Mackey functors over fusion systems

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Abstract

In this thesis we study the properties of Mackey functors over fusion systems as opposed to Mackey functors over groups. Given a fusion system $\mathcal{F}$ we start by defining the Mackey algebra of $\mathcal{F}$. We then use it in order to provide definitions for Mackey functors over $\mathcal{F}$ and $\mathcal{F}$-centric Mackey functors (also known as $\mathcal{F}^c$-restricted Mackey functors) which coincide with those in the literature. We go on to proving that several results such as Higman's criterion and the Green correspondence can be translated from Mackey functors over groups to $\mathcal{F}$-centric Mackey functors. We also show that the methods used to perform this translation cannot be used to prove similar results for Mackey functors over fusion systems in general.

In the second part of this thesis we focus our efforts on the sharpness conjecture for fusion systems. We do so by using spectral sequences in order to provide sufficient conditions (in terms of fusion subsystems of $\mathcal{F}$) for the conjecture to be satisfied for $\mathcal{F}$. We then use the developed tools in order to prove that the sharpness conjecture is satisfied for all Benson-Solomon fusion systems thus completing previous work of Henke, Libman and Lynd.

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Author's declaration

I declare that this thesis entitled *"Mackey functors over fusion systems"* is the result of my own research conducted under the supervision of Professor Nadia Mazza except as cited in references. This thesis has not been submitted in any form for the award of a higher degree elsewhere. However the article "*Green correspondence on centric Mackey functors over fusion systems*" comprising Chapter 2 has been published in the *"Journal of Algebra".* The article "*Sharpness for the Benson-Solomon fusion systems*" comprising Chapter 3 was submitted for publication at the time of the defense, however the author found a mistake in the proof of Proposition 3.5.10 (1) after the corrections to the thesis where accepted. This error invalidates the results of Proposition 3.5.10 and Theorems 3.A and 3.B.

Contents

Chapter 1

Introduction

This PhD thesis explores the properties that Mackey functors preserve when translated from the context of finite groups to the context of fusion systems.

The main results of the research conducted during the duration of the author's PhD are contained in the two articles that form Chapters 2 and 3. Chapter 4 briefly summarizes the results obtained during these articles and outlines a research project that could be pursued building on such results. Finally the present chapter serves to outline the mathematical context surrounding this thesis (see Section 1.1) and introduce the results obtained in Chapters 2 and 3 (see Section 1.2).

1.1 Fusion systems and Mackey functors

Fusion systems were introduced by Puig in [Pu06] as a common framework between fusion of p-subgroups in finite groups and p-blocks of finite groups. The most recurrent example of (saturated) fusion system is the following.

Example 1.1.1. Let p be a prime, let G be a finite group and let $S \in \mathrm{Syl}_p(G)$. The (saturated) fusion system $\mathcal{F} = \mathcal{F}_S(G)$ over S is the category with:

objects all the subgroups of S,

morphisms between any two subgroups $P, Q \leq S$ defined by

$$\mathrm{Hom}_{\mathcal{F}}(P,Q) := \left\{ c_x \,:\, x \in G \text{ s.t. } {}^{x}P := xPx^{-1} \leq Q \right\}$$

where $c_x(y) := {}^x y := xyx^{-1}$ for every $y \in P$, and

composition rule given by the standard composition of group morphisms. That is $c_y c_x = c_{yx}$.

A precise definition of fusion system is provided in Sections 2.2.1 and 3.2.1 and the interested reader is referred to [Li07] for introductory notes on them. For the purpose of this chapter it suffices to say that any fusion system over a finite p-group S is a category $\mathcal{F}$ with objects the subgroups of S and morphisms injective group morphisms. Additionally $\mathcal{F}$ must contain $\mathcal{F}_S(S)$ as a subcategory and its morphisms must satisfy two *saturation* axioms (see Definitions 2.2.6 and 3.2.6 for details).

Since their appearance fusion systems have received much interest in both algebra and topology (see [AKO11; Cr11]). Of particular relevance to this thesis is the work conducted by Broto, Levi and Oliver in [BLO03] where centric linking systems where introduced. These categories lead to topological spaces that "behave like" the "classifying space" of the underlying fusion system and it was proven by Chermak in [Ch13] that every fusion system has a unique associated centric linking system. These results are at the base of the homotopy theory for fusion systems.

Mackey functors are algebraic structures possessing operations which "behave like" the induction, restriction and conjugation mappings in group representation theory. These same operations seem to appear in a variety of different contexts such as group cohomology, algebraic K-theory of group rings and algebraic number theory amongst others. As a result the theory of Mackey functors can be applied to multiple branches of mathematics.

There are multiple equivalent methods for defining Mackey functors but the ones most relevant to this thesis are due to Dress (see [Dr71]) and to Thénevaz and Webb (see [TW95]). The former allows us to view Mackey functors as a pair of a covariant and a contravariant functors while the latter describes them as modules over a certain ring called the Mackey algebra. We provide precise definitions of Mackey functors in Sections 2.2.2 and 3.2.2 and refer the interested reader to [We00] for an introductory guide to Mackey functors. Such definitions however fall outside of the scope of this chapter. We instead provide here a list of examples of Mackey functors to aid in building an intuition.

Example 1.1.2.

- The contravariant functor $B(-)$ sending every finite group to its Burnside ring (seen as a $\mathbb{Z}$-module) is the contravariant part of a Mackey functor.
- For every integer $n \geq 0$ and every commutative ring $\mathcal{R}$ the contravariant functor $H^n(-, \mathcal{R})$ sending every finite group to its n^{th} cohomology group with coefficients in $\mathcal{R}$ is the contravariant part of a Mackey functor.
- With notation as before the covariant functor $H_n(-, \mathcal{R})$ sending every finite group to its n^{th} homology group with coefficients in $\mathcal{R}$ is the covariant part of a Mackey functor.

We know from Example 1.1.1 that fusion systems can convey the p-local structure of a finite group while, from Example 1.1.2, we know that Mackey functors can be used in order to study finite groups. With this in mind it is natural to ponder if Mackey functors can be used in order to study the p-local structure of a finite group by relating them with fusion systems. This thesis is born from this idea and explores some of the differences between Mackey functors over groups and Mackey functors over fusion systems (see Chapter 2) and makes some progress towards solving an open problem concerning the latter (see Chapter 3).

1.2 The articles

The core of this thesis resides in the two articles that form Chapters 2 and 3. In this subsection we briefly describe their content, with special emphasis on the results they prove.

1.2.1 Overview of "*Green correspondence on centric Mackey functors over fusion systems*"

Let p be a prime, let G be a finite group of order divisible by p, let P be a p subgroup of G and let $\mathcal{R}$ be a p-local ring. In [Gr64, Theorem 2] Green proves that there exists a one to one correspondence (later named Green correspondence) between finitely generated indecomposable $\mathcal{R}G$-modules with vertex P and finitely generated indecomposable $\mathcal{R}N_G(P)$-modules with vertex P. Here $N_G(P)$ denotes the normalizer of P in G. In [Gr71] Green is able to translate this result to Green functors, in [Sa82] Sasaki proves that a similar result applies to Mackey functors over groups and in [AK94] Auslander and Kleiner prove what, to our best knowledge, is the most general version of the Green correspondence.

Let $\mathcal{F}$ be a fusion system. In Chapter 2 we define Mackey functors over $\mathcal{F}$ and $\mathcal{F}$-centric Mackey functors (also known as $\mathcal{F}^c$-restricted Mackey functors). These definitions are in fact equivalent to those given in [DP15]. We then explore the differences between Mackey functors over fusion systems and Mackey functors over groups by translating the methods used in [Sa82] to fusion systems in order to obtain the following.

Theorem (Green correspondence). *Let $\mathcal{R}$ be a complete local and p-local (see Definition 2.2.40) PID , let P be a fully $\mathcal{F}$-normalized object in $\mathcal{F}$, let $M \in$ Mack$_{\mathcal{R}}(\mathcal{F}^c)$ (see Definition 2.2.29) be indecomposable with vertex P (see Definition 2.3.7 and Notation 2.4.9) and let $N \in$ Mack$_{\mathcal{R}}^{\mathcal{F}^c}(N_{\mathcal{F}}(P))$ (see Example 2.2.8) be indecomposable with vertex P. There exist unique (up to isomorphism) decompositions of $M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}(P)}$ and $N\uparrow^{\mathcal{F}}_{N_{\mathcal{F}}(P)}$ (see Definition 2.2.28) into direct sums of indecomposable Mackey functors. Moreover, writing these decompositions as*

$$M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}(P)} := \bigoplus_{i=0}^{n} M_i, \qquad\qquad N\uparrow^{\mathcal{F}}_{N_{\mathcal{F}}(P)} := \bigoplus_{j=0}^{m} N_j,$$

there exist unique $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$ such that both M_i and N_j have vertex P. We call these summands the ***Green correspondents*** *of M and N and denote them as $M_{N_{\mathcal{F}}(P)}$ and $N^{\mathcal{F}}$ respectively. Every indecomposable summand of $M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}(P)}$ other than $M_{N_{\mathcal{F}}(P)}$ has vertex in $\mathcal{Y}$ (see Notation 2.4.9) while every indecomposable summand of $N\uparrow^{\mathcal{F}}_{N_{\mathcal{F}}(P)}$ other than $N^{\mathcal{F}}$ has vertex $\mathcal{F}$-isomorphic to an element in $\mathcal{X}$ (see Notation 2.4.9). Finally, using the above notation for Green correspondents, we have the isomorphisms $(M_{N_{\mathcal{F}}})^{\mathcal{F}} \cong M$ and $\left(N^{\mathcal{F}}\right)_{N_{\mathcal{F}}} \cong N$.*

Since we necessarily have $P \in \mathcal{F}^c$ (see Definition 2.2.11), it follows from [Br05, Section 4] that the fusion system $N_{\mathcal{F}}(P)$ is in fact realizable (i.e. there exists a finite group G

such that $N_S(P) \in \mathrm{Syl}_p(G)$ and $N_{\mathcal{F}}(P) = \mathcal{F}_{N_S(P)}(G)$). Therefore, given any centric indecomposable Mackey functor over $\mathcal{F}$, the above theorem provides us with a unique Mackey functor over a realizable fusion system which, moreover, characterizes it.

