamma, v \in V$. That is to say the diagram,

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \rho(\gamma)\Big\downarrow & & \Big\downarrow\sigma(\gamma) \\ V & \xrightarrow{\varphi} & W \end{array}$$

commutes. An *isomorphism* is a homomorphism with an inverse, and we write $(V, \rho) \cong (W, \sigma)$.

We say $\rho, \sigma : \Gamma \to GL(V)$ are *equivalent* if there exists $A \in GL(V)$ such that $\rho(\gamma) = A^{-1}\sigma(\gamma)A$ for all $\gamma \in \Gamma$.

Given two representations $(V, \rho), (W, \sigma)$ of Γ, we may form a new representation $\rho \oplus \sigma : \Gamma \to GL(V \oplus W)$ with $\rho \oplus \sigma(\gamma) = \rho(\gamma) \oplus \sigma(\gamma)$. It is helpful to consider the matrix form of this representation, that is

$$\rho \oplus \sigma(\gamma) = \left(\begin{array}{c|c} \rho(\gamma) & 0 \\ \hline 0 & \sigma(\gamma) \end{array}\right).$$

If (V, ρ) is a representation of Γ, and W a linear subspace of V, we say W is *ρ-invariant* (or simply *invariant* when ρ is clear from the context) if $\rho(\gamma)W \subseteq W$ for all $\gamma \in \Gamma$.

Given (V, ρ) and a ρ-invariant subspace $W \leq V$, we can write

$$\rho(\gamma) = \left(\begin{array}{c|c} \rho^{(W)}(\gamma) & \sigma_1(\gamma) \\ \hline 0 & \sigma_2(\gamma) \end{array}\right),$$

where $\rho^{(W)}(\gamma)w \in W$ for all $w \in W$, and we say that $(W, \rho^{(W)})$ is a *subrepresentation* of (V, ρ).

A representation V is called *irreducible* (or simple) if $V \neq 0$ and there are no proper non-trivial subrepresentations.

Note that we know little about σ_1 and σ_2 in the above definition.

Theorem 2.6.1. *[43] A unitary representation has the property that the orthogonal complement of an invariant subspace is again invariant.*

Theorem 2.6.2. *(Maschke's Theorem) Let V be a representation of a finite group S with subrepresentation $W \subseteq V$. Assume that the characteristic of the field F does not divide the order of S. Then there is a subrepresentation $W' \subseteq V$ with $W \oplus W' = V$.*
Furthermore, any representation (V, ρ) can be written as a direct sum of irreducible representations, $(V, \rho) = \bigoplus_{i=1}^{d}(V_i, \rho_i)$.

Theorem 2.6.3. *(Schur's Lemma) Let $(V_1, \rho_1), (V_2, \rho_2)$ be irreducible representations of a finite group Γ. Any map $\varphi : V_1 \to V_2$ with $\varphi(\rho_1(\gamma)v) = \rho_2(\gamma)\varphi(v)$ for all $\gamma \in \Gamma, v \in V_1$, is either zero or an isomorphism.*

If $A = (a_{ij})$ is a square matrix then the *trace* of A is given by $tr(A) = \sum_i a_{ii}$. For a representation (V, ρ) of Γ, the *character* of ρ is the function $\chi_\rho : \Gamma \to F$ defined by $\chi_\rho(\gamma) = tr(\rho(\gamma))$.

Chapter 3

Symmetric Isostatic Frameworks

In this chapter, we will establish necessary conditions for a symmetric framework which has points restricted to lie on subsets of the space to be isostatic. To this end we first show that the rigidity matrix of a symmetric framework restricted to chosen surfaces embedded in three dimensions can be transformed into a block-decomposed form by using suitable symmetry-adapted bases. The necessary conditions are then obtained by comparing the number of rows and columns of each submatrix block. Using basic character theory, these conditions can be stated simply in terms of the number of structural components that remain unshifted under the various symmetries of the framework. We will repeat this process for the rigidity matrix of a symmetric linearly constrained framework, and establish equivalent conditions for linearly constrained frameworks in two, three, and higher dimensions.

3.1 Necessary Conditions for Symmetric Isostatic Frameworks on Surfaces

3.1.1 Block-diagonalization of the rigidity matrix

Let $G = (V, E)$ be a graph and Γ be a finite group. If A is a $m \times n$ matrix and $B = (b_i j)$ is a $p \times q$ matrix, the Kronecker product $A \otimes B$ is the $pm \times qn$ block matrix

$$A \otimes B = \begin{bmatrix} b_{11}A & \dots & b_{1q}A \\ \vdots & \ddots & \vdots \\ b_{p1}A & \dots & b_{pq}A \end{bmatrix}.$$

Then let $\tau(\gamma)$ denote the $3{\times}3$ matrix which represents γ with respect to the canonical basis of $\mathbb{R}^3$. Let $P_V(\gamma)$ and $P_E(\gamma)$ be the permutation matrix of V and E respectively, induced by γ. Here we define the permutation matrix to be the matrix so that $(e_i)^T P_V(\gamma) = (e_j)^T$ where $\phi(\gamma)(v_i) = v_j$. We have two important maps, $\tau \otimes P_V : \Gamma \to \mathbb{R}^{(3|V|)\times(3|V|)}$ and $\tilde{P}_E := P_E \oplus P_V : \Gamma \to \mathbb{R}^{(|E|+|V|)\times(|E|+|V|)}$.

In three dimensional rigidity, there is a 6-dimensional space of trivial motions of a framework of whose the points affinely span $\mathbb{R}^3$. We take special interest in a list of surfaces, one for each possible point symmetry group. The sphere removes all three translational trivial motions; the cylinder removes two translational and two rotational; the cone removes three translational and two rotational; the elliptical cylinder two translational and three rotational; and the ellipsoid removes all six trivial motions. We need to define each surface for our use. We remark that to define the surfaces in such a way requires using trivial motions to place the surface as so. In rigidity in free space, one might wish to place vertices at a position in space, using the trivial motions. However this is not possible in our case.

It is important for us to define which normal to each surface we will use, and make note of which symmetries preserve the surfaces. We define the *sphere* to be the surface $\mathcal{S} = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Then we define the normal to the sphere at a point to be $n_{\mathcal{S}}(x, y, z) = (x, y, z)$. The surface-preserving symmetry operations for $\mathcal{S}$ are rotations c_n, $n \in \mathbb{N}$, around any axis through the origin, reflections in a plane through the origin, denoted by σ, and improper rotations around an axis, denoted by s_n, $n \geq 2$, where s_2 is the inversion φ in the origin.

We define the *cylinder* to be the surface $\mathcal{Y} = \{(x, y, z) : x^2 + y^2 = 1\}$. Then we define the normal to the cylinder at a point to be $n_{\mathcal{Y}}(x, y, z) = (x, y, 0)$. The surface-preserving symmetry operations for $\mathcal{Y}$ are rotations c_n, $n \in \mathbb{N}$, around the z-axis, reflections in a plane containing the z-axis, denoted by σ, reflection in the xy-plane, denoted by σ', half-turn in an axis that is perpendicular to the z-axis (and goes through the origin), denoted by ${c_2}'$ and improper rotations around the z-axis, denoted by s_n, $n \geq 2$.

We define the *cone* to be the surface $\mathcal{C} = \{(x, y, z) : x^2 + y^2 = z^2\}$. Then we define the normal to the cone at a point to be $n_{\mathcal{C}}(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}(x, y, -z) = \frac{1}{\sqrt{2}|z|}(x, y, -z)$. The surface-preserving symmetry operations for $\mathcal{C}$ are rotations, c_n, $n \in \mathcal{N}$, about the z-axis, and a half turn rotation c_2' about an axis in the xy-plane, reflections σ in a plane containing the z-axis, and σ' in the xy-plane, and improper rotations, denoted s_n for $n \geq 2$ about the z-axis.

We define the *elliptical cylinder* to be the surface $\mathcal{L} = \{(x, y, z) : x^2+ay^2 = 1\}$ for some fixed $a > 0$. Then we define the normal to the elliptical cylinder at a point to be $n_{\mathcal{L}}(x, y, z) = \frac{1}{\sqrt{x^2+(ay)^2}}(x, ay, 0)$. The surface-preserving symmetry operations for $\mathcal{E}$ are rotations, c_2 about the z-axis, and c_2' about an axis in the xy-plane, reflections σ in the xz- or yz-plane, and σ' in the xy-plane, and the inversion φ through the origin.

We define the *ellipsoid* to be the surface $\mathcal{E} = \{(x, y, z) : x^2 + ay^2 + bz^2 = 1\}$ for some fixed $a, b > 0$. Then we define the normal to the ellipsoid at a point to be $n_{\mathcal{E}}(x, y, z) = \frac{1}{\sqrt{x^2+(ay)^2+(bz)^2}}(x, ay, bz)$. The surface-preserving symmetry operations for $\mathcal{E}$ are reflections in the xy-, xz-, yz-planes denoted $\sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ respectively, half-turn rotations about the x-, y-, z-axis, denoted c_{2x}, c_{2y}, c_{2z} respectively, and the inversion φ. For the reflections and half-turns, when it is clear that the choice of plane and axis is free, we may denote these by σ and c_2 respectively.

For the set of all these surfaces we write $\Psi = \{\mathcal{S}, \mathcal{Y}, \mathcal{C}, \mathcal{L}, \mathcal{E}\}$

Lemma 3.1.1. *Let $S \in \Psi$ and $\tau(\Gamma)$ be any isometry of $\mathbb{R}^3$ which preserves S. The normal of the image of a point is equal to the image of the normal of the point, that is to say, the equation $n(\tau(\gamma)p_k) = \tau(\gamma)n(p_k)$ holds for all $p_k \in p(V)$.*

Proof. On the sphere, $n_{\mathcal{S}}(x, y, z) = (x, y, z)$, so the equation holds. For our remaining surfaces, we use Schoenflies notation to check the equation holds for the symmetries of $\mathbb{R}^3$ which preserve the surface in question. We therefore consider where these symmetries map points to, and then consider these points for each surface and their respective normal. Let $p = p_k = (x, y, z)$ be a point on a surface. We check the following: (i) $\tau(\gamma) = c_n$, a rotation around the z-axis,

$$c_n p = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ y\cos\theta - x\sin\theta \\ z \end{pmatrix}$$

(ii) for $\tau(\gamma) = c_2'$, a rotation in the x-axis,

$$c_2' p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ -z \end{pmatrix}$$

(iii) for $\tau(\gamma) = \sigma$, a reflection in the yz-plane,

$$\sigma p = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$$

and any other reflections containing the z-axis can now be written as the combination of a rotation by angle θ and the reflection in the yz-plane. Since both of these satisfy the equation in the lemma, it is easy to see the product will too. (iv) for $\tau(\gamma) = \sigma'$, a reflection in the xy-plane,

$$\sigma' p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

(v) for $\tau(\gamma) = s_n$, the improper rotation about the $z - axis$,

$$s_n p = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ y\cos\theta - x\sin\theta \\ -z \end{pmatrix}$$

finishing our list of symmetries. We must now consider each surface individually.

We first look at the cylinder, which has normal to the surface $n_{\mathrm{y}}(x, y, z) :=$

$(x, y, 0)$. All five of the above symmetries preserve $\mathcal{Y}$, and now for a point $p \in \mathcal{Y}$ we have five calculations:

(i) $n_{\mathcal{Y}}(c_n p) = n_{\mathcal{Y}}(x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta, z) = (x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta, 0) = c_n(x, y, 0)$;

(ii) $n_{\mathcal{Y}}(c_2' p) = n_{\mathcal{Y}}(x, -y, -z) = (x, -y, 0) = c_2'(x, y, 0)$;

(iii) $n_{\mathcal{Y}}(\sigma p) = n_{\mathcal{Y}}(-x, y, z) = (-x, y, 0) = \sigma(x, y, 0)$;

(iv) $n_{\mathcal{Y}}(\sigma' p) = n_{\mathcal{Y}}(x, y, -z) = (x, y, 0) = \sigma'(x, y, 0)$;

(v) $n_{\mathcal{Y}}(s_n p) = n_{\mathcal{Y}}(x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta, -z) = (x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta, 0) = s_n(x, y, 0)$.

Hence this gives the desired result for any symmetry of the cylinder.

Now we must do the same for the cone, where $n_{\mathcal{C}}(x, y, z) := \frac{1}{\sqrt{2}|z|}(x, y, -z)$. Again all five of the symmetries preserve $\mathcal{C}$, giving us:

(i) $n_{\mathcal{C}}(c_n p) = n_{\mathcal{C}}(x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta, z) = \frac{1}{\sqrt{2}z}(x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta, -z) = c_n \frac{1}{\sqrt{2}|z|}(x, y, -z)$;

(ii) $n_{\mathcal{C}}(c_2' p) = n_{\mathcal{C}}(x, -y, -z) = \frac{1}{\sqrt{2}|z|}(x, -y, z) = c_2' \frac{1}{\sqrt{2}|z|}(x, y, -z)$; (iii) $n_{\mathcal{C}}(\sigma p) = n_{\mathcal{C}}(-x, y, z) = \frac{1}{\sqrt{2}|z|}(-x, y, -z) = \sigma \frac{1}{\sqrt{2}|z|}(x, y, -z)$;

(iv) $n_{\mathcal{C}}(\sigma' p) = n_{\mathcal{C}}(x, y, -z) = \frac{1}{\sqrt{2}|z|}(x, y, z) = \sigma' \frac{1}{\sqrt{2}|z|}(x, y, -z)$;

(v) $n_{\mathcal{C}}(s_n p) = n_{\mathcal{C}}(x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta, -z) = \frac{1}{\sqrt{2}|z|}(x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta, z) = s_n \frac{1}{\sqrt{2}|z|}(x, y, -z)$.

Hence this gives the desired result for any symmetry of the cone.

Next we consider the elliptical cylinder, and calculate the normal to be $n_{\mathcal{L}}(x, y, z) := \frac{1}{\sqrt{x^2 + (ay)^2}}(x, ay, 0)$. All five of the above symmetries preserve $\mathcal{L}$, although c_n and s_n each only hold for $n = 2$:

(i) $n_{\mathcal{L}}(c_2 p) = n_{\mathcal{L}}(-x, -y, z) = \frac{1}{\sqrt{x^2+(ay)^2}}(-x, -ay, 0) = c_2 \frac{1}{\sqrt{x^2+(ay)^2}}(x, ay, 0)$;

(ii) $n_{\mathcal{L}}(c_2' p) = n_{\mathcal{L}}(x, -y, -z) = \frac{1}{\sqrt{x^2+(ay)^2}}(x, -ay, 0) = c_2' \frac{1}{\sqrt{x^2+(ay)^2}}(x, ay, 0)$;

(iii) $n_{\mathcal{L}}(\sigma p) = n_{\mathcal{L}}(-x, y, z) = \frac{1}{\sqrt{x^2+(ay)^2}}(-x, ay, 0) = \sigma \frac{1}{\sqrt{x^2+(ay)^2}}(x, ay, 0)$;

(iv) $n_{\mathcal{L}}(\sigma' p) = n_{\mathcal{L}}(x, y, -z) = \frac{1}{\sqrt{x^2+(ay)^2}}(x, ay, 0) = \sigma' \frac{1}{\sqrt{x^2+(ay)^2}}(x, ay, 0)$;

(v) $n_{\mathcal{L}}(\varphi p) = n_{\mathcal{L}}(-x, -y, -z) = \frac{1}{\sqrt{x^2+(ay)^2}}(-x, -ay, 0) = \varphi \frac{1}{\sqrt{x^2+(ay)^2}}(x, ay, 0)$.

Hence this gives the desired result for any symmetry of the elliptical cylinder.

Finally for the ellipsoid, with normal to the surface $n_{\mathcal{E}}(x, y, z) := \frac{1}{\sqrt{x^2+(ay)^2+(bz)^2}}(x, ay, bz)$. We need only to consider the symmetries from (i), (v), with c_n and s_n each only holding for $n = 2$, and even though 3 mirrors are possible, we will only give one here in (iii):

(i) $n_{\mathcal{E}}(c_{2h}p) = n_{\mathcal{E}}(-x, -y, z) = \frac{1}{\sqrt{x^2+(ay)^2+(bz)^2}}(-x, -ay, bz) = c_{2h}\frac{1}{\sqrt{x^2+(ay)^2+(bz)^2}}(x, ay, bz)$;

(iii) $n_{\mathcal{E}}(\sigma p) = n_{\mathcal{E}}(-x, y, z) = \frac{1}{\sqrt{x^2+(ay)^2+(bz)^2}}(-x, ay, bz) = \sigma\frac{1}{\sqrt{x^2+(ay)^2+(bz)^2}}(x, ay, bz)$;

(v) $n_{\mathcal{E}}(\varphi p) = n_{\mathcal{E}}(-x, -y, -z) = \frac{1}{\sqrt{x^2+(ay)^2+(bz)^2}}(-x, -ay, -bz) = \varphi\frac{1}{\sqrt{x^2+(ay)^2+(bz)^2}}(x, ay, bz)$.

Hence this gives the desired result for any symmetry of the ellipsoid. □

Lemma 3.1.2. *Let (G, p) be a $\tau(\Gamma)$-symmetric framework on a surface $S \in \Psi$. If $R_S(G, p)u = z$, then for all $\gamma \in \Gamma$, we have*

$$R_S(G, p)(\tau \otimes P_V)(\gamma)u = \tilde{P}_E(\gamma)z.$$

Proof. Suppose $R_S(G, p)u = z$. Fix $\gamma \in \Gamma$ and let $\tau(\gamma)$ be the orthogonal matrix representing γ with respect to the canonical basis of $\mathbb{R}^3$. We enumerate the rows of $R_S(G, p)$ by the set $\{a_1, \dots, a_{|E|}, b_1, \dots, b_{|V|}\}$. By [40], we know that $(R_S(G, p)(\tau \otimes P_V)(\gamma)u)_{a_i} = (\tilde{P}_E(\gamma)z)_{a_i}$, for all $i \in [|E|]$. We are left to show the result holds for the rows of $R_S(G, p)$ which represent the normal vectors of the vertices on the surface.

Write $u \in \mathbb{R}^{3|V|}$ as $u = (u_1, \dots, u_{|V|})$, where $u_i \in R^3$ for all i, and let $\Phi(\gamma)(v_i) = v_k$. We first see that $(\tilde{P}_E(\gamma)z)_{b_k} = z_{b_i} = n(p_i) \cdot u_i$ by the definition of $P_V(\gamma)$. From $R_S(G, p)u = z$, we also get that $z_{b_i} = n(p_i) \cdot u_i$. Then $(\tau \otimes P_V)(\gamma)u = (\bar{u}_1, \dots, \bar{u}_{|V|})$, with $\bar{u}_l = \tau(\gamma)u_j$ when $\Phi(\gamma)(v_j) = v_l$. Therefore,

$$\begin{aligned}(R_S(G, p)(\tau \otimes P_V)(\gamma)u)_{b_k} &= n_1(p_k) \cdot (\tau(\gamma)u_i)_1 + n_2(p_k) \cdot (\tau(\gamma)u_i)_2 + n_3(v_k) \cdot (\tau(\gamma)u_i)_3 \\ &= n(p_k) \cdot (\tau(\gamma)u_i) \\ &= n(\tau(\gamma)p_k) \cdot (\tau(\gamma)u_i).\end{aligned}$$

Finally, using Lemma 3.1.1 plus the fact that the canonical inner product on $\mathbb{R}^d$ is invariant under the orthogonal transformation $\gamma \in \Gamma$ gives that $n(\tau(\gamma)p_k) \cdot (\tau(\gamma)u_i) = \tau(\gamma)n(p_k) \cdot (\tau(\gamma)u_i) = n(p_k) \cdot u_i = z_{b_i}$, finishing the proof. □

The following is an immediate corollary of Schur's lemma (see e.g. [43]) and the proposition above.

Corollary 3.1.3. *Let (G,p) be a $\tau(\Gamma)$-symmetric framework on $S \in \Psi$ and let $I_1, \dots, I_r$ be the pairwise non-equivalent irreducible linear representations of $\tau(\Gamma)$. Then there exists matrices A, B such that the matrices $B^{-1}R_S(G,p)A$ and $A^{-1}R_S(G,p)^T B$ are block-diagonalised and of the form*

$$\begin{pmatrix} R_1 & & & \mathbf{0} \\ & R_2 & & \\ & & \ddots & \\ & & & \\ \mathbf{0} & & & R_r \end{pmatrix}$$

where the submatrix R_i corresponds to the irreducible representation I_i.

This block decomposition corresponds to $\mathbb{R}^{3|V|} = X_1 \oplus \cdots \oplus X_r$, $\mathbb{R}^{|E|+|V|} = Y_1 \oplus \cdots \oplus Y_r$. Each of the X_i are $(\tau \otimes P_V)$-invariant subspaces, and each of the Y_i are $\tilde{P}_E$-invariant subspaces. Then, the submatrix R_i has size $(\dim(Y_i)) \times (\dim(X_i))$.

3.1.2 Additional necessary conditions

Using the block-decomposition of the rigidity matrix, we may follow the basic approach described in [11, 40] to derive added necessary conditions for a symmetric framework on a surface to be isostatic. We first need the following result.

Theorem 3.1.4. *The space of trivial motions of an affinely spanning $\tau(\Gamma)$-symmetric framework (G,p) on $S \in \Psi$, written $\mathcal{T}(G,p)$, is a $(\tau \otimes P_V)$-invariant subspace of $\mathbb{R}^{3|V|}$. Furthermore, the space of translational motions and the space of rotational motions of (G,p) are also $(\tau \otimes P_V)$-invariant subspaces of $\mathbb{R}^{3|V|}$.*

It is useful to introduce notation for the trivial motions of (G,p) before giving the proof of the above theorem. Let (G,p) be a symmetric framework on S, where $\{p(v) \mid v \in V\}$ span $\mathbb{R}^3$. Let $N = \ker(R(K_n, p))$. Then N is the subspace of $\mathbb{R}^{3|V|}$ consisting of all infinitesimal trivial motions of (G,p). Write $N = T \oplus R$ where T, R are the spaces of all translational and rotational trivial motions of (G,p), respectively. We wish to assign bases to T and R. Let $\{T_i : V(G) \to \mathbb{R}^3 \mid i = 1, \dots, 3\}$ where

$T_i(v) = e_i$, the ith canonical basis vector of $\mathbb{R}^3$, for all $v \in V$ and $i = 1, 2, 3$. Then the basis for R we give as $\{R_{ij} : V(G) \to \mathbb{R}^3 \mid 1 \leq i < j \leq 3\}$ with R_{ij} defined by $R_{ij}(v_k) = (p_k)_i e_j - (p_k)_j e_i$, for all $v_k \in V$.

Proof. Suppose that (G, p) affinely spans $\mathbb{R}^3$, so that the dimension of the trivial motion space of (G, p) on S is maximal. We first show that $N = \ker(R(K_n, p))$ is $(\tau \otimes P_V)$-invariant. By Lemma 3.1.2, if $R_S(K_n, p)u = z$ then $R_S(K_n, p)(\tau \otimes P_V)(\gamma)u = \tilde{P}_E(\gamma)z$. Let $u \in N$. Then $R_S(K_n, p)u = 0$, so

$$\begin{aligned} \tilde{P}_E(\gamma)R_S(K_n, p)u &= \tilde{P}_E(\gamma)z \\ &= \tilde{P}_E(\gamma)0 = 0 \end{aligned}$$

therefore $R_S(K_n, p)(\tau \otimes P_V)(\gamma)u = \tilde{P}_E(\gamma)R_S(K_n, p)u = 0$, giving $(\tau \otimes P_V)(\gamma)u \in \ker(R_S(K_n, p))$. Hence N is $(\tau \otimes P_V)$-invariant, as required for the first part of the theorem.

We note that the cylinder $\mathcal{Y}$ is the only surface in Ψ with non-trivial spaces for both translational motions and rotational motions. Therefore for any $S \in \Psi \setminus \{\mathcal{Y}\}$ the proof is complete. Hence from here we only consider the cylinder. To show that the space of translational motions is $(\tau \otimes P_V)$-invariant, first note that for $\mathcal{Y}$, this space is generated by the vector $t = (0, 0, 1, 0, 0, 1, \ldots, 0, 0, 1)^T$. We need to show that for each $\gamma \in \Gamma$, we have $(\tau \otimes P_V)(\gamma)t = \alpha t$ for some $\alpha \in \mathbb{R}$. By the definition of $\tau \otimes P_V$ this holds if $\tau(\gamma)(0, 0, 1)^T = \alpha(0, 0, 1)^T$ for all $\gamma \in \Gamma$. Since $\tau(\Gamma)$ preserves $\mathcal{Y}$, such an α does exist for each γ (specifically $\alpha = \pm 1$).

Finally we look at the space of rotational motions. For $\mathcal{Y}$, this space is generated by the vector $r = (r_1, \ldots, r_{|V|}) \in \mathbb{R}^{3|V|}$ defined as $r_k = (p_k)_1 e_2 - (p_k)_2 e_1 \in \mathbb{R}^3$, for all $k \in V$, where e_1 and e_2 are the standard basis vectors of $\mathbb{R}^3$ with 1 as the first and second coordinate, respectively. Note that r is perpendicular to t. Since for all $\gamma \in \Gamma$, $(\tau \otimes P_V)(\gamma)$ is an orthogonal matrix, $(\tau \otimes P_V)$ is a unitary representation (with respect to the canonical inner product on $R^{3|V|}$). Therefore the subrepresentation $H_e'^{(N)}$ of H_e' with representation space N is also unitary (with respect to the inner product obtained by restricting the canonical inner product on $R^{3|V|}$ to N). It follows that the space $\langle r \rangle$ is $(\tau \otimes P_V)$-invariant since it is the orthogonal complement to $\langle t \rangle$

in N. $\square$

Let $(\tau \otimes P_V)^{(\mathcal{T})}$ be the subrepresentation of $(\tau \otimes P_V)$ with representation space $\mathcal{T}(G,p)$. Then $\mathcal{T} = T_1 \oplus \cdots \oplus T_r$ where T_i is the $(\tau \otimes P_V)$-invariant subspace corresponding to the irreducible representation I_i.

Theorem 3.1.5. *Let (G,p) be a $\tau(\Gamma)$-symmetric framework on $S \in \Psi$. If (G,p) is isostatic, then*

$$\chi(\tilde{P}_E) = \chi(\tau \otimes P_V) - \chi((\tau \otimes P_V)^{(\mathcal{T})}).$$

Proof. By Maschke's Theorem, for the subrepresentation $(\tau \otimes P_V)^{(\mathcal{T})} \subseteq (\tau \otimes P_V)$, there exists a subrepresentation $(\tau \otimes P_V)^{(Q)} \subseteq (\tau \otimes P_V)$ with $(\tau \otimes P_V)^{(\mathcal{T})} \oplus (\tau \otimes P_V)^{(Q)} = \tau \otimes P_V$. Further, since $\tau \otimes P_V$ is unitary, we know that $Q(G,p)$ is the $(\tau \otimes P_V)$-invariant subspace of $\mathbb{R}^{3|V|}$ which is orthogonal to $\mathcal{T}(G,p)$.

Since (G,p) is isostatic, the restriction of the linear map given by the rigidity matrix to $Q(G,p)$ is an isomorphism onto $R^{|E|+|V|}$. Moreover if $R'_S(G,p)$ is the matrix corresponding to this linear map restricted to $Q(G,p)$, then, the statement for $R_S(G,p)$ in Lemma 3.1.2 also holds for $R'_S(G,p)$ and hence we have

$$R'_S(G,p)(\tau \otimes P_V)(\gamma)(R'_S(G,p))^{-1} = \tilde{P}_E(\gamma) \quad \text{for all } \gamma \in \Gamma.$$

Thus, $(\tau \otimes P_V)^{(Q)}$ and $\tilde{P}_E$ are isomorphic representations of Γ. Therefore, we have

$$\chi(\tilde{P}_E) = \chi((\tau \otimes P_V)^{(Q)}) = \chi(\tau \otimes P_V) - \chi((\tau \otimes P_V)^{(\mathcal{T})}). \qquad \square$$

3.2 The sphere

There is a precise geometric correspondence between infinitesimal rigidity in the plane and on the sphere (see [10] for details) and this extends to symmetric frameworks for any plane symmetry group [5]. There are however symmetries in the plane which are not solved, and symmetries which exist on the sphere which do not exist in the plane, therefore study here is warranted. There have been significant recent results on the sphere, with complete descriptions of isostatic graphs on concentric spheres [30], and in the forced symmetric case, the description of isostatic graphs with rotation, reflection, dihedral, inversion and improper rotation symmetry [32].

In this section and those which follow within this chapter, we look to decipher which of the point groups listed in Section 2.5 are impermissible. We will be left with a list of groups which may have symmetric isostatic frameworks, and provide conditions on the graphs for them to exist under such symmetries. We remark that this is not the same as claiming such graphs do indeed exist in all the cases.

We recall the surface-preserving symmetry operations for $\mathcal{S}$ are rotations c_n, $n \in \mathbb{N}$, around any axis through the origin, reflections σ, in a plane through the origin, improper rotations s_n, $n \geq 2$, around any axis through the origin, and that s_2 is the inversion φ. Write $\theta = \frac{2\pi}{n}$ for an anticlockwise rotation by angle around the axis of rotation. The values of the traces of the matrices for $\tilde{P}_E$ and $\tau \otimes P_V$ for each group element follow immediately from the definition. The following lemma provides the traces of the matrices for $(\tau \otimes P_V)^{(\mathcal{T})}$.

Lemma 3.2.1. *For the aforementioned symmetry operations on the sphere, the $\chi((\tau \otimes P_V)^{(\mathcal{T})})$ row of the character table is*

	id	$c_{n\geq 3}$	c_2	σ	s_n	φ
$\chi((\tau \otimes P_V)^{(\mathcal{T})})$	*3*	$2\cos\theta + 1$	-1	-1	$1 - 2\cos\theta$	*3*

Proof. We will show that, for all symmetry operation $\tau(\gamma)$ in our table, $(\tau \otimes P_V)(\gamma)$ acts linearly on the basis vectors of the trivial motion space. To do this, we find for each symmetry operation the coefficients α_k such that $(\tau \otimes P_V)(\gamma)b_j = \sum_k \alpha_k b_k$, where $\{b_k\}$ is the basis for the trivial motion space. Let (G, p) be a isostatic $\tau(\Gamma)$-symmetric framework, with $p = (x_1, y_1, z_1, \ldots, x_{|V|}, y_{|V|}, z_{|V|})$. Let the basis for the subspace of rotational trivial motions be $\{b_1, b_2, b_3\}$ where $b_1 = (-y_1, x_1, 0, \ldots, -y_{|V|}, x_{|V|}, 0) \in \mathbb{R}^{3|V|}$, $b_2 = (-z_1, 0, x_1, \ldots, -z_{|V|}, 0, x_{|V|}) \in \mathbb{R}^{3|V|}$, $b_3 = (0, -z_1, y_1, \ldots, 0, -z_{|V|}, y_{|V|}) \in \mathbb{R}^{3|V|}$. We will check the $3i-2, 3i-1, 3i$ coordinates of the vectors $(\tau \otimes P_V)(\gamma)b_j$, $j = 1, 2, 3$. Below we give these basic calculations. Throughout these calculations, we will be considering the point that is the preimage of $p(i) = (x_i, y_i, z_i)$ under $\tau(\gamma)$, for some vertex $v_i \in V$. This will be the preimage of the vertex v_i under the automorphism $\phi(\gamma)$, the vertex $v_{\hat{i}} := \phi(\gamma)^{-1}v_i$. Then $\tau(\gamma)^{-1}p_i = p_{\hat{i}} = (x_{\hat{i}}, y_{\hat{i}}, z_{\hat{i}})$.

For each group, $(\tau \otimes P_V)(id)b_j = b_j$ for $j = 1, 2, 3$, and so $\mathrm{tr}((\tau \otimes P_V)^{(\mathcal{T})}(id)) = 1 + 1 + 1 = 3$.

For $\tau(\gamma) = c_n$ an anticlockwise rotation by angle $\theta = \frac{2\pi}{n}$ around the z-axis, with $\tau(\gamma) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$(\tau \otimes P_V)(\gamma)b_1 = \begin{pmatrix} \vdots \\ -y_{\hat{i}}\cos(\theta) - x_{\hat{i}}\sin(\theta) \\ -y_{\hat{i}}\sin(\theta) + x_{\hat{i}}\cos(\theta) \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ -y_i \\ x_i \\ 0 \\ \vdots \end{pmatrix} = b_1$$

since $\tau(\gamma)(x, y, z) = (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta), z)$. Then,

$$(\tau \otimes P_V)(\gamma)b_2 = \begin{pmatrix} \vdots \\ -z_{\hat{i}}\cos(\theta) \\ -z_{\hat{i}}\sin(\theta) \\ x_{\hat{i}} \\ \vdots \end{pmatrix}$$

and

$$(\tau \otimes P_V)(\gamma)b_3 = \begin{pmatrix} \vdots \\ z_{\hat{i}}\sin(\theta) \\ -z_{\hat{i}}\cos(\theta) \\ y_{\hat{i}} \\ \vdots \end{pmatrix}.$$

Letting $\hat{i}$ be the index of $\phi(\gamma)^{-1}v_i$, we note that $\tau(\gamma^{-1})(x_i, y_i, z_i) = (x_i\cos(\theta) + y_i\sin(\theta), -x_i\sin(\theta) + y_i\cos(\theta), z_i) = (x_{\hat{i}}, y_{\hat{i}}, z_{\hat{i}})$ and observe the following two equations hold:

$$\begin{pmatrix} -z_{\hat{i}}\cos(\theta) \\ -z_{\hat{i}}\sin(\theta) \\ x_{\hat{i}} \end{pmatrix} = 0 \begin{pmatrix} -y_i \\ x_i \\ 0 \end{pmatrix} + \cos(\theta) \begin{pmatrix} -z_i \\ 0 \\ x_i \end{pmatrix} + \sin(\theta) \begin{pmatrix} 0 \\ -z_i \\ y_i \end{pmatrix} \qquad (3.2.1)$$

$$\begin{pmatrix} z_{\hat{i}}\sin(\theta) \\ -z_{\hat{i}}\cos(\theta) \\ y_{\hat{i}} \end{pmatrix} = 0 \begin{pmatrix} -y_i \\ x_i \\ 0 \end{pmatrix} - \sin(\theta) \begin{pmatrix} -z_i \\ 0 \\ x_i \end{pmatrix} + \cos(\theta) \begin{pmatrix} 0 \\ -z_i \\ y_i \end{pmatrix}. \qquad (3.2.2)$$

Therefore when we consider the diagonal entries of $(\tau \otimes P_V)^{(\mathcal{T})}(\gamma)$, the first entry is 1 since $(\tau \otimes P_V)$ maps the first basis vector to itself, and the second and third entries are both $\cos(\theta)$ by Equations 3.2.1 and 3.2.2 respectively. We can give $\mathrm{tr}((\tau \otimes P_V)^{(\mathcal{T})}(\gamma)) = 1 + 2\cos(\theta)$.

For $\tau(\gamma) = \sigma$ a reflection in a plane, say $\tau(\gamma) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$(\tau \otimes P_V)(\gamma)b_1 = (\dots, y_{\hat{i}}, x_{\hat{i}}, 0, \dots) = (\dots, y_i, -x_i, 0, \dots) = -b_1;$$

$$(\tau \otimes P_V)(\gamma)b_2 = (\dots, z_{\hat{i}}, 0, x_{\hat{i}}, \dots) = (\dots, z_i, 0, -x_i, \dots) = -b_2;$$

$$(\tau \otimes P_V)(\gamma)b_3 = (\dots, 0, -z_{\hat{i}}, y_{\hat{i}}, \dots) = (\dots, 0, -z_i, y_i, \dots) = b_3$$

since $\tau(\gamma)(x, y, z) = (-x, y, z)$. Hence, $\mathrm{tr}((\tau \otimes P_V)^{(\mathcal{T})}(\gamma)) = -1 - 1 + 1 = -1$. The same easily holds for $\tau(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\tau(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

For $\tau(\gamma) = s_n$ an anticlockwise improper rotation by angle $\theta = \frac{2\pi}{n}$, say around the z-axis, $\tau(\gamma) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and $\tau(\gamma)(x, y, z) = (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta), -z)$. Furthermore, $z_{\hat{i}} = -z_i$. Therefore $(\tau \otimes P_V)(\gamma)$

acts as follows:

$$(\tau \otimes P_V)(\gamma)b_1 = \begin{pmatrix} -y_{\hat{i}}\cos(\theta) - x_{\hat{i}}\sin(\theta) \\ -y_{\hat{i}}\sin(\theta) + x_{\hat{i}}\cos(\theta) \\ 0 \end{pmatrix} = \begin{pmatrix} \vdots \\ -y_i \\ x_i \\ 0 \\ \vdots \end{pmatrix} = b_1$$

$$(\tau \otimes P_V)(\gamma)b_2 = \begin{pmatrix} \vdots \\ -z_{\hat{i}}\cos(\theta) \\ -z_{\hat{i}}\sin(\theta) \\ -x_{\hat{i}} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ z_i\cos(\theta) \\ z_i\sin(\theta) \\ -x_{\hat{i}} \\ \vdots \end{pmatrix}$$

and

$$(\tau \otimes P_V)(\gamma)b_3 = \begin{pmatrix} \vdots \\ z_{\hat{i}}\sin(\theta) \\ -z_{\hat{i}}\cos(\theta) \\ -y_{\hat{i}} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ -z_i\sin(\theta) \\ z_i\cos(\theta) \\ -y_{\hat{i}} \\ \vdots \end{pmatrix}.$$

As in Equations 3.2.1 and 3.2.2, we write the second and third vectors above as a linear sum of the basis vectors,

$$\begin{pmatrix} z_i\cos(\theta) \\ z_i\sin(\theta) \\ -x_{\hat{i}} \end{pmatrix} = 0\begin{pmatrix} -y_i \\ x_i \\ 0 \end{pmatrix} - \cos(\theta)\begin{pmatrix} -z_i \\ 0 \\ x_i \end{pmatrix} - \sin(\theta)\begin{pmatrix} 0 \\ -z_i \\ y_i \end{pmatrix} \tag{3.2.3}$$

$$\begin{pmatrix} -z_i\sin(\theta) \\ z_i\cos(\theta) \\ -y_{\hat{i}} \end{pmatrix} = 0\begin{pmatrix} -y_i \\ x_i \\ 0 \end{pmatrix} + \sin(\theta)\begin{pmatrix} -z_i \\ 0 \\ x_i \end{pmatrix} - \cos(\theta)\begin{pmatrix} 0 \\ -z_i \\ y_i \end{pmatrix}. \tag{3.2.4}$$

The basis vector b_1 is preserved by $(\tau \otimes P_V)$, and with Equations 3.2.3 and 3.2.4, $\mathrm{tr}((\tau \otimes P_V)^{(\mathcal{T})}(\gamma)) = 1 - 2\cos(\theta)$. □

We are now able to give the full character table for $\tau(\Gamma)$-symmetric isostatic frameworks on $\mathbb{S}$ (see Table 3.1). We give these without calculation as they can be seen directly from the matrix representations of $\tau \otimes P_V$ and $\tilde{P}_E$.

For a Γ-symmetric graph $G = (V, E)$ with respect to $\phi : V \to \text{Aut}(G)$, we say that a vertex $v \in V$ is *fixed* by $\gamma \in \Gamma$ if $\phi(\gamma)(v) = v$. Similarly, an edge $uv \in E$ is *fixed* by $\gamma \in \Gamma$ if both u and v are fixed by γ or if $\phi(\gamma)(u) = v$ and $\phi(\gamma)(v) = u$. For groups of order two, we will often just say that a vertex or edge is fixed if it is fixed by the non-trivial group element.

Note that if (G, p) is a $\tau(\Gamma)$-symmetric framework on $\mathbb{S}$, then there is no vertex fixed for any improper rotation s_n. The number of vertices that are fixed by an element in $\tau(\Gamma)$ corresponding to rotations c_n, or reflections σ are denoted by v_n and v_σ, respectively. The number of edges that are fixed by the element in $\tau(\Gamma)$ corresponding to rotations c_n, reflections σ and improper rotations s_n are denoted by e_n, e_σ, and e_{sn}, respectively.

$\mathbb{S}$	id	c_n	σ	s_n
$\chi(\tilde{P}_E)$	$\lvert E\rvert + \lvert V\rvert$	$e_n + v_n$	$e_\sigma + v_\sigma$	e_{sn}
$\chi(\tau \otimes P_V)$	$3\lvert V\rvert$	$(2\cos\theta + 1)v_n$	v_σ	0
$\chi((\tau \otimes P_V)^{(\mathcal{T})})$	3	$2\cos\theta + 1$	-1	$1 - 2\cos\theta$

Table 3.1: Character table for symmetry operations of the sphere.

In the following proofs we shall use Theorem 3.1.5 to draw conclusions from Table 3.1.

Corollary 3.2.2. *If (G, p) is a $\tau(\Gamma)$-symmetric isostatic framework on $\mathbb{S}$, then $c_n \notin \tau(\Gamma)$ for any $n \geq 4$, and $s_n \notin \tau(\Gamma)$ for any $n \geq 2$. Moreover,*

- *if $c_2 \in \tau(\Gamma)$ then $e_2 = 1$ and $v_2 = 0$;*
- *if $c_3 \in \tau(\Gamma)$ then $e_3 = v_3 = 0$;*
- *if $\sigma \in \tau(\Gamma)$ then $e_\sigma = 1$.*

Furthermore, $\tau(\Gamma)$ is impermissible if it contains symmetries which cannot be generated by those listed above.

Proof. Let's first analyse isostatic frameworks with a c_n symmetry. From the table we have $e_n = 2v_n \cos\theta - 2\cos\theta - 1$. Note that by surface symmetry, $v_n = 0, 1, 2$.

Let $v_n = 0$. We have $e_n = -2\cos\theta - 1$. There are solutions when $n = 2, 3, 4$. We have that an isostatic framework with c_2 symmetry has no fixed vertex and one fixed edge, we have that an isostatic framework with c_3 symmetry would have no fixed vertex and no fixed edge, and an isostatic framework with c_4 symmetry is not permissible.

Let $v_n = 1$. We then have $e_n + 1 = 0$ which is not possible.

Lastly, let $v_n = 2$. Tthen $e_n = 2\cos\theta - 1$. This gives possible values of e_n as 0 or 1, when θ is $\frac{\pi}{3}$ or 0 respectively. However, an isostatic framework with c_6 symmetry must also have a c_3 symmetry, but the number of fixed vertices required in each contradicts. We may deduce that the only possible symmetry element of this kind is c_3.

Any isostatic framework with $s_{n\geq 5}$-symmetry will have a forbidden c_k symmetry. We must check s_3 and s_4. The table gives $e_{sn} = 2\cos\theta - 1$, and in both cases $2\cos\theta - 1$ is negative so this is not possible. □

The following corollary gives the possible non-trivial groups that can be constructed from the symmetries given in Corollary 3.2.2. Note that we will always consider the non-trivial groups, without explicitly stating from this point forward. These are the groups from Section 2.5 which can be constructed from only these symmetries, and do not contain c_n for any $n \geq 4$ or s_n for any $n \geq 2$.

Corollary 3.2.3. *If (G, p) is a $\tau(\Gamma)$-symmetric isostatic framework on $\mathcal{S}$, then*

$$\tau(\Gamma) = \begin{cases} C_s = \{id, \sigma\}; \\ C_2 = \{id, c_2\}; \\ C_3 = \{id, c_3, c_3^2\}; \\ C_{2v} = \{id, c_2, \sigma, c_2\sigma\}; \\ C_{3v} = \{id, c_3, c_3^2, \sigma, c_3\sigma, c_3^2\sigma\}; \\ D_3 = \{id, c_3, c_3^2, c_2', c_2'c_3, c_2'c_3^2\}; \\ T, \{id, c_2, c_3\} \in T. \end{cases}$$

Proof. We once again look for groups generated by $\{\mathrm{id}, c_2, c_3, \sigma\}$ which do not contain forbidden symmetry operations from Collorary 3.2.2. The groups C_s, C_2 and C_3 can easily be observed as permissible here. A group with a c_2 and c_3 symmetry operation with the same rotation axis would also have a c_6 which is forbidden. In D_2 the three half turn rotations would need to fix the same edge, since no edge can lie on an axis of rotation, but this is not possible. The restrictions of Corollary 3.2.2 allow D_3 as the half turns will fix edges not fixed by the threefold rotation. Similarly, the tetrahedral group T contains three two-fold rotations and four three-fold rotations. Each edge fixed by each of the half turn rotations will not be fixed by the other half turns, nor any of the three-fold rotations. The groups $D_{nh}, D_{nd}, T_d, T_h, O, O_h, I, I_h$ all preserve the sphere, but contain higher order rotations or improper rotations which do not have isostatic frameworks associated. This only leaves groups with one rotation axis and one reflection. If the rotation axis is perpendicular to the reflection plane, we would have φ or s_3 which is forbidden, so the axis of rotation axis must be contained in the plane of reflection. This leaves C_{2v} and C_{3v} as the only remaining groups. □

Theorem 3.2.4. *Let (G, p) be an isostatic $\tau(\Gamma)$-symmetric framework on $\mathcal{S}$. Then (G, ϕ) is Γ-symmetric, $(2, 3)$-tight and will satisfy the constraints given in Table 3.2:*

$\tau(\Gamma)$	Number of edges and vertices fixed by symmetry elements
C_s	$e_\sigma = 1$
C_2	$e_2 = 1$
C_3	$e_3 = v_3 = 0$
C_{2v}	$e_\sigma = e_2 = 1, v_2 = 0$
C_{3v}	$e_3 = 0, v_3 = 0, e_\sigma = e_{\sigma'} = e_{\sigma''} = 1$
D_3	$e_3 = 0, v_3 = v_2 = 0, e_{2'} = 1$
T	$v_2, v_3, e_3 = 0,\ e_2 = 1$

Table 3.2: Fixed edge and vertex counts for symmetry operations on the sphere.

Proof. The graph G must be Γ-symmetric and $(2, 3)$-tight (by [30]). The first three rows of our table come directly from Corollaries 3.2.2 and 3.2.3. For both C_{2v} and C_{3v} the vertices fixed by the reflections will not be fixed by the rotation, unless they lie on the axis of rotation. Edges fixed by the mirrors either lie on the plane of reflection, or are perpendicular to the plane. For C_{2v}, the edge fixed by the mirror must be fixed by the half-turn, or else its c_2-copy will be fixed by σ. When thinking about C_{3v}, the edges fixed by a reflection will not have c_3-copies also fixed by the same reflection, and it is possible to have edges fixed by the reflection which are not fixed by the rotation. Hence, both C_{2v} and C_{3v} are as in the table above. For D_3 and T, the single edge fixed by each half turn rotation has images under the other elements of D_3 and T which will not be fixed by that same rotation. □

Remark 3.2.5. Suppose the edge between p_i and p_j is fixed by c_2' in a D_3-symmetric framework. Then, the edges between c_3p_i and c_3p_j, and between $c_3^2p_i$ and $c_3^2p_j$ will be fixed by c_3c_2' and $c_3^2c_2'$ respectively. Similarly for tetrahedral group T, the edges fixed by each of the half turn rotation axis will be formed of an orbit under one of the c_3 rotation axis.

