Optimal Fare and Headway for a Demand Adaptive Paired-line Hybrid Transit System in a Rectangular Area with Elastic Demand

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Abstract: Demand adaptive paired-line hybrid transit systems that integrate fixed- and flex-route transit have emerged in the last decade and attracted increasing attention because of their potential to improve accessibility for passengers. To facilitate the operation of such a hybrid transit system, this study develops a model to determine the optimal fare and headways associated with fixed- and flex-route transit along a rectangular corridor. Compared with existing literature, the novelty of this study lies in designing the fare structure while simultaneously considering demand elasticity and passenger behaviour. A continuous approximation modelling approach is employed to derive the agency's and travellers' cost components. Using these, a nonlinear programming optimisation model is formulated to minimise the total user cost subject to the agency's nonnegative revenue constraints and passengers' route choice behaviour, which is characterised as a path-size logit model. Numerical experiments are performed using a stylised network to examine the properties of the model, in which the solution is obtained by combining a brute force method that enumerates headway and fare combinations and an iterative method that determines equilibrated passenger choices. The results show that as potential demand density increases, fare and headway fluctuate and drop, while the percentage of passengers choosing to ride on flex-route transit increases. In addition, there may be an optimal maximum offset distance, which is defined as the width of the corridor to be covered by the transit system, when the potential demand density is low, leading to minimum user cost and maximum travel demand within the service area.

Keywords: Public Transit; Transit Fare; Hybrid Transit System; Flex-route Transit; Elastic Demand

1. Introduction

Public transit is the backbone of urban transport and vital in promoting sustainable urban mobility. It can be broadly categorised into fixed-route and flex-route transit. The former refers to conventional transit, with predetermined fixed operational features, including fixed routes, stops, schedules, and frequencies (Amirgholy et al., 2017; Balke et al., 2000; Barnett, 1970; Eichler and Daganzo, 2006; Kocru and Hendrickson, 1983; Feng et al., 2024; Xu et al., 2023). In contrast, the latter allows transit vehicles to deviate from fixed settings to obtain various levels of flexibility and thus reduce operational costs and improve the level of service (Daganzo, 1978; Daganzo, 1984; Fu, 2002; Li et al., 2023; Pecherski Sabinik and Bar-Gera, 2022; Qiu et al., 2014; Quadrifoglio et al., 2008; Zhao and Dessouky, 2008; Zheng et al., 2019). The merit of fixed-route transit lies in its regularity, which commuters can learn to accommodate and memorise. The advantage of flex-route transit is its capacity to offer door-to-door services, rivalling or complementing emerging mobility services, such as ride-hailing and car-sharing. To achieve the benefits of both services, more and more public transit operators are deploying and supplementing flex-route services with conventional transit services, resulting in a hybrid transit system. This trend is attracting growing attention among those seeking to optimising the operation of flex-route and hybrid transit systems.

Hybrid transit is a combination of fixed-route and flex-route transit. There are various ways to integrate the two types of services. Stein (1978) divided a service area into zones and deployed fixed-route and flex-route transit for inter- and intra-zone services, respectively, so that passengers could use flex-route transit to reach an interchange station and then travel to another zone via fixed-route transit. In a similar setting, Aldaihani et al. (2004) determined the optimal number of zones to minimise a generalised cost. In the above two studies, flex-route transit can serve all passengers within a zone. In contrast, Chen and Nie (2017) proposed a demand adaptive paired-line hybrid transit (DAPL-HT) system that only allows flex-route transit to serve passengers whose distance from their origins/destinations to the nearest transfer stop exceeds a certain threshold. As a result, the entire service area is divided into non-overlapping sub-areas exclusively served by flex-route or fixed-route transit. Flex-route transit vehicles can be dispatched at intervals coordinated with fixed-route vehicles, which are paired in the DAPL-HT system. Transit operators can easily reallocate resources between fixed- and flex-route services to cater to varying demands when operating such a system. The DAPL-HT system was compared with the fixed-route transit proposed by Daganzo (2010) and the flex-route transit proposed by Nourbakhsh and Ouyang (2012), and the results demonstrated that the former outperforms the other two in various scenarios.

In terms of the methodologies used to optimise a transit system, mathematical programming models (e.g., bilevel programming, mixed integer nonlinear programming (MINP) and mixed integer linear programming (MILP)), and analytical continuous approximation (CA) models are prevalent. Table 1 chronologically summarises the literature on hybrid transit system optimisation and the corresponding methodologies. However, these studies focus on hybrid transit network design and aim to jointly optimise fixed-route and flex-route services to improve system performance without considering adjusting the transit fare, a fundamental variable in operating transit systems. Although Pinto et al. (2020) and Luo and Kang (2022) consider fares, they only set fares as input parameters, rather than variables, overlooking the possibility of adjusting the value of transit fares to impact passenger choices and the corresponding consequences for agency revenue. Practically speaking, a transit fare is more than a single monetary value; rather, how and to whom it is charged are also important. This is termed 'fare strategy' in this study. Commonly mentioned fare strategies in the literature and practice include a flat fare (Chien and Tsai, 2007; Kocru and Hendrickson, 1983; Lam and Zhou, 2000; Zhou et al., 2005), distance-based fare (Huang et al., 2016; Kilani et al., 2014; Li et al., 2009; Sun et al., 2019; Wang et al., 2018; Yook and Heaslip, 2015), time-based fare (Tirachini et al., 2014), service-based fare (Fleishman et al., 1996), sectional fare (Sun and Szeto, 2019; Wang and Qu, 2017), congestion fare (Liu et al. 2023), and free fare (Metaxatos, 2013). In brief, most existing studies focus on pricing for a fixed transit route (Bertsimas et al., 2020; Chien and Tsai, 2007; Guo et al., 2021; Huang et al., 2016; Huang, 2002; Sun and Szeto, 2019; Tirachini et al., 2014; Yang et al., 2023; Zhou et al., 2005). There is limited research on pricing flex-route transit (Kim and Schonfeld, 2015; Ma et al., 2022; Ni et al., 2021), not to mention the hybrid transit systems (Table 1). It is worth noting that the fixed-route fares in some countries, such as China, are controlled by the government, irrespective of demand densities and service quality. Given this reality, this study concentrates on designing the fare for flex-route transit in a hybrid transit system, for which, to the best of our knowledge, there lacks a systematic methodology.