Since the induction ($\uparrow_{N_{\mathcal{F}}}^{\mathcal{F}}$) and restriction ($\downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$) functors form an adjoint pair (see Remark 2.2.35) the above theorem is in fact a particular case of [AK94, Theorem 1.10] although the methods used for proving it differ and we believe they could be of some use on their own. Due to the, previously explained, relative simplicity of the fusion system $N_{\mathcal{F}}(P)$ respect to the fusion system $\mathcal{F}$ we believe Theorem 2.4.27 to be of particular interest since it allows to decompose products in $\mathcal{O}(\mathcal{F}^c)_{\sqcup}$ (see Definition 2.2.12) in terms of products in $\mathcal{O}((N_{\mathcal{F}})^c)_{\sqcup}$.

1.2.2 Overview of "*Sharpness for the Benson-Solomon fusion systems*"

Due to the work of Broto, Levi and Oliver (see [BLO03]) and of Chermak (see [Ch13]) we know that for every fusion system $\mathcal{F}$ there exists a unique (up to $\mathrm{mod}\ p$ homology isomorphism) topological space $B\mathcal{F}$ that "acts like" the classifying space of $\mathcal{F}$. That is $B\mathcal{F}$ is a topological space satisfying

$$B\mathcal{F} \simeq \mathrm{hocolim}_{\mathcal{O}(\mathcal{F}^c)}(B(-)) \qquad \text{and} \qquad \lim_{\mathcal{O}(\mathcal{F}^c)}(H^n(-, \mathbb{F}_p)) \cong H^n(B\mathcal{F}, \mathbb{F}_p)$$

where $\mathcal{O}(\mathcal{F}^c)$ is as in Definition 3.2.12. These results mimic the homotopy equivalence and isomorphism (see [Dw98]) that, for any finite group G, are given by

$$BG \simeq \mathrm{hocolim}_{\mathcal{O}_p^c(G)}(B(-)) \quad \mathrm{mod}\ p \quad \text{and} \quad \lim_{\mathcal{O}_p^c(G)}(H^n(-, \mathbb{F}_p)) \cong H^n(G, \mathbb{F}_p).$$

Motivated by the above parallelism, the identities $\lim^n_{\mathcal{O}_p^c(G)}(H^n(-, \mathbb{F}_p)) = 0$ for every $n \geq 1$, similar vanishings of higher limits (see [JM92]) and the fact that the cohomology functor is the contravariant part of a Mackey functor (see Example 1.1.2), Díaz and Park formulate in [DP15] the following

Conjecture (sharpness for fusion system). *Let p be a prime, let S be a finite p-group, let $\mathcal{F}$ be a fusion system over S and let $M = (M_*, M^*)$ be a Mackey functor over $\mathcal{F}$*

on $\mathbb{F}_p$ (see Definition 2.2.26). Then $\lim^n_{\mathcal{O}(\mathcal{F}^c)}\left(M^* \downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}(\mathcal{F}^c)}\right) = 0$ for every $n \geq 1$.

This conjecture has seen a lot of recent activity (see [GL23; GM22; HLL23; Pr23; Ya22]) and, following this trend, we prove the following

Theorem. *Let $\mathcal{F}$ be a fusion system over S, let I be a finite set, let $\boldsymbol{F} = \{\mathcal{F}_i\}_{i\in I}$ be a collection of fusion subsystems of $\mathcal{F}$ and for each $i \in I$ let $S_i \leq S$ be the finite p-group such that $\mathcal{F}_i$ is a fusion system over S_i. If the following 4 conditions are satisfied then the sharpness conjecture is satisfied for $\mathcal{F}$:*

(1) $\theta_{\boldsymbol{F}}$ (see Definition 3.4.2) is an epimorphism.

(2) For every $i \in I$ all $\mathcal{F}_i$-centric-radical subgroups of S_i are $\mathcal{F}$-centric (see Definitions 3.2.11 (1) and (3)).

(3) For every with $i \in I$ the sharpness conjecture is satisfied for $\mathcal{F}_i$.

(4) The family $\boldsymbol{F}$ satisfies the lifting property (see Definition 3.4.3).

Despite the seemingly large number of conditions needed to apply the above we prove in Section 3.4 that Conditions (1)-(3) are satisfied in numerous situations.

By studying the case where $I = \{1, 2\}$ we prove that if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems over S and they generate $\mathcal{F}$ then Condition 4 of the above is satisfied (see Theorem 3.C). Using this fact we are able to complete the work started in [HLL23, Theorems 1.1 and 1.4] by proving the following

Theorem. *The sharpness conjecture is satisfied for all Benson-Solomon fusion systems (see [LO05, Definition 1.6]).*

Since the Benson-Solomon fusion systems are the only known family of exotic fusions systems over 2 groups (see [AKO11, Section III.7]) then the above result together with [DP15, Theorem B] prove that sharpness is satisfied for all known fusion systems over 2-groups.

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Chapter 2

Green correspondence on centric Mackey functors over fusion systems

Abstract

In this paper we give a definition of (centric) Mackey functor over a fusion system (Definitions 2.2.26 and 2.2.29) which generalizes the notion of Mackey functor over a group. In this context we prove that, given some conditions on a related ring, the centric Burnside ring over a fusion system (as defined in [DL09]) acts on any centric Mackey functor (Proposition 2.2.43). We also prove that the Green correspondence holds for centric Mackey functors over fusion systems (Theorem 2.4.38). As a means to prove this we introduce a notion of relative projectivity for centric Mackey functors over fusion systems (Definition 2.3.1) and provide a decomposition of a particular product in $\mathcal{O}\left(\mathcal{F}^c\right)_\sqcup$ (Definition 2.2.12) in terms of the product in $\mathcal{O}\left(\left(N_\mathcal{F}\right)^c\right)_\sqcup$ (Theorem 2.4.27).

2.1 Introduction

A Mackey functor is an algebraic structure possessing operations which behave like the induction, restriction and conjugation maps in group representation theory. The concept of Mackey functor has been generalized to algebraic structures other than groups (see for example [We93]). We are particularly interested on their generalization to fusion systems.

Fusion systems, as defined by Puig in [Pu06] (where he calls them Frobenius Categories), are categories intended to convey the p-local structure of a finite group G. Not all fusion systems can however be derived from finite groups. This gives them an interest of their own.

When generalizing to fusion systems, Mackey functors inevitably lose some properties. One of these is the existence of a Green correspondence.

The Green correspondence first appeared in [Gr64] under the following form.

Theorem. *([Gr64, Theorem 2]) Let p be a prime, let $\mathcal{R}$ be a complete local PID with residue field of characteristic p, let G be a finite group and let P be a p-subgroup of G. There exists a one to one correspondence between finitely generated indecomposable $\mathcal{R}G$-modules with vertex P and finitely generated $\mathcal{R}N_G(P)$-modules with vertex P.*

This result was later generalized in [Gr71] and [Sa82] to Green functors and Mackey functors over groups respectively.

In this paper we prove that a Green correspondence like result can be found for centric Mackey functors over a fusion system (see Definition 2.2.29 and Theorem 2.4.38) although the same methods fail to provide such result for Mackey functors over fusion systems in general. This result can be used in order to study centric Mackey functors over a fusion system $\mathcal{F}$ in terms of Mackey functors over fusion systems of the form $N_{\mathcal{F}}(P)$ (see Example 2.2.8) with $P \in \mathcal{F}^c$ (see Definition 2.2.11) and fully $\mathcal{F}$-normalized (see Definition 2.2.4). It is known (see [Br05, Section 4]) that fusion systems of this form derive from finite groups. This makes them easier to work with than other fusion systems and, therefore, motivates the interest in proving that the Green correspondence holds for centric Mackey functors over fusion systems.

The paper is organized as follows.

In Section 2.2 we briefly recall the definitions of (saturated) fusion system (Definitions 2.2.1 and 2.2.6), of (centric) Mackey functor over a fusion system (Definitions 2.2.26 and 2.2.29) and of centric Burnside ring of a fusion system (Definition 2.2.38). In this section we also recall some well known properties regarding these concepts and prove 3 further results. The first one (Proposition 2.2.33) describes a decomposition of certain induced Mackey functors (see Definition 2.2.28). The second result (Lemma 2.2.36) allows us to rewrite the composition of certain induction and restriction functors (see Definition 2.2.28). The third one (Proposition 2.2.43) describes, under certain conditions concerning a related ring, an action of the centric Burnside ring over a fusion system on any centric Mackey functor over that fusion system.