3.3 The cylinder

The cylinder presents itself as the most likely surface for new characterisations of symmetric frameworks. The cylinder is well studied, with known characterisations of minimal rigidity on concentric cylinders [30], of global rigidity [19], and forced-symmetric isostatic graphs for rotation, reflection and inversion symmetry [32]. Further, the authors in the last article provided conjectures for characterisations of incidentally symmetric infinitesimally rigid/isostatic on the cylinder, in the presence of C_i and C_s symmetries.

In this section we calculate the characters of the representations appearing in the statement of Theorem 3.1.5 for the cylinder $\mathcal{Y}$ in order to establish necessary conditions for symmetric frameworks on the cylinder to be isostatic. In Chapter 4 we revisit the cylinder, to establish the combinatorial characterisation for the graphs which give rise to C_i-, C_s-, and C_2-symmetric frameworks, in doing so proving the above conjecture from [32]. This work on the cylinder has been submitted for publication [33].

We recall the surface-preserving symmetry operations for $\mathcal{Y}$ are rotations c_n, $n \in \mathbb{N}$, around the z-axis, reflections σ in a plane containing the z-axis, and σ' in the xy-plane, half-turn rotation c_2' in an axis that is perpendicular to the z-axis and improper rotations s_n, $n \geq 2$, around the z-axis. The values of the traces of the matrices for $\tilde{P}_E$ and $\tau \otimes P_V$ for each group element follow immediately from the definition. The following lemma provides the traces of the matrices for $(\tau \otimes P_V)^{(\mathcal{T})}$.

Lemma 3.3.1. *For the aforementioned symmetry operations on the cylinder, the $\chi((\tau \otimes P_V)^{(\mathcal{T})})$ row of the character table is*

	id	c_n	c_2'	σ	σ'	s_n	φ
$\chi((\tau \otimes P_V)^{(\mathcal{T})})$	*2*	*2*	-2	*0*	*0*	*0*	*0*

Proof. We will show that, for all symmetry operations $\tau(\gamma)$ in our table, $(\tau \otimes P_V)(\gamma)$ acts linearly on the basis vectors of the trivial motion space. To do this, we find for each symmetry operation the coefficients α_k such that $(\tau \otimes P_V)(\gamma)b_j = \sum_k \alpha_k b_k$, where $\{b_k\}$ is the basis for the trivial motion space. Let (G,p) be a isostatic $\tau(\Gamma)$-symmetric framework, with $p = (x_1, y_1, z_1, \ldots, x_{|V|}, y_{|V|}, z_{|V|})$. Let the basis of the subspace of translational trivial infinitesimal motions be $b_1 = (0,0,1,0,0,1,\ldots,0,0,1) \in \mathbb{R}^{3|V|}$, and the basis for the subspace of rotational trivial motions be $b_2 = (-y_1, x_1, 0, \ldots, -y_{|V|}, x_{|V|}, 0) \in \mathbb{R}^{3|V|}$. Lastly, recall for $\tau(\Gamma)$-symmetric frameworks, we have $\tau(\gamma)p_i = p_{\phi(\gamma)i}$ for all $i \in |V|$ and all $\gamma \in \Gamma$. We will check the $3i-2, 3i-1, 3i$ coordinates of the vectors $(\tau \otimes P_V)(\gamma)b_j$, $j = 1,2$. Below we give these basic calculations. As in Lemma 3.2.1, we use the notation $v_{\hat{i}} := \phi(\gamma)^{-1}v_i$ and $\tau(\gamma)^{-1}p_i = p_{\hat{i}} = (x_{\hat{i}}, y_{\hat{i}}, z_{\hat{i}})$.

For the identity, $(\tau \otimes P_V)(id)b_j = b_j$ for $j = 1,2$, and so $\operatorname{tr}((\tau \otimes P_V)^{(\mathcal{T})}(id)) = 1 + 1 = 2$.

For $\tau(\gamma) = c_n$ an anticlockwise rotation by angle $\theta = \frac{2\pi}{n}$ around the z-axis,

$$\tau(\gamma) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$(\tau \otimes P_V)(\gamma)b_1 = (\dots, 0, 0, 1, \dots) = b_1.$

$$(\tau \otimes P_V)(\gamma)b_2 = \begin{pmatrix} \vdots \\ -y_{\hat{i}}\cos(\theta) - x_{\hat{i}}\sin(\theta) \\ -y_{\hat{i}}\sin(\theta) + x_{\hat{i}}\cos(\theta) \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ -y_i \\ x_i \\ 0 \\ \vdots \end{pmatrix} = b_2$$

since $\tau(\gamma)(x, y, z) = (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta), z)$. Hence, $(\tau \otimes P_V)(\gamma)$ maps both basis vectors of $\mathfrak{T}$ to themselves, so $\mathrm{tr}((\tau \otimes P_V)^{(\mathfrak{T})}(\gamma)) = 1 + 1 = 2$.

For $\tau(\gamma) = c_2'$ a half-turn rotation perpendicular to the z-axis, wlog $\tau(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $(\tau \otimes P_V)(\gamma)b_1 = (\dots, 0, 0, -1, \dots) = -b_1$. Observe, $(\tau \otimes P_V)(\gamma)b_2 = (\dots, -y_{\hat{i}}, -x_{\hat{i}}, 0, \dots) = (\dots, y_i, -x_i, 0, \dots) = -b_2$ since $\tau(\gamma)(x, y, z) = (x, -y, -z)$. Hence, $(\tau \otimes P_V)(\gamma)$ maps both basis vectors of $\mathfrak{T}$ to the negative of themselves, so $\mathrm{tr}((\tau \otimes P_V)^{(\mathfrak{T})}(\gamma)) = -1 - 1 = -2$.

For $\tau(\gamma) = \sigma$ a reflection in a plane which contains the z-axis, wlog $\tau(\gamma) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ say, $(\tau \otimes P_V)(\gamma)b_1 = (\dots, 0, 0, 1, \dots) = b_1$. Then note that $(\tau \otimes P_V)(\gamma)b_2 = (\dots, y_{\hat{i}}, x_{\hat{i}}, 0, \dots) = (\dots, y_i, -x_i, 0, \dots) = -b_2$ since $\tau(\gamma)(x, y, z) = (-x, y, z)$. Hence, as reasoned previously, $\mathrm{tr}((\tau \otimes P_V)^{(\mathfrak{T})}(\gamma)) = 1 - 1 = 0$.

For $\tau(\gamma) = \sigma'$ a reflection in the xy-plane, $\tau(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, we have $(\tau \otimes P_V)(\gamma)b_1 = (\dots, 0, 0, -1, \dots) = -b_1$. Then, $(\tau \otimes P_V)(\gamma)b_2 = (\dots, -y_{\hat{i}}, x_{\hat{i}}, 0, \dots) = (\dots, -y_i, x_i, 0, \dots) = b_2$ since $\tau(\gamma)(x, y, z) = (x, y, -z)$. Hence, $\mathrm{tr}((\tau \otimes P_V)^{(\mathfrak{T})}(\gamma)) = -1 + 1 = 0$.

For $\tau(\gamma) = s_n$ an anticlockwise rotation by angle θ around the z-axis, followed by a reflection in the xy-plane, $\tau(\gamma) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $(\tau \otimes P_V)(\gamma)b_1 =$

$(\ldots, 0, 0, -1, \ldots) = -b_1$.

$$(\tau \otimes P_V)(\gamma)b_2 = \begin{pmatrix} \vdots \\ -y_{\hat{i}}\cos(\theta) - x_{\hat{i}}\sin(\theta) \\ -y_{\hat{i}}\sin(\theta) + x_{\hat{i}}\cos(\theta) \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ y_i \\ -x_i \\ 0 \\ \vdots \end{pmatrix} = b_2$$

since $\tau(\gamma)(x, y, z) = (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta), -z)$. Hence, $\mathrm{tr}((\tau \otimes P_V)^{(\mathcal{T})}(\gamma)) = -1 + 1 = 0$.

While covered by s_2, we check for $\tau(\gamma) = \varphi$ the inversion, $\tau(\gamma) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $(\tau \otimes P_V)(\gamma)b_1 = (\ldots, 0, 0, -1, \ldots) = -b_1$. Then, $(\tau \otimes P_V)(\gamma)b_2 = (\ldots, y_{\hat{i}}, -x_{\hat{i}}, 0, \ldots) = (\ldots, -y_i, x_i, 0, \ldots) = b_2$ since $\tau(\gamma)(x, y, z) = (-x, -y, -z)$. Hence, $\mathrm{tr}((\tau \otimes P_V)^{(\mathcal{T})}(\gamma)) = -1 + 1 = 0$. □

We are now able to give the full character table for $\tau(\Gamma)$-symmetric isostatic frameworks on $\mathcal{Y}$ (see Table 3.3). We give these without calculation as they can be seen directly from the matrix representations of $\tau \otimes P_V$ and $\tilde{P}_E$.

Note that if (G, p) is a Γ-symmetric framework on $\mathcal{Y}$ with respect to τ and ϕ, then there is no vertex fixed by an element of Γ corresponding to a rotation c_n about the z-axis or the inversion φ. The number of vertices that are fixed by the element in Γ corresponding to the half-turn c_2', or the reflections σ and σ' are denoted by $v_{2'}$, v_σ and $v_{\sigma'}$, respectively. An edge of G cannot be fixed by an element of Γ that corresponds to a rotation c_n, $n \geq 3$, or an improper rotation s_n, $n \geq 3$. Hence we have separate columns for c_2 and $\varphi = s_2$ below. The number of edges that are fixed by the element in Γ corresponding to the half-turns c_2 and c_2', the reflections σ and σ' and the inversion φ are denoted by e_2, $e_{2'}$, e_σ, $e_{\sigma'}$ and e_φ, respectively.

We note that the first column of Table 3.3 recovers the result from Theorem 2.3.1 that $|E| = 2|V| - 2$. In the following proofs we shall use Theorem 3.1.5 to draw conclusions from Table 3.3.

$\mathcal{Y}$	id	$c_{n\geq 3}$	c_2	c_2'	σ	σ'	$s_{n\geq 3}$	φ
$\chi(\tilde{P}_E)$	$\|E\|+\|V\|$	0	e_2	$e_{2'}+v_{2'}$	$e_\sigma+v_\sigma$	$e_{\sigma'}+v_{\sigma'}$	0	e_φ
$\chi(\tau\otimes P_V)$	$3\|V\|$	0	0	$-v_{2'}$	v_σ	$v_{\sigma'}$	0	0
$\chi((\tau\otimes P_V)^{(\mathcal{T})})$	2	2	2	-2	0	0	0	0

Table 3.3: Character table for symmetry operations of the cylinder.

Corollary 3.3.2. *If (G,p) is a $\tau(\Gamma)$-symmetric isostatic framework on $\mathcal{Y}$, then $c_n \notin \tau(\Gamma)$ for any $n\geq 2$, and $s_n\notin\tau(\Gamma)$ for any $n\geq 3$. Moreover,*

- *if $c_2'\in\tau(\Gamma)$ then $e_{2'}=2$ and $v_{2'}=0$, or $e_{2'}=0$ and $v_{2'}=1$;*
- *if $\sigma\in\tau(\Gamma)$ or $\sigma'\in\tau(\Gamma)$ then $e_\sigma=0$ and $e_{\sigma'}=0$;*
- *if $\varphi\in\tau(\Gamma)$ then $e_\varphi=0$.*

Furthermore, $\tau(\Gamma)$ is impermissible if it contains symmetries which cannot be generated by those listed above.

Proof. We will check for which symmetry operations the counts from the table are possible. Beginning with $c_{n\geq 3}$, we immediately see the equality $\chi(\tilde{P}_E)=\chi(\tau\otimes P_V)-\chi((\tau\otimes P_V)^{(\mathcal{T})})$ does not hold. For c_2, where fixed edges are possible, we have that $e_2=-2$ so there are no isostatic frameworks with a c_n $(n\geq 2)$ symmetry on the cylinder. Further to that, since any $s_{n\geq 3}$ symmetry would also imply a c_k symmetry for some $k\geq 2$, there are no isostatic frameworks which have a $s_{n\geq 3}$ symmetry on the cylinder.

Reading from the table we then draw the following conclusions. For an isostatic framework with a c_2' symmetry, $e_{2'}+2v_{2'}=2$, so there are either two fixed edges and no fixed vertex or one fixed vertex and no fixed edge. For both mirrors, $e_\sigma+v_\sigma=v_\sigma$, giving no restriction on the number of fixed vertices, but an isostatic framework must have no fixed edge. Finally for inversion, since there can be no fixed vertex with inversion symmetry on the cylinder, our table gives $e_\varphi=0$, so there are also no fixed edges. □

We recall from Section 2.5 the groups which can be constructed from our symmetry operations. In the following corollary, we show the groups that can be constructed from the symmetry operations above, which do not contain symmetries which are excluded in Corollary 3.3.2.

Corollary 3.3.3. *If (G, p) is a $\tau(\Gamma)$-symmetric isostatic framework on $\mathcal{Y}$, then*

$$\tau(\Gamma) = \begin{cases} C_i = \{id, \varphi\}; \\ C_s = \{id, \sigma\} \text{ or } \{id, \sigma'\}; \\ C_2 = \{id, c_2'\}; \\ C_{2v} = \{id, \sigma, \sigma', c_2'\}; \\ C_{2h} = \{id, \sigma, c_2', \varphi\}. \end{cases}$$

Proof. We need to find the groups which can be generated from the symmetry operations $\{\text{id}, \sigma, \sigma', c_2', \varphi\}$, which also do not contain the forbidden symmetry operations in Corollary 3.3.2. It is immediate that the groups of order 2 satisfy this, namely, $C_i = \{\text{id}, \varphi\}$, $C_s = \{\text{id}, \sigma\}$ or $\{\text{id}, \sigma'\}$, $C_2 = \{\text{id}, c_2'\}$. Observing that when the axis of rotation of c_2' lies within the mirror planes of σ and σ', we have that $\sigma\sigma' = \sigma'\sigma = c_2'$, and see the group $C_{2v} = \{\text{id}, \sigma, \sigma', c_2'\}$ is also allowable. Another group of order 4 can be found by now aligning the axis of rotation to be perpendicular to the plane of the mirror σ, then $C_{2h} = \{\text{id}, \sigma, c_2', \varphi\}$, with $\sigma c_2' = c_2'\sigma = \varphi$. Noting that $\varphi\sigma' = \sigma'\varphi = c_2$, we see $D_{2h} = \{\text{id}, \sigma, \sigma', c_2', \varphi, \varphi\sigma, \varphi\sigma', \varphi c_2'\}$ does not have a symmetric isostatic framework on the cylinder. Indeed, looking at point group tables, this is all of the possible groups from our symmetries, so our list is complete. □

We are now able to use Corollaries 3.3.2 and 3.3.3 to draw conclusions about $\tau(\Gamma)$-symmetric isostatic frameworks on the cylinder.

Theorem 3.3.4. *Let (G, p) be an isostatic $\tau(\Gamma)$-symmetric framework on $\mathcal{Y}$. Then (G, ϕ) is Γ-symmetric, $(2, 2)$-tight and will satisfy the constraints in Table 3.4.*

$\tau(\Gamma)$	Number of edges and vertices fixed by symmetry operations
C_i	$e_\varphi = 0$
C_s	$e_\sigma = 0$
C_2	$e_{2'} = 2, v_{2'} = 0$ or $e_{2'} = 0, v_{2'} = 1$
C_{2v}	$e_\sigma = e_{\sigma'} = 0, (e_{2'} = 2, v_{2'} = 0$ or $e_{2'} = 0, v_{2'} = 1)$
C_{2h}	$e_\sigma = 0, e_\varphi = 0, e_{2'} = 2, v_{2'} = 0$

Table 3.4: Fixed edge and vertex counts for symmetry operations on the cylinder.

Proof. The graph G must clearly be Γ-symmetric and $(2,2)$-tight (by [30]). The C_i, C_s, C_2-symmetric isostatic framework values follow immediately from Corollaries 3.3.2 and 3.3.3. For C_{2v}, C_{2h}, we must check that the values found in 3.3.2 and 3.3.3 do not contradict to further restrict. Firstly, for C_{2v}-symmetric isostatic frameworks, let's consider the case when the c_2' symmetry gives two fixed edges. The two edges fixed by c_2' will only be fixed by σ or σ' if the edges are perpendicular to the mirror planes, or contained in them. Therefore it is possible to have a framework with $e_\sigma = e_{\sigma'} = 0$ and $e_{2'} = 2$. In the $e_\sigma = e_{\sigma'} = 0, e_{2'} = 0, v_{2'} = 1$ case, the vertex fixed by c_2' will also be fixed by both mirrors (with the alignment we have chosen), so there is no contradiction here.

Secondly, for C_{2h}-symmetric isostatic frameworks, we again begin by considering the case when the c_2' symmetry gives two fixed edges. Here again, there is no requirement for the edges fixed by the c_2' symmetry to be fixed by either the inversion or the mirror perpendicular to the axis of rotation. However, say v is a vertex fixed by c_2', then $\sigma(v) = \varphi(v) \neq v$ will also be fixed by c_2', so the frameworks with $e_\sigma = 0, e_\varphi = 0, e_{2'} = 2, v_{2'} = 0$ are the only possible C_{2h}-symmetric isostatic frameworks. □

Remark 3.3.5. If a C_{2v}-symmetric isostatic framework has two edges fixed by the rotation, say f_1 and f_2, then $\sigma(f_1) = \sigma'(f_1) = f_2$. The two edges fixed by the rotation in a C_{2h}-symmetric isostatic framework, say u_1v_1 and u_2v_2, satisfy $\sigma(u_1v_1) = u_2v_2$ and if $\sigma(u_1) = u_2$, then $\varphi(u_1) = v_2$ and $\varphi(v_1) = u_2$.

3.4 The cone

When considering constructing frameworks on the cone (and similarly later on the elliptical cylinder), there are typically more difficulties in doing so than on the sphere or cylinder. An example of this added difficulty comes from considering the combinatorial reductions, which we introduce in Chapter 4. A common approach to these reductions, which we follow, is to consider the degree 3 vertices in the graph. The reason for this is to perform a $(2,1)$-reduction (recall from Section 2.1.2). A $(2,1)$-reduction will not be possible if the degree three vertex is in a K_4. In the class of $(2,3)$-tight graphs, there are no K_4 subgraphs. For $(2,2)$-tight graphs, K_4 subgraphs are themselves tight, and techniques (see Section 4.2) have been developed for this eventuality. In $(2,1)$-tight graphs, the vertices of a K_4 subgraph are sparse. However, recursive constructions of $(2,1)$-tight graphs have been established [29], and results for symmetric frameworks have been established in the forced-symmetric setting [32].

One could consider the torus instead of the cone and derive the same counts as below. The primary difference for our study is the possibility of a vertex at the origin, which when present will be fixed by multiple symmetries. We recall from Section 3.1 the surface-preserving symmetry operations for $\mathcal{C}$ are $\{c_n, c_2', \sigma, \sigma', s_n\}$ for all $n \geq 2$.

Remark 3.4.1. In the proof of Lemma 3.3.1 we considered how $(\tau \otimes P_v)(\gamma)$ acts on the basis vectors of the trivial motion spaces for both of the basis vectors of the trivial motions of the cone and elliptical cylinder. Namely, for the cone, we take the results of only the rotational basis vector, and for the elliptical cylinder we take the results only of the translational basis vector.

Corollary 3.4.2. *For the following symmetry groups on the cone, the* $\chi((\tau \otimes P_V)^{(\mathcal{T})})$ *row of the character table is*

$\mathcal{C}$	id	$c_{n\geq 3}$	c_2	c_2'	σ	σ'	$s_{n\geq 3}$	φ
$\chi((\tau \otimes P_V)^{(\mathcal{T})})$	1	1	1	-1	-1	1	1	1

Proof. This result is proved in the proof of Lemma 3.3.1, by excluding the translational trivial motion. □

Note that if (G,p) is a $\tau(\Gamma)$-symmetric framework on $\mathcal{C}$ with respect to τ and ϕ, then there is at most one vertex fixed by an element of Γ corresponding to a rotation c_n about the z-axis, c_2' about an axis in the xy-plane, an improper rotation s_n about the z-axis, or the inversion φ. The number of vertices that are fixed by the element in Γ corresponding to a rotation c_n, the half-turn c_2', the reflections σ and σ', the improper rotations s_n, or the inversion φ are denoted by v_n, $v_{2'}$, v_σ, $v_{\sigma'}$, v_{sn} and v_φ respectively. An edge of G cannot be fixed by an element of Γ that corresponds to a rotation c_n, $n \geq 3$, or an improper rotation s_n, $n \geq 3$. Hence we have separate columns for c_2 and $\varphi = s_2$ below. The number of edges that are fixed by the element in Γ corresponding to the half-turns c_2 and c_2', the reflections σ and σ' and the inversion φ are denoted by e_2, $e_{2'}$, e_σ, $e_{\sigma'}$ and e_φ, respectively. We have omitted the column for the identity, in which $\chi(\tilde{P}_E)(\text{id}) = |E| + |V|$, $\chi(\tau \otimes P_V)(\text{id}) = 3|V|$, and $\chi((\tau \otimes P_V)^{(\mathcal{T})}) = 1$.

$\mathcal{C}$	$c_{n\geq 3}$	c_2	c_2'	σ	σ'	$s_{n\geq 3}$	φ
$\chi(\tilde{P}_E)$	v_n	$e_2 + v_2$	$e_{2'} + v_{2'}$	$e_\sigma + v_\sigma$	$e_{\sigma'} + v_{\sigma'}$	v_{sn}	$e_\varphi + v_\varphi$
$\chi(\tau \otimes P_V)$	$(2\cos\theta + 1)v_n$	$-v_2$	$-v_{2'}$	v_σ	$v_{\sigma'}$	$(2\cos\theta - 1)v_{sn}$	$-3v_\varphi$
$\chi((\tau \otimes P_V)^{(\mathcal{T})})$	1	1	-1	-1	1	1	1

Table 3.5: Character table for symmetry operations of the cone.

In the following proofs we shall use Theorem 3.1.5 to draw conclusions from Table 3.5.

Corollary 3.4.3. *Let (G, p) be a $\tau(\Gamma)$-symmetric isostatic framework on the cone. Then $c_n \notin \tau(\Gamma)$ for any $n \geq 2$, $s_n \notin \tau(\Gamma)$ for $n \geq 2$, $\sigma' \notin \tau(\Gamma)$, and $\varphi \notin \tau(\Gamma)$. Furthermore, if an isostatic framework exists for the following symmetry operations, then it satisfies the following:*

- *if $c_2' \in \tau(\Gamma)$ then $e_{2'} = 1$ and $v_{2'} = 0$;*
- *$\sigma \in \tau(\Gamma)$ then $e_\sigma = 1$.*

Furthermore, $\tau(\Gamma)$ is impermissible if it contains symmetries which cannot be generated by those listed above.

Proof. We begin by looking to conclude which symmetries cannot have isostatic frameworks. For the cone, rotation about the axis has a count of $v_n = 2v_n \cos\theta + v_n - 1$, and we note that there can be at most one fixed vertex (a vertex at $(0, 0, 0)$). Note that if there is no fixed vertex, this equality does not hold, so we assume one fixed vertex. We can solve the equation with $v_n = 1$, to see that the symmetry operation would have to be c_6. However, a c_6-symmetric framework is also c_3 symmetric, but we have seen there is no c_3-symmetric isostatic framework. For c_2 symmetry, where fixed edges are possible, we have $e_2 + 2v_2 = -1$ which is a contradiction. A σ'-symmetric isostatic framework would have to satisfy $e_\sigma + v_\sigma = v_\sigma - 1$ which is not possible. As on the cylinder, any $s_{n\geq 3}$-symmetric framework would also be c_k-symmetric for some k. which is not possible.

This leaves three final groups operations, c_2', σ and φ. For c_2' the table gives $e_2 + 2v_2 = 1$, for σ we have $e_\sigma = 1$, and for φ we have $e_\varphi + 4v_\varphi = -1$, which has no solutions, as required. □

Corollary 3.4.4. *If (G, p) is a $\tau(\Gamma)$-symmetric isostatic framework on $\mathcal{C}$, then*

$$\tau(\Gamma) = \begin{cases} C_s = \{id, \sigma\}; \\ C_2 = \{id, c_2'\}. \end{cases}$$

Proof. We must find the groups which can be generated from the symmetry operations $\{\mathrm{id}, \sigma, c_2'\}$, which do not contain any forbidden symmetry operations found in Corollary 3.4.3. This gives us the groups of order 2. Since σ' is a forbidden operation, any group which contains both σ and c_2' must have the axis of rotation perpendicular to the plane of reflection. Furthermore, as with this alignment $\sigma c_2' = \varphi$, which is not contained in any groups with isostatic frameworks, and our list is complete. □

We are now ready to give the necessary conditions for a isostatic framework on the cone.

Theorem 3.4.5. *Let (G, p) be an isostatic $\tau(\Gamma)$-symmetric framework on $\mathcal{C}$. Then (G, ϕ) is Γ-symmetric, $(2, 1)$-tight and will satisfy the constraints given in Table 3.6:*

$\tau(\Gamma)$	Number of edges and vertices fixed by symmetry elements
C_s	$e_\sigma = 1$
C_2	$e_{2'} = 1, v_{2'} = 0$

Table 3.6: Fixed edge and vertex counts for symmetry operations on the cone.

Proof. The graph G must be Γ-symmetric and $(2, 1)$-tight (by [30]). Both rows of the table follow immediately from corollaries 3.4.3 and 3.4.4. □

3.5 The elliptical cylinder

Less analysis has been undertaken on the rigidity of frameworks on the elliptical cylinder than on the sphere, cylinder and cone. As we will see, isostatic frameworks without symmetry on the elliptical cylinder must be $(2,1)$-tight. Indeed, [30, Theorem 1.1] characterises rigidity for what the authors define as "Type 1 surfaces", which includes the elliptical cylinder. It is worth mentioning work in progress of Andrew Sainsbury that focuses on establishing necessary and sufficient conditions for generic global rigidity on the elliptical cylinder. We recall Remark 3.4.1, and the following may be taken as a corollary of Lemma 3.3.1. Unlike the cylinder, sphere or cone, the elliptical cylinder is not invariant for a general rotation about an axis. Therefore, we only consider $\mathrm{id}, c_2, c_2', \sigma, \sigma'$ and φ.

Corollary 3.5.1. *For the following symmetry groups on the elliptical cylinder, the $\chi((\tau \otimes P_V)^{(\mathcal{T})})$ row of the character table is*

$\mathcal{L}$	*id*	c_2	c_2'	σ	σ'	φ
$\chi((\tau \otimes P_V)^{(\mathcal{T})})$	*1*	*1*	-1	*1*	-1	-1

Proof. This result is proved in the proof of Lemma 3.3.1, by excluding the rotational trivial motion. □

Note that if (G, p) is a Γ-symmetric framework on $\mathcal{Y}$ with respect to τ and ϕ, then there is no vertex fixed by an element of Γ corresponding to a rotation c_n about the z-axis or the inversion φ. The number of vertices that are fixed by the element in Γ corresponding to the half-turn c_2', or the reflections σ and σ' are denoted by $v_{2'}$, v_σ and $v_{\sigma'}$, respectively. An edge of G cannot be fixed by an element of Γ that corresponds to a rotation c_n, $n \geq 3$, or an improper rotation s_n, $n \geq 3$. Hence we have separate columns for c_2 and $\varphi = s_2$ below. The number of edges that are fixed by the element in Γ corresponding to the half-turns c_2 and c_2', the reflections σ and σ' and the inversion φ are denoted by e_2, $e_{2'}$, e_σ, $e_{\sigma'}$ and e_φ, respectively.

$\mathcal{L}$	id	c_2	c_2'	σ	σ'	φ
$\chi(\tilde{P}_E)$	$\lvert E\rvert + \lvert V\rvert$	e_2	$e_{2'} + v_{2'}$	$e_\sigma + v_\sigma$	$e_{\sigma'} + v_{\sigma'}$	e_φ
$\chi(\tau \otimes P_V)$	$3\lvert V\rvert$	0	$-v_{2'}$	v_σ	$v_{\sigma'}$	0
$\chi((\tau \otimes P_V)^{(\mathcal{T})})$	1	1	-1	1	-1	-1

Table 3.7: Character table for symmetry operations of the elliptical cylinder.

The following result from Theorem 3.1.5 and Table 3.7 is immediate, so is given without proof.

Corollary 3.5.2. *Let (G,p) be a $\tau(\Gamma)$-symmetric framework on the elliptical cylinder. Then $c_2 \notin \tau(\Gamma)$ and $\sigma \notin \tau(\Gamma)$. Furthermore, if an isostatic framework exists for the following symmetry elements, then it satisfies the following:*

- *if $c_2' \in \tau(\Gamma)$ then $e_{2'} = 1$ and $v_{2'} = 0$;*
- *if $\sigma' \in \tau(\Gamma)$ then $e_{\sigma'} = 0$;*
- *if $\varphi \in \tau(\Gamma)$ then $e_\varphi = 1$.*

Furthermore, $\tau(\Gamma)$ is impermissible if it contains symmetries which cannot be generated by those listed above.

Corollary 3.5.3. *If (G,p) is a $\tau(\Gamma)$-symmetric isostatic framework on $\mathcal{L}$, then*

$$\tau(\Gamma) = \begin{cases} C_i = \{id, \varphi\}; \\ C_s = \{id, \sigma'\}; \\ C_2 = \{id, c_2'\}. \end{cases}$$

Proof. The possibility of C_i-, C_s-, and C_2-symmetric isostatic frameworks on $\mathcal{L}$ is immediate from Corollary 3.5.2. We therefore must check if any other groups can be generated from σ', c_2', and φ which do not include symmetries from Corollary 3.5.2 which do not permit isostatic frameworks. Here we are only given three choices for how to compose two of the elements: $\sigma' c_2'$ is a reflection containing the z-axis; $\sigma'\varphi$ is a half-turn rotation about the z-axis; $c_2'\varphi$ is also a reflection containing the z-axis (orthogonal to the reflection plane generated by $\sigma c_2'$). Hence the list in the corollary is complete. □

Theorem 3.5.4. *Let (G,p) be an isostatic $\tau(\Gamma)$-symmetric framework on $\mathcal{L}$. Then (G,ϕ) is Γ-symmetric, $(2,1)$-tight and will satisfy the constraints given in Table 3.8:*

3.6 The ellipsoid

The ellipsoid is an example of a surface where $\mathcal{T} = \emptyset$, hence Theorem 3.1.5 gives the equation $\chi(\tilde{P}_E) = \chi(\tau \otimes P_V)$. We recall from Section 3.1 that the surface-preserving

$\tau(\Gamma)$	Number of edges and vertices fixed by symmetry elements
C_i	$e_\varphi = 1$
C_s	$e_\sigma = 1$
C_2	$e_{2'} = 1, v_{2'} = 0$

Table 3.8: Fixed edge and vertex counts for symmetry operations on the elliptical cylinder.

symmetry operations we consider are half-turns, reflections and the inversion. We can therefore immediately give the full character table for $\tau(\Gamma)$-symmetric frameworks on $\mathcal{E}$ (Table 3.9). We note that on the ellipsoid, there can be no vertices fixed by the inversion. We remark that for a framework on $\mathcal{E}$, any graph $G = (V, E)$ must

$\mathcal{E}$	id	c_2	σ	φ
$\chi(\tilde{P}_E)$	$\|E\| + \|V\|$	$e_2 + v_2$	$e_\sigma + v_\sigma$	e_φ
$\chi(\tau \otimes P_V)$	$3\|V\|$	$-v_2$	v_σ	0

Table 3.9: Character table for symmetry operations of the ellipsoid.

satisfy $|E| = 2|V|$. This provides significant difficulty in establishing combinatorial results akin to those found in Chapter 4, as the graph may be 4-regular (that is all vertices are of degree 4). As a result, there is no known characterisation of generic rigidity or generic global rigidity on the ellipsoid [18]. One would expect similar difficulties would be found when attempting to find a characterisation of symmetric rigidity.

Corollary 3.6.1. *Let (G, p) be a $\tau(\Gamma)$-symmetric framework on the ellipsoid. If the following symmetry operations are in $\tau(\Gamma)$, then they satisfy the following:*

- *if $c_2 \in \tau(\Gamma)$ then $e_2 = v_2 = 0$;*
- *if $\sigma \in \tau(\Gamma)$ then $e_\sigma = 0$;*
- *if $\varphi \in \tau(\Gamma)$ then $e_\varphi = 0$.*

Furthermore, $\tau(\Gamma)$ is impermissible if it contains symmetries which cannot be generated by those listed above.

Proof. This is a consequence of Theorem 3.1.5, which we use to draw conclusions from Table 3.9. The results follow immediately. □

Corollary 3.6.2. *If (G, p) is a $\tau(\Gamma)$-symmetric isostatic framework on $\mathcal{E}$, then*

$$\tau(\Gamma) = \begin{cases} C_i = \{id, \varphi\}; \\ C_s = \{id, \sigma\}; \\ C_2 = \{id, c_2\}; \\ C_{2h} = \{id, \sigma, c_2, \varphi\}; \\ D_2 = \{id, c_{2x}, c_{2y}, c_{2z}\}; \\ D_{2h} = \{id, \sigma_1, \sigma_2, c_2, \varphi, \sigma_1\varphi, \sigma_2\varphi, c_2\varphi\}, \end{cases}$$

where c_{2x}, c_{2y}, c_{2z} are rotations about the x-, y-, z-axis respectively, and $\sigma_1, \sigma_2 \in \{\sigma_{xy}, \sigma_{xz}, \sigma_{yz}\}$.

Proof. None of the operations in Table 3.9 forbid isostatic frameworks, hence we are tasked to find which groups $\{\mathrm{id}, c_2, \sigma, \varphi\}$ generate. We observe choosing any two of $\{c_2, \sigma, \varphi\}$ will together generate the third. Hence we have C_{2h}, where for example, $\sigma c_2 = \varphi$, hence c_2 rotates about the axis which is perpendicular to the plane of the reflection. For D_2 we note that $c_{2r}^2 = \mathrm{id}$ for any $r = x, y, z$, and the composition of any two of c_{2x}, c_{2y}, c_{2z} generates the third. To generate D_{2h}, let $\sigma_1 \in \{\sigma_{xy}, \sigma_{xz}, \sigma_{yz}\}$ and $\sigma_2 \in \{\sigma_{xy}, \sigma_{xz}, \sigma_{yz}\} \setminus \sigma_1$. Then $\sigma_1\sigma_2 = c_2$, hence c_2 rotates about the line which is the intersection of the two planes of reflection of σ_1 and σ_2. One can check that both half turns not chosen as c_2 in C_{2h} can be generated from $\{c_2, \sigma, \varphi\}$, completing the proof. $\square$

Theorem 3.6.3. *Let (G, p) be an isostatic $\tau(\Gamma)$-symmetric framework on $\mathcal{E}$. Then (G, ϕ) is Γ-symmetric, $(2, 0)$-tight and will satisfy the constraints given in Table 3.10:*

$\tau(\Gamma)$	Number of edges and vertices fixed by symmetry elements
C_i	$e_\varphi = v_\varphi = 0$
C_s	$e_\sigma = 0$
C_2	$e_2 = v_2 = 0$
C_{2h}	$e_\sigma = e_2 = e_\varphi = v_2 = v_\varphi = 0,\ 2 \vert v_\sigma$
D_2	$e_2 = v_2 = 0$
D_{2h}	$e_\sigma = e_2 = e_\varphi = v_2 = v_\varphi = 0,\ 2 \vert v_\sigma$

Table 3.10: Fixed edge and vertex counts for symmetry operations on the ellipsoid.

Proof. The graph G must be Γ-symmetric and is $(2,0)$-tight from the count of the identity column in Table 3.9. Note that sparsity is a result of all subgraphs of the graph being symmetric to the identity group, and thus must too follow this count. The first four rows follow immediately from Corollaries 3.6.1 and 3.6.2. Since in Corollary 3.6.1, we have no fixed edges or vertices for c_2 and φ, and no fixed edges for σ, we only need to check if in C_{2h} and D_{2h} the vertices fixed by σ are restricted. Any point on the mirror would have its image point under the inversion on the mirror too. Since this point can never coincide with the original, the number of vertices fixed by the mirror must be divisible by 2. It is possible for the image under inversion and half turn rotation to coincide, so the number of mirror fixed vertices is not necessarily divisible by 4. Indeed, in D_{2h}, we have the same structure but now with two orthogonal mirrors. This does not impact the number of vertices fixed by each mirror. □

3.7 Necessary conditions for isostatic linearly constrained frameworks

For the remainder of the chapter we turn our attention to linearly constrained frameworks. We provide analogous results to those in Section 3.1. We require some altered definitions, where which term being used should be clear from the context.

Let $G = (V, E, L)$ be a graph and Γ be a finite group. Let $\phi : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^d)$ be homomorphisms. We say (G, p, q) is Γ-symmetric (with respect to ϕ and τ) if for every $\gamma \in \Gamma$, $\tau(\gamma)p_i = p_{\phi(\gamma)i}$ for all $i \in [|V|]$, and $\tau(\gamma)q_j = q_{\phi(\gamma)j}$ for all $j \in [|L|]$.

Let $\tau(\gamma)$ denote the $d \times d$ matrix which represents γ with respect to the canonical basis of $\mathbb{R}^d$. Let $P_V(\gamma)$, $P_E(\gamma)$ and $P_L(\gamma)$ be the permutation matrix of V, E and L respectively, induced by γ. We define a new matrix $P^*_L : \Gamma \to \mathbb{R}^{d|L|}$ with respect to τ by

$$P^*_L(\gamma)_{i,j} = \begin{cases} \tau(\gamma)\frac{q_i}{q_i} \text{ if } i = j \text{ and } l_i \in L \\ P_L(\gamma)_{i,j} \text{ otherwise,} \end{cases}$$

that is to say for a fixed loop l_i, the normal to the linear constraint q_i is either preserved or inverted by $\tau(\gamma)$, and the entry of the matrix P_L^* corresponding to that fixed loop is ± 1 respective to this preservation or inversion. We then have two important maps, $\tau \otimes P_V : \Gamma \to \mathbb{R}^{(d|V|)\times(d|V|)}$ and $P_{E,L} := P_E \oplus P_L^* : \Gamma \to \mathbb{R}^{(|E|+|L|)\times(|E|+|L|)}$.

3.7.1 Block-diagonalization of the rigidity matrix

Lemma 3.7.1. *Let G be a graph, $\tau(\Gamma)$ be a symmetry group, and $\phi : \Gamma \to \mathrm{Aut}(G)$ be a homomorphism. If $R(G, p, q)u = z$, then for all $\gamma \in \Gamma$, we have*

$$R(G, p, q)(\tau \otimes P_V)(\gamma)u = P_{E,L}(\gamma)z.$$

Proof. Suppose $R(G, p, q)u = z$. Fix $\gamma \in \Gamma$ and let $\tau(\gamma)$ be the orthogonal matrix representing γ with respect to the canonical basis of $\mathbb{R}^d$. We enumerate the rows of $R(G, p, q)$ by the set $\{a_1, \ldots, a_{|E|}, b_1, \ldots, b_{|L|}\}$. By [40], we know that $(R(G, p)(\tau \otimes P_V)(\gamma)u)_{a_i} = (\tilde{P}_E(\gamma)z)_{a_i}$, for all $i \in [|E|]$. We are left to show the result holds for the rows of $R(G, p, q)$ which represent the normal vectors of the vertices with loops.

Write $u \in \mathbb{R}^{d|L|}$ as $u = (u_1, \ldots, u_{|L|})$, where $u_i \in \mathbb{R}^d$ for all i, and let $\Phi(\gamma)(l_i) = l_k$. We first see that $(\tilde{P}_E(\gamma)z)_{b_k} = z_{b_i}$ by the definition of $P_L(\gamma)$. From $R(G, p, q)u = z$, we also get that $z_{b_i} = n(q_i) \cdot u_i$. Then $(\tau \otimes P_V)(\gamma)u = (\bar{u}_1, \ldots, \bar{u}_{|L|})$, with $\bar{u}_l = \tau(\gamma)u_j$ when $\Phi(\gamma)(v_j) = v_l$. Therefore,

$$\begin{aligned}
(R(G, p, q)(\tau \otimes P_V)(\gamma)u)_{b_k} &= n_1(p_k) \cdot (\tau(\gamma)u_i)_1 + \ldots n_d(p_k) \cdot (\tau(\gamma)u_i)_d \\
&= n(p_k) \cdot (\tau(\gamma)u_i) \\
&= n(\tau(\gamma)p_i) \cdot (\tau(\gamma)u_i).
\end{aligned}$$

Finally, the definition of symmetric looped graph and the fact that the canonical inner product on $\mathbb{R}^d$ is invariant under the orthogonal transformation $\tau(\gamma) \in O(\mathbb{R}^d)$ give that

$$n(\tau(\gamma)p_i) \cdot (\tau(\gamma)u_i) = \tau(\gamma)n(p_i) \cdot (\tau(\gamma)u_i) = n(p_i) \cdot u_i = z_{b_i},$$

finishing the proof. $\square$

The following is an immediate corollary of Schur's lemma (see e.g. [43]) and the lemma above.

Corollary 3.7.2. *Let (G, p, q) be a $\tau(\Gamma)$-symmetric framework and let $I_1, \ldots, I_r$ be the pairwise non-equivalent irreducible linear representations of $\tau(\Gamma)$. Then there exist matrices A, B such that the matrices $B^{-1}R(G, p, q)A$ and $A^{-1}R(G, p, q)^T B$ are block-diagonalised and of the form*

$$\begin{pmatrix} R_1 & & & \mathbf{0} \\ & R_2 & & \\ & & \ddots & \\ \mathbf{0} & & & R_r \end{pmatrix}$$

where the submatrix R_i corresponds to the irreducible representation I_i.

This block decomposition corresponds to $\mathbb{R}^{d|V|} = X_1 \oplus \cdots \oplus X_r$ and $\mathbb{R}^{|E|+|L|} = Y_1 \oplus \cdots \oplus Y_r$. The space X_i is the $(\tau \otimes P_V)$-invariant subspace of $\mathbb{R}^{d|V|}$ corresponding to I_i, and the space Y_i is the $\tilde{P}_E$-invariant subspace of $\mathbb{R}^{|E|}$ corresponding to I_i. Then, the submatrix R_i has size $(\dim(Y_i)) \times (\dim(X_i))$.

3.7.2 Additional necessary conditions

Using the block-decomposition of the rigidity matrix, we may follow the basic approach described in [11, 40] to derive added necessary conditions for a symmetric linearly constrained framework to be isostatic. We first need the following result.

If $A = (a_{ij})$ is a square matrix then the trace of A is given by $\mathrm{tr}(A) = \sum_i a_{ii}$. For a linear representation ρ of a group Γ and a fixed ordering $\gamma_1, \ldots, \gamma_{|\Gamma|}$ of the elements of Γ, the character of ρ is the $|\Gamma|$-dimensional vector $\chi(\rho)$ whose ith entry is $\mathrm{tr}(\rho(\gamma_i))$.

Theorem 3.7.3. *Let (G, p, q) be a $\tau(\Gamma)$-symmetric framework. If (G, p, q) is isostatic, then*

$$\chi(P_{E,L}) = \chi(\tau \otimes P_V).$$

Proof. Since (G, p, q) is isostatic, the rigidity matrix of (G, p, q) is a non-singular square matrix. Thus, by Lemma 3.7.1, we have

$$R(G, p, q)(\tau \otimes P_V)(\gamma)(R(G, p, q))^{-1} = P_{E,L}(\gamma) \quad \text{for all } \gamma \in \Gamma.$$

It follows that $\tau \otimes P_V$ and $P_{E,L}$ are isomorphic representations of Γ. Hence,

$$\chi(P_{E,L}) = \chi(\tau \otimes P_V).$$ □

3.8 Two-dimensional linearly constrained frameworks

Rigidity in the plane is well studied. There are classical results for generic frameworks [36, 23], symmetric frameworks [38, 39], linearly constrained frameworks [44], and globally rigid linearly constrained frameworks [15]. The setting of symmetric linearly constrained frameworks in the plane is conducive to new results. In the following section we calculate the characters of the representations appearing in the statement of Theorem 3.7.3 for the plane. The graphs satisfying these conditions for C_n-symmetric frameworks are further studied in Chapter 5: firstly Section 5.2 considers graphs with isostatic frameworks under the presence of a half-turn symmetry, and Section 5.3 considers graphs with isostatic frameworks under the presence of a n-fold symmetry for odd n. This work has been submitted for review in [34].

The surface-preserving symmetry operations for the plane are rotations c_n, $n \in \mathbb{N}$, around the origin, reflections (without loss of generality we take the mirror line to be the x-axis), denoted by σ. With these symmetries we will now give the full character table for $\tau(\Gamma)$-symmetric isostatic frameworks on the plane. We give the the row corresponding to $\tau \otimes P_V$ without calculation as the entries can be seen directly from the matrix representation.

For a Γ-symmetric graph $G = (V, E, L)$ with respect to $\phi : V \to \text{Aut}(G)$, we say that a loop $vv \in L$ is *fixed* by $\gamma \in \Gamma$ if $v \in V$ is fixed by γ. Loops fixed by γ correspond to linear constraints of vertices fixed by γ, where the linear constraint is also fixed by $\tau(\gamma)$. In the matrix $P_{E,L}$ there is two possibilities for the entry of a fixed loop, namely ± 1, representing if the normal of the linear constraint is preserved or inverted.

Neither an edge or a loop of G can be fixed by an element of Γ that corresponds to a rotation c_n, $n \geq 3$, so we have a separate column for c_2 below. The number of vertices that are fixed by the element in Γ corresponding to the half-turn c_2, a general n-fold rotation c_n, or the reflection σ are denoted by $v_{2'}$, v_n and v_σ, respectively. The number of edges that are fixed by the element in Γ corresponding to the half-turn c_2 and the reflection σ are denoted by e_2 and e_σ respectively. Finally, the number of loops that are fixed by c_2, c_n and σ are l_2, l_n, and $l_{\sigma,+}, l_{\sigma,-}$ respectively, where $l_{\sigma,+}$ counts linear constraints perpendicular to the mirror whose normals are preserved, and $l_{\sigma,-}$ counts those linear constraints parallel to the mirror with inverted normals.

Before giving the character table, we consider the values corresponding to the fixed linear constraints in $P_{E,L}(\gamma)$ for σ and c_n for all $n \geq 2$. For $n \geq 3$, no lines are fixed by c_n. A half turn which fixes a linear constraint would map a normal to its inverse, hence have a -1 entry representing that loop in $P_{E,L}(c_2)$. Linear constraints parallel to a mirror would have normals which are perpendicular, hence the normals would map to its inverse having a -1 entry in the matrix $P_{E,L}(\sigma)$, whereas the linear constraints perpendicular have normals parallel to the mirror, so have a 1 representing the loop in the matrix $P_{E,L}(\sigma)$. This gives all the information necessary to complete Table 3.11.

	id	$c_{n\geq 3}$	c_2	σ
$\chi(P_{E,L})$	$\lvert E\rvert + \lvert L\rvert$	0	$e_2 - l_2$	$e_\sigma + l_{\sigma,+} - l_{\sigma,-}$
$\chi(\tau \otimes P_V)$	$2\lvert V\rvert$	$2v_n \cos(\frac{2\pi}{n})$	$-2v_2$	0

Table 3.11: Character table for symmetry operations of the plane.