[Table 1 near here]

Moreover, Table 1 reveals that most studies of hybrid transit assign passengers to paths based on their origin-destination to achieve a certain objective, rather than embedding a choice model to capture passengers' choice behaviour in response to service changes, as well as that studies of hybrid transit services rarely consider demand elasticities. However, these two factors should not be ignored when optimising a hybrid transit system, as passenger behaviour and ridership are sensitive to fare and other factors, such as walking, waiting and in-vehicle travel times. In this study, we aim to minimise the total user travel cost from the passengers' perspective while ensuring that the operators' revenue is non-negative. This can be done when government-funded agencies operate public transport with the objective of providing user-centred transit services. To a certain extent, the transit system investigated in this study is similar to the DAPL-HT system proposed by Chen and Nie (2017), who established the superiority of a hybrid transit system. Nonetheless, in contrast to Chen and Nie (2017), the hybrid system examined in this study focuses on a transit corridor that contains one fixed route and one flex route, which resembles a segment of the MTA Line 646 flex-route service in Los Angeles (Qiu et al., 2014; Spasovic et al., 1994; Zheng et al., 2019). Although this study builds on the work of Chen and Nie (2017), several nontrivial revisions and extensions have been made to overcome the limitations of previous studies.

First, we adopt an elastic demand function, instead of assuming a constant demand. This captures the reality that a better transit service can attract users and *vice versa*. Then, to solve the model with elastic demand, an iterative approach is devised. Second, we allow all passengers in the service area to freely choose whether to ride in a flex-route vehicle, lifting the restriction that only passengers whose distance from their origins/destinations to the nearest fixed stop exceeds a certain threshold are served by flex-route transit. This is more general than the model developed by Chen and Nie (2017). Third, we consider four types of passengers instead of two, as in Chen and Nie (2017), depending on the user's route choice for transit travel within the service area. This offers a higher resolution and is more realistic, but it necessitates a new derivation of agency and user costs. Finally, this study proposes a fare strategy in which the four types of passengers using the hybrid service pay different fares depending on the service they use. Meanwhile, the fare associated with the flex-route transit is explicitly considered as a decision variable. To sum up, the main contributions of this study include the following:

(1) Propose a method via which to optimise the fare for flex-route transit in a DAPL-HT system, considering demand elasticity.
(2) Design a differentiated fare strategy for four types of passengers based on their choices.
(3) Devise corresponding user costs via continuous approximation modelling.
(4) Formulate a nonlinear optimisation model to determine the fare and headway and develop a solution methodology to solve the model.
(5) Conduct experiments to investigate the model properties and demonstrate the effect of various parameters.

The remainder of the paper is organised as follows. Section 2 formally describes the DAPL-HT system and the problem investigated in this study. Section 3 devises the optimisation model for the problem and examines the properties of the model. The results of the numerical experiments are presented and discussed in Section 4. Finally, Section 5 summarises the main findings of this paper and provides future research directions.

2. System Description and Problem Statement

In this study, a realistic road network can be approximately represented using rectilinear movement, as shown in Figure 1 (Quadrifoglio et al., 2008). The service area has a width of $2s$ (km) and a length of D (km) and is delimited by two terminals, of which one may be located close to a subway station and the other at a high-demand density site. In line with Chen and Nie (2017), the DAPL-HT system serving the two terminals consists of fixed- and flex-route (demand-adaptive lines) transit. The flex-route transit is designed to be paired with the fixed-route transit and pick up passengers at temporary stops within the study area based on passengers' requests. For the fixed-route transit, there are N stations, and the distance between the adjacent stops equals $2s$. For the flex-route transit, the maximum allowable deviation width from the fixed route is s. Accordingly, the area serviced by the system is divided into n square grids, with a constant spacing of $2s$, satisfying $n = D/2s$. The headway for the fixed- and flex-route transit is denoted as H_1 (h) and H_2 (h), respectively, and influenced by the demand level. Other operational characteristics of the DAPL-HT system are as follows: the average cruising speed for all transit vehicles is given by v^{veh} (km/h). The time lost per fixed stop due to deceleration and the acceleration is τ_{lost} (s/stop), and the additional pick-up and drop-off time required per passenger for flex-route transit is τ_{pick} (s/pax).

[Figure 1 near here]

Regarding the system's operation, it is assumed that a control centre assembles passengers' travel requests and dispatches flex-route vehicles in real time. When a flex-route transit vehicle drives within the area between fixed stops k and $k+1$, the vehicle receives the requests from the area between fixed stops $k+1$ and $k+2$ (Figure 1). Then, the vehicle formulates and executes an optimal route and visits optimal temporary stops to pick up the accepted passengers in the area between fixed stops $k+1$ and $k+2$. After all the requests are granted, the vehicle visits stop $k+2$, which is the last stop within the area between stops $k+1$ and $k+2$, and then begins serving the following area. The vehicle confirms the requests it can handle without spending more time travelling between k and $k+1$ than allowed. Additional requests will be passed on to the next vehicle in line.