In Section 2.3 we introduce the concept of relative projectivity of a Mackey functor over a fusion system (Definition 2.3.1) and prove that Higman's criterion holds for Mackey functors over fusion systems (Theorem 2.3.17). To do this we define the trace and restriction maps (Definition 2.3.8) and list some of the properties they satisfy (Proposition 2.3.9). These properties are needed in Subsections 2.4.3 and 2.4.4.

We conclude with Section 2.4 where we prove our two main results (Theorems 2.4.27 and 2.4.38). In Subsection 2.4.1 we state and prove Proposition 2.4.7 which generalizes [Gr71, Proposition 4.34] (see Example 2.4.8) and is key too proving that the Green correspondence holds for centric Mackey functors over fusion systems. Subsections 2.4.2-2.4.5 are dedicated to developing the tools necessary to prove that Proposition 2.4.7 can be applied in the context of centric Mackey functors over fusion systems. More precisely, during these subsections, we study different compositions of the induction and restriction functors (see Definition 2.2.28) and of trace and restriction maps (see Definition 2.3.8) and prove Theorem 2.4.27 which allows us to write certain products in $\mathcal{O}\left(\mathcal{F}^c\right)_\sqcup$ (Definition 2.2.12) in terms of products in $\mathcal{O}\left(N_{\mathcal{F}}\left(P\right)\right)_\sqcup$ for some fully $\mathcal{F}$-normalized (see Definition 2.2.4), $\mathcal{F}$-centric (see Definition 2.2.11) group P. Finally, we conclude in Subsection 2.4.6 where we use the developed tools in order to apply Proposition 2.4.7 in the context of centric Mackey functors over fusion systems and deduce from it Theorem 2.4.38 which shows that the Green correspondence holds in the context of centric Mackey functors over fusion systems.

We conclude this introduction with a brief summary of some common notation that we use throughout the paper

Notation 2.1.1.

- Given a unital ring $\mathcal{R}$ we denote by $1_{\mathcal{R}}$ its multiplicative identity element.
- Given a group G we denote by 1_G the neutral element of G.
- Given groups G, H such that $H \leq G$ we denote by ι_H^G (or simply by ι if H and G are clear) the natural inclusion map from H to G.
- All modules over rings are understood to be left modules unless otherwise specified.
- Given rings $\mathcal{R}$ and $\mathcal{S}$ such that $\mathcal{R} \subseteq \mathcal{S}$ and modules M and N over $\mathcal{R}$ and $\mathcal{S}$ respectively we write $M \cong_{\mathcal{R}} N$ to denote that M and N are equivalent as $\mathcal{R}$-modules.
- Given finite groups H and K and an (H, K)-biset X (e.g. we can take $X := G$ for some group G satisfying $H, K \leq G$) we denote by $[H \backslash X / K]$ any choice of representatives of (H, K)-orbits of X.
- Let $\mathcal{D} \subseteq \mathcal{C}$ be categories, unless otherwise specified, we write $X \in \mathcal{C}$ to denote that X is an object of $\mathcal{C}$ and $X \in \mathcal{C} \backslash \mathcal{D}$ to denote that $X \in \mathcal{C}$ and $X \notin \mathcal{D}$.
- Given a fusion system $\mathcal{F}$, objects $A, B \in \mathcal{F}$ and a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(A, B)$, we denote by $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(A, B)$ the morphism in $\mathcal{O}(\mathcal{F})$ with representative φ (see Definition 2.2.10).
- Given a category $\mathcal{C}$, objects $X, Y, Z \in \mathcal{C}$ and a morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ we denote by φ_* and φ^* the following induced maps between hom sets

$$\varphi_* := \mathrm{Hom}_{\mathcal{C}}(Z, -)(\varphi) : \underset{\psi}{\mathrm{Hom}_{\mathcal{C}}(Z, X)} \underset{\mapsto}{\rightarrow} \underset{\varphi\psi}{\mathrm{Hom}_{\mathcal{C}}(Z, Y)},$$

$$\varphi^* := \mathrm{Hom}_{\mathcal{C}}(-, Z)(\varphi) : \underset{\theta}{\mathrm{Hom}_{\mathcal{C}}(Y, Z)} \underset{\mapsto}{\rightarrow} \underset{\theta\varphi}{\mathrm{Hom}_{\mathcal{C}}(X, Z)}.$$

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2.2 Background and first results

In this section we review the concepts of fusion systems, of Mackey functors over a fusion system and of centric Burnside ring of a fusion system. The main results shown in this section are:

- Proposition 2.2.18: which provides us with ways of rewriting the sets $[P \times Q]$ (see Definition 2.2.17). These are very useful for future calculations.
- Propositions 2.2.24 and 2.2.33 and Lemma 2.2.36: which translate [TW95, Propositions 3.2 and 5.3] to the context of Mackey functors over fusion systems thus providing us with some insight concerning the Mackey algebra (see Definition 2.2.20) and the induction and restriction functors (see Definition 2.2.28).
- Proposition 2.2.43: which translates [TW95, Proposition 9.2] to the context of centric Mackey functors over fusion systems by first describing an action of the centric Burnside ring of a fusion system (see Definition 2.2.38) on any centric Mackey functor over a fusion system and then rewriting it in terms of the morphisms θ^P and θ_P (see Definition 2.2.37).

The reader already familiar with fusion systems, Mackey functors over fusion systems and centric Burnside ring of a fusion system may safely skip this section keeping in mind the results mentioned above.

2.2.1 Fusion systems

What follows is a brief introduction to fusion systems which mostly aims to establish some notation. For a more thorough introduction please refer to [Li07]. In this subsection we also report the main results of [Pu06, Section 4] which, given a saturated fusion system $\mathcal{F}$, prove constructively the existence of products and pullbacks in the category $\mathcal{O}\left(\mathcal{F}^c\right)_\sqcup$ (see Definition 2.2.12 and Propositions 2.2.15 and 2.2.16). We conclude this subsection with Proposition 2.2.18 which allows us to write products in $\mathcal{O}\left(\mathcal{F}^c\right)_\sqcup$ in terms of other products in the same category.

Definition 2.2.1. Let p be a prime and let S be a finite p-group. A **fusion system over** S is a category $\mathcal{F}$ having as objects subgroups of S and satisfying the following properties for every $P, Q \leq S$:

(1) Every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is an injective group homomorphism and the composition of morphisms in $\mathcal{F}$ is the same as the composition of morphisms in the category of groups.

(2) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q)$. That is, every group homomorphism from P to Q that can be described as conjugation by an element of S followed by inclusion is a morphism in $\mathcal{F}$.

(3) For every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ let $\tilde{\varphi} : P \to \varphi(P)$ be the isomorphism obtained by looking at φ as an isomorphism onto its image. Both $\tilde{\varphi}$ and $\tilde{\varphi}^{-1}$ are isomorphisms in $\mathcal{F}$.

Example 2.2.2. The most common example of fusion system is obtained by taking a finite group G containing a p-group S and defining $\mathcal{F}_S(G)$ as the fusion system over S whose morphisms are given by conjugation with elements of G followed by inclusion. When $S = G$ we often write $\mathcal{F}_S$ instead of $\mathcal{F}_S(S)$ although the latter is the more common notation in the literature.

Definition 2.2.1 and Example 2.2.2 motivate the introduction of the following notation.

Notation 2.2.3. From now on, unless otherwise specified, all introduced groups are understood to be finite, p **denotes a prime integer**, S **denotes a finite** p**-group** and $\mathcal{F}$ **denotes a fusion system over** S. Moreover, given subgroups $P, Q \leq S$ we write $P =_{\mathcal{F}} Q$ if P and Q are isomorphic in $\mathcal{F}$, $P \leq_{\mathcal{F}} Q$ if there exists $J \leq Q$ such that $P =_{\mathcal{F}} J$ and either $P \lneq_{\mathcal{F}} Q$ or $P <_{\mathcal{F}} Q$ if $P \leq_{\mathcal{F}} Q$ but $P \neq_{\mathcal{F}} Q$.

When the term fusion system appears in the literature it is usually in reference to a particular type of fusion system called saturated fusion system. These are fusion systems that are built to generalize Example 2.2.2 in the case where S is a Sylow p-subgroup of G.

Definition 2.2.4. Let $P \leq S$. We say that P is **fully** $\mathcal{F}$**-normalized** if for every $Q \leq S$ such that $Q =_{\mathcal{F}} P$ we have that $|N_S(Q)| \leq |N_S(P)|$.

Definition 2.2.5. Let $P, Q \leq S$ and let $\varphi \colon P \to Q$ be a morphism in $\mathcal{F}$. We define the φ-**normalizer** as the following subgroup of $N_S(P)$

$$N_{\varphi} := \{x \in N_S(P) : \exists z \in N_S(\varphi(P)) \text{ such that } \varphi({}^x y) = {}^z\varphi(y) \; \forall y \in P\}.$$

Definition 2.2.6. A fusion system $\mathcal{F}$ is said to be **saturated** if the following 2 conditions are satisfied:

(1) $\mathrm{Aut}_S(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$.

