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Uniquely realisable graphs in analytic normed planes

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A framework (G,p) in Euclidean space $\mathbb{E}^d$ is globally rigid if it is the unique realisation, up to rigid congruences, of G with the edge lengths of (G,p). Building on key results of Hendrickson [28] and Connelly [14], Jackson and Jordán [29] gave a complete combinatorial characterisation of when a generic framework is global rigidity in $\mathbb{E}^2$. We prove an analogous result when the Euclidean norm is replaced by any norm that is analytic on $\mathbb{R}^2 \setminus \{0\}$. Specifically, we show that a graph $G = (V,E)$ has an open set of globally rigid realisations in a non-Euclidean analytic normed plane if and only if G is 2-connected and $G - e$ contains 2 edge-disjoint spanning trees for all $e \in E$. We also prove that the analogous necessary conditions hold in d-dimensional normed spaces.

1 Introduction

A *bar-joint framework* (G,p) in $\mathbb{E}^d$ is an ordered pair consisting of a finite, simple graph $G = (V,E)$ and a map $p : V \to \mathbb{R}^d$. Given a framework (G,p) in $\mathbb{E}^d$, a fundamental question is whether the edge lengths of (G,p) determine a unique framework up to rigid congruences of $\mathbb{E}^d$. This is the concept of global rigidity which has been researched intensively over the last 40 years (e.g. [13, 26, 29]).

The study of global rigidity is motivated by a number of fundamental questions in various areas of applied mathematics, including protein structure determination [39] and sensor network localisation [3]. These applications often involve distance measures other than the standard Euclidean norm (see, for example, [6] or [17] and the references therein), which motivates the topic of this paper; the study of global rigidity for frameworks in normed spaces. A research program in this direction was initiated by the first and third author in [24]. A (real) normed plane is *analytic* if it is non-Euclidean and the norm restricted to the non-zero points is a real analytic function. While analytic norms are special, we can approximate any norm by a uniformly convergent sequence of analytic norms. Hence it is reasonable to expect that the analytic case is representative of the general situation. In this article we give a complete combinatorial description of global rigidity for the class of analytic normed planes.

One may also consider the related concept of local rigidity, where there must exist only a finite number of realisations (up to rigid congruences) in the given normed space. This has also been studied intensively for many years in the Euclidean case (see, for example, [4, 38, 51, 49]), and research in the non-Euclidean setting was recently initiated by Kitson and Power [36]. Indeed the main result of [36], which we describe below, will be important in our analysis. Their work influenced subsequent study of a number of related rigidity problems, including more general normed spaces [21, 23, 33, 34], and in the presence of symmetry [32, 35].

Many papers have also considered the ostensibly similar concepts of rigidity and global rigidity under non-Euclidean metrics, such as hyperbolic and Minkowski metrics [27, 50, 52]. However these contexts are sufficiently similar to the Euclidean setting for the same combinatorial techniques to be applied. Moreover, at the level of infinitesimal rigidity there is an elegant projective invariance [49, 52]. In the case of non-Euclidean norms these conveniences are unavailable, and alternative combinatorial ideas are required to study new classes of graphs and matroids. Furthermore, since distance constraints are not quadratic in the non-Euclidean case, unlike in the Euclidean case, we do not have the luxury of an obviously accessible “equilibrium stress matrix” approach ([13, 14, 26] inter alia). We overcome these issues by restricting to analytic normed planes, deploying combinatorial tools from [22] and [31], and making use of the analytic geometry techniques developed in [24]. In

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particular we will apply the theory of generalised rigid body motions introduced by the first and third author in [24].

The main result of the paper is as follows. (The definitions of global rigidity is formalised in Section 2.2.)

Theorem 1.1. Let $G = (V, E)$ be a graph with $|V| \geq 2$ and X be an analytic normed plane. Then G has a non-empty open set of globally rigid realisations in X if and only if G is 2-connected and $G - e$ contains 2 edge-disjoint spanning trees for all $e \in E$. □

To prove the combinatorial part of the theorem we make use of the strategy developed in [30, 31]. Those papers concern the global rigidity of frameworks in $\mathbb{R}^3$ that are restricted to lie on the surface of a cylinder. While we know of no direct equivalence, it turns out that, for both rigidity and global rigidity, the "generic" situation with any analytic norm is equivalent to the generic situation on the surface of a cylinder with the Euclidean norm. That is, the graphs underlying generically rigid frameworks on the surface of the cylinder with the Euclidean norm are the same as those underlying "generically" rigid frameworks under any analytic norm. The equivalence for infinitesimal rigidity can be read off by comparing the main results of [19, 36] and [48], while the corresponding equivalence for global rigidity follows in the same manner by comparing our main result to the main result of [31].

There is one major combinatorial complication that we face in extending the strategy from [31]. In that context the key difficulty was to establish global rigidity for "circuits" in an appropriate matroid; the general case then followed by a relatively straightforward argument. We may establish global rigidity for the appropriate circuits in analytic normed planes using their combinatorial result (stated as Theorem 4.5 below) in combination with the results of [22, 24] and Sections 3 and 7. However in Section 4 we will explain why their argument for the general case cannot (to our knowledge) be applied to analytic normed planes. Instead we use "ear decompositions" to develop a more detailed combinatorial theory of the class of redundantly rigid (see Section 2.2) and 2-connected graphs in analytic normed planes. Our main technical combinatorial result is a recursive construction, Corollary 6.15, which is a generalisation of [31, Theorem 2.1] from the special case of circuits to arbitrary redundantly rigid and 2-connected graphs.

We conclude the introduction with an outline of the paper. In Section 2 we introduce the necessary background from rigidity theory, survey the Euclidean case and what is known in other normed planes, and provide key background results concerning analytic normed planes. We then prove precise analogues to the so-called Hendrickson conditions (see [28] for the Euclidean case) in Section 3; these are graph theoretic conditions necessary for a framework to be globally rigid in an arbitrary dimension analytic normed space with finitely many linear isometries. The remainder of the paper is restricted to the 2-dimensional case. Sections 4, 5 and 6 form the main technical part of the paper. In these sections we analyse the simple $(2, 2)$-sparse matroid $\mathcal{M}(2, 2)$, equate the necessary conditions of Section 3 with a connectivity condition in $\mathcal{M}(2, 2)$, deduce key "gluing" properties of the graphs for which $\mathcal{M}(2, 2)$ satisfies the connectivity condition, and use three graph operations to give a recursive construction of all such graphs. We then use this construction in Section 7 to prove that the necessary graph conditions are also sufficient, completing the proof of our characterisation. The geometric extension part of our inductive proof relies on the results of [22, 24] and one new geometric argument which we develop.

2 Normed space rigidity theory

In this section we introduce the concepts and results from rigidity theory in normed spaces.

2.1 Normed space geometry

All normed spaces will be finite-dimensional real linear spaces. There are two main types: *Euclidean spaces*, where the norm satisfies the parallelogram law, and *non-Euclidean spaces*, where the norm does not. As Euclidean spaces of the same dimension are isometrically isomorphic, we shall denote any Euclidean space of dimension d by $\mathbb{E}^d$. We shall be particularly interested in *analytic normed spaces*; i.e., non-Euclidean normed spaces where the norm is a real analytic function when restricted to the set of non-zero points in the space. Although the norm of a Euclidean space will also be a real analytic function when restricted to its set of non-zero points, we opt to consider Euclidean spaces as entirely separate entities.

We require the following terminology. The dual of a normed space X will be denoted by X^* and will have the norm $\|f\|^* := \sup_{\|z\|=1} f(z)$. A point x in a normed space X is *supported by* $f \in X^*$ (equivalently, f is a *support functional of* x) if $f(x) = \|x\|^2$ and $\|f\|^* = \|x\|$. It follows from the Hahn-Banach theorem that every point has at least one support functional, and every linear functional of X is a support functional of a point in X. A non-zero point $x \in X$ is said to be *smooth* if it has exactly one support functional, which we will denote by φ_x; we will also fix φ_0 to be the zero function of X. We say a normed space X is *smooth* if every non-zero point is smooth, and we say X is *strictly convex* if every element of X^* is the support functional of exactly one point of X. Given a smooth normed space X, we define $\varphi : X \to X^*,\ x \mapsto \varphi_x$ to be the *support map of* X. The map φ is continuous and homogeneous (i.e., $\varphi_{cx} = c\varphi_x$ for all $c \in \mathbb{R}$ and $x \in X$), and is a homeomorphism if and only if X is strictly convex (see for example [5, Part III] or [12, Ch. II]). All analytic normed spaces are smooth and strictly convex [24, Lemma 3.1].

2.2 Rigidity in normed spaces

For a finite set V and normed space X, a *placement of* V *in* X is any element $p = (p_v)_{v\in V}$ in the vector space X^V. We denote the restriction of p to a subset $U \subset V$ by $p|_U := (p_v)_{v\in U}$. For a given placement p and any map $g : X \to X$, we define the placement $g \circ p := (g(p_v))_{v\in V}$. A placement is *spanning* if the affine span of the set $\{p_v : v \in V\}$ is X; equivalently, p is spanning if the only affine map $g : X \to X$ where $g \circ p = p$ is the identity map, denoted by I. Given that $\operatorname{Isom}(X)$ is the set of isometries of X, we say a placement is *isometrically full* if, given $g \in \operatorname{Isom}(X)$, we have $g \circ p = p$ if and only if $g = I$. If $|V| \geq \dim X + 1$, then the set of placements of V that are not spanning form a *null* subset of X^V (i.e., a set with Lebesgue measure zero) and hence the set of spanning placements is a *conull* subset of X^V (i.e., a set with a null complement set)*. Since all isometries of a normed space are affine [44], the set of isometrically full placements forms a conull subset of X^V.

A *framework in* X is a pair (G, p) where $G = (V, E)$ is a (finite, simple) graph and p is a placement of V in X; here we also say that p is a *placement of* G *in* X. We denote the set of placements of G with non-zero edge lengths by

$$X_G := \left\{p \in X^V : p_v \neq p_w \text{ for all } vw \in E\right\}.$$

X_G is an open conull subset of X^V; further, if $\dim X > 1$ then X_G is path-connected. We define the *rigidity map of* G *in* X to be the map

$$f_G : X^V \to \mathbb{R}^E,\ p = (p_v)_{v\in V} \mapsto \left(\frac{1}{2}\|p_v - p_w\|^2\right)_{vw\in E}.$$

Two frameworks (G, p) and (G, q) in a normed space X are *equivalent* (denoted $(G, p) \sim (G, p')$) if $f_G(p) = f_G(p')$, and are *congruent* (or $p \sim p'$) if there exists $g \in \operatorname{Isom}(X)$ such that $p' = g \circ p$. A framework (G, p) in X is *locally rigid* if there exists $\epsilon > 0$ such that every framework (G, q) in X that is equivalent to (G, p) and has $\|p_v - q_v\| < \epsilon$ for each $v \in V$ is necessarily congruent to (G, p); otherwise (G, p) is *locally flexible.* (See Figures 1 and 2 for basic examples.) (G, p) is *globally rigid* if every framework in X that is equivalent to (G, p) is also congruent to (G, p).

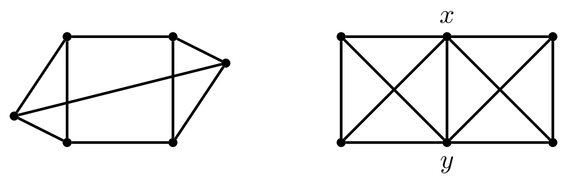


Fig. 1. (Left): A locally rigid but not globally rigid framework in the Euclidean plane. (Right): A locally rigid framework in ℓ_p^2 for any $2 < p < \infty$; see [20, Proposition 6.6] for a direct proof. Although the underlying graph does have globally rigid placements for all even values of p (Lemma 7.1), this particular framework is not globally rigid since the labelled vertices x, y lie on a line of reflection in any ℓ_p^2.

Determining whether a framework in $\mathbb{E}^d$ is locally rigid is NP-hard when $d \geq 2$[1]. One would expect that this is also true for (almost all) normed spaces of dimension two and higher. This motivates us to consider whether a framework can be deformed by infinitesimal motions. For suitable frameworks this will provide a sufficient condition for local rigidity, and it allows us to discuss when local rigidity is a property of a graph.

Our change in approach requires some additional definitions. For a graph $G = (V, E)$, a placement p of G in X and the corresponding framework (G, p) are said to be *well-positioned* if f_G is differentiable at p and $p \in X_G$; equivalently, (G, p) is well-positioned if $p_v - p_w$ is smooth for all $vw \in E$. It is immediate that, given G has at least one edge, the space X is smooth if and only if the set of well-positioned placements of G is exactly the set X_G. If (G, p) is well-positioned, we define the *rigidity operator of* (G, p) to be the derivative

$$df_G(p) : X^V \to \mathbb{R}^E,\ (x_v)_{v\in V} \mapsto (\varphi_{p_v - p_w}(x_v - x_w))_{vw\in E}$$

*Any conull subset is dense, and any closed null subset is nowhere dense, but the converse of these statements are not true; e.g., $\mathbb{Q}$ is null and dense in $\mathbb{R}$, and any fat Cantor set is closed and nowhere dense in $\mathbb{R}$ but not null.

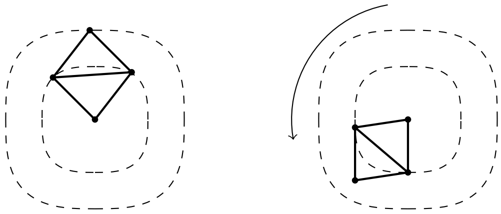


Fig. 2. A flexible framework in the analytic plane ℓ_4^2 (left) and an equivalent but non-congruent framework that can be reached by a continuous motion (right).

of f_G at p. Any element $u \in \ker df_G(p)$ is called an *infinitesimal flex of* (G,p). An infinitesimal flex $u \in X^V$ is *trivial* if there exists a linear map $T : X \to X$ and a point $z \in X$ where $u_v = T(p_v) + z$ for all $v \in V$, and for every point $x \in X$ with support functional $f \in X^*$, we have $f \circ T(x) = 0$.†

We say that a well-positioned framework (G,p) in a normed space X is *infinitesimally rigid* if every infinitesimal flex of (G,p) is trivial. The following result outlines a relationship between local and infinitesimal rigidity. We first recall that a well-positioned framework (G,p) is *regular* if the rigidity operator $df_G(p)$ has maximal rank over all well-positioned placements of G in X.

Theorem 2.1. Let (G,p) be a well-positioned framework in a normed space X.

(i) [20, Theorem 3.9] If (G,p) is infinitesimally rigid, then it is also locally rigid.
(ii) [21, Theorem 1.1 & Lemma 4.4] If (G,p) is regular and locally rigid, and the set of smooth points of X is open, then (G,p) is infinitesimally rigid.

□

Given a well-positioned framework (G,p) in a normed space X, the following three potential properties of (G,p) are of particular interest: (G,p) is *independent* if $\operatorname{rank} df_G(p) = |E|$; (G,p) is *minimally (infinitesimally) rigid* if (G,p) is infinitesimally rigid and independent; and (G,p) is *redundantly (infinitesimally) rigid* if $(G-e,p)$ is infinitesimally rigid for every edge $e \in E$.

2.3 Regular frameworks and placements

A well-positioned framework (G,p) is *strongly regular* if every framework equivalent to (G,p) is well-positioned and has maximal rank. The placement p of a set V is *completely regular* if for all graphs H with $V(H) := V$, (H,p) is well-positioned and regular, and *completely strongly regular* if for all graphs H with $V(H) := V$, (H,p) is well-positioned and strongly regular. We refer to a framework as completely (strongly) regular if its placement is completely (strongly) regular. We denote the set of regular placements of G in X by $\operatorname{Reg}(G;X)$, the set of strongly regular placements of G in X by $\operatorname{Str}(G;X)$, the set of completely regular placements of a set V in X by $\operatorname{Com}(V;X)$, and the set of completely strongly regular placements of a set V in X by $\operatorname{ComStr}(V;X)$. If a framework is strongly or completely regular then it is regular, however, Figure 3 highlights that complete regularity and strong regularity do not obey a hierarchy.

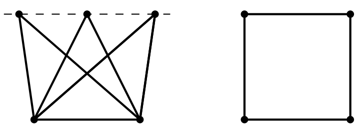


Fig. 3. Two frameworks in the Euclidean plane. The framework on the left is strongly regular but not completely regular as it has a colinear triple. The framework on the right is completely regular but not strongly regular as we can flatten the framework into a colinear non-regular framework.

Remark 2.2. If (G,p) is a strongly regular framework in a normed space then the point $f_G(p)$ is a regular value of the map f_G. This implies that the set $f_G^{-1}(f_G(p))$ of equivalent placements forms a differentiable manifold (see [43, Theorem 3.5.4]). □

†The map T is forced to be a tangent vector of the linear isometry group of X at the identity map I. To see this, first consider the map T as a linear vector field. For any integral curves $\alpha, \beta : \mathbb{R} \to X$ (i.e., differentiable paths where $\alpha'(t) = T(\alpha(t))$), the distance $\|\alpha(t) - \beta(t)\|$ is constant as t varies. This is as the support functional equation satisfied by T at every point in X implies the map $t \mapsto \|\alpha(t) - \beta(t)\|$ is constant; see [20] for details on how this can be proven via generalised derivatives. Hence T is the derivative of a differentiable path in the linear isometry group of X.

The set $\mathrm{Reg}(G;X)$ is an open subset of the set of well-positioned placements of G in X (see [21, Lemma 4.4]), which in turn is a conull subset of X^V. Hence the set $\mathrm{Reg}(G;X)$ is not a null set. Fortunately, the situation is drastically improved for analytic normed spaces.

Proposition 2.3 ([24, Propositions 3.2 and 3.6])**.** Let X be an analytic normed space and $G = (V,E)$ a graph. Then $\mathrm{Reg}(G;X)$, $\mathrm{Str}(G;X)$, $\mathrm{Com}(V;X)$ and $\mathrm{ComStr}(V;X)$ are open conull subsets of X^V. □

Similarly to analytic normed spaces, the sets of regular, strongly regular, completely regular and completely strongly regular placements in any Euclidean space are all open conull sets. Many results in Euclidean rigidity theory are worded in terms of *generic placements*; i.e., a placement where the coordinates form an algebraically independent set. In our language, any generic placement gives rise to a regular value (see [14, Proposition 3.3]) and hence is completely strongly regular, however not every completely strongly regular framework will be generic; this is as $\mathrm{ComStr}(V;\mathbb{E}^d)$ is open and conull, while the set of generic placements is only conull.

2.4 Combinatorial rigidity in normed planes

Given a connected graph, the placement of this graph in X that takes every vertex to the same point will give a locally rigid framework. The existence of such degenerate placements motivates us to consider, rather than whether there exists a placement giving a locally rigid framework, whether there exists an open set of placements corresponding to locally rigid frameworks. If a framework (G,p) is infinitesimally rigid in a normed space X, then there exists an open set $U \subset X^V$ where for all $q \in U$, the framework (G,q) is locally rigid [20, Corollary 3.10]. This fact underlines the usefulness of infinitesimal rigidity and leads us to give the following definitions. A graph $G = (V,E)$ is *rigid* (respectively, *minimally rigid*, *redundantly rigid*) in a normed space X if there exists a well-positioned framework (G,p) in X which is infinitesimally rigid (respectively, minimally rigid, redundantly rigid). A graph that is not rigid in X is said to be *flexible*.

Given a non-negative integer k, a graph $G = (V,E)$ is $(2,k)$-*sparse* if $|E'| \leq 2|V'| - k$ for every subgraph (V',E') of G (with $|E'| > 0$). Further G is $(2,k)$-*tight* if it is $(2,k)$-sparse and $|E| = 2|V| - k$. The following result of Pollaczek-Geiringer [51] exactly characterises which graphs are minimally rigid in the Euclidean plane.

Theorem 2.4. A graph is minimally rigid in $\mathbb{E}^2$ if and only if it is isomorphic to K_1 or it is $(2,3)$-tight. □

The next result (proved in full generality in [19], building on the special case of ℓ_p^2 [36]) gives an analogous characterisation of the minimally rigid graphs for any non-Euclidean normed plane.

Theorem 2.5. A graph is minimally rigid in a non-Euclidean normed plane if and only if it is $(2,2)$-tight. □

By a classical result of Nash-Williams [46], a graph is $(2,2)$-tight if and only if it is the edge-disjoint union of two spanning trees. Such graphs are well studied, for example in tree packing problems in combinatorial optimisation [25]. By comparing Theorem 2.4 and Theorem 2.5 we observe that there are graphs which are rigid in non-Euclidean normed planes that are flexible in $\mathbb{E}^2$, an example being two copies of K_4 intersecting in a single vertex, and vice versa, an example being the complete bipartite graph $K_{3,3}$.