In the following proofs we shall use Theorem 3.7.3 to draw conclusions from Table 3.11. The first column (for the identity element) simply recovers the result from Theorem 2.4.2 that $|E| + |L| = 2|V|$ for an isostatic linearly constrained framework in the plane. The other columns provide further conditions for isostaticity in the presence of symmetry. It is easy to see that if the counts in Corollary 3.8.1 are satisfied for a $\tau(\Gamma)$-symmetric framework, then the corresponding counts are also satisfied for any $\tau(\Gamma')$-symmetric subframework with $\Gamma' \subseteq \Gamma$.

Corollary 3.8.1. *If (G, p, q) is a $\tau(\Gamma)$-symmetric isostatic linearly constrained framework in the plane, then the following hold,*

- *for n odd or $n \geq 6$ even, if $c_n \in \tau(\Gamma)$, then $v_n = e_n = l_n = 0$;*
- *if $c_2 \in \tau(\Gamma)$ then $v_2 = e_2 = l_2 = 0$ or $v_2 = 1, l_2 = 2$;*
- *if $c_4 \in \tau(\Gamma)$ then $v_4 = 0, 1$ and $e_4 = l_4 = 0$;*
- *if $\sigma \in \tau(\Gamma)$ then $e_\sigma + l_{\sigma,+} = l_{\sigma,-}$.*

Furthermore, $\tau(\Gamma)$ is impermissible if it contains symmetries which cannot be generated by those listed above.

Proof. Recall from Theorem 3.7.3 that if (G, p, q) is a $\tau(\Gamma)$-symmetric isostatic framework, then $\chi(P_{E,L}) = \chi(\tau \otimes P_V)$. We now consider each of the columns in Table 3.11. From the second column, we obtain $v_n \cos(\frac{2\pi}{n}) = 0$, hence either $v_n = 0$ or $\cos(\frac{2\pi}{n}) = 0$. The latter is only possible for a positive integer n when $n = 4$. Hence, for any symmetry group containing c_4, we have $v_4 = 0$ or $v_4 = 1$, since p is injective. For all other $n \geq 3$, there are no fixed vertices, edges or loops.

In the second column, the equation $\chi(P_{E,L}) = \chi(\tau \otimes P_V)$ can only hold for $c_2 \in \tau(\Gamma)$ if $2v_2 = l_2 - e_2$. Again recalling that for rotations we may have at most one fixed vertex, this implies that either $v_2 = 0$ then necessarily we have $l_2 = 0$ and hence $e_2 = 0$, or $v_2 = 1$ then $e_2 = l_2 - 2$. Considering the condition given by id at the single fixed vertex, $l_2 \leq 2$. This gives when $v_2 = 1$, $l_2 = 2$ and $e_2 = 0$. Any group containing c_4 necessarily contains c_2, and the two conditions above are not mutually exclusive. Instead, a fixed point in a C_4-symmetric framework is constrained by two lines which must be perpendicular.

For the reflection, from the table, we immediately have $e_\sigma + l_{\sigma,+} = l_{\sigma,-}$. □

Note that from these symmetry operations *any* symmetry group in the plane is possible. This contrasts with the situation for bar-joint frameworks in the plane where isostatic symmetric frameworks are only possible for a small number of symmetry groups: see [7, 38, 39] for details.

Corollary 3.8.2. *If (G, p, q) is a $\tau(\Gamma)$-symmetric isostatic linearly constrained framework in the plane, then*

$$\tau(\Gamma) = \begin{cases} C_s = \{id, \sigma\} \\ C_n = \{id, c_n, \dots, c_n^{n-1}\}; \\ C_{nv} = \{id, c_n, \dots, c_n^{n-1}, \sigma, \dots, \sigma c_n^{n-1}\}. \end{cases}$$

We are now able to use Corollary 3.8.1 to summarize the conclusions about $\tau(\Gamma)$-symmetric isostatic linearly constrained frameworks for each possible symmetry group $\tau(\Gamma)$. We say that (G, ϕ) is $\tau(\Gamma)$-*tight* if it is tight, Γ-symmetric and satisfies the relevant constraints in Table 3.12.

$\tau(\Gamma)$	Number of edges, loops and vertices fixed by symmetry operations
C_s	$e_\sigma + l_{\sigma,+} = l_{\sigma,-}$
C_n	$v_n, e_n, l_n = 0$
C_2	$v_2, e_2, l_2 = 0$ or $v_2 = 1, e_2 = 0, l_2 = 2$
C_4	$v_2, v_4, e_2, e_4, l_2, l_4 = 0$ or $v_2, v_4 = 1, e_2, e_4 = 0, l_2 = 2, l_4 = 0$
C_{2v}	$e_\sigma + l_{\sigma,+} = l_{\sigma,-}$, ($v_2, e_2, l_2 = 0$ or $v_2 = 1, e_2 = 0, l_2 = 2$)
C_{4v}	$e_\sigma + l_{\sigma,+} = l_{\sigma,-}$, ($v_2, v_4, e_2, e_4, l_2, l_4 = 0$ or $v_2, v_4 = 1, e_2, e_4 = 0, l_2 = 2, l_4 = 0$)
C_{nv}	$e_\sigma + l_{\sigma,+} = l_{\sigma,-}$, $v_n, e_n, l_n = 0$

Table 3.12: Fixed edge, loop and vertex counts for symmetry operations on the plane. Note that n in the above table is a positive integer, not equal to 2 or 4.

Theorem 3.8.3. *Let (G, p, q) be an isostatic $\tau(\Gamma)$-symmetric framework on the plane. Then (G, ϕ) is $\tau(\Gamma)$-tight.*

Proof. Clearly G is Γ-symmetric and tight by Theorem 2.4.2. That the constraints in Table 3.12 are satisfied follows immediately from Corollary 3.8.1 for the groups C_s and C_n. For dihedral groups we check if any of the constraints from Corollary 3.8.1 are mutually exclusive. A vertex fixed by c_2 will be a point at the origin, therefore lie on any mirror. The two c_2-fixed loops incident to this vertex can be σ symmetric to each other or, one must lie on the mirror and the other perpendicular. With an additional c_4 symmetry operation, these linear constraints must be perpendicular. Either of the two cases above is still possible, with the mirrors bisecting the angle between the linear constraints or one line lying on each mirror. For C_{nv} there will be no point at the origin. Any vertices, edges or loops fixed by a mirror will have an orbit of c_n symmetric copies, all fixed by their own respective mirrors. None of this contradicts c_n counts. □

3.9 Three-dimensional linearly constrained frameworks

We turn our attention to three-dimensional linearly constrained frameworks. The symmetry operations to consider in $\mathbb{R}^3$ are rotations around an axis by an angle $\frac{2\pi}{n}$, c_n for $n \in \mathbb{N}$, reflections in a plane, denoted by σ, and improper rotations around an axis by an angle $\frac{2\pi}{n}$, denoted by s_n, $n \geq 2$. Recall that for $n = 2$, s_n is the inversion φ in the origin.

We begin with comments on $\chi(P_{E,L})$ for each symmetry operation. We recall that loops represent constraining a point to a hyperplane, which in this case restricts the point to a plane. For c_2, a fixed loop would lie on a vertex along the axis of rotation. Any linear constraint could contain this axis of rotation, or be perpendicular to it. If the linear constraint contained the axis, any normal to that linear constraint would be inverted by c_2. Count these fixed loops with $l_{2,-}$, noting that these would have negative entry in P_L^*. Conversely, a constraint plane perpendicular to the axis has its normal preserved by c_2, and we count these loops with $l_{2,+}$. For higher order rotations, fixed vertices again lie on the axis of rotation. A fixed loop would need the constraint plane to be perpendicular to the axis of rotation, and the normal to this plane would be preserved by c_n. For improper rotations, the only fixed point in $\mathbb{R}^3$ is the origin. For the inversion any plane through the origin would be fixed, with the the normal being inverted. Higher order improper rotations only fix the plane through the origin perpendicular to the axis of rotation, and s_n inverts the normal of this plane. A plane can be fixed by a reflection if the constraint plane is the mirror plane, which inverts the normal and we count with $l_{\sigma,-}$. Alternatively when the constraint plane is orthogonal to the mirror plane the normal would be preserved, and we count these fixed loops with $l_{\sigma,+}$.

We can give the entries for $\chi(\tau \otimes P_V)$ by reading immediately from the matrices, and so we are ready to give the character table for symmetry operations of $\mathbb{R}^3$.

$\mathbb{R}^3$	id	c_2	c_n	s_n	σ
$\chi(P_{E,L})$	$\lvert E\rvert + \lvert L\rvert$	$e_2 + l_{2,+} - l_{2,-}$	$e_n + l_n$	$e_{s,n} - l_{s,n}$	$e_\sigma + l_{\sigma,+} - l_{\sigma,-}$
$\chi(\tau \otimes P_V)$	$3\lvert V\rvert$	$-v_2$	$v_n(2\cos(\frac{2\pi}{n}) + 1)$	$v_{s,n}(2\cos(\frac{2\pi}{n}) - 1)$	v_σ

Table 3.13: Character table for symmetry operations of $\mathbb{R}^3$.

Corollary 3.9.1. *If (G, p, q) is a $\tau(\Gamma)$-symmetric isostatic linearly constrained framework in $\mathbb{R}^3$, then the following hold,*

- *if $c_2 \in \tau(\Gamma)$ then $v_2 + e_2 + l_{2,+} = l_{2,-}$;*
- *if $c_3 \in \tau(\Gamma)$ then $e_3, l_3 = 0$;*
- *if $c_4 \in \tau(\Gamma)$ then $v_4 = e_4 + l_4$;*
- *if $c_6 \in \tau(\Gamma)$ then $v_6, e_6, l_6 = 0$;*
- *if $\varphi \in \tau(\Gamma)$ then $e_\varphi = 0$ and $3v_\varphi = l_\varphi$;*
- *if $s_3 \in \tau(\Gamma)$ then $v_{s,3}, e_{s,3}, l_{s,3} = 0$;*
- *if $s_4 \in \tau(\Gamma)$ then $e_{s,4} = 0$ and either $l_{s,4} = v_{s,4} = 0$ or $l_{s,4} = v_{s,4} = 1$;*
- *if $s_6 \in \tau(\Gamma)$ then $e_{s,6}, l_{s,6} = 0$ and $v_{s,6} = 0, 1$;*
- *if $\sigma \in \tau(\Gamma)$ then $e_\sigma + l_{\sigma,+} = v_\sigma + l_{\sigma,-}$.*

Furthermore, $\tau(\Gamma)$ is impermissible if it contains symmetries which cannot be generated by those listed above.

Proof. We will first consider rotations. The result of c_2 follows immediately from the table. For $n \geq 3$, we check for solutions to the equation $e_n + l_n = v_n(2\cos(\frac{2\pi}{n}) + 1)$. As e_n, l_n, v_n are integers, we require $\cos(\frac{2\pi}{n})$ to be rational. By Theorem 2.5.1, this happens if and only if $n = 1, 2, 3, 4, 6$. Solving the equation for $n = 3, 4, 6$, we deduce the equations $e_3 + l_3 = 0$, $v_4 = e_4 + l_4$, and $2v_6 = e_6 + l_6$. Since e_3, l_3 are positive, this gives the required result for c_3 and c_4. Then, any edge or loop fixed by c_6 will be fixed by c_3 (since the edge must be between two fixed vertices and the loop represents a plane perpendicular to the axis of rotation), since $e_3, l_3 = 0$ implies $e_6, l_6 = 0$, and this gives that $v_6 = 0$.

Considering improper rotations, for the inversion Table 3.13 gives $e_\varphi - l_\varphi = -3v_\varphi$. With the inversion, the only possibility for a fixed vertex is at the origin, hence $v_\varphi = 0, 1$. Recall that fixed loops are incident to fixed vertices. When $v_\varphi = 0$, $l_\varphi = 0$, and so $e_\varphi = 0$. Alternatively, $v_\varphi = 1$ and $e_\varphi - l_\varphi = -3$. An isostatic linearly constrained framework in $\mathbb{R}^3$ may have at most 3 linear constraints at a point, and so $e_\varphi - 3 \leq -3$. For improper rotations of order at least 3, there can be at most one fixed loop, with linear constraint a plane perpendicular to the axis of rotation. For $n = 3$, $e_{s,3} - l_{s,3} = -2v_{s,3}$. Again, $v_{s,3} = 0, 1$ and $l_{s,3} \leq 3$, so the possible solutions for $(e_{s,3}, l_{s,3}, v_{s,3})$ are $(0, 0, 0), (0, 2, 1)$ and $(1, 3, 1)$. In the latter two cases, there are more loops than permissible, so $(e_{s,3}, l_{s,3}, v_{s,3}) = (0, 0, 0)$ is the unique solution. For $n = 4$, $e_{s,4} - l_{s,4} = -v_{s,4}$. Here, $(e_{s,4}, l_{s,4}, v_{s,4}) = (0, 0, 0), (0, 1, 1)$. As $(s_4)^2 = c_2$, we consider the interactions of edges loops and vertices fixed by s_4 with c_2. Indeed, neither of these cases conflict with the conditions of c_2. In the latter case, the vertex fixed by s_4 would be fixed by c_2, the loop fixed by s_4 would be fixed by c_2 and have its normal preserved. Therefore an isostatic framework would require $l_{2,-} = 2$, which could happen with one loop the image of the other under s_4. For $n = 6$, $e_{s,6} - l_{s,6} = 0$. We note that $(s_6)^2 = c_3$ and $(s_6)^3 = \varphi$. As $l_3 = 0$, this forces $l_{s,6} = 0$, which leads to $e_{s,6} = 0$. If $v_{s,6} = 1$, then $v_\varphi = 1$ and $l_\varphi = 3$, and s_6 and c_3 must map these constraint planes to each other.

Finally, the result for the reflection follows immediately from Table 3.13. □

The symmetries named in Corollary 3.9.1 are ones which occur naturally in the real world. For example, these symmetries are commonly seen in the structure of crystals. In this setting, the groups they produce are called the crystallographic point groups. For further reading on point groups, the reader is recommended [2]. In the following corollary, c_n and s_n will denote rotations about the 'primary' axis of the group, and c_2' is a half turn rotation about an axis perpendicular to the primary axis. To denote reflections, σ is a reflection in a plane containing the primary axis, and σ' a reflection in a plane perpendicular to this axis. Schoenflies notation can cause difficulties in describing the largest of the groups, especially when multiple axes of rotation are involved (and there is no clear candidate for the 'primary' axis), as well as multiple mirror planes which are not perpendicular or containing the principal rotation axis. For this reason, in the following corollary, the symmetry groups $D_{nd}, T, T_d, T_h, O, O_h$ are listed with the symmetry operations they contain from Corollary 3.9.1.

Corollary 3.9.2. *If (G, p, q) is a $\tau(\Gamma)$-symmetric isostatic linearly constrained framework in $\mathbb{R}^3$, then*

$$\tau(\Gamma) = \begin{cases} C_s = \{id, \sigma\}; \\ C_i = \{id, \varphi\}; \\ C_n = \{id, c_n, \ldots, c_n^{n-1}\} \text{ for } n \in \{2,3,4,6\}; \\ S_n = \{id, s_n, \ldots, s_n^{n-1}\} \text{ for } n \in \{4,6\}; \\ C_{nv} = \{id, c_n, \ldots, c_n^{n-1}, \sigma, \ldots, \sigma c_n^{n-1}\} \text{ for } n \in \{2,3,4,6\}; \\ C_{nh} = \{id, c_n, \ldots, c_n^{n-1}, \sigma', \ldots, \sigma' c_n^{n-1}\} \text{ for } n \in \{2,3,4,6\}; \\ D_n = \{id, c_n, \ldots, c_n^{n-1}, c_2', \ldots, c_2' c_n^{n-1}\} \text{ for } n \in \{2,3,4,6\}; \\ D_{nh} = \{id, c_n, \ldots, c_n^{n-1}, c_2', \ldots, c_2' c_n^{n-1}, \sigma, \ldots, \sigma c_2' c_n^{n-1}\} \text{ for } n \in \{2,3,4,6\}; \\ D_{nd}, \{id, c_n, c_2', \sigma, s_{2n}\} \in D_{nd} \text{ for } n \in \{2,3\}; \\ T, \{id, c_2, c_3\} \in T; \\ T_d, \{id, c_2, c_3, \sigma, s_4\} \in T_d; \\ T_h, \{id, c_2, c_3, \sigma, \varphi\} \in T_h; \\ O, \{id, c_2, c_3, c_4\} \in O; \\ O_h, \{id, c_2, c_3, c_4, \sigma\} \in O_h. \end{cases}$$

Proof. We begin by noting that $S_3 = C_{3v}$ so does not appear in the above. Further, both $s_8 \in D_{4d}$ and $s_{12} \in D_{6d}$ so do not have isostatic frameworks associated with them. Then, every symmetry operation in Corollary 3.9.1 can have no associated fixed elements. Hence, any group which contains only symmetries generated by elements of $\{c_2, c_3, c_4, c_6, \varphi, s_3, s_4, s_6, \sigma\}$ may permit an isostatic linearly constrained framework. □

As in previous sections, the identity column of the character table gives $|E| + |L| = 3|V|$. For linearly constrained frameworks in $\mathbb{R}^d$ for $d \geq 3$, there is no

$\tau(\Gamma)$	Number of edges, loops and vertices fixed by symmetry operations
C_s	$e_\sigma + l_{\sigma,+} = v_\sigma + l_{\sigma,-}$
C_i	$e_\varphi = 0,\ 3v_\varphi = l_\varphi$
C_2	$v_2 + e_2 + l_{2,+} = l_{2,-}$
C_3	$e_3, l_3 = 0$
C_4	$v_2 = v_4 = e_4 + l_4,\ l_{2,+} = l_4,\ v_2 + e_2 + l_{2,+} = l_{2,-}$
C_6	$e_i, l_i, v_i = 0$ for $i = 2, 3, 6$
C_{2v}	$e_\sigma + l_{\sigma,+} = v_\sigma + l_{\sigma,-},\ v_2 + e_2 + l_{2,+} = l_{2,-}$
C_{3v}	$e_\sigma + l_{\sigma,+} = v_\sigma + l_{\sigma,-},\ e_3, l_3 = 0$
C_{4v}	$e_\sigma + l_{\sigma,+} = v_\sigma + l_{\sigma,-},\ v_2 = v_4 = e_4 + l_4,\ l_{2,+} = l_4,\ v_2 + e_2 + l_{2,+} = l_{2,-}$
C_{6v}	$e_\sigma + l_{\sigma,+} = v_\sigma + l_{\sigma,-},\ e_i, l_i, v_i = 0$ for $i = 2, 3, 6$
C_{2h}	$e_{\sigma'} + l_{\sigma',+} = v_{\sigma'} + l_{\sigma',-},\ v_2 + e_2 + l_{2,+} = l_{2,-}$
C_{3h}	$e_{\sigma'} + l_{\sigma',+} = v_{\sigma'} + l_{\sigma',-},\ e_3, l_3 = 0$
C_{4h}	$e_{\sigma'} + l_{\sigma',+} = v_{\sigma'} + l_{\sigma',-},\ v_2 = v_4 = e_4 + l_4,\ l_{2,+} = l_4,\ v_2 + e_2 + l_{2,+} = l_{2,-}$
C_{6h}	$e_{\sigma'} + l_{\sigma',+} = v_{\sigma'} + l_{\sigma',-},\ e_i, l_i, v_i = 0$ for $i = 2, 3, 6$
S_4	$e_{s,4} = 0,\ l_{s,4} = v_{s,4} = 0, 1,\ v_2 + e_2 + l_{2,+} = l_{2,-}$
S_6	$e_{s,6}, e_3, e_\varphi, l_{s,6}, l_3 = 0,\ v_{s,6} = v_\varphi = 0, 1,\ l_\varphi = 3v_\varphi$

Table 3.14: Fixed edge, loop and vertex counts for some symmetry operations in $\mathbb{R}^3$.

analogous result to Theorem 2.4.2. However, we can extend Theorem 2.4.3 to apply under the presence of the symmetries in Corollary 3.9.2. We say that (G, ϕ) is $\tau(\Gamma)$-3-tight if it is 3-tight, Γ-symmetric and satisfies the relevant constraints in Table 3.14. We omit the rows of the table for $D_n, D_{nh}, D_{nd}, T, T_d, T_h, O$ and O_h, which as a result are not included in the following theorem. Note that a marginal improvement may be made here, using a result akin to Lemma 2.2.2, to improve upon the 3-tight condition. However, it is likely the constructive proof, should it follow that in [20], would not be enough to prove any strengthening of the theorem.

Theorem 3.9.3. *Let (G, ϕ) be a Γ-symmetric looped simple graph with the property that every vertex of G is incident with at least 1 loop, and (G, p, q) be an isostatic $\tau(\Gamma)$-symmetric framework in $\mathbb{R}^3$. Then (G, ϕ) is $\tau(\Gamma)$-3-tight and K_5-free.*

Proof. We know G is Γ-symmetric and that it is 3-sparse and K_5-free by Theorem 2.4.3. That the constraints in Table 3.12 are satisfied follows immediately from Corollary 3.9.1 for the groups C_s, C_2, C_3, C_i. For C_4, we recall any vertex fixed by either c_2 or c_4 is fixed by both, and lies on the axis of rotation. Any edge fixed by

c_4 must have endpoints both fixed, and thus is fixed by c_2. Any loop fixed by c_4 will represent a plane perpendicular to the rotation axis; such planes will be fixed by, and have their normals preserved by, c_2. Hence $l_{2,+} = l_4$. The other conditions in the table come from the corollary. For C_6, since $v_6 = 0$, there are no vertices on the axis of rotation, so v_2 and v_3 are also zero. As a result, there are no fixed loops, and therefore the equation for c_2 in Corollary 3.9.1 gives $e_2 = 0$.

For dihedral groups we check if any of the constraints from Corollary 3.8.1 are mutually exclusive. For groups C_{nv}, it is reasonably easy to check that the constraints do not conflict, as there is no requirement for the edges, loops and vertices fixed by a rotation to be fixed by the reflection. For groups C_{nh}, any vertices fixed by the rotation will be fixed by the reflection. One should consider whether loops fixed by the rotation will have mirror image loops, or are fixed by σ'. In the later case, one would need to consider whether the normals are preserved or inverted. Additionally, $s_n \in C_{nh}$ so the fixed elements of improper rotations should also be considered. However, after inspection it can be seen that the constraints are not mutually exclusive.

For rotation-reflection groups S_4 and S_6, we first observe the possible loop and vertex fixed by s_4 would be fixed by c_2, and the image of any c_2 fixed element under s_4 would still be fixed by c_2. This does not lead to any further constraints. Recall $s_6^2 = c_3$ and $s_6^3 = \varphi$. The vertex fixed at the origin must have 3 loops fixed by the inversion. These loops must be formed from a single orbit by s_6 or c_3. Vertices on the axis of rotation fixed by c_3 will have an image under φ which is also fixed by c_3. None of these conditions are mutually exclusive. □

We remark that the fixed element conditions for a C_4-3-tight graph have non-zero solutions. For example, $v_2 = 2$, $e_2 = e_4 = 1$, $l_{2,+} = l_4 = 1$, $l_{2,-} = 4$. Indeed, we may construct a graph with exactly these fixed elements (see Figure 3.1).

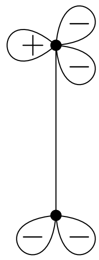

Figure 3.1: A C_4-3-tight, $\Gamma = \{\text{id}, \gamma, \gamma^2, \gamma^3\}$-graph, with $\tau(\gamma) = c_4$. All elements are fixed by γ^2. We note γ fixes the vertices, the edge, and the + marked loop, and transposes the − marked loops at each vertex.

3.10 Linearly constrained frameworks in higher dimensions

We finally look at d-dimensional linearly constrained frameworks for some symmetry operations and a selection of the symmetry groups they generate. In this section, we use Schoenflies notation outside its intended application of symmetries in two and three dimensions. Indeed, this limits the symmetries that can be discussed. For example in $\mathbb{R}^4$, we would not be able to describe the symmetry given in the matrix

$$M = \begin{bmatrix} \cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3}) & 0 & 0 \\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ 0 & 0 & \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix},$$

representing a double rotation, with one threefold rotation in the xy-plane and a fourfold rotation in the zw-plane. For a complete description of symmetry operations in higher dimensions, one may turn to Coxeter notation. The symmetry operations we consider are rotations around an axis by an angle $\frac{2\pi}{n}$, c_n for $n \in \mathbb{N}$, reflections in a hyperplane, denoted by σ, and improper rotations around an axis by an angle $\frac{2\pi}{n}$, denoted by s_n, $n \geq 2$. Unlike for $d = 3$, s_2 is not the inversion φ in the origin.

Again we begin with comments about $\chi(P_{E,L})$ for each symmetry operation. We recall that loops represent constraining a point to a hyperplane. We further note that an axis of rotation in d-dimensional space would be a $(d-2)$-dimensional subspace. For c_n, a fixed loop would lie on a vertex along the axis of rotation. Any $(d-1)$-dimensional constraint could contain this axis of rotation, or contain the plane of rotation. If the hyperplane contained the axis, the normal would lie in the plane of rotation and therefore would be inverted by c_2; again we count these fixed loops with $l_{2,-}$. Higher order rotations do not preserve the hyperplane in this case. Conversely, when the constraint hyperplane contains the rotation plane, its normal is preserved by c_2 and c_n generally, and we count these loops with $l_{2,+}$ and l_n respectively. For higher order rotations, fixed vertices again lie on the axis of rotation. For improper rotations, the only fixed point is the origin. Higher order improper rotations only fix hyperplanes through the origin containing the plane of rotation, and s_n inverts the normal of these planes. The inversion fixes any hyperplane through the origin, with the the normal being inverted. A hyperplane can be fixed by a reflection if the constraint plane is the mirror plane, which inverts the normal and we count with $l_{\sigma,-}$. Alternatively when the constraint plane is orthogonal to the mirror plane the normal would be preserved, and we count these fixed loops with $l_{\sigma,+}$.

We can give the entries for $\chi(\tau \otimes P_V)$ by reading immediately from the matrices, and so we are ready to give the character table for symmetry operations of $\mathbb{R}^d$. We have omitted the column for the identity, in which $\chi(P_{E,L})(\text{id}) = |E| + |L|$ and $\chi(\tau \otimes P_V)(\text{id}) = d|V|$.

$\mathbb{R}^d$	c_2	c_n	s_n	φ	σ
$\chi(P_{E,L})$	$e_2 + l_{2,+} - l_{2,-}$	$e_n + l_n$	$e_{s,n} - l_{s,n}$	$e_\varphi - l_\varphi$	$e_\sigma + l_{\sigma,+} - l_{\sigma,-}$
$\chi(\tau \otimes P_V)$	$v_2(d-4)$	$v_n(2\cos(\frac{2\pi}{n}) + d - 2)$	$v_{s,n}(2\cos(\frac{2\pi}{n}) + d - 4)$	$-dv_\varphi$	$v_\sigma(d-2)$

Table 3.15: Character table for some symmetry operations of $\mathbb{R}^d$.

Corollary 3.10.1. *If (G, p, q) is a $\tau(\Gamma)$-symmetric isostatic framework in $\mathbb{R}^d$, then the following hold,*

- *if $c_2 \in \tau(\Gamma)$ then $e_2 + l_{2,+} = (d-4)v_2 + l_{2,-}$;*
- *if $c_3 \in \tau(\Gamma)$ then $e_3 + l_3 = (d-3)v_3$;*
- *if $c_4 \in \tau(\Gamma)$ then $e_4 + l_4 = (d-2)v_4$;*

- *if* $c_6 \in \tau(\Gamma)$ *then* $e_6, l_6, v_6 = 0$*;*
- *if* $s_2 \in \tau(\Gamma)$ *then* $e_{s,2} - l_{s,2} = (d-6)v_{s,2}$*;*
- *if* $s_3 \in \tau(\Gamma)$ *then* $e_{s,3} - l_{s,3} = (d-5)v_{s,3}$*;*
- *if* $s_4 \in \tau(\Gamma)$ *then* $e_{s,4} - l_{s,4} = (d-4)v_{s,4}$*;*
- *if* $s_6 \in \tau(\Gamma)$ *then* $e_{s,6} - l_{s,6} = (d-3)v_{s,6}$*;*
- *if* $\sigma \in \tau(\Gamma)$ *then* $e_\sigma + l_{\sigma,+} = (d-2)v_\sigma + l_{\sigma,-}$*;*
- *if* $\varphi \in \tau(\Gamma)$ *then* $e_\varphi = 0$*,* $l_\varphi = dv_\varphi$*.*

Furthermore, $\tau(\Gamma)$ *is impermissible if it contains symmetries which cannot be generated by those listed above.*

Proof. By Theorem 2.5.1, the rational values for $\cos(\frac{2\pi}{n})$ are $n = 1, 2, 3, 4, 6$, hence these are the orders of the rotations we take interest in ($n \geq 2$). We can read the relationships between fixed edges loops and vertices from Table 3.15 immediately for $c_2, c_3, s_2, \sigma, \varphi$ since they do not generate other symmetry operations. However, for an isostatic framework in $\mathbb{R}^d$ there can be at most d loops at any vertex, rearranging the equation for the inversion gives $l_\varphi = e_\varphi + dv_\varphi$, and so $e_\varphi = 0$. As $(c_4)^2 = c_2$, anything fixed by c_4 must be fixed by c_2. Vertices and hyperplanes counted by $l_{2,+}$ fixed by c_2 must be fixed by c_4, however hyperplanes counted by $l_{2,-}$ and edges with endpoints not fixed are not fixed by c_4. Reading from the table, with the additional information that $e_2 + l_{2,+} \geq e_4 + l_4$, gives the added condition that $2v_4 \leq l_{2,-}$, and the conditions are not contradictory. $(c_6)^2 = c_3$, the table gives that $e_6 + l_6 = (d-1)v_6$. We recall that any vertex, edge or loop fixed by a rotation of order $n \geq 3$ will be fixed by all rotations of order $n \geq 3$. Therefore $e_3 + l_3 = e_6 + l_6$, and $(d-1)v_6 = (d-3)v_6$, which gives us $e_6, l_6, v_6 = 0$. We note that $(s_3)^4 = c_3$, any hyperplane, vertex or edge fixed by s_3 will also be fixed by c_3. On the other hand, there can be loops, vertices and edges fixed by c_3 which are not fixed by s_3. The conditions in the table for s_3 and c_3 then are not mutually exclusive. Further, $(s_3)^3 = \sigma$, with $e_{s,3} \leq e_\sigma, l_{s,3} \leq l_{\sigma,-}, v_{s,3} \leq v_\sigma$, which also does not restrict the conditions from the table. We then have that $(s_4)^2 = c_2$, and by geometry $e_{s,4} \leq e_2, l_{s,4} \leq l_{2,+}, v_{s,4} \leq v_2$, which does not further restrict the equations from Table 3.15. Finally $(s_6)^2 = c_3$ and $(s_6)^3 = s_2$. Any element fixed by s_6 will be fixed by s_2 and c_3, and any vertex fixed by s_2 is also fixed by s_6. However, s_2 can fix edges and loops which s_6 does not, so the equations are not mutually exclusive. Likewise, elements fixed by c_3 need not be fixed by s_6, so we do not obtain further constraints on the s_6 symmetry. □

We note that for s_n there are $(d-2)$ possible hyperplanes that can be fixed, hence the can be at most one fixed vertex at the origin, and $l_{s,n} \leq (d-2)v_{s,n}$.

Unlike in the previous sections where we gave all groups that could be generated from our symmetries, and containing only symmetries which have isostatic frameworks, this is not possible for general dimensions. Instead we give some examples of symmetry groups most likely to be of interest in future research, due to their commonly occurring nature in lower dimensions, smaller group size, and potentially simpler to establish combinatorial characterisations. Therefore, we will consider $\tau(\Gamma)$-symmetric isostatic framework in $\mathbb{R}^d$, with

$$\tau(\Gamma) = \begin{cases} C_s = \{\text{id}, \sigma\} \\ C_i = \{\text{id}, \varphi\} \\ C_2 = \{\text{id}, c_2\} \\ C_3 = \{\text{id}, c_3, c_3^2\} \\ C_4 = \{\text{id}, c_4, c_2, c_4^3\}; \\ C_6 = \{\text{id}, c_6, c_3, c_2, c_6^4, c_6^5\}; \\ S_4 = \{\text{id}, s_4, c_2, s_4^3\}. \end{cases}$$

As in Section 3.9, we extend Theorem 2.4.3 to apply under the presence of the symmetries listed above. We say that (G, ϕ) is $\tau(\Gamma)$-d-tight if it is d-tight, Γ-symmetric and satisfies the relevant constraints in Table 3.16.

Theorem 3.10.2. *Let $d \geq 4$ be an integer, (G, ϕ) be a Γ-symmetric looped simple graph with the property that every vertex of G is incident with at least $\lfloor \frac{d}{2} \rfloor$ loops, and (G, p, q) be an isostatic $\tau(\Gamma)$-symmetric framework in $\mathbb{R}^d$. Then (G, ϕ) is $\tau(\Gamma)$-d-tight and K_{d+2}-free.*

$\tau(\Gamma)$	Number of edges, loops and vertices fixed by symmetry operations
C_s	$e_\sigma + l_{\sigma,+} = (d-2)v_\sigma + l_{\sigma,-}$
C_i	$e_\varphi = 0,\ l_\varphi = dv_\varphi$
C_2	$v_2 + e_2 + l_{2,+} = l_{2,-}$
C_3	$e_3, l_3 = 0$
C_4	$v_2 = v_4 = e_4 + l_4,\ l_{2,+} = l_4,\ v_2 + e_2 + l_{2,+} = l_{2,-}$
C_6	$e_i, l_i, v_i = 0$ for $i = 2, 3, 6$
S_4	$e_{s,4} - l_{s,4} = (d-4)v_{s,4},\ e_2 + l_{2,+} = (d-4)v_2 + l_{2,-}$

Table 3.16: Fixed edge, loop and vertex counts for symmetry operations in $\mathbb{R}^d$.

Proof. We know G is Γ-symmetric and that it is d-sparse and K_{d+2}-free by Theorem 2.4.3. That the constraints in Table 3.16 are satisfied follows immediately from Corollary 3.10.1 for the groups C_s, C_2, C_3, C_i. For C_4, we recall any vertex fixed by either c_2 or c_4 is fixed by both, and lies on the axis of rotation (which we recall is a $(d-2)$-dimensional subspace). Any edge fixed by c_4 must have endpoints both fixed, and thus is fixed by c_2. Any loop fixed by c_4 will represent a hyperplane orthogonal to the rotation axis; such planes will be fixed by, and have their normals preserved by, c_2. Hence $l_{2,+} = l_4$. The other conditions in the table come from the corollary. For C_6, since $v_6 = 0$, there are no vertices on the axis of rotation, so v_2 and v_3 are also zero. As a result, there are no fixed loops, and therefore the equation for c_2 in Corollary 3.10.1 gives $e_2 = 0$.

For the rotation-reflection groups S_4, we first observe any loop or vertex fixed by s_4 would be fixed by c_2, and the image of any c_2 fixed element under s_4 would still be fixed by c_2. This does not lead to any further constraints. □

We conclude with a remark on the limitations of the approach of using looped simple graphs as above. In our work we remained consistent with previous literature on the topic. However, once combining linearly constrained frameworks with a symmetry action on the graph, cases arise which appear special in this setting, although might not be special for symmetric linearly constrained frameworks in general. In particular, this special case concerns points fixed by the symmetry, which also have linear constraints present. In our view of loops at a vertex, the symmetry forces the number of loops to be a factor of the order of the group. For example, in Section 3.8, C_4 presents as a special case for the number of edges, loops and vertices fixed by symmetry operations.

An alternate approach could be to keep a linear space of equations at each vertex, rather than loops in the graph structure, and look at maps between them as a Γ-representation. This or a similar view, will lead to a generalisation where the linear constraints are not symmetric and would add a level of complexity warranting study as a new project.

Chapter 4

Isostatic Frameworks on the Cylinder

4.1 Rigidity preserving operations

Given a $\tau(\Gamma)$-symmetric isostatic framework on $\mathcal{Y}$, in this section we will construct larger $\tau(\Gamma)$-symmetric isostatic frameworks on $\mathcal{Y}$. To do this we introduce symmetry-adapted Henneberg-type graph operations. These operations are depicted in Figures 4.1, 4.2 and 4.3.

Where it is reasonable to do so, we will work with a general group $\Gamma = \{\text{id} = \gamma_0, \gamma_1, \ldots, \gamma_{t-1}\}$ and we will write $\gamma_k v$ instead of $\phi(\gamma_k)(v)$ and often $\gamma_k(x, y, z)$ or $(x^{(k)}, y^{(k)}, z^{(k)})$ for $\tau(\gamma_k)(p(v))$ where $p(v) = (x, y, z)$. For a group of order two, it will be common to write $v' = \gamma(v)$ for $\gamma \in \Gamma \setminus \{\text{id}\}$.

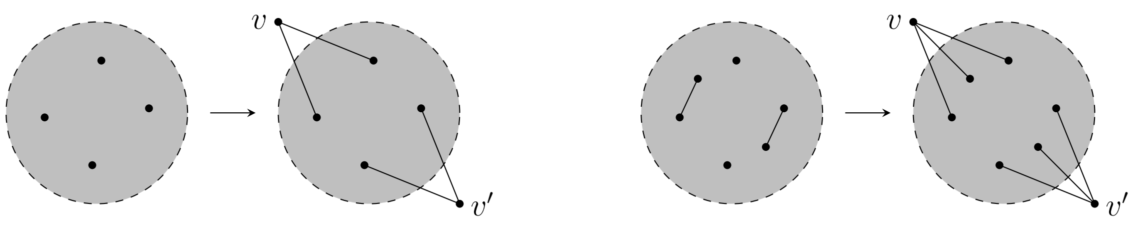

Figure 4.1: Symmetrised 0- and 1-extensions adding new vertices v and v' in each case.

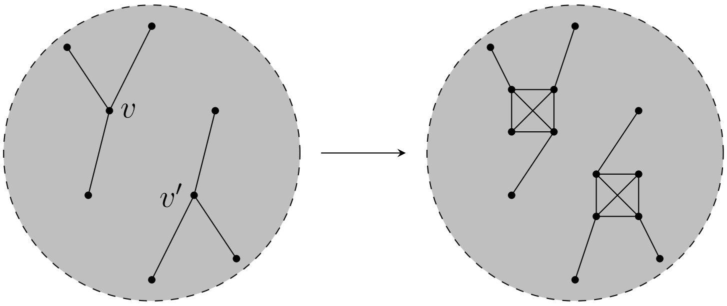

Figure 4.2: The symmetrised vertex-to-K_4 operation (in this case expanding the degree 3 vertices v and v').

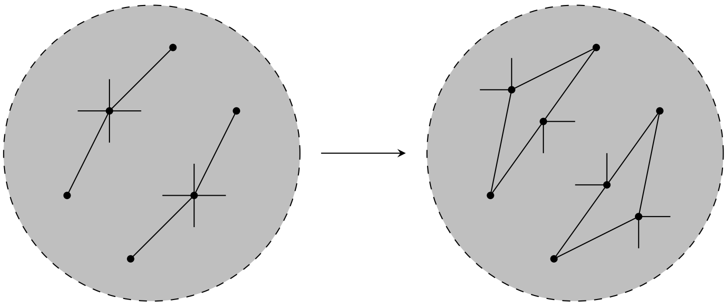

Figure 4.3: The symmetrised vertex-to-C_4 operation. In this example each of the split vertices had degree 6 and the corresponding two new vertices have degree 4 each.

In each of the following operations we have a Γ-symmetric graph (G, ϕ) for a group Γ of order t and define a new Γ-symmetric graph (G^+, ϕ^+). We write $G = (V, E)$ and $G^+ = (V^+, E^+)$. For all $\gamma \in \Gamma$ and $v \in V$, $\phi^+(\gamma)v = \phi(\gamma)v$. A *symmetrised 0-extension* creates a new Γ-symmetric graph G^+ by adding the t vertices $\{v, \gamma v, \ldots, \gamma_{t-1}v\}$ with v adjacent to two vertices, say $v_i, v_j \in V$, and for each $k \in \{1, \ldots, t-1\}$, $\gamma_k v$ adjacent to $\gamma_k v_i, \gamma_k v_j$. Let $e_i = x_i y_i$, $0 \leq i \leq t-1$ be an edge orbit of G of size t under the action of Γ. Further let $z_0 \neq x_0, y_0$ and let $z_i = \gamma_i z_0$ for $i = 1, \ldots, t-1$. A *symmetrised 1-extension* creates a new Γ-symmetric graph by deleting all the edges e_i from G and adding t vertices $\{v, \gamma v, \ldots, \gamma_{t-1}v\}$ with v adjacent to x_0, y_0 and z_0, and $\gamma_i v$ adjacent to x_i, y_i and z_i for $i = 1, \ldots, t-1$. A *symmetrised vertex-to-C_4* operation at the vertices $w, \gamma_1 w, \ldots, \gamma_{t-1}w$, creates a new Γ-symmetric graph $G^+ = (V^+, E^+)$ where $V^+ = V \cup \{u, \ldots, \gamma_{t-1}u\}$. The edge set changes such that if w is adjacent to $v_1, \ldots, v_r$ in G, v_1, v_2 are adjacent to both w and the new vertex u, with $v_3, \ldots, v_r$ adjacent to one of w or u in E^+, so that symmetry is preserved. Similarly $\gamma_k v_1, \gamma_k v_2$ are adjacent to both $\gamma_k w$ and $\gamma_k u$ and $\gamma_k v_3, \ldots, \gamma_k v_r$ are adjacent to one of $\gamma_k w$ or $\gamma_k u$ in G^+. A *symmetrised vertex-to-*K_4 operation at the vertices $w, \gamma_1 w, \ldots, \gamma_{t-1} w$, creates a new Γ-symmetric graph G^+ with $V^+ = V \cup \{a_0, b_0, c_0, \ldots, a_{t-1}, b_{t-1}, c_{t-1}\}$, where for each $1 \leq i \leq t-1$, $\gamma_i a_0 = a_i, \gamma_i b_0 = b_i, \gamma_i c_0 = c_i$. If in G the vertex w is adjacent to $v_1, \ldots, v_r$, then v_i is adjacent to some $d_i \in \{w, a, b, c\}$ in G^+ for each i. Similarly $\gamma_k v_i$ is adjacent to $\gamma_k d_i$ for all k. Finally, we let $G^+[w, a_0, b_0, c_0] \cong K_4$ and $G^+[\gamma_i w, a_i, b_i, c_1] \cong K_4$ for all i.

For $\Gamma = \mathbb{Z}_2$, we introduce special cases of symmetrised extensions above. A *symmetrised fixed-vertex* 0*-extension*, adds a single degree two vertex v that is fixed. The neighbours of the new vertex are not fixed, but are images of each other under the non-trivial group element. A *symmetrised fixed-vertex-to-*C_4 operation at the fixed vertex w creates a new graph $G^+ = G + u$, where u is also a fixed vertex. The edge set changes such that if w is adjacent to $v_1, \ldots, v_r$ in G, then v_1, v_2 are adjacent to both w and the new vertex u, with $v_3, \ldots, v_r$ adjacent to one of w or u in E^+.

4.1.1 Henneberg extensions

To make the geometric statements in this section as general as possible, we sometimes show that the graph operations preserve $\tau(\Gamma)$-independence and sometimes $\tau(\Gamma)$-rigidity depending on the proof strategy. Note that for some symmetry groups $\tau(\Gamma)$, there are no $\tau(\Gamma)$-isostatic graphs and hence this distinction is important.

Lemma 4.1.1. *Suppose (G, ϕ) is Γ-symmetric. Let (G^+, ϕ^+) be obtained from (G, ϕ) by a symmetrised 0-extension such that v_i and v_j are not fixed vertices and $v_i \neq \gamma_k v_j$ for any k. If G is $\tau(\Gamma)$-independent (isostatic) on $\mathcal{Y}$, then G^+ is $\tau(\Gamma)$-independent (isostatic) on $\mathcal{Y}$.*

Proof. Write $G^+ = G + \{v, \ldots, \gamma_{t-1} v\}$, and let $v \in V^+$ be adjacent to v_i, v_j, and for each $k \in \{1, \ldots, t-1\}$, $\gamma_k v$ adjacent to $\gamma_k v_i, \gamma_k v_j$. Since G is $\tau(\Gamma)$-independent on $\mathcal{Y}$ we may choose p so that $R_{\mathcal{Y}}(G, p)$ has linearly independent rows. Define $p^+ : V^+ \to \mathbb{R}^3$ by $p^+(w) = p(w)$ for all $w \in V$, $p^+(v) = (x, y, z)$, and $p^+(\gamma_k v) = (x^{(k)}, y^{(k)}, z^{(k)})$. Write $p(v_i) = (x_i, y_i, z_i)$, $p(v_j) = (x_j, y_j, z_j)$. Then,

$R_{\mathcal{Y}}(G^+, p^+) =$

$$\begin{bmatrix} R_{\mathcal{Y}}(G,p) & & & & & & & \\ & x-x_i & y-y_i & z-z_i & & & & \\ * & x-x_j & y-y_j & z-z_j & & \mathbf{0} & & \\ & x & y & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & x^{(k)}-x_i^{(k)} & y^{(k)}-y_i^{(k)} & z^{(k)}-z_i^{(k)} \\ * & & \mathbf{0} & & & x^{(k)}-x_j^{(k)} & y^{(k)}-y_j^{(k)} & z^{(k)}-z_j^{(k)} \\ & & & & & x^{(k)} & y^{(k)} & 0 \\ & & & & & & & \ddots \end{bmatrix}$$

and hence the fact that $R_{\mathcal{Y}}(G^+, p^+)$ has linearly independent rows will follow once each 3×3 submatrix indicated above is shown to be invertible. For the first such submatrix, one can see that is the case unless $p(v_j)$ lies on the intersection between the cylinder and the plane $A = \{(x, y, z) + a_1(x, y, 0) + a_2(x - x_i, y - y_i, z - z_i)\}$. Note that the hypotheses of the lemma guarantee that p^+ can be chosen to avoid this case. Since each $\tau(\gamma_k)$ is an isometry, all of the other $t - 1$ remaining submatrices are also invertible, and so rank $R_{\mathcal{Y}}(G^+, p^+) =$ rank $R_{\mathcal{Y}}(G, p) + 3t$. Hence, if G is $\tau(\Gamma)$-independent on the cylinder, so is G^+. As the operation preserves sparsity counts, the above holds for isostaticity □

We note that $p(v_j)$ could belong to the plane A in the above proof when v, v_i, v_j are in special positions. Hence when some of v, v_i, v_j are fixed by the symmetry or are images of one another under the symmetry, a symmetrised 0-extension may not preserve rigidity. In the following remark we note two cases when such symmetry exists but $R_{\mathcal{Y}}(G^+, p^+)$ has full rank.

Remark 4.1.2. For a $\mathbb{Z}_2$-symmetric graph G and symmetry group $\tau(\Gamma) = C_s$, let G^+ be defined in either of the following ways:

- let $G^+ = G + \{v\}$ be obtained by a symmetrised fixed-vertex 0-extension,
- let $G^+ = G + \{v, v'\}$ be obtained by a symmetrised 0-extension, where $N(v) = \{v_i, v_j\} = N(v')$.