To simplify the analysis, we introduce the following assumptions. Except for the assumption related to elastic demand, other assumptions are adopted from Chen and Nie (2017) and Daganzo (2010):

A1. Passengers always use the stops closest to their origin and destination.

A2. The service area generates λ_0 potential passenger requests per hour per unit area, and the potential number of travellers is exogenous and independent of services within the area.

A3. An elastic demand function is adopted to represent travellers' responses to changes in the travel utility (i.e., the actual number of passengers using DAPL-HT is subject to the expected utility, which varies under different transit fare and headway settings).

A4. Passengers send their requests for the flex-route service to a control centre prior to their desired departure time, and these requests are accepted on a first-come first-serve basis.

A5. If there is no demand, the flex-route vehicle moves along a fixed route.

A6. There is no further transfer penalty between fixed- and flex-route transit services.

Based on the preceding description, we will determine the optimal headways associated with the two transit types and determine a fare strategy for passengers using the hybrid transit system to minimise total user cost, given the constraint that the agency's revenue is nonnegative. Regarding fare strategy, a flat fare is proposed for fixed-route transit, which is in line with prevailing practices in most cities in China. For flex-route transit, the fare is explicitly determined by the model, as optimising flex-route transit fare is identified as a research gap. The agency's profit is the difference between total fare income and operational cost. The former depends on the actual number of passengers served, and the latter is determined by vehicle travel distance and the fleet size required.

3. Modelling

This section begins by defining four types of passengers and an elastic demand function. Then, the agency cost associated with the DAPL-HT system is calculated. The corresponding cost components for each passenger type are derived via continuous approximation. Finally, an optimisation model is formulated to determine the optimal headways for the two transit modes, H_1 and H_2, and the fare for flex-route transit, f_2, to minimise total user cost. The main notations used are listed in Table 2.

[Table 2 near here]

3.1. Passenger classification and elastic demand

3.1.1. Four types of passengers

The passengers travelling in the area served by the DAPL-HT system can be divided into four types based on their route choice (see Figure 2). In the following description, the fixed stops closest to the passengers' origins (destinations) are called origin (destination) stops. Meanwhile, fixed-route transit is simplified to fixed, and flex-route transit is simplified to flex.

- Type I: Origin-*walk*-Origin Stop-*fixed*-Destination Stop-*walk*-Destination.
- Type II: Origin-*walk*-Origin Stop-*fixed*-Destination Stop-*flex*-Destination.

- Type III: Origin-*flex*-Origin Stop-*fixed*-Destination Stop-*walk*-Destination.
- Type IV: Origin-*flex*-Origin Stop-*fixed*-Destination Stop-*flex*-Destination.

[Figure 2 near here]

Denoting the proportions of the four types of passengers as p_{I} , p_{II} , p_{III} and p_{IV}, we have $p_{\mathrm{I}} + p_{\mathrm{II}} + p_{\mathrm{III}} + p_{\mathrm{IV}} = 1$. Type I passengers only ride the fixed-route transit line once during their journeys. Type II and Type III passengers travel via the fixed-route and flex-route transit lines each once. Type IV passengers use the flex-route transit line twice and the fixed-route transit line once along their journeys (see Figure 2). Note that except for those of Type I passengers, trips may involve transfers between fixed- and flex-route services.

User route choices and their probabilities can be characterised by the well-established multinomial logit (MNL) model (Ben-Akiva et al., 1985). However, this model may not suit this study, as explained below. As shown in Figure 2, a passenger's travel path is divided into three segments: the first and the third segments involve the option of walking or taking a flex-route vehicle, and the second involves taking a fixed-route vehicle. Clearly, there are overlaps among the four options. As a result, there is a correlation, leading to a violation of the independence of irrelevant alternatives assumption, which undergirds the MNL model. To address this, the path size logit (PSL) model is adopted to estimate the passengers' distribution among the four options (e.g., Ben-Akiva and Bierlaire, 1999; Duncan et al., 2020). Mathematically, it is given by the following equations:

$$p_j = \frac{\exp\left(-\theta u_j + \beta \ln\left(\gamma_j\right)\right)}{\sum_{k\in\Omega} \exp\left(-\theta u_k + \beta \ln\left(\gamma_k\right)\right)} = \frac{1}{\sum_{k\in\Omega} \left(\gamma_k / \gamma_j\right)^{\beta} \exp\left(-\theta\left(u_k - u_j\right)\right)} , \tag{1}$$

where

$$u_j = \left(w_A A_j + w_W W_j + w_T T_j \right) c^{\text{time}} + w_F F_j \ . \tag{2}$$

$$\gamma_j = \sum_{b \in \Gamma_j} \frac{t_b}{t_j} \frac{1}{\sum_{j \in \Omega} \delta_{bj}} \tag{3}$$

In Equation (1), θ and β are the scaling parameters, γ_j is the correction term for Type $j \in \Omega$ passenger routes. In Equation (2), A_j, W_j , H_j and F_j denote the average walking time, average waiting time, average in-vehicle travel time and fare, respectively, for Type j passengers. The cost component is calculated in Section 3.3, where w_A , w_W , w_T and w_F are the coefficients of the corresponding units. In Equation (3), t_b is the travel cost of segment b along a Type j route, t_j is the sum of the travel costs for all segments of a Type j route and the time component is calculated in Section 3.3. δ_{bj} is 1 if segment b is on a Type j route and 0 otherwise.

