KERNELS OF OPERATORS ON BANACH SPACES INDUCED BY ALMOST DISJOINT FAMILIES

BENCE HORVÁTH AND NIELS JAKOB LAUSTSEN

In memoriam: H. G. Dales (1944–2022)

Abstract. Let $\mathcal{A}$ be an almost disjoint family of subsets of an infinite set Γ, and denote by $X_{\mathcal{A}}$ the closed subspace of $\ell_\infty(\Gamma)$ spanned by the indicator functions of intersections of finitely many sets in $\mathcal{A}$. We show that if $\mathcal{A}$ has cardinality greater than Γ, then the closed subspace of $X_{\mathcal{A}}$ spanned by the indicator functions of sets of the form $\bigcap_{j=1}^{n+1} A_j$, where $n \in \mathbb{N}$ and $A_1, \ldots, A_{n+1} \in \mathcal{A}$ are distinct, cannot be the kernel of any bounded operator $X_{\mathcal{A}} \to \ell_\infty(\Gamma)$. As a consequence, we deduce that the subspace

$$\big\{x \in \ell_\infty(\Gamma) : \text{the set } \{\gamma \in \Gamma : |x(\gamma)| > \varepsilon\} \text{ has cardinality smaller than } \Gamma \text{ for every } \varepsilon > 0\big\}$$

of $\ell_\infty(\Gamma)$ is not the kernel of any bounded operator on $\ell_\infty(\Gamma)$; this generalises results of Kalton and of Pełczyński and Sudakov.

The situation is more complex for the Banach space $\ell_\infty^c(\Gamma)$ of countably supported, bounded functions defined on an uncountable set Γ. We show that it is undecidable in ZFC whether every bounded operator on $\ell_\infty^c(\omega_1)$ which vanishes on $c_0(\omega_1)$ must vanish on a subspace of the form $\ell_\infty^c(A)$ for some uncountable subset A of ω_1.

To appear in *Houston Journal of Mathematics.*

1. Introduction and statement of the main result

The problem of classifying all complemented subspaces — that is, the closed subspaces which arise as the kernels of bounded idempotent operators — of a given Banach space is fundamental in the study of Banach spaces. This problem has a very natural generalisation, obtained by omitting the word “idempotent”, originally formulated in [9]:

Question 1.1. Which closed subspaces of a Banach space X can be realised as the kernel of a bounded operator $X \to X$? In particular, for which Banach spaces X is it true that every closed subspace of X is the kernel of some bounded operator $X \to X$?

In the positive direction, White and the second author [9, Proposition 2.1] observed that whenever W is a closed subspace of a Banach space X such that X/W is separable, there is a bounded operator $X \to X$ whose kernel is W. Consequently, Question 1.1 is of interest only for non-separable Banach spaces X.

The focus in [9] was on reflexive spaces; its main result [9, Corollary 2.8] states that the dual space X^* of Wark's non-separable reflexive Banach space X with “few operators” introduced in [15] contains a closed subspace which is not the kernel of any bounded

2020 *Mathematics Subject Classification.* Primary 46B26, 46E15, 47B01; Secondary 18A30 47B38.

Key words and phrases. Non-separable Banach space, bounded operator, kernel, almost disjoint family.

operator $X^* \to X^*$. More recently, Arnott and the second author [2] have shown that for $1 < p < \infty$ and an arbitrary index set Γ, every closed subspace of $X = \ell_p(\Gamma)$ and $X = c_0(\Gamma)$ is the kernel of a bounded operator $X \to X$, whereas $\ell_1(\Gamma)$ contains closed subspaces which are not the kernel of any bounded operator $\ell_1(\Gamma) \to \ell_1(\Gamma)$ whenever Γ is uncountable.

Many years prior to this work, Kalton [8, Proposition 4] showed that c_0 is not the kernel of any bounded operator $\ell_\infty \to \ell_\infty$, thereby generalising Phillips' Theorem that c_0 is not complemented in ℓ_∞. More precisely, building on the approach Whitley [16] used in his simplified proof of Phillips' Theorem, Kalton proved that whenever the kernel of a bounded operator $T\colon \ell_\infty \to \ell_\infty$ contains c_0, there is an infinite subset M of $\mathbb{N}$ such that $\ell_\infty(M) \subseteq \ker T$, where we have identified $\ell_\infty(M)$ with the subspace $\{x \in \ell_\infty : x(n) = 0\ (n \in \mathbb{N} \setminus M)\}$ of ℓ_∞.

Our main theorem generalises Kalton's result in two ways: first, it replaces the index set $\mathbb{N}$ with an arbitrary infinite set Γ, and second, it applies to operators defined on certain $C(K)$-subspaces of $\ell_\infty(\Gamma)$ that have found a number of significant applications recently. Before we can state it precisely, we must introduce some notation and terminology.

Since our results depend only on the cardinality of the index set Γ, throughout this paper Γ will denote an infinite cardinal number unless otherwise stated, and we use the notation

$$[\Gamma]^\Gamma = \{A \subseteq \Gamma : |A| = \Gamma\} \qquad \text{and} \qquad [\Gamma]^{<\Gamma} = \{A \subseteq \Gamma : |A| < \Gamma\},$$

where $|A|$ denotes the cardinality of the set A. A family $\mathcal{A} \subset [\Gamma]^\Gamma$ is called *almost disjoint* if $A \cap B \in [\Gamma]^{<\Gamma}$ whenever $A, B \in \mathcal{A}$ are distinct. A famous result of Sierpiński [14] states that $[\Gamma]^\Gamma$ contains an almost disjoint family of cardinality greater than Γ.

We write $\mathbb{1}_A$ for the indicator function of a subset A of Γ, considered as an element of $\ell_\infty(\Gamma)$. The following closed subspace of $\ell_\infty(\Gamma)$ induced by a family $\mathcal{A}$ of subsets of Γ is at the heart of our work:

$$X_{\mathcal{A}} = \overline{\operatorname{span}}\,\{\mathbb{1}_{\bigcap_{j=1}^n A_j} : n \in \mathbb{N},\ A_1, \ldots, A_n \in \mathcal{A}\} = \overline{\operatorname{span}}\,\{\mathbb{1}_A : A \in \mathcal{A} \cup \mathcal{A}_{\Cap}\}, \tag{1.1}$$

where we have introduced the symbol

$$\mathcal{A}_{\Cap} = \left\{\bigcap_{j=1}^{n+1} A_j : n \in \mathbb{N},\ A_1, \ldots, A_{n+1} \in \mathcal{A},\ A_j \neq A_k \text{ for } j \neq k\right\} \tag{1.2}$$

for the collection of finite intersections of at least two distinct sets in $\mathcal{A}$. With this notation at hand, we can state our main result as follows.