If G is C_s-independent (isostatic) on $\mathcal{Y}$ then G^+ is C_s-independent (isostatic) on $\mathcal{Y}$.

Lemma 4.1.3. *Let (G, ϕ) be a Γ-symmetric graph, and (G^+, ϕ^+) be obtained from (G, ϕ) by a symmetrised 1-extension. If G is $\tau(\Gamma)$-rigid (isostatic) on $\mathcal{Y}$, then G^+ is $\tau(\Gamma)$-rigid (isostatic) on $\mathcal{Y}$.*

Proof. Let G^+ be obtained from a symmetrised 1-extension on G, that is by deleting the edges $\{v_1v_2, \ldots, \gamma_{t-1}(v_1v_2)\}$, and adding the vertices $\{v_0, \ldots, \gamma_{t-1}v_0\}$ where v_0 is adjacent to v_1, v_2, v_3 and each $\gamma_i v_0$ is adjacent to $\gamma_i v_1, \gamma_i v_2, \gamma_i v_3$. Let (G, p) be completely Γ-regular on $\mathcal{Y}$ and define $p^+ = (p_0,\, p_{-1} = \gamma_1(p_0), \ldots, p_{-t+1} = \gamma_{t-1}(p_0), p)$, where (G^+, p^+) is completely Γ-regular. Suppose for a contradiction (G^+, p^+) is not infinitesimally rigid on $\mathcal{Y}$. Then any $\tau(\Gamma)$-symmetric framework of G^+ on $\mathcal{Y}$ will be infinitesimally flexible. We will use a sequence of $\tau(\Gamma)$-symmetric frameworks, moving only the points $\{p_0, \ldots, p_{-t+1}\}$. First let a, b be tangent vectors at p_1, with b orthogonal to $p_1 - p_2$ and a orthogonal to b. Let $((G^+, p^j))_{j=0}^{\infty}$ where $p^j = (p_0^j, \ldots, p_{-t+1}^j = \gamma_{t-1}(p_0^j), p)$ is so that

$$\frac{\gamma_i(p_1) - \gamma_i(p_0^j)}{||\gamma_i(p_1) - \gamma_i(p_0^j)||} \to \gamma_i a$$

as $j \to \infty$, for each $i \in 0, \ldots, t-1$. The frameworks (G^+, p^j) have a unit norm infinitesimal motion u^j which is orthogonal to the space of trivial motions. By the Bolzano-Weierstrass theorem there is a subsequence of (u^j) which converges to a vector, u^∞ say, also of unit norm. We can discard and relabel parts of the sequence to assume this holds for the original sequence. Looking at the limit (G^+, p^∞), write $u^\infty = (u_0^\infty, \ldots, u_{-t+1}^\infty, u_1, u_2, \ldots, u_n)$, $p^\infty = (p_0^\infty, \ldots, p_{-t+1}^\infty, p_1, p_2, \ldots, p_n)$ with $\gamma_i(p_0^\infty) = \gamma_i(p_1)$ for each i.

We show that (u_1, u_2) is an infinitesimal motion of the bar joining $p(v_1)$ and $p(v_2)$. Since p_0^j converges to p_1 in the a direction, the velocities u_1 and u_0^∞ have the same component in this direction, so $(u_1 - u_0^\infty) \cdot a = 0$. Then $u_1 - u_0^\infty$ is tangential to $\mathcal{Y}$ at p_1, and orthogonal to a, so it is orthogonal to $p_1 - p_2$. Also, $u_2 - u_0^\infty$ is orthogonal to $p_1 - p_2$. Subtracting one from the other gives $u_1 - u_2$ is orthogonal to $p_1 - p_2$, which is the required condition for an infinitesimal motion.

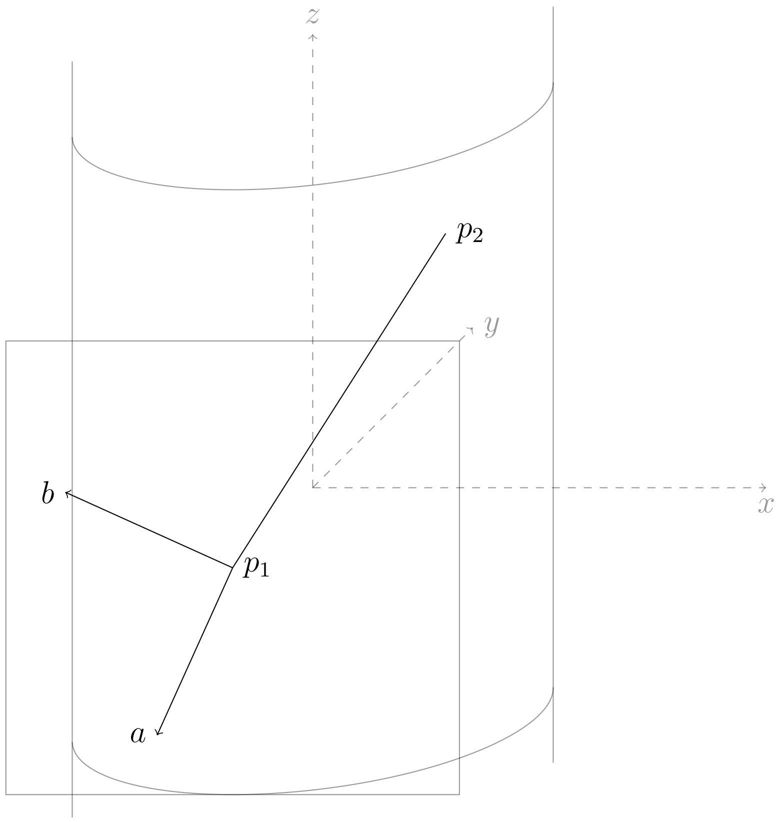

Figure 4.4: Tangent vectors a, b in relation to the edge v_1v_2.

Once again looking at (G, p), we know the infinitesimal motion $u = (u_1, u_2, \ldots, u_n)$ is a trivial motion. In order to preserve the distances $d(p_0^\infty, p_2)$ and $d(p_0^\infty, p_3)$, u_0^∞ is determined by u_2 and u_3. Similarly u_{-i}^∞ is determined by the motion vectors of u which are present on the neighbours of $\gamma_i p_0$, for all $1 \leq i \leq t-1$. We now see that u_0^∞ agrees with u_1 and so u^∞ is a trivial motion for (G^+, p^∞). However, since u^∞ is a unit norm infinitesimal motion and orthogonal to the space of trivial motions, we have reached a contradiction. □

4.1.2 Further operations

For a graph G and pairwise vertex disjoint subgraphs $H_1, \ldots, H_k$ of G, write $G//\{H_i\}_{i=1}^k$ for the graph derived from G by contracting each of the subgraphs $H_1, \ldots, H_k$ to their own single vertex. The resultant graph $G//\{H_i\}_{i=1}^k$ will have $|V(G)| - \sum_{i=1}^k (|V(H_i)| - 1)$ vertices and $|E(G)| - \sum_{i=1}^k |E(H_i)|$ edges. When $k = 1$ we will sometimes use the more common notation G/H_1.

Lemma 4.1.4. *Suppose (G,ϕ) is Γ-symmetric and $H \leq G$ is a copy of K_4. Further, suppose for all $\gamma \in \Gamma \setminus \{id\}$, we have that $V(H) \cap V(\gamma H) = \emptyset$. If $G//\{\gamma_i(H)\}_{i=0}^{t-1}$ is $\tau(\Gamma)$-isostatic on $\mathcal{Y}$, then G is $\tau(\Gamma)$-isostatic on $\mathcal{Y}$.*

Proof. Let $|V| = n$ and (G,p) be a $\tau(\Gamma)$-symmetric framework on $\mathcal{Y}$ which is completely Γ-regular. Further, let the vertices of H be x,y,z,w. Suppose $p = (p(v_1),\dots,p(v_n))$, labelling so that

$$V(\gamma_i(H)) = \{\gamma_i x = v_{4i+1}, \gamma_i y = v_{4i+2}, \gamma_i z = v_{4i+3}, \gamma_i w = v_{4i+4}\}$$

for each $i = 1,\dots t-1$. Define a set of graphs $\{G_j\}_{j=0}^{t}$ by

$$G_j = \begin{cases} G//\{\gamma_i(H)\}_{i=j}^{t-1} & \text{if } j = 0,\dots,t-1; \\ G & \text{if } j = t. \end{cases}$$

where $\gamma_0 = \text{id}$. We want to show by induction that for $0 \leq j \leq t-1$, if G_j is isostatic on $\mathcal{Y}$, then G_{j+1} is isostatic on $\mathcal{Y}$. Then repeating this method, we show $G_t := G$ will be isostatic and $\tau(\Gamma)$-symmetric on $\mathcal{Y}$. For each $0 \leq i \leq t-1$, let the vertices $v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}$ in G contract to v_{4i+1} in $\{G_0,\dots,G_{i-1}\}$.[i] We start by writing

$$R_{\mathcal{Y}}(G_1, p|_{G_1}) = \begin{pmatrix} R_{\mathcal{Y}}(\gamma_0(H), p|_{\gamma_0(H)}) & 0 \\ M_1(p) & M_2(p) \end{pmatrix}$$

where $M_2(p)$ is a square matrix of size $3(n-3t-1)$, since $|V(G_1)| = n-3(t-1)$ and so $M_2(p)$ has $3(n-3(t-1))-12$ columns, and $|E(G_1)| = |E|-6(t-1) = 2n-6t+4$ so $M_2(p)$ has $2n-6t+4+(n-3(t-1))-(6+4)$ rows. For a contradiction, suppose that G_1 is not $\tau(\Gamma)$-isostatic. Then there exists a non-trivial infinitesimal motion m of $(G_1, p|_{G_1})$. Since $(H, p|_H)$ is infinitesimally rigid on $\mathcal{Y}$, we may suppose that

$$m = (0,0,0,0,m_5,m_9,\dots,m_{4t+1},m_{4t+2},\dots,m_n).$$

[i]In the graph G_j, j can be seen as a count on the number of K_4 copies of H that are not contracted.

Consider the realisation $(G_1, \hat{p})$ such that

$$\hat{p} = (p(v_1), p(v_1), p(v_1), p(v_1), p(v_5), p(v_9), \ldots, p(v_{4t+1}), p(v_{4t+2}), \ldots, p(v_n))$$

and define (G_0, p^*) by letting

$$p^* = (p(v_1), p(v_5), p(v_9), \ldots, p(v_{4t+1}), p(v_{4t+2}), \ldots, p(v_n)).$$

By construction (G_0, p^*) is completely Γ-regular, so it is $\tau(\Gamma)$-isostatic on $\mathcal{Y}$. Now, $M_2(p)$ is square with the nonzero vector $(m_5, m_9, \ldots, m_{4t+1}, m_{4t+2}, \ldots, m_n) \in \ker M_2(p)$. Hence $\text{rank} M_2(p) < 3(n-3t-1)$. Since (G, p) is completely Γ-regular, we also have $\text{rank} M_2(\hat{p}) < 3(n-3t-1)$ and hence there exists a nonzero vector $\hat{m} \in \ker M_2(\hat{p})$. Therefore we have

$$R_{\mathcal{Y}}(G_0, p^*) \begin{pmatrix} 0 \\ \hat{m} \end{pmatrix} = \begin{pmatrix} p(v_1) & 0 \\ * & M_2(\hat{p}) \end{pmatrix} \begin{pmatrix} 0 \\ \hat{m} \end{pmatrix} = 0,$$

contradicting the infinitesimal rigidity of (G_0, p^*). We continue the above process inductively, writing $R_{\mathcal{Y}}(G_j, p)$ as

$$\begin{pmatrix} R_{\mathcal{Y}}(\gamma_{j-1}(H), p|_{\gamma_{j-1}(H)}) & 0 \\ L_1(p) & L_2(p) \end{pmatrix}$$

where $L_2(p)$ is a square matrix of size $3(n-3(t-j)-4)$. From the same contradiction argument as before, we have that (G_j, p) is isostatic, and by noting that G_t will be $\tau(\Gamma)$-symmetric, we finish the proof. □

The proof of the following lemma works with a similar strategy as is applied in Lemma 4.1.4. For the first bullet point of the lemma, for a C_i-symmetric graph, we additionally need to perform a (non-symmetric) 0-reduction on the vertex resulting from the contraction of G_1.

Lemma 4.1.5. *Suppose (G_1, φ_1) and (G_2, φ_2) are Γ-symmetric graphs with $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.*

- *For $\tau(\Gamma) = C_i$, let (G, ϕ) be the Γ-symmetric graph with $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2 \cup \{e_1, e_2\}$, and ϕ defined so that $\phi(\gamma)|_{V_i} = \phi_i(\gamma)$ for $i = 1, 2$ and all $\gamma \in \Gamma$; additionally $e_1 = xy, e_2 = x'y'$ for any $x \in V_1, y \in V_2$.*

- *For $\tau(\Gamma) \in \{C_2, C_s\}$, suppose G_2 has a fixed vertex v with neighbours $x_1, x'_1, \dots, x_k, x'_k$. Define (G, ϕ) to be the Γ-symmetric graph with vertex set $V = V_1 \cup V_2 \setminus \{v\}$, and edge set E obtained from $E_1 \cup E_2$ by deleting the edges $vx_1, vx'_1, \dots, vx_k, vx'_k$ and replacing them with the edges $x_1y_1, x'_1y'_1, \dots, x_ky_k, x'_ky'_k$ for some not necessarily distinct $y_1, y'_1, \dots, y_k, y'_k \in V_1$, and ϕ being induced by φ_1, φ_2, similar to the above.*

If G_1 and G_2 are $\tau(\Gamma)$-rigid (isostatic) on $\mathcal{Y}$, then G is $\tau(\Gamma)$-rigid (isostatic) on $\mathcal{Y}$.

Proof. We prove the two statements simultaneously. Let $|V| = n$ and (G, p) be a completely $\tau(\Gamma)$-regular framework on $\mathcal{Y}$. Put $p = (p(v_1), \dots, p(v_n))$ labelling so that $V_1 = \{v_1, \dots, v_r\}$ and $V_2 = \{v_{r+1}, \dots, v_n\}$. As in Lemma 4.1.4, we write

$$R_{\mathcal{Y}}(G, p) = \begin{pmatrix} R_{\mathcal{Y}}(G_1, p|_{G_1}) & 0 \\ M_1(p) & M_2(p) \end{pmatrix}$$

where $M_2(p)$ is a $3(n-r)$ square matrix. We repeat the same arguments as before to show G is rigid. For a contradiction, suppose that G is not rigid. Then there exists some non-trivial infinitesimal motion m of (G, p). Since $(G_1, p|_{G_1})$ is $\tau(\Gamma)$-rigid on $\mathcal{Y}$, we may suppose that $m = (0, \dots, 0, m_{r+1}, \dots, m_n)$. Consider the realisation $(G, \hat{p})$ such that $\hat{p} = (p(v_1), \dots, p(v_1), p(v_{r+1}), \dots, p(v_n))$ and define $(G/G_1, p^*)$ by letting $p^* = (p(v_1), p(v_{r+1}), \dots, p(v_n))$. By construction $(G/G_1, p^*)$ is completely regular, so $(G/G_1, p^*)$ is independent on $\mathcal{Y}$.

Now, $M_2(p)$ is square with the nonzero vector $(m_1, m_{r+1}, \dots, m_n) \in \ker M_2(p)$. Hence $\text{rank} M_2(p) < 3(n-r)$. Since $(G/G_1, p^*)$ is completely $\tau(\Gamma)$-regular, we also have $\text{rank} M_2(\hat{p}) < 3(n-r)$ and hence there exists a nonzero vector $\hat{m} \in \ker M_2(\hat{p})$. Therefore we have

$$(R_{\mathcal{Y}}(G/G_1, p^*)) \begin{pmatrix} 0 \\ \hat{m} \end{pmatrix} = \begin{pmatrix} p(v_1) & 0 \\ * & M_2(\hat{p}) \end{pmatrix} \begin{pmatrix} 0 \\ \hat{m} \end{pmatrix} = 0,$$

contradicting the rigidity of $(G/G_1, p^*)$. Note that in the C_i-symmetric case, G/G_1 is the graph obtained from G_2 by a (non-symmetrised) 0-extension. Hence, we know that if G_1 and G_2 are $\tau(\Gamma)$-rigid on $\mathcal{Y}$, then G/G_1 is rigid and so G is $\tau(\Gamma)$-rigid. As the operation preserves sparsity, the above also preserves isostaticity. □

Recall that the normal to the cylinder, which we write as $n_{\mathcal{Y}}$, acts on a point (x, y, z) of the cylinder by $n_{\mathcal{Y}}(x, y, z) = (x, y, 0)$. As our focus in this chapter is on the cylinder, we will simply write $n(w)$ for the normal at the point w.

Lemma 4.1.6. *Let (G, p) be a $\tau(\Gamma)$-symmetric and independent (isostatic) framework. Let $w \in V$ be adjacent to $v_1, \dots, v_r$. Suppose that $p(w) - p(v_1)$, $p(w) - p(v_2)$, and $n(w)$ are linearly independent. Let (G^+, ϕ^+) be obtained by performing a symmetrised vertex-to-C_4 operation at the vertices $w, \gamma_1 w, \dots, \gamma_{t-1} w$. Let $p^+(v) = p(v)$ for all $v \in V \setminus \{\gamma_k w \mid k \in \{0, \dots, t-1\}\}$, and $p^+(\gamma_k w) = p^+(\gamma_k u) = p(\gamma_k w)$ for all k. Then (G^+, p^+) is independent (isostatic).*

Proof. We will construct $R_{\mathcal{Y}}(G^+, p^+)$ from $R_{\mathcal{Y}}(G, p)$ by a series of matrix row operations. We first add $3t$ zero columns to $R_{\mathcal{Y}}(G, p)$ for the new vertices $\{\gamma_k u\}$. Then add $3t$ rows to this matrix, for the edges $\gamma_k u \gamma_k v_1, \gamma_k u \gamma_k v_2$, and the normal vectors to the surface at the points $p(\gamma_k u)$. Since $p(w) - p(v_1)$, $p(w) - p(v_2)$, $n(w)$ are linearly independent (and, hence, so are each of the $p(\gamma_k w) - p(\gamma_k v_1)$, $p(\gamma_k w) - p(\gamma_k v_2)$, $n(\gamma_k w)$), $\text{rank} R_{\mathcal{Y}}(G^+, p^+) = \text{rank} R_{\mathcal{Y}}(G, p) + 3t$. This gives the matrix M of the form:

$$M = \left[\begin{array}{ccc} * & p(w)-p(v_1) & 0 \\ * & p(w)-p(v_2) & 0 \\ & \vdots & \\ * & p(w)-p(v_i) & 0 \\ & \vdots & \\ * & 0 & p(u)-p(v_1) \\ * & 0 & p(u)-p(v_2) \\ & \vdots & \\ \hline * & n(w) & 0 \\ * & 0 & n(u) \\ & \vdots & \end{array}\right],$$

where the columns given are for the vertices w and u, and rows given for the edges $wv_1, wv_2, wv_i, uv_1, uv_2$ and normal vectors to the surface at w and u. There would be similar columns for each pair $\gamma_k w$ and $\gamma_k u$. This is the rigidity matrix for a graph generated from G by a $\tau(\Gamma)$-symmetric vertex-to-C_4 operation where $v_i w$ is an edge for all $3 \leq i \leq r$. We wish to show that removing the edges $\{\gamma_k w \gamma_k v_i : k = 0, \dots, t-1\}$ and replacing them with the edges $\{\gamma_k u \gamma_k v_i : k = 0, \dots, t-1\}$ preserves $\tau(\Gamma)$-independence.

Since $p(w) - p(v_1)$, $p(w) - p(v_2)$, and $n(w)$ are linearly independent and span $\mathbb{R}^3$, there exists $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$p(w) - p(v_i) = \alpha(p(w) - p(v_1)) + \beta(p(w) - p(v_2)) + \gamma n(w).$$

Hence we perform row operations as follows. From the row of wv_i, subtract α multiples of the row of wv_1, β multiples of the row of wv_2, and γ multiples of the row for the normal vector of w. Then to the row of wv_i, add α multiples of the row of uv_1, β multiples of the row of uv_2, and γ multiples of the row for the normal vector of u. Since $p(w) = p(u)$, when we do this to every neighbour v_i of u, and similarly $\gamma_k v_i$ of $\gamma_k u$ (since all $\tau(\gamma_k)$ are isometries of $\mathbb{R}^3$ that preserve the cylinder, the same α, β, γ work for the symmetric copies) in G^+, we obtain $R_{\mathcal{Y}}(G^+, p^+)$. The row operations preserve $\tau(\Gamma)$-independence, giving the desired result. As the operation preserves sparsity counts, the above preserves isostaticity. □

When considering C_s-symmetric frameworks, we will use a special case of Lemma 4.1.6 which we record in the following remark.

Remark 4.1.7. Let (G, p) be a C_s-symmetric and independent (isostatic) framework with $w \in V$ fixed by σ and adjacent to $v_1, \dots, v_r$. Suppose that $p(w) - p(v_1)$, $p(w) - p(v_1')$, and $n(w)$ are linearly independent. Let G^+ be obtained by performing a symmetrised fixed-vertex-to-C_4 operation at w, so that v_1, v_1' are adjacent to both w and the new vertex u also fixed by σ in G^+. Let $p^+(v) = p(v)$ for all $v \in V$, and $p^+(u) = p(w)$. Then (G^+, p^+) is independent (isostatic).

For the case when the group is C_2, we will need one more operation. A *double 1-extension* on a $\mathbb{Z}_2$-symmetric graph G is the combination of two non-symmetric

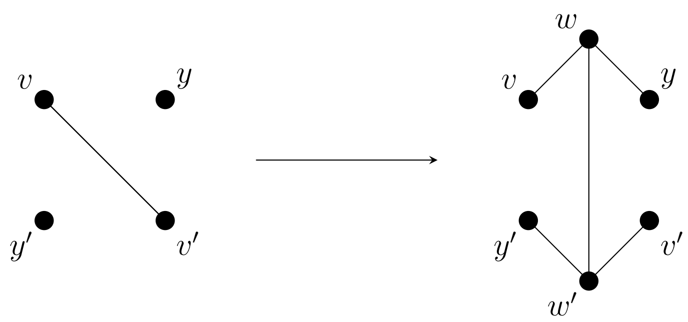

Figure 4.5: A double 1-extension which deletes a fixed edge, and adds a new fixed edge between two degree 3 vertices.

1-extensions: the first creates a new graph G^+ by removing a fixed edge $e = vv'$ of G, adding a new vertex, say w, of degree three adjacent to v, v' and some other vertex y; followed by another non-symmetric 1-extension on G^+, namely removing wv' and adding a new vertex w' with 3 incident edges chosen so that $v' = \varphi(v)$. See Figure 4.5.

Lemma 4.1.8. *Let (G, ϕ) be a Γ-symmetric graph (where $\Gamma = \mathbb{Z}_2$), with fixed edge vv'. Let (G^+, ϕ^+) be the graph with vertex set $V^+ = V + \{w, w'\}$, and edge set $E^+ = E - vv' + \{wv, wy, w'v', w'y', ww'\}$, $\phi^+(\gamma)|_V = \phi(\gamma)$ for all $\gamma \in \mathbb{Z}_2$. If G is C_2-rigid (isostatic) on the cylinder then G^+ is too.*

Proof. Let G^+ be obtained from a double 1-extension on G, that is by deleting the edge vv', and adding the vertices w, w' where w is a node adjacent to v, y, w' and w' is adjacent to v', y', w. Let $c = \tau(\gamma)$ be the half-turn in $\tau(\Gamma)$ (recall that previously c was called either c_2 or c_2' depending on the position of the rotational axis relative to the cylinder). Let p_0 and $c(p_0)$ be the positions of the vertex w and its symmetric copy. Let (G, p) be completely Γ-regular on $\mathcal{Y}$ and define $p^+ = (p_0, p_{-1}, p)$, so that (G^+, p^+) is completely Γ-regular. We let $p(v) = p_1$, $p(v') = p_2 = c(p_1)$, $p(y) = p_3$, and $p(y') = p_4 = c(p_3)$.

Suppose for a contradiction that (G^+, p^+) is not infinitesimally rigid on $\mathcal{Y}$. Then any $\tau(\Gamma)$-symmetric framework of G^+ on $\mathcal{Y}$ will be infinitesimally flexible. We will use a sequence of $\tau(\Gamma)$-symmetric frameworks, moving only the points $\{p_0, c(p_0)\}$. Let $\mathcal{T}$ denote the tangent plane to $\mathcal{Y}$ at p_1. Choose a and b to be orthogonal vectors in $\mathcal{T}$ such that b is orthogonal to $p_1 - p_2$ and a orthogonal to b. Let $((G^+, p^j))_{j=0}^{\infty}$ be a sequence of frameworks where $p^j = (p_0^j, c(p_0^j), p)$ is taken so that

$$\frac{c^i(p_1) - c^i(p_0^j)}{||c^i(p_1) - c^i(p_0^j)||} \to c^i a$$

as $j \to \infty$, for each $i \in 0, 1$. The frameworks (G^+, p^j) have a unit norm infinitesimal motion u^j which is orthogonal to the space of trivial motions of $\mathcal{Y}$. By the Bolzano-Weierstrass theorem there is a subsequence of (u^j) which converges to a vector, u^∞ say, also of unit norm. We can discard and relabel parts of the sequence to assume this holds for the original sequence. For convenience, in an infinitesimal motion u, we will denote the instantaneous velocity at $c(p_0)$ by u_{-1}. Looking at the limit (G^+, p^∞), write $u^\infty = (u_0^\infty, u_{-1}^\infty, u_1, u_2, \dots, u_n)$, $p^\infty = (p_0^\infty, c(p_0^\infty), p_1, p_2, \dots, p_n)$ with $p_0^\infty = p_1$ and $c(p_0^\infty) = p_2$.

We show that (u_1, u_2) is an infinitesimal motion of the bar joining $p(v)$ and $p(v')$. Since p_0 converges to p_1 in the a direction, the velocities u_1 and u_0^∞ have the same component in this direction, so $(u_1 - u_0^\infty) \cdot a = 0$. Then $u_1 - u_0^\infty$ is tangential to $\mathcal{Y}$ at p_1, and orthogonal to a, it must be orthogonal to $p_1 - p_2$. Similarly, $c(p_0)$ converges to $c(p_1) = p_2$ in the $c(a)$ direction, the velocities u_2 and u_{-1}^∞ have the same component in this direction, so $(u_2 - u_{-1}^\infty) \cdot c(a) = 0$. Then $u_2 - u_{-1}^\infty$ is tangential to $\mathcal{Y}$ at p_2, and orthogonal to $c(a)$. Hence, $u_2 - u_{-1}^\infty$ must be equal to $\pm c(b)$, and orthogonal to $p_2 - p_1$. As there is a bar joining $p(w) = p_0^\infty$ and $p(w') = c(p_0^\infty)$, $u_0^\infty - u_{-1}^\infty$ is orthogonal to $p_1 - p_2$. We may express this as

$$\langle u_1 - u_0^\infty, p_1 - p_2 \rangle = \langle u_0^\infty - u_{-1}^\infty, p_1 - p_2 \rangle = \langle u_{-1}^\infty - u_2, p_1 - p_2 \rangle = 0.$$

It follows from summation of the above, that $\langle u_1 - u_2, p_1 - p_2 \rangle = 0$, which is the required condition for an infinitesimal motion.

Once again looking at (G, p), we know the infinitesimal motion $u = (u_1, u_2, \dots, u_n)$ is a trivial motion. In order to preserve the distance $d(p_0^\infty, p_3)$, u_0^∞ takes one of two values, representing rotating or translating the bar between $p(w)$ and $p(y)$. Additionally, $(u_1 - u_0^\infty) \cdot a = 0$ determines u_0^∞. Similarly, u_{-1}^∞ is determined by $d(c(p_0^\infty), p_4)$ and $(u_2 - u_{-1}^\infty) \cdot \gamma_2 a = 0$. Finally, since $\langle u_0^\infty - u_{-1}^\infty, p_0^\infty - c(p_0^\infty) \rangle = 0$, $\langle u_1 - u_2, p_1 - p_2 \rangle = 0$, and $p_0^\infty = p_1, c(p_0^\infty) = p_2$, we have that u_0^∞ agrees with u_1 and u_{-1}^∞ agrees with u_2, so u^∞ is a trivial motion for (G^+, p^∞). This gives a contradiction since u^∞ is a unit norm infinitesimal motion orthogonal to the space of trivial motions of $\mathcal{Y}$. $\square$

4.2 Symmetric isostatic graphs

In the next four sections we prove our main results on the cylinder. These are combinatorial characterisations of when a symmetric graph is isostatic on $\mathcal{Y}$ for the symmetry groups $C_i = \{\text{id}, \varphi\}$, $C_2 = \{\text{id}, c_2'\}$ and $C_s = \{\text{id}, \sigma\}$. These results give a precise converse to the necessary conditions developed in Section 3.3 and utilise the geometric operations of the previous section. In order to prove the results we need to develop some combinatorics. In this section we work as generally as possible among the three groups. Then the three subsequent sections specialise one by one to the specific symmetry groups.

4.2.1 Base graphs

Consider the inversion symmetry group C_i. It follows from Theorem 3.3.4 that the graphs we need to understand are C_i-symmetric graphs which are $(2,2)$-tight and have no edges or vertices fixed by the inversion φ. Henceforth we shall refer to such graphs as $(2,2)$-C_i*-tight* graphs. Similarly, graphs which are $(2,2)$-sparse and C_i-symmetric shall be referred to as $(2,2)$-C_i*-sparse*. Figure 4.6 shows the two base graphs for the class of $(2,2)$-C_i-tight graphs; we will call the graph on six vertices (F_1, ϕ_1), and the graph on eight vertices (F_2, ϕ_0), where for $\gamma \in \mathbb{Z}_2 \setminus \{\text{id}\}$, $\phi_1(\gamma)$ and $\phi_0(\gamma)$ do not fix any vertices or edges of F_1 and F_2 respectively.

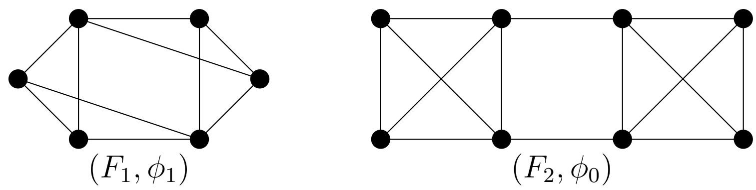

Figure 4.6: The C_i-symmetric base graphs, with inversion through the centre of each graph.

Instead consider the half-turn symmetry group C_2. By Theorem 3.3.4, a C_2-isostatic graph is $(2,2)$-tight and has two fixed edges and no fixed vertex, or no fixed edge and one fixed vertex. Hence a graph is called $(2,2)$-C_2-*tight* if it is $(2,2)$-tight, C_2-symmetric and contains either two fixed edges and no fixed vertex, or no fixed edge and one fixed vertex. Similarly, graphs which are $(2,2)$-sparse and C_2-symmetric shall be referred to as $(2,2)$-C_2-*sparse*. In Figure 4.7, we show four small C_2-symmetric graphs that are $(2,2)$-tight. These are, reading left to right, top to bottom: (K_4, ϕ_3) with two fixed edges and no fixed vertex, (W_5, ϕ_4) with one fixed vertex and no fixed edge, $(Wd(4,2), \phi_5)$ with one fixed vertex and no fixed edge, and (F_2, ϕ_2). These will turn out to be the base graphs of our recursive construction.

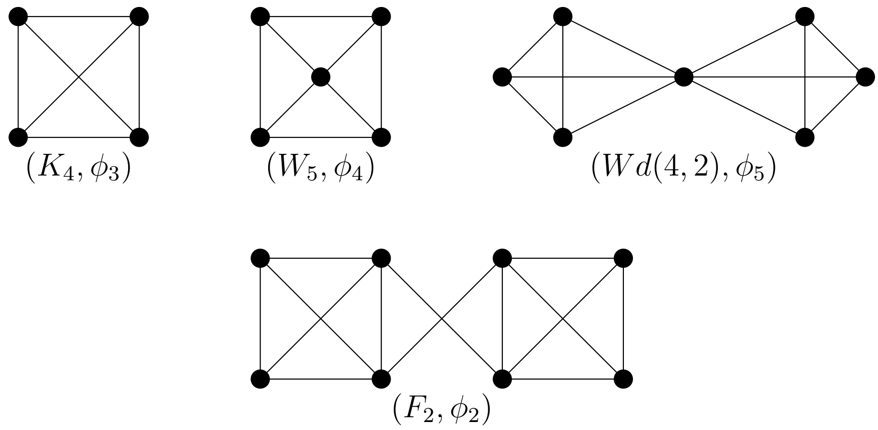

Figure 4.7: The C_2-symmetric base graphs, with rotation at the centre of each graph.

Finally consider the reflection symmetry group C_s. By Theorem 3.3.4, a C_s-isostatic graph is $(2,2)$-tight and has no fixed edge and any number of fixed vertices. Hence a graph is called $(2,2)$-C_s-*tight* if it is $(2,2)$-tight, C_s-symmetric and contains no fixed edge. Similarly, graphs which are $(2,2)$-sparse, C_s-symmetric and have no fixed edge shall be referred to as $(2,2)$-C_s-*sparse*. In Figure 4.8, we show six small C_s-symmetric graphs that are $(2,2)$-tight. These are, reading left to right, top to bottom: (F_2, ϕ_2), (W_5, ϕ_4), $(Wd(4,2), \phi_5)$, (F_1, ϕ_1), (F_1, ϕ_6) with two fixed vertices and no fixed edge, and $(K_{3,4}, \phi_7)$ with three fixed vertices and no fixed edge. These will be the base graphs of our recursive construction.

4.2.2 Reduction operations

We will consider *reduction operations*: these are the reverse of the extension operations described in Section 4.1. While the operations we require vary slightly for

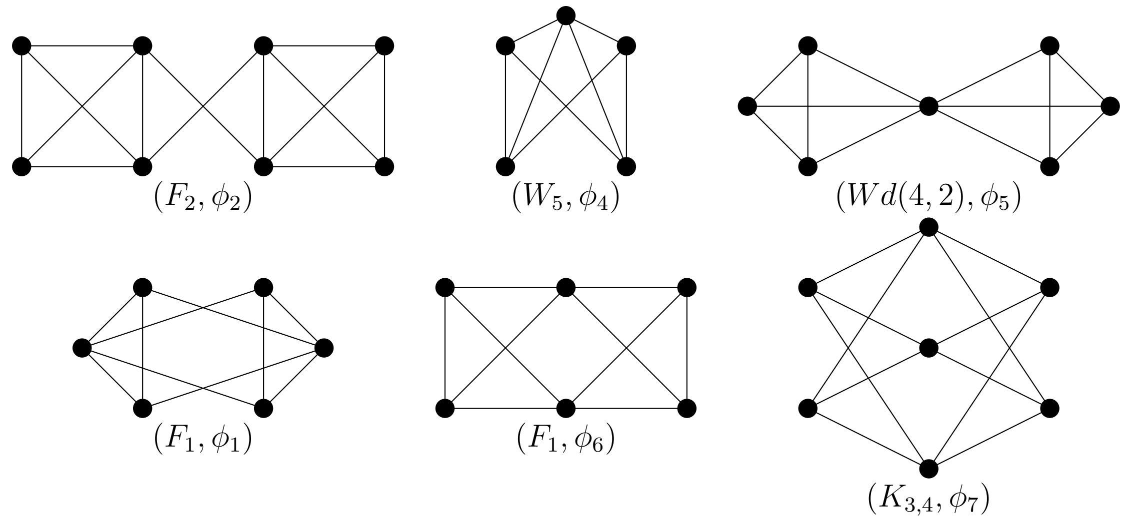

Figure 4.8: The C_s-symmetric base graphs, with the mirror vertically aligned on the page.

each symmetry group, the following are required across the three symmetries we will provide characterisations for: symmetrised 0-reduction, symmetrised 1-reduction, symmetrised C_4 contraction, symmetrised K_4 contraction.

Lemma 4.2.1. *Let (G, ϕ) be $(2,2)$-C-tight for $C \in \{C_i, C_2, C_s\}$ and suppose $v \in V$ is a vertex of degree 2. Then either $C = C_s$, $v = \sigma(v) = v'$ and $H = G - \{v\}$ is $(2,2)$-C-tight or $v \neq v'$ and $H = G - \{v, v'\}$ is $(2,2)$-C-tight.*

Proof. The case when $C = C_s$ and $v = v'$ is trivial. Moreover if $C = C_2$ then any degree two vertex v in a $(2,2)$-C-tight graph G satisfies $v' = c_2'(v) \neq v$, for otherwise the subgraph $G - v$ would be $(2,2)$-tight but have no fixed edges or vertices, contradicting the fact that G is $(2,2)$-C-tight. For any C, $vv' \notin E$ for otherwise $H = G - \{v, v'\}$ would have $|V(H)| = |V| - 2$ but $|E(H)| = |E| - 3$, violating the $(2,2)$-sparsity of G. Then, any subgraph of H is a subgraph of G, so as G is $(2,2)$-tight, H is. Also H will be C-symmetric, and we do not remove any fixed edges or vertices. □

Most of the technical work in the next four sections involves analysing when we can remove a vertex of degree 3. Hence, for brevity, we will say that a vertex of degree 3 is called a *node*.

Nodes in C_2-symmetric graphs will often require extra attention. A method we will repeatedly use and can be seen in the below lemma, involves finding a tight C_2-symmetric subgraph that does not contain fixed elements; such a subgraph contradicts Theorem 3.3.4.

Lemma 4.2.2. *Let (G, ϕ) be $(2,2)$-C-tight for $C \in \{C_i, C_2, C_s\}$ and suppose $v \in V$ is a node so that $x, y \in N(v)$ with $xy \notin E$ and $\{x, y\} \neq \{x', y'\}$. Then $G' = G - \{v, v'\} + \{xy, x'y'\}$ is not $(2,2)$-C-tight if and only if at least one of the following hold:*

1. *there exists a 2-critical set U with $x, y \in U$;*
2. *there exists a 3-critical set W with $x, y, x', y' \in W$;*
3. *$C = C_2$ and there exists a 4-critical set T with $x, y, x', y' \in T$ and $G[T]$ is C_2-symmetric with no fixed vertex or edges.*

Proof. Suppose that x, y (resp. x', y') are contained in a 2-critical set U, or x, y, x', y' are contained in a 3-critical set W. Then U and W would, with the new edges, create subgraphs $G'[U] = (U, E_1)$ and $G'[W] = (W, E_2)$ where $|E_1| = 2|U| - 1$ and $|E_2| = 2|W| - 1$ respectively. This proves the first two conditions imply G' is not $(2,2)$-C-tight. Additionally for $(2,2)$-C_2-tight graphs, all C_2-symmetric tight subgraphs must have the fixed vertex or edge constraint. Any reduction cannot create a tight subgraph which does not satisfy this fixed count. Therefore a 4-critical C_2-symmetric vertex set T where $G[T]$ does not contain fixed edges or vertices has $G'[T]$ a C_2-symmetric $(2,2)$-tight subgraph of G', which is not $(2,2)$-C_2-tight. Hence the third condition implies G' is not $(2,2)$-C-tight.

Conversely if conditions (1)-(3) hold then the facts that G is $(2,2)$-C-tight, G' is obtained from a subgraph of G by adding 2 distinct edges, and C_i and C_s do not have fixed vertex or edge constraints that need to be preserved in the reduction imply that G' is $(2,2)$-C-tight. □

Lemma 4.2.3. *Let (G, ϕ) be $(2,2)$-C-tight for $C \in \{C_i, C_2, C_s\}$ with no fixed edge and suppose $v \in V$ is a node with $N(v) = \{x, y, z\}$. If the pair x, y is not contained in any 2-critical subset of $V \setminus \{v, v'\}$, then there does not exist $W \subseteq V \setminus \{v, v'\}$ with $x, x', y, y' \in W$ and $i_G(W) = 2|W| - 3$.*

Proof. Suppose for a contradiction that there exists some $W \subseteq V \backslash \{v, v'\}$, $x, x', y, y' \in W$ with $i(W) = 2|W| - 3$. Observe $i(W') = i(W)$, $i(W \cup W') \leq 2|W \cup W'| - 3$ and $i(W \cap W') \leq 2|W \cap W'| - 3$ (since $x, x', y, y' \in W \cap W'$). Now we have

$$\begin{aligned} 2|W| - 3 + 2|W'| - 3 = i(W) + i(W') &= i(W \cup W') + i(W \cap W') - d(W, W') \\ &\leq 2|W \cup W'| - 3 + 2|W \cap W'| - 3 - d(W, W') \\ &= 2|W| + 2|W'| - 6 - d(W, W'). \end{aligned} \tag{4.2.1}$$

It follows that we have equality throughout and $d(W, W') = 0$. However $W \cup W'$ is C-symmetric with no fixed edges, so $i(W \cup W')$ is even, a contradiction. □

Remark 4.2.4. The following is a result analogous to [27, Lemma 2.2]. Similar counting arguments to Equation (4.2.1) can be used to give the following (and other similar observations) on the union and intersection of k-critical sets that we use repeatedly. Let (G, ϕ) be $(2, 2)$-tight. Take $X, Y \subseteq V$. If $X, Y \subseteq V$ are 2-critical and $X \cap Y \neq \emptyset$ then $X \cup Y$ and $X \cap Y$ are 2-critical and $d(X, Y) = 0$.

Further if X is 2-critical, Y is 3-critical and $X \cap Y \neq \emptyset$, then either:

- $d(X, Y) = 0$, $i(X \cap Y) = 2|X \cap Y| - 3$ and $i(X \cup Y) = 2|X \cup Y| - 2$; or
- $d(X, Y) = 0$, $i(X \cap Y) = 2|X \cap Y| - 2$ and $i(X \cup Y) = 2|X \cup Y| - 3$; or
- $d(X, Y) = 1$ and $X \cap Y$ and $X \cup Y$ are 2-critical.

Recall for a closed neighbourhood of a vertex v, we write $N[v] = N(v) \cup \{v\}$.

Lemma 4.2.5. *Let (G, ϕ) be $(2, 2)$-C-tight for $C \in \{C_i, C_s\}$ and suppose $v \in V$ is a node with $N(v) \cap N(v') = \emptyset$. Then either $G[N[v]] = K_4$, or there exists $x, y \in N(v)$ such that $xy \notin E$, and $G^- = G - \{v, v'\} + \{xy, x'y'\}$ is $(2, 2)$-C-tight.*

Proof. Assume that $G[N[v]] \neq K_4$. By Lemma 4.2.3, we only need to show that for one pair of non-adjacent vertices in $N(v)$, there is no 2-critical set containing them. We consider cases based on $i(N(v))$. Let $N(v) = \{x, y, z\}$. Firstly, where there are no edges on the neighbours of v, if all of the pairs $\{x, y\}, \{x, z\}, \{y, z\}$ are contained in 2-critical sets $U_1, U_2, U_3 \subseteq V - \{v, v'\}$ say, then by Remark 4.2.4,

$U_1 \cup U_2$ is 2-critical and so $U_1 \cup U_2 \cup \{v\}$ breaks $(2,2)$-sparsity of G. Similarly when $i(N(v)) = 1$. Now suppose $i(N(v)) = 2$, and say $xy \notin E$. If there existed a 2-critical $U \subseteq V - \{v, v'\}$ with $x, y \in U$, then $i_G(U \cup \{v, z\}) = 2|U \cup \{v, z\}| - 1$ which contradicts $(2,2)$-sparsity of G. Hence $G^- = G - \{v, v'\} + \{xy, x'y'\}$ is $(2,2)$-C-tight. □

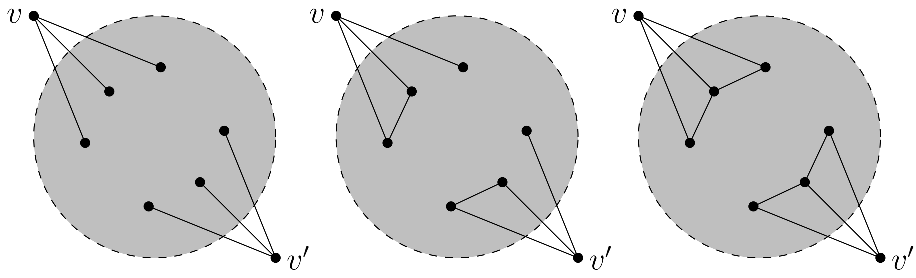

Figure 4.9: The local structure of the cases in Lemma 4.2.5.

Lemma 4.2.6. *Let (G, ϕ) be $(2,2)$-C-tight for $C \in \{C_i, C_s\}$ and suppose $v \in V$ is a node such that $N(v) = \{x, y, z\}$ and $N(v) \cap N(v') = \{x, y\}$, with $x' = y$ or $C = C_s$ and x and y are fixed vertices. Then one of the following hold:*

1. *$G[\{v, v', x, y, z, z'\}] \cong (F_1, \phi_1)$;*
2. *$C = C_s$ and $G[\{v, v', x, y, z, z'\}] \cong (F_1, \phi_6)$;*
3. *there exists some $v_1 \in \{x, y\}$ such that $G^- = G - \{v, v'\} + \{v_1 z, v_1' z'\}$ is $(2,2)$-C-tight.*

Proof. Suppose $\{xz, yz, xz', yz'\} \subset E$. If $x' = y$, then $G[\{v, v', x, y, z, z'\}] \cong (F_1, \phi_1)$ as in (1), otherwise x and y are fixed and $G[\{v, v', x, y, z, z'\}] \cong (F_1, \phi_6)$ as in (2).

When one of the edge pairs $\{xz, yz'\}, \{xz', yz\}$ is present, without loss of generality say $\{xz, yz'\} \in E$. Suppose there exists a $U \subseteq V - v$, with $y, z \in U$ which is 2-critical. If $U \cap U' \neq \emptyset$, then $U \cup U'$ is 2-critical by Remark 4.2.4, and $U \cup U' \cup \{v\}$ violates $(2,2)$-sparsity of G. If $U \cap U' = \emptyset$, then $xz', yz' \in d(U, U')$ so $U \cup U'$ is 2-critical and $U \cup U' \cup \{v\}$ again breaks $(2,2)$-sparsity. By Lemma 4.2.3, since there is no 2-critical set on x, y, z, z', we have that $i_G(W) \leq 2|W| - 4$ for all $W \subseteq V \setminus \{v, v'\}$ such that $x, y, z, z' \in W$, so $G^- = G - \{v, v'\} + \{xz', yz\}$ is $(2,2)$-C-tight.