Definition 2.6. We define a graph G to be *weakly globally rigid* in a normed space X if the set

$$\mathrm{GRig}(G;X) := \big\{p \in X^V : (G,p) \text{ is globally rigid}\big\}$$

has a non-empty interior. □

In the literature it is not uncommon to define a graph G to be globally rigid in $\mathbb{E}^d$ if (and only if) there exists a globally rigid generic framework (G,p) in $\mathbb{E}^d$. It follows from the "averaging theorem" of Connelly and Whiteley [15] that this definition implies weak global rigidity in Euclidean spaces. Furthermore, global rigidity in Euclidean spaces is a "generic property" (i.e., if the property holds for a single generic placement then it holds for all of them) [14, 26]. Hence the interior of the set $\mathrm{GRig}(G;\mathbb{E}^d)$ will either be empty (in which case G is not globally rigid) or an open conull set (in which case G is globally rigid); i.e., weak global rigidity is equivalent to global rigidity. However, in the context of normed spaces, it may not be the case that this holds. Specifically, it is conceivably possible that the set $\mathrm{GRig}(G;X)$ have non-empty interior but not be conull (or even dense). Because of this, we now make the distinction between weak global rigidity, as defined above, and *strong global rigidity*, where the set $\mathrm{GRig}(G;X)$ is both an open and conull set. When strong and weak global rigidity are equivalent (i.e., in Euclidean spaces), we will forgo this distinction.

We next state two key results for global rigidity in Euclidean spaces. The main purpose of this paper is to prove precise analogues of both results for analytic normed spaces.

Theorem 2.7 ([28])**.** If a graph G is globally rigid in $\mathbb{E}^d$, then G is either a complete graph on at most $d+1$ vertices or G is $(d+1)$-connected and redundantly rigid in $\mathbb{E}^d$. □

Theorem 2.8 ([14, 28, 29])**.** A graph G is globally rigid in $\mathbb{E}^2$ if and only if G is either a complete graph on at most 3 vertices or G is 3-connected and redundantly rigid in $\mathbb{E}^2$. □

It is easy to find examples of graphs that are globally rigid in the Euclidean plane but not in other normed planes. In particular, any wheel graph is globally rigid in $\mathbb{E}^2$ by [7], but in any non-Euclidean normed plane such graphs are minimally rigid and hence not redundantly rigid. In due course we will see that redundant rigidity is also a necessary condition for weak global rigidity in analytic normed planes (see Theorem 3.8). Conversely, it was shown in [24] that a graph formed by taking two large complete graphs and gluing them at a single edge is weakly globally rigid in any analytic normed plane, whereas in $\mathbb{E}^2$ there is an obvious violation of global rigidity obtained by reflecting one of the parts through the line defined by the common edge.

3 Necessary conditions for weak global rigidity in normed spaces

In this section we develop necessary Hendrickson-type conditions [28] for graphs to be weakly globally rigid. Although we work with arbitrary normed spaces, we occasionally require the assumption that the normed space has only finitely many linear isometries. After this section we focus solely on non-Euclidean normed planes, which always have finitely many linear isometries [55, pg. 83].

3.1 Weakly globally rigid graphs are 2-connected

We begin with the following result.

Theorem 3.1. Let $G = (V, E)$, with $|V| \geq 3$, be weakly globally rigid in a non-Euclidean normed space. Then G is 2-connected. □

Proof. Suppose for a contradiction that G is weakly globally rigid in X but it is not 2-connected. First suppose that G is a complete graph with two vertices v, w. Fix (G, p) to be a globally rigid framework in X. By applying translations to p, we may suppose that $p_w = 0$, and hence $p_v \in B_r := \{x \in X : \|x\| = r\}$. For any point $x \in B_r$, note that the framework (G, q), where $q_w = 0$ and $q_v = x$, will be equivalent to (G, p). As (G, p) is globally rigid, there exists a linear isometry of X that maps p_v to x. Hence the set of linear isometries of X act transitively on B_r. However, this implies X is Euclidean (see [55, Corollary 3.3.5]), a contradiction.

Now suppose that G is not the complete graph with 2 vertices. Hence there exists $u \in V$ and a partition V_1, V_2 of $V \setminus \{u\}$ so that there is no edge of G connecting a vertex in V_1 to a vertex in V_2. Choose any well-positioned placement p of G with $p_u = 0$ so that p lies in the interior of $\mathrm{GRig}(G; X)$. By perturbing if necessary, we may also assume that: (i) for any distinct vertices v, w we have $p_v \neq p_w$, and (ii) there exist vertices $v_1 \in V_1$ and $v_2 \in V_2$ where $\|p_{v_1} - p_{v_2}\| \neq \|p_{v_1} + p_{v_2}\|$. To see why we may assume point (ii), we note that if $\|p_{v_1} - p_{v_2}\| = \|p_{v_1} + p_{v_2}\|$, we will exchange p for the placement $q \in \mathrm{GRig}(G; X)$ where $q_{v_1} = p_{v_1} + \delta(p_{v_1} + p_{v_2})$ and $q_{v_2} = p_{v_2} + \delta(p_{v_1} + p_{v_2})$ for sufficiently small $\delta > 0$ and $q_v = p_v$ otherwise, as then $\|q_{v_1} - q_{v_2}\| = \|p_{v_1} - p_{v_2}\|$ and $\|q_{v_1} + q_{v_2}\| = (1 + \delta)\|p_{v_1} + p_{v_2}\|$; indeed if $\|p_{v_1} - p_{v_2}\| = (1 + \delta)\|p_{v_1} + p_{v_2}\|$ also, then $p_{v_1} = p_{v_2}$, contradicting our assumption that p is an injective map.

Define the placement p' of G with $p'_v = p_v$ if $v \in V_1 \cup \{u\}$ and $p'_v := -p_v$ if $v \in V_2$. Then $(G, p) \sim (G, p')$ but $p \not\sim p'$ since $\|p'_{v_1} - p'_{v_2}\| = \|p_{v_1} + p_{v_2}\| \neq \|p_{v_1} - p_{v_2}\|$. Thus (G, p) is not globally rigid, contradicting that $p \in \mathrm{GRig}(G; X)$. ■

Compared to Euclidean spaces, this is a relatively low connectivity requirement. Indeed, any graph with at least $d + 2$ vertices that is weakly globally rigid in $\mathbb{E}^d$ must be $(d + 1)$-connected by Theorem 2.7. Figure 4 shows both an example and non-example of this sufficient condition.

Proposition 3.2. Let X be a normed space with finitely many linear isometries. Suppose there exists a graph $G = (V, E)$ that is weakly globally rigid in X with $|V| \geq 2$. Then there exists a graph G' that is weakly globally rigid in X but not 3-connected. □

Proof. Choose an edge $v_1v_2 \in E$. Define $G' = (V', E')$ to be the graph formed from gluing two copies $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G at the edge v_1v_2. As X only has finitely many linear isometries, there exists an open dense set of points that are not invariant under any non-trivial linear isometry of X. Hence we may choose an open set $U \subset \mathrm{GRig}(G; X)$ where for each $p \in U$, the vector $p_{v_1} - p_{v_2}$ is not invariant under any non-trivial linear isometry of X. Define U' to be the set of placements of G' in X where for each $p \in U'$ we have $p|_{V_i} \in U$ for each $i \in \{1, 2\}$. Since the set U is open, it follows that U' is also open. Choose a placement $p \in U'$, and choose a placement $q \in X^{V'}$ so that $(G, q) \sim (G, p)$ and $q_{v_1} = p_{v_1}$. By applying translations, we may assume $p_{v_1} = 0$. Since both $(G_1, p|_{V_1})$ and $(G_2, p|_{V_2})$ are globally rigid, there exist linear isometries T_1, T_2 of X such that $T_i(p_v) = q_v$ for all $v \in V_i$. Importantly, $T_1(p_{v_2}) = T_2(p_{v_2}) = q_{v_2}$. As $p_{v_2} - p_{v_1} = p_{v_2}$ is invariant under linear isometries and $p_{v_2} = T_2^{-1} \circ T_1(p_{v_2})$, we must have $T_1 = T_2$. Hence $q \sim p$ and (G, p) is globally rigid. ■

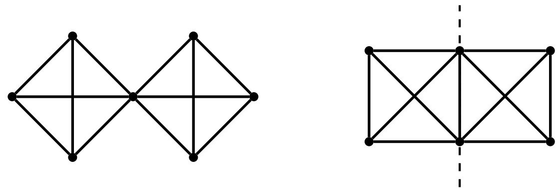


Fig. 4. (Left): A minimally rigid graph in any non-Euclidean normed plane that is dependent and flexible in $\mathbb{E}^2$. It is not weakly globally rigid in any normed plane as it is not 2-connected. (Right): A graph that is rigid in any normed plane. This graph is also weakly globally rigid in all analytic normed planes [24] (see Theorem 7.1 below). It is not globally rigid in $\mathbb{E}^2$; for almost any placement, the left two vertices may be reflected across the dashed line to obtain an equivalent but non-congruent framework. (This reflection does not exist in most normed planes.)

3.2 Global rigidity implies redundant rigidity

Recalling the relevant notation from Section 2, we define for a normed space X and a graph G the set

$$X_G^{IF} := \{p \in X_G : \text{ for all } g \in \mathrm{Isom}(X), g \circ p = p \Rightarrow g = I\}.$$

Note that if $p \in X_G$ is spanning, then $p \in X_G^{IF}$. This fact leads to our next result.

Lemma 3.3. Let X be a d-dimensional normed space and $G = (V, E)$. If $|V| \geq d + 1$ then X_G^{IF} is an open conull set. □

We can also determine whether a placement lies in X_G^{IF} solely by the dimension of its kernel.

Lemma 3.4 ([24, Lemma 3.9])**.** Let X be a d-dimensional smooth normed space, $G = (V, E)$ and $p \in X_G$ be a placement of G with $\dim \ker df_G(p) < d - 1 + |V|$. Then $p \in X_G^{IF}$. □

Given a normed space X and two placements p, p' of a set V, we remember that $p \sim p'$ if and only if there exists an isometry $g \in \mathrm{Isom}(X)$ with $p' = g \circ p$. It is immediate that if $p \in X_G$ and $g \in \mathrm{Isom}(X)$ then $g \circ p \in X_G$. Define the quotient space $X^V/\sim$ with equivalence classes $\tilde{p} := \{q \in X^V : q \sim p\}$ and quotient map $\pi : X^V \to X^V/\sim$.

Lemma 3.5. Let X be a normed space and M be a differentiable submanifold of X_G^{IF}, where $g \circ p \in M$ for each $p \in M$ and $g \in \mathrm{Isom}(X)$. Then the set $\pi(M)$ is a differentiable submanifold of $X^V/\sim$ with $\dim \pi(M) = \dim M - \dim \mathrm{Isom}(X)$, and the map π restricted to M and $\pi(M)$ is a submersion (i.e., a differentiable map with surjective derivative everywhere). □

Proof. For each point $p \in M \subset X_G^{IF}$, the map $g \mapsto g \circ p$ is injective. The result now follows from [21, Lemma 3.3] and [2, Proposition 4.1.23]. ■

We note that if $p \sim q$ then $f_G(p) = f_G(q)$ and hence the *quotient rigidity map*

$$\tilde{f}_G : X_G^{IF}/\sim \to \mathbb{R}^E, \ \tilde{p} \mapsto \left(\frac{1}{2}\|p_v - p_w\|^2\right)_{vw \in E}$$

is well-defined. Define the set

$$\mathcal{C}(G, p) := \tilde{f}_G^{-1}\left(\tilde{f}_G(\pi(p))\right).$$

If f_G is differentiable at p then $\tilde{f}_G$ is differentiable at $\pi(p)$.

We now focus on d-dimensional smooth normed spaces with finite linear isometries. Since all isometries are affine [44], the isometry group is d-dimensional.

Lemma 3.6. Let X be a d-dimensional smooth normed space with finitely many linear isometries, (G, p) be a strongly regular framework in X and $k := \dim \ker df_G(p) - d$. If $k < |V| - 1$ and G is connected, then $\mathcal{C}(G, p)$ is a compact differentiable submanifold of $X_G^{IF}/\sim$ of dimension k. □

Proof. By our assumption that (G,p) is strongly regular, for every $p' \in f_G^{-1}(f_G(p))$ we have $\dim\ker df_G(p') = \dim\ker df_G(p) < d-1+|V|$. Hence $f_G^{-1}(f_G(p))$ is a subset of X_G^{IF} by Lemma 3.4. As (G,p) is strongly regular the set $f_G^{-1}(f_G(p))$ is a differentiable submanifold of X_G^{IF} (see Remark 2.2). Since $\mathcal{C}(G,p) = \pi(f_G^{-1}(f_G(p)))$, the set $\mathcal{C}(G,p)$ is a differentiable submanifold of $X_G^{IF}/\sim$ with dimension k by Lemma 3.5. Choose any $v_0 \in V$ and define the closed set $S := \{q \in X^V : f_G(q) = f_G(p),\ q_{v_0} = 0\}$. Since G is connected, the set S is compact. As $\pi(S) = \mathcal{C}(G,p)$ and the map π is continuous, the set $\mathcal{C}(G,p)$ must also be compact. ■

We are now ready to prove the following sufficient condition for global rigidity.

Theorem 3.7. Let (G,p) be a completely strongly regular globally rigid framework in a smooth non-Euclidean d-dimensional normed space X with finitely many linear isometries. If G has at least two vertices, then (G,p) is redundantly rigid. □

Proof. Assume G is not redundantly rigid in X, i.e., there exists $vw \in E$ such that $G - vw$ is flexible. If G is flexible then (G,p) is not globally rigid by Theorem 2.1. Suppose G is rigid (and so $|V| > 2$ as K_2 is flexible, see [21, Theorem 5.8]). Since $\dim\ker df_G(p) = d$, we have that $\dim\ker df_{G-vw}(p) - d = 1 < |V| - 1$. By Lemma 3.6, the set $\mathcal{C}(G,p)$ is finite and $\mathcal{C}(G-vw,p)$ is a compact 1-dimensional differentiable manifold; i.e., the connected component of $\mathcal{C}(G-vw,p)$ containing $\tilde{p}$ is diffeomorphic to a circle (see [40, Theorem 5.27]). Hence there exists a continuously differentiable map $\phi : [0,1] \to \mathcal{C}(G-vw,p)$ with $\phi(0) = \phi(1) = \tilde{p}$ and $\phi(t_1) \neq \phi(t_2)$ if $0 < t_1 < t_2 < 1$. Define the continuously differentiable function

$$h_{vw} : f_{G-vw}^{-1}(f_{G-vw}(p)) \to \mathbb{R},\ q \mapsto \|q_v - q_w\|^2.$$

If $\pi(q) = \pi(q')$ then $h_{vw}(q) = h_{vw}(p)$ and hence the map

$$\tilde{h}_{vw} : \mathcal{C}(G-vw,p) \to \mathbb{R},\ \tilde{q} \mapsto \|q_v - q_w\|^2$$

is well-defined. Since both h_{vw} and π are continuously differentiable and $\tilde{h}_{vw} \circ \pi = h_{vw}$, the map $\tilde{h}_{vw}$ is continuously differentiable. Define the continuously differentiable function

$$g : [0,1] \to \mathbb{R},\ t \mapsto \tilde{h}_{vw}(\phi(t)).$$

We will denote $t_{\max}$ and $t_{\min}$ to be the points where g attains its maximum and minimum values respectively. We cannot have $t_{\max} = t_{\min}$ (i.e., g is constant), as this would imply the set $\mathcal{C}(G,p)$ is not finite, contradicting that (G,p) is locally rigid and p is strongly regular.

First suppose that either $t_{\max} = 0$ or $t_{\min} = 0$. As 0 is a local minimum/maximum of g, we must have $g'(0) = 0$, i.e., given $\tilde{u}$ is the unique (up to scalar) vector in the tangent space of $\mathcal{C}(G-vw,p)$ at $\tilde{p}$, we have $d\tilde{h}_{vw}(\tilde{p})(\tilde{u}) = 0$. As π restricted to $\mathcal{C}(G-vw,p)$ and $\pi(\mathcal{C}(G-vw,p))$ is a submersion (Lemma 3.5), there exists a vector $u \in X^V$ in the tangent space of $f_{G-vw}^{-1}(f_{G-vw}(p))$ at p such that $dh_{vw}(p)(u) = 0$ and $d\pi(p)(u) \neq 0$. However this implies u is a non-trivial infinitesimal flex of $(G-vw,p)$ where $\varphi_{p_v - p_w}(u_v - u_w) = 0$, contradicting that (G,p) is infinitesimally rigid.

Without loss of generality, we may now assume $0 < t_{\min} < t_{\max} < 1$. By the Intermediate Value Theorem, there exists $t_{\min} < t_0 < t_{\max}$ such that $g(t_0) = g(0)$. Hence $\tilde{f}_G(\phi(t_0)) = \tilde{f}_G(\tilde{p})$ but $\phi(t_0) \neq \tilde{p}$. It follows that for any placement q of G with $\pi(q) = \phi(t_0)$, we have $(G,q) \sim (G,p)$ but $q \not\sim p$, thus (G,p) is not globally rigid. ■

3.3 Weak global rigidity in analytic normed spaces

We now extend our necessary global rigidity conditions from frameworks to graphs by restricting to analytic normed spaces.

While every globally and infinitesimally rigid framework is strongly regular, not every such framework is completely strongly regular. In fact some normed spaces have no completely strongly regular frameworks at all (see Section 7.1 for more details). Fortunately this not the case for analytic normed spaces (Proposition 2.3), allowing us to obtain our main result of the section.

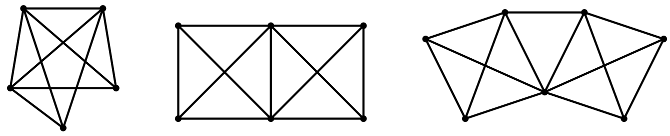
Theorem 3.8. Let $G = (V,E)$ be weakly globally rigid in an analytic normed space X with finitely many linear isometries. If $|V| \geq 2$, then G is redundantly rigid and 2-connected. □

Proof. By Theorem 3.1, G is 2-connected. Fix $U \subset X^V$ to be an open set of globally rigid placements of G. Using Proposition 2.3, we choose a completely strongly regular placement $p \in U$. As X is smooth, (G,p) (and hence also G) is redundantly rigid in X by Theorem 3.7. ■

We conjecture that the theorem remains valid in a wider family of normed spaces including, for example, all smooth ℓ_p-spaces. The remainder of the paper will be devoted to showing that for analytic normed planes these conditions are sufficient as well as necessary.

4 Sparse graphs and circuits

In the next three sections we study the graphs relevant to global rigidity in analytic normed planes. We begin by describing the circuits in the following matroid. Let $\mathcal{M}(2,2)$ denote the *simple* $(2,2)$*-sparse matroid*, that is, the matroid on $E(K_n)^\ddagger$ in which a set $E \subseteq E(K_n)$ is independent if $E = \emptyset$ or $E \neq \emptyset$ and the subgraph induced by E is $(2,2)$-sparse. We will slightly abuse notation by using $(2,2)$*-circuit* to refer both to a circuit in the matroid $\mathcal{M}(2,2)$ as well as the graph induced by a circuit in this matroid. In graph theoretic terms, $G = (V,E)$ is a $(2,2)$-circuit if and only if $|E| = 2|V| - 1$ and $|E'| \leq 2|V'| - 2$ for all proper subgraphs (V', E') of G. Three examples of $(2,2)$-circuits are illustrated in Figure 5.



K_5^- B_1 B_2

Fig. 5. Three small $(2,2)$-circuits.

4.1 Recursive constructions

A recursive construction of $(2,2)$-circuits was first proved in [47]. To describe it we require decomposition results for $(2,2)$-circuits which uses the graph operations illustrated in Figure 6. These operations are as follows for a pair of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ that may share edges or vertices. We denote the neighbourhood and degree of a vertex v in a graph G by $N_G(v)$ and $d_G(v)$ respectively; when the graph in question is clear we shall drop the subscript.