Theorem 1.2. *Let $\mathcal{A} \subset [\Gamma]^\Gamma$ be an almost disjoint family of cardinality greater than Γ, and suppose that $T\colon X_{\mathcal{A}} \to \ell_\infty(\Gamma)$ is a bounded operator for which $T\mathbb{1}_A = 0$ for every $A \in \mathcal{A}_{\Cap}$. Then $\mathcal{A}$ contains a subset $\mathcal{B}$ for which $|\mathcal{A} \setminus \mathcal{B}| \leqslant \Gamma$ and $T\mathbb{1}_B = 0$ for every $B \in \mathcal{B}$.*

Note that $\mathbb{1}_B \notin \overline{\operatorname{span}}\,\{\mathbb{1}_A : A \in \mathcal{A}_{\Cap}\}$ for every $B \in \mathcal{A}$ (see Lemma 2.1 below for details), so Theorem 1.2 implies that $\overline{\operatorname{span}}\,\{\mathbb{1}_A : A \in \mathcal{A}_{\Cap}\}$ is a closed subspace of $X_{\mathcal{A}}$ which cannot be the kernel of any bounded operator $X_{\mathcal{A}} \to \ell_\infty(\Gamma)$. This answers the second part of Question 1.1 in the negative for both $X = X_{\mathcal{A}}$ (by applying Theorem 1.2 to the composition of an operator $X_{\mathcal{A}} \to X_{\mathcal{A}}$ with the inclusion map $X_{\mathcal{A}} \to \ell_\infty(\Gamma)$) and $X = \ell_\infty(\Gamma)$ (by applying Theorem 1.2 to the restriction to $X_{\mathcal{A}}$ of an operator $\ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$).

In fact, with a small amount of extra effort, we can prove a variant of the latter conclusion that will simultaneously generalise the previously mentioned result of Kalton [8, Proposition 4] to uncountable index sets and a theorem of Pełczyński and Sudakov [12, Theorem 1] stating that the closed subspace

$$\ell_\infty^{<}(\Gamma) = \bigl\{x \in \ell_\infty(\Gamma) : \{\gamma \in \Gamma : |x(\gamma)| > \varepsilon\} \in [\Gamma]^{<\Gamma} \text{ for every } \varepsilon > 0\bigr\} \tag{1.3}$$

is not complemented in $\ell_\infty(\Gamma)$ for any infinite cardinal number Γ.

Corollary 1.3. *Let $T\colon \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ be a bounded operator whose kernel contains $\ell_\infty^{<}(\Gamma)$. Then there exists an almost disjoint family $\mathcal{B} \subset [\Gamma]^\Gamma$ of cardinality greater than Γ such that $X_{\mathcal{B}} \subseteq \ker T$. In particular, $\ell_\infty^{<}(\Gamma)$ is a closed subspace of $\ell_\infty(\Gamma)$ which is not the kernel of any bounded operator $\ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$.*

Remark 1.4. We observe that $\ell_\infty^{<}(\mathbb{N}) = c_0$, so Corollary 1.3 is a genuine generalisation of Kalton's result, which corresponds to $\Gamma = \mathbb{N}$.

We shall prove Theorem 1.2 and Corollary 1.3 in Section 2, using an elementary combinatorial identity (2.5) as our key tool. In Section 3 we explore how our results might carry over to the Banach space $\ell_\infty^c(\Gamma)$ of countably supported, bounded functions defined on an uncountable set Γ. Our most interesting conclusion is that the following statement is independent of ZFC:

(1.4) *Let $T\colon \ell_\infty^c(\omega_1) \to \ell_\infty^c(\omega_1)$ be a bounded operator whose kernel contains $c_0(\omega_1)$. Then $\ell_\infty^c(A) \subseteq \ker T$ for some uncountable subset A of ω_1.*

Remark 1.5. (i) Johnson and Lindenstrauss [7] initiated the study of Banach spaces induced by an almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$. In this case it is conventional to consider a slightly larger space than our space $X_{\mathcal{A}}$, namely $\overline{X_{\mathcal{A}} + c_0}$. Clearly, the two spaces are equal if and only if every singleton $\{n\}$, for $n \in \mathbb{N}$, arises as the intersection of finitely many sets in $\mathcal{A}$. Otherwise, by treating the atoms of $\mathcal{A}_{\Cap}$ as single points, one can replace $\mathcal{A}$ with an almost disjoint family for which equality holds. Therefore, we shall not distinguish between the two approaches and focus on the space $X_{\mathcal{A}}$ defined by (1.1).

(ii) In the general case where $\mathcal{A} \subset [\Gamma]^\Gamma$ is an almost disjoint family for an infinite cardinal number Γ (possibly uncountable), the Banach space $X_{\mathcal{A}}$ has a standard representation as a $C(K)$-space.

The easiest way to see this is to begin with the case of complex scalars: then $\ell_\infty(\Gamma)$ is a commutative C^*-algebra with respect to the pointwise operations, and $X_{\mathcal{A}}$ is a closed, self-adjoint subalgebra of it. Therefore the commutative Gelfand–Naimark Theorem implies that $X_{\mathcal{A}}$ is isometrically $*$-isomorphic to $C_0(K_{\mathcal{A}})$ for some locally compact Hausdorff space $K_{\mathcal{A}}$, which can be (abstractly) described as either the maximal ideal space of the algebra $X_{\mathcal{A}}$ or the Stone space of the Boolean subring generated by $\mathcal{A}$ within the power set of Γ.

The analogous conclusion for real scalars follows by considering the self-adjoint parts of the above C^*-algebras because a real-valued function in either $\ell_\infty(\Gamma)$, $X_{\mathcal{A}}$ or $C_0(K_{\mathcal{A}})$ is simply a self-adjoint element of the corresponding complex algebra.

Combining the isomorphism $X_{\mathcal{A}} \cong C_0(K_{\mathcal{A}})$ with Theorem 1.2, we conclude that $C_0(K_{\mathcal{A}})$ contains a closed subspace which is not the kernel of any bounded operator $C_0(K_{\mathcal{A}}) \to \ell_\infty(\Gamma)$, thereby answering the second part of Question 1.1 in the negative for $X = C_0(K_{\mathcal{A}})$.