Now assume we have no edges on $N(v)$. We want to show that we can add either xz, yz' or yz, xz' to $G - \{v, v'\}$. Suppose we can add neither xz or $x'z$, that is, there are 2-critical sets $U_1, U_2 \subseteq V - v$ with $x, z \in U_1$ and $x', z \in U_2$. Then $U_1 \cap U_2 \neq \emptyset$, so $U_1 \cup U_2$ is 2-critical by Remark 4.2.4. Thus the subgraph induced by $U_1 \cup U_2 \cup \{v\}$ contradicts G being $(2, 2)$-tight. We recall Lemma 4.2.3 gives for any W containing $\{x, y, z, z'\}$, that $i_G(W) \leq 2|W| - 4$, giving us the required result. □

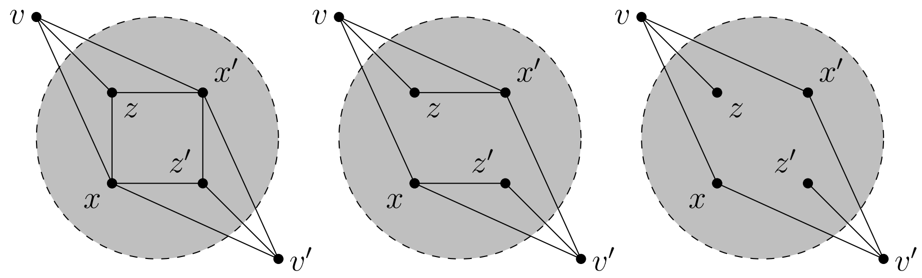

Figure 4.10: The local structure of the cases in Lemma 4.2.6.

Lemma 4.2.7. *Let (G, ϕ) be $(2, 2)$-C-tight for $C \in \{C_2, C_s\}$ and suppose $v \in V$ is a node so that $N[v] \cap N[v'] = \{t\}$, where t is a fixed vertex in G. Let $N(v) = \{x, y, t\}$. Then either $G[N[v] \cup N[v']] = (Wd(4, 2), \phi_5)$ or one of $G_1 = G - \{v, v'\} + \{xt, x't\}$, $G_2 = G - \{v, v'\} + \{yt, y't\}$, or $G_3 = G - \{v, v'\} + \{xy, x'y'\}$ is $(2, 2)$-C-tight.*

Proof. Since G has no fixed edges, Lemma 4.2.3 implies that if x, y, x', y' are in a 3-critical set then they are in a 2-critical set too. Hence, for the remainder of the proof, we only consider 2-critical or 4-critical sets in the case when $C = C_2$.

We break up the proof into cases by considering the number of edges induced by the neighbours of v. Firstly, when all 3 edges xy, xt, yt are present in the graph, we have a copy of $Wd(4, 2)$. Now, when two edges are present, without loss of generality, we may assume either $xy \notin E$ or $yt \notin E$. If $xy \notin E$ (resp. $yt \notin E$), suppose there exists a 2-critical $U \subset V$ with $x, y \in U$ (resp. $t, y \in U$). Then the subgraph induced by $U \cup \{v, t\}$ (resp. $U \cup \{v, x\}$) violates the $(2, 2)$-sparsity of G. There is no 4-critical C_2-symmetric set T containing x, y and not v, t since the subgraph induced by $T \cup \{v, v', t\}$ violates $(2, 2)$-sparsity.

Consider now the case where one or zero edges are induced by $\{x, y, t\}$. No two of the pairs $\{x, y\}, \{x, t\}, \{y, t\}$ can each be contained in a 2-critical set, as if any two were contained in 2-critical sets U_1, U_2, then by Remark 4.2.4, $U_1 \cup U_2$ is 2-critical but the subgraph induced by $U_1 \cup U_2 + v$ violates the $(2,2)$-sparsity of G. For $C = C_2$, to complete the proof we need to confirm that one of these pairs and its symmetric copy is not in a 4-critical set which contains no fixed vertex. However, for any two sets from $\{x, y, x', y'\}, \{x, x', t\}, \{y, y', t\}$, at least one contains the fixed vertex of G. Hence we may reduce symmetrically unless $G[N[v] \cup N[v']] \cong (Wd(4,2), \phi_5)$. □

Lemma 4.2.8. *Let (G, ϕ) be $(2,2)$-C-tight for $C \in \{C_2, C_s\}$ and suppose $v \in V$ is a node chosen so that $N[v] \cap N[v'] = \{t, x, x'\}$, where t is fixed. Then either $G[N[v] \cup N[v']] = (W_5, \phi_4)$, or $G' = G - \{v, v'\} + \{xt, x't\}$ is $(2,2)$-C-tight.*

Proof. Since t is a fixed vertex, the edge xx' does not exist. We therefore only have to consider whether xt and $x't$ are edges of G. If $xt, x't \in E$, then $G[N[v] \cup N[v']] = (W_5, \phi_4)$. So suppose $xt, x't \notin E$. Suppose there exist sets $W_1, W_2 \subset V$ that are both 2-critical, with $x, t \in W_1$, $x', t \in W_2$. Then $W_1 \cup W_2$ is 2-critical and the subgraph induced by $W_1 \cup W_2 \cup \{v, v'\}$ contradicts the $(2,2)$-sparsity of G. Similarly, any 3-critical blocking set U containing x, x', t would induce a subgraph that breaks $(2,2)$-sparsity after adding v, v' and their incident edges. Finally, for $C = C_2$, xt cannot be blocked by a 4-critical set T, as they cannot contain fixed vertices and t itself is fixed. □

4.2.3 Contraction operations

Lemma 4.2.9. *Let (G, ϕ) be $(2,2)$-C-tight for $C \in \{C_i, C_2, C_s\}$. Suppose G contains a copy of K_4 with vertices $\{x_1, x_2, x_3, x_4\} = X$, and put $\{x_1', x_2', x_3', x_4'\} = X'$ where $X \neq X'$. Let G^- denote the graph obtained from G by contracting X to w and X' to w' so that, for any $v \in V \setminus (X \cup X')$ with $vx_i \in E$ (resp. $vx_i' \in E$), we have $vw \in E(G^-)$ (resp. $vw' \in E(G^-)$). Then either*

1. *G^- is $(2,2)$-C-tight,*
2. *there exists $y \in V \setminus X$ such that $yx_i, yx_j \in E$ for some $1 \leq i < j \leq 4$,*
3. *$C = C_i, C_2, C_s$ and $G[X, X'] \cong (F_2, \phi_0)$ or $G[X, X'] \cong (F_2, \phi_2)$, or*

4. *$C = C_2, C_s$ and $G[X, X'] \cong (Wd(4,2), \phi_5)$.*

Proof. First note that for any C, $|X \cap X'| \leq 1$ since G is $(2,2)$-tight. If $|X \cap X'| = 1$, this vertex must be fixed by any of the symmetries, so $C = C_2$ or C_s, and $G[X, X'] \cong (Wd(4,2), \phi_5)$, which is condition (4). We may therefore suppose $X \cap X' = \emptyset$. Let G^- be as above. Observe that C-symmetry is preserved in the reduction operation. We have $|V(G^-)| = |V| - 6$ and $|E(G^-)| = |E| - 12$. We first show that if G^- is simple, then it is $(2,2)$-tight. By construction,

$$|E(G^-)| = |E| - 12 = 2|V| - 2 - 12 = 2(|V| - 6) - 2 = 2|V(G^-)| - 2.$$

Now consider $F \leq G^-$. If $w, w' \notin V(F)$, then F is a subgraph of G. Since G is $(2,2)$-tight, $|E(F)| \leq 2|V(F)| - 2$. Any subgraph containing w or w' can be compared to a subgraph $F' \leq G$, by replacing w, w' with X, X' respectively, as well as making the appropriate edge set adjustment. From F' being a subgraph of G it easily follows that F is $(2,2)$-sparse, so G^- is $(2,2)$-C-tight.

We next consider when the operation could create multiple edges. Let t denote the number of neighbours in X of a vertex $v \in V \setminus X$. Note that $t \leq 2$ as $i_{G^-}(\{x_1, x_2, x_3, x_4, v\}) = 6 + t \leq 8$. If $t = 2$, we create an edge of multiplicity two between v and w. This gives condition (2). The other possibility is for a multiple edge between w and w'. This will happen when $d(X, X') \geq 2$. Since $i_{G^-}(\{x_1, x_2, x_3, x_4, x_1', x_2', x_3', x_4'\}) \leq 14$, there can be at most two such edges. When this is an equality, $G[X, X'] \cong (F_2, \phi_0)$ or $G[X, X'] \cong (F_2, \phi_2)$, depending on fixed edges, giving condition (3). In other cases we may perform the reduction operation and the resulting graph G^- is $(2,2)$-C-tight, which is condition (1) and completes the proof. □

Lemma 4.2.10. *Let (G, ϕ) be $(2,2)$-C-tight for $C \in \{C_i, C_2, C_s\}$ and let X be a copy of K_4 in G which contains a node v and $X \cap X' = \emptyset$. Suppose we cannot contract X since there exists $y \in V$ with two edges to distinct vertices, say a, b in X. Then there is a C-symmetric C_4 contraction that results in a $(2,2)$-C-tight graph.*

Proof. Label the final vertex of X as c. We write $H = G[\{a, b, c, v, y\}]$. Note that $vy \notin E$, and so $G[\{a, b, v, y\}] \cong K_4 - e$. Hence there is a potential C_4 contraction, with $v \to y$. We claim that this C_4 contraction results in a smaller $(2,2)$-tight graph and hence the C-symmetric C_4 contraction results in a $(2,2)$-C-tight graph. We begin by noting that there is no 2-critical set U containing v, y and at most one of a, b (otherwise adding the vertices of H not contained in U and their incident edges violate $(2,2)$-sparsity). Similarly there is no 3-critical set containing v, y but not a, b.

Since v is a node, and $a, b \in N(y)$, $c \notin N(y)$, the subgraphs of the contracted graph of small criticality we are interested in will contain one or both of the edges $cy, c'y'$. Suppose there exists a 2-critical set U with $\{c, y\} \in U$ and $a, b, v \notin U$. Then $U \cup v$ is 3-critical and hence does not exist as above. Similarly there is no 2-critical set containing $\{c', y'\}$. To complete the proof we check that there is no 3-critical set W containing c, y, c' and y'. Let $L = W + \{a, b, a', b'\}$. Since $ac, ay, bc, by, a'c', a'y', b'c', b'y' \in E$, we have $i_G(L) \geq 2|L| - 3$. However, we then see that

$$i_G(L + \{v, v'\}) \geq 2|L| - 3 + 6 = 2|L + \{v, v'\}| - 1,$$

contradicting G being $(2,2)$-tight. □

Lemma 4.2.11. *Let (G, ϕ) be $(2,2)$-C-tight for $C \in \{C_2, C_s\}$ with no fixed edges, $\delta(G) \geq 3$ and let $H \leq G$ be a proper subgraph. If H is $(2,2)$-C-tight then there exists a proper tight subgraph F of G, with $H \leq F$, such that G/F is $(2,2)$-C-tight.*

Proof. We begin by noting that unless there exists a $y_1 \in V \setminus V(H)$ that is adjacent to two vertices of H, we can contract H to a fixed vertex to create a simple graph G/H, and

$$|E(G/H)| = |E| - |E(H)| = 2|V| - 2 - 2|V(H)| + 2 = 2(|V(G/H)| - 1).$$

Any subgraph of G/H which breaks $(2,2)$-sparsity either does not contain the contracted vertex and hence trivially breaks the $(2,2)$-sparsity of G, or does and the obvious corresponding subgraph breaks the $(2,2)$-sparsity of G since K_1 and H are both $(2,2)$-tight. If G is C_2-symmetric, for H to be $(2,2)$-C_2-tight it must contain the fixed vertex of G, and since H contracts to a fixed vertex in G/H it would be the only such fixed vertex. This contraction preserves C-symmetry, so G/H would be $(2,2)$-C-tight. If such a y_1 exists, then let H_1 be the subgraph of G including H and y_1 (and y_1' if y_1 is not fixed). Note that H_1 is also $(2,2)$-C-tight. By the same reasoning as above, H_1 can be contracted to a fixed vertex unless there exists y_2 adjacent to two vertices of H_1. This sequence must end with a proper tight subgraph $F = H_k$ as $\delta(G) \geq 3$, completing the proof. □

4.3 C_i-symmetric isostatic graphs

We now focus exclusively on C_i symmetry and put together the combinatorial analysis to this point to prove a recursive construction. From this we then deduce our characterisation of completely C_i-regular isostatic frameworks. We need one final lemma first.

Lemma 4.3.1. *Let (G, ϕ) be a $(2,2)$-C_i-tight graph distinct from (F_1, ϕ_1) and (F_2, ϕ_2). If all nodes are in copies of (F_1, ϕ_1) or (F_2, ϕ_2), then G contains a 2-edge-separating set S. Further, let G_1, G_2 be the connected components of $G - S$. Then both G_1 and G_2 are $(2,2)$-C_i-tight with one of the G_i being isomorphic to (F_1, ϕ_1) or (F_2, ϕ_2).*

Proof. Let k be the number of $(2,2)$-C_i-tight subgraphs which are isomorphic to (F_1, ϕ_1) or (F_2, ϕ_0). We first show that these k $(2,2)$-C_i-tight subgraphs cannot have intersecting vertex sets. Two C_i-symmetric subgraphs cannot have an intersection of size 1, since the intersection is C_i-symmetric and there are no fixed vertices. Since G is $(2,2)$-sparse the intersection of any two F_i is 2-critical. Hence the intersection is of size at least four. Since each F_i is C_i-symmetric, their intersection must be, so $H = F_i \cap F_j$ is a proper $(2,2)$-C_i-tight subgraph with $4 \leq |V(H)| \leq 6$. Since F_1 is not a subgraph of F_2, and K_4 is not C_i-symmetric, this means that all of the F_i are pairwise vertex disjoint. Let v_0 be the number of vertices of G in these k $(2,2)$-C_i-tight subgraphs, $r = |V| - v_0$, $e_0 = 2v_0 - 2k$ be the number of edges of G in these k subgraphs, and $s = |E| - e_0$.

Since $|E| = 2|V| - 2$, we can now deduce, with substitutions from above, that

$s + e_0 = 2r + 2v_0 - 2$, and hence

$$s + 2v_0 - 2k = 2r + 2v_0 - 2.$$

This implies that $s = 2r + 2k - 2$. Let $H_1, H_2, \dots, H_k$ denote the k copies of F_1, F_2. For any $1 \leq j \leq k$, $G \setminus H_j$ is $(2,2)$-C_i-tight and $d(H_j, G \setminus H_j)$ is even, since no edges of G are fixed by the inversion. Each of the r vertices not in some H_j are of degree at least four. Counting incidences, we see $2s \geq 4r + \sum_{i=1}^{k} a_i$ where for each i, $a_i \in \{2, 4, \dots\}$ is counting the number of edges incident to each H_i. We can substitute s from the above to obtain

$$2(2r + 2k - 2) \geq 4r + \sum_{i=1}^{k} a_i,$$

and cancelling gives $4k - 4 \geq \sum_{i=1}^{k} a_i$. This means at least two of the a_i are equal to two, so at least two $(2,2)$-C_i-tight subgraphs can be separated from G with the removal of two edges, hence G contains a 2-edge-separating set S.

Let G_1, G_2 be the components of $G - S$. We know from the above that one component is isomorphic to (F_1, ϕ_1) or (F_2, ϕ_0), without loss of generality say G_2. Then G_1 is $(2,2)$-tight and contains a copy of F_1 or F_2 which is $(2,2)$-C_i-tight. This gives us that $\varphi(G_1) \cap G_1 \neq \emptyset$. Further, we note that G_1 inherits inversion symmetry from G and $S \cap \varphi(G_1) = \emptyset$. Since $\varphi(G_1)$ is connected, this implies $\varphi(G_1) = G_1$. Since φ fixes no vertices or edges of G, it will not fix any vertices or edges of G_i. Hence G_1 is $(2,2)$-C_i-tight. □

Theorem 4.3.2. *A graph (G, ϕ) is $(2,2)$-C_i-tight if and only if (G, ϕ) can be generated from (F_1, ϕ_1) or (F_2, ϕ_0) by symmetrised 0-extensions, 1-extensions, vertex-to-K_4 operations, vertex-to-C_4 operations, and joining such a graph to a copy of (F_1, ϕ_1) or (F_2, ϕ_0) by two new distinct edges that are images of each other under φ.*

Proof. We first show that if G can be generated from the stated operations, then it is $(2,2)$-C_i-tight. Note that (F_1, ϕ_1) and (F_2, ϕ_0) are independent and $(2,2)$-C_i-tight. In Section 4.1 we showed that the named operations preserve independence. It is clear these operations introduce two edges for each new vertex and do not introduce fixed edges. Thus, if we apply these operations to an independent and $(2,2)$-C_i-tight graph, the result will also be independent and have the correct edge counts described in Section 3.3. Thus the new graph must be $(2,2)$-C_i-tight from Theorem 3.3.4.

For the converse, we show by induction that any $(2,2)$-C_i-tight graph G can be generated from a copy of (F_1, ϕ_1) or (F_2, ϕ_0). Suppose the induction hypothesis holds for all graphs with $|V| < n$. Now let $|V| = n$ and suppose G is not isomorphic to either of the base graphs (F_1, ϕ_1) and (F_2, ϕ_0). We wish to show that there is an operation from our list taking G to a $(2,2)$-C_i-tight graph $G^- = (V^-, E^-)$ with $|V^-| < n$. Then we know that G^- can be generated from a copy of (F_1, ϕ_1) or (F_2, ϕ_0), and hence so can G. We first note that any $(2,2)$-C_i-tight graph G has $2 \leq \delta(G) \leq 3$. There is no $v \in V$ with $d(v) = 0, 1$, as then $G - v$ would break sparsity. By the handshaking lemma, if all vertices are at least degree 4, then $|E| \geq 2|V|$. If $\delta(G) = 2$, then we remove any degree 2 vertex and its symmetric copy. This yields a $(2,2)$-C_i-tight graph by Lemma 4.2.1, and this graph $G^- = (V^-, E^-)$ has $|V^-| = n - 2$ as required. Otherwise $\delta(G) = 3$.

If there exists a degree three vertex $v \in V$ with $N(v) \cap N(v') = \emptyset$, with $G[N[v]] \not\cong K_4$, then we perform a C_i-symmetric 1-reduction, which is possible by Lemma 4.2.5. If $N(v) \cap N(v') \neq \emptyset$ and $G[N[v] \cup N[v']] \not\cong F_1$, then we again perform a symmetrised 1-reduction which is possible by Lemma 4.2.6. In both cases, the new graph $G^- = (V^-, E^-)$ also has $|V^-| = n - 2$ as required. Otherwise, all nodes are in copies of K_4 or (F_1, ϕ_1).

Now suppose G contains a subgraph isomorphic to K_4 and consider a contraction of this K_4. By Lemma 4.2.9, this K_4 can be reduced unless there is a vertex with two neighbours in the K_4, or the K_4 is part of a subgraph isomorphic to (F_2, ϕ_0). In the former case, we use Lemma 4.2.10, and $G^- = (V^-, E^-)$ is a $(2,2)$-C_i-tight graph with $|V^-| < n$. In the latter case, all nodes are in $(2,2)$-C_i-tight subgraphs isomorphic to (F_1, ϕ_1) or (F_2, ϕ_0) and we recall G is not isomorphic to (F_2, ϕ_0). Hence we may apply Lemma 4.3.1 to deduce that G contains a two edge seperating set S, so that $G - S$ has two connected components G_1, G_2, where without loss of generality G_1 is $(2,2)$-C_i-tight and G_2 is isomorphic to (F_1, ϕ_1) or (F_2, ϕ_0). Writing $G_1 = (V_1, E_1)$, we have $|V_1| < n$ so G_1 and by extension G can be generated from

(F_1, ϕ_1) or (F_2, ϕ_0). Finally, since G is not isomorphic to (F_1, ϕ_1) or (F_2, ϕ_0), we are finished. □

Theorem 4.3.3. *A graph (G, ϕ) is C_i-isostatic if and only if it is $(2, 2)$-C_i-tight.*

Proof. Necessity was proved in Theorem 3.3.4. It is easy to check using any computer algebra package that the base graphs (F_1, ϕ_1) and (F_2, ϕ_0) are C_i-isostatic. Sufficiency follows from Theorem 4.3.2 and the results of Section 4.1, namely Lemmas 4.1.1–4.1.6, by induction on $|V|$. □

4.4 C_2-symmetric isostatic graphs

In this section we turn our attention to C_2-symmetric graphs on the cylinder. In our recursive construction we will take care to maintain the number of fixed edges and vertices in each operation, and hence we will essentially view the case of two fixed edges and no fixed vertex as disjoint from the case of no fixed edge and one fixed vertex.

4.4.1 Reduction operations

In the C_i-symmetric case, when looking at 1-reductions, we considered the induced subgraphs on open neighbourhoods of the vertex we wished to remove. However, for C_2 symmetry, we must consider closed neighbourhoods, as we may have fixed edges. The options for the intersection of the closed neighbour sets of a node, say v, and its image v' are: empty intersection; one vertex in the intersection, where the vertex in the intersection will be fixed; two vertices in the intersection, where v and v' are both adjacent to a vertex and its image under the half-turn or where $vv' \in E$; three vertices in the intersection, with one vertex fixed and no fixed edges; four vertices in the intersection, and the vertices form either K_4 or $K_4 - e$ as an induced subgraph. Note that the two cases above with fixed vertices were shown to be reducible in Section 4.2.

We recall from Lemma 4.2.2 for C_i and C_s symmetry, that we had to consider 2- and 3-critical sets which prevent a symmetrised 1-reduction. These both need to be considered with C_2 symmetry, but the conditions that a $(2,2)$-C_2-tight graph has one fixed vertex and no fixed edges or no fixed vertex and two fixed edges means we must now also consider 4-critical sets which do not have any fixed edges or vertices. Performing a symmetrised 1-reduction which adds two edges to such a set would violate our conditions for $(2,2)$-C_2-tightness.

Lemma 4.4.1. *Let (G,ϕ) be $(2,2)$-C_2-tight and suppose $v \in V$ is a node with $N[v] \cap N[v']$ either empty or consisting of only one fixed vertex and suppose $i_G(N(v)) \leq 1$. If there is a 4-critical C_2-symmetric subset $T \subset V - \{v,v'\}$, with $G[T]$ containing no fixed edges or vertices, and containing two non-adjacent vertices of $N(v)$, then there exists a C_2-symmetric 1-reduction at v that results in a $(2,2)$-C_2-tight graph.*

Proof. Let $N(v) = \{x,y,z\}$ and let $T \subset V - \{v,v'\}$ be as in the proposition statement. Without loss of generality, we may suppose $xy, xz \notin E$ and $x,y \in T$. Note that $z \notin T$. We show that either $G' = G - \{v,v'\} + \{xz, x'z'\}$ is $(2,2)$-C_2-tight or $yz \notin E$ and $G' = G - \{v,v'\} + \{yz, y'z'\}$ is $(2,2)$-C_2-tight. We first prove that there cannot exist a 4-critical C_2-symmetric set T_1 such that $G[T_1]$ contains no fixed edges or vertices and $x,z \in T_1$. Suppose to the contrary, that T_1 exists. As $T_1 \cap T \neq \emptyset$, and both $T_1 \cap T$ and $T_1 \cup T$ are C_2-symmetric and the induced subgraphs do not contain fixed edges or vertices, we have $i(T_1 \cap T) \leq 2|T_1 \cap T| - 4$ and $i(T_1 \cup T) \leq 2|T_1 \cup T| - 4$. Then

$$\begin{aligned}2|T_1| - 4 + 2|T| - 4 = i(T_1) + i(T) &= i(T_1 \cup T) + i(T_1 \cap T) - d(T_1,T)\\ &\leq 2|T_1 \cup T| - 4 + 2|T_1 \cup T| - 4 = 2|T_1| + 2|T| - 8.\end{aligned}$$

Hence equality holds and $T_1 \cap T$ and $T_1 \cup T$ are 4-critical. This is a contradiction as $T_1 \cup T \cup \{v,v'\}$ would be 2-critical with no fixed edge and no fixed vertex induced by this set. Similarly if $yz \notin E$, then there does not exist a 4-critical C_2-symmetric set T_2 such that $G[T_2]$ contains no fixed edges or vertices and $y,z \in T_2$.

Assume now that there exist two 2-critical sets U_1 and U_2 containing $\{x,z\}$ and $\{x',z'\}$ respectively. We may assume $U_2 = U_1'$, for otherwise we could consider $U_3 = U_1 \cup U_2'$ and $U_3' = U_1' \cup U_2$. Let $U = U_1 \cup U_2$. Note that if $U_1 \cap U_2 = \emptyset$ then U is 4-critical. Otherwise, by Remark 4.2.4, U is 2-critical, and since $G[U]$ is C_2-

symmetric, it contains the fixed edges or vertex. It follows that $T \cup U = T \cup U_1 \cup U_2$ is 4-critical. Writing a for the criticality of U, we have

$$\begin{aligned} 2|T| - 4 + 2|U| - a = i(T) + i(U) &= i(T \cup U) + i(T \cap U) - d(T, U) \\ &\leq 2|T \cup U| - 4 + 2|T \cap U| - 2 = 2|T| + 2|U| - 6. \end{aligned}$$

If U is 4-critical this would imply that $G[T \cup U \cup \{v, v'\}]$ is $(2,2)$-tight and C_2-symmetric but does not contain fixed elements, which contradicts Theorem 3.3.4. So we may suppose U is 2-critical and $d(T, U) = 0$, implying $yz, y'z' \notin E$. Then there cannot exist a 2-critical set on $\{y, z\}$ or $\{y', z'\}$, as if say $y, z \in X$ was 2-critical, $G[U \cup X \cup \{v, v'\}]$ would not be sparse.

Finally, assume there exists a 3-critical set W containing $\{x, z, x', z'\}$, or when $yz \notin E$, $\{y, z, y', z'\}$. We can assume this set is C_2-symmetric by taking $W \cup W'$. Since the induced subgraph contains only one fixed edge, $i(W \cup T) \leq 2|W \cup T| - 3$, and $i(W \cap T) \leq 2|W \cap T| - 4$. By similar calculations as we did for 4 and 2-critical sets, we see that in the equations above equality holds throughout, and hence $T \cup W \cup \{v, v'\}$ breaks $(2,2)$-sparsity of G. Then by Lemma 4.2.2, either $G' = G - \{v, v'\} + \{xz, x'z'\}$ or $yz \notin E$ and $G' = G - \{v, v'\} + \{yz, y'z'\}$ is $(2,2)$-C_2-tight as required. □

Lemma 4.4.2. *Let (G, ϕ) be $(2,2)$-C_2-tight and suppose $v \in V$ is a node with $N[v] \cap N[v'] = \emptyset$. Then either $G[N[v]] = K_4$, or there exists $x, y \in N(v)$ such that $xy \notin E$, and $G^- = G - \{v, v'\} + \{xy, x'y'\}$ is $(2,2)$-C_2-tight.*

Proof. We break up this proof into cases by looking at the number of edges amongst the neighbours of v. Label the neighbours of v by x, y, z. Firstly, when all 3 edges xy, xz, yz are present in the graph, we have a K_4. Next suppose two edges are present, say without loss of generality $xy \notin E$. Suppose there exists a 2-critical set $U \subset V - v$ with $x, y \in U$. Then the subgraph induced by $U \cup \{v, z\}$ violates the $(2,2)$-sparsity of G. To do the 1-reduction symmetrically, we must check that there is no $W \subset V - v$ with $x, y, x', y' \in W$ such that $|E(W)| = 2|V(W)| - 3$. This follows since the subgraph induced by $W \cup \{v, z, v', z'\}$ breaks $(2,2)$-sparsity. If there exists a 4-critical C_2-symmetric subset $T \subset V - \{v, v'\}$ containing x, y, x', y' then $T \cup \{v, z, v', z'\}$ is 2-critical and C_2-symmetric, so all fixed edges and/or vertices are contained in $G[T]$.

For the case with one or zero edges amongst x, y, z, we begin by noting that no two of the pairs $\{x, y\}, \{x, z\}, \{y, z\}$ can each be contained in a 2-critical set, as if any two were contained in 2-critical sets U_1, U_2, then, by Remark 4.2.4, $U_1 \cup U_2$ is 2-critical and $U_1 \cup U_2 + v$ violates the $(2, 2)$-sparsity of G. If there exists a 4-critical C_2-symmetric subset $T \subset V - \{v, v'\}$ containing v_1, v_2, v_1', v_2' then $T \cup \{v, v_3, v', v_3'\}$ is 2-critical and C_2-symmetric. Hence by Lemma 4.4.1, if $\{v_1, v_2\}$ was the only pair not in a 2-critical set we can perform a C_2-symmetric 1-reduction at v in this case.

Finally we must consider when there do not exist 2-critical sets containing $\{v_1, v_2\}$ and $\{v_2, v_3\}$ respectively with $\{v_1, v_2, v_3\} = \{x, y, z\}$. (Whether there is a 2-critical set containing v_1, v_3 is not important for the argument that follows.) Assume for a contradiction that $W_1, W_2 \subset V - v$ are 3-critical with $\{v_1, v_2, v_1', v_2'\} \in W_1$, $\{v_2, v_3, v_2', v_3'\} \in W_2$. By counting similar to Remark 4.2.4, the union and intersection of two 3-critical sets are either both 3-critical or one is 2-critical and the other is 4-critical. Since $W_1 \cap W_1'$ and $W_1 \cup W_1'$ contain $\{v_1, v_2\}$ neither are 2-critical (similarly $W_2 \cap W_2'$ and $W_2 \cup W_2'$ are not 2-critical since they both contain $\{v_2, v_3\}$). Hence both $W_1 \cup W_1'$ and $W_2 \cup W_2'$ are 3-critical and C_2-symmetric, so the subgraphs induced by these sets must each contain exactly 1 fixed edge. If they do not contain the same fixed edge, $(W_1 \cup W_1') \cap (W_2 \cup W_2')$ is C_2-symmetric and contains no fixed edges or vertices, so must be 4-critical, which would imply $(W_1 \cup W_1') \cup (W_2 \cup W_2')$ is 2-critical but then $\{v_1, v_2\}$ is contained in a 2-critical set. If the induced subgraphs do contain the same fixed edge, both $(W_1 \cup W_1') \cap (W_2 \cup W_2')$ and $(W_1 \cup W_1') \cup (W_2 \cup W_2')$ would be 3-critical, but then the subgraph induced by $(W_1 \cup W_1') \cup (W_2 \cup W_2') + \{v, v'\}$ would violate $(2, 2)$-sparsity. Hence one of the pairs v_i, v_j is not contained in a 2-critical or a 3-critical subset of $V - v$. It remains to deal with the case when this pair v_i, v_j is blocked by a 4-critical subset $T \subset V - \{v, v'\}$. It follows from Lemma 4.4.1 that we can reduce v symmetrically and the proof is complete. □

Lemma 4.4.3. *Let (G, ϕ) be $(2, 2)$-C_2-tight and suppose $v \in V$ is a node with $N(v) = \{x, y, v'\}$ and $N[v] \cap N[v'] = \{v, v'\}$.*

1. *Suppose $xx', yy' \notin E$. Then either $G_1' = G - \{v, v'\} + \{xx'\}$ or $G_2' = G -$*

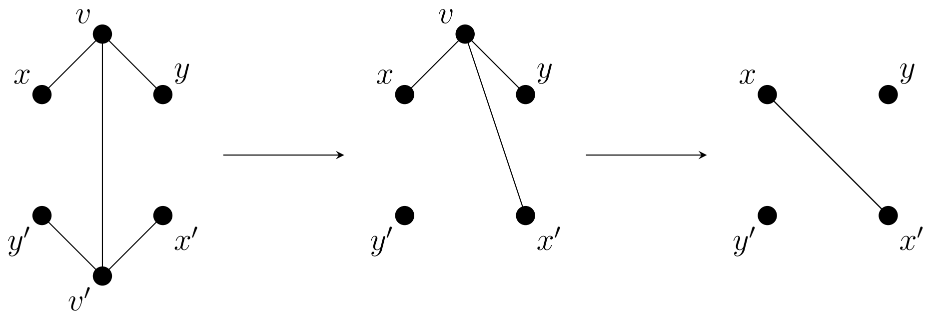

Figure 4.11: Reduction schematic when the degree three vertex is adjacent to its symmetric image.

$\{v, v'\} + \{yy'\}$ *is* $(2, 2)$-C_2-*tight.*

2. *Suppose* $xx' \in E$ *or* $yy' \in E$. *Then there is another node in* G *and it is not of this type.*

The following proof has two cases, firstly assuming the edges xx' and yy' are not present among the neighbours of v and v', and secondly assuming one is. (Note that it is not possible for both to be since $vv' \in E$ would give three fixed edges.)

Proof. For (1), we may perform a non-symmetric 1-reduction at v' as it cannot happen that $\{v, x'\}$ and $\{v, y'\}$ can be in 2-critical blocking sets, or else the union of these sets, say W, is 2-critical and $W + v'$ breaks sparsity. To perform a second non-symmetric 1-reduction at v, we see that neither $\{x, x'\}$ or $\{y, y'\}$ can be contained in a 2-critical set. If there were such a set, without loss of generality call it U and let it contain x, x', then $U \cup U'$ is 2-critical ($x, x' \in U \cap U'$), C_2-symmetric, but cannot contain both of the fixed edges of G, which is a contradiction.

For (2), assume without loss of generality that $xx' \in E$. Since G is $(2, 2)$-tight and $\delta(G) = 3$ there are at least four nodes in G. If v, v', x, x' are the only nodes, then $G - \{v, v', x, x'\}$ is $(2, 2)$-C_2-tight. We now simply note that this arrangement can only appear once in each graph, since it has both of the fixed edges. □

In the above proof, for the second 1-reduction we are still considering G, rather than $G - v + xx'$. We can do this since the blocking set in the reduced graph does not use v, therefore it does not include the edge vv' so the blocking set without v' would still be 2-critical.

Lemma 4.4.4. *Let (G, ϕ) be $(2,2)$-C_2-tight, suppose $v \in V$ is a node such that $N[v] \cap N[v'] = \{x, x'\}$ and let the other neighbour of v be z. Then $G_1' = G - \{v, v'\} + \{xz, x'z'\}$ or $G_2' = G - \{v, v'\} + \{x'z, xz'\}$ is $(2,2)$-C_2-tight.*

Proof. We prove this by case analysis, counting if the edges xz, yz, and xy are present. Firstly, $xz, x'z$ and xx' cannot all be present, as the subgraph induced by $N[v] \cup N[v']$ would break $(2,2)$-sparsity. Further we do not have $xz, x'z \in E$ and $xx' \notin E$, as $N[v] \cup N[v']$ is 2-critical, C_2-symmetric and the induced subgraph does not contain the correct fixed elements. Our first case where a 1-reduction is possible is when one edge of $xz, x'z$ is present with xx', say $x'z, xx' \in E$, $xz \notin E$. If there exists a 2-critical set U containing x, z, not containing v, then the subgraph induced by $U \cup \{v, x'\}$ contradicts the $(2,2)$-sparsity of G. If there exists a 3-critical set W, with $x, x', z, z' \in U$, $v, v' \notin W$, then the subgraph induced by $W \cup \{v, v'\}$ also breaks the $(2,2)$-sparsity of G. For any 4-critical T containing x, z, x', z', $G[T]$ contains a fixed edge, namely xx'. By counting, if $G[T]$ contains one fixed edge and T is 4-critical, it must contain both fixed edges, therefore there is no 4-critical blocking set for the 1-reduction at v and v'.

Consider the case when one of the edges $xz, x'z$ is present, say $x'z \in E$, $xz, xx' \notin E$. There does not exist 2-critical U_1, U_2 with $x, z \in U_1$, $x, x' \in U_2$, as this contradicts Remark 4.2.4 as $d(U_1, U_2) \neq 0$. We therefore know that one of $G_1 = G - v + xx'$ or $G_2 = G - v + xz$ is $(2,2)$-tight, although not C_2-symmetric. We want to show that it is always the case that we can perform a (non-symmetric) 1-reduction at v by adding the edge xz. Suppose we add xx'. Consider 1-reductions at $v' \in G_1$. If there exists a 2-critical set U containing $\{x', z'\}$, then the subgraph induced by $W \cup \{x', z'\}$ contradicts the $(2,2)$-sparsity of G_1. Hence we can perform a 1-reduction at v' in G_1 adding the edge $x'z'$. Since we could perform this 1-reduction in G_1, we know a 2-critical set U^* in G preventing a 1-reduction adding the edge $x'z'$ must contain v. However, then the subgraph H induced by $U^* \cup \{x, v'\}$ contradicts the $(2,2)$-sparsity of G, as H contains the edges $xv, xz', v'x', v'z', v'x$. Hence, we may perform a 1-reduction at v' in G by adding the edge $x'z'$.

Now when $xz, x'z \notin E$, if both $\{x, z\}$ and $\{x', z\}$ are in 2-critical sets U_1 and U_2 respectively, $U_1 \cup U_2$ is 2 critical so $U_1 \cup U_2 \cup \{v\}$ contradicts $(2,2)$-sparsity of G.

There is no 3-critical set W or 4-critical set T containing x, z, x', z'. Observe that such a set $W + \{v, v'\}$ would break sparsity, and a set $T + \{v, v'\}$ would be 2-critical, C_2-symmetric, but $G[T \cup \{v, v'\}]$ contains no fixed vertex or edge. □

Lemma 4.4.5. *Let (G, ϕ) be $(2,2)$-C_2-tight and suppose $v \in V$ is a node such that $N[v] \cap N[v'] = \{v, v', x, x'\}$ and $xx' \notin E$. Then $G' = G - \{v, v'\} + \{xx'\}$ is $(2,2)$-C_2-tight.*

Proof. G' is not $(2,2)$-C_2-tight if and only if there exists a 2-critical set X in $G - \{v, v'\}$ containing x and x'. However vv' is not in $G[X]$ so such a set X cannot exist and the lemma follows. □

4.4.2 Combinatorial characterisation

We can now put together the combinatorial results of this section to prove the following recursive construction and then apply this result alongside the results of Section 4.1 to deduce our characterisation of C_2-isostatic graphs.

Theorem 4.4.6. *A graph (G, ϕ) is $(2,2)$-C_2-tight if and only if (G, ϕ) can be generated from $(K_4, \phi_3), (W_5, \phi_4), (Wd(4,2), \phi_5), (F_2, \phi_2)$ (these graphs were depicted in Figure 4.7) by symmetrised 0-extensions, 1-extensions, vertex-to-K_4 operations and vertex-to-C_4 operations.*

Proof. Each of the base graphs are independent and tight and Section 4.1 showed the symmetrised 0-extension, 1-extension, double 1-extension, vertex-to-K_4, vertex-to-C_4 and vertex-to-$(2,2)$-C_2-tight operations preserve independence. It is easy to see the operations also preserve $(2,2)$-tightness and the number of fixed elements. We can therefore apply Theorem 3.3.4 to the extended graph.

Conversely, we show by induction that any $(2,2)$-C_2-tight graph G can be generated from our base graphs. Suppose the induction hypothesis holds for all graphs with $|V| < n$. Now let $|V| = n$ and suppose G is not isomorphic to one of the base graphs in Figure 4.7. Obviously any $(2,2)$-C_2-tight graph contains a vertex of degree 2 or 3. The former case is dealt with by Lemma 4.2.1. Hence suppose $\delta(G) = 3$ and v is a vertex of minimum degree. There are 6 cases depending on the closed neighbourhood of v, namely with labelling from this section,

$N[v] \cap N[v'] \in \{\emptyset, \{t\}, \{v, v'\}, \{x, x'\}, \{t, x, x'\}, \{v, v', x, x'\}\}$. By Lemmas 4.2.7, 4.2.8, and 4.4.2–4.4.5 we see that the only blocks to reducing any given node are K_4 (either C_2-symmetric or non-symmetric) and the base graphs $(Wd(4, 2), \phi_5)$ and (W_5, ϕ_4). (Note that if the option in Lemma 4.4.3(2) occurs then we may reduce the other node unless it is contained in a non-symmetric K_4.)

Suppose one of the base graphs in Figure 4.7 is a subgraph of G, denoted by H. If $H \cong (K_4, \phi_3)$ or (F_2, ϕ_2), H contains all the fixed edges of G and there can be no other base graph copy. Otherwise, $H \cong (W_5, \phi_4)$ or $(Wd(4, 2), \phi_5)$ and if another copy of either (W_5, ϕ_4) or $(Wd(4, 2), \phi_5)$ exist, call it H_1, then note that $H_1 \cap H$ is precisely the fixed vertex. Then H_1 is a proper $(2, 2)$-C_2-tight subgraph of G and we apply Lemma 4.2.11. We may now suppose that H is the only subgraph of G which is a copy of a base graph depicted in Figure 4.7.

We will show there is a node in G not contained in H. Note that H has at least four degree three vertices. Observing that $d(V(H), V(G \setminus H)) \geq 2$, the sum of the degrees in H increases by at least two in G compared to H, meaning there must be two nodes in $G \setminus H$. Hence we may assume all vertices of degree three are in a unique $(2, 2)$-C_2-tight base graph or a K_4 copy which is not C_2-symmetric. We may now, in all cases, suppose that G has a degree 3 vertex that is contained in a K_4. We now apply Lemma 4.2.9 and 4.2.10 to complete the proof. □

Theorem 4.4.7. *A graph (G, ϕ) is C_2-isostatic if and only if it is $(2, 2)$-C_2-tight.*

Proof. Since C_2-isostatic graphs are $(2, 2)$-tight, necessity follows from Theorem 3.3.4. It is easy to check using any computer algebra package that the base graphs depicted in Figure 4.7 are C_2-isostatic. Hence the sufficiency follows from Theorem 4.4.6 and Lemmas 4.1.1, 4.1.3, 4.1.4, 4.1.6, 4.1.8 by induction on $|V|$. □

4.5 C_s-symmetric isostatic graphs

We turn our attention to C_s-symmetric graphs on the cylinder. Here C_s is generated by a single reflection σ which could contain the cylinder axis or be perpendicular to it.

4.5.1 Reduction operations

For a $(2,2)$-C_s-tight graph, there are 6 possible cases for the structure of $N(v) \cap N(v')$, namely $N(v) \cap N(v') \in \{\emptyset, \{t\}, \{x, x'\}, \{t_1, t_2\}, \{x, x', t\}, \{t_1, t_2, t_3\}\}$, where vertices fixed by the non-trivial element are denoted t, and those not fixed x. In Section 4.2, Lemmas 4.2.5 -4.2.8 dealt with the first five of these cases. These lemmas showed the reduction is possible, or the node is contained in a $(2,2)$-C_s-tight subgraph of G. This leaves only the toughest case when all three neighbours of a node lie on the mirror.

Hence, for the remainder of this section we assume that all nodes have all neighbours on the mirror. The following lemmas require some new notation for describing our graphs. We will consider a vertex partition $V = V^r \cup V^b \cup V^g$ into red, blue and green vertices. The partition is chosen so that a vertex which is fixed by the mirror symmetry is red, any vertex which is adjacent to a red vertex is blue, and the remaining vertices are green. This also gives us a notion of edge colouring. We colour an edge red-blue if its endpoints are one red and one blue, blue-blue if its endpoints are blue, blue-green if its endpoints are one blue and one green, and green-green if its endpoints are green. Note that red-red edges are not possible in a $(2,2)$-C_s-tight graph, and red-green edges are not possible by the choice of the partition. We can therefore write $E = E^{rb} \cup E^{bb} \cup E^{bg} \cup E^{gg}$.

It will also be useful to consider the subgraphs of G which consist of red-blue and blue-blue edges. We will call these *red-blue connected components* or *rb-components* for shorthand. We label the rb-components of a graph $A_1, \dots, A_k$, so the component $A_i = (V_i, E_i)$ is k_i-critical, has red vertex set V_i^r and blue vertex set V_i^b, and has red-blue edges E_i^{rb} and blue-blue edges E_i^{bb} as in G. A natural extension of this is to say that the subset of the edges E^{bg} that are incident to a vertex in A_i form a new set denoted E_i^{bg}. Lastly, let $S \subset V^b$ be the nodes with all three neighbours on the mirror. Let $s = |S|$ and $s_i = |S \cap V_i|$. We illustrate these definitions in Figure 4.12.

Lemma 4.5.1. *Let (G, ϕ) be $(2,2)$-C_s-tight with $\delta(G) \geq 3$. Then $G[V^r \cup V^b]$ is $(2,2)$-C_s-tight if and only if $V^g = \emptyset$. Moreover, if $V^g \neq \emptyset$ then there exists an*

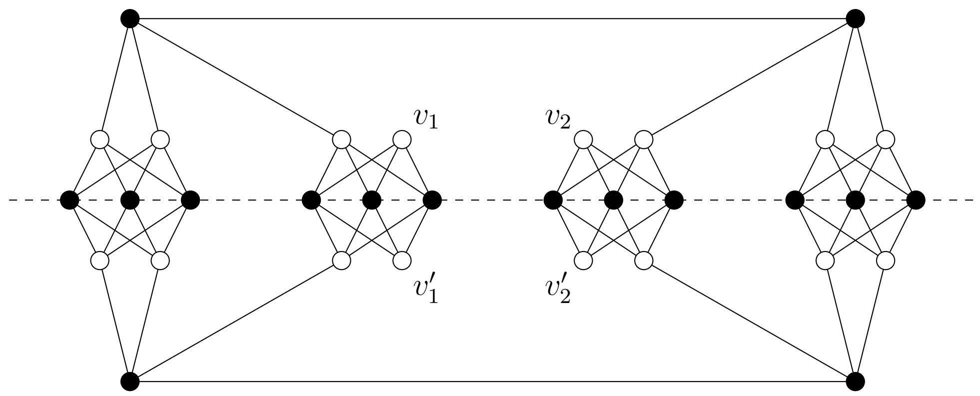

Figure 4.12: A $(2,2)$-C_s-tight graph G. The red vertices lie on the mirror line, the blue vertices are depicted as unfilled circles and the green vertices are the filled vertices not on the mirror. Each copy of $K_{3,4}$ in G is a rb-component and $S = \{v_1, v_1', v_2, v_2'\}$.

$i \in \{1, \dots, k\}$ *such that* $|E_i^{bg}| \leq 2k_i - 2$ *and* $s_i \geq 2k_i - |E_i^{bg}|$.

Proof. If $V^g = \emptyset$ then $G[V^r \cup V^b] = G$ and hence it is $(2,2)$-C_s-tight. Conversely, we begin by noting that

$$|E^{rb}| + |E^{bb}| + |E^{bg}| + |E^{gg}| = 2|V^r| + 2|V^b| + 2|V^g| - 2. \tag{4.5.1}$$

Then, for each $i \in \{1, \dots, k\}$, $|E_i^{rb}| + |E_i^{bb}| = 2|V_i^r| + 2|V_i^b| - k_i$. Summing gives

$$|E^{rb}| + |E^{bb}| = 2|V^r| + 2|V^b| - \sum_{i=1}^{k} k_i \tag{4.5.2}$$

and then, by subtracting (4.5.2) from (4.5.1), we obtain $|E^{bg}| + |E^{gg}| = 2|V^g| - 2 + \sum_{i=1}^{k} k_i$. Counting vertex degrees gives $|E^{bg}| + 2|E^{gg}| = \sum_{v \in V^g} d_G(v) \geq 4|V^g|$. Therefore, $2|E^{bg}| + 2|E^{gg}| = 4|V^g| - 4 + 2\sum_{i=1}^{k} k_i \leq |E^{bg}| + 2|E^{gg}| - 4 + 2\sum_{i=1}^{k} k_i$. Rearranging and simplifying gives

$$|E^{bg}| \leq 2\sum_{i=1}^{k} k_i - 4, \tag{4.5.3}$$

which, for $k = 1$ and $k_1 = 2$, completes the first statement of the proof.