(2) For every $P \leq S$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ such that $\varphi(P)$ is fully $\mathcal{F}$-normalized there exists $\hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_{\varphi}, S)$ such that $\hat{\varphi}\iota_P^{N_{\varphi}} = \varphi$.

Example 2.2.7. The fusion system $\mathcal{F}_S(G)$ of Example 2.2.2 is saturated if S is a Sylow p subgroup of G.

Example 2.2.8. Given a saturated fusion system $\mathcal{F}$ and a fully $\mathcal{F}$-normalized subgroup $P \leq S$, we define the saturated fusion system $N_{\mathcal{F}}(P)$ over $N_S(P)$ by setting for every $A, B \leq N_S(P)$

$$\mathrm{Hom}_{N_{\mathcal{F}}(P)}(A, B) := \left\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(A, B) \mid \exists \hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(AP, BP) \text{ s.t. } \iota_B^{BP}\varphi = \hat{\varphi}\iota_A^{AP}\right\}.$$

Definition 2.2.6 motivates the introduction of the following notation.

Notation 2.2.9. From now on, unless otherwise specified, **all introduced fusion systems are understood to be saturated**. In particular $\mathcal{F}$ denotes a saturated fusion system over a finite p-group S.

When dealing with Mackey functors over fusion systems (as we do throughout this paper) it is convenient not to work with the fusion system directly but rather with its orbit category.

Definition 2.2.10. We define the **orbit category of a fusion system** $\mathcal{F}$ as the category $\mathcal{O}(\mathcal{F})$ having as objects the same objects as $\mathcal{F}$ and as morphisms

$$\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P, Q) := \mathrm{Aut}_Q(Q) \backslash \mathrm{Hom}_{\mathcal{F}}(P, Q),$$

for every $P, Q \leq S$. Here $\mathrm{Aut}_Q(Q)$ is acting on $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ by composing on the left.

An important subcategory of $\mathcal{O}(\mathcal{F})$ which we often work with is the centric subcategory.

Definition 2.2.11. Let $P \leq S$. We say that P is $\mathcal{F}$-**centric** if $C_S(Q) \leq Q$ for every $Q \leq S$ such that $Q =_{\mathcal{F}} P$. The **centric subcategory of** $\mathcal{F}$ (denoted by $\mathcal{F}^c$) is defined as the full subcategory of $\mathcal{F}$ having as objects $\mathcal{F}$-centric subgroups of S. Likewise, the **centric subcategory of** $\mathcal{O}(\mathcal{F}^c)$ (denoted by $\mathcal{O}(\mathcal{F}^c)$) is the full subcategory of $\mathcal{O}(\mathcal{F})$ having as objects the $\mathcal{F}$-centric subgroups of S.

We are in fact particularly interested in the additive extension of $\mathcal{O}(\mathcal{F}^c)$.

Definition 2.2.12. (see [JM92, Section 4]) Let $\mathcal{F}$ be a fusion system. We denote by $\mathcal{O}(\mathcal{F}^c)_{\sqcup}$ the **additive extension of** $\mathcal{O}(\mathcal{F}^c)$. That is $\mathcal{O}(\mathcal{F}^c)_{\sqcup}$ is the category having as objects formal finite (possibly empty) coproducts of the form $\bigsqcup_{i=1}^n P_i$, where each P_i is an object in $\mathcal{O}(\mathcal{F}^c)$, and as morphisms $\boldsymbol{f} : \bigsqcup_{i=1}^n P_i \to \bigsqcup_{j=1}^m Q_j$ tuples of the form $\boldsymbol{f} := \left(\sigma, \{f_i\}_{i=1,\ldots,n}\right)$ where $\sigma\colon \{1,\ldots,n\} \to \{1,\ldots,m\}$ is any map and $f_i \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F}^c)}\left(P_i, Q_{\sigma(i)}\right)$ for every $i = 1,\ldots,n$. Composition is given by

$$\left(\tau, \{g_j\}_{i=j,\ldots,m}\right)\left(\sigma, \{f_i\}_{i=1,\ldots,n}\right) = \left(\tau\sigma, \left\{g_{\sigma(i)} f_i\right\}_{i=1,\ldots,n}\right).$$

Whenever σ is clear (for example when $m = 1$), we simply write

$$\bigsqcup_{i=1}^n f_i := \left(\sigma, \{f_i\}_{i=1,\ldots,n}\right).$$

We often abuse notation and consider objects in $\mathcal{O}(\mathcal{F}^c)$ as objects in $\mathcal{O}(\mathcal{F}^c)_{\sqcup}$ via the natural inclusion of categories $\mathcal{O}(\mathcal{F}^c) \hookrightarrow \mathcal{O}(\mathcal{F}^c)_{\sqcup}$.

Remark 2.2.13. The additive extension can in fact be defined for any small category $\mathcal{C}$. In this situation we can define the additive extension $\mathcal{C}_{\sqcup}$of $\mathcal{C}$ can be defined as the full subcategory of the category of diagrams $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ having as objects the constant functor to the empty set and finite coproducts of contravariant functors of the form $\mathrm{Hom}_{\mathcal{C}}(-, X)$ with X an object in $\mathcal{C}$. Yoneda's Lemma assures us that the category obtained in this way is equivalent to the one described in Definition 2.2.12 in the case $\mathcal{C} = \mathcal{O}(\mathcal{F}^c)$.

Remark 2.2.14. Despite what the name might suggest the additive extension of a category is not necessarily an additive category but rather a category in which all finite coproducts exist.

In [Pu06] Puig proves constructively that the category $\mathcal{O}(\mathcal{F}^c)_\sqcup$ admits both products and pullbacks. We report these results below.

Proposition 2.2.15 ([Pu06, Proposition 4.8]). *The category $\mathcal{O}(\mathcal{F}^c)_\sqcup$ admits pullbacks which are distributive with respect to its coproducts. Moreover, given $P, Q, J \in \mathcal{F}^c$ such that $P, Q \leq J$ the pullback of the diagram $P \overset{\overline{\iota_P^J}}{\to} J \overset{\overline{\iota_Q^J}}{\leftarrow} Q$ is given by*

$$P \times_J Q := \bigsqcup_{\substack{x\in[P\backslash J/Q] \\ P^x\cap Q\in\mathcal{F}^c}} P^x \cap Q, \qquad \pi_P^{P\times_J Q} := \bigsqcup_{\substack{x\in[P\backslash J/Q] \\ P^x\cap Q\in\mathcal{F}^c}} \overline{\iota c_x}, \qquad \pi_Q^{P\times_J Q} := \bigsqcup_{\substack{x\in[P\backslash J/Q] \\ P^x\cap Q\in\mathcal{F}^c}} \overline{\iota}.$$

Proposition 2.2.16 ([Pu06, Proposition 4.7]). *The category $\mathcal{O}(\mathcal{F}^c)_\sqcup$ admits products which are distributive with respect to its coproducts.*

In [Pu06], Puig explicitly describes the product of Proposition 2.2.16. Since products are distributive with respect to coproducts, in order to define the products in $\mathcal{O}(\mathcal{F}^c)_\sqcup$, it suffices to describe the product of any two objects $P, Q \in \mathcal{O}(\mathcal{F}^c)$.

This product, denoted by $P \times_\mathcal{F} Q$, can be built as follows:

First take all pairs $(A, \overline{\varphi})$ with $A \in \mathcal{O}(\mathcal{F}^c)$ satisfying $A \leq P$ and $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F}^c)}(A, Q)$.

Then define the preorder $\precsim_P$ on the set of all such pairs by setting $(A, \overline{\varphi}) \precsim_P \left(B, \overline{\psi}\right)$ if and only if there exists $x \in P$ such that $A^x \leq B$ and $\overline{\varphi c_x} = \overline{\psi \iota_{A^x}^B}$.

Then take all pairs that are maximal under such preorder and define among them the equivalence relation

$$(A, \overline{\varphi}) \sim \left(B, \overline{\psi}\right) \overset{\text{def}}{\iff} (A, \overline{\varphi}) \precsim_P \left(B, \overline{\psi}\right) \text{ and } \left(B, \overline{\psi}\right) \precsim_P (A, \overline{\varphi}) \tag{2.1}$$

Finally fix any set $[P \times_\mathcal{F} Q]$ containing exactly one representative for each equivalence class of maximal elements under this relation and define

$$P \times_\mathcal{F} Q := \bigsqcup_{(A,\overline{\varphi})} A, \qquad \pi_P^{P\times_\mathcal{F} Q} := \bigsqcup_{(A,\overline{\varphi})} \overline{\iota_A^P}, \qquad \pi_Q^{P\times_\mathcal{F} Q} := \bigsqcup_{(A,\overline{\varphi})} \overline{\varphi}, \tag{2.2}$$

where the tuples $(A, \overline{\varphi})$ run over the set $[P \times_\mathcal{F} Q]$ and $\pi_P^{P\times_\mathcal{F} Q} : P \times_\mathcal{F} Q \to P$ and $\pi_Q^{P\times_\mathcal{F} Q} : P \times_\mathcal{F} Q \to Q$ denote the natural projections associated to the product. The definition of the equivalence $\sim$ ensures us that any choice of the set $[P \times_\mathcal{F} Q]$ leads to isomorphic constructions of $P \times_{\mathcal{F}} Q$. Whenever the fusion system $\mathcal{F}$ is clear we simply write $P \times Q$ and $[P \times Q]$.