(iii) For $\Gamma = \mathbb{N}$, the topological space $K_{\mathcal{A}}$ has a long and illustrious history going back nearly a century to the work of Alexandroff and Urysohn [1]. It has since appeared under many names, including *Mrówka space, Ψ-space, AU-compactum* and *Isbell–Mrówka space*. One reason the case $\Gamma = \mathbb{N}$ is attractive is that $K_{\mathcal{A}}$ has a nice concrete description: it contains two kinds of points, labelled x_n for $n \in \mathbb{N}$ and y_A for $A \in \mathcal{A}$, with the former being isolated, while a neighbourhood basis for the latter consists of all sets of the form $\{x_n : n \in A \setminus F\} \cup \{y_A\}$, where $F \subset A$ is finite.

2. The proofs of Theorem 1.2 and Corollary 1.3

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ be the scalar field. As usual, the *support* of an element $x \in \ell_\infty(\Gamma)$ is $\operatorname{supp} x = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$. For an almost disjoint family $\mathcal{A} \subset [\Gamma]^\Gamma$, define

$$V_{\mathcal{A}} = \operatorname{span}\{\mathbb{1}_A : A \in \mathcal{A}_\Cap\} \qquad \text{and} \qquad W_{\mathcal{A}} = \overline{V_{\mathcal{A}}}. \tag{2.1}$$

We shall use the following simple fact several times, so we state it formally for ease of reference.

Lemma 2.1. *Let $\mathcal{A} \subset [\Gamma]^\Gamma$ be an almost disjoint family. Then*

$$\operatorname{supp} v \in [\Gamma]^{<\Gamma} \qquad (v \in V_{\mathcal{A}}). \tag{2.2}$$

Suppose that $|\mathcal{A}| > \Gamma$. Then $W_{\mathcal{A}} \neq X_{\mathcal{B}}$ for every subset $\mathcal{B}$ of $\mathcal{A}$.

Proof. The almost disjointness of $\mathcal{A}$ implies that $\mathcal{A}_\Cap \subseteq [\Gamma]^{<\Gamma}$. This proves (2.2) because every element of $V_{\mathcal{A}}$ vanishes outside a finite union of sets in $\mathcal{A}_\Cap$.

Suppose that $|\mathcal{A}| > \Gamma$. Then $\mathcal{A}$ cannot consist of disjoint sets, so $W_{\mathcal{A}} \neq \{0\} = X_\emptyset$. Now consider a non-empty subset $\mathcal{B}$ of $\mathcal{A}$, and take $B \in \mathcal{B}$. The indicator function $\mathbb{1}_B$ belongs to $X_{\mathcal{B}}$ by definition. We claim that $\mathbb{1}_B \notin W_{\mathcal{A}}$, and therefore $X_{\mathcal{B}} \not\subseteq W_{\mathcal{A}}$. More precisely, we shall show that $\mathbb{1}_B$ has distance 1 to the dense subspace $V_{\mathcal{A}}$ of $W_{\mathcal{A}}$. Indeed, the first part of the proof shows that $\operatorname{supp} v \in [\Gamma]^{<\Gamma}$ for every $v \in V_{\mathcal{A}}$, whereas $|B| = \Gamma$, so B contains an element β such that $v(\beta) = 0$. Consequently $\|\mathbb{1}_B - v\|_\infty \geqslant |(\mathbb{1}_B - v)(\beta)| = 1$, which proves the claim. □

Our next lemma can be viewed as a generalisation of the fact that $c_0(\mathbb{R})$ embeds isometrically into ℓ_∞/c_0 (see, *e.g.,* [3, Theorem 1.25]). Before we state it, recall from Remark 1.5(ii) that $X_{\mathcal{A}}$ is a closed subalgebra of $\ell_\infty(\Gamma)$. Since $W_{\mathcal{A}}$ is a closed ideal of $X_{\mathcal{A}}$, the quotient $X_{\mathcal{A}}/W_{\mathcal{A}}$ is a commutative Banach algebra (in fact a C^*-algebra in the complex case). We write $(e_A)_{A\in\mathcal{A}}$ for the unit vector basis of $c_0(\mathcal{A})$; that is, $e_A = \mathbb{1}_{\{A\}} \in \ell_\infty(\mathcal{A})$ for $A \in \mathcal{A}$. We use this notation to ensure a clear distinction between the elements $e_A \in c_0(\mathcal{A})$ and $\mathbb{1}_A \in X_{\mathcal{A}}$.

Lemma 2.2. *Let $\mathcal{A} \subset [\Gamma]^\Gamma$ be an almost disjoint family. Then the map given by*

$$\psi(e_A) = \mathbb{1}_A + W_{\mathcal{A}} \qquad (A \in \mathcal{A}) \tag{2.3}$$

extends uniquely to an isometric algebra isomorphism $\psi\colon c_0(\mathcal{A}) \to X_{\mathcal{A}}/W_{\mathcal{A}}$.

In the proof we require the following elementary combinatorial lemma, which may be thought of as a weighted version of the Inclusion–Exclusion Principle, as it is stated in [11, Proposition 6.62], for example. Results of this kind are well-known in the literature, but since we have not been able to find this particular variant anywhere or deduce it from known results, we include a short, self-contained proof of it in Appendix A.

Lemma 2.3. *Let $A_1, \ldots, A_m$ be distinct subsets of Γ for some $m \in \mathbb{N}$, set*

$$A_j^d = A_j \setminus \bigcup_{\substack{k=1\\ k\neq j}}^{m} A_k \qquad (j \in \{1, \ldots, m\}), \tag{2.4}$$

and take $\sigma_1, \ldots, \sigma_m \in \mathbb{K}$. Then

$$\sum_{j=1}^{m} \sigma_j \mathbb{1}_{A_j^d} = \sum_{\emptyset \neq N \subseteq \{1,\ldots,m\}} (-1)^{|N|+1} \Big(\sum_{j\in N} \sigma_j\Big) \mathbb{1}_{\bigcap_{j\in N} A_j}. \tag{2.5}$$

Proof of Lemma 2.2. We can define a linear map $\psi\colon \operatorname{span}\{e_A : A \in \mathcal{A}\} \to X_{\mathcal{A}}/W_{\mathcal{A}}$ by (2.3). Since its domain $\operatorname{span}\{e_A : A \in \mathcal{A}\}$ is a dense subalgebra of $c_0(\mathcal{A})$, it will suffice to show that this map is multiplicative, isometric and has dense range.