(i) Suppose that for each $i \in \{1,2\}$ there exists a subgraph $A_i = (U_i, F_i)$ of G_i such that A_1 is the complete graph with $U_1 = \{a,b\}$, and A_2 is the complete graph with $U_2 = \{a,b,c,d\}$. If $V_1 \cap V_2 = \{a,b\}$ and $d_{G_2}(c) = d_{G_2}(d) = 3$, then the *1-join* of (G_1, G_2) is the graph $G = (V,E)$ where

$$V = V_1 \cup (V_2 \setminus \{c,d\}), \qquad E = (E_1 \setminus F_1) \cup (E_2 \setminus F_2).$$

(ii) Suppose that for each $i \in \{1,2\}$ there exists a subgraph $A_i = (U_i, F_i)$ of G_i such that A_1 is the complete graph with $U_1 = \{a,b,c_1,d_1\}$, and A_2 is the complete graph with $U_2 = \{a,b,c_2,d_2\}$. If $V_1 \cap V_2 = \{a,b\}$ and $d_{G_2}(c_i) = d_{G_2}(d_i) = 3$ for each $i \in \{1,2\}$, then the *2-join* of (G_1, G_2) is the graph $G = (V,E)$ where

$$V = (V_1 \setminus \{c_1,d_1\}) \cup (V_2 \setminus \{c_2,d_2\}), \qquad E = (E_1 \setminus F_1) \cup (E_2 \setminus F_2) + ab.$$

(iii) Suppose that for each $i \in \{1,2\}$, there exists a vertex v_i with neighbourhood $\{a_i, b_i, c_i\}$ in G_i, and set $F_i = \{a_i v_i, b_i v_i, c_i v_i\}$. If $V_1 \cap V_2 = \emptyset$, then the *3-join* of (G_1, G_2) is the graph $G = (V,E)$ where

$$V = (V_1 - v_1) \cup (V_2 - v_2), \qquad E = (E_1 \setminus F_1) \cup (E_2 \setminus F_2) \cup \{a_1a_2, b_1b_2, c_1c_2\}.$$

For each join operation we can define a separation operation that acts as an inverse. To define these operations, we recall two types of separation for a graph G. A *k-vertex-separation* is a pair of induced subgraphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ that share exactly k vertices where $G = G_1 \cup G_2$, and neither $V_1 \setminus V_2$ nor $V_2 \setminus V_1$ are empty; the set $V_1 \cap V_2$ is then called a *vertex cut*. In a k-edge-connected graph, a *k-edge-separation* is a pair of non-empty vertex-disjoint induced subgraphs G_1, G_2 where $G_1 \cup G_2 = G - S$ for some edge set $S \subseteq E$ with $|S| = k$; the set S is then called an *edge cut*. We will mainly be interested in 2-vertex-separations and 3-edge-separations. We define a 2-vertex-separation (G_1, G_2) to be *non-trivial* if neither G_1 nor G_2 are isomorphic to K_4, and we define a 3-edge-separation (G_1, G_2) to be *non-trivial* if the corresponding edge cut S is independent (i.e., no two edges in S share a vertex). Using these definitions, we now define the following operations for a graph $G = (V,E)$ and pair of subgraphs $H_1 = (U_1, F_1), H_2 = (U_2, F_2)$. For any disjoint non-empty sets $S, S' \subseteq V$, we denote the complete graph with vertex set S by $K[S]$, and the complete bipartite graph with parts S, S' by $K[S, S']$.

‡It is common in the literature for the $(2,2)$-sparse matroid to be defined on the complete multigraph (i.e., the multigraph with two parallel edges between every pair of vertices) since parallel edges may be independent. Hence, while all graphs in this article are simple, we use the word simple here to emphasise this distinction.

(i) Suppose that (H_1, H_2) is a 2-vertex-separation with $U_1 \cap U_2 = \{a, b\}$ and $F_1 \cap F_2 = \emptyset$. A *1-separation of G (with respect to (H_1, H_2))* is an ordered pair (G_1, G_2) with $G_1 = H_1 \cup K[\{a, b\}]$ and $G_2 = H_2 \cup K[\{a, b, c, d\}]$ for two new vertices c, d.

(ii) Suppose that (H_1, H_2) is a 2-vertex-separation with $U_1 \cap U_2 = \{a, b\}$ and $F_1 \cap F_2 = \{ab\}$. A *2-separation of G (with respect to (H_1, H_2))* is an ordered pair (G_1, G_2) with $G_1 = H_1 \cup K[\{a, b, c_1, d_1\}]$ and $G_2 = H_2 \cup K[\{a, b, c_2, d_2\}]$ for four new vertices c_1, c_2, d_1, d_2.

(iii) Suppose that (H_1, H_2) is a non-trivial 3-edge-separation where the edges a_1a_2, b_1b_2, c_1c_2 connect H_1 and H_2. A *3-separation of G (with respect to (H_1, H_2))* is an ordered pair (G_1, G_2) with $G_1 = H_1 \cup K[\{v_1\}, \{a_1, b_1, c_1\}]$ and $G_2 = H_2 \cup K[\{v_2\}, \{a_2, b_2, c_2\}]$ for two new vertices v_1, v_2.

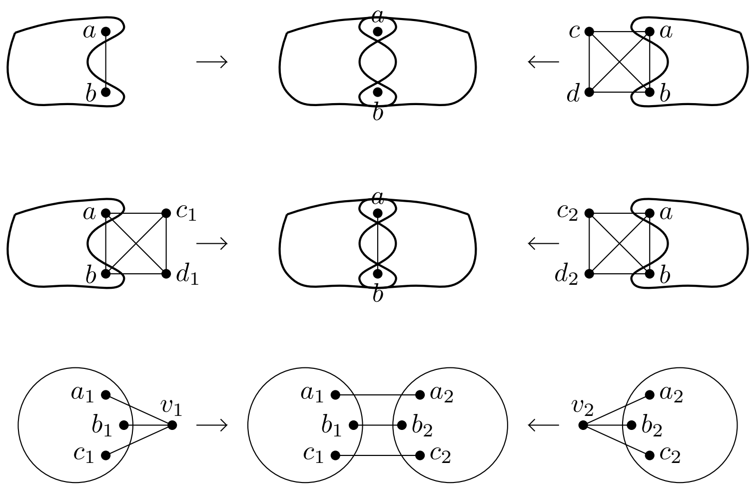


Fig. 6. The 1-, 2- and 3-join operations.

For $(2, 2)$-circuits, the 1-join operation can be defined based on the density of each "side" of the 2-vertex-separation. However in the next section we will apply these operations to a broader family of graphs where that convenience is unavailable. For this reason we state the next three lemmas differently to how they appeared in [47].

Lemma 4.1 ([47, Lemmas 3.1, 3.2, 3.3])**.** Let G_1 and G_2 be graphs and suppose that G is a j-join of G_1 and G_2 for some $1 \leq j \leq 3$. If G_1 and G_2 are $(2, 2)$-circuits then G is a $(2, 2)$-circuit. □

Lemma 4.2 ([47, Lemmas 3.2, 3.3])**.** Let G be a graph and suppose that (G_1, G_2) is a j-separation of G for some $j \in \{2, 3\}$. If G is a $(2, 2)$-circuit then G_1 and G_2 are $(2, 2)$-circuits. □

Lemma 4.3 ([47, Lemma 3.1])**.** Let G be a graph with 2-vertex-separation (H_1, H_2). Suppose that (G_1, G_2) is a 1-separation of G with respect to (H_1, H_2), and (G'_2, G'_1) is a 1-separation of G with respect to (H_2, H_1). If G is a $(2, 2)$-circuit then either G_1 and G_2 are $(2, 2)$-circuits and G'_1 and G'_2 are not $(2, 2)$-circuits, or G'_1 and G'_2 are $(2, 2)$-circuits and G_1 and G_2 are not $(2, 2)$-circuits. □

The recursive construction of $(2, 2)$-circuits in [47] begins with the three $(2, 2)$-circuits shown in Figure 5 and uses the j-join operations as well as the well-known 1-extension operation. Given a graph $G = (V, E)$, a *1-extension* forms a new graph by deleting an edge $xy \in E$ and adding one new vertex v and three new edges xv, yv, zv for some $z \in V \setminus \{x, y\}$. Conversely, a *1-reduction* deletes a vertex v of degree 3 and adds an edge between any two non-adjacent neighbours of v.

Theorem 4.4 ([47, Theorem 1.1])**.** Suppose G is a $(2, 2)$-circuit. Then G can be obtained from disjoint copies of K_5^-, B_1, and B_2 by recursively applying 1-extension operations within connected components and 1-, 2- and 3-join operations of connected components. □

An alternative recursive construction of $(2, 2)$-circuits was provided in [31]. To describe this construction we first define two local graph operations. The first operation, *K_4^--extension*, deletes an edge v_1v_2, adds new vertices u_1 and u_2, and adds new edges $v_1u_1, v_1u_2, v_2u_1, v_2u_2, u_1u_2$. Conversely, a *$K_4^-$-reduction* deletes two adjacent vertices u_1, u_2 of degree 3, where $N_G(u_1) \cap N_G(u_2) = \{v_1, v_2\}$ and $v_1v_2 \notin E$, and adds the edge v_1v_2. The second operation, the *generalised vertex split*, is defined as follows: choose $v \in V$ and a partition N_1, N_2 of the neighbours of v; then delete v from G and add new vertices v_1 and v_2, joined to N_1 and N_2 respectively; finally add new edges v_1v_2 and v_1x for some $x \in V \setminus N_1$. Conversely, an *edge-reduction* contracts an edge to a vertex and then deletes the resulting loop and one other edge to a neighbour of the new vertex. These operations are illustrated in Figures 7 and 8 respectively.

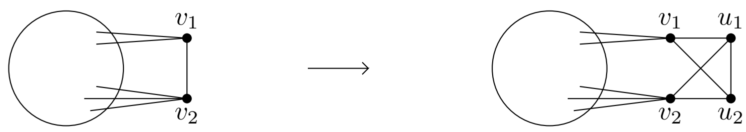


Fig. 7. The K_4^--extension. The K_4^--reduction is the inverse of the K_4^--extension.

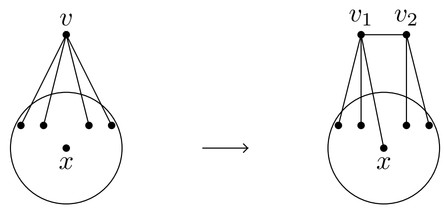


Fig. 8. Generalised vertex split. An edge-reduction is the inverse of the generalised vertex split.

Note that the usual vertex splitting operation (see [56]) is the special case when x is chosen to be a neighbour of v_2. Also, the special case when v_1 has degree 3 (and $v_2 = v$) is a 1-extension.

Theorem 4.5 ([31, Theorem 2.1])**.** Every $(2,2)$-circuit can be obtained from K_5^- or B_1 by recursively applying the operations of K_4^--extension and generalised vertex splitting such that each intermediate graph is a $(2,2)$-circuit. □

Later we will show that the operations in the previous theorem preserve weak global rigidity in any analytic normed plane X. It then quickly follows that every $(2,2)$-circuit is weakly globally rigid in X. It is tempting to think that combining this with a simple gluing argument suffices to prove that 2-connected and redundantly rigid graphs[§] are weakly globally rigid in X. We can easily show that gluing globally rigid frameworks, when done correctly, will create a globally rigid of framework.

Proposition 4.6. Let $H = (U, F)$ and $H' = (U', F')$ be graphs with $|U \cap U'| \geq d$ and let X be a d-dimensional normed space. Let (G, p) be a completely regular framework in X where $G = H \cup H'$, and define the placements $q := p|_U$, $q' := p|_{U'}$ and $q \cap q' := p|_{U \cap U'}$. Suppose that both (H, q) and (H', q') are globally rigid in X and $q \cap q'$ is isometrically full. Then (G, p) is globally rigid in X. □

Proof. Let $(G, \tilde{p})$ be a framework in X that is equivalent to (G, p). Since both (H, q) and (H', q') are globally rigid in X, there exist isometries g, g' where $g \circ q = \tilde{p}|_U$ and $g' \circ q' = \tilde{p}|_{U'}$. In particular $\tilde{p}_v = (g \circ p)_v = (g' \circ p)_v$ for $v \in U \cap U'$. Since $q \cap q'$ is isometrically full, we have that $g = g'$. Hence $\tilde{p} = g \circ p$ and $(G, \tilde{p})$ is congruent to (G, p). ■

Unfortunately, this does not show that gluing weakly globally rigid graphs over a sufficient number of vertices will preserve weak global rigidity. This is a consequence of the distinction between weak and strong global rigidity and the fact that it is currently unknown whether they coincide. That is, whether global rigidity is a *generic property*, i.e., a graph is globally rigid in a normed space X if and only if $\mathrm{GRig}(G; X)$ is an open and dense set. For example, suppose that H, H' are weakly globally rigid graphs in a normed space X, and suppose that we can find isometrically full placements of their intersection. Even knowing that both H and H' have open sets of globally rigid realisations in X is not sufficient to apply Proposition 4.6, as there may not exist globally rigid placements of both H and H' that agree on the vertices in their intersection.

[§]In Section 5 we will show that all such graphs are obtained by an "ear decomposition" into $(2,2)$-circuits.

4.2 Properties of $(2,2)$-circuits

We require a number of technical results about $(2,2)$-circuits from [47] and some mild extensions of these results. For convenience we refer to vertices of degree 3 as *nodes*. We say a node v (and the corresponding 1-reduction) in a $(2,2)$-circuit G is *admissible* if there exists a 1-reduction of G at v resulting in another $(2,2)$-circuit. A key theorem used in the proof of Theorem 4.4 establishes connectivity conditions sufficient to guarantee the existence of admissible nodes. That result was strengthened in [31] to the following.

Theorem 4.7 ([31, Theorem 2.2])**.** Suppose G is a $(2,2)$-circuit which is distinct from K_5^-, B_1 and B_2. If G has no non-trivial 2-vertex-separation and no non-trivial 3-edge-separation then G has at least two admissible nodes. □

We omit the proof of the next elementary result.

Lemma 4.8. Let G' be obtained from G by a 1-extension. If G is a $(2,2)$-circuit then so is G'. □

All the remaining results in this section will concer a graph $G=(V,E)$. Given a non-empty subset X of V, $G[X]$ is the subgraph of G induced by X and $i_G(X)$ is the number of edges of $G[X]$. The set X is *critical in* G (or simply *critical* if the context is clear) if $i_G(X)=2|X|-2$. For disjoint subsets X and Y of V, we define $d_G(X,Y):=|\{xy\in E : x\in X, y\in Y\}|$. When the graph G is clear from the context, we abbreviate the above to $i(X)$ and $d(X,Y)$ respectively.

Lemma 4.9 ([47, Lemma 2.2])**.** Let G be a $(2,2)$-circuit and let $X,Y\subsetneq V$ be critical such that $|X\cap Y|\geq 1$ and $X\cup Y\subsetneq V$. Then $X\cap Y$ and $X\cup Y$ are both critical, and $d(X\setminus Y, Y\setminus X)=0$. □

Lemma 4.10 ([47, Lemma 2.3])**.** Let G be a $(2,2)$-circuit and let $X\subsetneq V$ be critical. Then G is 2-connected and 3-edge-connected, and $G[X]$ is connected. □

Note that the last lemma implies the minimum degree of a $(2,2)$-circuit G, denoted by $\delta(G)$, is at least 3. Since $|E|=2|V|-1$, it follows that $\delta(G)=3$.

Lemma 4.11 ([47, Lemma 2.4])**.** Let G be a $(2,2)$-circuit and let $X\subsetneq V$ be critical. Then $V\setminus X$ contains a node in G. □

Lemma 4.12 ([47, Lemma 2.5])**.** Let G be a $(2,2)$-circuit and let v be a node in G with neighbourhood $\{x,y,z\}$. Then the 1-reduction at v adding xy is not admissible if and only if either $xy\in E$, or there is a critical set $X\subsetneq V$ with $x,y\in X$ and $v,z\notin X$. □

For a graph G, let $V_3=\{v\in V : v \text{ is a node}\}$. Let $V_3^*\subseteq V_3$ be the set of nodes which are not contained in copies of K_4 in G. Later we will use the elementary fact that if $v\in V_3^*$ and u is a neighbour of v then $u\in V_3$ if and only if $u\in V_3^*$. We call a node v with $d_{G[V_3^*]}(v)\leq 1$ a *leaf* node and a node with $d_{G[V_3^*]}(v)=2$ a *series* node. Any node v with $d_{G[V_3^*]}(v)=3$ can never be admissible since any 1-reduction will give a graph with minimum degree 2.

Lemma 4.13. Let G be a $(2,2)$-circuit. Then $G[V_3]$ contains no cycles. □

Proof. Suppose $C\subseteq V_3$ induces a cycle in $G[V_3]$ and hence in G. As G is not a cycle, $V\setminus C\neq\emptyset$. Also, $d(C,V\setminus C)=|C|$ since $C\subseteq V_3$. Now

$$i(V\setminus C)=2|V|-1-i(C)-d(C,V\setminus C)=2|V|-1-2|C|=2(|V|-|C|)-1=2|V\setminus C|-1,$$

a contradiction. Hence $G[V_3]$ contains no cycles. ■

For a node v in a $(2,2)$-circuit G with $N(v)=\{x,y,z\}$, we say that a critical set X in G is *v-critical* if $x,y\in X$ and $v,z\notin X$. If X is a v-critical set containing $\{x,y\}$ for some node v with neighbourhood $\{x,y,z\}$ and $d_G(z)\geq 4$, then X is *node-critical.*

The next lemma is a strengthening of [47, Lemma 2.10].

Lemma 4.14. Let G be a $(2,2)$-circuit containing a node v with neighbourhood $\{x,y,z\}$. Suppose $d_G(z)\geq 4$ and there exists a v-critical set X in G containing x,y. Further suppose that there exists a non-admissible node $u\in V\setminus(X+v)$ such that either:

(i) u is a series node with precisely one neighbour w in X, and w is a node, or
(ii) u is a leaf node with no edges between its neighbours.

Then there is a node-critical set X' in G such that $X \subsetneq X'$. □

Proof. Suppose (i) holds and let $N_G(u) = \{w, a, b\}$. As u is a series node we may suppose without loss of generality that $d_G(a) = 3$ and $d_G(b) \geq 4$. Lemma 4.13 implies that $G[V_3]$ contains no cycles, so $wa \notin E$. As u is non-admissible and $wa \notin E$, Lemma 4.12 implies there exists a u-critical set Y in G containing $\{w, a\}$. As $w \in X \cap Y$ and $u, b \notin X \cup Y$, Lemma 4.9 implies that $X \cup Y$ is a u-critical set in G containing $\{w, a\}$. Moreover, $X \cup Y$ is node-critical as $d_G(b) \geq 4$. Finally, as $a \in Y \setminus X$, we have $X \subsetneq X \cup Y$.

Now suppose (ii) holds and let $N_G(u) = \{a, b, c\}$. As $v \notin X + u$ we must have $|N_G(u) \cap X| \leq 2$, since otherwise $G[X + u]$ is a proper subgraph in G with $2|X + u| - 1$ edges. If $|N_G(u) \cap X| = 2$ then $X + u$ is a node-critical v-critical set in G properly containing X and we are done. So we may suppose that $|N_G(u) \cap X| \leq 1$. As u is a leaf node we may suppose without loss of generality that $d_G(a), d_G(c) \geq 4$. As u is non-admissible and there are no edges between the neighbours of u, Lemma 4.12 implies there exist u-critical sets, Y_1 and Y_2, in G on $\{a, b\}$ and $\{b, c\}$ respectively. As $b \in Y_1 \cap Y_2$ and $u \notin Y_1 \cup Y_2$, Lemma 4.9 implies $Y_1 \cup Y_2$ is a critical set. Moreover, as $N_G(u) \subseteq Y_1 \cup Y_2$, it follows that $Y_1 \cup Y_2 = V - u$. Hence $Y_1 \cap X \neq \emptyset$ or $Y_2 \cap X \neq \emptyset$. We may suppose, without loss of generality, that $Y_1 \cap X \neq \emptyset$.

As $Y_1 \cap X \neq \emptyset$ and $u \notin Y_1 \cup X$, Lemma 4.9 implies $Y_1 \cup X$ is critical in G. Moreover, as $a, b \in Y_1 \cup X$ and $|N_G(u) \cap X| \leq 1$ we have that $X \subsetneq Y_1 \cup X$. If $c \notin Y_1 \cup X$ then $Y_1 \cup X$ is node-critical in G and we are done. On the other hand, if $c \in Y_1 \cup X$ then $c \in X$. So $Y_2 \cap X \neq \emptyset$ and $N_G(u) \cap X = \{c\}$. As $u, a \notin Y_2 \cup X$, the set $Y_2 \cup X$ is a node-critical u-critical set in G by Lemma 4.9. Finally, as $b \in Y_2 \setminus X$, we have $X \subsetneq Y_2 \cup X$. ■

The final result in this section strengthens [47, Lemma 2.8]. Similarly to 1-reductions, we define a K_4^--reduction of a $(2, 2)$-circuit to be *admissible* if the resulting graph is also a $(2, 2)$-circuit.