3.1.2. Elastic demand function

The actual demand density of flex-route transit services is affected by in-vehicle time, waiting time and fare (Kim and Schonfeld, 2015; Yang et al., 2021). In addition to flex-route transit, some passengers (Type I, Type II, and Type III) can walk rather than take flex-route vehicles to fixed stops to take fixed-route transit in the DAPL-HT system. Therefore, the actual demand density of DAPL-HT services is also affected by walking time. Overall, passenger demand is subject to the utility of the user cost, including walking time, waiting time, in-vehicle time and fare. Accordingly, following Sun and Szeto (2019), the logit-based elastic demand function is as follows:

$$\lambda = \lambda_0 - \psi\left(-\frac{1}{\theta}\ln\sum_{j\in\Omega}\left(\exp\left(-\theta\cdot u_j\right)\right)\right), \tag{4}$$

where u_j is given by Equation (2).

3.2. Agency cost

The agency's hourly operation costs for the DAPL-HT system depend on the fleet size and total travel distance. Mathematically, the operation cost per hour is defined by as follows:

$$C_{\text{oper}} = \sum_{i\in\Phi} c^{\text{veh}} m_i + c^{\text{dist}} d_i \ . \tag{5}$$

Essentially, we adopt a cost similar to that of Chen and Nie (2017) and Daganzo (1984) to compute the total expected service distance and fleet size. In what follows, we briefly elaborate on the computational process of the two components.

3.2.1. Service distance

The total expected service distance is calculated by dividing the round-trip distance by the headway. The calculation of the round-trip distance varies between the two types of services. For fixed-route transit, given round trip distance $2D$ and headway H_1, the expected hourly travel distance is computed as $2D/H_1$. For flex-route transit, the total distance per round trip includes the transverse and longitudinal distances that must be travelled to pick up and drop off passengers. The expected transverse distance is $2D$, which is the same as the round-trip distance for the fixed-route service. The expected longitudinal distance depends on the number of passenger trips served. It can be derived that the expected longitudinal distance per passenger is $2s/3$ and the number of passenger trips generated along the route per round trip is $2Ds\lambda H_2$. Meanwhile, only Type II, Type III, and Type IV passengers will be served by flex-route transit.

Thus, the total expected longitudinal distance per round trip for flex-route transit is given by $\frac{4}{3} D s^2 H_2 \lambda \sum_{j\in\{\mathrm{II},\mathrm{III},\mathrm{IV}\}} p_j$, which is consistent with that in Daganzo (1984). In addition, because the flex-route vehicles will stop at each stop, they will incur additional longitudinal distance. The average longitudinal distance between a flex-route vehicle and a fixed stop is $s/2$, and the number of fixed stops is N. Therefore, the average additional longitudinal distance is $2(N-1)s/2 = D/2$ per round trip.

Based on the above analysis, the average distance travelled per hour via fixed-route transit and flex-route transit can be calculated as follows:

$$d_i = \begin{cases} \dfrac{2D}{H_i}, & i=1 \\ \dfrac{2.5D}{H_i} + \dfrac{4Ds^2}{3} \sum\limits_{j\in\{\mathrm{II},\mathrm{III},\mathrm{IV}\}} \lambda p_j, & i=2 \end{cases}. \tag{6}$$

3.2.2. Fleet size

Given the total expected service distance per hour, the corresponding number of vehicles required can be obtained by dividing the total distance by the average distance travelled per vehicle within one hour, as expressed by the following:

$$m_i = \frac{d_i}{\overline{v}_i}, \forall i \in \Phi \ , \tag{7}$$

$$\frac{d_i}{\overline{v}_i} = \begin{cases} \dfrac{d_i}{v^{\mathrm{veh}}} + \dfrac{2(N-1)\tau_{\mathrm{lost}}}{H_i}, & i=1 \\ \dfrac{d_i}{v^{\mathrm{veh}}} + 4\tau_{\mathrm{pick}} Ds \sum\limits_{j\in\{\mathrm{II},\mathrm{III},\mathrm{IV}\}} \lambda p_j, & i=2 \end{cases}, \tag{8}$$

where $\overline{v}_i$ in Equation (7), to a certain extent, can be interpreted as an effective speed after considering all the time consumed during a round trip. Given such an interpretation, Equation (8) is derived from the perspective of time consumption. On the right-hand side of this equation, the first term computes the cruising time, and the second term computes the dwell time at the station and the time required for picking up and dropping off passengers, respectively, for fixed- and flex-route services (Chen and Nie, 2017; Daganzo, 2010). Then, by substituting Eq. (8) into Eq. (7), we can obtain the fleet size.

3.3. User cost

3.3.1. User trip time cost

The expected total user time is defined as the sum of three attributes: expected walking time (A), waiting time (W), and in-vehicle travel time (T). Mathematically, it is written as follows:

$$Z = A + W + T \tag{9}$$

In the following, we compute the cost attribute for each passenger type separately and then derive the total cost.

3.3.1.1. Walking time

The walking time for a certain type of passenger is computed according to their walking distance. The total expected walking time for a trip is expressed by the following:

$$A = \sum_{j \in \Omega} \frac{l_j}{v^{\text{walk}}} p_j = \sum_{j \in \Omega} A_j p_j \quad , \tag{10}$$

where l_j is the expected walking distance for a passenger of Type j, v^{walk} is the walking speed, A_j is the average walking time for a passenger of type j and p_j is the proportion of passenger type j. Because the proposed system operates in a rectangular corridor and the service area is divided into multiple square grids with a constant spacing of 2*s*, the walking distance between a point (origin or destination of passenger) and the nearest fixed stop is approximately the sum of the transverse and longitudinal distances. Given that the trip origins and destinations are uniformly and independently distributed in the service area, the expected walking distance for passengers is the distance from the centre of mass of the service area to the nearest fixed stop (i.e., $l = 2 \cdot s/2 = s$). For Type I passengers who only use fixed-route transit, we have $l_{\mathrm{I}} = 2l/v^{\mathrm{walk}}$. Type II and Type III passengers may walk once from the origin/destination to the nearest fixed stop, and thus, $l_{\mathrm{II}} = l_{\mathrm{III}} = l/v^{\mathrm{walk}}$. Because Type IV passengers take flex-route transit between a fixed stop and their origins/destinations without walking, their walking distance is zero. Finally, we can obtain the average walking time per passenger via Equation (10).