The multiplicativity of ψ follows from the fact that $\mathbb{1}_A \cdot \mathbb{1}_B = \mathbb{1}_{A\cap B} \in V_{\mathcal{A}}$ whenever $A, B \in \mathcal{A}$ are distinct. By definition, the image of ψ is

$$\operatorname{span}\{\psi(e_A) : A \in \mathcal{A}\} = \operatorname{span}\{\mathbb{1}_A + W_{\mathcal{A}} : A \in \mathcal{A}\},$$

and this subspace is dense in $X_{\mathcal{A}}/W_{\mathcal{A}}$ because the definitions (1.1) and (2.1) of $X_{\mathcal{A}}$ and $V_{\mathcal{A}}$ imply that the subspace spanned by $\{\mathbb{1}_A : A \in \mathcal{A}\} \cup V_{\mathcal{A}}$ is dense in $X_{\mathcal{A}}$.

It remains to show that ψ is an isometry. To this end, take distinct sets $A_1, \ldots, A_m \in \mathcal{A}$ for some $m \in \mathbb{N}$, define $A_1^d, \ldots, A_m^d$ by (2.4), and let $\sigma_1, \ldots, \sigma_m \in \mathbb{K}$. Then we have

$$\psi\Big(\sum_{j=1}^{m} \sigma_j e_{A_j}\Big) = \sum_{j=1}^{m} \sigma_j \mathbb{1}_{A_j} + W_{\mathcal{A}} = \sum_{j=1}^{m} \sigma_j \mathbb{1}_{A_j^d} + W_{\mathcal{A}}$$

because Lemma 2.3 implies that

$$\sum_{j=1}^{m} \sigma_j \mathbb{1}_{A_j} - \sum_{j=1}^{m} \sigma_j \mathbb{1}_{A_j^d} = \sum_{\substack{N \subseteq \{1,\ldots,m\}\\ |N|\geqslant 2}} (-1)^{|N|} \Big(\sum_{j\in N} \sigma_j\Big) \mathbb{1}_{\bigcap_{j\in N} A_j} \in V_{\mathcal{A}} \subseteq W_{\mathcal{A}}.$$

Therefore, showing that ψ is an isometry is equivalent to showing that

$$\Big\|\sum_{j=1}^{m} \sigma_j \mathbb{1}_{A_j^d} + W_{\mathcal{A}}\Big\| = \max_{1\leqslant j\leqslant m} |\sigma_j|. \tag{2.6}$$

The inequality $\leqslant$ is clear because the sets $A_1^d, \ldots, A_m^d$ are disjoint.

To verify the opposite inequality, choose $i \in \{1, \ldots, m\}$ such that $|\sigma_i| = \max_{1\leqslant j\leqslant m}|\sigma_j|$. For each $v \in V_{\mathcal{A}}$, Lemma 2.1 shows that $\operatorname{supp} v \in [\Gamma]^{<\Gamma}$. Consequently, since $|A_i| = \Gamma$ and $\bigcup_{k=1,\, k\neq i}^{m} A_k \cap A_i \in [\Gamma]^{<\Gamma}$, we can find $\gamma \in A_i^d$ such that $v(\gamma) = 0$. By disjointness, we have $\gamma \notin A_j^d$ for each $j \in \{1, \ldots, m\} \setminus \{i\}$, and therefore

$$\Big\|\sum_{j=1}^{m} \sigma_j \mathbb{1}_{A_j^d} - v\Big\|_\infty \geqslant \Big|\sum_{j=1}^{m} \sigma_j \mathbb{1}_{A_j^d}(\gamma) - v(\gamma)\Big| = |\sigma_i|.$$

This implies that $\big\|\sum_{j=1}^{m} \sigma_j \mathbb{1}_{A_j^d} + W_{\mathcal{A}}\big\| \geqslant |\sigma_i|$ because $v \in V_{\mathcal{A}}$ was arbitrary and $V_{\mathcal{A}}$ is dense in $W_{\mathcal{A}}$; hence (2.6) follows. □

Remark 2.4. In the case of complex scalars, the map ψ given by (2.3) is a $*$-isomorphism.

Lemma 2.5. *Let $U\colon E \to X^*$ be a bounded operator, where $E = c_0(J)$ or $E = \ell_p(J)$ for some infinite set J and some $p \in (1, \infty)$, and X is a non-zero Banach space. Then the set $\{j \in J : Ue_j \neq 0\}$ has cardinality no greater than the density character of X, where $(e_j)_{j\in J}$ denotes the unit vector basis for E.*

Proof. We begin by showing that the set $J_{n,x} = \{j \in J : |\langle Ue_j, x\rangle| \geqslant 1/n\}$ is finite for each $n \in \mathbb{N}$ and $x \in X$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality bracket between X and its dual space X^*. Assume the contrary, and take $n \in \mathbb{N}$ and $x \in X$ such that $J_{n,x}$ contains an infinite sequence $(j_k)_{k\in\mathbb{N}}$ of distinct elements. For each $k \in \mathbb{N}$, choose a scalar σ_k of modulus one such that $\sigma_k\langle Ue_{j_k}, x\rangle \geqslant 1/n$. Then $y = \sum_{k=1}^{\infty}(\sigma_k/k)e_{j_k}$ defines an element of E (recall that we do not allow $E = \ell_1(J)$). However, we have

$$|\langle Uy, x\rangle| = \Big|\sum_{k=1}^{\infty} \frac{\sigma_k}{k}\langle Ue_{j_k}, x\rangle\Big| \geqslant \frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k} = \infty,$$

which is absurd. This contradiction proves that $J_{n,x}$ is finite for each $n \in \mathbb{N}$ and $x \in X$. It follows that the set $J_x = \bigcup_{n\in\mathbb{N}} J_{n,x} = \{j \in J : \langle Ue_j, x\rangle \neq 0\}$ is countable for each $x \in X$.

Now take a dense subset D of X. Then $\bigcup_{x\in D} J_x = \{j \in J : Ue_j \neq 0\}$, and therefore $|\{j \in J : Ue_j \neq 0\}| \leqslant |D|$ because D is infinite (otherwise it could not be dense in a non-zero Banach space) and the sets J_x are countable. □

Proof of Theorem 1.2. Suppose that $T\colon X_{\mathcal{A}} \to \ell_\infty(\Gamma)$ is a bounded operator for which $T\mathbb{1}_A = 0$ for every $A \in \mathcal{A}_{⋒}$, and define $\mathcal{B} = \{A \in \mathcal{A} : T\mathbb{1}_A = 0\}$. Then trivially $T\mathbb{1}_B = 0$ for every $B \in \mathcal{B}$, so the conclusion will follow provided that we can show that $|\mathcal{A} \setminus \mathcal{B}| \leqslant \Gamma$.