Lemma 4.15. Let G be a $(2, 2)$-circuit containing a node v with neighbourhood $\{x, y, z\}$. If $yz \notin E$ and $xz \in E$, then either:

(i) $xy \in E$, $\{y, z\}$ is not a vertex cut of G, and the 1-reduction of G at v adding yz is admissible,
(ii) $xy \in E$, $\{y, z\}$ is a vertex cut of G, and either the 1-reduction of G at v adding yz is admissible or the K_4^--reduction of G deleting v and x and adding yz is admissible, or
(iii) $xy \notin E$ and there is an admissible 1-reduction of G at v.

□

Proof. First suppose that $xy \in E$. If there does not exist a v-critical set X containing $\{y, z\}$ in G, then the 1-reduction of G at v adding yz is admissible by Lemma 4.12. So suppose there exists a v-critical set X containing $\{y, z\}$ in G. The induced subgraph on the set $X \cup \{v, x\}$ has at least $2|X \cup \{v, x\}| - 1$ edges, it follows that $V = X \cup \{v, x\}$ and $N_G(x) = \{v, y, z\}$. This implies $\{y, z\}$ is a vertex cut of G. Define H_1, H_2 to be the induced subgraphs of G on the vertex sets $N_G(v) + v$ and $V \setminus \{v, x\}$ respectively. Let (G_1, G_2) be the 1-separation of G on (H_1, H_2) and let (G_1', G_2') be the 1-separation of G on (H_2, H_1). As $G_1 \cong K_4$, Lemma 4.3 implies that G_1' is a $(2, 2)$-circuit. The K_4^--reduction of G that deletes v and x and adds yz gives G_1' and hence is admissible. The final case, i.e. when $xy \notin E$, was proved in [47]. ■

In the next two sections we will show that the characterisation in Theorem 4.5 essentially remains true for arbitrary 2-connected and redundantly rigid graphs in analytic normed planes. Since redundantly rigid graphs in analytic normed planes need not contain spanning $(2, 2)$-circuits, we will need some additional combinatorial concepts and results to prove it. An example is the complete bipartite graph $K_{3,6}$. Note that $|E(K_{3,6})| = 2|V(K_{3,6})|$ and so $K_{3,6}$ is not a $(2, 2)$-circuit. Since every vertex in the part of size 6 has degree 3 and every edge is incident to some such vertex, $K_{3,6}$ does not contain a spanning $(2, 2)$-circuit. Nevertheless it is redundantly rigid in any analytic normed plane by Theorem 2.5.

5 $\mathcal{M}(2, 2)$-connected graphs

Recall that a matroid $M = (E, r)$ with ground set E and rank function r is *connected* if every pair $e, f \in E$ is contained in a common circuit. Equivalently, we may define a relation on E by saying that $e, f \in E$ are related if $e = f$ or there is a circuit C in M with $e, f \in C$. It is well-known that this is an equivalence relation. If M has at least two elements, then M is connected if and only if there is only one equivalence class. A graph G is $\mathcal{M}(2, 2)$-*connected* if G has no isolated vertices, $|E(G)| \geq 2$ and every pair of elements of $E(G)$ belongs to a common $(2, 2)$-circuit in G. It is easy to see that any $\mathcal{M}(2, 2)$-connected graph contains a spanning $(2, 2)$-tight subgraph and hence is rigid in any non-Euclidean normed plane (Theorem 2.5). In fact more is true.

Lemma 5.1. Let (G, p) be a completely regular framework in a non-Euclidean normed plane X. Then G is $\mathcal{M}(2,2)$-connected if and only if G is 2-connected and (G, p) is redundantly rigid. □

Proof. Since p is a completely regular placement, (G, p) is redundantly rigid if and only if $G - e$ is rigid in X for every edge $e \in E$. When combined with Theorem 2.5, it follows that (G, p) is redundantly rigid if and only if G contains a $(2,2)$-tight spanning subgraph and every edge is contained in a $(2,2)$-circuit. We now apply the following two equivalences: (i) G is redundantly rigid on the cylinder if and only G contains a $(2,2)$-tight spanning subgraph and every edge is contained in a $(2,2)$-circuit (see [48, Theorem 5.4]), and (ii) G is redundantly rigid on the cylinder and 2-connected if and only if G is $\mathcal{M}(2,2)$-connected (see [47, Theorem 5.4]). ■

It follows that every $\mathcal{M}(2,2)$-connected graph is 3-edge-connected.

5.1 Graph operations preserving $\mathcal{M}(2,2)$-connectivity

We begin with two standard graph operations; 1-extension and edge addition.

Lemma 5.2. Let $G' = (V', E')$ be an $\mathcal{M}(2,2)$-connected graph and suppose that G' is obtained from $G = (V, E)$ by a 1-extension or an edge addition. Then G' is $\mathcal{M}(2,2)$-connected. □

Proof. Suppose that G' is obtained from G by adding a new edge e. As G is $\mathcal{M}(2,2)$-connected there exists a $(2,2)$-circuit in G containing the endpoints of e. Hence there exists a $(2,2)$-circuit C' in G' with $e \in C'$. The $\mathcal{M}(2,2)$-connectivity of G' now follows from the transitivity of the relation that defines $\mathcal{M}(2,2)$-connectivity.

Suppose that G' is obtained from G by a 1-extension operation that deletes an edge e and adds a new vertex v with three incident edges e_1, e_2, e_3. For $1 \le i \le 3$, let $e_i = vu_i$ and suppose that $e = u_1u_2$. Since G is $\mathcal{M}(2,2)$-connected, $d_G(u_3) \ge 3$. Thus we can choose $f \in E \cap E'$ such that f is incident to u_3. Now, take $f' \in E' - f$. If $f' \in E$ then there exists a $(2,2)$-circuit C in G containing f and f'. If e is not contained in C then C is also in G', so there exists a $(2,2)$-circuit $C' = C$ in G' containing f and f'. If $e \in C$ then, by Lemma 4.8, there exists a $(2,2)$-circuit C' in G' containing f and f'. So, for all $f' \in (E \cap E') - f$ there exists a $(2,2)$-circuit C' in G' containing f and f'. If $f' \notin E$ then $f' \in \{e_1, e_2, e_3\}$. As there exists a $(2,2)$-circuit C in G containing e and f, Lemma 4.8 implies there exists a $(2,2)$-circuit C' in G' containing f, e_1, e_2, and e_3. Therefore, for all $f' \in E' - f$ there exists a $(2,2)$-circuit C' in G' containing f and f'. Hence, by the transitivity of the relation that defines $\mathcal{M}(2,2)$-connectivity, G' is $\mathcal{M}(2,2)$-connected. ■

We next extend Lemmas 4.1, 4.2 and 4.3 to $\mathcal{M}(2,2)$-connected graphs. To prove these extensions, we use two technical results. For a graph $G = (V, E)$ and a non-empty subset $F \subseteq E$, define $V[F] = \{v \in V : vw \in F\}$ and $G[F] = (V[F], F)$.

Lemma 5.3. Let $G = (V, E)$ be a graph with subgraphs $H_1 = (U_1, F_1)$, $H_2 = (U_2, F_2)$. Suppose that (H_1, H_2) is a 2-vertex-separation of G and $F_1 \cap F_2 = \{f\}$. If G is $\mathcal{M}(2,2)$-connected then for all $e_1 \in F_1 \setminus F_2$ and all $e_2 \in F_2 \setminus F_1$, there exists a $(2,2)$-circuit $C \subseteq E$ containing $\{f, e_1, e_2\}$. □

Proof. Fix the edges $e_1 \in F_1 \setminus F_2$, $e_2 \in F_2 \setminus F_1$, and choose a $(2,2)$-circuit C such that $e_1, e_2 \in C$. As $G[C]$ is 2-connected by Lemma 4.10, $U_1 \cap U_2 \subseteq V[C]$. If $f \in C$ then we are done, so suppose that $f \notin C$. Let $G' = G[C + f]$. Lemma 5.2 implies G' is $\mathcal{M}(2,2)$-connected, so for each $i \in \{1, 2\}$ we fix a $(2,2)$-circuit $C_i \subseteq C + f$ such that $\{f, e_i\} \subseteq C_i$. If $e_1 \in C_2$ or $e_2 \in C_1$ we are done, so suppose that $e_1 \notin C_2$ and $e_2 \notin C_1$.

Take a $(2,2)$-circuit $C_1' \subseteq C + f$ such that $f \in C_1'$. If $C_1' \ne C_1$ then the circuit exchange axiom implies there exists a $(2,2)$-circuit $D \subseteq (C_1 \cup C_1') - f$. If $e_2 \notin C_1'$ then $D \subseteq (C_1 \cup C_1') - f \subsetneq C$, contradicting the fact that no proper subset of a circuit is a circuit. Hence $e_2 \in C_1'$. Since C_1' was arbitrary, C_1 is the unique circuit in $C + f$ containing f but not e_2. Similarly C_2 is the unique circuit in $C + f$ containing f but not e_1.

As $f \in C_1 \cap C_2$ and $C_1 \ne C_2$, there exists a $(2,2)$-circuit C' such that $C' \subseteq (C_1 \cup C_2) - f \subseteq C$. Hence $C' = (C_1 \cup C_2) - f = C$ and $|C_1 \cup C_2| = |C| + 1 = 2|V[C]|$. Now, if $C_1 \cap C_2 = \{f\}$ then

$$
\begin{aligned}
2|V[C]| = |C_1 \cup C_2| &= |C_1| + |C_2| - |C_1 \cap C_2| \\
&= (2|V[C_1]| - 1) + (2|V[C_2]| - 1) - 1 \\
&= 2|V[C]| + 2(|V[C_1]| + |V[C_2]| - |V[C]|) - 3,
\end{aligned}
$$

a contradiction. Hence there exists $e \in (C_1 \cap C_2) - f$. Note that, as $(C_1 \cup C_2) - f = C$, $e \in C$.

The circuit exchange axiom implies there exists a $(2,2)$-circuit $C'' \subseteq C + f$ such that $C'' \subseteq (C \cup C_1) - e$. As $e \notin C''$ we have $C'' \notin \{C, C_1, C_2\}$, and as $C'' \ne C$ and no proper subset of a circuit is a circuit we have $f \in C''$. By the uniqueness of C_1 and C_2 it follows that $f, e_1, e_2 \in C''$. Hence C'' is a $(2,2)$-circuit contained in E such that $\{f, e_1, e_2\} \subseteq C''$. ■

Lemma 5.4. Let G be a graph, and suppose $H \cong K_4$ is a subgraph of G containing exactly two vertices of degree 3 in G. If G is $\mathcal{M}(2,2)$-connected, then for all $e \in E(G)$ there exists a $(2,2)$-circuit $C \subseteq E(G)$ such that $E(H) \cup \{e\} \subseteq C$. □

Proof. Since any $(2,2)$-circuit has minimum degree 3, this follows from Lemma 5.3. ■

Lemma 5.5. Let G_1 and G_2 be graphs and suppose that G is the j-join of (G_1, G_2) for some $j \in \{1,2,3\}$. If G_1 and G_2 are $\mathcal{M}(2,2)$-connected then G is $\mathcal{M}(2,2)$-connected. □

Proof. Let $G = (V, E)$, and $G_i = (V_i, E_i)$ for $i \in \{1,2\}$. By transitivity of the equivalence relation that defines $\mathcal{M}(2,2)$-connectivity, it suffices to show there exists $e \in E$ such that for all $f \in E - e$ there exists a $(2,2)$-circuit $C \subseteq E$ such that $e, f \in C$. We consider each type of j-join in turn.

Firstly suppose G is the 1-join of (G_1, G_2). Let a, b be the unique vertices shared by G_1, G_2 and let c, d be the vertices of G_2 deleted by the 1-join. Fix $H_2 = (U_2, F_2) = K[\{a,b,c,d\}]$ and fix an edge $e \in E \cap E_1$. Choose any edge $f \in E - e$. Now, either $f \in E_1$ or $f \in E_2$. If $f \in E_1$ then there exists a $(2,2)$-circuit $C_1 \subseteq E_1$ such that $e, f \in C_1$. If $ab \notin C_1$ then we are done. If $ab \in C_1$ then Lemma 5.4 implies there exists a $(2,2)$-circuit $C_2 \subseteq E_2$ such that $F_2 \subseteq C_2$. The 1-join of $(G_1[C_1], G_2[C_2])$, denoted by $G^* = (V^*, E^*)$, is a $(2,2)$-circuit by Lemma 4.1. Moreover, G^* is a subgraph of G and $e, f \in E^*$. If $f \in E_2$ then there exists a $(2,2)$-circuit $C_3 \subseteq E_1$ such that $e, ab \in C_3$. Lemma 5.4 implies there exists a $(2,2)$-circuit $C_4 \subseteq E_2$ such that $F_2 + f \subseteq C_4$. The 1-join of $(G_1[C_3], G_2[C_4])$, denoted by G', is a $(2,2)$-circuit by Lemma 4.1. Moreover, G' is a subgraph of G and $e, f \in E(G')$.

Next suppose G is the 2-join of (G_1, G_2). Let a, b be the unique vertices shared by G_1, G_2, and for each $i \in \{1,2\}$ let c_i, d_i be the vertices of G_i deleted by the 2-join and fix $H_i = (U_i, F_i) = K[\{a, b, c_i, d_i\}]$. Let $e = ab$ and choose any $f \in E$. By relabelling if necessary, we may assume $f \in E_1$. As G_1 and G_2 are $\mathcal{M}(2,2)$-connected, Lemma 5.4 implies that there exist $(2,2)$-circuits $C_1 \subseteq E_1$ and $C_2 \subseteq E_2$ such that $F_1 + f \subseteq C_1$ and $F_2 \subsetneq C_2$. The 2-join of $(G_1[C_1], G_2[C_2])$, denoted by G^*, is a $(2,2)$-circuit by Lemma 4.1. Moreover, G^* is a subgraph of G and $e, f \in E(G^*)$.

Finally, suppose G is the 3-join of (G_1, G_2). For each $i \in \{1,2\}$ let $\{v_i\} = V_i \setminus V$, let $N_{G_i}(v_i) = \{a_i, b_i, c_i\}$, and let $F := E \setminus (E_1 \cup E_2) = \{a_1a_2, b_1b_2, c_1c_2\}$. Let $e = a_1a_2$ and for $i \in \{1,2\}$ take any $f_i \in E \cap E_i$. As G_1 and G_2 are $\mathcal{M}(2,2)$-connected, for $i \in \{1,2\}$ there exists a $(2,2)$-circuit $C_i \subseteq E_i$ such that $a_iv_i, f_i \in C_i$. For $i \in \{1,2\}$, since $\delta(G_i[C_i]) = 3$ we have that $\{a_iv_i, b_iv_i, c_iv_i\} \subseteq C_i$. The 3-join of $(G_1[C_1], G_2[C_2])$, denoted by G^*, is a $(2,2)$-circuit by Lemma 4.1. Moreover, G^* is a subgraph of G and $F \cup \{f_1, f_2\} \subseteq E(G^*)$. As $E = (E \cap E_1) \cup (E \cap E_2) \cup F$ we have shown that for all $f \in E - e$ there exists a $(2,2)$-circuit $C \subseteq E$ such that $e, f \in C$. ■

Lemma 5.6. Let (G_1, G_2) be a j-separation of a graph G for some $j \in \{2,3\}$. If G is $\mathcal{M}(2,2)$-connected then G_1 and G_2 are $\mathcal{M}(2,2)$-connected. □

Proof. Let $G = (V, E)$, and $G_i = (V_i, E_i)$ for $i \in \{1,2\}$. First suppose (G_1, G_2) is a 2-separation of G with respect to the 2-vertex-separation (H_1, H_2), where $H_i = (U_i, F_i)$ for each $i \in \{1,2\}$. By definition, for each $i \in \{1,2\}$ we have that $G_i = H_i \cup K[\{a, b, c_i, d_i\}]$ for some vertices $c_i, d_i \notin V$. For each $i \in \{1,2\}$ choose $f_i \in F_i - ab$. As G is $\mathcal{M}(2,2)$-connected, Lemma 5.3 implies there exists a $(2,2)$-circuit $C \subseteq E$ such that $ab, f_1, f_2 \in C$. For $i \in \{1,2\}$ let $H_i' = H_i[C \cap F_i]$. Then (H_1', H_2') is a 2-vertex-separation of $G[C]$ and $ab \in E(H_1') \cap E(H_2')$. Let (G_1', G_2') be the 2-separation of $G[C]$ with respect to (H_1', H_2'). Then $E(K[\{a, b, c_i, d_i\}]) + f_i \subseteq E(G_i')$ for each $i \in \{1,2\}$. Lemma 4.2 implies that G_i' is a $(2,2)$-circuit for each $i \in \{1,2\}$. Hence for each $i \in \{1,2\}$, for all $e_i \in E_i - ab$ there exists a $(2,2)$-circuit $C_i \subseteq E_i$ such that $e_i, ab \in C_i$. By transitivity of the equivalence relation that defines $\mathcal{M}(2,2)$-connectivity it follows that G_1 and G_2 are $\mathcal{M}(2,2)$-connected.

Now suppose (G_1, G_2) is a 3-separation of G with respect to the non-trivial 3-edge-separation (H_1, H_2), where $H_i = (U_i, F_i)$ for each $i \in \{1,2\}$. By definition, for each $i \in \{1,2\}$ we have that $G_i = H_i \cup K[\{v_i\}, \{a_i, b_i, c_i\}]$ for some vertices $a_i, b_i, c_i \in V$, $v_i \notin V$. For each $i \in \{1,2\}$ choose $f_i \in F_i$. As G is $\mathcal{M}(2,2)$-connected, there exists a $(2,2)$-circuit $C \subseteq E$ such that $f_1, f_2 \in C$. Now $a_1a_2, b_1b_2, c_1c_2 \in C$ by Lemma 4.10. For each $i \in \{1,2\}$ let $H_i' = H_i[C \cap F_i]$. Then (H_1', H_2') is a non-trivial 3-edge-separation of $G[C]$ with corresponding edge cut $\{a_1a_2, b_1b_2, c_1c_2\}$. Let (G_1', G_2') be the 3-separation of $G[C]$ with respect to (H_1', H_2'). Then $\{f_i, a_iv_i, b_iv_i, c_iv_i\} \subset E(G_i')$ for each $i \in \{1,2\}$. Lemma 4.2 implies G_i' is a $(2,2)$-circuit for each $i \in \{1,2\}$. Hence for each $i \in \{1,2\}$, for all $e_i \in E_i - a_iv_i$ there exists a $(2,2)$-circuit $C_i \subseteq E_i$ such that $e_i, a_iv_i \in C_i$. By transitivity of the equivalence relation that defines $\mathcal{M}(2,2)$-connectivity it follows that G_1 and G_2 are $\mathcal{M}(2,2)$-connected. ■

Lemma 5.7. Let G be a graph with a 2-vertex-separation (H_1, H_2). Suppose that (G_1, G_2) is a 1-separation of G with respect to (H_1, H_2), and (G_2', G_1') is a 1-separation of G with respect to (H_2, H_1). If G is $\mathcal{M}(2,2)$-connected then

(i) G_1' and G_2 are $\mathcal{M}(2,2)$-connected, and
(ii) G_1 is $\mathcal{M}(2,2)$-connected or G_2' is $\mathcal{M}(2,2)$-connected.

□

Proof. Let $G = (V, E)$ and for $i \in \{1,2\}$ let $G_i = (V_i, E_i)$ and $H_i = (U_i, F_i)$. Let $\{a,b\} = U_1 \cap U_2$. By definition of 1-separation, $ab \notin E$. By Lemma 5.2, $G + ab$ is $\mathcal{M}(2,2)$-connected. Now we may take the 2-separation of G with respect to the 2-vertex-separation $(H_1 + ab, H_2 + ab)$, which gives the ordered pair (G_1', G_2). Theorem 5.6 gives us that G_1' and G_2 are $\mathcal{M}(2,2)$-connected.