3.3.1.2. Waiting time

The total expected waiting time is computed via the following:

$$W = \sum_{j \in \Omega} W_j p_j \quad , \tag{11}$$

where W_j denotes the waiting time for passenger Type j . In general, we follow the prevailing literature and approximate the expected waiting time for both types of services as half of the headway (Aldaihani et al., 2004; Nourbakhsh and Ouyang, 2012). For passengers only using fixed-route transit (Type I passengers), this can be computed straightforwardly and is given by $W_{\mathrm{I}} = H_1/2$. For Type II and Type III passengers, because they travel via fixed-route transit and flex-route transit once during the journey, respectively, the waiting time is equal to the sum of the expected waiting time at the temporary origin stop and the destination fixed stop, indicating that $W_{\mathrm{II}} = W_{\mathrm{III}} = \left(H_1/2 + H_2/2\right)$. Finally, for Type IV passengers, the waiting time equals the sum of the expected waiting time at the temporary origin stop, origin fixed stop, and destination fixed stop, leading to $W_{\mathrm{IV}} = \left(H_1/2 + H_2\right)$.

3.3.1.3. In-vehicle travel time

The total expected in-vehicle travel time is computed as follows:

$$T = \sum_{i \in \Phi, j \in \Omega} \frac{r_j}{\overline{v}_i} p_j = \sum_{j \in \Omega} T_j p_j \ , \tag{12}$$

where r_j is the in-vehicle travel distance of passenger Type j and $\overline{v}_i$ is the vehicle travel speed for the two forms of transits. To derive the travel distance for each passenger type, we first denote the expected in-vehicle passenger travel distances for the fixed-route transit and flex-route transit as r_1 and r_2, respectively. Then, for Type I passengers, we have $r_1 p_{\mathrm{I}} = r_1 p_{\mathrm{I}}$, where the left-hand side represents the total expected passenger travel distance for Type I passengers and the right-hand side represents the total passenger in-vehicle travel distance for the fixed transit route for Type I passengers based on its proportion. Similarly, for the other three types, we have $r_j p_j = \left(r_1 + r_2\right) p_j, \forall j \in \left\{\mathrm{II}, \mathrm{III}\right\}$ and $r_{\mathrm{IV}} p_{\mathrm{IV}} = \left(r_1 + 2r_2\right) p_{\mathrm{IV}}$. In what follows, we will focus on computing r_1 and r_2.

To compute the expected in-vehicle travel distance for fixed-route transit, r_1, we adopt the method developed by Aldaihani et al. (2004). The number of all possible results from the origin fixed stop to the destination fixed stop requested by the passenger is denoted as n_1 [1]. d_n

[1] Refers to the number of sets of all fixed origin and destination stops that passengers may choose. For example, when there are three fixed stops in the system, the set of all the possible outcomes for a passenger is $\left\{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\right\}$, for example, $n_1 = 6$.

expresses the cumulative distance of all possible results of the passengers' requests on a fixed-route line with n zones. Thus, the expected in-vehicle travel distance for fixed-route transit can be calculated as follows:

$$r_1 = \frac{d_n D}{n_1} = \frac{\frac{D}{n}\sum_{k=1}^{n-1} 2k\left(n-k\right)}{\sum_{k=1}^{n-1} 2k} \quad . \tag{13}$$

To compute the expected in-vehicle travel distance of flex-route r_2, we adopt the method developed by Chen and Nie (2017). Computing the average in-vehicle travel distance is similar to computing the average walking distance, including average transverse and longitudinal distances. The average transverse distance between the origin/destination and the origin/destination fixed stop is $s/2$. The average longitudinal distance of flex-route transit depends on the number of Type II–IV (Type II, Type III, and Type IV) passengers. It is difficult to estimate the average longitudinal distance. For convenience, it is assumed that the ratio of the transverse distance to the total travel distance per hour for flex-route transit can represent the ratio of the passengers' expected in-vehicle travel distance to the total vehicle travel distance. In this way, it is possible to avoid directly calculating the offset longitudinal distance. Then, the transverse distance in a flex-route round trip is $2D$, and the total distance per hour of flex-route transit is $d_2 H_2$. Therefore, the ratio of the total distance and the average transverse distance for a flex-route vehicle is $d_2 H_2 / 2D$. Thus, the expected in-vehicle travel distance for flex-route transit can be estimated as follows:

$$r_2 = \frac{d_2 H_2}{2D}\frac{s}{2}, \tag{14}$$

where d_2 is defined in Equation (6). Therefore, the expected in-vehicle travel time per passenger trip for the four types of passengers is as follows:

$$T_j = \begin{cases} \dfrac{r_1}{\overline{v}_1}, & j = \mathrm{I} \\ \dfrac{r_1}{\overline{v}_1} + \dfrac{r_2}{\overline{v}_2}, & j = \mathrm{II} / \mathrm{III} \\ \dfrac{r_1}{\overline{v}_1} + 2\dfrac{r_2}{\overline{v}_2}, & j = \mathrm{IV} \end{cases} . \tag{15}$$

where $\overline{v}_1, \overline{v}_2$ are defined in Equation (7).Thus, the total in-vehicle travel time per passenger trip can be calculated based on Equation (12) accordingly.

3.3.2. Fare strategy

The average fare per passenger is given by the following:

$$F = \sum_{j \in \Omega} F_j p_j \quad , \tag{16}$$

where F_j denotes the fare for passenger Type j.