Since $W_{\mathcal{A}} \subseteq \ker T$ by hypothesis, the Fundamental Isomorphism Theorem implies that we can define a bounded operator $\widetilde{T}\colon X_{\mathcal{A}}/W_{\mathcal{A}} \to \ell_\infty(\Gamma)$ by $\widetilde{T}(x + W_{\mathcal{A}}) = Tx$. Composing it with the isometric isomorphism $\psi\colon c_0(\mathcal{A}) \to X_{\mathcal{A}}/W_{\mathcal{A}}$ from Lemma 2.2, we obtain an operator $U = \widetilde{T} \circ \psi\colon c_0(\mathcal{A}) \to \ell_\infty(\Gamma)$ which satisfies $Ue_A = T\mathbb{1}_A$ for every $A \in \mathcal{A}$, and therefore

$$\{A \in \mathcal{A} : Ue_A \neq 0\} = \{A \in \mathcal{A} : T\mathbb{1}_A \neq 0\} = \mathcal{A} \setminus \mathcal{B}.$$

Lemma 2.5 shows that the set on the left-hand side of this equation has cardinality at most Γ because $\ell_\infty(\Gamma) \cong \ell_1(\Gamma)^*$ and $\ell_1(\Gamma)$ has density character Γ. The result follows. □

Proof of Corollary 1.3. As previously mentioned, Sierpiński [14] has shown that $[\Gamma]^\Gamma$ contains an almost disjoint family $\mathcal{A}$ of cardinality greater than Γ. The almost disjointness of $\mathcal{A}$ means that $\mathbb{1}_A \in \ell_\infty^{<}(\Gamma)$ for every $A \in \mathcal{A}_{\Cap}$. Hence, given a bounded operator $T\colon \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ with $\ell_\infty^{<}(\Gamma) \subseteq \ker T$, we can apply Theorem 1.2 to the restriction of T to $X_{\mathcal{A}}$ to obtain a subset $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{A} \setminus \mathcal{B}| \leqslant \Gamma$ and $X_{\mathcal{B}} \subseteq \ker T$. We must have $|\mathcal{B}| > \Gamma$ because $\Gamma < |\mathcal{A}| = \max\{|\mathcal{B}|, |\mathcal{A} \setminus \mathcal{B}|\}$. This establishes the first part of the result. The second part follows because $\mathbb{1}_B \notin \ell_\infty^{<}(\Gamma)$ for every $B \in [\Gamma]^\Gamma$. □

We conclude this section with a question that we are grateful to the referee for drawing our attention to. For any filter $\mathcal{F}$ on Γ, the set

$$c_{\mathcal{F}}(\Gamma) = \Big\{x \in \ell_\infty(\Gamma) : \lim_{\gamma \to \mathcal{F}} x(\gamma) = 0\Big\}$$

defines a closed subspace of $\ell_\infty(\Gamma)$. In the case where $\mathcal{F} = \{A \subseteq \Gamma : \Gamma \setminus A \in [\Gamma]^{<\Gamma}\}$, we have $c_{\mathcal{F}}(\Gamma) = \ell_\infty^{<}(\Gamma)$. In view of Corollary 1.3, this raises the following question:

Question 2.6. For which filters $\mathcal{F}$ on Γ is it possible to realise the subspace $c_{\mathcal{F}}(\Gamma)$ of $\ell_\infty(\Gamma)$ as the kernel of a bounded operator $\ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$?

We note that $c_{\mathcal{F}}(\Gamma) = c_0(\Gamma)$ if $\mathcal{F}$ is the Fréchet filter consisting of all cofinite subsets of Γ. Let us also remark that in the case where $\mathcal{F} = \{A \subseteq \Gamma : \alpha \in A\}$ is the principal ultrafilter determined by some $\alpha \in \Gamma$, $c_{\mathcal{F}}(\Gamma)$ is the kernel of a bounded operator, namely the rank-one operator $x \otimes \delta_\alpha\colon \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)$ given by $y \mapsto y(\alpha)x$, where x can be any non-zero element of $\ell_\infty(\Gamma)$.

3. Kernels of operators on $\ell_\infty^c(\Gamma)$

A natural variant of the Banach space $\ell_\infty(\Gamma)$ and its subspace $\ell_\infty^{<}(\Gamma)$ defined by (1.3) is

$$\ell_\infty^c(\Gamma) = \{x \in \ell_\infty(\Gamma) : \operatorname{supp} x \text{ is countable}\}.$$

This is readily seen to be a closed subspace of $\ell_\infty(\Gamma)$ and hence a Banach space in its own right. Since $\ell_\infty^c(\Gamma) = \ell_\infty(\Gamma)$ when Γ is countable, one can view $\ell_\infty^c(\Gamma)$ as a "smaller" generalisation of ℓ_∞ to uncountable index sets Γ than $\ell_\infty(\Gamma)$ itself.

From a certain perspective, this generalisation is nicer than $\ell_\infty(\Gamma)$. Indeed, Johnson, Kania and Schechtman [6, Theorem 1.4] have given a complete classification of the complemented subspaces of $\ell_\infty^c(\Gamma)$, something that has been achieved for $\ell_\infty(\Gamma)$ only when Γ is countable. However, $\ell_\infty^c(\Gamma)$ does not enjoy the same universal properties as $\ell_\infty(\Gamma)$ when Γ is uncountable because $\ell_\infty(\Gamma)$ is injective, whereas $\ell_\infty^c(\Gamma)$ is not due to a result of Pełczyński and Sudakov [12, Corollary, page 87], which states that $\ell_\infty^c(\Gamma)$ is not complemented in $\ell_\infty(\Gamma)$. We remark that $\ell_\infty^c(\Gamma)$ has the weaker property of being separably injective (see [3, Definition 2.1 and Example 2.4] for details).

This section is motivated by the following question, which asks whether a natural generalisation of Kalton's result for ℓ_∞ holds true for $\ell_\infty^c(\Gamma)$.

Question 3.1. Let Γ be an uncountable cardinal number, and suppose that $T\colon \ell_\infty^c(\Gamma) \to \ell_\infty^c(\Gamma)$ is a bounded operator with $c_0(\Gamma) \subseteq \ker T$. Does $\ker T$ contain the subspace

$$\ell_\infty^c(A) = \{x \in \ell_\infty^c(\Gamma) : \operatorname{supp} x \subseteq A\}$$

for some $A \in [\Gamma]^\Gamma$?