Now let us suppose, for contradiction, that neither G_1 nor G_2' are $\mathcal{M}(2,2)$-connected. By transitivity of the equivalence relation that defines $\mathcal{M}(2,2)$-connectivity, there exists $e_1 \in E_1 \cap E$ such that for each $(2,2)$-circuit $C \subseteq E_1$, if $ab \in C$ then $e_1 \notin C$. Similarly, there exists $e_2 \in E_2' \cap E$ such that for all $(2,2)$-circuits $C \subseteq E_2'$, if $ab \in C$ then $e_2 \notin C$. As G is $\mathcal{M}(2,2)$-connected, there exists a $(2,2)$-circuit $C^* \subseteq E$ such that $e_1, e_2 \in C^*$. As $e_1 \in E_1 - ab$ and $e_2 \in E_2' - ab$ it follows that for $i \in \{1,2\}$, $e_i \in F_i \cap C^*$. Fix $G^* = G[C^*]$. As G^* is 2-connected (Lemma 4.10) and contains vertices in both $U_1 \setminus \{a,b\}$ and $U_2 \setminus \{a,b\}$, we have that $\{a,b\}$ is a vertex cut of G^* and $(H_1 \cap G^*, H_2 \cap G^*)$ is the corresponding 2-vertex-separation. Hence we may take the 1-separations of G^* with respect to both $(H_1 \cap G^*, H_2 \cap G^*)$ and $(H_2 \cap G^*, H_1 \cap G^*)$, and we denote the resulting ordered pairs of graphs by (G_1^*, G_2^*) and $(G_2^{*\prime}, G_1^{*\prime})$ respectively.

By our choice of G^*, we have that G_1^* is a subgraph of G_1 containing e_1, ab and $G_2^{*\prime}$ is a subgraph of G_2' containing e_2, ab. However, Lemma 4.3 implies that one of G_1^*, $G_2^{*\prime}$ must be a $(2,2)$-circuit which gives a contradiction. Hence G_1 is $\mathcal{M}(2,2)$-connected or G_2' is $\mathcal{M}(2,2)$-connected. ■

We next consider the effect of generalised vertex splits and K_4^--extensions on $\mathcal{M}(2,2)$-connected graphs. If G' is obtained from a $\mathcal{M}(2,2)$-connected graph by a generalised vertex split, then G' need not be $\mathcal{M}(2,2)$-connected (indeed, G' may have a vertex of degree less than 3). Consequently, when analysing the reduction step of our recursive construction, we will need to take care when applying edge-reductions. However, this complication does not arise for K_4^--extensions and K_4^--reductions.

Lemma 5.8. Let G be a $\mathcal{M}(2,2)$-connected graph. Then any K_4^--extension of G is also $\mathcal{M}(2,2)$-connected. Conversely, any K_4^--reduction of G that adds an edge $e \notin E(G)$ is $\mathcal{M}(2,2)$-connected. □

Proof. A K_4^--extension of G is the 1-join of (G, H), where $H \cong B_1$, and so preserves $\mathcal{M}(2,2)$-connectivity by Lemma 5.5. Suppose G' is a K_4^--reduction of G that adds e. Consider the 1-separations, (G', H_1) and $(H_2, G + e)$, of G where $H_1 \cong B_1$ and $H_2 \cong K_4$. As K_4 is not $\mathcal{M}(2,2)$-connected, G' is $\mathcal{M}(2,2)$-connected by Lemma 5.7. ■

5.2 Ear decompositions

Given a non-empty sequence of circuits $C_1, \dots, C_t$ in a matroid $M = (E, r)$, we define the sets $D_i = C_1 \cup \dots \cup C_i$ and $\tilde{C}_i = C_i \setminus D_{i-1}$ for each $1 \le i \le m$. The sequence $C_1, \dots, C_m$ is a *partial ear decomposition* of M if, for all $2 \le i \le m$,

(E1) $C_i \cap D_{i-1} \neq \emptyset$,
(E2) $C_i \setminus D_{i-1} \neq \emptyset$, and
(E3) no circuit C_i' satisfying (E1) and (E2) has $C_i' \setminus D_{i-1} \subsetneq C_i \setminus D_{i-1}$.

A partial ear decomposition is an *ear decomposition* of M if $D_m = E$. The following standard result [16] shows the close relationship between matroid connectivity and ear decompositions.

Lemma 5.9. Let $M = (E, r)$ be a matroid with $|E| \ge 2$. Then:

(i) M is connected if and only if it has an ear decompostion.
(ii) If M is connected then every partial ear decomposition is extendable to an ear decomposition.
(iii) If $C_1, \dots, C_t$ is an ear decomposition of M then $r(D_i) - r(D_{i-1}) = |\tilde{C}_i| - 1$ for all $2 \le i \le t$.

□

We say that a graph G with no isolated vertices has a *(partial) ear decomposition* if there is a (partial) ear decomposition of the matroid $\mathcal{M}(2,2)$ restricted to the edge set of G. The lack of explicit reference to the matroid in the terminology should not cause confusion, as the only matroid we consider is $\mathcal{M}(2,2)$.

Example 5.10. Recall that $K_{3,6}$ is redundantly rigid in any analytic normed plane but does not contain a spanning $(2,2)$-circuit. Let the part of size 6 be denoted $\{v_1, v_2, \dots, v_6\}$ and the part of size 3 be denoted $\{u_1, u_2, u_3\}$. An ear decomposition for $K_{3,6}$ is given by (C_1, C_2), where C_1 is the edge set of the $(2,2)$-circuit $K_{3,6} - v_1 \cong K_{3,5}$ and C_2 is the the edge set of $K_{3,6} - v_2$. Then $\tilde{C}_2 = \{v_1u_1, v_1u_2, v_1u_3\}$ and $D_2 = E(K_{3,6})$. □

6 A recursive construction of 2-connected, redundantly rigid graphs

The purpose of this section is to derive the following recursive construction of $\mathcal{M}(2,2)$-connected graphs. This, combined with the geometric results of Section 7, will be used to prove Theorem 1.1.

Theorem 6.1. A graph G is $\mathcal{M}(2,2)$-connected if and only if G can be generated from K_5^- or B_1 by generalised vertex splits, K_4^--extensions and edge additions such that each generalised vertex split preserves $\mathcal{M}(2,2)$-connectivity. □

The recursive construction of Theorem 6.1 is illustrated for a specific example in Figure 9.

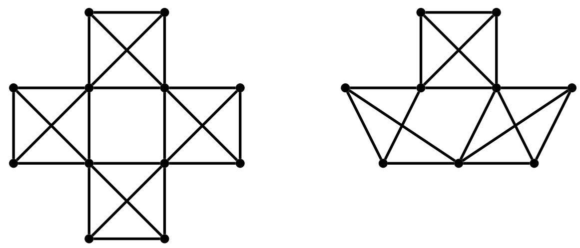


Fig. 9. The graph on the left is both 2-connected and redundantly rigid in any analytic normed plane. To reduce this graph, the first operation must be an edge-deletion on one of the four "inner" edges, then we may apply a K_4^--reduction on the K_4 we deleted the edge from. This can be followed by an edge-reduction that contracts the edge added by the K_4^--reduction and deletes one of the three remaining "inner" edges resulting in the graph on the right. Now a K_4^--reduction gives B_2 and finally an edge-reduction gives B_1. To construct this graph using Theorem 6.1 we reverse this sequence of operations.

It follows from Lemma 5.1 that it suffices to consider $\mathcal{M}(2,2)$-connected graphs. Given an $\mathcal{M}(2,2)$-connected graph G, we say that an edge-reduction of G is *admissible* if the resulting graph is $\mathcal{M}(2,2)$-connected; if the edge-reduction is equivalent to a 1-reduction on a node, then we say that the node is *admissible*. Similarly, we say that an edge-deletion of G (and the removed edge) is *admissible* if the resulting graph is $\mathcal{M}(2,2)$-connected. A K_4^--reduction is *admissible* if the resulting graph is $\mathcal{M}(2,2)$-connected. Lemma 5.8 implies that a K_4^--reduction adding e is admissible if and only if e is not an edge of G. We will show that every $\mathcal{M}(2,2)$-connected graph has an operation (edge-reduction, edge deletion, or K_4^--reduction) that is admissible. We require three lemmas.

Lemma 6.2. Suppose G is $\mathcal{M}(2,2)$-connected with an ear decomposition $C_1, C_2 \dots, C_t$ such that $t \geq 2$, and for all $1 \leq i \leq t$ let $H_i = (V_i, C_i) = G[C_i]$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and $X = V_t \setminus Y$. The following statements hold.

(i) Either $Y = \emptyset$ and $|\tilde{C}_t| = 1$ or $Y \neq \emptyset$ and every edge $e \in \tilde{C}_t$ is incident to Y.
(ii) $|\tilde{C}_t| = 2|Y| + 1$.
(iii) If $Y \neq \emptyset$ then X is critical in H_t.
(iv) If $Y \neq \emptyset$ then $G[Y]$ is connected.
(v) $|X| \geq 4$.

□

Proof. Part (i) is an easy consequence of (E3). Part (ii) follows from Lemma 5.9(iii) and the observation that each set D_i has rank $2|\bigcup_{j=1}^{i} V_j| - 2$ in $\mathcal{M}(2,2)$ as $\bigcup_{j=1}^{i} H_j$ is $\mathcal{M}(2,2)$-connected. For part (iii), as $Y \neq \emptyset$, (i) and (ii) together imply that

$$i_{H_t}(X) = |C_t| - |\tilde{C}_t| = 2|V_t| - 1 - (2|Y| + 1) = 2|X| - 2.$$

Hence X is critical in H_t.

For part (iv), as $Y \neq \emptyset$ the induced subgraph $G[Y]$ exists. Let $Y_1, \dots, Y_k$ be the vertex sets of the components of $G[Y]$. As H_t is a $(2,2)$-circuit, $|E'| \leq 2|V'| - 2$ for every proper subgraph (V', E') of H_t. So, since X is critical, if $k \geq 2$ then $i_{H_t}(Y_i) + d(X, Y_i) \leq 2|Y_i|$ for all $1 \leq i \leq k$. Hence, part (ii) implies that

$$2|Y| + 1 = |\tilde{C}_t| = \sum_{i=1}^{k} (i_{H_t}(Y_i) + d(X, Y_i)) \leq \sum_{i=1}^{k} 2|Y_i| = 2|Y|$$

which is a contradiction. Therefore $k = 1$ and so $G[Y]$ is connected.

Finally, for part (v), if $Y = \emptyset$ then since H_t is a $(2,2)$-circuit, $|X| = |V_t| \geq 5$. If $Y \neq \emptyset$ then part (iii) implies that X is critical in H_t, so $|X| = 1$ or $|X| \geq 4$. As $C_1, \dots, C_t$ is an ear decomposition of G, we have $C_t \cap D_{t-1} \neq \emptyset$ which implies $i_{H_t}(X) \neq 0$ and hence $|X| \neq 1$. ■

Before the next rather technical result, we make the following observation about 1-reductions. Let H be a $(2,2)$-circuit, let $v \in V(H)$ be a node with $N(v) = \{x, y, z\}$, and suppose that $xy \notin E(H)$. Since $H - vz$ is $(2,2)$-tight, $H - v$ is $(2,2)$-tight and so $H - v + xy$ contains a unique $(2,2)$-circuit J. We have $J = H - v + xy$ if and only if the 1-reduction of H at v adding xy is admissible. If $J \neq H - v + xy$ then $V(J)$ is the minimal v-critical set in H containing $\{x, y\}$.

Lemma 6.3. Suppose G is $\mathcal{M}(2,2)$-connected with an ear decomposition $C_1, C_2, \ldots, C_t$ such that $t \geq 2$, and for all $1 \leq i \leq t$ let $H_i = (V_i, C_i) = G[C_i]$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and $X = V_t \setminus Y$. Let v be a node in G contained in Y, take $x, y \in N_G(v)$, and suppose $xy \notin E$. Let J be the unique $(2,2)$-circuit in $H_t - v + xy$ and $C = E(J)$. If $E(H_t - v + xy) \setminus E(H_t[X]) \subsetneq C$ then v is admissible in G. □

Proof. We show that any pair of edges of $G - v + xy$ is contained in a $(2,2)$-circuit. Note $E(H_t[X]) \subseteq D_{t-1}$ by Lemma 6.2(i). Since $E(H_t - v + xy) \setminus E(H_t[X]) \subsetneq C$, $E(G - v + xy) = E(H_t - v + xy) \cup D_{t-1}$ and $D_{t-1} \cup C \subseteq E(G - v + xy)$, we have $D_{t-1} \cup C = E(G - v + xy)$. Hence $G - v + xy = (\bigcup_{i=1}^{t-1} H_i) \cup J$. As $C \subseteq E(H_t - v + xy)$ and $E(H_t - v + xy) \setminus E(H_t[X]) \subsetneq C$, we have $C \cap E(H_t[X]) \neq \emptyset$. Fix an edge $e \in C \cap E(H_t[X])$ and choose any edge $f \in E(G - v + xy) - e$. If $f \in D_{t-1}$ then, since Lemma 5.9(i) implies that $\bigcup_{i=1}^{t-1} H_i$ is $\mathcal{M}(2,2)$-connected, there exists a $(2,2)$-circuit contained in $G - v + xy$ and containing e and f. If $f \in C$ then, since J is clearly $\mathcal{M}(2,2)$-connected, there exists a $(2,2)$-circuit containing e and f. So, there exists $e \in E(G - v + xy)$ such that for all $f \in E(G - v + xy) - e$ there exists a $(2,2)$-circuit containing e and f. The result now follows by the transitivity of the equivalence relation defining $\mathcal{M}(2,2)$-connectivity. ■

Lemma 6.4. Suppose G is $\mathcal{M}(2,2)$-connected with an ear decomposition $C_1, C_2, \ldots, C_t$ such that $t \geq 2$, and for all $1 \leq i \leq t$ let $H_i = (V_i, C_i) = G[C_i]$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and $X = V_t \setminus Y$. Suppose $Y \neq \emptyset$ and no 3-edge-separation (F_1, F_2) of H_t has the property that $|V(F_1)|, |V(F_2)| \geq 2$ and $F_i \subset H_t[Y]$ for some $i \in \{1, 2\}$. Choose a set X_0 that contains X and is critical in H_t¶. Let $X_1, \ldots, X_n$ be critical sets in H_t with $|X_i| \geq 2$ for $1 \leq i \leq n$ and let $\mathcal{Y} = V_t \setminus \bigcup_{i=0}^{n} X_i$. If $|\mathcal{Y}| \geq 2$ or $\bigcup_{i=0}^{n} H_t[X_i]$ is disconnected, then $\mathcal{Y}$ contains at least two nodes of G. □

Proof. Let $Z_1, \ldots, Z_m$ be the vertex sets of the connected components of $\bigcup_{i=0}^{n} H_t[X_i]$. By Lemma 4.10 each X_i is contained in exactly one set Z_j. By reordering if necessary we may assume that $X_0 \subseteq Z_1$, which implies $Z_j \subseteq Y$ for each $2 \leq j \leq m$. Lemma 6.2(v) implies that $|X_0| \geq |X| \geq 4$. So, as X_i is critical in H_t for each $1 \leq i \leq n$, we have $|X_i| \geq 4$ for all $0 \leq i \leq n$. As $|\mathcal{Y}| \geq 2$ or $\bigcup_{i=0}^{n} H_t[X_i]$ is disconnected, we have $|V_t \setminus Z_j| \geq 2$ for all $1 \leq j \leq m$. Since no 3-edge-separation (F_1, F_2) of H_t has the property that $|V(F_1)|, |V(F_2)| \geq 2$ and $F_i \subset H_t[Y]$ for some $i \in \{1, 2\}$ and since $V_t \setminus Z_1 \subseteq Y$ and $Z_j \subseteq Y$ for each $2 \leq j \leq m$, we have $d_{H_t}(Z_j, V_t \setminus Z_j) \geq 4$ for each $1 \leq j \leq m$. By Lemma 4.9, Z_j is critical in H_t for all $1 \leq j \leq m$. So for all $1 \leq j \leq m$,

$$\sum_{v \in Z_j} (4 - d_{H_t[Z_j]}(v)) = 4|Z_j| - 2i_{H_t}(Z_j) = 4.$$

Hence

$$\sum_{v \in Z_j} (4 - d_{H_t}(v)) = 4|Z_j| - (2i_{H_t}(Z_j) + d_{H_t}(Z_j, V_t \setminus Z_j)) = 4 - d_{H_t}(Z_j, V_t \setminus Z_j) \leq 0.$$

Therefore $\sum_{j=1}^{m} \sum_{v \in Z_j} (4 - d_{H_t}(v)) \leq 0$.

Since $|C_t| = 2|V_t| = 1$ we have $\sum_{v \in V_t} (4 - d_{H_t}(v)) = 4|V_t| - 2|C_t| = 2$. Combining this with the previous inequality gives us that

$$2 = \sum_{v \in V_t} (4 - d_{H_t}(v)) \leq \sum_{v \in \mathcal{Y}} (4 - d_{H_t}(v)).$$

As $\delta(H_t) = 3$, $\mathcal{Y}$ contains at least two nodes of H_t. As $\mathcal{Y} \subseteq Y$, these are also nodes of G. ■

We can now prove our first main combinatorial result which establishes that admissible reductions always exist under a technical connectivity hypothesis.

Theorem 6.5. Suppose G is $\mathcal{M}(2,2)$-connected with an ear decomposition $C_1, C_2 \ldots, C_t$ such that $t \geq 2$, and for all $1 \leq i \leq t$ let $H_i = (V_i, C_i) = G[C_i]$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and $X = V_t \setminus Y$. Suppose no 3-edge-separation (F_1, F_2) of H_t has the property that $|V(F_1)|, |V(F_2)| \geq 2$ and $F_i \subset H_t[Y]$ for some $i \in \{1, 2\}$. Then there is an admissible edge-reduction, edge-deletion or K_4^--reduction of G. □

¶As $Y \neq \emptyset$, Lemma 6.2(iii) implies X is critical and hence such an X_0 exists.

Proof. We proceed by contradiction. Let $G=(V,E)$ and suppose that G has no admissible edge-reductions, admissible edge-deletions or admissible K_4^--reductions. If $Y=\emptyset$ then Lemma 6.2(i) implies that $\tilde{C}_t=\{e\}$ and Lemma 5.9(i) implies $G-e$ is $\mathcal{M}(2,2)$-connected, and so e is admissible. Hence $Y\neq\emptyset$. By Lemma 6.2(iii) and (v), X is critical in H_t and $|X|\geq 4$. Lemma 4.11 now implies that Y contains a node v in H_t. Note that since C_t is the last $(2,2)$-circuit in the ear decomposition, $N_G(v)=N_{H_t}(v)$). Label the vertices in $N(v)$ by x,y,z.

First suppose that $N(v)\subseteq X$. By Lemma 6.2(iv) we have $Y=\{v\}$. Lemma 6.2(i) implies that $C_1,\dots,C_{t-1}$ is an ear decomposition of $G-v$, and so $G-v$ is $\mathcal{M}(2,2)$-connected by Lemma 5.9(i). If $xy\notin E$ then Lemma 5.2 implies that $G-v+xy$ is $\mathcal{M}(2,2)$-connected and hence v is admissible in G, a contradiction. If $xy\in E$ then $G-xy$ is a 1-extension of $G-v$ and so Lemma 5.2 implies that $G-xy$ is $\mathcal{M}(2,2)$-connected and hence xy is admissible in G, a contradiction.

Suppose next that $|N(v)\cap X|=2$, say $x,y\in X$ and $z\in Y$. If $xz,yz\in E$ then $xz,yz\in C_t$ and so, as X is critical in H_t, we have $i_{H_t}(X\cup\{v,z\})=2|X\cup\{v,z\}|-1$. Hence $Y=\{v,z\}$ and $d_G(X,\{v,z\})=4$. This in turn implies that $(G[V\setminus\{v,z\}],G[\{v,x,y,z\}])$ is a 2-vertex-separation with vertex cut $\{x,y\}$. If $xy\notin E$ then $G[V\setminus\{v,z\}]+xy$ is an admissible K_4^--reduction of G by Lemma 5.8, a contradiction. Hence $xy\in E$. Note that $G[V\setminus\{v,z\}]=\bigcup_{i=1}^{t-1}H_i$ by Lemma 6.2(i), and so $G[V\setminus\{v,z\}]$ is $\mathcal{M}(2,2)$-connected by Lemma 5.9(i). As $G-xy$ is a K_4^--extension of $G[V\setminus\{v,z\}]$, Lemma 5.8 implies $G-xy$ is $\mathcal{M}(2,2)$-connected which contradicts the assumption that G has no admissible edge-deletions.

Alternatively we have that $|\{xz,yz\}\cap E|\leq 1$. Then we may suppose, without loss of generality, that $xz\notin E$. Let us denote the graph obtained by the 1-reduction of H_t at v adding the edge xz by $H_t'=(V_t',C_t')$. Suppose first that this 1-reduction of H_t is admissible. Then $E(H_t[X])\subsetneq C_t'$. As X is critical in H_t and $|X|\geq 4$, we have $E(H_t[X])\neq\emptyset$. Therefore $E(H_t[X])\cap C_t'\neq\emptyset$ and so $C_t'\setminus E(H_t[X])\subsetneq C_t'$. So v is admissible in G by Lemma 6.3, a contradiction.