In order to reference the previous construction, it is worth introducing the following.

Definition 2.2.17. For every $P, Q \in \mathcal{F}^c$ we denote by $[P \times_{\mathcal{F}} Q]$ (or simply $[P \times Q]$ if $\mathcal{F}$ is clear) any choice of the set of representatives built as above. In other words $[P \times_{\mathcal{F}} Q]$ is any set of tuples $(A, \overline{\varphi})$ such that $A \in \mathcal{F}^c$, $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F}^c)}(A, Q)$ and Equation (2.2) is satisfied.

We conclude this subsection with a series of identities that allow us to write $P \times Q$ in terms of other products in $\mathcal{O}(\mathcal{F}^c)_{\sqcup}$.

Proposition 2.2.18. *For every $P, Q \in \mathcal{F}^c$*

(1) We can take

$$[Q \times P] = \left\{ \left(\varphi(A), \overline{\iota\varphi^{-1}} \right) \,:\, (A, \overline{\varphi}) \in [P \times Q] \right\}.$$

Where we are viewing the representative φ of $\overline{\varphi}$ as an isomorphism onto its image.

(2) If $\mathcal{F} = \mathcal{F}_S$ we can take

$$[P \times_{\mathcal{F}_S} Q] = \bigsqcup_{\substack{x \in [Q \backslash S / P] \\ Q^x \cap P \in \mathcal{F}_S^c}} \{ (Q^x \cap P, \overline{\iota c_x}) \}.$$

(3) For every isomorphism $\psi \colon Q \to \psi(Q)$ we can take

$$[P \times \psi(Q)] = \left\{ \left(A, \overline{\psi\varphi} \right) \,:\, (A, \overline{\varphi}) \in [P \times Q] \right\}.$$

(4) For every isomorphism $\psi \colon P \to \psi(P)$ we can take

$$[\psi(P) \times Q] = \left\{ \left(\psi(A), \overline{\varphi\psi^{-1}} \right) \,:\, (A, \overline{\varphi}) \in [P \times Q] \right\}.$$

Where we are viewing ψ as an isomorphism between the appropriate restrictions.

(5) For every $J \in \mathcal{F}^c$ such that $J \leq Q$ we can take

$$[P \times J] = \bigsqcup_{(A, \overline{\varphi}) \in [P \times Q]} \bigsqcup_{\substack{x \in [J \backslash Q / \varphi(A)] \\ J^x \cap \varphi(A) \in \mathcal{F}^c}} \left\{ \left(\varphi^{-1}(J^x \cap \varphi(A)), \overline{\iota c_x \varphi} \right) \right\}.$$

Where we are fixing a representative φ of $\overline{\varphi}$ and viewing it as an isomorphism between the appropriate restrictions.

(6) *For every $J \in \mathcal{F}^c$ such that $J \leq P$ we can take*

$$[J \times Q] = \bigsqcup_{(A,\overline{\varphi})\in[P\times Q]} \bigsqcup_{\substack{x\in[A\backslash P/J] \\ A^x\cap J\in\mathcal{F}^c}} \{(A^x \cap J, \overline{\varphi\iota c_x})\}.$$

(7) *For every $J \in \mathcal{F}^c$ we can take*

$$\bigsqcup_{(A,\overline{\varphi})\in[P\times Q]} \bigsqcup_{(B,\overline{\psi})\in[J\times A]} \{(B, \overline{\iota\psi})\} = \bigsqcup_{\substack{(C,\overline{\theta})\in[J\times P] \\ (D,\overline{\gamma})\in[J\times Q]}} \bigsqcup_{\substack{x\in[D\backslash J/C] \\ D^x\cap C\in\mathcal{F}^c}} \{(D^x \cap C, \overline{\theta\iota})\}.$$

Proof.

(1) With the notation of Item (1) we have that $\varphi(A) \leq Q$ and, since $A \leq P$, we can conclude that $\overline{\iota\varphi^{-1}} \in \operatorname{Hom}_{\mathcal{O}(\mathcal{F})}(\varphi(A), P)$. We can now define

$$f := \bigsqcup_{(A,\overline{\varphi})\in[P\times Q]} \overline{\varphi^{-1}} : \bigsqcup_{(A,\overline{\varphi})\in[P\times Q]} \varphi(A) \to P \times Q := \bigsqcup_{(A,\overline{\varphi})\in[P\times Q]} A.$$

With this setup we have that

$$\pi_P^{P\times Q} f = \bigsqcup_{(A,\overline{\varphi})\in[P\times Q]} \overline{\iota_A^P} f = \bigsqcup_{(A,\overline{\varphi})\in[P\times Q]} \overline{\iota\varphi^{-1}},$$

and

$$\pi_Q^{P\times Q} f = \bigsqcup_{(A,\overline{\varphi})\in[P\times Q]} \overline{\varphi} f = \bigsqcup_{(A,\overline{\varphi})\in[P\times Q]} \overline{\iota_{\varphi(A)}^Q}.$$

Since $Q \times P \cong P \times Q$ and f is an isomorphism then the above identities prove Item (1).

(2) All morphisms in $\mathcal{F}_S$ are, by definition, of the form c_x for some $x \in S$. Thus, for any element $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}_S} Q]$, there exists $x \in S$ such that ${}^xA \leq Q$ and $\overline{\varphi} = \overline{\iota c_x}$. In particular we have that $A \leq Q^x$. Since, by construction, $A \leq P$ we can conclude that $A \leq Q^x \cap P$. Therefore we can take $\overline{\iota_A^{Q^x\cap P}} \in \operatorname{Hom}_{\mathcal{O}(\mathcal{F}_S^c)}(A, Q^x \cap P)$ and, viewing c_x as an isomorphism from $Q^x \cap P$ to $Q \cap {}^xP$,

we have that $\overline{\varphi} = \overline{\iota_{Q\cap^x P}^{Q} c_x}\, \overline{\iota_A^{Q^x\cap P}}$. From maximality of the pair $(A, \overline{\varphi})$ we can conclude that $A = Q^x \cap P$. In other words, all elements in $[P \times_{\mathcal{F}_S} Q]$ are of the form $(Q^x \cap P, \overline{\iota c_x})$ for some $x \in S$. Notice now that, for every $y \in Q$, we have $Q^{yx} \cap P = Q^x \cap P$ and $\overline{\iota c_{yx}} = \overline{\iota c_x}$. Moreover, we know that $[P \times_{\mathcal{F}_S} Q]$ contains exactly one representative for each of the equivalence classes given by the relation of Equation (2.1). It is therefore possible to choose $[P \times_{\mathcal{F}_S} Q]$ and $[Q\backslash S/P]$ so that

$$[P \times_{\mathcal{F}_S} Q] = \bigcup_{\substack{x\in[Q\backslash S/P]\\ Q^x\cap P\in\mathcal{F}_S^c}} \{(Q^{xz_x} \cap P, \overline{\iota c_{xz_x}})\}. \tag{2.3}$$

For some appropriate $z_x \in P$. Assume now that there exist $x, y \in S$ and $z \in P$ such that $Q^x \cap P = Q^{yz} \cap P \in \mathcal{F}_S^c$ and that $\overline{\iota c_x} = \overline{\iota c_{yz}}$. From this last identity we can deduce that there exist $u \in Q$ and $v \in C_S(Q^x \cap P)$ such that $x = uyzv$. Since $Q^x \cap P \in \mathcal{F}_S^c$ then we have that $C_S(Q^x \cap P) \leq Q^x \cap P$ and, in particular, $v \in P$. We can therefore conclude that $y \in QxP$ and, therefore, that the union in Equation (2.3) is disjoint. Item (2) then follows by taking an appropriate choice of the representatives $[Q\backslash S/P]$ (i.e. taking xz_x instead of x).

(3) Let $\mathcal{C}$ be a category, let $X, Y, Z \in \mathcal{C}$ be objects and let $\alpha\colon Y \to Z$ be an isomorphism in $\mathcal{C}$. We know from category theory that, if the product $X \times Y$ exists in $\mathcal{C}$, then the product $X \times Z$ also exists in $\mathcal{C}$ and satisfies

$$X \times Z = X \times Y, \qquad \pi_Z^{X\times Z} = \alpha\pi_Y^{X\times Y}, \qquad \pi_X^{X\times Z} = \pi_X^{X\times Y}.$$

where $\pi_A^{A\times B}$ denote the natural projections. With the notation of Item (3) we have that $\overline{\psi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F}^c)}(K, \psi(K))$ is an isomorphism in $\mathcal{O}(\mathcal{F}^c)_{\sqcup}$ and for every $(A, \overline{\varphi}) \in [P \times Q]$ we have that $A \leq P$ and that $\overline{\psi\varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(A, \psi(Q))$. We can therefore apply the previous result taking $\mathcal{C} := \mathcal{O}(\mathcal{F}^c)_{\sqcup}$, $X := P$, $Y := Q$, $Z := \psi(Q)$ and $\alpha = \overline{\psi}$ thus proving Item (3).