This question turns out to be undecidable in ZFC for $\Gamma = \omega_1$, as we already stated in Section 1. In one direction, this depends on a deep result of Baumgartner [4, statement (1) on page 403], who showed that the following statement is independent of ZFC and the negation of the Continuum Hypothesis:

(3.1) There exists a family $\mathcal{A} \subset [\omega_1]^{\omega_1}$ of cardinality ω_2 such that $A \cap B$ is finite whenever $A, B \in \mathcal{A}$ are distinct.

Theorem 3.2. *Assume Baumgartner's statement* (3.1)*, and let* $T\colon \ell_\infty^c(\omega_1) \to \ell_\infty(\omega_1)$ *be a bounded operator whose kernel contains* $c_0(\omega_1)$*. Then* $\ell_\infty^c(A) \subseteq \ker T$ *for some* $A \in [\omega_1]^{\omega_1}$*.*

Proof. Take a family $\mathcal{A} \subset [\omega_1]^{\omega_1}$ of cardinality ω_2 such that $A \cap B$ is finite whenever $A, B \in \mathcal{A}$ are distinct, and assume towards a contradiction that no set $A \in \mathcal{A}$ satisfies $\ell_\infty^c(A) \subseteq \ker T$. Then, for every $A \in \mathcal{A}$, we can pick a unit vector $x_A \in \ell_\infty^c(A)$ such that $Tx_A \neq 0$. Set $\mathcal{A}_{\gamma,n} = \{A \in \mathcal{A} : |(Tx_A)(\gamma)| \geqslant 1/n\}$ for each $n \in \mathbb{N}$ and $\gamma \in \omega_1$. Since

$$\bigcup_{\gamma\in\omega_1,\, n\in\mathbb{N}} \mathcal{A}_{\gamma,n} = \{A \in \mathcal{A} : Tx_A \neq 0\} = \mathcal{A},$$

which has cardinality ω_2, we must have $|\mathcal{A}_{\gamma_0,n_0}| = \omega_2$ for some $n_0 \in \mathbb{N}$ and $\gamma_0 \in \omega_1$. In fact, it will suffice for our purposes to know that $\mathcal{A}_{\gamma_0,n_0}$ contains m distinct sets $A_1, \ldots, A_m$ for some integer $m > n_0\|T\|$.

For each $i \in \{1, \ldots, m\}$, choose $\sigma_i \in \mathbb{K}$ with $|\sigma_i| = 1$ such that $\sigma_i(Tx_{A_i})(\gamma_0) \geqslant 1/n_0$, and set $x = \sum_{i=1}^m \sigma_i x_{A_i} \in \ell_\infty^c(\omega_1)$. We shall express this vector as $x = y + z$, where $\|y\|_\infty \leqslant 1$ and $Tz = 0$. To this end, set

$$B = \bigcup_{1\leqslant i<j\leqslant m} A_i \cap A_j,$$

which is a finite subset of ω_1 by the hypothesis on $\mathcal{A}$. It follows that $z = x \cdot \mathbb{1}_B$ has finite support, so $z \in c_0(\omega_1) \subseteq \ker T$. Furthermore, we have $y = x - z = \sum_{i=1}^m \sigma_i x_{A_i} \cdot \mathbb{1}_{A_i\setminus B}$, where the terms $\sigma_i x_{A_i} \cdot \mathbb{1}_{A_i\setminus B}$ are disjointly supported by the choice of B, and they have norm at most 1 by the choices of x_{A_i} and σ_i. This implies that $\|y\|_\infty \leqslant 1$, and therefore

$$\|T\| \geqslant \|Ty\|_\infty = \|Ty + Tz\|_\infty = \|Tx\|_\infty \geqslant |(Tx)(\gamma_0)| = \Big|\sum_{i=1}^m \sigma_i(Tx_{A_i})(\gamma_0)\Big| \geqslant \frac{m}{n_0} > \|T\|,$$

which is obviously absurd. The result follows. □

To prove that it is also consistent with ZFC that the answer to Question 3.1 is negative, we require a set-theoretic axiom known as ♣, which is independent of ZFC. It is defined as follows (see, *e.g.,* [5, Definition 4.35]).

Definition 3.3. Let $(\lambda_\alpha)_{\alpha\in\omega_1}$ denote the increasing enumeration of the countably infinite limit ordinals. Then ♣ is the statement that there exists a transfinite sequence $(B_\alpha)_{\alpha\in\omega_1}$ such that:

(i) for each $\alpha \in \omega_1$, $B_\alpha = \{\beta_\alpha^{(k)} : k \in \mathbb{N}\}$, where $(\beta_\alpha^{(k)})_{k\in\mathbb{N}}$ is a strictly increasing sequence of ordinal numbers converging to λ_α;
(ii) every uncountable subset of ω_1 contains B_α for some $\alpha \in \omega_1$.

Theorem 3.4. *Assume ♣. Then a bounded operator $T\colon \ell_\infty^c(\omega_1) \to \ell_\infty^c(\omega_1)$ exists for which $c_0(\omega_1) \subseteq \ker T$, but $\ell_\infty^c(A) \nsubseteq \ker T$ for every $A \in [\omega_1]^{\omega_1}$.*

Proof. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, take a transfinite sequence $(B_\alpha)_{\alpha\in\omega_1}$ as in Definition 3.3, and let $\alpha \in \omega_1$. Writing $B_\alpha = \{\beta_\alpha^{(k)} : k \in \mathbb{N}\}$ as in Definition 3.3(i), we can define a bounded linear functional $\varphi_\alpha\colon \ell_\infty^c(\omega_1) \to \mathbb{K}$ of norm 1 by

$$\varphi_\alpha(x) = \lim_{k\to\mathcal{U}} x(\beta_\alpha^{(k)}).$$

We note that $\varphi_\alpha(\mathbb{1}_{B_\alpha}) = 1$ because $\mathbb{1}_{B_\alpha}(\beta_\alpha^{(k)}) = 1$ for each $k \in \mathbb{N}$. Furthermore, the fact that $\varphi_\alpha(\mathbb{1}_{\{\gamma\}}) = 0$ for each $\gamma \in \omega_1$ implies that

$$\ker\varphi_\alpha \supseteq \overline{\operatorname{span}}\,\{\mathbb{1}_{\{\gamma\}} : \gamma \in \omega_1\} = c_0(\omega_1).$$

Hence we can define a bounded operator $T\colon \ell_\infty^c(\omega_1) \to \ell_\infty(\omega_1)$ of norm 1 with $c_0(\omega_1) \subseteq \ker T$ by

$$(Tx)(\alpha) = \varphi_\alpha(x) \qquad (x \in \ell_\infty^c(\omega_1),\ \alpha \in \omega_1).$$