Hence the 1-reduction of H_t at v adding the edge xz is not admissible. By switching x and y, we also have that if $yz\notin E$ then the 1-reduction of H_t at v adding the edge yz is not admissible. Since $xz\notin E$, Lemma 4.12 implies that there exists a minimal v-critical set X_1 of H_t containing $\{x,z\}$ (but not containing v or y). Lemma 4.9 then implies that $X\cup X_1$ and $X\cap X_1$ are critical in H_t, and $d_{H_t}(X,X_1)=0$. As $d_G(\{v\},X\cup X_1)=d_{H_t}(\{v\},X\cup X_1)=3$, it follows that $X\cup X_1=V_t-v$ and so $Y-v\subseteq X_1\setminus X$. Let J be the unique $(2,2)$-circuit in H_t' and let $C=E(J)$. Note that the minimality of X_1 implies that $V(J)=X_1$. Hence $C_t'\setminus E(H_t[X])\subseteq C$. Since $X\cap X_1$ is critical, the graph $H_t[X\cap X_1]$ is connected by Lemma 4.10. If $|X\cap X_1|=1$ then $X\cap X_1=\{x\}$, $yz\notin E$ and $\{v,x\}$ is a 2-vertex-separation of H_t. Consider the 1-reduction of H_t that deletes v and adds yz. As such a 1-reduction must be non-admissible, there exists a critical set X_2 containing y and z but not x or v. However $\{x\}$ is 1-vertex-separation of $H_t[X\cup X_1]$, contradicting Lemma 4.10. Hence $|X\cap X_1|\geq 2$ and thus $E(H_t[X])\cap C\neq\emptyset$. It now follows that

$$C_t'\setminus E(H_t[X])=C\setminus E(H_t[X])\subsetneq C.$$

Therefore v is admissible in G by Lemma 6.3, a contradiction. Hence $|N(v)\cap X|\leq 1$, i.e., v has at most one neighbour in X. Since v was chosen arbitrarily, this holds for every node in Y.

Claim 6.6. There exists a node in Y that is not contained in a subgraph of G isomorphic to K_4. □

Proof of claim. Let $X_1,\dots,X_n$ be all the subsets of V_t that induce a K_4 subgraph in H_t and let $X=X_0$. Let $\mathcal{Y}=V_t\setminus\bigcup_{i=0}^n X_i$. If $|\mathcal{Y}|\geq 2$ or $\bigcup_{i=0}^n H_t[X_i]$ is disconnected then, as no 3-edge-separation (F_1,F_2) of H_t has the property that $|V(F_1)|,|V(F_2)|\geq 2$ and $F_i\subset H_t[Y]$ for some $i\in\{1,2\}$, we can apply Lemma 6.4 to deduce the claim. So suppose otherwise. As $\bigcup_{i=0}^n H_t[X_i]$ is connected, by relabelling the subscripts from 1 to n (if necessary) we may assume that $X_0,\dots,X_n$ are ordered such that $X_j\cap(\bigcup_{i=0}^{j-1}X_i)\neq\emptyset$ for all $1\leq j\leq n$. Fix $s=\min\{j:\bigcup_{i=0}^j X_i=\bigcup_{i=0}^n X_i\}$.

Lemma 4.11 implies that Y contains a node, so we may assume $s\geq 1$. Then $\bigcup_{i=0}^{s-1}X_i$ is a proper subset of V_t. Lemma 4.9 implies that, for each $1\leq j\leq s-1$, the set $\bigcup_{i=0}^j X_i$ is critical in H_t and

$$d_{H_t}\left(\bigcup_{i=0}^{j-1}X_i\setminus X_j,\ X_j\setminus\bigcup_{i=0}^{j-1}X_i\right)=0. \tag{1}$$

If $|(\bigcup_{i=0}^{s-1}X_i)\cap X_s|\geq 2$ then, as $H_t[X_s]\cong K_4$, the induced subgraph of H_t on the vertex set $(\bigcup_{i=0}^{s-1}X_i)\cap X_s$ has at least one edge. It follows from Equation (1) that any edge in the aforementioned induced subgraph must be induced by one of the sets $X_0,\dots,X_{s-1}$, hence there exists $0\leq j\leq s-1$ such that $|X_j\cap X_s|\geq 2$. However, as $|X_j\cap X_s|<|X_s|=4$, we note that $X_j\cap X_s$ is not critical in H_t and hence Lemma 4.9 implies that $X_j\cup X_s=V_t$. So, $X_j=X$ and $Y=X_s\setminus X$ contains either 1 or two nodes. This contradicts our assumption that every node in Y has at most one neighbour in X. Hence $|(\bigcup_{i=0}^{s-1} X_i) \cap X_s| = 1$. Fix $Z = \bigcup_{i=0}^{s-1} X_i$, so Z is critical in H_t. Also fix w to be the unique vertex in the set $Z \cap X_s$. As $|\mathcal{Y}| \leq 1$ we now have that H_t is one of the two graphs shown in Figure 10.

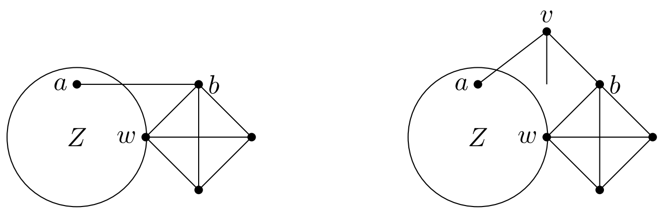


Fig. 10. The two possibilities for the structure of H_t in Claim 6.6 when $Z \cap X_s = \{w\}$; $\mathcal{Y} = \emptyset$ (left) and $\mathcal{Y} = \{v\}$ (right).

Suppose that $\mathcal{Y} = \emptyset$ (i.e., H_t is the graph on the left in Figure 10). Then $Z \cup X_s = \bigcup_{i=0}^{s} X_i = V_t$. As H_t is a $(2,2)$-circuit, there exists $a \in Z$ and $b \in X_s \setminus Z$ such that $ab \in C_t$. Then $(H_t[Z], H_t[X_s + a])$ is a non-trivial 2-vertex-separation of H_t. Moreover, as $X_s - w \subseteq Y$, $(G[V \setminus (X_s - w)], G[X_s + a])$ is a non-trivial 2-vertex-separation of G.

Suppose that $aw \in E$. Let (G_1, G_2) be the 2-separation of G with respect to the 2-vertex-separation $(G[V \setminus (X_s - w)], G[X_s + a])$. By Lemma 5.6, G_1 and G_2 are $\mathcal{M}(2,2)$-connected. The edge-reduction of G that deletes bw and contracts ab gives the graph G_1, so there is an admissible edge-reduction of G. This is a contradiction, so we must have $aw \notin E$. Let (G_1, G_2) be the 1-separation of G with respect to the 2-vertex-separation $(G[V \setminus (X_s - w)], G[X_s + a])$ and let (G_2', G_1') be the 1-separation of G with respect to the 2-vertex-separation $(G[X_s + a], G[V \setminus (X_s - w)])$. As $\delta(G_2') \leq 2$, Lemma 5.7 implies G_1 and G_2 are $\mathcal{M}(2,2)$-connected. The edge-reduction of G that deletes wb and contracts ab gives a graph isomorphic to a K_4^--extension of G_1. Lemma 5.8 implies that this edge-reduction gives an $\mathcal{M}(2,2)$-connected graph. However this also implies that there is an admissible edge-reduction of G, another contradiction.

Hence $\mathcal{Y} = \{v\}$ (i.e., H_t is the graph on the right in Figure 10). Then $Z \cup X_s = \bigcup_{i=0}^{s} X_i$ is critical in H_t. As H_t is a $(2,2)$-circuit, it follows from Lemma 4.10 that $d_{H_t}(v) = 3$, and v has neighbours a, b in H_t where $a \in Z$ and $b \notin Z$. Since $a \in Z$, $b \in X_s \setminus Z$ and $v \notin Z \cup X_s$, Lemma 4.9 implies $ab \notin C_t$. As $b \in Y$, we have $ab \notin E$ and the claim is established. ■

Recall that V_3 is the set of nodes in G[‖] and V_3^* is the set of nodes in G that are not contained in a subgraph of G isomorphic to K_4. By Claim 6.6, we have $Y \cap V_3^* \neq \emptyset$. For an arbitrary node $v \in Y \cap V_3^*$, given that $N(v) = \{x, y, z\}$ and $|N(v) \cap X| \leq 1$, we will assume without loss of generality that $y, z \in Y$. If the 1-reduction of H_t at v adding the edge $e \in \{xy, xz, yz\}$ is admissible then, by a similar argument as in the case that $|N(v) \cap X| = 2$, it follows that v is admissible in G, a contradiction. So v is non-admissible in H_t. Note also that if there is an admissible K_4^--reduction of H_t, say deleting v, y and adding the edge xz, then the corresponding K_4^--reduction of G is also admissible (since $|N(v) \cap X| \leq 1$). Hence, by Lemma 4.15, $xy, xz, yz \notin C_t$, and so $xy, xz, yz \notin E$ also.

Claim 6.7. There exists a node $v \in Y \cap V_3^*$ and a v-critical set X^* in H_t such that X^* is node-critical in H_t and $X \subseteq X^*$. □

Proof of claim. Since $Y \cap V_3^*$ is non-empty, Lemma 4.13 implies that $H_t[Y \cap V_3^*]$ is a forest. Hence we may fix $v \in Y \cap V_3^*$ to be a leaf. As $xy, xz, yz \notin C_t$ and v is not admissible in H_t, it follows from Lemma 4.12 that there exist minimal v-critical sets X_1, X_2 and X_3 in H_t containing $\{x, z\}$, $\{y, z\}$ and $\{x, y\}$ respectively. If $x \in X$ then Lemma 4.9 implies that both $X \cup X_1$ and $X \cup X_3$ are v-critical sets containing X in H_t. As v is a leaf of the forest $H_t[Y \cap V_3^*]$ and $y, z \in Y$, we have $d_{H_t}(y) \geq 4$ or $d_{H_t}(z) \geq 4$. Hence one of $X \cup X_1$ and $X \cup X_3$ is node-critical as required. Suppose instead that $x \notin X$, i.e., $x, y, z \in Y$. As v is a leaf in $H_t[Y \cap V_3^*]$, we may assume, without loss of generality, that $d_{H_t}(x) \geq 4$ and $d_{H_t}(y) \geq 4$. By Lemma 4.9, $X_1 \cup X_2$ is a critical set in H_t. The subgraph of H_t induced by $X_1 \cup X_2 + v$ is a $(2,2)$-circuit, and so $X_1 \cup X_2 = V_t - v$. Hence $X \cap X_i$ is non-empty for some $i \in \{1, 2\}$ and therefore Lemma 4.9 implies that $X \cup X_i$ is the required v-critical, node-critical set in H_t containing X. ■

[‖] This is technically an abuse of notation, since the $(2,2)$-circuit H_3 has vertex set V_3. There is no ambiguity, however, since we only ever refer to arbitrary $(2,2)$-circuits (i.e., $H_i = (V_i, C_i)$) or H_t.

We may chose a node $v \in Y \cap V_3^*$ and a v-critical and node-critical set $X^* \subsetneq V_t$ such that $X \subseteq X^*$ and $|X^*|$ is as large as possible over all such choices of v and X^*. Fix z to be the unique neighbour of v that is not contained in X^*. Since X^* is node-critical, $d_{H_t}(z) \geq 4$, and so $d_G(z) \geq 4$. By applying Lemma 4.11 to the set $X^* + v$, which is critical in H_t, we deduce that the set $Z := V_t \setminus (X^* + v)$ contains a node.

Claim 6.8. Every node in Z is contained in a subgraph of H_t isomorphic to K_4. □

Proof of claim. By Lemma 4.13, the graph $H_t[Z \cap V_3]$ is a forest. Choose a leaf w of $H_t[Z \cap V_3]$. The node w has at most one neighbour in X^*; if w had three neighbours in X^* then the induced subgraph of H_t on $X^* + w \subsetneq V_t$ would be a $(2,2)$-circuit, and if w had two neighbours in X^* then $X^* + w$ would be a larger v-critical and node-critical set than X^*. Hence either: (i) w is a leaf node in H_t, (ii) w is a series node in H_t with exactly one neighbour in X^*, and said neighbour is also a node, or (iii) w is contained in a subgraph of H_t (and hence also G) isomorphic to K_4. If (i) or (ii) hold (i.e., $w \in Y \cap V_3^*$) then (as noted earlier) there are no edges of H_t between the neighbours of w. By Lemma 4.14, there exists a node critical set in H_t strictly containing X^*, contradicting the maximality of X^*. Hence every leaf of $H_t[Z \cap V_3]$ is contained in a subgraph of H_t isomorphic to K_4.

Suppose that Z contains a node that is not a leaf of $H_t[Z \cap V_3]$. Then there exists a node $w \in Z$ that is not a leaf of $H_t[Z \cap V_3]$ with neighbours $a, b \in N(w) \cap Z \cap V_3$ where a is a leaf of $H_t[Z \cap V_3]$. Then a is contained in a subgraph of H_t isomorphic to K_4. Since a and w are nodes, $ab \in C_t$. This contradicts the fact that $H_t[Z \cap V_3]$ is a forest. Therefore every node in Z is a leaf of $H_t[Z \cap V_3]$ and hence is contained in a subgraph isomorphic to K_4. ■

Let $X_0 = X^*$ and let $X_1, \dots, X_k$ be the critical sets in H_t such that for $1 \leq i \leq k$, $|X_i| = 4$ and $X_i \nsubseteq X^*$. As Z contains a node, we can fix a node $w \in Z$. Claim 6.8 then implies that $k \geq 1$. Suppose that the induced subgraph $\bigcup_{i=0}^k H_t[X_i]$ is disconnected. By Lemma 6.4, the set $V_t \setminus (\bigcup_{i=0}^k X_i) \subseteq Y$ contains at least two nodes of H_t. So $V_t \setminus (\bigcup_{i=0}^k X_i + v) \subseteq Z$ contains a node, say w', of H_t. As $w' \notin \bigcup_{i=0}^k X_i$, w' is not contained in a subgraph of H_t isomorphic to K_4, however this contradicts Claim 6.8. Therefore $\bigcup_{i=0}^k H_t[X_i]$ is connected.

By reordering $X_1, \dots, X_k$, we may assume that the induced subgraph $\bigcup_{i=0}^s H_t[X_i]$ is connected for each $1 \leq s \leq k$. By the maximality of X^*, the fact that $k \geq 1$, and Lemma 4.9, it follows that $X^* \cup X_1$ must contain z. As H_t is a $(2,2)$-circuit and all three neighbours of v are in $X^* \cup X_1$, it follows that $X^* \cup X_1 = V_t - v$. Hence $X_1 = \{a, b, w, z\}$ and $N(w) = \{a, b, z\}$, where $a \in X^*$ and $b \notin X^*$. Further, Lemma 4.9 implies $N(z) = \{v, a, b, w\}$ and $d_{H_t}(b) = 3$ as depicted in Figure 11.

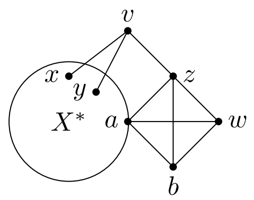


Fig. 11. The only possible configuration of H_t.

First note that, given $J_1 = G[V \setminus \{b, w, z\}]$ and $J_2 = G[X_1 + v]$, the pair (J_1, J_2) is a 2-vertex separation of G. Let (G_1, G_2) be the 1-separation of G with respect to (J_1, J_2), and let (G_2', G_1') be the 1-separation of G with respect to (J_2, J_1). As $\delta(G_2') \leq 2$, Lemma 5.7 implies that G_1 is $\mathcal{M}(2,2)$-connected. The edge-reduction of G that deletes za and contracts vz gives a graph isomorphic to a K_4^--extension of G_1. Lemma 5.8 implies that this edge-reduction gives an $\mathcal{M}(2,2)$-connected graph, so there is an admissible edge-reduction of G. ■

An *atom* of G is a subgraph F such that F is an element of a non-trivial 2-vertex-separation or non-trivial 3-edge-separation of G and no proper subgraph of F has this property. Note that if (F_1, F_2) is a non-trivial 2-vertex-separation or non-trivial 3-edge-separation of G, then both F_1 and F_2 must contain atoms.

Theorem 6.9. If G is an $\mathcal{M}(2,2)$-connected graph, distinct from K_5^- and B_1, then there is an admissible edge-reduction, K_4^--reduction, or edge-deletion of G. □

Proof. The case where G is a $(2,2)$-circuit is Theorem 4.5. Suppose G is not a $(2,2)$-circuit. Furthermore, suppose that every $\mathcal{M}(2,2)$-connected graph H for which $|E(H)| + |V(H)| < |E| + |V|$ is either K_5^- or B_1, or contains an admissible edge-reduction, K_4^--reduction, or edge-deletion (we will not require this assumption until later in Case 6.14). By Theorem 6.5 and in the notation set up in the statement of that result, it remains to consider the case where there exists a 3-edge-separation (F_1, F_2) of H_t such that $|V(F_1)|, |V(F_2)| \geq 2$ and

$F_1 \subset H_t[Y]$. As G is $\mathcal{M}(2,2)$-connected, G is 2-connected by Lemma 5.1 and so G has the structure of one of the four cases shown in Figure 12.

(i) (ii) (iii) (iv)

Fig. 12. The four types of 3-edge-separation.

Firstly, we show there exists a non-trivial 2-vertex-separation or 3-edge-separation, (F_1', F_2'), of H_t such that $|V(F_1') \cap X| \leq 1$ and F_1' is an atom if H_t contained in either F_1 or $H_t[V(F_1) + v_2]$. In case (i), F_1 contains an atom of H_t as (F_1, F_2) is non-trivial. For case (ii)–(iv), label the following vertices: for case (ii), let v_2 be the vertex of F_2 with two neighbours in F_1 and let v_1 be the vertex of F_1 not adjacent to v_2; for case (iii), let v_1 be the vertex of F_1 with two neighbours in F_2 and let v_2 be the vertex of F_2 not adjacent to v_1; for case (iv), let v_2 be the vertex of F_2 with two neighbours in F_1 and let v_1 be the vertex of F_1 with two neighbours in F_2. Then, in cases (ii)–(iv), we note that $(H_t[V(F_1) + v_2], H_t[V(F_2) + v_1])$ is a non-trivial 2-vertex-separation of H_t. Hence $H_t[V(F_1) + v_2]$ contains an atom of H_t. In each of cases (i)–(iv), let F_1' be the atom of H_t contained in F_1 or $H_t[V(F_1) + v_2]$ respectively, and let (F_1', F_2') be the corresponding 2-vertex-separation or 3-edge-separation of H_t.

Note that, in any of the cases (ii)–(iv), if (F_1', F_2') is a non-trivial 3-edge-separation, then $v_2 \notin V(F_1')$. So, if (F_1', F_2') is a non-trivial 3-edge-separation then $V(F_1') \subseteq Y$. If (F_1', F_2') is a non-trivial 2-vertex-separation of H_t, then $|V(F_1') \setminus V(F_1)| \leq 1$, so $|V(F_1') \cap X| \leq 1$. Hence, whether (F_1', F_2') is a non-trivial 2-vertex-separation or a non-trivial 3-edge-separation of H_t, the subgraph F_1' is also an atom of G. We now split the proof into three cases.

Case 6.10. (F_1', F_2') is a non-trivial 3-edge-separation of H_t. □

As $V(F_1') \subseteq Y$, $(F_1', G[V \setminus V(F_1')])$ is a non-trivial 3-edge-separation of G. Let $F_3' = G[V \setminus V(F_1')]$ and $S = E \setminus (E(F_1') \cup E(F_3')) = \{xa, yb, zc\}$, where $x, y, z \in V(F_1')$. Let (G_1, G_2) be the 3-separation of H_t with respect to (F_1', F_2') and let (G_1, G_3) be the 3-separation of G with respect to (F_1', F_3'). Lemma 4.2 implies G_1 is a $(2,2)$-circuit and Lemma 5.6 implies G_3 is $\mathcal{M}(2,2)$-connected. As F_1' is an atom of H_t, there are no non-trivial 2-vertex-separations or 3-edge-separations of G_1 and $G_1 \not\cong B_2$. Hence Theorem 4.7 implies that $G_1 \cong K_5^-$, $G_1 \cong B_1$, or G_1 contains two admissible nodes.