If $|E_i^{bg}| \geq 2k_i$ for all i, we would contradict Equation (4.5.3). Again counting vertex degrees,

$$|E_i^{rb}| = \sum_{v \in V_i^r} d_G(v) \geq 4|V_i^r| \tag{4.5.4}$$

and since the vertices of S_i are nodes,

$$|E_i^{rb}| + 2|E_i^{bb}| + |E_i^{bg}| = \sum_{v \in V_i^b} d_G(v) \geq 4|V_i^b| - s_i. \tag{4.5.5}$$

Adding Equations (4.5.4) and (4.5.5), we see that $2|E_i^{rb}| + 2|E_i^{bb}| + |E_i^{bg}| \geq 4|V_i^r| + 4|V_i^b| - s_i$. Now recalling Equation (4.5.2) (restricted to A_i), we obtain

$$4|V_i^r| + 4|V_i^b| - s_i - |E_i^{bg}| \leq 2|E_i^{rb}| + 2|E_i^{bb}| \leq 4|V_i^r| + 4|V_i^b| - 2k_i,$$

which completes the proof. □

Lemma 4.5.2. *Let (G, ϕ) be a $(2,2)$-C_s-tight graph, distinct from $K_{3,4}$, with $\delta(G) = 3$. Suppose that the neighbour set of every node consists only of fixed vertices and that no proper subgraph H of G is $(2,2)$-C_s-tight. Then there exists a C_4-contraction (which contracts two fixed vertices) that results in a $(2,2)$-C_s-tight graph.*

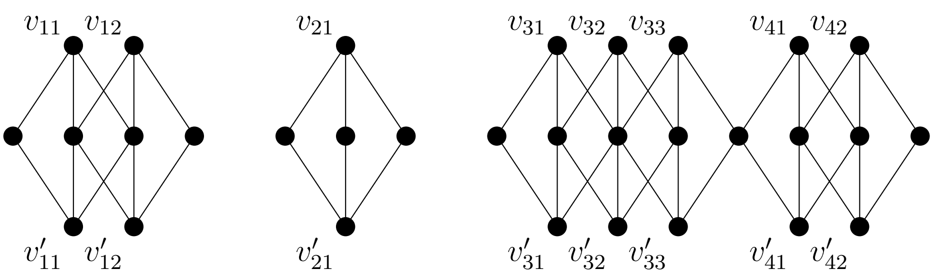

Figure 4.13: The depicted graph H is a subgraph of some $(2,2)$-C_s-tight graph G. All nodes in G have all their neighbours on the mirror and H is induced by S and the neighbours of vertices in S. We have labeled so that $v_{ji}, v'_{ji} \in S_j$. Note that v_{31} and v'_{33} are in the same set of the partition since there is a 4-cycle containing v_{31} and v_{32} and another containing v_{32} and v'_{33}, but v_{33} and v_{41} are in different sets since no two common neighbours exist.

Proof. If $V^g \neq \emptyset$, then by Lemma 4.5.1 there exists a rb-component A_i with $|E_i^{bg}| \leq 2k_i - 2$ and $s_i \geq 2k_i - |E_i^{bg}|$. Suppose $S \cap V_i = \{u_1, u_2, \ldots, u_r\}$. We define S_1 recursively. Let u_1 be in S_1. For any $u_q \in S \cap V_i$, $u_q \in S_1$ if there exists $t_1, t_2 \in V^r$

and $u_p \in S_1$ so that $u_q t_1 u_p t_2$ is a 4-cycle in G. If $S_1 \neq S \cap V_i$, take $u_k \in (S \cap V_i) \setminus S_1$ and put it in S_2, then define S_2 analogously. In this manner we obtain the partition $S \cap V_i = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_l$. (See Figure 4.13 for an illustration.) Since there is no $K_{3,4}$ subgraph of G we may assume, for a contradiction, that all red pairs of neighbours of nodes are contained in at least two 4-cycles. Take a pair of red vertices that are in two 4-cycles. The 4 neighbouring vertices have at least 3 neighbours on the mirror. As there is no $K_{3,4}$ subgraph, there must be at least two new red vertices adjacent to the 4 blue vertices mentioned. This creates new 4-cycles which involve the original two red vertices. Hence the degree of any red vertex adjacent to a node is at least six. It is possible that a vertex of S_j shares exactly one neighbour with a vertex of S_k for $j \neq k$. Let there be p such vertices. All such vertices have degree at least 12.

For each S_j there are $\frac{1}{2}|S_j| + 2$ red vertices of degree at least six (this double counts the p vertices of minimum degree 12), and at least $|S_j| + 4$ blue vertices of degree at least 4. Let r and b be the number of red and blue vertices respectively of V_i not already counted. Then, $|V_i| = \frac{1}{2}s_i + 2l - p + r + s_i + 4l + b + s_i$, the first four summands representing red vertices, the next three blue vertices, and the last summand the nodes (which are also blue). Once again we turn to counting degrees. We have

$$2|E_i| + |E_i^{bg}| \geq 6(\tfrac{1}{2}s_i + 2l - 2p) + 12p + 4r + 4(s_i + 4l) + 4b + 3s_i = 10s_i + 28l + 4r + 4b. \tag{4.5.6}$$

Also, since each A_i has $|E_i^{bg}| \leq 2k_i - 2$ and $|V_i| = \frac{5}{2}s_i + 6l + r + b - p$, we have

$$2|E_i| + |E_i^{bg}| \leq 4|V_i| - 2k_i + 2k_i - 2 = 10s_i + 24l + 4r + 4b - 4p - 2. \tag{4.5.7}$$

This implies that $4l + 4p \leq -2$, contradicting our assumption that all pairs of neighbours of a node are in two C_4. If $V^g = \emptyset$, the proof is unchanged except that $|E^{bg}| = 0$ and $|E| = 2|V| - 2$, so Equation (4.5.7) is

$$2|E| \leq 4|V| - 4 = 10s + 24l + 4r + 4b - 4p - 4$$

and so $4l + 4p \leq -4$ instead.

Finally, we need to check that this C_4-contraction preserves sparsity. Label the vertices v, v', t, t_1 where t, t_1 are fixed and are contracted and labelled t in the new graph, and let the final neighbour of v be t_2. Indeed, if a subgraph H of the reduced graph breaks sparsity, then $H = (V_t, E_t)$ has $|E_t| \geq 2|V_t| - 1$. If H is C_s-symmetric this must be $|E_t| \geq 2|V_t|$, and if H is not symmetric, H has at least one fixed vertex (namely t), so $V_t \cap \sigma V_t \neq \emptyset$, and by similar counting arguments to Remark 4.2.4, one of $H \cap H'$ and $H \cup H'$ has $|\tilde{E}| = 2|\tilde{V}|$. We may therefore assume H is C_s-symmetric. Noting that $E_t \subset E$, we draw the following conclusions.

If $v, v' \in V_t$ then $i(V_t + \{t_1\}) = |E_t| + 2$ so $V_t + \{t_1\}$ breaks sparsity. Else we have $v, v' \notin V_t$ and $i(V_t + \{t_1, v, v'\}) = |E_t| + 4$ if $t_2 \notin V_t$ and $i(V_t + \{t_1, v, v'\}) = |E_t| + 6$ if $t_2 \in V_t$. Therefore $V_t + \{t_1, v, v'\}$ breaks sparsity unless $t_2 \notin V_t$ and $|E_t| = 2|V_t|$. However, in this final case $G[V_t + \{t_1, v, v'\}]$ is a $(2,2)$-C_s-tight proper subgraph of G contradicting the conditions of the lemma. □

4.5.2 Combinatorial characterisation

We can now put together the combinatorial results of this section to prove the following recursive construction and then apply this result alongside the results of Section 4.1 to deduce our characterisation of C_2-isostatic graphs.

Theorem 4.5.3. *A graph (G, ϕ) is $(2,2)$-C_s-tight if and only if (G, ϕ) can be generated from the graphs $(F_2, \phi_2), (W_5, \phi_4), (Wd(4,2), \phi_5), (F_1, \phi_1), (F_1, \phi_6), (K_{3,4}, \phi_7)$ (these graphs were depicted in Figure 4.8) by fixed-vertex 0-extensions, fixed-vertex-to-C_4 and symmetrised 0-extensions, 1-extensions, vertex-to-K_4, vertex-to-C_4, and vertex-to-$(2,2)$-C_s-tight operations.*

Proof. Each of the base graphs are independent and tight and it can be seen that the fixed-vertex 0-extension, fixed-vertex-to-C_4 and symmetrised 0-extension, 1-extension, vertex-to-K_4, vertex-to-C_4 and vertex-to-$(2,2)$-C_s-tight operations preserve independence, tightness and do not introduce fixed edges. By Theorem 3.3.4, any graph after such operations is $(2,2)$-C_s-tight.

Conversely, we show by induction that any $(2,2)$-C_s-tight graph G can be generated from our base graphs. Suppose the induction hypothesis holds for all graphs with $|V| < n$. Now let $|V| = n$ and suppose G is not isomorphic to one of the base graphs in Figure 4.8. Obviously any $(2,2)$-C_s-tight graph contains a vertex of degree 2 or 3. The former case is dealt with by Lemma 4.2.1. We can also apply Lemma 4.2.11 to assume there are no $(2,2)$-C_s-tight proper subgraphs of G. Hence suppose $\delta(G) = 3$ and v is a vertex of minimum degree. There are 6 cases depending on the closed neighbourhood of v, namely with labelling from this section, $N(v) \cap N(v') \in \{\emptyset, \{t\}, \{x, x'\}, \{t_1, t_2\}, \{t, x, x'\}, \{t_1, t_2, t_3\}\}$. By Lemmas 4.2.5 ($\emptyset$), 4.2.7 ($\{t\}$), 4.2.6 ($\{x, x'\}$ and $\{t_1, t_2\}$), 4.2.8 ($\{t, x, x'\}$) we see that the only remaining blocks to reducing any given node is K_4 or all three neighbours being fixed vertices. If G has a degree 3 vertex that is contained in a K_4 then by Lemmas 4.2.9 and 4.2.10 we may assume that the K_4 and its symmetric copy intersect non-trivially. Since there are no $(2,2)$-C_s-tight proper subgraphs of G this gives a contradiction. Finally we may suppose that all nodes have all neighbours on the mirror, and by Lemma 4.5.2 there exists a C_4 contraction, completing the proof. □

Theorem 4.5.4. *A graph (G, ϕ) is C_s-isostatic if and only if it is $(2,2)$-C_s-tight.*

Proof. Since C_s-isostatic graphs are $(2,2)$-tight, necessity follows from Theorem 3.3.4. It is easy to check using any computer algebra package that the base graphs depicted in Figure 4.8 are C_s-isostatic. Hence the sufficiency follows from Theorem 4.5.3, Lemmas 4.1.1, 4.1.3, 4.1.4, 4.1.5, 4.1.6, and Remarks 4.1.2, 4.1.7 by induction on $|V|$. □

4.6 Additional Results

An immediate consequence of Theorems 4.3.3, 4.4.7, and 4.5.4 is that there are efficient, deterministic algorithms for determining whether a given graph is C_i-, C_2-, or C_s-isostatic since the $(2,2)$-sparsity counts can be checked using the standard pebble game algorithm [16, 24] and the additional symmetry conditions for the number of fixed vertices and edges can be checked in constant time, from the group action ϕ.

4.6.1 Tree decomposition

It is classical [26] that every $(2,2)$-tight graph can be decomposed into the edge-disjoint union of two spanning trees, and such packing or decomposition results are often of interest in combinatorial optimisation [12]. We derive symmetric decomposition results for C_2, C_i and C_s in the following corollaries.

Corollary 4.6.1. *A graph (G, ϕ) is C_2-isostatic if and only if it is the edge-disjoint union of two $\mathbb{Z}_2$-symmetric spanning trees $(T_1, \phi), (T_2, \phi)$.*

Proof. To show sufficiency, note that $(T_1, \phi), (T_2, \phi)$ can be labelled so that if u is the symmetric copy of v in T_1, then they are symmetric copies in T_2. By parity, each tree will either have one fixed vertex, which will be the same vertex in G, or one fixed edge. Since the spanning trees are edge-disjoint, G will either have one fixed vertex and no fixed edge, or no fixed vertex and two fixed edges. Further, it is known that the edge-disjoint union of two spanning trees is $(2,2)$-tight. The fact that G is $(2,2)$-C_2-tight now follows from the C_2-symmetry of the two spanning trees.

We prove the necessity of the symmetric decomposition by applying Theorem 4.4.6. It will be convenient to think of the edges of the two trees as being coloured red and blue respectively. We illustrate appropriate colourings of the base graphs in Figure 4.14. To check that the operations preserve the coloured trees, we describe the edge colourings for each operation.

Firstly, the symmetrised 0-extension has one edge coloured red and the other blue, with the symmetric edges coloured the same as their preimage. For a symmetrised 1-extension, in which say xy and $x'y'$ in G are deleted and the new vertices added in G^+ are v and v', then colour $vx, vy, v'x', v'y'$ in G^+ the colour of xy in G, and set the third edge incident to v (resp. v') as the other colour. A double 1-extension can be thought of in the same way; if vv' was deleted in G, the path containing v, w, w', v' will be coloured the same as ww'. In a symmetrised vertex-to-K_4 operation, the two new K_4 subgraphs should be coloured as in Figure 4.14 in such a way to preserve the symmetry. A vertex-to-$(2,2)$-C_2-tight subgraph operation replaces a fixed vertex with a $(2,2)$-C_2-tight subgraph. As seen in Lemma

4.2.11, the new subgraph can be constructed from either (W_5, ϕ_4) or $(Wd(4,2)\phi_5)$ with a series of symmetrised 0-extensions. We therefore colour the subgraph starting with the (W_5, ϕ_4) or $(Wd(4,2), \phi_5)$ copy, and colour the edges of the 0-extensions as previously described.

Finally, we note that we do not perform fixed-vertex-to-C_4 operations when considering $(2,2)$-C_2-tight graphs. A symmetrised vertex-to-C_4 operation can have two possibilities. The path of length 2 on v_1, w, v_2 (with w to be split into w and u in the operation, $N_G(w) = v_1, v_2, \ldots, v_r$ and v_1, v_2 becoming adjacent to both) can be coloured with both edges the same colour, or each edge different. In both cases, colour the edges of $\hat{G} = G^+ \setminus \{wv_1, wv_2, uv_1, uv_2, w'v_1', w'v_2', u'v_1', u'v_2'\}$ as in G. Now suppose first that wv_1 is red and wv_2 is blue in G. Then in G^+, we colour wv_1, uv_1 red and wv_2, uv_2 blue, and μv_i the same colour as wv_i for all $\mu \in \{w, u\}$ and $i \in \{3, \ldots, r\}$ (colouring the edges in the orbit analogously).

Hence we may suppose both wv_1 and wv_2 are coloured red in G. We claim that for any arrangement of the edges from $v_3, \ldots, v_r$ to either w or u in G^+, there is a colouring in G^+ of wv_1, wv_2, uv_1, uv_2 with three red edges and one blue edge that will result in G^+ being the edge-disjoint union of two C_2-symmetric spanning trees. Note that such a colouring gives $|V(G^+)| - 1$ blue and red edges. Necessarily, w and u are in different connected components of the $\hat{G}$ induced by the blue edges, say X_w and X_u respectively. The vertex v_1 will be in one of these components, without loss of generality say X_w. Colouring the edge uv_1 blue will connect these two components and hence give a blue spanning tree. Since wv_1 and wv_2 are coloured red in G it is easy to see that colouring the edges uv_2, wv_1, wv_2 red in G^+ will produce a red spanning tree. Applying this colouring symmetrically completes the proof. □

With a similar proof, we can establish the following result. We highlight the difference in the two corollaries, namely that in Corollary 4.6.1 the non-identity element of Γ fixes the colouring, whereas in Corollary 4.6.2 the non-identity element of Γ reverses the colouring. We illustrate appropriate colourings of the base graphs in Figures 4.15 and 4.16.

Corollary 4.6.2. *For $\tau(\Gamma) \in \{C_i, C_s\}$, a Γ-symmetric graph (G, ϕ) is $\tau(\Gamma)$-isostatic*

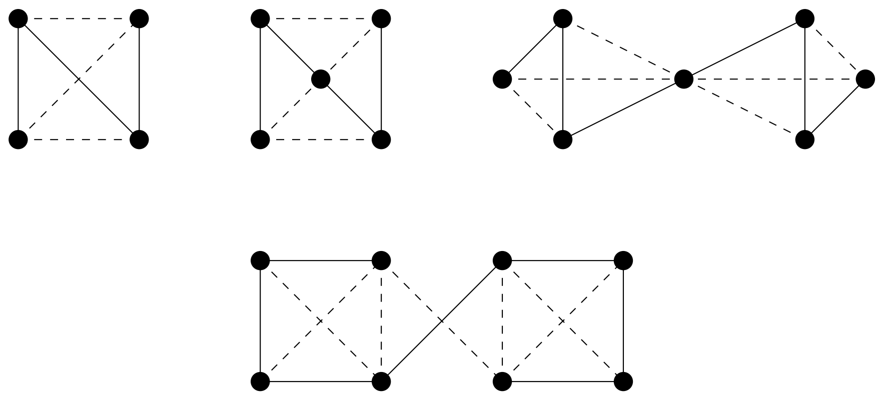

Figure 4.14: The C_2-symmetric base graphs decomposed into two C_2-symmetric edge disjoint trees, coloured red and blue (depicted with dashed and solid edges respectively).

if and only if it is the edge-disjoint union of two spanning trees T_1, T_2*, where* $\phi(\gamma)T_1 = T_2$ *for the non-trivial element* γ *of* Γ*.*

Proof. To show sufficiency, note that for the non-trivial element γ of Γ, $\phi(\gamma)$ does not fix any edges of (G, ϕ), since T_1, T_2 are edge-disjoint. It is possible for vertices to be fixed by $\phi(\gamma)$; in particular there may be no fixed vertices as is required for C_i-isostaticity. It is known that the edge-disjoint union of two spanning trees is $(2, 2)$-tight. Hence G is $(2, 2)$-$\tau(\Gamma)$-tight.

We prove the necessity of the symmetric decomposition by applying Theorems 4.3.2 and 4.5.3. As in Corollary 4.6.1, it will be convenient to think of the edges of the two trees as being coloured red and blue respectively. We illustrate appropriate colourings of the base graphs in Figures 4.15 and 4.16 for $\tau(\Gamma) = C_i$ and $\tau(\Gamma) = C_s$ respectively. To check that the operations preserve the coloured trees, we describe the edge colourings for each operation, beginning with the shared operations described in Section 4.2.

Firstly, the symmetrised 0-extension has one edge coloured red and the other blue, with the symmetric edges coloured the opposite to their preimage. For a symmetrised 1-extension, say xy (a red coloured edge) and $x'y'$ (a blue coloured edge) in G are deleted and the new vertices added in G^+ are v and v'. Let the third neighbour of v be z. Then colour $vx, vy, v'z'$ in G^+ the colour of xy in G (red), and $v'x', v'y', vz$ in G^+ the colour of $x'y'$ in G (blue). In a symmetrised vertex-to-K_4 operation, the two new K_4 subgraphs are coloured so that each vertex of the K_4 is incident to an edge from a red tree and an edge from a blue tree, and the symmetric copy so that $\phi(\gamma)$ exchanges red and blue. Such a colouring is the K_4 in Figure 4.14. The same logic applies for a vertex-to-$(2,2)$-C_s-tight subgraph operation.

Finally we consider vertex-to-C_4 operations. First we consider fixed-vertex-to-C_4 operations for $(2,2)$-C_s-tight graphs. The path of length 2, v_1, w, v_1', will be coloured one edge blue and the other red. The new vertex u must colour the edge to v_1 with a different colour to the edge to v_1'. A symmetrised vertex-to-C_4 operation can have two possibilities. The path of length 2 on v_1, w, v_2 (with w to be split into w and u in the operation, $N_G(w) = v_1, v_2, \dots, v_r$ and v_1, v_2 becoming adjacent to both) can be coloured with both edges the same colour, or each edge different. In both cases, colour the edges of $\hat{G} = G^+ \setminus \{wv_1, wv_2, uv_1, uv_2, w'v_1', w'v_2', u'v_1', u'v_2'\}$ as in G. Now suppose first that wv_1 is red and wv_2 is blue in G. Then in G^+, we colour wv_1, uv_1 red and wv_2, uv_2 blue, and colouring the edges in the orbit the alternate colour.

Hence we may suppose both wv_1 and wv_2 are coloured red in G. We claim that for any arrangement of the edges from $v_3, \dots, v_r$ to either w or u in G^+, there is a colouring in G^+ of wv_1, wv_2, uv_1, uv_2 with three red edges and one blue edge that will result in G^+ being the edge-disjoint union of two $\tau(\Gamma)$-symmetric spanning trees. Note that such a colouring gives $|V(G^+)| - 1$ blue and red edges. Necessarily, w and u are in different connected components of the $\hat{G}$ induced by the blue edges, say X_w and X_u respectively. The vertex v_1 will be in one of these components, without loss of generality say X_w. Colouring the edge uv_1 blue will connect these two components and hence give a blue spanning tree. Since wv_1 and wv_2 are coloured red in G it is easy to see that colouring the edges uv_2, wv_1, wv_2 red in G^+ will produce a red spanning tree. Applying this colouring with three blue edges and one red on $u'v_1', u'v_2', w'v_1', w'v_2'$ completes the proof. □

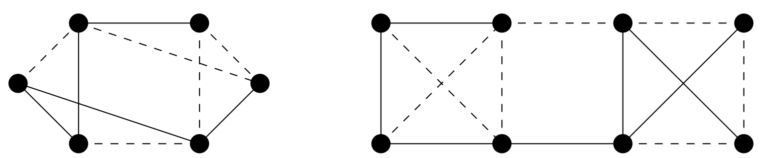

Figure 4.15: The C_i-symmetric base graphs decomposed into two edge disjoint spanning trees, coloured red and blue (depicted with dashed and solid edges respectively), which are images of one another under the map $\phi(\gamma)$.

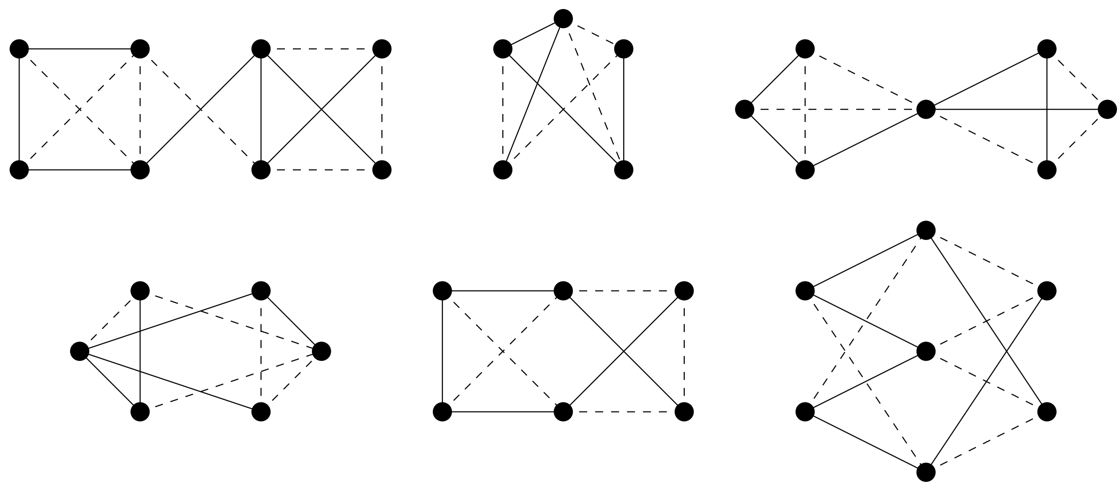

Figure 4.16: The C_s-symmetric base graphs decomposed into two edge disjoint spanning trees, coloured red and blue (depicted with dashed and solid edges respectively), which are images of one another under the map $\phi(\gamma)$.

The next obvious challenge would be to extend the characterisations in Theorems 4.3.3, 4.4.7, and 4.5.4 to deal with the remaining groups described in Theorem 3.3.4. While it is conceivable these groups could be handled by an elaboration of our techniques there will be many more cases and technical details to consider due to the multiple symmetry conditions. Moreover the corresponding problems in the Euclidean plane (see [38, 39]) remain open, providing a note of caution.

4.6.2 $\tau(\Gamma)$-symmetric infinitesimal rigidity

Analogous to the situation for frameworks in the Euclidean plane, an infinitesimally rigid C_2-symmetric framework on $\mathcal{Y}$ does not necessarily have a spanning isostatic subframework with the same symmetry. An example is depicted in Figure 4.17. Thus, for symmetric frameworks on $\mathcal{Y}$, infinitesimal rigidity can in general not be characterised in terms of symmetric isostatic subframeworks. To analyse symmetric frameworks for infinitesimal rigidity, rather than isostaticity, a different approach (similar to the one in [41], for example) may be needed. Surprisingly, it turns out that for C_i and C_s the situation is special and a simplified version of the approach in [41] may be applied in combination with Theorems 4.3.3 and 4.5.4 to deduce the following characterisation of symmetric infinitesimal rigidity. We present the reader a sketch of the proof, since a formal proof requires ideas and methods not visited in

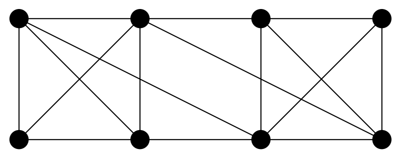

Figure 4.17: A C_2-rigid graph where no vertex or edge is fixed by the half-turn. There is no $(2,2)$-C_2-tight spanning subgraph.

the rest of the thesis. We first introduce terminology akin to that found in [41].

Let (G, ϕ) be a simple Γ-symmetric graph. For $v \in V$, let $\Gamma_v := \{\phi(\gamma)v : \gamma \in \Gamma\}$ be the *vertex orbit* of v. For $e \in E$, let $\Gamma_e := \{\phi(\gamma)e : \gamma \in \Gamma\}$ be the *edge orbit* of e. We write G/Γ for the *quotient graph* of G by Γ, which is a multigraph with vertex set V_0 the set of all vertex orbits of G under Γ and edge set E_0 the set of all edge orbits of G under Γ. That is $V_0 = \{\Gamma_v : v \in V\}$ and $E_0 = \{\Gamma_e : e \in E\}$. The quotient graph then has incidence relation $\Gamma_e = \Gamma_u\Gamma_v$ if some (and therefore every) edge in Γ_e is incident with a vertex in Γ_u and a vertex in Γ_v.

We will create a directed graph called the gain graph. For our purposes, a *directed graph* is a multigraph (with possible duplicate edges) with *edge orientation* (i.e. $uv \neq vu$). In G, fix a representative v^* for every vertex orbit Γ_v. Fix an orientation on the edges of G/Γ. Then for the directed edge $e = uv$, there is a unique $\gamma \in \Gamma$ so that $(u^*, \gamma v^*) \in E(G)$. This γ is the *gain* of e. Let $G_0 = G/\Gamma$ be a directed quotient graph. Then the pair (G_0, ψ) is a *gain graph* if $\psi : E_0 \to \Gamma$ maps each edge of G_0 to its gain. This definition assumes that there are no fixed vertices (each vertex has trivial stabiliser).

Before we are ready to give our theorem we require one more piece of terminology. Let $C \leq G_0$ be a cycle, with vertices $\{v_1, \ldots, v_n\}$ and directed edges $\{e_1, \ldots, e_n\}$. The *gain of a cycle* is $\sum_i \psi(e_i)^{\epsilon_i}$ where $\epsilon_i = 1$ if the cycle follows the directed edge and $\epsilon_i = -1$ otherwise. A subgraph $H \leq (G, \psi)$ is *balanced* if the gain of every cycle is the identity of Γ.

Theorem 4.6.3. *For $\tau(\Gamma) \in \{C_i, C_s\}$, a graph (G, ϕ) is $\tau(\Gamma)$-infinitesimally rigid if and only if (G, ϕ) has a spanning subgraph H that is $(2,2)$-$\tau(\Gamma)$-tight.*

Proof. If (G, ϕ) has a spanning subgraph H that is $(2,2)$-$\tau(\Gamma)$-tight, then (G, ϕ) is $\tau(\Gamma)$-infinitesimally rigid by Theorems 4.3.3 and 4.5.4. We first consider C_i symmetry. We want to show that if (G, p) is C_i-symmetric and infinitesimally rigid, then G contains a $(2,2)$-C_i-tight spanning subgraph.

Since (G, p) is C_i-symmetric, its rigidity matrix block-decomposes into two blocks R_0 and R_1, where R_0 corresponds to the trivial (fully-symmetric) representation ρ_0 and R_1 to the non-trivial (anti-symmetric) representation ρ_1. Each of these blocks has $|V_0|$ columns, where V_0 is the set of vertex orbits under C_i. No vertex can be fixed by inversion, since the framework lies on the cylinder.

Since (G, p) is infinitesimally rigid, it must be "fully-symmetric infinitesimally rigid" and "anti-symmetric infinitesimally rigid". That is to say, the quotient framework of G must have a spanning ρ_0-isostatic subframework and a spanning ρ_1-isostatic subframework. Here, ρ_i-isostatic means that there is no non-trivial infinitesimal motion or self-stress that is ρ_i-symmetric (i.e. it is in the $(P_V \otimes \tau)$-invariant subspace and P_E-invariant subspace corresponding to ρ_i, respectively).

Since the rotation about the cylinder axis is a ρ_0-symmetric trivial motion, the spanning ρ_0-symmetric isostatic subframework must satisfy: $|E_0| = 2|V_0| - 1$ (where E_0 and V_0 are the sets of edge and vertex orbits under C_i) and also $|E_0'| \leq 2|V_0'| - 1$ for all subgraphs and $|E_0'| \leq 2|V_0'| - 2$ for all balanced subgraphs (V_0', E_0'). Call such a group-labelled (multi-)graph $(2,2,1)$-tight.

Since the translation along the cylinder axis is a ρ_1-symmetric trivial motion, the spanning ρ_1-symmetric isostatic subframework must satisfiy the same $(2,2,1)$-tight count.

Now, notice that the lifting of a $(2,2,1)$-tight quotient gain graph with no loops is a $(2,2)$-tight C_i-symmetric graph which has no vertex or edge that is fixed by inversion. We still need to show that there is a $(2,2,1)$-tight spanning subgraph that has no loop edges (so that the lifting has no edge fixed by the inversion).

Note that a loop corresponds to a single edge in the lifted graph and that edge contributes a single row to R_0 and no row to R_1. So the $(2,2,1)$-tight gain graph that we showed above must exist for ρ_1-isostaticity cannot have a loop. So its lifting is $(2,2)$-C_i-tight, as desired.

For C_s the proof is similar, but slightly more subtle, as there can be vertices fixed by the reflection. This would require a modification to the definition of the gain graph when there are vertices with non-trivial stabilisers. An example of such a modification can be found in a recent paper by La Porta and Schuzle [22]. It's easy to see that each fixed vertex contributes one column to R_0 and one column to R_1, as it has a fully-symmetric degree of freedom (along the mirror plane) and an anti-symmetric degree of freedom (perpendicular to the mirror plane).

The trivial motion space still splits as above, so there is a 1-dimensional trivial motion space of symmetry ρ_0 and ρ_1.

So for both ρ_0- and ρ_1-isostaticity, we must have a spanning subgraph satisfying $|E_0| = 2|V_0| + |V_0^{\text{fixed}}| - 1$ and the corresponding subgraph counts. Call this $(2, 1, 2, 1)$-tight. As above for C_i, a loop edge is redundant in the $\rho - 1$-symmetric infinitesimal rigidity matroid, so the $(2, 1, 2, 1)$-tight subgraph guaranteed to exist for ρ_1-isostaticity cannot have a loop. Similarly, it cannot have an edge between vertices that are fixed by the reflection. So this $(2, 1, 2, 1)$-tight quotient graph lifts to a $(2, 2)$-tight graph with no fixed edges (but possibly fixed vertices), as desired. □

Chapter 5

Linearly Constrained Isostatic Frameworks

5.1 Rigidity preserving operations

Given a $\tau(\Gamma)$-symmetric isostatic linearly constrained framework in $\mathbb{R}^2$, we next introduce several construction operations and prove that their application results in larger $\tau(\Gamma)$-symmetric isostatic linearly constrained frameworks in $\mathbb{R}^2$. Moreover we use this to show that a certain infinite family of C_n-symmetric linearly constrained frameworks are isostatic. These construction operations are symmetry-adapted looped Henneberg-type graph operations. The operations are depicted in Figures 5.1 and 5.2 for specific symmetry groups.

We will work with an arbitrary finite group $\Gamma = \{\text{id} = \gamma_0, \gamma_1, \dots, \gamma_{t-1}\}$ and we will write $\gamma_k v$ instead of $\phi(\gamma_k)(v)$ and often $\gamma_k(x, y)$ or $(x^{(k)}, y^{(k)})$ for $\tau(\gamma_k)(p(v))$ where $p(v) = (x, y)$. For a group of order two, it will be common to write $v' = \gamma v$ for $\gamma \in \Gamma \setminus \{\text{id}\}$.

Let $G = (V, E, L)$ be a Γ-symmetric looped simple graph for a group Γ of order t. Then a *symmetrised 0-extension* creates a new Γ-symmetric looped simple graph $G^+ = (V^+, E^+, L^+)$ by adding the t vertices $\{v, \gamma v, \dots, \gamma_{t-1} v\}$ with either: v adjacent to v_i, v_j, and for each $k \in \{1, \dots, t-1\}$, $\gamma_k v$ adjacent to $\gamma_k v_i, \gamma_k v_j$; or v adjacent to v_i and incident to the loop (v, v), and for each $k \in \{1, \dots, t-1\}$, $\gamma_k v$ adjacent to $\gamma_k v_i$ and incident to $(\gamma_k v_i, \gamma_k v_i)$.

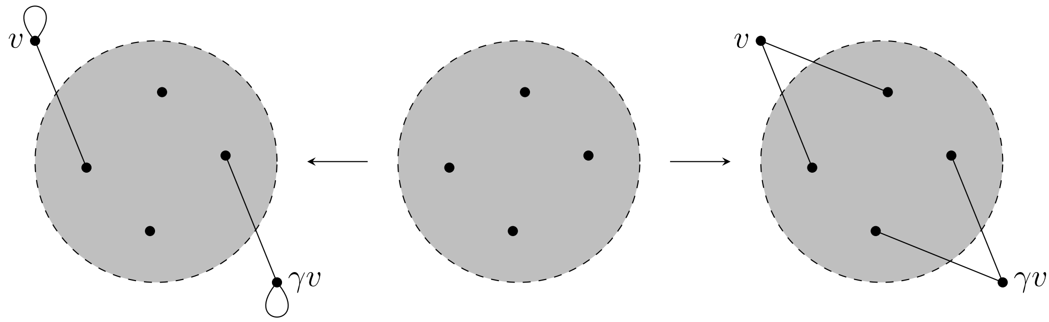

Figure 5.1: C_n-symmetric 0-extensions adding new vertices v and γv in each case; $n = 2$ shown.

Let $e_i = x_i y_i \in E$, $i = 0 \leq i \leq t-1$ be an edge orbit of G of size t under the action of Γ. Further let $z_0 \neq x_0, y_0$ and let $z_i = \gamma_i z_0$ for $i = 1, \ldots, t-1$. A *symmetrised 1-extension* creates a new Γ-symmetric looped simple graph by adding t vertices $\{v, \gamma v, \ldots, \gamma_{t-1} v\}$ and deleting all the edges e_i from G, and with v adjacent to x_0, y_0 and z_0, and $\gamma_i v$ adjacent to x_i, y_i and z_i for $i = 1, \ldots, t-1$. Alternatively, let $l_i = x_i x_i \in L$ for $i = 0 \leq i \leq t-1$ be a loop orbit of G of size t under the action of Γ. A *symmetrised looped 1-extension* creates a new Γ-symmetric looped simple graph by adding t vertices $\{v, \gamma v, \ldots, \gamma_{t-1} v\}$ and t loops $l_i^* = (\gamma_i v, \gamma_i v)$ for $i = 0 \leq i \leq t-1$ and deleting all the loops l_i from G, and with v adjacent to x_0 and y_0, and $\gamma_i v$ adjacent to x_i and y_i for $i = 1, \ldots, t-1$.

Lemma 5.1.1. *Suppose G is Γ-symmetric. Let G^+ be obtained from G by a symmetrised 0-extension. If (G, p, q) is $\tau(\Gamma)$-isostatic in $\mathbb{R}^2$, then for appropriate maps p^+, q^+, (G^+, p^+, q^+) is $\tau(\Gamma)$-isostatic in $\mathbb{R}^2$.*

Proof. There are two cases to consider here for the two variants of 0-extensions. We first consider a new orbit of vertices adjacent to two vertices. Write $G^+ = G + \{v, \ldots, \gamma_{t-1} v\}$, and let $v \in V^+$ be adjacent to v_1, v_2, and for each $k \in \{1, \ldots, t-1\}$, $\gamma_k v$ adjacent to $\gamma_k v_1, \gamma_k v_2$. Define $p^+ : V^+ \to \mathbb{R}^2$ by $p^+(z) = p(z)$ for all $z \in V$, $p^+(v) = (x, y)$, and $p^+(\gamma_k v) = (x^{(k)}, y^{(k)})$. Write $p(v_1) = (x_1, y_1)$, $p(v_2) = (x_2, y_2)$.

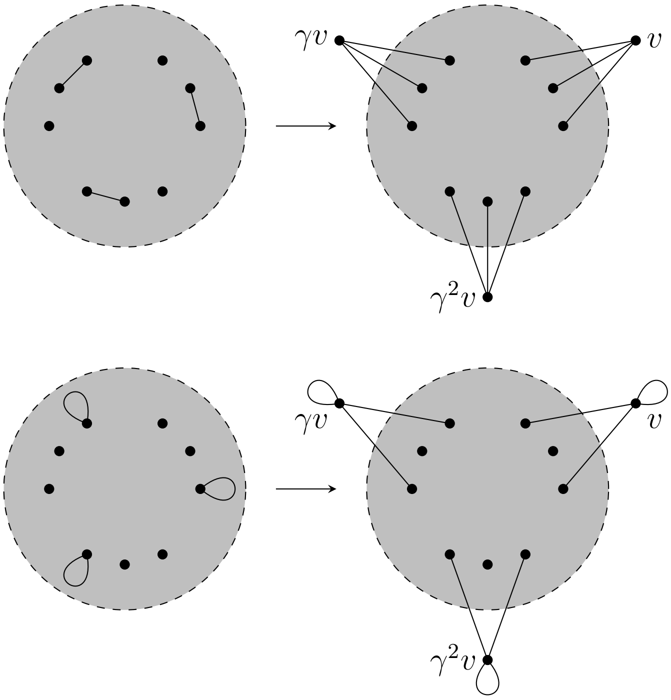

Figure 5.2: C_n-symmetric 1-extensions adding new vertices v, γv and $\gamma^2 v$ in each case; $n = 3$ shown.

Then,

$$R(G^+, p^+, q) = \begin{bmatrix} R(G,p,q) & & & & & & \\ & x - x_1 & y - y_1 & & & & \\ * & x - x_2 & y - y_2 & & \mathbf{0} & & \\ & & & \ddots & & & \\ & & & & x^{(k)} - x_1^{(k)} & y^{(k)} - y_1^{(k)} & \\ * & \mathbf{0} & & & x^{(k)} - x_2^{(k)} & y^{(k)} - y_2^{(k)} & \\ & & & & & & \ddots \end{bmatrix},$$

and hence the fact that $R(G^+, p^+, q)$ has linearly independent rows will follow once each 2×2 submatrix indicated above is shown to be invertible. Choose (x, y) so that $p^+(v), p^+(v_1), p^+(v_2)$ are not collinear, that is $(x, y) \neq (x_1, y_1) + \lambda(x_2 - x_1, y_2 - y_1)$ for any $\lambda \in \mathbb{R}$, which is exactly the requirement for the first submatrix to be invertible. Since each $\tau(\gamma_k)$ is an isometry, all of the other $t - 1$ remaining submatrices are also invertible, and so $\operatorname{rank} R(G^+, p^+, q) \geq \operatorname{rank} R(G, p, q) + 2t$. Hence, if G is $\tau(\Gamma)$-

independent so is G^+.

Alternatively, consider a new orbit of vertices with one loop incident, adjacent to one vertex. Write $G^+ = G + \{v, \ldots, \gamma_{t-1}v\}$ and let $v \in V^+$ be adjacent to v_1 and incident to the loop $l' = vv$, and for each $k \in \{1, \ldots, t-1\}$, $\gamma_k v$ adjacent to $\gamma_k v_1$ and incident to $\gamma_k l$. Define $p^+ : V^+ \to \mathbb{R}^2$ by $p^+(z) = p(z)$ for all $z \in V$, $p^+(v) = (x, y)$, and $p^+(\gamma_k v) = (x^{(k)}, y^{(k)})$ and $q^+ : L^+ \to \mathbb{R}^2$ by $q^+ = q(l)$ for all $l \in L$, $q^+(l') = (u, w)$, and $q^+(\gamma_k l') = (u^{(k)}, w^{(k)})$. Write $p(v_1) = (x_1, y_1)$. Then,

$$R(G^+, p^+, q^+) = \begin{bmatrix} R(G,p,q) & & & & & & \\ & x - x_1 & y - y_1 & & & & \\ * & u & w & & \mathbf{0} & & \\ & & & \ddots & & & \\ & & & & x^{(k)} - x_1^{(k)} & y^{(k)} - y_1^{(k)} & \\ * & \mathbf{0} & & & u^{(k)} & w^{(k)} & \\ & & & & & & \ddots \end{bmatrix},$$

and hence the fact that $R(G^+, p^+, q^+)$ has linearly independent rows will follow once each 2×2 submatrix indicated above is shown to be invertible. Choose $p^+(v), q^+(l')$ so that $(u, w) \neq \lambda(x - x_1, y - y_1)$ for any $\lambda \in \mathbb{R}$ (equality in the equation corresponds to the linear constraint being perpendicular to the edge vv_1), hence the first submatrix is invertible. Since each $\tau(\gamma_k)$ is an isometry, all of the other $t-1$ remaining submatrices are also invertible, and so $\operatorname{rank} R(G^+, p^+, q^+) \geq \operatorname{rank} R(G, p, q) + 2t$. Hence, if G is $\tau(\Gamma)$-independent so is G^+. □

Remark 5.1.2. For a C_s-symmetric graph G, let $G^+ = G + v$ be obtained by a 0-extension with a vertex v fixed by σ. Then, by letting v be adjacent to $v_1, v_2 \in V$, $y_2 = -y_1$, the relevant 2×2 matrix (see the proof above) is still invertible, hence if G is C_s-independent, then G^+ is C_s-independent.

Lemma 5.1.3. *Let G be a Γ-symmetric graph, and G^+ be obtained from G by a symmetrised 1-extension. If G is $\tau(\Gamma)$-isostatic, then G^+ is $\tau(\Gamma)$-isostatic.*

Proof. Suppose first that the 1-extension adds a new set of vertices, $\{v, \gamma v, \ldots, \gamma^{t-1}v\}$, where for each $i \in \{0, \ldots, t-1\}$, $\gamma^i v$ is adjacent to three vertices, say $\gamma^i v_1, \gamma^i v_2, \gamma^i v_3$, and the edges $(\gamma^i v_1, \gamma^i v_2)$ are deleted. Call this new graph G^+. Write $p(v_1) = (x_1, y_1)$, $p(v_2) = (x_2, y_2)$, $p(v_3) = (x_3, y_3)$. Define $p^+ : V^+ \to \mathbb{R}^2$ by putting $p^+(z) = p(z)$ for all $z \in V$, and choosing special positions so that $p^+(v) = \frac{1}{2}(p(v_1) + p(v_2)) = (x, y)$, and $p^+(\gamma_k v) = (x^{(k)}, y^{(k)})$. Now consider the rigidity matrix of the realisation of K_3 with vertex positions $p^+(v), p^+(v_1), p^+(v_2)$. Then,

$$R(K_3, p^+) = \begin{bmatrix} p^+(v_1) - p^+(v_2) & p^+(v_2) - p^+(v_1) & \mathbf{0} \\ p^+(v_1) - p^+(v) & \mathbf{0} & p^+(v) - p^+(v_1) \\ \mathbf{0} & p^+(v_2) - p^+(v) & p^+(v) - p^+(v_2) \end{bmatrix}$$
$$= \begin{bmatrix} p^+(v_1) - p^+(v_2) & p^+(v_2) - p^+(v_1) & \mathbf{0} \\ \frac{1}{2}(p^+(v_1) - p^+(v_2)) & \mathbf{0} & \frac{1}{2}(p^+(v_2) - p^+(v_1)) \\ \mathbf{0} & \frac{1}{2}(p^+(v_2) - p^+(v_1)) & \frac{1}{2}(p^+(v_1) - p^+(v_2)) \end{bmatrix}$$

has rank 2 and the linear dependence is non-zero on all 3 rows. We note that the $\tau(\gamma)$ preserves this linear dependence. Then, since $(G^+ + \bigcup_{i=0}^{t-1}\{(\gamma^i v_1, \gamma^i v_2)\} \setminus \bigcup_{i=0}^{t-1}\{(\gamma^i v, \gamma^i v_2)\}, p^+, q)$ is obtained from (G, p, q) by a symmetrised 0-extension,

$$\operatorname{rank} R(G^+ + \bigcup_{i=0}^{t-1}\{(\gamma^i v_1, \gamma^i v_2)\} \setminus \bigcup_{i=0}^{t-1}\{(\gamma^i v, \gamma^i v_2)\}, p^+, q) = \operatorname{rank} R(G, p, q) + 2t.$$

We observe from $R(K_3, p^+)$ that

$$\operatorname{rank} R(G^+ + \bigcup_{i=0}^{t-1}\{(\gamma^i v_1, \gamma^i v_2)\}, p^+, q) = \operatorname{rank} R(G, p, q) + 2t$$

too, and further, since any row can be written as a linear combination of the other two, we can delete any edge orbit from the orbit of the K_3 subgraphs and preserve infinitesimal rigidity. Since (G^+, p^+, q) is infinitesimally rigid in this special position, G^+ is $\tau(\Gamma)$-isostatic.