To see that the range of T is contained in $\ell_\infty^c(\omega_1)$, let $x \in \ell_\infty^c(\omega_1)$, and take $\alpha_0 \in \omega_1$ such that $\operatorname{supp} x \subseteq [0, \lambda_{\alpha_0}]$. Then, for every countable ordinal number $\alpha > \alpha_0$, we have

$$\lambda_{\alpha_0} < \lambda_\alpha = \lim_{k\to\infty} \beta_\alpha^{(k)},$$

so there are at most finitely many $k \in \mathbb{N}$ such that $\beta_\alpha^{(k)} \leqslant \lambda_{\alpha_0}$, and therefore at most finitely many $k \in \mathbb{N}$ such that $x(\beta_\alpha^{(k)}) \neq 0$. Hence $\varphi_\alpha(x) = 0$ whenever $\alpha > \alpha_0$. This shows that $\operatorname{supp} Tx \subseteq [0, \alpha_0]$, and consequently $Tx \in \ell_\infty^c(\omega_1)$.

Finally, we verify that $\ell_\infty^c(A) \nsubseteq \ker T$ for every $A \in [\omega_1]^{\omega_1}$. Definition 3.3(ii) implies that we can find $\alpha \in \omega_1$ such that $B_\alpha \subseteq A$. Then $\mathbb{1}_{B_\alpha} \in \ell_\infty^c(A)$ because B_α is countable, and we have $(T\mathbb{1}_{B_\alpha})(\alpha) = \varphi_\alpha(\mathbb{1}_{B_\alpha}) = 1$, so $\mathbb{1}_{B_\alpha} \notin \ker T$. The conclusion follows. □

Combining Theorems 3.2 and 3.4, we see that the statement (1.4) is independent of ZFC, as previously claimed.

This raises the question: within ZFC, what can we say about kernels of bounded operators on $\ell_\infty^c(\Gamma)$? To address it, we require the following piece of terminology: a bounded operator $T\colon X \to Y$ between Banach spaces X and Y *fixes a copy* of a Banach space Z if there is a bounded operator $U\colon Z \to X$ such that TU is an isomorphic embedding.

Proposition 3.5. *Let Γ be a cardinal number with uncountable cofinality, let X be a Banach space, and suppose that $T\colon \ell_\infty^c(\Gamma) \to X$ is a bounded operator which does not fix a copy of $c_0(\Gamma)$. Then $\ell_\infty^c(A) \subseteq \ker T$ for some $A \in [\Gamma]^\Gamma$.*

Proof. We shall prove the contrapositive statement, so suppose that $\ell_\infty^c(A) \nsubseteq \ker T$ for every $A \in [\Gamma]^\Gamma$. Since Γ is infinite, we can take a family $(A_\gamma)_{\gamma\in\Gamma}$ of mutually disjoint sets in $[\Gamma]^\Gamma$. The hypothesis implies that we can choose a unit vector $x_\gamma \in \ell_\infty^c(A_\gamma) \setminus \ker T$ for each $\gamma \in \Gamma$. Then the transfinite sequence $(x_\gamma)_{\gamma\in\Gamma}$ consists of disjointly supported unit vectors in $\ell_\infty^c(\Gamma)$, so we can define a linear isometry $S\colon c_0(\Gamma) \to \ell_\infty^c(\Gamma)$ by setting $Se_\gamma = x_\gamma$ for each $\gamma \in \Gamma$.

Combining the fact that Γ has uncountable cofinality with the observation that

$$\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n, \quad \text{where} \quad \Gamma_n = \Big\{\gamma \in \Gamma : \|Tx_\gamma\|_\infty \geqslant \frac{1}{n}\Big\},$$

we conclude that $|\Gamma_n| = \Gamma$ for some $n \in \mathbb{N}$. Then $\inf_{\gamma\in\Gamma_n} \|TSe_\gamma\|_\infty \geqslant 1/n > 0$, so a famous result of Rosenthal stated in [13, Remark 1 (page 30)] implies that we can find $\Gamma' \in [\Gamma_n]^\Gamma$ such that the restriction of TS to $c_0(\Gamma') \subseteq c_0(\Gamma)$ is an isomorphic embedding. This shows that T fixes a copy of $c_0(\Gamma)$. ☐

Proposition 3.5 should be compared with the following result of Wójtowicz [17, Theorem 1].

Theorem 3.6 (Wójtowicz)**.** *Let $T\colon \ell_\infty(\Gamma)/c_0(\Gamma) \to X$ be a bounded operator, where Γ is infinite and X is a Banach space. If T does not fix a copy of ℓ_∞, then $\ker T$ contains a closed subspace which is isomorphic to c_0.*

Remark 3.7. Let $X = \ell_\infty^c(\Gamma)$, and write $\mathscr{B}(X)$ for the Banach algebra of bounded operators $X \to X$. Johnson, Kania and Schechtman [6, Theorems 1.1 and 3.14] have shown that the set $\mathscr{S}_X(X) = \{T \in \mathscr{B}(X) : T \text{ does not fix a copy of } X\}$ is the unique maximal ideal of $\mathscr{B}(X)$ and that $T \in \mathscr{S}_X(X)$ if and only if T does not fix a copy of $c_0(\Gamma)$. Suppose that Γ has uncountable cofinality. Then we can combine this result with Proposition 3.5 to deduce that for every $T \in \mathscr{S}_X(X)$, there exists $A \in [\Gamma]^\Gamma$ such that $\ell_\infty^c(A) \subseteq \ker T$.

Appendix A. The proof of Lemma 2.3

Proof of Lemma 2.3*.* We begin by recalling the statement of the lemma and introduce some additional notation that will help us present its proof concisely. The set-up is that $\sigma_1, \ldots, \sigma_m$ are scalars, while $A_1, \ldots, A_m$ are distinct subsets of Γ for which we define

$$A_j^d = A_j \setminus \bigcup_{\substack{k=1\\k\neq j}}^m A_k \qquad (j \in \{1, \ldots, m\}).$$

Then the claim is that the functions lhs, rhs: $\Gamma \to \mathbb{K}$ given by

$$\text{lhs} = \sum_{j=1}^m \sigma_j \mathbb{1}_{A_j^d} \qquad \text{and} \qquad \text{rhs} = \sum_{\emptyset\neq N\subseteq\{1,\ldots,m\}} (-1)^{|N|+1}\Big(\sum_{j\in N}\sigma_j\Big)\mathbb{1}_{\bigcap_{j\in N} A_j}$$

are equal.