If $G_1 \cong K_5^-$ then $F_1' \cong K_4$. Let G' be the graph given by the edge-reduction of G that contracts xa and deletes xy. Then G' is isomorphic to a graph given by applying two 1-extensions to G_3, so G' is $\mathcal{M}(2,2)$-connected by Lemma 5.2. Hence G has an admissible edge-reduction. If $G_1 \cong B_1$ then one of the vertices x, y, z has degree 3 in G_1 and the other two have degree 5 in G_1. Without loss of generality we may suppose that $d_{G_1}(x) = 3$, and so $d_{F_1'}(x) = 2$. Let G' be the graph given by the edge-reduction of G that contracts xa and deletes xy. Then G' is isomorphic to a 1-join of (G_3, B_2), so G' is $\mathcal{M}(2,2)$-connected by Lemma 5.5. Hence G has an admissible edge-reduction.

Suppose that G_1 contains two admissible nodes. Set v to be the vertex added to F_1' to form G_1 (i.e., $V(G_1) = V(F_1') + v$ and $N_{G_1}(v) = \{x, y, z\}$). Since G_1 contains two admissible nodes, one of the admissible nodes, denoted u, is contained in $V(F_1')$. Set $N_{G_1}(u) = \{r, s, t\}$. Suppose the admissible 1-reduction of G_1 at u adds the edge rs. The resulting graph is isomorphic to the one given by the edge-reduction of G_1 that contracts ur and deletes ut, which we call G_1'. Since this 1-reduction of the $(2,2)$-circuit G_1 is admissible (and so cannot create a degree 2 vertex), it follows that $v \neq t$. Without loss of generality we shall suppose that $s \neq v$ also. Let $N_G(u) = \{r', s, t\}$, where $r' = r$ if $r \neq v$, and $r' \in \{a, b, c\}$ if $r = v$. Let G' be the graph given by the edge-reduction of G that contracts ur' and deletes ut. Then G' is a 3-join of (G_1', G_3) and so G' is $\mathcal{M}(2,2)$-connected by Lemma 5.5. Hence G has an admissible edge-reduction.

Case 6.11. (F_1', F_2') is a non-trivial 2-vertex-separation of H_t and $E(F_1') \cap E(F_2') \neq \emptyset$. □

Let $V(F_1') \cap V(F_2') = \{x, y\}$ and $F_3' = G[(V \setminus V(F_1')) \cup \{x, y\}])$. As $|V(F_1') \cap X| \leq 1$, (F_1', F_3') is a non-trivial 2-vertex-separation of G and $E(F_1') \cap E(F_3') = \{xy\}$. Let (G_1, G_2) be the 2-separation of H_t with respect to (F_1', F_2') and let (G_1, G_3) be the 2-separation of G with respect to (F_1', F_3'). Lemma 4.2 implies G_1 is a $(2,2)$-circuit and Lemma 5.6 implies G_3 is $\mathcal{M}(2,2)$-connected.

As F_1' is an atom of G, there is no non-trivial 3-edge-separation of G_1. If there exists a non-trivial 2-vertex-separation of G_1, then $\{x, y\}$ must be the corresponding vertex cut. As F_1' is an atom of G, one of the components of the 2-vertex-separation must be a K_4 subgraph of F'_1 containing xy. Hence B_1 is a subgraph of G_1 where $\{x, y\}$ is a vertex cut of B_1. As G_1 is a $(2, 2)$-circuit, this implies $G_1 \cong B_1$. However, then $F'_1 \cong K_4$, which contradicts the fact that F'_1 is an atom of H_t. Hence there exist no non-trivial 2-vertex-separations or 3-edge-separations of G_1 and moreover $G_1 \ncong B_1$. Also, $G_1 \ncong K_5^-$, since G_1 is not 3-connected. So, Theorem 4.7 implies that $G_1 \cong B_2$ or G_1 contains two admissible nodes.

If $G_1 \cong B_2$ then we may suppose without loss of generality that $d_{F'_1}(x) = 2$ and set $N_{F'_1}(x) = \{y, a\}$. Let G' be the graph given by the edge-reduction of G that contracts xa and deletes xy. Then $G' \cong G_3$, so G' is $\mathcal{M}(2, 2)$-connected. Hence G has an admissible edge-reduction.

If G_1 contains two admissible nodes then let $\{w, z\} = V(G_1) \setminus V(F'_1)$ and let v be an admissible node in G_1. Let G'_1 be the graph given by the admissible 1-reduction of G_1 that deletes v and adds the edge e. As $v \notin \{w, x, y, z\}$ and $v \in Y$, v is a node in G and $e \notin E$. Now let G' be the graph given by the 1-reduction of G that deletes v and adds the edge e. Then G' is isomorphic to a 2-join of (G'_1, G_3) and so G' is $\mathcal{M}(2, 2)$-connected by Lemma 5.5. Therefore G has either an admissible 1-reduction or an admissible edge-reduction.

Case 6.12. (F'_1, F'_2) is a non-trivial 2-vertex-separation of H_t and $E(F'_1) \cap E(F'_2) = \emptyset$. □

Let $V(F'_1) \cap V(F'_2) = \{x, y\}$. Let $F'_3 = G[(V \setminus V(F'_1)) \cup \{x, y\}])$. As $|V(F'_1) \cap X| \leq 1$, (F'_1, F'_3) is a non-trivial 2-vertex-separation of G and $E(F'_1) \cap E(F'_3) = \emptyset$. Let (G_1, G_2) be the 1-separation of H_t with respect to (F'_1, F'_2) and let $(\tilde{G}_2, \tilde{G}_1)$ be the 1-separation of H_t with respect to (F'_2, F'_1). Lemma 4.3 implies either G_1 and G_2 are $(2, 2)$-circuits and $\tilde{G}_1$ and $\tilde{G}_2$ are not, or $\tilde{G}_1$ and $\tilde{G}_2$ are $(2, 2)$-circuits and G_1 and G_2 are not. We consider these possibilities as separate subcases.

Case 6.13. $\tilde{G}_1$ and $\tilde{G}_2$ are $(2, 2)$-circuits. □

Let $(\tilde{G}_3, \tilde{G}_1)$ be the 1-separation of G with respect to (F'_3, F'_1). By our assumption, G_1 is not a $(2, 2)$-circuit. Since $\tilde{G}_1$ is a $(2, 2)$-circuit that contains G_1, it follows that G_1 is $(2, 2)$-sparse and so not $\mathcal{M}(2, 2)$-connected. Hence, Lemma 5.7 implies that $\tilde{G}_3$ is $\mathcal{M}(2, 2)$-connected.

As F'_1 is an atom of G, there is no non-trivial 3-edge-separation of G_1. If there exists a non-trivial 2-vertex-separation of G_1, then $\{x, y\}$ must be the corresponding vertex cut. As F'_1 is an atom of G, one of the components of the 2-vertex-separation must be a K_4 subgraph of F'_1 containing xy, however this contradicts that $xy \notin E$. Hence there are no non-trivial 2-vertex-separations or 3-edge-separations of $\tilde{G}_1$. Also, as $\tilde{G}_1$ is not 3-connected, $\tilde{G}_1 \ncong K_5^-$. Hence Theorem 4.7 implies that $\tilde{G}_1 \cong B_1$ or B_2, or $\tilde{G}_1$ contains two admissible nodes.

If $\tilde{G}_1 \cong B_1$ then $F'_1 \cong K_4^-$ with $F'_1 = \{x, y, r, s\}$ for some vertices r, s. Let G' be the graph given by the K_4^--reduction of G that deletes r and s and adds the edge xy. Then $G' = \tilde{G}_3$, so G' is $\mathcal{M}(2, 2)$-connected and the K_4^--reduction is admissible. If $\tilde{G}_1 \cong B_2$ then F'_1 is isomorphic to K_4 with an added vertex connected to exactly one vertex in the K_4. Furthermore, the vertex with degree 1 in F'_1 must be either x or y. We may suppose without loss of generality that $d_{F'_1}(x) = 1$ and set a to be the single neighbour of x in F'_1. Let G' be the graph given by the edge-reduction of G that contracts xa and deletes ya. Then G' is isomorphic to a K_4^--extension of $\tilde{G}_3$, so G' is $\mathcal{M}(2, 2)$-connected by Lemma 5.8 and this edge-reduction is admissible.

If $\tilde{G}_1$ contains two admissible nodes then let $\{w, z\} = V(G_1) \setminus V(F'_1)$ and let v be an admissible node in $\tilde{G}_1$. Let $\tilde{G}'_1$ be the graph given by the admissible 1-reduction of $\tilde{G}_1$ that deletes v and adds the edge e. As $v \notin \{w, x, y, z\}$ and $v \in Y$, v is a node in G and $e \notin E$. Now let G' be the graph given by the 1-reduction of G that deletes v and adds the edge e. Then G' is isomorphic to a 1-join of $(\tilde{G}_3, G'_1)$ and so G' is $\mathcal{M}(2, 2)$-connected by Lemma 5.5, completing this subcase.

Case 6.14. G_1 and G_2 are $(2, 2)$-circuits. □

Let (G_1, G_3) be a 1-separation of G with respect to the non-trivial 2-vertex-separation (F'_1, F'_3). By Lemma 5.7, G_3 is $\mathcal{M}(2, 2)$-connected. As F'_1 is an atom of H_t, there are no non-trivial 2-vertex-separations or 3-edge-separations of G_1. Therefore Theorem 4.7 implies that $G_1 \cong K_5^-, B_1$, or B_2, or G_1 contains two admissible nodes.

If $G_1 \cong K_5^-$ then F'_1 is isomorphic to one of two graphs (see Figure 13). Let $V(F'_1) = \{x, y, a, b, c\}$. We may suppose without loss of generality that $d_{F'_1}(c) = 4$. If $ab \in E(F'_1)$ then we may suppose without loss of generality that $ay \notin E(F'_1)$. In either case, let G' be the graph given by the edge-reduction of G that contracts by and deletes bc. Then $G' \cong G_3$, and hence this is an admissible edge-reduction.

If $G_1 \cong B_1$ then, as F'_1 is an atom of H_t, F'_1 is isomorphic to one of two graphs (see Figure 14). To see this note that if x and y were the middle vertices of B_1, then the subgraph of F'_1 induced by $\{x, y, a, b\}$, where a and b are adjacent so that $B_1[x, y, a, b]$ is isomorphic to K_4^-, would be an element of a non-trivial 2-vertex-separation of H_t which contradicts the fact that F'_1 is an atom of H_t. We may now suppose without loss of generality that $d_{F'_1}(x) = 2$. Let $N_{F'_1}(x) = \{a, b\}$ such that $d_{F'_1}(a) \leq d_{F'_1}(b)$. In either case, let G' be the graph given by the edge-reduction of G that contracts xa and deletes xb. Then G' is isomorphic to a 2-join of (B_2, G_3) and hence G' is $\mathcal{M}(2, 2)$-connected by Lemma 5.5. So G has an admissible edge-reduction.

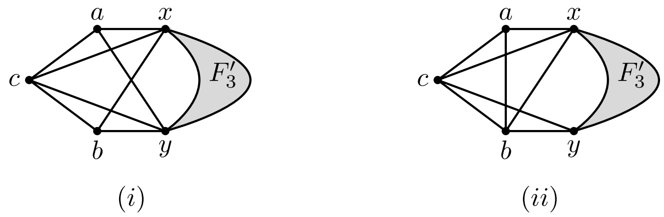


Fig. 13. The graph G in the subcase $G_1 \cong K_5^-$ of Case 6.14.

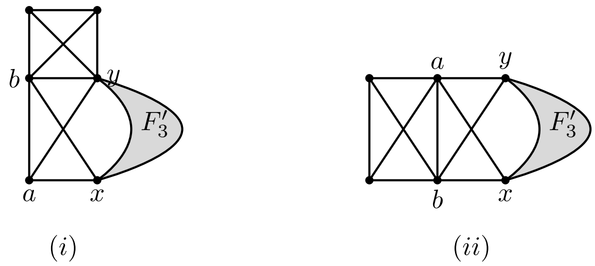


Fig. 14. The graph G in the subcase $G_1 \cong B_1$ of Case 6.14.

If G_1 contains two admissible nodes, then either there exists $v \in V(F_1') \setminus \{x, y\}$ such that v is an admissible node in G_1, or x and y are the only admissible nodes of G_1. In the first case, let G_1' be given by an admissible 1-reduction of G_1 at v that adds the edge e. Importantly, $e \neq xy$, since xy is an edge of G_1. As $v \notin \{x, y\}$, v is a node in G also. Let G' be given by the 1-reduction of G at v that adds the edge e. Then G' is isomorphic to a 1-join of (G_1', G_3), so G' is $\mathcal{M}(2,2)$-connected by Lemma 5.5. Hence G has an admissible edge-reduction. Alternatively, suppose that x and y are the only admissible nodes of G_1. Let $N_{G_1}(x) = \{a, b, y\}$ and let G_1' be given by the admissible 1-reduction of G_1 that removes x and adds the edge e (see Figure 15). As this 1-reduction is admissible, $d_{G_1'}(y) \geq 3$, so $e \in \{ya, yb\}$. Without loss of generality we may suppose that $e = ya$, so $ya \notin E$. Let G' be the edge-reduction of G that contracts xa and deletes xb. Then, as $xy, ya \notin E$, G' is isomorphic to a 1-join of (G_1', G_3) and hence is $\mathcal{M}(2,2)$-connected by Lemma 5.5. Therefore G has an admissible edge-reduction.

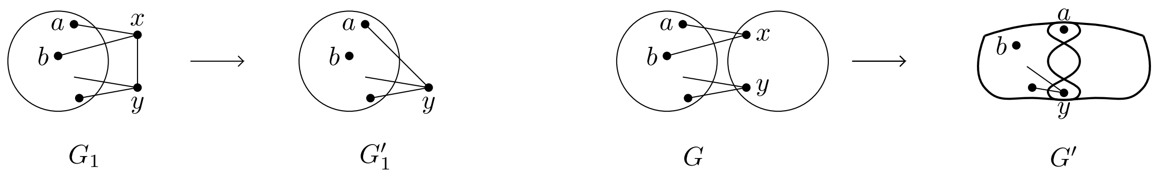


Fig. 15. The subcase of Case 6.14 when G_1 is not isomorphic to K_5^-, B_1 or B_2. Here G' is the 1-join of G_1' and G_3.

If $G_1 \cong B_2$ then, as F_1' is an atom of H_t, G is isomorphic to the graph shown in Figure 16. Let $\{u\} = N_{F_1'}(x) \cap N_{F_1'}(y)$ and let

$$G_1' = (V(G_1) \cup \{w, z\}, (E(G_1) \setminus \{xu\}) \cup \{xw, xz, yw, yz, wz\});$$

see Figure 16. Then G_1' is isomorphic to a 1-join of (B_2, B_1) so G_1' is $\mathcal{M}(2,2)$-connected by Lemma 5.5. Now consider G_3. As $|V(G_3)| + |E(G_3)| = |V| + |E| - 9$, it follows from our initial induction assumption (see the first paragraph of the proof) that either G_3 is isomorphic to K_5^- or B_1, or there exists an admissible K_4^--reduction, edge-reduction, or edge-deletion of G_3. As G_3 is not 3-connected, $G_3 \ncong K_5^-$. If $G_3 \cong B_1$ then G is isomorphic to the graph on the right of Figure 9, and so has an admissible K_4^--reduction. Suppose there exists an admissible K_4^--reduction, edge-reduction, or edge-deletion of G_3. Label the vertices in $V(G_3) = V(F_3')$ by w', z'.

If there exists an admissible K_4^--reduction of G_3 then the same K_4^- is present in G as an induced subgraph. Let G_3' be the graph given by the K_4^--reduction of G_3 and let G' be the graph given by the K_4^--reduction of G. Then G' is isomorphic to a 1-join of (G_1, G_3') and so G' is $\mathcal{M}(2,2)$-connected by Lemma 5.5. Hence G has an admissible K_4^--reduction.

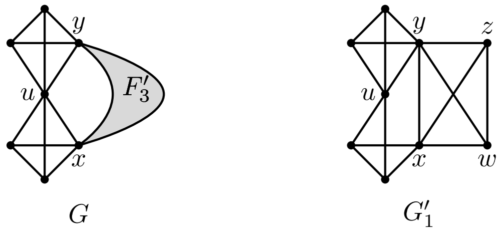


Fig. 16. The graphs G and G_1' in the subcase $G_1 \cong B_2$ of Case 6.14.

If there exists an admissible edge-reduction of G_3, let this contract e and delete f and call the resulting graph G_3'. As this edge-reduction is admissible, $e \notin \{xy, xw', xz', yw', yz', w'z'\}$ and $f \notin \{xw', xz, yw', yz', w'z'\}$. Hence $e \in E$, and $f \in E$ if and only if $f \neq xy$. Suppose that $f \neq xy$ and let G' be given by the edge-reduction of G that contracts e and deletes f. Then G' is isomorphic to a 1-join of (G_1, G_3') and so G' is $\mathcal{M}(2,2)$-connected by Lemma 5.5. Hence G has an admissible edge-reduction. Now suppose $f = xy$. Without loss of generality we suppose that $e = xa$ for some $a \in V(F_3') - y$ and denote the vertex resulting from the contraction of e by x'.

Let G' be given by the edge-reduction of G that contracts e and deletes xu. If $ya \in E(G_3)$ then G' is isomorphic to a 2-join of (G_1', G_3') with x' relabelled as x. Since both G_1' and G_3' are $\mathcal{M}(2,2)$-connected, G' is $\mathcal{M}(2,2)$-connected also by Lemma 5.5. If $ya \notin E(G_3)$ then $x'y \notin E(G_3')$. Let (G_4, G_5) be a 1-separation of G_3' on $(G_3'[V(G_3') \setminus \{w', z'\}], G_3'[\{x', y, w', z',\}])$ and let $(\tilde{G}_5, \tilde{G}_4)$ be a 1-separation of G_3' on $(G_3'[\{x', y, w', z'\}], G_3'[V(G_3') \setminus \{w', z'\}])$. Then G' is isomorphic to a 1-join of (G_4, G_1'). As $\tilde{G}_5 \cong K_4$, Lemma 5.7 implies that G_4 is $\mathcal{M}(2,2)$-connected. So Lemma 5.5 gives that G' is $\mathcal{M}(2,2)$-connected. Therefore G has an admissible edge-reduction.

Lastly, if there exists an admissible edge-deletion of G_3 that deletes the edge e, then $e \notin \{xw', xz', yw', yz', w'z'\}$. If $e \neq xy$ then $G - e$ is isomorphic to a 1-join of $(G_1, G_3 - e)$ and so $G - e$ is $\mathcal{M}(2,2)$-connected by Lemma 5.5. Suppose $e = xy$. Given $G_{3,1} := G_3[V(G_3') \setminus \{w', z'\}] - e$ and $G_{3,2} := G_3[\{x, y, w', z'\}] - e$, let (G_4, G_5) be a 1-separation of $G_3 - e$ on $(G_{3,1}, G_{3,2})$ and let $(\tilde{G}_5, \tilde{G}_4)$ be a 1-separation of $G_3 - e$ on $(G_{3,2}, G_{3,1})$. As $\tilde{G}_5 \cong K_4$, Lemma 5.7 implies G_4 is $\mathcal{M}(2,2)$-connected. Since $G - xu$ is isomorphic to a 1-join of (G_4, G_1'), Lemma 5.5 implies $G - xu$ is $\mathcal{M}(2,2)$-connected. Thus G has an admissible edge-deletion. This concludes the proof. ■

Proof of Theorem 6.1. The case when G is a $(2,2)$-circuit is Theorem 4.5. The case when G is not a $(2,2)$-circuit follows, in one direction, from Lemmas 5.2 and 5.5. The converse follows from Theorem 6.9. ■

Corollary 6.15. A graph G is 2-connected and redundantly rigid in an analytic normed plane X if and only if G can be generated from K_5^- or B_1 (see Figure 5) by K_4^--extensions, edge additions and generalised vertex splits such that each intermediate graph is 2-connected and redundantly rigid in X. □

Proof. By Proposition 2.3 and Lemma 5.1, a graph is $M(2,2)$-connected if and only if it is redundantly rigid in X and 2-connected. We now apply Theorem 6.1. ■

7 Characterising weakly globally rigid graphs in analytic normed planes

The main result of the paper (Theorem 1.1) is proved in this section. We begin by stating two results from [24]. The first concerns the base graphs from Corollary 6.15 (see Figure 5)** and the second is an analogue of the "averaging theorem" of Connelly and Whiteley [15].