Let f_1 and f_2 denote the fares for fixed-route and flex-route transit. We propose the following differentiated fare strategies for the four types of passengers. Type I passengers only pay fares for fixed-route transit because they only use fixed-route services. Type II and Type III passengers pay fares for fixed- and flex-route transit because fixed- and flex-route transit are each used once. Type IV passengers travel via flex-route services twice and flex-route transit once. Mathematically, we have we following:

$$F_{\mathrm{I}} = f_1 , \tag{17}$$

$$F_{\mathrm{II}} = F_{\mathrm{III}} = f_1 + f_2 \text{, and} \tag{18}$$

$$F_{\mathrm{IV}} = f_1 + 2 f_2 . \tag{19}$$

3.4. Formulation

We now present the model with which to design the optimal headway for the hybrid transit system and fare for the hybrid transit service as follows:

$$\min C_{\text{user}}(H_1, H_2, f_2) = c^{\text{time}} Z + F \tag{20}$$

subject to

$$0 < H_1 \leq 1 \text{ , } 0 < H_2 \leq 1 \text{ , } f_{\min} \leq f_2 \leq f_{\max} \text{ ,} \tag{21}$$

$$2Ds\lambda F \geq C_{\text{oper}} \text{ .} \tag{22}$$

The objective function, Equation (20), is intended to minimise total user travel cost, where c^{time} denotes the value of time used to convert the user trip time cost components defined in Equation (9) into a single monetary value. Constraints (21) are the headway or frequency constraints for the three decision variables. Constraint (22) requires the agency's fare revenue, the left-hand side of the equation, to be greater than or equal to its operation cost.

4. Solution Algorithm

The proposed nonlinear programming model can be solved via heuristic algorithms (Chen and Nie, 2017). Because the focus of this study is gaining insight into the setting of fares and headway, instead of developing an advanced solution method, and the proposed model does not have that many variables, this study employs the brute-force method, which enumerates potential combinations of the decision variables at a small incremental interval. Then, after evaluating all enumerated solutions, those that satisfy the nonnegative agency revenue constraint are filtered and sorted to determine the optimal solution. This method can ensure that an acceptable optimal solution is obtained for practical implementation and analytical purposes, given that it would be unusual for an operator to set a transit fare and frequency with three or four decimal point accuracy.

Given an enumerated solution, the evaluation of the solution contains two major components: one is to compute the elastic demand function to obtain λ, and the other is to calculate the equilibrated flow distribution, following Equation (1), and the corresponding passenger travel cost. Considering that the demand density affects the setting of the transit fare and headway and, in turn, the resultant user cost under various fare and headway combinations impacts travel demand, this study devises an iterative algorithm, as described below.

Step 1. Initialisation.

Set iteration counter $k = 0$.

Set initial passenger flows distribution by evenly distributing the demand to the four types of passengers (i.e., $p_j^0 = 0.25, j \in \Omega$).

Step 2. Update demand.

Compute u_j^k based on p_j^k via Equation (2).

Compute λ^k under u_j^k via Equation (4).

Step 3. Update passenger flow, cost, and demand.

Compute equilibrated passenger demand distribution that satisfies Equation (1) using the method of successive averages (Liu et al., 2009), which terminates when the number of iterations reaches 100 or the absolute value of the difference between the current iteration's demand and the updated demand is less than 0.1.

Obtain the resultant flow distributions and corresponding utility, which are denoted by p_j^{k+1} and u_j^{k+1}, respectively.

Compute λ^{k+1} under u_j^{k+1} via Equation (4).

Step 4. Check convergence.

If $\left|\lambda_k - \lambda_{k-1}\right| < 0.001$ or $k > 100$, then stop the iteration and output the solutions.

Otherwise, set $k = k+1$ and go to Step 2.

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5. Numerical Studies

In this section, we first illustrate the effects of the headway and transit fare on the user cost and agency revenue. Then, we vary the demand density to examine the changes in the optimal headway and transit fare. Finally, a sensitivity analysis is conducted to investigate how the result is affected by the key input parameters, including offset s, line length D, parameter θ and fixed-route fare f_1. In all experiments, we set the range of fares for flex-route transit to $f_{\min} = 1$ CNY and $f_{\max} = 10$ CNY based on current practice (Shang et al., 2022). The fare for fixed-route transit, which is regulated by the local government in China and does not change with user travel time or demand density, is set at 2 CNY. w_T is considered as half of w_W ; w_A and w_F are the same as w_W and the default weight value w_W is 1. The value of time c^{time} generally represents a balance between the two interest groups (Daganzo, 2010). The maximum offset distance for a flex-route vehicle is typically set at 0.4–1.2 km (Zheng et al., 2021), with 0.6 km being initially chosen by default and a subsequent sensitivity analysis being performed for various offset distances. In addition, the operating cost per distance c^{dist} for flex-route transit vehicles was set to 80% of the value in Table 3, taking into account the fact that flex-route transit service can be provided with smaller vehicles. Without further specifications, the default values of the input parameters, which are taken from Chen and Nie (2017) and Sun and Szeto (2019), are listed in Table 3. The code for the experiment can be found at our GitHub repository: ronguo2/Hybrid_Opt_Rec (github.com).

[Table 3 near here]

5.1. User cost and agency revenue under variable potential demand density

To demonstrate the properties of the proposed model, we first visualise the combined effects of H_1, H_2, and f_2 on user cost and agency revenue under various potential demand densities. The results are plotted in Figures 3 and 4.