(4) The same arguments used to prove Item (3) can be used to prove Item (4).

(5) Let $\mathcal{C}$ be a category admitting products and pullbacks, let $X, Y, Z \in \mathcal{C}$ be objects, let $\alpha\colon Z \to Y$ be a morphism in $\mathcal{C}$ and let $(X \times Y) \times_Y Z$ be the pullback of the

diagram $X \times Y \overset{\pi_Y^{X\times Y}}{\longrightarrow} Y \overset{\alpha}{\longleftarrow} Z$. We know from category theory that

$$(X \times Y) \times_Y Z = X \times Z, \ \ \pi_Z^{X\times Z} = \pi_Z^{(X\times Y)\times_Y Z}, \ \ \pi_X^{X\times Z} = \pi_X^{X\times Y}\pi_{X\times Y}^{(X\times Y)\times_Y Z}.$$

Where $\pi_A^{A\times B}$ and $\pi_A^{A\times_C B}$ denote the natural projections. With the notation of Item (5) we have that for every $(A, \overline{\varphi}) \in [P \times Q]$ and every $x \in [J\backslash Q/\varphi(A)]$ then $\varphi^{-1}(J^x \cap \varphi(A)) \leq A \leq P$ and $\overline{\iota c_x \varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(\varphi^{-1}(J^x \cap \varphi(A)), J)$. Using Proposition 2.2.15 we can now apply the previous result taking $Y := Q$, $X := P$, $Z := J$ and $\alpha := \overline{\iota_J^Q}$ thus proving Item (5).

(6) The same arguments used to prove Item (5) can be used to prove Item (6).

(7) Let $\mathcal{C}$ be a category admitting products and pullbacks, let $X, Y, Z \in \mathcal{C}$ be objects and let $(X \times Y) \times_Y (Y \times Z)$ be the pullback of the diagram $Y \times X \overset{\pi_Y^{Y\times X}}{\longrightarrow} Y \overset{\pi_Y^{Y\times Z}}{\longleftarrow} Y \times Z$. We know from category theory that

$$(Y \times X) \times_Y (Y \times Z) = Y \times (X \times Z), \quad \pi_Y^{Y\times X}\pi_{Y\times X}^{(Y\times X)\times_Y(Y\times Z)} = \pi_Y^{Y\times(X\times Z)},$$
$$\pi_X^{Y\times X}\pi_{Y\times X}^{(Y\times X)\times_Y(Y\times Z)} = \pi_X^{X\times Z}\pi_{X\times Z}^{Y\times(X\times Z)}.$$

Where $\pi_A^{A\times B}$ and $\pi_A^{A\times_C B}$ denote the natural projections. For every $(A, \overline{\varphi}) \in [P \times Q]$, every $(B, \overline{\psi}) \in [J \times A]$, every $(C, \overline{\theta}) \in [J \times P]$, every $(D, \overline{\gamma}) \in [J \times Q]$ and every $x \in [D\backslash J/C]$ we have that $B, D^x \cap C \leq J$, that $\overline{\iota\psi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F}^c)}(B, P)$ and that $\overline{\theta\iota} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F}^c)}(D^x \cap C, P)$. Using Propositions 2.2.15 and 2.2.16 we can now apply the previous result taking $\mathcal{C} := \mathcal{O}(\mathcal{F}^c)_\sqcup$, $X := P$, $Z := Q$ and $Y := J$ thus proving Item (7).

□

2.2.2 Mackey functors over fusion systems

In this subsection we define (centric) Mackey functor over a fusion system (Definitions 2.2.26 and 2.2.29) and the (centric) induction, restriction and conjugation functors (Definition 2.2.28 and Proposition 2.2.30). Moreover we provide some tools for studying certain induced Mackey functors (Proposition 2.2.33) and certain compositions of the induction and restriction functors (Lemma 2.2.36).

Through this subsection we will be using Notations 2.1.1, 2.2.3 and 2.2.9.

Let us start by defining the Mackey algebra of a fusion system. In order to do that we use methods similar to those used in [Bo15; HTW10].

Definition 2.2.19. Throughout this definition we denote with an overline $\overline{X}$ the isomorphism class of a biset X. Let P, Q be subgroups of S such that $P \leq Q$ and denote by ${}_PQ_Q$ and ${}_QQ_P$ the group Q seen as a (P, Q)-biset and a (Q, P)-biset respectively. We define the **restriction from Q to P** and the **induction from P to Q** as the following isomorphism class of (P, Q)-bisets and (Q, P)-bisets respectively

$$R_P^Q := \overline{{}_PQ_Q}, \qquad I_P^Q := \overline{{}_QQ_P}.$$

On the other hand, given an isomorphism $\varphi\colon P \to \varphi(P)$, we can view P as a $(\varphi(P), P)$-biset by setting for every $x, y \in P$ and every $z \in \varphi(P)$ the actions $z \cdot x = \varphi^{-1}(z)\, x$ and $x \cdot y = xy$. Let ${}_{\varphi(P)}P_P$ be the group P viewed as a $(\varphi(P), P)$-biset biset in this manner. We define the **conjugation by φ** as the isomorphism class of $(\varphi(P), P)$-bisets given by

$$c_{\varphi,P} := \overline{{}_{\varphi(P)}P_P}.$$

If the group P is clear we simply write c_φ instead of $c_{\varphi,P}$.

We want the Mackey algebra to be an algebra over a commutative ring $\mathcal{R}$ that is generated by elements of the form R_P^Q, I_P^Q and c_φ with $P \leq Q \leq S$ and φ an isomorphism in $\mathcal{F}$. To do this we will follow ideas similar to those used by Bouc in [Bo10; Bo15; HTW10].

Define $\boldsymbol{C}$ to be the category whose objects are subgroups of S and whose morphism sets are defined inductively via the following rules:

- For every $P \leq Q \leq S$ and every isomorphism $\varphi : P \to \varphi(P)$ in $\mathcal{F}$ then

$$R_P^Q \in \mathrm{Hom}_{\boldsymbol{C}}(Q, P), \quad I_P^Q \in \mathrm{Hom}_{\boldsymbol{C}}(P, Q), \quad c_{\varphi,P} \in \mathrm{Hom}_{\boldsymbol{C}}(P, \varphi(P)).$$

- With notation as in Definition 2.2.19, for every $P, Q \leq S$ and every (Q, P)-bisets X and X' if $\overline{X}, \overline{X'} \in \mathrm{Hom}_{\boldsymbol{C}}(P, Q)$ then $\overline{X \sqcup X'} \in \mathrm{Hom}_{\boldsymbol{C}}(P, Q)$. That is the isomorphism class of the disjoint union of X and X' is also a morphism from P to Q in $\boldsymbol{C}$.

- With notation as in Definition 2.2.19, for every $P, J, Q \leq S$, every (J,P)-biset X such that $\overline{X} \in \mathrm{Hom}_{\boldsymbol{C}}(P,J)$ and every (Q,J)-biset Y such that $\overline{Y} \in \mathrm{Hom}(J,Q)$ then $\overline{Y \times_J X} \in \mathrm{Hom}_{\boldsymbol{C}}(P,Q)$. Here $Y \times_J X$ is the (Q,P)-biset obtained as a quotient of the (Q,P)-biset $Y \times X$ modulo the equivalence relation

$$(y \cdot j) \times x \sim y \times (j \cdot x),$$

where $x \in X$, $y \in Y$ and $j \in J$.

We can now use the category $\boldsymbol{C}$ in order to define the Mackey algebra.

Definition 2.2.20. Let $\mathcal{R}$ be a commutative ring with unit, let $\boldsymbol{C}$ be the category defined above and, using the notation of Definition 2.2.19 define I as the two sided ideal of the category algebra $\mathcal{R}\boldsymbol{C}$ generated by elements of the form $\overline{X} + \overline{X'} - \overline{X \sqcup X'}$ where X and X' are (Q,P)-bisets (for some $P, Q \leq S$) such that $\overline{X}, \overline{X'} \in \mathrm{Hom}_{\boldsymbol{C}}(P,Q)$. The **Mackey algebra of $\mathcal{F}$ on $\mathcal{R}$** is the quotient algebra $\mu_{\mathcal{R}}(\mathcal{F}) := \mathcal{R}\boldsymbol{C}/I$.

The previous definitions motivate the introduction of the following notation.

Notation 2.2.21. From now, unless otherwise specified, $\mathcal{R}$ **denotes a commutative ring with unit**.

The following relations on the elements of the Mackey algebra are useful in what follows.

Lemma 2.2.22. *The elements I_P^Q, R_P^Q and c_φ of the Mackey algebra $\mu_{\mathcal{R}}(\mathcal{F})$ satisfy relations analogous to the similarly denoted elements in the Mackey algebra of a group (see [TW95, Section 3]). More precisely, the following are satisfied:*

(1) Let P be a subgroup of S, and let $x \in P$. We have that $I_P^P = R_P^P = c_{c_x,P}$. Moreover I_P^P is an idempotent in $\mu_{\mathcal{R}}(\mathcal{F})$.