Consider now a looped 1-extension that creates a new graph G^+ from G by adding vertices $\{v, \gamma v, \dots, \gamma^{t-1} v\}$, where for each $i \in \{0, \dots, t-1\}$, $\gamma^i v$ is adjacent to two vertices, say $\gamma^i v_1, \gamma^i v_2$, and incident to the new loop $\gamma^i l'$, with the loops $\gamma^i l = (\gamma^i v_1, \gamma^i v_1)$ deleted. Write $p(v_1) = (x_1, y_1)$, $p(v_2) = (x_2, y_2)$. Define $p^+ : V^+ \to \mathbb{R}^2$ by putting $p^+(z) = p(z)$ for all $z \in V$, and choosing special positions

$p^+(v) = p(v_1)+q(l) = (x, y)$, and $p^+(\gamma_k v) = (x^{(k)}, y^{(k)})$. Further define $q^+ : L^+ \to \mathbb{R}^2$ by $q^+(h) = q(h)$ for all $h \in L$ and the special position $q^+(l') = q^+(l) = (c, d)$ and symmetrically the loops $\{\gamma l', \dots, \gamma^{t-1} l'\}$. We then consider the rigidity matrix for the realisation of $H = G[\{v, v_1\}]$ with the vertex positions $p^+(v), p^+(v_1)$ and linear constraints $q^+(l), q^+(l')$. Then,

$$R(H, p^+, q^+) = \begin{bmatrix} p^+(v_1) - p^+(v) & p^+(v) - p^+(v_1) \\ q^+(l) & \mathbf{0} \\ \mathbf{0} & q^+(l') \end{bmatrix} = \begin{bmatrix} (-c, -d) & (c, d) \\ (c, d) & \mathbf{0} \\ \mathbf{0} & (c, d) \end{bmatrix}$$

has rank 2 and the linear dependence is non-zero on all 3 rows. We note that $\tau(\gamma)$ preserves this linear dependence. Then since

$$(G^+ + \bigcup_{i=0}^{t-1} \{(\gamma^i v_1, \gamma^i v_1)\} \setminus \bigcup_{i=0}^{t-1} \{(\gamma^i v, \gamma^i v_1)\}, p^+, q^+)$$

is obtained from (G, p, q) by a symmetrised 0-extension,

$$\operatorname{rank} R(G^+ + \bigcup_{i=0}^{t-1} \{(\gamma^i v_1, \gamma^i v_1)\} \setminus \bigcup_{i=0}^{t-1} \{(\gamma^i v, \gamma^i v_1)\}, p^+, q^+) = \operatorname{rank} R(G, p, q) + 2t.$$

We then see from $R(H, p^+, q^+)$ that

$$\operatorname{rank} R(G^+ + \bigcup_{i=0}^{t-1} \{(\gamma^i v_1, \gamma^i v_1)\}, p^+, q) = \operatorname{rank} R(G, p, q) + 2t$$

too, and further since any row can be written as a linear combination of the other two, we can delete the orbit of one of $\{l, l', vv_1\}$ and preserve infinitesimal rigidity. Since (G^+, p^+, q^+) is infinitesimally rigid in this special position, G^+ is $\tau(\Gamma)$-isostatic.

□

We next show some specific families of graphs are Γ-isostatic for certain groups. In the next sections these families will turn out to be base graphs for our construction arguments. A *pinned graph* on n vertices is the graph with 2 loops incident to each vertex and $E = \emptyset$, which we denote $\mathcal{P}_n$. A *looped n-cycle* is a cycle on n vertices with a single loop incident to each vertex (see Figure 5.3), which we denote LC_n. Define the symmetric graph $(\mathcal{P}_1, \phi_0)$ to have $\phi_0 : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathcal{P}_1)$ fix all elements of $\mathcal{P}_1$ for all $\gamma \in \mathbb{Z}/2\mathbb{Z}$. Furthermore, define $(\mathcal{P}_1, \phi_1)$ to have $\phi_1 : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathcal{P}_1)$ fix the vertex and transpose the loops for the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$. Let $(\mathcal{P}_n, \phi_n)$ be defined by $\phi_n : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}(\mathcal{P}_n)$ having a single orbit of the n vertices. Lastly, (LC_n, ψ_n) has $\psi_n : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}(LC_n)$ form a single orbit of the n vertices.

Lemma 5.1.4. *The following graphs or graph classes are $\tau(\Gamma)$-isostatic:*

- $(\mathcal{P}_1, \phi_0)$ *is C_2-isostatic;*
- $(\mathcal{P}_1, \phi_1)$ *is C_4-isostatic;*
- *for $n \geq 2$, $(\mathcal{P}_n, \phi_n)$ is C_n-isostatic;*
- *for odd $n \geq 3$, (LC_n, ψ_n) is C_n-isostatic.*

Proof. In the first two bullet points, we have a single vertex restricted by two linear constraints. This will be pinned unless the corresponding loops are c_2 images of each other, in which case the linear constraints must coincide. In the third bullet point, every vertex is pinned. Therefore in the first three bullet points the graphs are $\tau(\Gamma)$-isostatic. In the final bullet point, we put the framework in special position, so that the linear constraints pass through the origin (that is the vertices can only move radially). Any infinitesimal motion of a vertex can therefore be described as "inward" or "outward" from the origin, depending on whether the radial distance decreases or increases. In order for the edge lengths to be preserved, any inward moving vertex must be adjacent to two outward moving vertices, and likewise any outward moving vertex must be adjacent to two inward moving vertices. As a result of this, inward and outward moving vertices must alternate in the cycle. Thus there are as many outward as inward moving vertices, which is not possible with n odd. Hence, this special position is infinitesimally rigid, and so the graph is $\tau(\Gamma)$-isostatic. □

Using another special position argument, Lemma 5.1.4 can be extended to show all C_n-symmetric looped n-cycles are $\tau(\Gamma)$-isostatic. While we are only able to

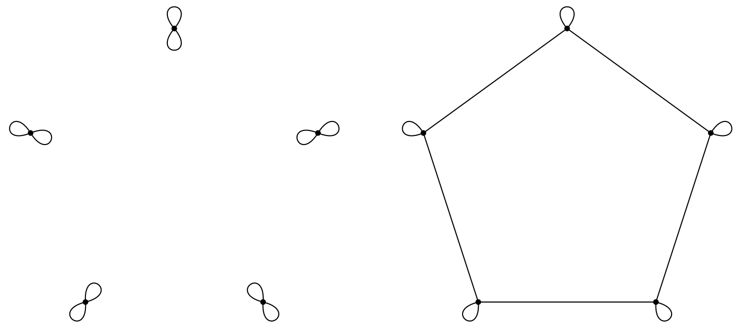

Figure 5.3: Base graphs for the construction of C_n-symmetric ($n \geq 3$) linearly constrained isostatic frameworks; $(\mathcal{P}_5, \phi_5)$ and (LC_5, ψ_5) depicted.

construct the set of all $\tau(\Gamma)$-isostatic graphs for rotational groups of order 2 or of odd order, we give the extended result below.

Lemma 5.1.5. *All C_n-symmetric looped n-cycles are rigid.*

Proof. The case for odd looped n-cycles was shown in Lemma 5.1.4 and for simplicity is omitted here, but can also be shown to be rigid by a small elaboration of this method. Let $n = 2k$ be even. Let $G_0 = (\{v_0\}, \emptyset, \{l_0, l'\})$ be a graph with one pinned vertex, and define $p_0 : V \to \mathbb{R}^2$ by $p_0(v_0) = (1, 0)$. Note that as long as $q_0(l_0)$ and $q_0(l')$ are not linearly dependent, the framework (G_0, p_0, q_0) is rigid. We set $q_0(l') = (\sin \frac{\pi}{2k}, \cos \frac{\pi}{2k})$ and $q_0(l_0) = (-\sin \frac{\pi}{2k}, \cos \frac{\pi}{2k})$. For $1 \leq i \leq n-1$, construct G_i from a looped 0-extension, with new vertex v_i adjacent to v_{i-1} and loop l_i at v_i. The maps p_i, q_i act on $\{v_0, \dots, v_{i-1}\}$ and $\{l_0, \dots, l_{i-1}\}$ as p_{i-1} and q_{i-1} respectively, with $p_i(v_i) = \tau(c_{2k})(p_{i-1}(v_{i-1}))$ and $q_i(l_i) = \tau(c_{2k})(q_{i-1}(l_{i-1}))$. Since $p_i(v_i) - p_i(v_{i-1}) \neq 0$, the 0-extension preserves rigidity. Write $p = p_{2k-1}$ and $q = q_{2k-1}$. Let G be a looped $2k$-cycle. By construction, $G_{2k-1} = G - \{v_0v_{2k-1} + l'\}$, and in the framework (G_{2k-1}, p, q), the constraint from l' is the line through $p(v_0)$ and $p(v_{k-1})$, and the constraint from l_{2k-1} is the line through $p(v_{2k-1})$ and $p(v_k)$ (see Figure 5.4). As with (H, p^+, q^+) in Lemma 5.1.3, we can show that the rigidity matrix on $(\{v_0, v_{2k-1}\}, v_0v_{2k-1}, \{l_{2k-1}, l'\})$ has rank 2 with linear dependence non-zero on all 3 rows, hence $\operatorname{rank} R(G_{2k-1}, p, q) = \operatorname{rank} R(G, p, q)$ and so G, which is C_{2k}-symmetric, is rigid. □

In the following lemmas we consider only the C_s-tight graphs, as we require 0-extensions and 1-extensions which involve fixed vertices in this setting.

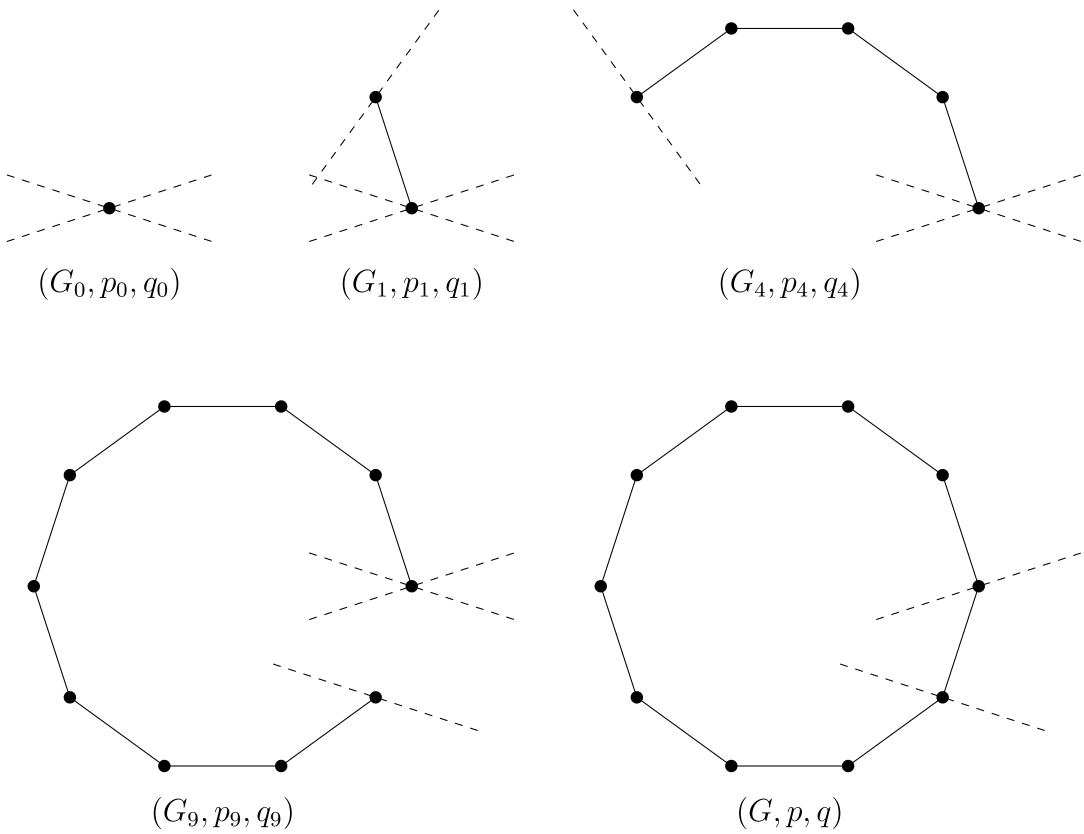

Figure 5.4: Process of generating a looped 10-cycle (G, p, q). At each stage shown the framework is depicted with edges of the graph included and linear constraints of interest included as dotted lines.

Lemma 5.1.6. *Suppose G is C_s-symmetric. Let G^+ be obtained from G by a looped 0-extension at a fixed vertex, which fixes the added vertex, edge and loop, or a 0-extension adding a fixed vertex adjacent to two symmetric vertices. If (G, p, q) is C_s-isostatic in $\mathbb{R}^2$, then for appropriate maps p^+, q^+, (G^+, p^+, q^+) is C_s-isostatic in $\mathbb{R}^2$.*

Proof. Write $G^+ = G + \{v\}$ and let v be adjacent to v_1 and incident to the loop $l' = vv$. Without loss of generality let the mirror line be the x-axis. Define $p^+ : \hat{V} \to \mathbb{R}^2$ by $p^+(z) = p(z)$ for all $z \in V$, $p^+(v) = (x, 0)$, and $q^+ : \hat{L} \to \mathbb{R}^2$ by $q^+ = q(l)$ for all $l \in L$, $q^+(l') = (0, 1)$. Write $p(v_1) = (x_1, 0)$. Then, $R(G^+, p^+, q^+) =$

$$\begin{bmatrix} R(G,p,q) & \mathbf{0} & \\ & x - x_1 & 0 \\ * & 0 & 1 \end{bmatrix},$$

and hence the fact that $R(G^+, p^+, q^+)$ has linearly independent rows follows if the 2×2 submatrix indicated above is invertible. This happens as long as $x \neq x_1$.

Alternatively when $G^+ = G + \{v\}$, let v be adjacent to v_1 and v_1'. We retain the definition of p^+ and q^+, now letting $p^+(v) = (x, 0)$, $p^+(v_1) = (x_1, y)$ and $p^+(v_1') = (x_1, -y)$. Now, $R(G^+, p^+, q^+) =$

$$\begin{bmatrix} R(G,p,q) & \mathbf{0} & \\ & x - x_1 & x - x_1 \\ * & -y & y \end{bmatrix},$$

with the 2×2 submatrix indicated above is invertible if and only if $x \neq x_1$ and $y \neq 0$.

In each case we may choose a special position so that $\operatorname{rank} R(G^+, p^+, q^+) \geq \operatorname{rank} R(G, p, q) + 2$, and therefore this holds for the vertices in C_s-generic position. Hence, if G is C_s-isostatic so is G^+. □

Lemma 5.1.7. *Let G be a Γ-symmetric graph, and G^+ be obtained from G by a 1-extension on a fixed edge adding a new fixed vertex incident to a fixed edge. If G is $\tau(\Gamma)$-isostatic, then G^+ is $\tau(\Gamma)$-isostatic.*

Proof. Suppose that the 1-extension adds a new vertex, v, where v is adjacent to three vertices, say v_1, v_1', v_2, and the edge (v_1, v_1') is deleted. Call this new graph G^+. Write $p(v_1) = (x_1, y_1)$, $p(v_2) = (x_2, y_2)$, and by symmetry $p(v_1') = (x_1, -y_1)$. Define $p^+ : V^+ \to \mathbb{R}^2$ by putting $p^+(z) = p(z)$ for all $z \in V$, and choosing special positions so that $p^+(v) = (p(v_1) + p(v_1')) = (x_1, 0)$. Now consider the rigidity matrix of the realisation of K_3 with vertex positions $p^+(v), p^+(v_1), p^+(v_1')$. Then,

$$R(K_3, p^+) = \begin{bmatrix} p^+(v_1) - p^+(v_1') & p^+(v_1') - p^+(v_1) & \mathbf{0} \\ p^+(v_1) - p^+(v) & \mathbf{0} & p^+(v) - p^+(v_1) \\ \mathbf{0} & p^+(v_1') - p^+(v) & p^+(v) - p^+(v_1') \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -2y_1 & 0 & 2y_1 & 0 & 0 \\ 0 & -y_1 & 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & -y_1 & 0 & y_1 \end{bmatrix}$$

has rank 2 and the linear dependence is non-zero on all 3 rows. Then, since $(G^+ + (v_1, v_1') - (v, v_1')\}, p^+, q)$ is obtained from (G, p, q) by a 0-extension adding a fixed vertex,

$$\operatorname{rank} R(G^+ + (v_1, v_1') - (v, v_1')\}, p^+, q) = \operatorname{rank} R(G, p, q) + 2t.$$

We observe from $R(K_3, p^+)$ that $\operatorname{rank} R(G^+ + (v_1, v_1')\}, p^+, q) = \operatorname{rank} R(G, p, q) + 2t$ too, and further, since any row can be written as a linear combination of the other two, we can delete any edge from the the K_3 subgraph and preserve infinitesimal rigidity. Since (G^+, p^+, q) is infinitesimally rigid in this special position, G^+ is $\tau(\Gamma)$-isostatic. □

Having seen fixed vertex variants of 0- and 1-extensions, and the looped 0-extension, it is natural to ask whether there is such a fixed vertex move for looped 1-extensions. An extension on two vertices off the mirror would require deleting two loops to preserve symmetry. To resolve this, one may think to perform two looped 1-extensions, adding two new looped vertices with one loop at each. However then each of these loops would be fixed. In order to preserve the C_s-tightness condition given in Table 3.12, one loop would correspond to a linear constraint perpendicular to the mirror, and the other correspond to one along the mirror. Alternatively, one

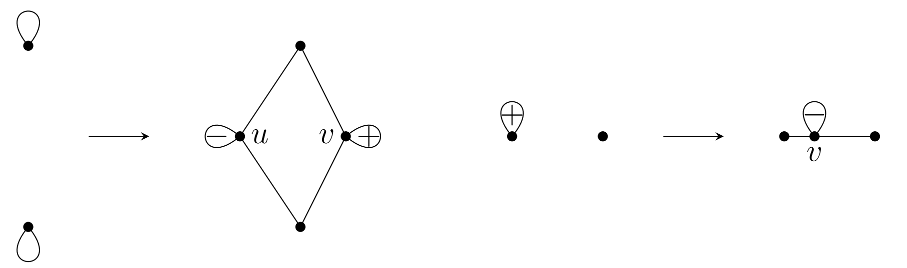

Figure 5.5: Two ideas for σ-fixed looped 1-extensions adding new vertices u, v and v respectively.

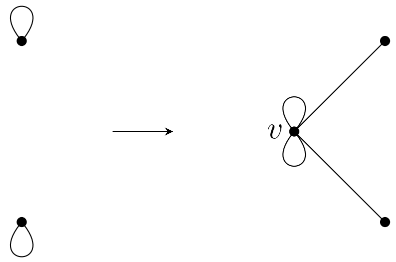

Figure 5.6: Fixed 1-extension variant adding a new fixed vertex v.

may consider two fixed vertices, one incident to a loop. A 1-extension adding a fixed vertex incident to two fixed edges and one fixed loop, in order to preserve tightness counts, would require the deleted loop to correspond to a perpendicular constraint, and the new loop correspond to a linear constraint along the mirror. Figure 5.5 depicts the result of such ideas. Both of these however do not preserve infinitesimal rigidity. In the following lemma we present a variant to a looped fixed vertex 1-extension which does preserve infinitesimal rigidity. It is unlikely such a move would be required to classify C_s-tight graphs, but we present it here for interest.

Lemma 5.1.8. *Let G be a Γ-symmetric graph. Let G^+ be obtained from G by: adding a new pinned fixed vertex; adding two new edges from this new vertex to any vertex of G incident to a loop and its σ image; and deleting a loop at this vertex of G and its image under σ. If G is $\tau(\Gamma)$-isostatic, then G^+ is $\tau(\Gamma)$-isostatic.*

Proof. We create the new graph G^+ from G by adding a new vertex $\{v\}$, where v is adjacent to two vertices each the mirror image of the other, say w, w', and incident to the new loops l_v and l'_v which are σ images of each other, with the loops $l = (w, w)$

and $\sigma l = (w', w')$ deleted (see Figure 5.6). Write $p(w) = (x, y)$, $p(w') = (x, -y)$. Define $q^+ : L^+ \to \mathbb{R}^2$ by $q^+(h) = q(h)$ for all $h \in L$ and the special position $q^+(l_v) = q^+(l) = (c, d)$ and symmetrically the loop $q^+(l'_v) = q^+(l') = (c, -d)$. Further define $p^+ : V^+ \to \mathbb{R}^2$ by putting $p^+(z) = p(z)$ for all $z \in V$, and choosing the unique special position $p^+(v) = (x_0, 0)$ so that $(x_0, 0) = (x, y) + \lambda(c, d)$ for some real $\lambda \neq 0$. We then consider the rigidity matrix for the realisation of $H = G[\{v, w\}] \backslash \{l'_v\}$ with the vertex positions $p^+(v), p^+(w)$ and linear constraints $q^+(l_v), q^+(l)$. Then,

$$R(H, p^+, q^+) = \begin{bmatrix} p^+(w) - p^+(v) & p^+(v) - p^+(w) \\ q^+(l) & \mathbf{0} \\ \mathbf{0} & q^+(l_v) \end{bmatrix} = \begin{bmatrix} \lambda(-c, -d) & \lambda(c, d) \\ (c, d) & \mathbf{0} \\ \mathbf{0} & (c, d) \end{bmatrix}$$

has rank 2 and the linear dependence is non-zero on all 3 rows. We note that this linear dependence also holds on $H' = G[\{v, w'\}] \setminus \{l_v\}$. Then since $(G^+ + \{l, l'\} \setminus \{l_v, l'_v\}, p^+, q^+)$ is obtained from (G, p, q) by a fixed vertex 0-extension,

$$\operatorname{rank} R(G^+ + \{l, l'\} \setminus \{l_v, l'_v\}, p^+, q^+) = \operatorname{rank} R(G, p, q) + 2.$$

We then see from $R(H, p^+, q^+)$ that $\operatorname{rank} R(G^+ + \{l, l'\}, p^+, q) = \operatorname{rank} R(G, p, q) + 2$ too, and further since any row can be written as a linear combination of the other two, we can delete one of $\{l_v, l, vw\}$ and its image in $\{l'_v, l', vw'\}$ and preserve infinitesimal rigidity. Since (G^+, p^+, q^+) is infinitesimally rigid in this special position, G^+ is $\tau(\Gamma)$-isostatic. □

Lemma 5.1.9. *Let G be $\tau(\Gamma)$-isostatic. Let $w \in V$ be adjacent to $v_1, \ldots, v_r$ and incident to the loops l_1, l_2. Suppose that both $\{p(w) - p(v_1), p(w) - p(v_2)\}$ and $\{q(l_1), q(l_2)\}$ are linearly independent sets in $\mathbb{R}^2$. Let G^+ be obtained by forming a vertex-to-C_4 operation at the vertices w (and σw if $\sigma \in \Gamma$ and $\sigma w \neq w$), so that v_1, v_2 are adjacent to both w and the new vertex u in G^+ (and similarly $\sigma v_1, \sigma v_2$ are adjacent to both σw and σu when they exist). Then G^+ is $\tau(\Gamma)$-isostatic.*

Proof. We will construct $R(G^+, p^+, q^+)$ from $R(G, p, q)$ by a series of matrix row operations. We first add 2 zero columns to $R(G, p, q)$ for the vertex $\{u, \sigma u\}$ and another 2 zero columns for the vertex σu when $\sigma \in \Gamma$ and $\sigma u \neq u$ (which for the remainder of the proof we will assume). Then add 4 rows to this matrix, for the edges $uv_1, uv_2, u'v'_1, u'v'_2$. Since this is a pair of 0-extensions, $\operatorname{rank} R(G^+, p^+, q^+) = \operatorname{rank} R(G, p, q) + 2$. This gives the matrix M of the form:

$$M = \left[\begin{array}{ccc} * & p(w) - p(v_1) & 0 \\ * & p(w) - p(v_2) & 0 \\ & \vdots & \\ * & p(w) - p(v_i) & 0 \\ & \vdots & \\ * & 0 & p(u) - p(v_1) \\ * & 0 & p(u) - p(v_2) \\ & \vdots & \\ \hline * & q(l_1) & 0 \\ * & q(l_2) & 0 \\ & \vdots & \end{array}\right],$$

where the columns given are for the vertices w and u, and rows given for the edges $wv_1, wv_2, wv_i, uv_1, uv_2$ and loops l_1, l_2. There would be similar columns for each pair σw and σu. This is the rigidity matrix for a graph generated from G by a $\tau(\Gamma)$-symmetric vertex-to-C_4 operation where $v_i w$ is an edge for all $3 \leq i \leq k$. We wish to show that removing the edges $\{wv_i\}$ and loops l_1, l_2 at w and replacing them with the edges $\{uv_i\}$ and loops l_3, l_4 at u preserves the rank of the rigidity matrix.

Since $p(w) - p(v_1)$ and $p(w) - p(v_2)$ are linearly independent and span $\mathbb{R}^2$, there exists $\alpha, \beta \in \mathbb{R}$ such that $p(w) - p(v_i) = \alpha(p(w) - p(v_1)) + \beta(p(w) - p(v_2))$. Hence we perform row operations as follows. From the row of wv_i, subtract α multiples of the row of wv_1, and β multiples of the row of wv_2. Then to the row of wv_i, add α multiples of the row of uv_1, and β multiples of the row of uv_2. Since $p(w) = p(u)$, when we do this to every neighbour v_i of u, and similarly $\gamma_k v_i$ of $\gamma_k u$ (since all $\tau(\gamma_k)$ are isometries of $\mathbb{R}^3$ that preserve the cylinder, the same α, β, γ work for the symmetric copies in G^+), we obtain $R_y(G^+, p^+)$. The row operations preserve $\tau(\Gamma)$-independence, giving the desired result. □

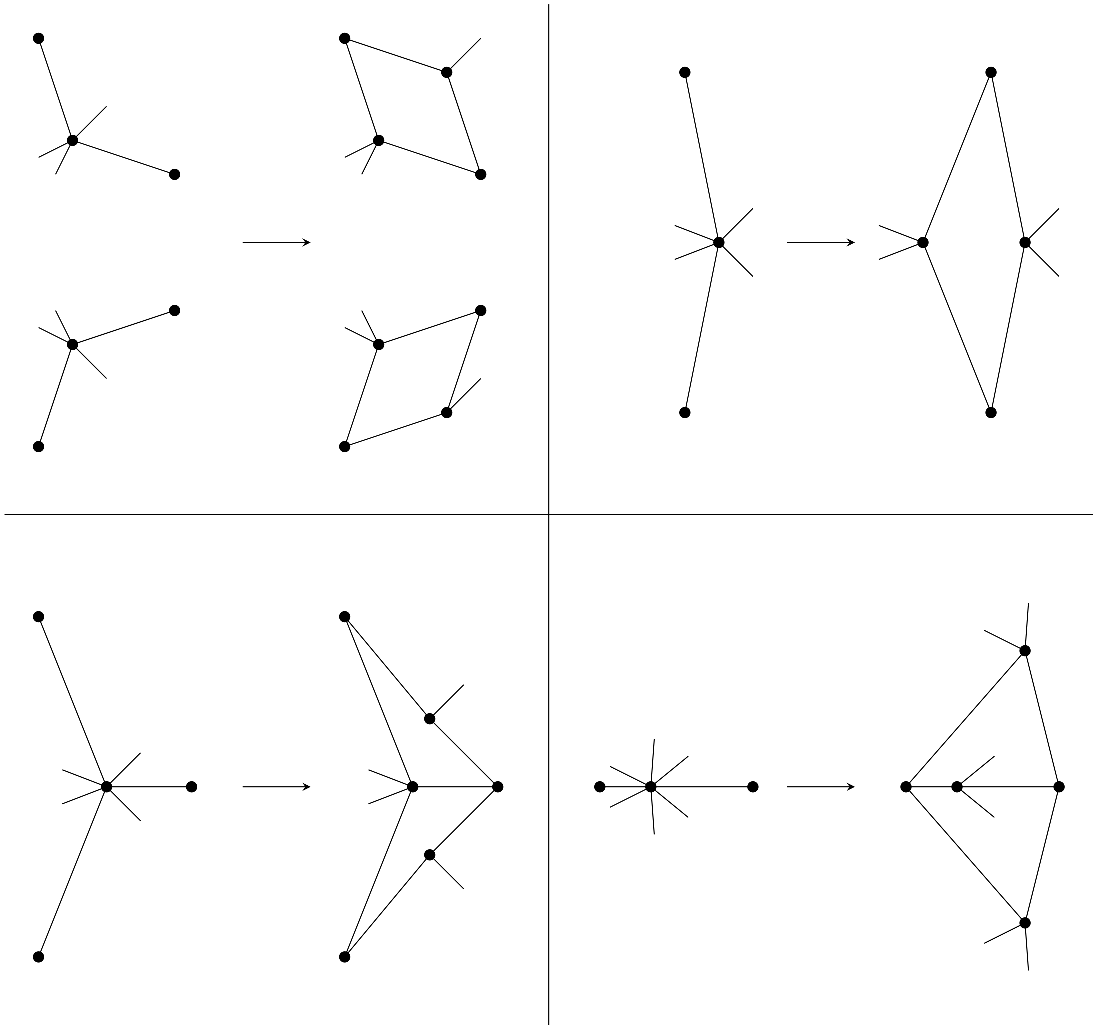

Figure 5.7: The four possibilities for vertex-to-C_4 operations on C_s-tight graphs. Lemma 5.1.9 shows these operations preserve rigidity except in the case involving 3 fixed vertices (bottom right).

5.2 C_2-symmetric isostatic graphs

In the following sections we prove the main results of this chapter. These results show that the standard sparsity counts, together with the necessary conditions derived in Section 3.8, are also sufficient for generic symmetric frameworks to be isostatic. The proofs are based on a recursive construction using the Henneberg-type moves discussed in the previous section. We first consider C_2 rotational symmetry. Recall $G = (V, E, L)$ is: *sparse* if $|E'| + |L'| \le 2|V'|$ for all subgraphs (V', E') of G and $|E'| \le 2|V'| - 3$ for all simple subgraphs with $|E'| > 0$; and *tight* if it is sparse and $|E|+|L| = 2|V|$. Furthermore, a C_2-tight graph is a tight graph with $v_2 = e_2 = l_2 = 0$ or $v_2 = 1, e_2 = 0, l_2 = 2$.

We refer the reader to Section 2.1 for a reminder of the definition of k-critical and k-edge-critical sets. Note that if X is $\tau(\Gamma)$-symmetric and for each $\gamma \in \Gamma \setminus \{\text{id}\}$, γ has no fixed edges or loops, then $|\Gamma|$ divides k_X and $\bar{k}_X$. Additionally, we note that a C_2-tight graph with fixed edge, loop, and vertex counts described in Table 3.12, has either no fixed loops or two fixed loops incident to a fixed vertex. Therefore any symmetric vertex set will induce a subgraph with no fixed loops or two fixed loops, and so $|C_2| = 2$ will still divide k_X. This fact will be crucial in what follows.

Lemma 5.2.1. *Let G be a graph and suppose $A, B \subseteq V$ have non-empty intersection. Then $k_A + k_B = k_{A\cup B} + k_{A\cap B} + d(A, B)$ and $\bar{k}_A + \bar{k}_B = \bar{k}_{A\cup B} + \bar{k}_{A\cap B} + d(A, B)$.*

Proof. Since $|A| + |B| = |A \cup B| + |A \cap B|$, we have

$$\begin{aligned} 2|A| - k_A + 2|B| - k_B &= i_{E+L}(A) + i_{E+L}(B) = i_{E+L}(A \cup B) + i_{E+L}(A \cap B) - d(A, B) \\ &= 2|A \cup B| - k_{A\cup B} + 2|A \cap B| - k_{A\cap B} - d(A, B) \\ &= 2|A| + 2|B| - k_{A\cup B} - k_{A\cap B} - d(A, B), \end{aligned} \tag{5.2.1}$$

giving the result. The edge-critical variant is identical. □

Most of the technical work in the next two sections involves analysing when we can remove a vertex incident to 3 edges or 2 edges and 1 loop. In this chapter, we will call such a vertex, in either case, a *node*. We will consider *reduction operations*:

these are the reverse of the extension operations described in Section 5.1, namely symmetrised (looped) 0-reductions and symmetrised (looped) 1-reductions.

5.2.1 Reduction operations

Lemma 5.2.2. *Let (G, ϕ) be a C_2-tight graph containing a vertex v incident to two edges. Then $G \setminus \{v, v'\}$ is C_2-tight.*

Note that this lemma includes the cases when v has degree two and is adjacent to two distinct vertices and when v has degree three with a loop at v (recall Figure 5.1).

Proof. If either $G - v$ or $G \setminus \{v, v'\}$ breaks sparsity, G would not be tight. □

In the proof of the following lemma, spefically the second paragraph of Case 2, we prove the following remark which we will reference in future proofs.

Remark 5.2.3. Suppose (G, ϕ) is C_2-tight containing a node $v \in V$ with $N(v) = \{x, y, z\}$. Then it is impossible for x, y to be in a 0-critical set and x, z, x', z' to be in a 4-edge-critical set. This conclusion is independent of the number of edges induced by $N(v)$.

Lemma 5.2.4. *Let (G, ϕ) be a C_2-tight graph and suppose $v \in V$ is a node with $N(v) = \{x, y, z\}$ and $N(v) \cap N(v') = \emptyset$ or $\{t\}$ where $t \in \{x, y, z\}$ is a fixed vertex. For a pair $x_1, x_2 \in \{x, y, z\}$ with $x_1x_2 \notin E$, the following holds: x_1, x_2 is not contained in any 0-critical set or any 3-edge-critical set, and x_1, x_2, x_1', x_2' is not contained in any 1-critical set or any 4-edge-critical set.*

Proof. In the following argument, we leave it possible that one of the vertices of $\{x, y, z\}$ is fixed, and for example $x' = x$. We split up the proof based on the number of edges of G induced by $N(v)$. If three edges were present $G[N(v) \cup \{v\}] \cong K_4$ which is not sparse, so we may assume $xy \notin E$. Suppose the pair x, y is not in a 0-critical set and suppose there exists a 1-critical set W containing x, y, x', y'. Then $W \cup W'$ and $W \cap W'$ are both C_2-symmetric so have even criticality since G is C_2-tight. By Lemma 5.2.1 one must be 0-critical, a contradiction since both contain

x, y. Hence we know that if $x, y \in N(v)$ are not in a 0-critical set, then x, y, x', y' are not in a 1-critical set.

Case 1: If there are two edges on the vertices of $N(v)$, say $x, y \notin E$, then x, y cannot be contained in a 3-edge-critical subset and x, y, x', y' are not in a 4-edge critical subset as these sets with v, z and v, z, v', z' added respectively would not be sparse. It's easy to see that there is no 0-critical set on x, y, and there is no 1-critical set on x, y, x', y' from the paragraph above, finishing Case 1.

Case 2: Suppose exactly one edge, say xz, is present on the vertices of $N(v)$. First assume x, y is contained in a 0-critical set, X. Then it is easy to check that y, z is not in a 0-critical set. Assume there exists a 3-edge-critical set U on y, z. This means there are $l_U \in \{0, 1, 2, 3\}$ loops on the vertices of U, with U being $(3 - l_U)$-critical. By Lemma 5.2.1,

$$k_{X \cup U} = k_X + k_U - k_{X \cap U} - d(X, U).$$

We know $k_X = 0$, $X \cap U$ has edge-criticality of at least 2 and $X \cap U \subset U$ so there are $l_{X \cap U} \leq l_U$ loops on the vertices of $X \cap U$. Therefore,

$$k_U - k_{X \cap U} = 3 - l_U - \bar{k}_{X \cap U} + l_{X \cap U} \leq 1,$$

and xz gives that $d(X, U)$ is at least 1, so $X \cup U$ is 0-critical. But then $X \cup U + \{v\}$ is not sparse in G, a contradiction.

To show y, z, y', z' is not contained in a 4-edge-critical set, first notice that $X \cup X'$ is 0-critical, containing x, y, x', y'. Relabel this union as X. Assume there is a 4-edge-critical set W on y, z, y', z'. We take the symmetric sets $W \cup W'$ and $W \cap W'$, which by symmetry have even edge-criticality. Additionally, $y, z, y', z' \in W \cap W'$, so that $\bar{k}_{W \cap W'} \geq 3$. Hence by Lemma 5.2.1, $\bar{k}_{W \cup W'} + \bar{k}_{W \cap W'} \leq \bar{k}_W + \bar{k}_{W'} = 8$, and so $\bar{k}_{W \cap W'}, \bar{k}_{W \cup W'} = 4$. As a result of this, we may always take 4-edge-critical sets of C_2-tight graphs to be symmetric (in particular we may assume W is symmetric).

There are either 0, 2, or 4 loops on the vertices of W (by symmetry and G being C_2-tight), with W being 4, 2, or 0-critical respectively in those cases. Again, since $k_{X \cup W} = k_X + k_W - k_{X \cap W} - d(X, W)$, any looped vertex of W is also a vertex of

$X \cap W$, giving $k_W - k_{X\cap W} = 0$ ($X \cap W$ contains at least 2 vertices and is symmetric, so has at least 4 for its edge-critical value) and $X \cup W \cup \{v\}$ is not sparse. Hence the lemma holds in this case with $x_1, x_2 = y, z$.

Now assume neither x, y nor y, z are contained in 0-critical sets. If both x, y, x', y' and y, z, y', z' are in 4-edge-critical sets W_1, W_2, then we may suppose W_1 and W_2 are symmetric as before. Both $W_1 \cup W_2$ and $W_1 \cap W_2$ are C_2-symmetric and have at least 2 vertices, so are also both 4-edge-critical. In particular, $W_1 \cup W_2 \cup \{v, v'\}$ violates the sparsity of G. When only one of x, y, x', y' and y, z, y', z' are in a 4-edge-critical set, say x, y, x', y' in W, assume there exists a 3-edge-critical set U containing y, z. Lemma 5.2.1 for edge-criticality says

$$\bar{k}_{U\cup W} = \bar{k}_U + \bar{k}_W - \bar{k}_{U\cap W} - d(U, W).$$

Since $\bar{k}_{U\cap W} \geq 2$ and $d(U, W) \geq 1$ (due to the presence of the edge xz), we have $\bar{k}_{U\cup W} \leq 4$. Similarly, $\bar{k}_{U\cup U'\cup W} \leq 4$, but adding v, v' contradicts the sparsity of G. Finally, if no such 4-edge-critical set exists, x, y and y, z cannot both be in 3-edge-critical sets, say U_1, U_2, by considering $U_1 \cup U_2 \cup \{v\}$, as xz again gives $d(U_1, U_2) \geq 1$. Hence the lemma holds when one edge is present on the neighbours of v.

Case 3: Lastly suppose there are no edges induced by the vertices of $N(v)$. It is easy to show that there exists a 0-critical set on at most one pair of neighbours of v, while there can be 3-edge-critical sets on at most two pairs.

First assume x, y is contained in a 0-critical set, X. Assume x, z and y, z are contained in 3-edge-critical sets U_1, U_2 and note that $(U_1 \cup U_2) \cap X$ and any supersets thereof (since they contain x, y) cannot be 3-edge-critical else all 3 pairs of neighbours of v are in 3-edge critical sets. Consider the equations,

$$\bar{k}_{U_1\cup U_2} = \bar{k}_{U_1} + \bar{k}_{U_2} - \bar{k}_{U_1\cap U_2} - d(U_1, U_2) \tag{5.2.2}$$

$$k_{(U_1\cup U_2)\cup X} = k_{U_1\cup U_2} + k_X - k_{(U_1\cup U_2)\cap X} - d(U_1 \cup U_2, X). \tag{5.2.3}$$

A contradiction occurs when

$$k_{U_1\cup U_2} \leq k_{(U_1\cup U_2)\cap X} + d(U_1 \cup U_2, X),$$

as this would violate the sparsity of G. Let $l_a \geq l_b$ count the number of loops on vertices of $U_1 \cup U_2, (U_1 \cup U_2) \cap X$ respectively. Equation (5.2.2) along with the inequalities $\bar{k}_{U_1\cup U_2} \geq 4$ and $\bar{k}_{U_1\cap U_2} \geq 2$ gives that these two inequalities are in fact equalities. Then, since $\bar{k}_{(U_1\cup U_2)\cap X} \geq 4 = \bar{k}_{(U_1\cup U_2)}$, we have

$$k_{(U_1\cup U_2)\cap X} = \bar{k}_{(U_1\cup U_2)\cap X} - l_b \geq \bar{k}_{(U_1\cup U_2)} - l_a = k_{(U_1\cup U_2)},$$

which, with Equation (5.2.3), gives the desired contradiction. So there is a pair x_1, x_2 not contained in a 0-critical or 3-edge critical set. By Remark 5.2.3, there is no 4-edge-critical or 1-critical set containing x_1, x_2, x_1', x_2', completing this case.

Finally, suppose there is no 0-critical set on any pair of neighbours of $N(v)$. We know, from above, that at most one pair of neighbours of v with their symmetric copies can be contained in a 4-edge-critical set. Hence in this case we only obtain a contradiction to the lemma if say x, y, x', y' is in a 4-edge-critical set W while x, z and y, z are in 3-edge-critical sets, say U_1 and U_2 respectively (see Figure 5.8). Recall that x, y cannot also be in a 3-edge-critical set and since $\bar{k}_{U_1\cap U_2} \geq 2$ in Equation (5.2.2), we have $\bar{k}_{U_1\cup U_2} = 4$. Further, $x, y \in (U_1 \cup U_2) \cap W$, so $\bar{k}_{(U_1\cup U_2)\cup W} = 4$. Now instead of considering the 3-edge-critical sets U_1 and U_2 with the 4-edge-critical set W, consider the 3-edge-critical sets U_1' and U_2' with the 4-edge-critical set $(U_1 \cup U_2) \cup W$. The same methods show $\bar{k}_{(U_1\cup U_2)\cup W\cup(U_1'\cup U_2')} = 4$. However the set $(U_1 \cup U_2) \cup W \cup (U_1' \cup U_2')$ with $\{v, v'\}$ added is not sparse. This exhausts all cases, completing the proof. □

Lemma 5.2.5. *Let (G, ϕ) be C_2-tight and suppose $v \in V$ is a node such that $N(v) = \{x, y, x'\}$ and $N(v) \cap N(v') = \{x, x'\}$. For a pair $(x_1, x_2) \in \{(x, y), (x', y)\}$, with $x_1x_2 \notin E$, we have:*

x_1, x_2 is not contained in any 0-critical set or any 3-edge-critical set; and

x_1, x_2, x_1', x_2' is not contained in any 1-critical set or any 4-edge-critical set.

Proof. Without loss of generality, one of the following hold:

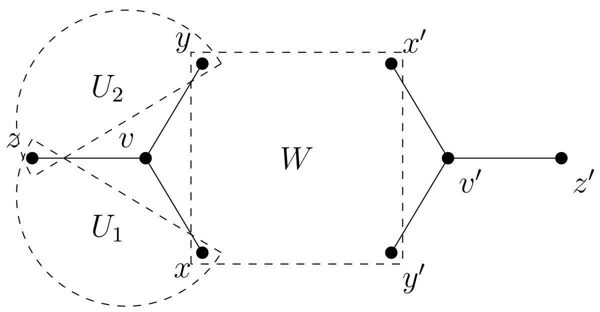

Figure 5.8: Diagram of part of a C_2-tight graph with hypothetical 3-edge-critical sets U_1, U_2 and 4-edge-critical set W, with $x, z \in U_1$, $y, z \in U_2$, $x, y, x', y' \in W$.

1. $xy, x'y' \in E$, $x'y, xy' \notin E$.

2. $xy, x'y, xy', x'y' \notin E$.

The edge sets described in (1) and (2) describe all possibilities when $N(v) \cap N(v') = \{x, x'\}$. Suppose the edges present are as in (1) or (2), and at least one of the following exists: (i) there exists X with $x', y \in X$ which is 0-critical; or (ii) there exists W with $x', y, x, y' \in W$ which is 1-critial or 4-edge-critical. Note that $X \cup X'$ is 0-critical and contains all the neighbours of v and v', as with such a W, $X \cup \{v, v'\}$ and $W \cup \{v, v'\}$ break sparsity of G immediately.

Hence for the remainder of the proof, we need only consider 3-edge-critical sets. First in case (1), let U be 3-edge-critical with $x', y \in U$. From Lemma 5.2.1,

$$\bar{k}_{U \cup U'} \leq \bar{k}_U + \bar{k}_{U'} - d(U, U') = 3 + 3 - 2 = 4.$$

Then $U \cup U' \cup \{v, v'\}$ violates the sparsity of G, completing the first case. Now assume we have no edges as in (2). We want to show that we can add either $xy, x'y'$ or $x'y, xy'$ to $G - \{v, v'\}$. Suppose there exists two 3-edge-critical sets, U_1, U_2, with $x, y \in U_1$ and $x', y \in U_2$. Relabel $Y := U_1 \cup U_2$. Note that Y is not 3-edge-critical else $Y + \{v\}$ is not sparse. On the other hand $\bar{k}_{U_1 \cap U_2} \geq 2$, so by counting $\bar{k}_Y \leq 4$, hence Y must be 4-edge-critical. Symmetrically, $\bar{k}_{Y'} = 4$. Then as $x, x' \in Y \cap Y'$, $\bar{k}_{Y \cap Y'} \geq 3$, and since $Y \cap Y'$ is symmetric, the number of edges on the set must be even, so $\bar{k}_{Y\cap Y'} \geq 4$. Hence

$$\bar{k}_{Y\cup Y'} = \bar{k}_Y + \bar{k}_{Y'} - \bar{k}_{Y\cap Y'} - d(Y, Y') \leq 4 + 4 - 4 = 4,$$

so adding $\{v, v'\}$ to $Y \cup Y'$ breaks sparsity of G. □

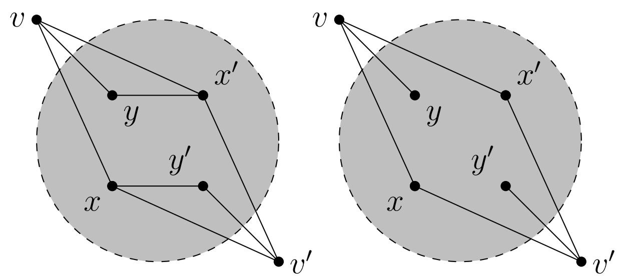

Figure 5.9: The local structure of cases (1) and (2) in Lemma 5.2.5.

Lemma 5.2.6. *Let (G, ϕ) be a C_2-tight graph containing a node v with three distinct neighbours. Then $G \setminus \{v, v'\} + \{x_1x_2, x_1'x_2'\}$ is C_2-tight for some $x_1, x_2 \in N(v)$.*

Proof. From the definition of a C_2-tight graph, we know there are no fixed edges and at most 1 fixed vertex in G. Therefore $vv' \notin E$ and $|N(v) \cap N(v')|$ cannot be 3. Lemmas 5.2.4 and 5.2.5 show that when $N(v) \cap N(v')$ equals $\emptyset$, $\{t\}$ or $\{x, x'\}$, the reduction will preserve sparsity, as required. □

Lemma 5.2.7. *Let (G, ϕ) be a C_2-tight graph. Suppose v is a node adjacent to distinct vertices $\{x, y\}$, and v is incident to a loop. There exists some $x_1 \in \{x, y\}$ such that $G \setminus \{v, v'\} + \{(x_1, x_1), (x_1', x_1')\}$ is C_2-tight.*

Proof. Suppose there exist 0-critical sets X, Y, with $x \in X$ and $Y \in Y$. Then $X \cup Y + \{v\}$ is not sparse. Hence, without loss of generality we may take it that x is not in a 0-critical set. Instead now suppose x, x' is in a 1-critical set, say W. Then $W \cup W$ and $W \cap W$ are both contain x, x' and are C_2-symmetric, hence have an even critical value. From the equation in Lemma 5.2.1,

$$2 = k_W + k_{W'} = k_{W\cup W'} + k_{W\cap W'} + d(W, W').$$

Thus one of $W \cup W$ and $W \cap W$ is 0-critical, label this 0-critical set U. If y is in a 0-critical set Y, $U \cup Y \cup \{v\}$ breaks sparsity of G. If instead y, y' is in a 1-critical set, we may do the same steps as above to find a 0-critical set containing both y, y', leading to the same contradiction. Hence there is a $x_1 \in \{x, y\}$ which is not in a 0-critical set with x_1, x_1' not in a 1-critical set, completing the proof. □

We put together the combinatorial analysis to this point to prove the following recursive construction. From this we then deduce our characterisation of C_2-isostatic graphs. We need one more lemma which we prove for arbitrary cyclic groups as we will use it again later in the thesis.