Lemma 7.1 ([24, Theorems 5.3 & 5.4])**.** The graphs K_5^- and B_1 are weakly globally rigid in any analytic normed plane. □

Theorem 7.2 ([24, Theorem 3.10])**.** Let (G, p) be a globally rigid and infinitesimally rigid framework in a smooth normed space X with finitely many linear isometries. Then there exists an open neighbourhood U of p where (G, q) is globally and infinitesimally rigid in X for all $q \in U$. □

We prove that K_4^--extensions preserve weak global rigidity in analytic normed planes. We require the following lemma.

**It is worth pointing out that for a given norm, weakly global rigidity for such small graphs could be checked computationally. However, giving a rigorous proof for all analytic norms simultaneously is non-trivial.

Lemma 7.3. Let X be a non-Euclidean normed plane. Then for all but a finite number of points $x \in X$ with $\|x\| = 1$, if g is an isometry and $g(x) = x$ then g is the identity map. □

Proof. Let A be the set of points $x \in X$ where $\|x\| = 1$ and there exists a non-trivial linear isometry g_x with $g_x(x) = x$. We note that if $x, y \in A$ are linearly independent then $g_x(y) \neq y$ (as otherwise g_x must be the identity map), thus the cardinality of A is less than twice the cardinality of the group of linear isometries of X. The result now follows as all non-Euclidean normed planes have a finite amount of linear isometries (see [19, Proposition 2.5(ii)]). ■

Let X be an analytic normed plane, let $z \in X$ be any non-zero point, and define $\mathcal{H}$ and $\mathcal{H}'$ to be the two open half-planes formed by the line through 0 and z. It was shown in [24, Theorem 4.3] that there exists a unique map $R_z : X \to X$ (called the *z-reflection*) defined by the following properties: (i) if $x \in X$ does not lie on the line through 0 and z, then $R_z(x)$ is the unique point distinct from x where $\|R_z(x)\| = \|x\|$ and $\|R_z(x) - z\| = \|x - z\|$, and (ii) if $x \in X$ lies on the line through 0 and z, then $R_z(x) = x$. The map R_z has additional useful properties that we will require. Firstly, it is a homeomorphism and an involution, and it is an analytic diffeomorphism when its domain and codomain are restricted to the set of points that do not lie on the line through 0 and z. Secondly, any point in $\mathcal{H}$ will be mapped to a point in $\mathcal{H}'$ and vice versa.

Lemma 7.4. Let X be an analytic normed plane, $z \in X$ be any non-zero point and $\mathcal{H}$ be an open half plane formed by the line through 0 and z. Suppose that R_z is not an isometry. Then for any two points x, y that lie in the closure of $\mathcal{H}$ and for any $\varepsilon > 0$, there exists $x', y' \in \mathcal{H}$ such that $\|x - x'\| < \varepsilon$, $\|y - y'\| < \varepsilon$ and

$$\|R_z(x') - R_z(y')\| \neq \|x' - y'\|, \qquad \|x' - R_z(y')\| \neq \|x' - y'\|, \qquad \|R_z(x') - y'\| \neq \|x' - y'\|.$$

□

Proof. Define $U \subset \mathcal{H} \times \mathcal{H}$ to be the connected subset of all points (x, y) where $x \neq y$. It suffices to prove that the following two sets are open conull subsets of U:

$$D := \{(x, y) \in U : \|R_z(x) - R_z(y)\| \neq \|x - y\|\}, \qquad D' := \{(x, y) \in U : \|x - R_z(y)\| \neq \|x - y\|\}.$$

Define $f : U \to \mathbb{R}$ to be the analytic function that maps $(x, y) \in U$ to $\|R_z(x) - R_z(y)\| - \|x - z\|$, and define $g : U \to \mathbb{R}$ to be the analytic function that maps $(x, y) \in U$ to $\|x - R_z(y)\| - \|x - z\|$. It is immediate that the zero sets of f and g are the sets $U \setminus D$ and $U \setminus D'$ respectively. As both f and g are analytic and the set U is connected, their zero sets are either closed null sets or the entirety of U (see [45]); i.e., either D (respectively D') is empty or D (respectively D') is an open conull set. If $D = \emptyset$ then R_z is an isometry (as it is an involution and invariant on the line through 0 and z), hence $U \setminus D$ is a closed conull set. As $\|x - R_z(x)\| - \|x - x\| \neq 0$ and R_z is continuous, D' is non-empty. Hence $g(x, y) \neq 0$ for sufficiently close x and y. ■

We now use z-reflections to prove that K_4^--extensions preserve weak global rigidity in analytic normed planes. (See Section 3.2 for the definition of the compact differentiable manifold $\mathcal{C}(G, p)$.)

Theorem 7.5. Let $G = (V, E)$ be a weakly globally rigid graph in an analytic normed plane X and let $G' = (V', E')$ be formed from G by a K_4^--extension. Then G' is weakly globally rigid in X. □

Proof. Suppose G' is formed from G by the K_4^--extension applied to the edge v_1v_2 that adds vertices u_1, u_2. Since G is weakly globally rigid and X is analytic, we use Proposition 2.3 to choose a completely strongly regular globally rigid framework (G, p) with $p_{v_1} = 0$ and $p_{v_2} = z$ for some $\|z\| = 1$; by Lemma 7.3, we may assume that $g(z) \neq z$ given any non-trivial linear isometry g. By Theorem 3.7, (G, p) is redundantly rigid. Since (G, p) is completely strongly regular, the set $\mathcal{C}(G - v_1v_2, p)$ is finite. Since (G, p) is globally rigid, for any placement $q \in X^V$ where $f_{G-v_1v_2}(q) = f_{G-v_1v_2}(p)$, we have $\|q_{v_1} - q_{v_2}\| = \|p_{v_1} - p_{v_2}\|$ if and only if $q \sim p$. For any equivalence class $\tilde{q} \in \mathcal{C}(G, p)$ we have $\|q_{v_1} - q_{v_2}\| = \|q'_{v_1} - q'_{v_2}\|$ for all $q, q' \in \tilde{q}$. Since $\mathcal{C}(G - v_1v_2, p)$ is finite,

$$t := \min \left\{ \left| \|q_{v_1} - q_{v_2}\| - \|p_{v_1} - p_{v_2}\| \right| : f_{G-v_1v_2}(q) = f_{G-v_1v_2}(p), \ q \not\sim p \right\}.$$

is well-defined and non-zero.

Choose $\varepsilon < t/4$. Let $\mathcal{H}, \mathcal{H}'$ be the open halfplanes defined by the line through $0, z$. By our choice of z, the z-reflection R_z is not an isometry. By Lemma 7.4, there exists $x_1, x_2 \in \mathcal{H}$ such that $\|x_1\| < \varepsilon$, $\|x_2 - z\| < \varepsilon$, $\|R_z(x_1) - R_z(x_2)\| \neq \|x_1 - x_2\|$, $\|x_1 - R_z(x_2)\| \neq \|x_1 - x_2\|$ and $\|R_z(x_1) - x_2\| \neq \|x_1 - x_2\|$. Choose p' to be a placement of G' where $p'_v = p_v$ for all $v \in V$ and $p'_{u_i} = x_i$ for each $i \in \{1,2\}$. As $\|p'_{v_i} - p'_{u_i}\| < \varepsilon$ for $i \in \{1,2\}$, for each $q \in f_{G'}^{-1}(f_{G'}(p'))$ we have

$$\Big| \|q_{v_1} - q_{v_2}\| - \|q_{u_1} - q_{u_2}\| \Big| \leq \|(q_{v_1} - q_{u_1}) + (q_{u_2} - q_{v_2})\| \leq \|q_{v_1} - q_{u_1}\| + \|q_{v_2} - q_{u_2}\| < 2\varepsilon$$

Hence, for any $q \in f_{G'}^{-1}(f_{G'}(p'))$ we have

$$\begin{aligned}\Big|\|q_{v_1} - q_{v_2}\| - \|p'_{v_1} - p'_{v_2}\|\Big| &= \Big|\|q_{v_1} - q_{v_2}\| - \|q_{u_1} - q_{u_2}\| + \|p'_{u_1} - p'_{u_2}\| - \|p'_{v_1} - p'_{v_2}\|\Big| \\ &\leq \Big|\|q_{v_1} - q_{v_2}\| - \|q_{u_1} - q_{u_2}\|\Big| + \Big|\|p'_{u_1} - p'_{u_2}\| - \|p'_{v_1} - p'_{v_2}\|\Big| < 4\varepsilon < t.\end{aligned}$$

By our choice of t it follows that for all $q \in f_{G'}^{-1}(f_{G'}(p'))$ we have $\|q_{v_1} - q_{v_2}\| = \|p'_{v_1} - p'_{v_2}\|$, and thus (G', p') is globally rigid if and only if $(G' + v_1v_2, p')$ is globally rigid.

Fix a placement $q \in X^V$ where $(G' + v_1v_2, q) \sim (G' + v_1v_2, p')$. As (G, p) is a globally rigid subframework of $(G' + v_1v_2, p')$, we may suppose we chose q such that $q_v = p'_v$ for all $v \in V$. By the uniqueness R_z, we have $q_{u_i} \in \{p'_{u_i}, R_z(p'_{u_i})\}$ for each $i \in \{1,2\}$. However, by our choice of x_1, x_2 we have $q_{u_i} = p'_{u_i}$ for each $i \in \{1,2\}$. Thus $(G' + v_1v_2, p')$ (and so (G', p')) is globally rigid.

We now prove (G', p') is infinitesimally rigid; the result then follows from Theorem 7.2. Choose an infinitesimal flex $a \in X^{V'}$ of (G', p'). Since (G, p) is redundantly rigid, a restricted to the vertices of G is a trivial infinitesimal flex. As every non-Euclidean normed plane has finitely many linear isometries ([19, Proposition 2.5(ii)]), we may suppose $a_v = 0$ for all $v \in V$. By our choice of x_1, x_2, the vectors $p'_{u_i} - p'_{v_1}, p'_{u_i} - p'_{v_2}$ are linearly independent for each $i \in \{1,2\}$. As X is smooth and strictly convex, the vectors $p'_{u_i} - p'_{v_1}, p'_{u_i} - p'_{v_2}$ have unique support functionals and the functionals $\varphi_{p'_{u_i} - p'_{v_1}}, \varphi_{p'_{u_i} - p'_{v_2}}$ are linearly independent. For each $i, j \in \{1,2\}$ we have

$$\varphi_{p'_{u_i} - p'_{v_j}}(a_{u_i} - a_{v_j}) = \varphi_{p'_{u_i} - p'_{v_j}}(a_{u_i}) = 0.$$

Hence $a_{u_1} = a_{u_2} = 0$, i.e., a is trivial and (G', p') is infinitesimally rigid. ■

Generalised vertex splits also preserve weak global rigidity when certain criteria are met.

Theorem 7.6 ([22, Theorem 5.4])**.** Let G be a weakly globally rigid graph in an analytic normed plane X. Let G' be a generalised vertex split of G at the vertex z with new vertices u, v and suppose that $G' - uv$ is rigid in X. Then G' is weakly globally rigid in X. □

With this, we are now finally ready to prove our main result (Theorem 1.1). Recall that this states that, for a graph $G = (V, E)$ with $|V| \geq 5$ and an analytic normed plane X, G is weakly globally rigid in X if and only if G is 2-connected and redundantly rigid in X.

Proof of Theorem 1.1. If G is weakly globally rigid in X then it is 2-connected and redundantly rigid in X by Theorem 3.8. Conversely, suppose G is 2-connected and redundantly rigid in X. By Corollary 6.15, G can be generated from K_5^- or B_1 by generalised vertex splits that preserve redundant rigidity in X, K_4^--extensions and edge additions. The graphs K_5^- and B_1 are weakly globally rigid in X by Lemma 7.1. By Theorem 7.5, K_4^--extensions preserve weak global rigidity in X, and by Theorem 7.6 generalised vertex splits that preserve redundant rigidity in X also preserve weak global rigidity in X. Since edge additions certainly preserve weak global rigidity, the proof is complete. ■

Note that the characterisation is effective since there exist efficient algorithms that can check redundant rigidity in X [8, 41] and 2-connectivity [54].

7.1 Non-analytic norms

It would be interesting to extend Theorem 1.1 to non-analytic normed planes, for example with the ℓ_∞ norm. We give brief comments on some of the difficulty of such an endeavour.

Firstly, our definition for redundant rigidity with graphs finds some difficulties. Surprisingly, it is not true for any normed space X that a graph G is redundantly rigid in X if and only if $G - e$ is rigid for all $e \in E$ in X. For example, take the normed space $\ell_\infty^2 := (\mathbb{R}^2, \|\cdot\|_\infty)$, where $\|(a,b)\|_\infty := \max\{|a|, |b|\}$. For any well-positioned framework (G, p), we can define a partition $E_1 \cup E_2$ of the edges of $G = (V, E)$, where $vw \in E_1$ (respectively, $vw \in E_2$) if and only if, given $(x_1, x_2) = p_v - p_w$, we have $|x_1| > |x_2|$ (respectively, $|x_2| > |x_1|$). It was shown in [33] that rank $df_G(p)$ is equal to the sum of the ranks of the oriented incidence matrices of the *monochromatic subgraphs* $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ of (G, p). Hence a well-positioned framework is infinitesimally rigid in

ℓ_∞^2 if and only if both its monochromatic subgraphs are connected, and redundantly rigid in ℓ_∞^2 if and only if both its monochromatic subgraphs are 2-edge-connected. Consider the graph K_5^-, the complete graph on five vertices minus an edge. While $K_5^- - e$ is rigid in ℓ_∞^2 for any edge $e \in E(K_5^-)$ (see Theorem 2.5 below), any placement will not be redundantly rigid, as the edges of K_5^- cannot be decomposed into two 2-edge-connected spanning subgraphs.

An important geometric fact we rely on is that almost all placements in an analytic normed space are completely regular. However, there exist normed spaces with no completely regular placements for certain graphs. Take, for example, ℓ_∞^2. There exist no completely regular placements of any set with five or more elements in ℓ_∞^2, as one of the monochromatic subgraphs of any K_5 subgraph must contain a cycle, and the framework restricted to that monochromatic subgraph will not be regular.

7.2 Sufficient conditions

We conclude the paper by deriving some combinatorial sufficient conditions from Theorem 1.1. First there is an immediate link between weak global rigidity in analytic normed planes and global rigidity in the Euclidean plane.

Corollary 7.7. Let $G = (V, E)$ be a globally rigid graph in $\mathbb{E}^2$ with $|V| \geq 2$ and let X be an analytic normed plane. Then G is weakly globally rigid in X if and only if $|E| > 2|V| - 2$. □

Proof. If G is weakly globally rigid in X then $|E| > 2|V| - 2$ by Theorems 1.1 and 2.5. For the converse suppose G is globally rigid in $\mathbb{E}^2$ with $|E| > 2|V| - 2$. Since G is simple, $|V| \geq 5$, and so G is 2-connected by Theorem 2.8. Choose any edge e of G. By Theorems 2.8 and 2.4, there exists a spanning $(2, 3)$-tight subgraph $H \subset G - e$. As $|E| > 2|V| - 2$, there exists an edge f contained in $G - e$ but not H. The graph $H + f$ is $(2, 2)$-tight, hence $G - e$ is rigid in X by Theorem 2.5. Since e was arbitrary, the result follows from Theorem 1.1. ■

Corollary 7.8. Any 2-connected and 4-edge-connected graph is weakly globally rigid in any analytic normed plane. □

Proof. A standard result in graph theory (see [10]) states that graph $G = (V, E)$ is 4-edge-connected if and only if for all $F \subseteq E$ with $|F| \leq 2$, the graph $G - F$ contains 2 edge-disjoint spanning trees. The theorem now follows from Theorems 1.1 and 2.5. ■

It is easy to see that these connectivity conditions are best possible since 2-connectivity is a necessary condition for weak global rigidity (Theorem 3.1) and there exist graphs that are both 2-connected and 3-edge-connected but do not contain a $(2, 2)$-tight spanning subgraph (e.g., the complete bipartite graph $K_{3,3}$). We go out on a limb and conjecture that every $2d$-edge-connected and 2-connected graph is weakly globally rigid in any d-dimensional normed space with a finite number of linear isometries. (Recall that when $d > 2$ and the norm is non-Euclidean, the weaker property of rigidity is only understood in certain special cases [23].)

Define the *algebraic connectivity* of a graph to be the second smallest eigenvalue of the Laplacian matrix of the graph.

Corollary 7.9. Let G be a 2-connected graph with minimum degree $\delta \geq 5$ and algebraic connectivity $\mu > \frac{4}{\delta+1}$. Then G is weakly globally rigid in any analytic normed plane. □

Proof. By [11, Theorems 4.10 & 4.12(ii)] and Theorem 2.5, G is redundantly rigid in any analytic normed plane. The result now follows from Theorem 1.1. ■

An alternative sufficient condition for global rigidity in d-dimensional Euclidean space was provided by Tanigawa [53]. Our next corollary shows that an analogous statement is true for analytic normed planes.

Corollary 7.10. Let $G = (V, E)$ be a graph, $|V| \geq 3$ and X be an analytic normed plane. If $G - v$ is rigid in X for all $v \in V$ then G is weakly globally rigid in X. □

Proof. Since rigid graphs are connected and $|V| \geq 3$, G is 2-connected. Moreover by considering any edge incident to a given vertex, G must be redundantly rigid in X. Hence G is weakly globally rigid in X by Theorem 1.1. ■

We conclude the paper by applying Corollary 7.8 to two natural random graph models. An *Erdös-Rényi random graph* $G_{n,p}$ is a graph with n vertices where each edge exists with probability p.

Corollary 7.11. Let $G_{n,p}$ be an Erdös-Rényi random graph with

$$p = \frac{\log(n) + k \log\log(n) + c_n}{n}$$

for some positive integer k and some real sequence $(c_n)_{n\in\mathbb{N}}$. Then the following holds for any analytic normed plane X.

(i) If $k = 1$ then:
   (a) $G_{n,p}$ is rigid in X a.a.s. if $c_n \to \infty$.
   (b) $G_{n,p}$ is flexible in X a.a.s. if $c_n \to -\infty$.

(ii) If $k = 2$ then:
   (a) $G_{n,p}$ is weakly globally rigid in X a.a.s. if $c_n \to \infty$.
   (b) $G_{n,p}$ is not weakly globally rigid in X a.a.s. if $c_n \to -\infty$.

□

Proof. We first consider the case where $c_n \to \infty$. By [42, Corollary 1.2], $G_{n,p}$ is rigid in $\mathbb{E}^{k+1}$ a.a.s. First suppose $k = 1$. As $G_{n,p}$ is rigid in $\mathbb{E}^2$ a.a.s., Theorem 2.4 implies $G_{n,p}$ contains a spanning $(2, 3)$-tight subgraph H a.a.s. Since the expected number of edges of $G_{n,p}$ is $\frac{1}{2}(n-1)(\log(n) + \log\log(n) + c_n)$ which is strictly larger than $2n - 3$ for sufficiently large n, there exists an edge $e \in E(G_{n,p}) \setminus E(H)$ a.a.s. As $H + e$ (when both H and e exist) is a $(2, 2)$-tight graph, $G_{n,p}$ is rigid in X a.a.s. by Theorem 2.5. Now suppose $k = 2$. By [26, Theorem 1.18], $G_{n,p}$ is globally rigid in $\mathbb{E}^2$ a.a.s. Since the expected number of edges of $G_{n,p}$ is $\frac{1}{2}(n-1)(\log(n) + \log\log(n) + c_n)$ which is strictly larger than $2n - 2$ for sufficiently large n, $G_{n,p}$ is weakly globally rigid in X a.a.s. by Corollary 7.7.

We now consider the case where $c_n \to -\infty$. Here we have that $G_{n,p}$ has minimum degree at most k a.a.s.; see [9, Section 7] for more details. If $k = 1$ then $G_{n,p}$ is flexible in X a.a.s. If $k = 2$ then $G_{n,p}$ is not redundantly rigid in X a.a.s., and hence is not weakly globally rigid in X a.a.s., by Theorem 1.1. ■

Let $\mathcal{G}_{n,k}$ be the set of all k-regular graphs with n vertices. By equipping the set $\mathcal{G}_{n,k}$ with the uniform probability measure, we define a graph chosen at random from $\mathcal{G}_{n,k}$ to be a *random k-regular graph.*

Corollary 7.12. Let G be a random k-regular graph and let X be an analytic normed plane. If $k \geq 4$ then G is weakly globally rigid in X a.a.s. □

Proof. A random k-regular graph is k-connected a.a.s. [57], hence G is weakly globally rigid in X a.a.s. by Corollary 7.8. ■

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