Figure 3 shows that the user cost is reduced with a decrease in H_1, H_2, and f_2 for cases of low potential demand (Figure 3a) and high potential demand (Figure 3b). It is obvious that a higher headway results in a longer waiting time for passengers, which, in turn, results in a higher travel cost, and a high fare certainly increases user cost. Comparing Figures 3a and 3b, the user costs are lower at high potential demand levels than at low potential demand levels when the combinations of headway and fare are the same. Moreover, the plots reveals that various combinations of flex-route transit fares and headways for the DAPL-HT system can lead to the same user cost under the same potential demand.

Figure 4 plots the agency revenue under low (Figure 4a) and high (Figure 4b) potential demand densities. It is evident that total revenue grows with an increase in potential demand density. This indicates that the agency's revenue can always be positive when the potential demand density is sufficiently high. It is worth noting that agency revenue does not increase with headway and fare (see Figure 4b). This is because although increasing headway reduces the operational cost of fixed-route transit, it may cause flex-route vehicles to travel a greater distance to pick up and drop off passengers, which increases operating costs. In addition, because the optimisation objective is constrained by agency revenue, it is necessary to filter out the combinations of H_1, H_2, and f_2 that lead to a negative agency revenue. In other words, we filter the user cost corresponding to a nonnegative operator benefit and then compare the user cost and find the minimum value.

[Figure 3 near here]

[Figure 4 near here]

5.2. Headway and fare combination

Figure 5 plots the combinations of headway and fare under various potential passenger demand densities. Figure 5a shows that H_1, H_2, and f_2 generally increase, with fluctuations, as potential demand density increases. The reasons for this are as follows: for the operator, this could be attributed to the scale of economics. Within a certain range of potential demand density levels (e.g., $\lambda_0 < 100$ pax/h/km^2), the increase in the demand density allows the agency to operate more fixed- and flex-route vehicles, as represented by a lower H_1 and H_2, with a lower price for Type II–IV passengers. However, as the potential demand density level continues to increase, lower headway may lead to higher operating costs. Therefore, even with higher demand density, it is necessary to increase the fare for flex-route transit to maintain nonnegative revenue. This is shown in Figure 5a, where a point of inflection is observed at $\lambda_0 = 160$ pax/h/km^2. Similarly, there are change points in which the change in headway differs from the overall trend ($\lambda_0 = 160 - 180$ pax/h/km^2).

For users, the expected average cost is always reduced, regardless of the increment in demand density (see Figure 5b). Correspondingly, the ratio of actual demand to potential demand increases. This is the joint effect on the part of fluctuating declines in the headway of the two transit types in the hybrid system and the basic fares for flex-route transit. Figure 5c shows how the proportions of the four types of passengers change with potential demand. As the potential demand density increases, the proportion of Type IV passengers increases, that of

Type I passengers decreases, and that of Type II and Type III passengers remain essentially the same. Overall, the percentage of passengers choosing flex-route transit increases as demand density grows (Type II–IV). Figure 5d shows the changes in fares for the four types of passengers. It is clear that Type II–IV passengers face the same increasing trend in fares with demand density, as shown in Figure 5a. This is related to the fare strategy developed (i.e., Type I passengers only pay for fares for fixed-route transit, Type II and Type III passengers pay fares for fixed- and flex-route transit, and Type IV passengers will pay both fixed- and two flex-route transit fares) and the fact that the fixed-route transit fares are the given values.

[Figure 5 near here]

5.3. Sensitivity analysis

In this section, we conduct a sensitivity analysis to examine how headway, the flex-route fare and total user cost in DAPL-HT systems are affected by key input parameters. The other input parameters used for the following tests are listed in Table 3.

5.3.1. Effect of flex-route transit offset distance

This section investigates the influence of the maximum vehicle offset dimensions s on the optimal result for the DAPL-HT system. We varied s from 0.4 to 1.2 km and plotted the results in Figure 6. Figure 6a shows that the user cost first decreases and then increases with increasing offset distance. The trend for the actual demand density λ is opposite that of user cost at a low potential demand density (Figure 6b). This clearly demonstrates the existence of an optimal setting for the side length of the corridor at low demand densities, which generates the most potential demand density. In contrast, as the potential demand density increases, user cost is positively correlated with offset distance for flex-route transit, while the corresponding actual demand is negatively correlated with offset distance.

The corresponding changes in H_1, H_2, and fares for the four types of passengers are plotted in Figures 6c–e. At a low level of demand density, the optimal headway in DAPL-HT systems decrease with increasing vehicle offset distance, while the optimal fare for Type II–IV passengers fluctuates. Type II–IV passenger fares vary more significantly with offset distance at low demand levels and less at medium and high demand levels. This is because at high demand levels, lower fares are sufficient to satisfy the agency revenue constraint. Under the setting specified in this situation, the optimal maximum offset distance s is 0.6 km, the corresponding headways are $H_1 = 0.20$ h, $H_2 = 0.16$ h and the fare for flex-route transit is 5.8 CNY (i.e., Type I passengers fare is 2 CNY, Type II/Type III passengers fare is 7.8 CNY, Type IV passengers fare is 13.6 CNY). At higher demand density levels, the trend for optimal headway for fixed- and flex-route transit decreases with increasing offset distance. In addition, Figure 6f shows that the percentage of riders on flex-route transit (Type II–IV passengers) increased with s. In other words, the larger the set offset distance is, the more popular flex-route transit is in the hybrid system.

[Figure 6 near here]

5.3.2. Effect of the side length of the DAPL-HT system service area

This section investigates the influence of the side length of the DAPL-HT system service area, D, on the optimal result for the system. We varied line length D from 5 to 20 km and plotted the results in Figure 7.