(2) Let P, Q and J be subgroups of S such that $P \leq Q \leq J$ and let $\varphi\colon P \to \varphi(P)$ and $\psi\colon \varphi(P) \to \psi(\varphi(P))$ be isomorphisms in $\mathcal{F}$. We have that

$$R_P^Q R_Q^J = R_P^J, \qquad I_Q^J I_P^Q = I_P^J, \qquad c_{\psi,\varphi(P)} c_{\varphi,P} = c_{\psi\varphi,P}.$$

(3) Let P and Q be subgroups of S such that $P \leq Q$ and let $\theta\colon Q \to \theta(Q)$ be an isomorphism in $\mathcal{F}$. We have that

$$c_{\theta,Q} I_P^Q = I_{\theta(P)}^{\theta(Q)} c_{\theta_{|P},P}, \qquad\qquad c_{\theta_{|P},P} R_P^Q = R_{\theta(P)}^{\theta(Q)} c_{\theta,Q}.$$

Where $\theta_{|P} : P \to \theta(P)$ is the restriction of θ to P.

(4) Let P, Q and J be subgroups of S such that $P, Q \leq J$. We have that

$$R_Q^J I_P^J = \sum_{x \in [Q \backslash J / P]} I_{(Q \cap {}^x P)}^Q c_{c_x,(Q^x \cap P)} R_{(Q^x \cap P)}^P.$$

(5) All other combinations of induction restriction and conjugation are 0.

Proof. See [Bo10, Section 2.3]. Alternatively notice that the elements I_P^Q, R_P^Q and c_φ of the Mackey algebra are, by definition, isomorphism classes of the bisets Ind_x, Res_x and $\mathfrak{L}_x$ of [HTW10, Definition 6.8]. With this in mind the above relations follow from [HTW10, Proposition 6.9 and Theorem 5.3]. □

As an immediate consequence of Lemma 2.2.22 we have the following.

Corollary 2.2.23. *Let P and Q be subgroups of S, let $x \in Q$ and let $\varphi\colon P \to \varphi(P)$ be an isomorphism in $\mathcal{F}$ such that $\varphi(P) \leq Q$. Then $I_{{}^x\varphi(P)}^Q c_{c_x\varphi} = I_{\varphi(P)}^Q c_\varphi$ and $c_{\varphi^{-1} c_{x^{-1}}} R_{{}^x\varphi(P)}^Q = c_{\varphi^{-1}} R_{\varphi(P)}^Q$. In particular, given $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P, Q)$ and representatives $\varphi_1, \varphi_2 \in \overline{\varphi}$ then, seeing φ_1 and φ_2 as isomorphisms onto their images, we can define*

$$I_{\overline{\varphi}(P)}^Q c_{\overline{\varphi}} := I_{\varphi_1(P)}^Q c_{\varphi_1} = I_{\varphi_2(H)}^Q c_{\varphi_2}, \qquad c_{\overline{\varphi^{-1}}} R_{\overline{\varphi}(P)}^Q := c_{\varphi_1^{-1}} R_{\varphi_1(P)}^Q = c_{\varphi_2^{-1}} R_{\varphi_2(P)}^Q.$$

Moreover, given $J \leq S$ and $\overline{\psi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(Q, J)$, we have that

$$I_{\overline{\psi\varphi}(P)}^J c_{\overline{\psi\varphi}} = I_{\overline{\psi}(Q)}^J c_{\overline{\psi}} I_{\overline{\varphi}(P)}^Q c_{\overline{\varphi}}, \qquad c_{\overline{(\psi\varphi)^{-1}}} R_{\overline{\psi\varphi}(P)}^J = c_{\overline{\varphi^{-1}}} I_{\overline{\varphi}(P)}^Q c_{\overline{\psi^{-1}}} R_{\overline{\psi}(Q)}^J.$$

Proof. We only prove the statement for the case involving induction. The proof for the case involving restriction is analogous. The first part of the statement follows from Lemma 2.2.22 (1)-(3) via the identities below

$$I_{{}^x\varphi(P)}^Q c_{c_x\varphi} = I_{{}^x\varphi(P)}^Q c_{c_x} c_\varphi = c_{c_k} I_{\varphi(P)}^Q c_\varphi = I_Q^Q I_{\varphi(P)}^Q c_\varphi = I_{\varphi(P)}^Q c_\varphi.$$

The second part of the statement follows from Items (2) and (3) of Lemma 2.2.22 via the identities below

$$I_{\overline{\psi\varphi}(P)}^J c_{\overline{\psi\varphi}} = I_{\psi(Q)}^J I_{\psi(\varphi(P))}^{\psi(Q)} c_{\psi_{|\varphi(P)}} c_\varphi = I_{\psi(Q)}^J c_\psi I_{\varphi(P)}^Q c_\varphi = I_{\overline{\psi}(Q)}^J c_{\overline{\psi}} I_{\overline{\varphi}(P)}^Q c_{\overline{\varphi}}.$$

□

Another important consequence of Lemma 2.2.22 is the following result which translates [TW95, Proposition 3.2] to the context of Mackey functors over fusion systems.

Proposition 2.2.24. *The Mackey algebra $\mu_{\mathcal{R}}(\mathcal{F})$ admits an $\mathcal{R}$-basis of the form $\mathcal{B} := \bigsqcup_{A,B\leq S}\mathcal{B}_{(A,B)}$, where*

$$\mathcal{B}_{(A,B)} := \bigsqcup_{\substack{C\leq A \\ \text{up to } A\text{-conj}}} \bigsqcup_{\varphi\in[\mathrm{Aut}_B(B)\backslash \mathrm{Hom}_{\mathcal{F}}(C,B)/\mathrm{Aut}_A(C)]} \left\{I^B_{\overline{\varphi}(C)}c_{\overline{\varphi}}R^A_C\right\}.$$

In particular, $\mu_{\mathcal{R}}(\mathcal{F})$ is finitely generated as an $\mathcal{R}$-module.

Proof. From Items (1), (2) and (5) of Lemma 2.2.22 we know that $1_{\mu_{\mathcal{R}}(\mathcal{F})} = \sum_{P\leq S} I^P_P$ and that the I^P_P are mutually orthogonal idempotents. With this in mind we can obtain the following $\mathcal{R}$-module decomposition of $\mu_{\mathcal{R}}(\mathcal{F})$

$$\mu_{\mathcal{R}}(\mathcal{F}) \cong_{\mathcal{R}} \bigoplus_{A,B\leq S} I^A_A\mu_{\mathcal{R}}(\mathcal{F})\,I^B_B.$$

Fix now $A,B\leq S$. From the above it suffices to prove that $\mathcal{B}_{(A,B)}$ is an $\mathcal{R}$-basis of $I^A_A\mu_{\mathcal{R}}(\mathcal{F})\,I^B_B$. Using Lemma 2.2.22 we can write any element in $I^A_A\mu_{\mathcal{R}}(\mathcal{F})\,I^B_B$ as an $\mathcal{R}$-linear combination of elements of the form $I^B_{\varphi(C)}c_{\varphi,C}R^A_C$ with $C\leq A$ and $\varphi\colon C\to\varphi(C)$ an isomorphism in $\mathcal{F}$ satisfying $\varphi(C)\leq B$. For $i=1,2$ let C_i be a subgroup of A, let $\varphi_i\in \mathrm{Hom}_{\mathcal{F}}(C_i,\varphi_i(C_i))$ be an isomorphism in $\mathcal{F}$ such that $\varphi_i(C_i)\leq B$, view $\Delta(C_i,\varphi_i)$ as a subgroup of $B\times A$ and define the representative $X_i := (B\times A)/\Delta(C_i,\varphi_i)$ of $I^B_{\varphi_i(C_i)}c_{\varphi_i}R^A_{C_i}$. We know (see [Bo10, Lemma 2.3.4 (1)]) that each X_i is a transitive biset. Therefore we can use [Bo10, Lemma 2.1.9 and Definition 2.3.1] in order to deduce that $I^B_{\varphi_1(C_1)}c_{\varphi_1,C_1}R^A_{C_1} = I^B_{\varphi_2(C_2)}c_{\varphi_2,C_2}R^A_{C_2}$ if and only if there exist $a\in A$ and $b\in B$ such that $C_2 = C_1^a$ and $\varphi_2 = c_b\varphi_1c_a$. We also know (see [Bo10, Lemmas 2.1.9 and 2.2.2]) that any finite (A,B)-biset can be written in a unique way (up to isomorphism) as a disjoint union of finite transitive (A,B)-bisets. Let now $M_{(A,B)}$ be the commutative monoid generated by isomorphism classes of (A,B)-bisets of the form $I^B_{\varphi(C)}c_{\varphi,C}R^A_C$ with addition given by $\overline{X}+\overline{Y} = \overline{X\sqcup Y}$ (see the notation of Definition 2.2.19). We can deduce from the above that $\mathcal{B}_{(A,B)}$ (viewed as a subset of $M_{(A,B)}$) is an $\mathbb{N}$-basis of $M_{(A,B)}$. Recall now that $I_A^A \mu_{\mathbb{Z}}(\mathcal{F}) I_B^B$ is, by definition, the Grothendieck group of $M_{(A,B)}$. Thus, we can deduce that $\mathcal{B}_{(A,B)}$ (viewed as a subset of $I_A^A \mu_{\mathbb{Z}}(\mathcal{F}) I_B^B$) is a $\mathbb{Z}$-basis of $I_A^A \mu_{\mathbb{Z}}(\mathcal{F}) I_B^B$. Since tensor product preserves direct sum decomposition and $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathcal{R}$ for any commutative ring $\mathcal{R}$, then we can deduce that $\mathcal{B}_{(A,B)}$ (viewed as a subset of $I_A^A \mu_{\mathcal{R}}(\mathcal{F}) I_B^B$) is an $\mathcal{R}$-basis of $I_A^A \mu_{\mathcal{R}}(\mathcal{F}) I_B^B$ thus concluding the proof. □