We verify this equality pointwise. Take $\gamma \in \Gamma$, define $M_\gamma = \big\{j \in \{1, \ldots, m\} : \gamma \in A_j\big\}$, and observe that $\gamma \in \bigcap_{j\in N} A_j$ if and only if $N \subseteq M_\gamma$. Clearly $\text{lhs}(\gamma) = 0 = \text{rhs}(\gamma)$ if

$M_\gamma = \emptyset$, while $\mathrm{lhs}(\gamma) = \sigma_j = \mathrm{rhs}(\gamma)$ if $M_\gamma = \{j\}$ for some $j \in \{1,\ldots,m\}$. Hence we may suppose that M_γ has at least two elements. In this case $\gamma \in \bigcup_{k=1,\,k\neq j}^{m} A_k$ for every $j \in \{1,\ldots,m\}$, so $\mathrm{lhs}(\gamma) = 0$. On the other hand, using the observation above, we find

$$\mathrm{rhs}(\gamma) = \sum_{\emptyset\neq N\subseteq M_\gamma} (-1)^{|N|+1}\Big(\sum_{j\in N}\sigma_j\Big) = \sum_{k=1}^{|M_\gamma|}(-1)^{k+1}\sum_{\substack{N\subseteq M_\gamma\\ |N|=k}}\Big(\sum_{j\in N}\sigma_j\Big) = \sum_{k=1}^{|M_\gamma|}(-1)^{k+1}\sum_{j\in M_\gamma} a_{k,j}\sigma_j,$$

where $a_{k,j}$ denotes the number of subsets N of M_γ of size k such that $j \in N$. Since $a_{k,j} = \binom{|M_\gamma|-1}{k-1}$, by re-indexing and applying the binomial formula, we conclude that

$$\mathrm{rhs}(\gamma) = \sum_{\ell=0}^{|M_\gamma|-1}(-1)^\ell\binom{|M_\gamma|-1}{\ell}\sum_{j\in M_\gamma}\sigma_j = (-1+1)^{|M_\gamma|-1}\sum_{j\in M_\gamma}\sigma_j = 0,$$

as desired. □

Remark A.1. The aim of this remark is to explain why we may view Lemma 2.3 as a weighted variant of the Inclusion–Exclusion Principle. Let μ be a scalar-valued, finitely additive measure defined on a Boolean subalgebra $\mathcal{F}$ of the power set of Γ. Then, taking $\sigma_1 = \cdots = \sigma_m = 1$ in Lemma 2.3 and integrating both sides of (2.5) with respect to μ, we obtain

$$\sum_{j=1}^{m}\mu(A_j^d) = \sum_{\emptyset\neq N\subseteq\{1,\ldots,m\}}(-1)^{|N|+1}\cdot|N|\cdot\mu\Big(\bigcap_{j\in N}A_j\Big) \qquad (m\in\mathbb{N},\ A_1,\ldots,A_m\in\mathcal{F}).$$

Acknowledgements. The first-named author is grateful to Max Arnott for bringing this problem to his attention. He acknowledges with thanks the financial support provided by the Jagiellonian Scholarship "Initiative of Excellence — Jagiellonian University" awarded by the Faculty of Mathematics and Computer Science. We would also like to thank Yemon Choi and Tommaso Russo for some enlightening conversations, and Jesús Castillo for mentioning the original reference [8] to us. Finally, we thank the referee for their careful reading of our paper and helpful suggestions.

References

1. P. Alexandroff and P. Urysohn, *Memoire sur les espaces topologiques compacts dedie a Monsieur D. Egoroff*, 1929.
2. M. Arnott and N. J. Laustsen, Kernels of bounded operators on the classical transfinite Banach sequence spaces, preprint, 2023.
3. A. Avilés, F. Cabello Sánchez, J. M. F. Castillo, M. González and Y. Moreno, *Separably injective Banach spaces*, Lect. Notes Math., Springer-Verlag, 2016.
4. J. E. Baumgartner, Almost-disjoint sets, the dense set problem and the partition calculus, *Ann. Math. Logic* **10** (1976), 401–439.
5. P. Hájek, V. Montesinos, J. Vanderwerff and V. Zizler, *Biorthogonal systems in Banach spaces*, CMS Books in Mathematics, Springer-Verlag, 2008.
6. W. B. Johnson, T. Kania and G. Schechtman, Closed ideals of operators on and complemented subspaces of Banach spaces of functions with countable support, *Proc. Amer. Math. Soc.* **144** (2016), 4471–4485.

7. W. B. Johnson and J. Lindenstrauss, Some remarks on weakly compactly generated Banach spaces, *Israel J. Math.* **17** (1974), 219–230.
8. N. J. Kalton, Spaces of compact operators, *Math. Ann.* **208** (1974), 267–278.
9. N. J. Laustsen and J. T. White, Subspaces that can and cannot be the kernel of a bounded operator on a Banach space, in *Banach algebras and applications. Proceedings of the 23rd international conference, University of Oulu, Finland, July 3–11, 2017* (ed. M. Filali), *De Gruyter Proceedings in Mathematics* (2020), 189–196.
10. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, 1977.
11. C. Mariconda and A. Tonolo, *Discrete Calculus: Methods for Counting*, Springer-Verlag, 2016.
12. A. Pełczyński and V. N. Sudakov, Remark on non-complemented subspaces of the space $m(S)$, *Colloq. Math.* **9** (1962), 85–88.
13. H. Rosenthal, On relatively disjoint families of measures, with some applications to Banach space theory, *Studia Math.* **37** (1970), 13–36.
14. W. Sierpiński, Sur une décomposition d'ensembles, *Monatsh. Math. Phys.* **35** (1928), 239–242.
15. H. M. Wark, A non-separable reflexive Banach space on which there are few operators, *J. London Math. Soc.* **64** (2001), 675–689.
16. R. Whitley, Projecting m onto c_0, *Amer. Math. Monthly* **73** (1966), 285–286.
17. M. Wójtowicz, Operators preserving ℓ_∞, *RACSAM* **108** (2014), 511–517.

(B. Horváth) Baloise Insurance Ltd, Aeschengraben 21, Postfach 2275, Basel 4002, Switzerland
Email address: hotvath@gmail.com

(N. J. Laustsen) School of Mathematical Sciences, Fylde College, Lancaster University, Lancaster LA1 4YF, United Kingdom
Email address: n.laustsen@lancaster.ac.uk