[Figure 7 near here]

As expected, Figures 7a and b show that user cost increases with line length at variable demand densities, with the opposite being true for actual demand density. The corresponding changes in passenger fares and ridership ratios for flex-route (Type II–IV passengers) are plotted in Figures 7c and d. At a low demand density, the optimal fare for Type II–IV passengers always increases with line length, and the proportion of Type II–IV passengers decreases, while at medium and high demand densities, fares and ridership ratios remain stable. This is because the longer the line is, the higher the operational cost is. Thus, to secure non-negative returns for the agency, fares must be high at low demand densities. In contrast, at high demand densities, the operator's revenue neutrality can be achieved with a low fare. Therefore, at high demand densities, fares, increase, and at medium and high demand densities, they remain stable, which affects the proportions of Type II–IV passengers.

5.3.3. Effect of θ

This section investigates how the changes in the parameter of the logit model affect the solution. θ was varied from 0.1 to 0.5 at $\lambda_0 = 100$ pax/h/km^2. The results are plotted in Figure 8.

[Figure 8 near here]

Figure 8a shows that the fares for Type II–IV decrease as θ increases. This is because the larger θ is, the greater the proportion of passengers who choose flex-route transit service in DAPL-HT systems, and also because the lower fare reduces user cost, as expected and shown in Figure 8b. The differences in minimum user cost under different values of θ can be significant. For example, when $\theta = 0.1$, the minimum user cost is 29.07 CNY; when θ increases to 0.5, the user cost is 23.70 CNY, which represents a difference of about 18%. A higher value of θ can be understood as passengers having more information about travel costs (Sun and Szeto, 2019). That is, the more trip information is available to passengers, the lower the cost to the user will be.

5.3.4. Effect of fixed-route transit fare

The section investigates how changes in the fare for fixed-route transit affect passenger fares overall, passenger choice and user cost. The results are plotted in Figure 9, which shows that adjusting the fixed-route fare affects passengers' route choice and fares overall, especially at low demand densities ($\lambda_0 \le 40$ pax/h/km^2). Given a certain potential demand density, when the fixed-route fare is low ($f_1 = 0.1$ CNY), the proportion of Type I passengers increases, while the proportion of Type II–IV passengers decreases (Figure 9b). In this case, fares for Type II–IV passengers must be raised to satisfy the agency revenue constraint (Figure 9a), leading to a larger total user cost (Figure 9b). Therefore, a low fixed-route transit fare is not desirable from a total user cost perspective.

[Figure 9 near here]

6. Conclusions

This study investigated paired-line hybrid transit systems, which integrate fixed-route and flex-route transit. Given that the transit fare had not yet been considered as a decision viable under elastic demand, a nonlinear optimisation model was devised via continuous approximation. Specifically, we consider a transit corridor and divide passengers into four types depending on the services they use. Accordingly, a fare strategy such that the four types of passengers pay differentiated fares is proposed. Then, user cost components are derived to compute the utilities associated with each type of service. The utilities are adopted to compute the passenger flow distribution, in which passengers' travel choices are characterised by using a path-size logit model that considers the overlap between various types of services. Afterwards, a nonlinear programming model is formulated to determine the flex-route transit fare and headways for both flex and fixed transit. Finally, we employ a brute-force method to obtain the solution and thus analyse the properties of the model via various numerical studies.

This study has had several interesting findings. First, we demonstrate the impact of potential demand density on the setting of transit fares and ridership. As potential demand density increases, both the headway and optimal transit fare decrease, with fluctuations. Second, the demand density level significantly affects passengers' route choices. As potential demand density increases, the proportion of passengers who do not choose flex-route transit decreases (Type I), the proportion of passengers who choose to ride flex-route vehicles once remains stable (Type II and Type III) and the proportion of passengers who choose to use flex-route transit twice increases (Type IV). Third, under a low potential demand density, there is an optimal offset distance that helps achieve the maximum actual travel demand. In contrast, at high potential demand densities, the actual demand density decreases with increasing offset distance for flex-route transit, and user cost increases. Regardless of the level of potential demand density, the proportion of Type II–IV passengers increases with increasing offset distance for flex-route transit. Fourth, as the side length of the DAPL-HT system service area increases, the proportion of Type II–IV passengers decreases, and fares increase at low potential demand densities, while remaining essentially unchanged at medium and high potential demand densities. Regardless of the level of potential demand density, actual demand density decreases as side length increases. Fifth, both the optimal flex-route transit fare and headways of the two transit modes do not necessarily monotonically change with potential travel demand increases. Finally, setting a low fixed-route transit fare would increase total user cost, and the impact of this choice would be more significant at a lower potential demand density.

This study's results present several avenues for new research. First, some assumptions can be relaxed to model more complicated and realistic scenarios. For example, passenger demand may not be uniformly distributed within the area (Qian et al., 2024). Meanwhile, when modelling passengers' choice behaviour, it is possible to explore other features, such as bounded rationality (e.g., Jiang and Ceder, 2021; Jiang et al., 2022; Jiang, 2024) and reliability (e.g., Szeto et al., 2011; Szeto et al., 2013; Jiang and Szeto, 2016; Jiang, 2022). Second, different fare structures and interchange discounts can be further investigated. For example, we can design and examine other fare strategies and embrace fare incentives (e.g., Tang et al., 2020; Yang et al., 2023). Third, the solution algorithm used in this study can be applied in the planning stage, and a more efficient algorithm could allow real-time applications, such as synchronising the timetable for transfer passengers (e.g., Lee et al., 2022). Fourth, other network route structures, such as radial network structures (Chen and Nie, 2018) and chet networks (Pravinvongvuth and Matarage, 2023), can be further investigated. Finally, the linkage mechanism between government subsidies and fares is also a direction that can be further explored in the future (e.g., Shu and Durango-Cohen, 2021).

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