Two vertices $u, v \in V$ are connected if there exists a path from u to v. We define two vertices $u, v \in V$ to be *γ-symmetrically connected* if $u = \gamma v$ or $v = \gamma u$, or if there exists a path from u to γv or γu to v. A *Γ-symmetrically connected component* is a set of vertices which are pairwise γ-symmetrically connected for some $\gamma \in \Gamma$, and a graph is *Γ-symmetrically connected* if it has only one Γ-symmetrically connected component.

Lemma 5.2.8. *A graph (G, ϕ) is C_n-tight if and only if every C_n-symmetrically connected component of G is C_n-tight.*

Proof. Label the C_n-symmetrically connected components of G with $H_1, \ldots, H_r$, with $H_i = (V_i, E_i, L_i)$. By sparsity, we know each of the subgraphs H_i are sparse. We have $|E_i| + |L_i| \leq 2|V_i|$ and $|E_i| \leq 2|V_i| - 3$, which gives G being tight if and only if equality holds in the first equation for each i, which is to say each C_n-symmetrically connected component is tight. □

We recall the following C_2-tight graphs: $(\mathcal{P}_1, \phi_0)$ with one fixed vertex and two fixed loops; $(\mathcal{P}_2, \phi_2)$ with two vertices and four loops and no vertices or loops fixed. These graphs are depicted in Figure 5.10.

Theorem 5.2.9. *A graph (G, ϕ) is C_2-tight if and only if (G, ϕ) can be generated from disjoint copies of $(\mathcal{P}_1, \phi_0)$ and $(\mathcal{P}_2, \phi_2)$ by symmetrised 0-extensions and symmetrised 1-extensions.*

Figure 5.10: The base graphs, left $(\mathcal{P}_1, \phi_0)$ and right $(\mathcal{P}_2, \phi_2)$, for C_2-symmetric linearly constrained frameworks.

Proof. It is easy to see that any graph generated from $(\mathcal{P}_1, \phi_0)$ or $(\mathcal{P}_2, \phi_2)$ by symmetrised 0-extensions and 1-extensions is C_2-symmetrically connected and C_2-tight. Hence G is C_2-tight by Lemma 5.2.8.

We show by induction that any C_2-tight graph G can be generated from symmetrically connected copies of $(\mathcal{P}_1, \phi_0), (\mathcal{P}_2, \phi_2)$. We may assume by Lemma 5.2.8 that G is C_2-symmetrically connected. Suppose the induction hypothesis holds for all graphs with $|V| < m$. Now let $|V| = m$ and suppose G is not isomorphic to $(\mathcal{P}_1, \phi_0)$ or $(\mathcal{P}_2, \phi_2)$. Since G is C_2-tight, the minimum degree is at least 2 and at most 4.

A degree 2 vertex in a tight graph must be adjacent to two vertices and hence is reducible by Lemma 5.2.2. A degree 3 vertex can have a loop and an edge incident to it or be adjacent to three vertices. The former is reducible by Lemma 5.2.2 and the latter by Lemma 5.2.6.

Hence suppose $\delta(G) = 4$ and v is a vertex of minimum degree. We claim there is such a v that is incident to a loop. Suppose not, then every vertex has at least 4 neighbours and so $2|E| \leq \sum_{v \in V} \deg_E(v)$ which violates the definition of tight. Since v is incident to a loop and has degree 4 either it has two incident loops or is incident to two edges and a loop. In the former case the orbit of such a vertex would be its own C_2-symmetrically connected component. In the latter case w is reducible by Lemma 5.2.7. This exhausts the possible cases and completes the proof. □

Theorem 5.2.10. *A graph (G, ϕ) is C_2-isostatic if and only if it is C_2-tight.*

Proof. Necessity was proved in Theorem 3.8.3. Lemma 5.1.4 implies the base graphs $(\mathcal{P}_1, \phi_0)$ and $(\mathcal{P}_2, \phi_2)$ are C_2-isostatic. Sufficiency follows from Theorem 5.2.9 and Lemmas 5.2.2, 5.2.6 and 5.2.7 by induction on $|V|$. □

5.3 C_n-symmetric isostatic graphs

In this section we analyse C_n rotational symmetry, for all odd $n \geq 3$. The arguments will build on the proofs in the previous section.

5.3.1 Reduction operations

Lemma 5.3.1. *Let (G, ϕ) be a C_n-tight graph containing a vertex v incident to two edges. Then $G \setminus \bigcup_{i=0}^{n}\{\gamma^i v\}$ is C_n-tight.*

Note that this lemma includes the cases when v is degree two adjacent to two distinct vertices and when v is degree three with a loop at v, and is depicted in Figure 5.1.

Proof. For fixed $1 \leq k \leq n$ and any $0 \leq i_1 < \cdots < i_k \leq n-1$, if $G \setminus \bigcup_{j=0}^{k}\{\gamma^{i_j} v\}$ breaks sparsity, G would not be sparse. □

Let us first show the following key technical lemma for C_3, and then follow it with the general case. Both proofs are similar, but technical; the easier case is presented first for the reader's convenience.

Lemma 5.3.2. *Let (G, ϕ) be a C_3-tight graph containing a v adjacent to two distinct vertices $\{v_1, v_2\}$ and incident to a loop. Let $\gamma \in C_3 \setminus \{id\}$. Then there exists some $x \in \{v_1, v_2\}$ such that $G \setminus \bigcup_{i=0}^{2}\{\gamma^i v\} + \bigcup_{i=0}^{2}\{\gamma^i x \gamma^i x\}$ is C_3-tight.*

Proof. We prove the lemma by first checking there is a neighbour of v not contained in a 0-critical subset of V, and then (for any $k \in \{1, 2\}$) that any $k+1$ symmetric copies of that neighbour are not contained in a k-critical subset of V.

If both of v_1, v_2 were in 0-critical sets, say V_1, V_2 respectively, then $V_1 \cup V_2 \cup \{v\}$ would break sparsity. Label the vertex which is not in a 0-critical set x. Note that any symmetric copy of x is also not in a 0-critical set.

Let $0 \leq i_1 < i_2 \leq 2$ and W be a set which contains $\{\gamma^{i_1}x, \gamma^{i_2}x\}$. First we note that W and $\gamma^{i_2-i_1}W$ are not 0-critical since they each contain a copy of x. Suppose for a contradiction that W and $\gamma^{i_2-i_1}W$ are 1-critical. Then $W \cup \gamma^{i_2-i_1}W$ contains $\{\gamma^{i_1}x, \gamma^{i_2}x, \gamma^{2i_2-i_1}x\}$ and is not 0-critical; and $W \cap \gamma^{i_2-i_1}W$ contains $\{\gamma^{i_2}x\}$ and is also not 0-critical. By Lemma 5.2.1, $k_{W\cup\gamma^{i_2-i_1}W}$ and $k_{W\cap\gamma^{i_2-i_1}W}$ must both be 1. Observe that $\gamma^{2i_2-i_1}$ is the third and final element of C_3, and $\gamma^{2i_2-i_1}W$ is 1-critical, so again, by Lemma 5.2.1, $k_{(W\cup\gamma^{i_2-i_1}W)\cup\gamma^{2i_2-i_1}W}$ and $k_{(W\cup\gamma^{i_2-i_1}W)\cap\gamma^{2i_2-i_1}W}$ must both be 1. However, since G is C_3-symmetric, 3 divides both $|W\cup\gamma W\cup\gamma^2 W|$ and $i_{E+L}(W\cup\gamma W\cup\gamma^2 W)$. Then since $i_{E+L}(A)=2|A|-k_A$, 3 would have to divide the criticality of $W\cup\gamma W\cup\gamma^2 W$, which is a contradiction. Therefore any $\{\gamma^{i_1}x,\gamma^{i_2}x\}$ with distinct $i_1,i_2\in\{0,1,2\}$ is not contained in a 1-critical set.

Let W be a set which contains $\{x,\gamma x,\gamma^2 x\}$. We know from the above W is not 0 or 1-critical. Assume for contradiction it is 2-critical. Then $W\cup\gamma W$ contains $\{x,\gamma x,\gamma^2 x\}$ and so is not 1-critical; and $W\cap\gamma W$ contains $\{x,\gamma x,\gamma^2 x\}$ which is also not 1-critical. By Lemma 5.2.1, $k_{W\cup\gamma W}$ and $k_{W\cap\gamma W}$ must both be 2. Similarly, $W\cup\gamma W\cup\gamma^2 W$ is 2-critical. However, as before G is C_3-symmetric, so 3 would have to divide the criticality of $W\cup\gamma W\cup\gamma^2 W$, which is a contradiction. This proves any set containing $\{x,\gamma x,\gamma^2 x\}$ is not a 2-critical set. Hence, there are no sets which would break sparsity conditions of our graph class by performing a 1-reduction at v, adding a loop at a neighbour. □

Having established this result for C_3-symmetric graphs, we extend this to odd order cyclic symmetric groups.

The two 1-reductions we perform have similarities in their proofs, where we build an inductive argument on the number of symmetric copies of neighbours of v. Since some of these ideas overlap, they will be shared between the two proofs where possible. The significant difference between the two proofs is that adding an edge means we must check both edge sparsity and sparsity are preserved, whereas adding a loop does not require the edge sparsity condition to be checked. To begin with we derive a technical lemma which is used in the inductive step of both arguments.

Let $A\subset V$ and j be a positive integer less than or equal to n. Define $\mathcal{X}_j(A)$ to be the set with elements being j copies of A under action γ, written $X_j(A)=\{\gamma^{i_1}A,\ldots,\gamma^{i_j}A : i_1<\cdots<i_j\in\{0,\ldots,n-1\}\}$. We will write X_j for $X_j(A)$ and $\mathcal{X}_j$ for $\mathcal{X}_j(A)$ where the context is clear. Let $\phi:\mathcal{X}_j\to\mathbb{Z}_n$, $\phi(\{\gamma^{i_1}A,\ldots,\gamma^{i_j}A\})=\{i_1,\ldots,i_j\}$.

Lemma 5.3.3. *Let n be a positive odd integer, (G, ϕ) be a C_n-tight graph and let X_j be defined as above. If for $A \subset N(v)$, X_1 is not in a 0-critical set for all $X_1 \in \mathcal{X}_1$, then X_j is not contained in a $(j-1)$-critical set for all $j \leq n$.*

Proof. We proceed by induction on j and begin by noting that the case when $j = 1$ is trivial. Assume no X_j is contained in a $(j-1)$-critical set for all $j \leq k$. We will show any X_{k+1} is not contained in a k-critical set. Fix $X_{k+1} \in \mathcal{X}_{k+1}$ and write $X = \phi(X_{k+1})$. For notation, write $\gamma^i X = \phi(\gamma^i X_{k+1})$ and $\gamma^i X \cup \gamma^j X = \phi(\gamma^i X_{k+1} \cup \gamma^j X_{k+1})$ and $\gamma^i X \cap \gamma^j X$ similarly. By the induction hypothesis, any set containing X_{k+1} also contains k or fewer copies of x, so is not j-critical for any $j < k$. Suppose for a contradiction that X_{k+1} is contained in a k-critical set, say W. We will show that $\bigcup_{i=0}^{n-1} \gamma^i W$ has $2|\bigcup_{i=0}^{n-1} \gamma^i W| - a$ edges for some $a < n$. Since $\bigcup_{i=0}^{n-1} \gamma^i W$ is symmetric, n divides $|\bigcup_{i=0}^{n-1} \gamma^i W|$. As our graph class has no fixed edges or loops, this gives a contradiction. Initially, we assume that X generates $\mathbb{Z}_n$.

Observe that if $\gamma^{i_t} X \subseteq X \cup \cdots \cup \gamma^{i_{t-1}} X$ and $X \cup \cdots \cup \gamma^{i_{t-1}} X = X \cup \cdots \cup \gamma^{i_t} X$ then

$$k_{W \cup \cdots \cup \gamma^{i_{t-1}} W} = k_{W \cup \cdots \cup \gamma^{i_t} W}.$$

Therefore, to bound $k_{W \cup \cdots \cup \gamma^{n-1} W}$ our main focus is the case when $\gamma^{i_t} X \not\subseteq X \cup \cdots \cup \gamma^{i_{t-1}} X$. First, when $\gamma^{i_1} X \not\subseteq X$ we have $|X \cap \gamma^{i_1} X| < |X| < |X \cup \gamma^{i_1} X|$. By the induction hypothesis, any set containing $\phi^{-1}(X \cap \gamma^{i_1} X)$, such as $W \cap \gamma^{i_1} W$, is not contained in a $(|X \cap \gamma^{i_1} X| - 1)$-critical set. This implies $k_{W \cap \gamma^{i_1} W} \geq |X \cap \gamma^{i_1} X|$. Lemma 5.2.1 implies that

$$\begin{aligned} k_{W \cup \gamma^{i_1} W} &= k_W + k_{\gamma^{i_1} W} - k_{W \cap \gamma^{i_1} W} - d(W, \gamma^{i_1} W) \\ &\leq |X| - 1 + |\gamma^{i_1} X| - 1 - |X \cap \gamma^{i_1} X| - d(W, \gamma^{i_1} W) \qquad (5.3.1) \\ &\leq |X \cup \gamma^{i_1} X| - 2 \end{aligned}$$

and hence the critical value for $W \cup \gamma^{i_1} W$ is at most $(|X \cup \gamma^{i_1} X| - 2)$.
We repeat this process with $\gamma^{i_2} X \not\subseteq X \cup \gamma^{i_1} X$, and hence investigate $(X \cup \gamma^{i_1} X) \cup \gamma^{i_2} X$ and $(X \cup \gamma^{i_1} X) \cap \gamma^{i_2} X$. We know that $k_{W \cup \gamma^{i_1} W} = k$ or $k_{W \cup \gamma^{i_1} W} \leq |X \cup \gamma^{i_1} X| - 2$

and $k_{\gamma^{i_2}W} = k$. Since $\gamma^{i_2}X \nsubseteq X \cup \gamma^{i_1}X$ we have

$$|(X \cup \gamma^{i_1}X) \cap \gamma^{i_2}X| < |X \cup \gamma^{i_1}X| < |(X \cup \gamma^{i_1}X) \cup \gamma^{i_2}X|.$$

Noting that $\phi^{-1}((X\cup\gamma^{i_1}X)\cap\gamma^{i_2}X)$ is not contained in any $(|(X\cup\gamma^{i_1}X)\cap\gamma^{i_2}X|-1)$-critical set, (as $(X \cup \gamma^{i_1}X) \cap \gamma^{i_2}X \subset \gamma^{i_2}X$ we can apply the induction hypothesis), so

$$k_{(W\cup\gamma^{i_1}W)\cap\gamma^{i_2}W} \geq |(X \cup \gamma^{i_1}X) \cap \gamma^{i_2}X|$$

(as $\phi^{-1}((X \cup \gamma^{i_1}X) \cap \gamma^{i_2}X) \subseteq (W \cup \gamma^{i_1}W) \cap \gamma^{i_2}W$), therefore

$$\begin{aligned} k_{W\cup\gamma^{i_1}W\cup\gamma^{i_2}W} &= k_{W\cup\gamma^{i_1}W} + k_{\gamma^{i_2}W} - k_{(W\cup\gamma^{i_1}W)\cap\gamma^{i_2}W} - d(W \cup \gamma^{i_1}W, \gamma^{i_2}W) \\ &\leq |X \cup \gamma^{i_1}X| - 2 + |\gamma^{i_2}X| - 1 - |(X \cup \gamma^{i_1}X) \cap \gamma^{i_2}X| \\ &- d(W \cup \gamma^{i_1}W, \gamma^{i_2}W) \\ &\leq |X \cup \gamma^{i_1}X \cup \gamma^{i_2}X| - 3. \end{aligned} \tag{5.3.2}$$

Thus the critical value for $W \cup \gamma^{i_1}W \cup \gamma^{i_2}W$ is at most $(|X \cup \gamma^{i_1}X \cup \gamma^{i_2}X| - 3)$. Recalling that $\gamma^{i_t}X \subseteq X \cup \cdots \cup \gamma^{i_{t-1}}X$ implies that

$$k_{W\cup\cdots\cup\gamma^{i_{t-1}}W} = k_{W\cup\cdots\cup\gamma^{i_t}W},$$

and noting that $|(X \cup \cdots \cup \gamma^{n-1}X) \setminus X| = n - k - 1$, the case when $\gamma^{i_t}X \nsubseteq X \cup \cdots \cup \gamma^{i_{t-1}}X$ can happen at most $n - k - 1$ times. Therefore, we obtain that

$$k_{W\cup\cdots\cup\gamma^{n-1}W} \leq |X \cup \cdots \cup \gamma^{n-1}X| - n + k.$$

Finally since $|X\cup\cdots\cup\gamma^{n-1}X| = n$, $W\cup\cdots\cup\gamma^{n-1}W$ cannot be n-critical. However, since G is C_n-symmetric, n divides $|W \cup \cdots \cup \gamma^{n-1}W|$ and n divides $i_{E+L}(W \cup \cdots \cup \gamma^{n-1}W)$. However then, since $i_{E+L}(A) = 2|A| - k_A$, n divides the criticality of $W \cup \cdots \cup \gamma^{n-1}W$, a contradiction.

Instead assume that X generates a subgroup of $\mathbb{Z}_n$, let m be such that $m \cdot |\langle X\rangle| = n$ and $l = |\langle X\rangle| \geq k + 1$. Let $j_1, \ldots, j_l$ be distinct elements of $\langle X\rangle$ so that $\gamma^{j_r} \cap (\gamma^{j_1}W \cup \cdots \cup \gamma^{j_{r-1}}W)$ is non-empty for all $2 \leq r \leq l$. We can apply the above argument, instead noting that $|(\gamma^{j_1}X \cup \cdots \cup \gamma^{j_l}X) \setminus X| = l - k - 1$, so that

$$k_{\gamma^{j_1}W\cup\cdots\cup\gamma^{j_l}W} \leq |\gamma^{j_1}X \cup \cdots \cup \gamma^{j_l}X| - l + k = k.$$

Let $W^* = \gamma^{j_1}W \cup \cdots \cup \gamma^{j_l}W$. By construction we have

$$W^* \cup \gamma W^* \cup \cdots \cup \gamma^{m-1}W^* = W \cup \gamma W \cup \cdots \cup \gamma^{n-1}W.$$

Therefore, since the criticality of the union of sets is less than the sum of the criticalities of those sets,

$$k_{W\cup\cdots\cup\gamma^{n-1}W} \leq mk_{W^*} \leq mk < m(k+1) \leq ml = n,$$

and we arrive at a contradiction as before. □

Lemma 5.3.4. *Let (G, ϕ) be a C_n-tight graph. Suppose v is a node adjacent to distinct vertices $\{v_1, v_2\}$, and v has a loop. Let $\gamma \in C_n$ be a generator of C_n. There exists some $x \in \{v_1, v_2\}$ such that $G \setminus \bigcup_{i=0}^{n-1}\{\gamma^i v\} + \bigcup_{i=0}^{n-1}\{\gamma^i x \gamma^i x\}$ is C_n-tight.*

Proof. Since $\gamma \in C_n$ is a generator of C_n and no element is fixed by c_n, for any $u \in V$, $u, \gamma u, \ldots, \gamma^{n-1}u$ are all distinct. We will show that there is a neighbour of v not contained in a 0-critical subset of V, and that (for any $k \in \{1, \ldots, n-1\}$) any $k+1$ symmetric copies of that neighbour are not contained in a k-critical subset of V.

To see there exists an $x \in \{v_1, v_2\}$ which is not in a 0-critical set, suppose both v_1 and v_2 are in 0-critical sets. Let V_1, V_2 denote the 0-critical sets containing v_1, v_2 respectively. Then $V_1 \cup V_2 \cup \{v\}$ would break sparsity. The base case of induction is now complete since, by extension, $\gamma^i x$ is not in a 0-critical set for any $i \in [n-1]$. Indeed, we now have a set $\{x, \gamma x, \ldots, \gamma^{n-1}x\}$ where no one element is contained in a 0-critical set. Hence, we have, for $\{x\} \subset N(v)$, shown that any $X_1(\{x\}) \in \mathcal{X}_1(\{x\})$ is not contained in a 0-critical set, so Lemma 5.3.3 implies that no X_j is contained in a $(j-1)$-critical set for all $j \leq n$. Hence, the 1-reduction will not break sparsity of G, completing the proof. □

Lemma 5.3.5. *For a positive odd integer n, let (G, ϕ) be a C_n-tight graph and let $v \in V$ be a node adjacent to distinct vertices $\{v_1, v_2, v_3\}$. Let $\gamma \in C_n$ be a generator of C_n. There exists some $x, y \in \{v_1, v_2, v_3\}$ such that $G \setminus \bigcup_{i=0}^{n-1}\{\gamma^i v\} + \bigcup_{i=0}^{n-1}\{\gamma^i x \gamma^i y\}$ is C_n-tight.*

Proof. With the same arguments used in Lemma 5.2.4, we may show that there exists a pair $\{x_0, y_0\} \in \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}$ which is not contained in a 0-critical set or a 3-edge-critical set. We claim that for some $i, j \in \{1, 2, 3\}$, and any choice of $k+1$ elements from $\{\{v_i, v_j\}, \{\gamma v_i, \gamma v_j\}, \dots, \{\gamma^{n-1} v_i, \gamma^{n-1} v_j\}\}$, there is no $(k-1)$-critical set containing those k elements. We prove this by induction on k.

We have that $\{x_0, y_0\}$ is not contained in a 0-critical set. We write $x_i = \gamma^i x_0$, and similarly y_i. By the symmetry of G, $\{x_i, y_i\}$ is not in a 0-critical set for any $i \in \{0, \dots, n-1\}$. Hence the basis of induction is complete.

Write $X_k(\{x_0, y_0\}) = \{\{x_{i_1}, y_{i_1}\}, \dots, \{x_{i_k}, y_{i_k}\} : i_1 < \cdots < i_k \in \{0, \dots, n-1\}\}$. Assume no X_j is contained in a $(j-1)$-critical set for all $j \leq k$. Then, Lemma 5.3.3 completes the induction. Hence, we have shown that there is a pair $\{x_0, y_0\}$ not in a 0-critical set, and for any such pair, the union of any k symmetric copies of that pair is not contained in a $(k-1)$-critical set. It remains to show is that the 1-reduction does not violate the inequality $|E'| \leq 2|V'| - 3$.

We require an analogous reduction to Lemma 5.3.3 with edge-criticality. To this end, for a given k-edge-critical set W, we consider the union $\bigcup_{i=0}^{n-1} \gamma^i W$, which is a C_n-symmetric set. In this case it is possible for the order of the group to divide the number of edges in the set, so instead we will show that $\bigcup_{i=0}^{n-1} \gamma^i W$ has $2|\bigcup_{i=0}^{n-1} \gamma^i W| - a$ edges for some $a < 2n$. Since $\bigcup_{i=0}^{n-1} \gamma^i W$ is symmetric we know n divides $|\bigcup_{i=0}^{n-1} \gamma^i W|$. Hence it must be that $a = n$.

Assume no X_j is contained in a $(j+2)$-edge-critical set for all $j \leq k$. Then, for a contradiction, suppose X_{k+1} is contained in a $(k+3)$-edge-critical set. That is, there exists a W containing $k+1$ copies of $\{x_0, y_0\}$ such that $\bar{k}_W = k+3 = |X| + 3 - 1$. We follow a similar approach to Lemma 5.3.3, however it is now vital that the appropriate intersections be non-empty.

First take $|X \cap \gamma^{i_2 - i_1} X| < |X| < |X \cup \gamma^{i_2 - i_1} X|$. By the induction hypothesis, any set containing $\phi^{-1}(X \cap \gamma^{i_2 - i_1} X)$, such as $W \cap \gamma^{i_2 - i_1} W$, is not contained in a

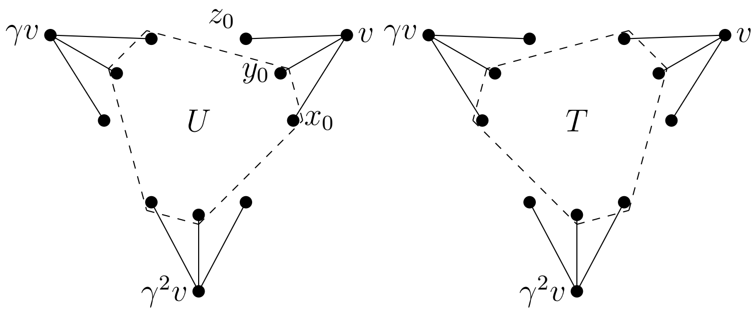

Figure 5.11: Two C_n-symmetric sets U and T, with U n-critical and T 0-critical. The case when $n = 3$ is shown.

$(|X \cap \gamma^{i_2-i_1}X| + 2)$-critical set. This implies $k_{W\cap\gamma^{i_2-i_1}W} \geq |X \cap \gamma^{i_2-i_1}X| + 3$. Lemma 5.2.1 implies that

$$\begin{aligned} \bar{k}_{W\cup\gamma^{i_2-i_1}W} &= \bar{k}_W + \bar{k}_{\gamma^{i_2-i_1}W} - \bar{k}_{W\cap\gamma^{i_2-i_1}W} - d(W, \gamma^{i_2-i_1}W) \\ &\leq |X| + 3 - 1 + |\gamma^{i_2-i_1}X| + 3 - 1 - |X \cap \gamma^{i_2-i_1}X| - 3 \\ &\leq |X \cup \gamma^{i_2-i_1}X| + 3 - 2. \end{aligned} \tag{5.3.3}$$

Hence the critical value for $W \cup \gamma^{i_2-i_1}W$ is at most $|X \cup \gamma^{i_2-i_1}X| + 3 - 2$. We repeat this process with $\gamma^{i_3-i_1}X \not\subseteq X \cup \gamma^{i_2-i_1}X$, and hence take $(X \cup \gamma^{i_2-i_1}X) \cup \gamma^{i_3-i_1}X$ and $(X \cup \gamma^{i_2-i_1}X) \cap \gamma^{i_3-i_1}X$. Continuing, we obtain

$$\bar{k}_{W\cup\gamma^{i_2-i_1}W\cup\cdots\cup\gamma^{i_k-i_1}W} \leq |X \cup \cdots \cup \gamma^{i_k-i_1}X| + 3 - k.$$

If X generates $\mathbb{Z}_n$, with the same reasoning as in Lemma 5.3.3, we can choose an ordering $j_1, \dots, j_n$ of $\mathbb{Z}_n$ such that

$$\bar{k}_{\gamma^{j_1}W\cup\cdots\cup\gamma^{j_n}W} \leq |\gamma^{j_1}X \cup \cdots \cup \gamma^{j_n}X| + 3 - n + k = k + 3.$$

Since $\gamma^{j_1}W\cup\cdots\cup\gamma^{j_n}W$ is c_n-symmetric, we have a contradiction unless $\bar{k}_{\gamma^{j_1}W\cup\cdots\cup\gamma^{j_n}W} = n$. Suppose this equality holds, and write $U = \bigcup_{i=0}^{n-1} \gamma^i W$.

Without loss of generality we may suppose $\{x_0, y_0\} = \{v_1, v_2\}$. If both edges v_1v_3, v_2v_3 are present, then $\bigcup_{i=0}^{n-1}(\gamma^i W \cup \{z_i, v_i\})$ breaks sparsity. So suppose otherwise. We will show that one pair from $\{\{v_1, v_3\}, \{v_2, v_3\}\}$ is in neither a 0-critical or a 3-edge-critical set. Take a pair from $\{\{v_1, v_3\}, \{v_2, v_3\}\}$, say $\{y_0, z_0\}$. If $\{y_0, z_0\}$ is in a 0-critical set T_0, then $T = \bigcup_{i=0}^{n-1} \gamma^i T_0$ is 0-critical. We have already shown that U is not p-critical for $p \in \{0, \dots, n-1\}$, hence $n = \bar{k}_U \geq k_U \geq n$, therefore U must be n-critical and $i_L(U) = 0$. Then,

$$k_{U \cup T} \leq k_U + k_T - k_{U \cap T} = n - k_{U \cap T}.$$

Since $i_L(U) = 0$, $i_L(U \cap T) = 0$, so $k_{U \cap T} = \bar{k}_{U \cap T} \geq 3$. This gives $k_{U \cup T} \leq n - 3$. Then, $\bigcup_{i=0}^{n-1} \gamma^i v \cup U \cup T$ violates the sparsity of G (see Figure 5.11 for an illustration). Similarly, if $x_0 z_0 \notin E$, then $\{x_0, z_0\}$ is not in a 0-critical set, and we can apply the inductive argument from the beginning of the proof to deduce that no k copies of either pair is in a $(k-1)$-critical set.

Now suppose $\{y_0, z_0\}$ is in a 3-edge-critical set T_0. If $x_0 z_0 \in E$, then

$$\bar{k}_{U \cup T_0} = \bar{k}_U + \bar{k}_{T_0} - \bar{k}_{U \cap T_0} - d(U, T_0) \leq n + 3 - 2 - 1 = n.$$

Repeating this with $T_1 = \gamma T_0$, so $\bar{k}_{U \cup T_0 \cup T_1} \leq n$, until $T_{n-1} = \gamma^{n-1} T_0$ gives $\bar{k}_{U \cup T_0 \cup \dots \cup T_{n-1}} \leq n$. This union with $\bigcup_{i=0}^{n-1} \{\gamma^i v\}$ breaks sparsity of G. We note that the above contradiction would hold with $x_0 z_0 \notin E$ and $\bar{k}_{U \cap T_0} \geq 3$. Therefore assume $x_0 z_0 \notin E$ and $\bar{k}_{U \cap T_0} = 2$. Similar to the above, we arrive at a contradiction if $\{x_0, z_0\}$ is in a 3-edge-critical S_0 unless $\bar{k}_{U \cap S_0} = 2$. Then

$$\bar{k}_{S_0 \cup T_0} = \bar{k}_{S_0} + \bar{k}_{T_0} - \bar{k}_{S_0 \cap T_0} - d(S_0, T_0) \leq 3 + 3 - 2 = 4.$$

By definition, edge-criticality equal to 2 implies the vertex set is a singleton. Hence $U \cap T_0 = \{y_0\}$ and $U \cap S_0 = \{x_0\}$. Then $U \cap (S_0 \cup T_0) = \{x_0, y_0\}$, and since $x_0 y_0 \notin E$, $\{x_0, y_0\}$ is 4-edge-critical (as depicted in Figure 5.12).

We have

$$\bar{k}_{U \cup (S_0 \cup T_0)} \leq \bar{k}_U + \bar{k}_{S_0 \cup T_0} - \bar{k}_{U \cap (S_0 \cup T_0)} \leq n + 4 - 4 = n.$$

Repeating with $(S_i, T_i) = (\gamma^i S_0, \gamma^i T_0)$ for $i = 1, \dots n-1$ implies that $\bigcup_{i=0}^{n-1} (S_i \cup T_i) \cup U$

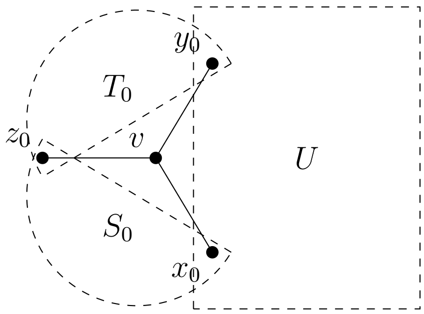

Figure 5.12: C_n-symmetric and n-edge-critical U with 3-edge-critical sets S_0 and T_0.

is n-edge-critical. Then adding $v, \dots, \gamma^{n-1}v$ breaks sparsity of G. Therefore one of the pairs $\{x_0, z_0\}$ and $\{y_0, z_0\}$ are not contained in a 3-edge-critical set. Without loss of generality, say that it is $\{x_0, z_0\}$. We can now build an inductive argument, assuming q copies of $\{x_0, z_0\}$ are not contained in a $(q+2)$-critical set for $1 \leq q \leq k_1$. As before with $\{x_0, y_0\}$, suppose $k_1 + 1$ copies of $\{x_0, z_0\}$ are contained in a (k_1+2)-edge-critical set R. As with W, $\gamma^{h_1}R \cup \cdots \cup \gamma^{h_n}R$ is C_n-symmetric. Hence we have a contradiction unless $\bar{k}_{\gamma^{h_1}R\cup\cdots\cup\gamma^{h_n}R} = n$ for some ordering $h_1, \dots, h_n$ of $\mathbb{Z}_n$. Recall that $U = \gamma^{j_1}W \cup \cdots \cup \gamma^{j_n}W$ and put $R^* = \gamma^{h_1}R \cup \cdots \cup \gamma^{h_n}R$. Then R^* is C_n-symmetric. For any set the edge-criticality is at least 2, hence by sparsity and symmetry $\bar{k}_{U\cap R^*} \geq n$. Therefore,

$$\bar{k}_{U\cup R^*} \leq \bar{k}_U + \bar{k}_{R^*} - \bar{k}_{U\cap R^*} \leq n + n - n = n,$$

which, on adding $\bigcup_{i=0}^{n-1}\{\gamma^i v\}$, violates the sparsity of G. This completes the induction for one of $\{x_0, y_0\}$ and $\{x_0, z_0\}$.

If X generates a subgroup of $\mathbb{Z}_n$, let m be such that $m \cdot |\langle X\rangle| = n$. Then, with $l = |\langle X\rangle| \geq k+1$, there is an ordering $j_1, \dots, j_l$ of $\langle X\rangle$ such that

$$\bar{k}_{\gamma^{j_1}W\cup\cdots\cup\gamma^{j_l}W} \leq |\gamma^{j_1}X \cup \cdots \cup \gamma^{j_l}X| + 3 - l + k = k + 3.$$

Let $W^* = \gamma^{j_1}W \cup \cdots \cup \gamma^{j_l}W$. By construction we have

$$W^* \cup \gamma W^* \cup \cdots \cup \gamma^{m-1}W^* = W \cup \gamma W \cup \cdots \cup \gamma^{n-1}W.$$

Whenever $n \geq 3$ is odd, we have that

$$m(k+3) = m(k+1) + 2m \leq ml + 2m = n + 2m < 2n.$$

Hence,

$$\bar{k}_{W\cup\cdots\cup\gamma^{n-1}W} = m\bar{k}_{\gamma^{j_1}W\cup\cdots\cup\gamma^{j_l}W} < 2n,$$

and since $W \cup \cdots \cup \gamma^{n-1}W$ is C_n-symmetric we may repeat the argument in the paragraph above to obtain a contradiction. □

It was only in the final paragraph of the proof where the proof does not apply to an arbitrary cyclic group. Consider a C_{2m}-tight graph G, and the set $X_2 = \{\{x_0, y_0\}, \{x_m, y_m\}\}$ which is contained in a 4-edge-critical set W. Then assuming each of the sets are disjoint, $W \cup \cdots \cup \gamma^{m-1}W$ is $4m = 2n$-edge-critical. A different approach is therefore required in these groups.

We can now present our main results.

Theorem 5.3.6. *Let n be a positive odd integer. A graph (G, ϕ) is C_n-tight if and only if every C_n-symmetrically connected component of G can be generated from $(\mathcal{P}_n, \phi_n)$ or (LC_n, ψ_n) by symmetrised 0-extensions and 1-extensions.*

Proof. Any C_n-symmetrically connected component generated from one of the base graphs by symmetrised 0-extensions and 1-extensions is C_n-tight. For the converse, we show by induction that any C_n-tight graph G can be generated from symmetrically connected copies of our base graphs. We may assume by Lemma 5.2.8 that G is C_n-symmetrically connected. Suppose the induction hypothesis holds for all graphs with $|V| < m$. Now let $|V| = m$ and suppose G is not isomorphic to one of the base graphs in Figure 5.3. For a tight graph, $2 \leq \delta(G) \leq 4$.

A degree 2 vertex is reducible by Lemma 5.3.1. A degree 3 vertex can have a loop and an edge incident to it or be adjacent to three vertices. The former is reducible by Lemma 5.3.1 and the latter by Lemma 5.3.5. When $\delta(G) = 4$, it can be shown (see the proof of Theorem 5.2.9) that there exists a vertex $w \in V$ incident to a loop and adjacent to two vertices. Then w is reducible by Lemma 5.3.4, completing the proof. □

Theorem 5.3.7. *Let n be a positive odd integer. A graph (G, ϕ) is C_n-isostatic if and only if it is C_n-tight.*

Proof. Since C_n-isostatic graphs are tight, necessity follows from Theorem 3.8.3. In Lemma 5.1.4, the base graphs $(\mathcal{P}_n, \phi_n)$ and (LC_n, ψ_n) ($n = 5$ depicted in Figure 5.3) are C_n-isostatic. Hence the sufficiency follows from Theorem 5.3.6 and Lemmas 5.1.1 and 5.1.3 by induction on $|V|$. □

Chapter 6

Further Research

In many respects the most natural continuation of my studies would have been to consider a reflection symmetry for linearly constrained frameworks in $\mathbb{R}^2$. The extension operations that we expect to be required for a classification were given in Section 5.1. We will discuss the difficulties that arise from our methods of classifying linearly constrained C_s-symmetric isostatic graphs. We consider reduction operations for C_s-tight graphs. We recall C_s-tight graphs have two alignments of fixed linear constraints, one perpendicular to the mirror which is represented positively in P_L^* and one along the mirror which is represented negatively in P_L^*, and the total number of such loops representing each case is counted by $l_{\sigma,+}$ and $l_{\sigma,-}$ respectively. From Table 3.12 we have (with e_σ, v_σ the number of fixed edges and vertices by the reflection) that $e_\sigma + l_{\sigma,+} = l_{\sigma,-}$ and there is no restriction on v_σ besides that each fixed loop must be incident to a fixed vertex, trivially giving $v_\sigma \geq \max\{l_{\sigma,+}, l_{\sigma,-}\}$.

In order to differentiate between the different possibilities for the image of a linear constraint by the reflection in our figures, we will identify linear constraints not fixed by the mirror by loops without signage, those fixed with normals which are preserved by the mirror by loops with a plus sign within, and those fixed with normals inverted by the mirror by loops with a minus sign in.

Here we introduce our base graphs for C_s-tight graph with such notation (see Figure 6.1). We show five small C_s-symmetric graphs that are tight. These are, reading left to right, top to bottom: $(\mathcal{P}_1, \omega_0)$ with two fixed loops, one which has its normal preserved and the other having its normal inverted; $(\mathcal{P}_1, \omega_1)$ with no

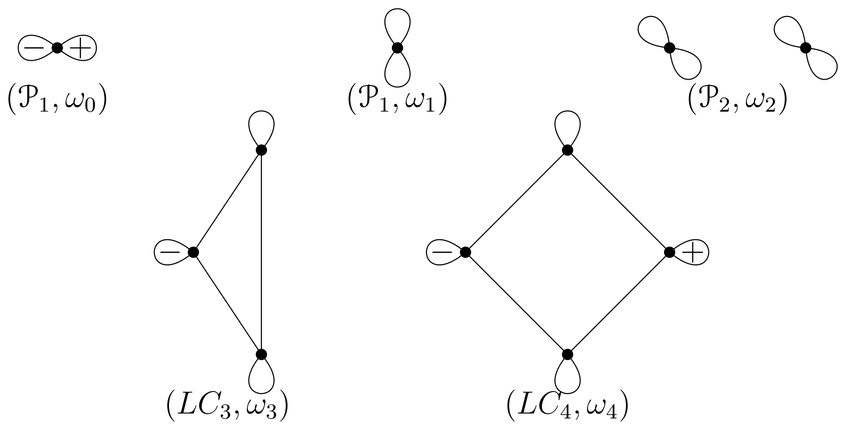

Figure 6.1: Some C_s-tight base graphs.

fixed loops; $(\mathcal{P}_2, \omega_2)$ with no fixed elements; (LC_3, ω_3) with one fixed edge, one fixed vertex, and one fixed loop which has its normal inverted; and (LC_4, ω_4) with two fixed vertices and two fixed loops, one of each kind. These could form the base graphs of a recursive construction.

We will again consider reduction operations previously seen, namely symmetrised 0-reduction, symmetrised 1-reduction, and additionally for C_s-symmetry, fixed 0-reductions and C_4-reductions. Like in $(2,2)$-C_s-tight graphs, much extra consideration is required around fixed vertices. In this setting, we also have fixed edges and loops, which prove to be a significant problem for a graph classification like in Section 4.5. As is often the case one of the most difficult part of this problem is reducing vertices of degree 3. In Figure 6.2 we give some of the variations in neighbours of a degree 3 vertex. In this figure we omit a fixed vertex with 3 neighbours all fixed since we believe this case should not arise once 0-reductions have been performed. In Figure 6.3 we demonstrate a C_s-tight graph in which all nodes lie on the mirror. This was not encountered when we studied the $(2,2)$-C_s-tight graphs. In this particular graph, a reduction of a fixed vertex to a fixed edge would preserve the $e_\sigma + l_{\sigma,+} = l_{\sigma,-}$ constraint, since the node is already incident to a fixed edge.

Another natural question to ask, is if there exists analogous results to Theorem 4.6.3 for our characterisations of isostatic linearly constrained frameworks in the plane. If C_n acts freely on the vertices, edges and loops, we *conjecture* that an infinitesimally rigid C_n-symmetric linearly constrained framework in $\mathbb{R}^2$ will always

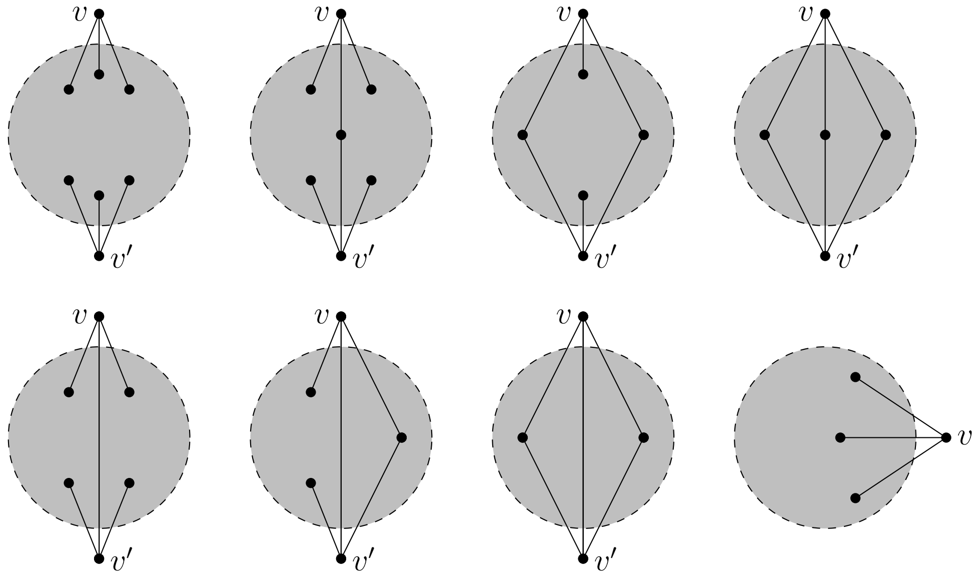

Figure 6.2: Some possibilities for nodes in a C_s-tight graph. In the bottom right graph v is a fixed vertex.

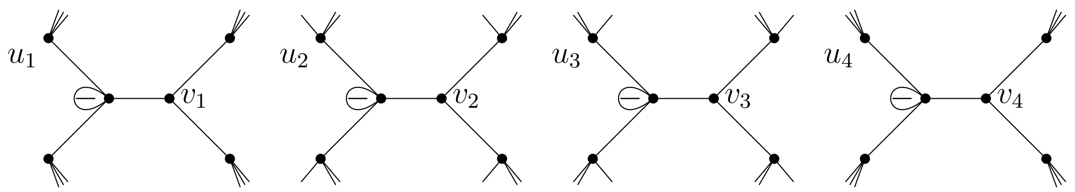

Figure 6.3: A C_s-tight graph where each node $\{v_1, v_2, v_3, v_4\}$ lies on the mirror. Some edges are shown incomplete, with u_1, u_2, u_3, u_4 forming a K_4, and similarly the other vertices not on the mirror form a K_4 with their translational copies (as drawn).

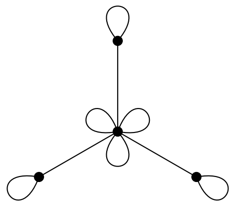

Figure 6.4: A C_3-symmetric rigid graph with no spanning isostatic C_3-symmetric subgraph.

have a spanning isostatic subframework with the same symmetry. If so, generic infinitesimal rigidity of C_n-symmetric linearly constrained frameworks in $\mathbb{R}^2$, where C_n acts freely, can be characterised in terms of symmetric isostatic subframeworks. This is in general not true; Figure 6.4 provides a small counterexample.

Further to this, when we first undertook our study of symmetric linearly constrained frameworks, we planned to investigate rigidity in $\mathbb{R}^d$. Hence given more time, symmetric linearly constrained frameworks in $\mathbb{R}^d$ would likely have been the next point of study in this thesis. Here, there is the widest selection of groups to choose from. Once again the inversion group appears the most likely to give promising results. There are no fixed edges to consider, and either there is no fixed vertex or loop, or the only fixed vertex will be pinned at the origin by d hyperplanes. While it would be natural to begin the study with the three dimensional case, one would hope that the results of [8, 20] could be used to give analogous results for similarly defined graphs in d dimensions.

One may instead choose to return to the symmetry of isostatic frameworks on surfaces. While new complexity would likely arise, it is possible that to give characterisations of $(2,2)$-C_{2v}-tight and $(2,2)$-C_{2h}-tight graphs requires no new techniques from those used in Chapter 4. One likely source of further difficulties comes when performing 1-reductions on such graphs. Checking for k-critical sets, for $k \in \{3,4,5,6\}$, which contain 1, 2, 3 and 4 (respectively) of the pairs of endpoints for the edges to be added in the reduction will require a careful approach. However, this is similar to the work undertaken in Section 5.3 considering the k-edge-critical sets.

Both the cone and elliptical cylinder present as natural best options to classify the class of $(2,1)$-$\tau(\Gamma)$-tight graphs (notation to specify which surface would appear useful). From our research, order two groups are the least problematic to work with, and all of the groups with isostatic frameworks in this setting are order 2. All of the groups contain one fixed edge (an odd number is guaranteed given the total edge parity and groups being order 2), which is different to anything studied in this thesis. The inversion and half-turn present as the best groups to first approach given the difficulties that can arise with fixed vertices. In particular, two fixed vertices can be adjacent which will change the problem from the cylinder case.

As seen in [10], there is a precise geometric correspondence between infinitesimal rigidity in the plane and that on the sphere, which has been extended to symmetric frameworks. A similar geometric correspondence between infinitesimal rigidity of normed planes and the cylinder looks likely to exist. It would be interesting to investigate whether an extension to symmetric frameworks can also be made here. Indeed, for $q \in (1, \infty)$, $q \neq 2$, isostatic frameworks in the normed plane ℓ_q^2 have a 'generic' characterisation which is $(2,2)$-tight just as the cylinder does. In the plane setting, the isometries differ to those on the cylinder, which one would expect to allow different groups to express minimally rigid graphs, and the representation theory to give different fixed edge and vertex counts.

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