Corollary 2.2.25. *Let $P \leq S$ and let $\mathcal{F}'$ be a fusion system over P satisfying $\mathcal{F}' \subseteq \mathcal{F}$. There exists a natural inclusion of Mackey algebras $\mu_{\mathcal{R}}(\mathcal{F}') \subseteq \mu_{\mathcal{R}}(\mathcal{F})$ and this inclusion preserves unit if and only if $P = S$.*

Proof. From Proposition 2.2.24 we know that $\mu_{\mathcal{R}}(\mathcal{F}')$ is generated, as an $\mathcal{R}$-module, by elements of the form $I_{\varphi(C)}^B c_\varphi R_C^A$ such that $A, B, C \leq P$ and φ is an isomorphism in $\mathcal{F}'$. Since $\mathcal{F}' \subseteq \mathcal{F}$ we know that any isomorphism in $\mathcal{F}'$ is also an isomorphism in $\mathcal{F}$ and since $P \leq S$ we know that every subgroup of P is also a subgroup of S. Therefore, with A, B, C and φ as before, we have that $I_{\varphi(C)}^A c_\varphi R_C^B \in \mu_{\mathcal{R}}(\mathcal{F})$. This gives us the natural inclusion $\mu_{\mathcal{R}}(\mathcal{F}') \subseteq \mu_{\mathcal{R}}(\mathcal{F})$. For this inclusion to preserve the unit we need to have $1_{\mu_{\mathcal{R}}(\mathcal{F})} = \sum_{Q \leq S} I_Q^Q = \sum_{Q' \leq P} I_{Q'}^{Q'} = 1_{\mu_{\mathcal{R}}(\mathcal{F}')}$ which happens if and only if $P = S$. □

We are now ready to define what a Mackey functor over a fusion system is.

Definition 2.2.26. A **Mackey functor over $\mathcal{F}$ on $\mathcal{R}$** (or simply **Mackey functor** if $\mathcal{F}$ and $\mathcal{R}$ are clear) is a finitely generated left $\mu_{\mathcal{R}}(\mathcal{F})$-module. The **category of Mackey functors over $\mathcal{F}$ on $\mathcal{R}$** (denoted by $\mathbf{Mack}_{\mathcal{R}}(\mathcal{F})$) is the category $\mu_{\mathcal{R}}(\mathcal{F})\text{-mod}$.

Example 2.2.27. Any globally defined Mackey functor (see [We93, Section 1]) inherits a structure of Mackey functor over any fusion system. Any conjugation invariant Mackey functor over a finite group G with Sylow p subgroup S leads naturally to a Mackey functor over $\mathcal{F}_S(G)$ (see [Bo15]). The Mackey algebra $\mu_{\mathcal{R}}(\mathcal{F})$ is itself a Mackey functor over $\mathcal{F}$.

This definition of Mackey functor over a fusion system allows us to use some well known results of ring theory in order to define the induction, restriction and conjugation functors.

Definition 2.2.28. Let $P \leq S$ and let $\mathcal{F}' \subseteq \mathcal{F}$ be a fusion system over P. From Corollary 2.2.25 we have that $\mu_{\mathcal{R}}(\mathcal{F}') \subseteq \mu_{\mathcal{R}}(\mathcal{F})$. This allows us to define the

restriction from $\mathcal{F}$ to $\mathcal{F}'$ functor as the functor $\downarrow_{\mathcal{F}'}^{\mathcal{F}}\colon \mu_{\mathcal{R}}(\mathcal{F})\text{-mod} \to \mu_{\mathcal{R}}(\mathcal{F}')\text{-mod}$, that sends any $\mu_{\mathcal{R}}(\mathcal{F})$-module M to the $\mu_{\mathcal{R}}(\mathcal{F}')$-module

$$M\downarrow_{\mathcal{F}'}^{\mathcal{F}} := \downarrow_{\mathcal{F}'}^{\mathcal{F}}(M) := 1_{\mu_{\mathcal{R}}(\mathcal{F}')}M.$$

Here $1_{\mu_{\mathcal{R}}(\mathcal{F}')} = \sum_{Q\leq P} I_Q^Q$ denotes the identity of $\mu_{\mathcal{R}}(\mathcal{F}')$ seen as an element of $\mu_{\mathcal{R}}(\mathcal{F})$ via the natural inclusion of Corollary 2.2.25.

Analogously, we can define the **induction from $\mathcal{F}'$ to $\mathcal{F}$ functor** as the functor $\uparrow_{\mathcal{F}'}^{\mathcal{F}}\colon \mu_{\mathcal{R}}(\mathcal{F}')\text{-mod} \to \mu_{\mathcal{R}}(\mathcal{F})\text{-mod}$, that sends any $\mu_{\mathcal{R}}(\mathcal{F}')$-module N to the $\mu_{\mathcal{R}}(\mathcal{F})$-module

$$N\uparrow_{\mathcal{F}'}^{\mathcal{F}} := \uparrow_{\mathcal{F}'}^{\mathcal{F}}(N) := \mu_{\mathcal{R}}(\mathcal{F})\,1_{\mu_{\mathcal{R}}(\mathcal{F}')} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}')} N.$$

Finally, let $Q\leq S$ and let $\varphi\colon P \hookrightarrow\!\!\!\!\rightarrow Q$ be a group isomorphism (not necessarily in $\mathcal{F}$). φ induces an isomorphism of $\mathcal{R}$-algebras $\hat{\varphi}: \mu_{\mathcal{R}}(\mathcal{F}_P) \hookrightarrow\!\!\!\!\rightarrow \mu_{\mathcal{R}}(\mathcal{F}_Q)$ obtained by setting

$$\hat{\varphi}\left(I_{{}^xC}^{B} c_{c_x} R_C^A\right) := I_{{}^{\varphi(x)}\varphi(C)}^{\varphi(B)} c_{c_{\varphi(x)}} R_{\varphi(C)}^{\varphi(A)},$$

for every $A,B,C\leq P$ and $x\in P$ such that $C\leq A$ and ${}^xC\leq B$. This allows us to define the **conjugation by φ^{-1} functor** as the invertible functor

$$^{\varphi^{-1}}\cdot := \hat{\varphi}^* : \mu_{\mathcal{R}}(\mathcal{F}_Q)\text{-mod} \to \mu_{\mathcal{R}}(\mathcal{F}_P)\text{-mod},$$

that sends any Mackey functor L over $\mathcal{F}_Q$ to the Mackey functor ${}^{\varphi^{-1}}L$ over $\mathcal{F}_P$ which equals L as an $\mathcal{R}$-module and such that for every $I_{{}^xC}^{B} c_{c_x} R_C^A \in \mu_{\mathcal{R}}(\mathcal{F}_P)$ as before and every $y\in {}^{\varphi^{-1}}L$

$$I_{{}^xC}^{B} c_{c_x} R_C^A \cdot y := I_{{}^{\varphi(x)}\varphi(C)}^{\varphi(B)} c_{c_{\varphi(x)}} R_{\varphi(C)}^{\varphi(A)} y.$$

Where, on the right hand side, we are viewing y as an element of L in order to apply the action of $\mu_{\mathcal{R}}(\mathcal{F}_Q)$ on it but we are viewing the result of this action as an element of ${}^{\varphi^{-1}}L$.

Let's now take a moment to notice a key difference between Mackey functors over groups and Mackey functors over fusion systems. Let G be a finite group, let $H,K\leq G$ and let M be a Mackey functor over G on $\mathcal{R}$. It is a well known result (see [We00, Section

3]) that

$$M\downarrow_H^G\uparrow_H^G(K)\cong_{\mathcal{R}}\bigoplus_{x\in[K\backslash G/H]}M\left(K^x\cap H\right). \tag{2.4}$$

It is also well known (see [TW95, Proposition 5.3]) that for every Mackey functor N over H the following equivalence of Mackey functors over K holds

$$N\uparrow_H^G\downarrow_K^G\cong\bigoplus_{x\in[K\backslash G/H]}\left({}^{x}\left(N\downarrow_{K^x\cap H}^H\right)\right)\uparrow_{K\cap{}^xH}^H. \tag{2.5}$$

Equations (2.4) and (2.5) play a key role in the arguments used in [Sa82] in order to obtain a Green correspondence for Mackey functors over groups. However, when trying to obtain similar results in the context of Mackey functors over fusion systems, the author was met with many complications. All of them can be traced back to the fact that the category $\mathcal{O}(\mathcal{F})_{\sqcup}$ does not in general admit products. In order to avoid such complications, Proposition 2.2.16 suggests that we should introduce the following.

Definition 2.2.29. Let $P\leq S$, let $\mathcal{F}'\subseteq\mathcal{F}$ be a fusion system over P and let M be a Mackey functor over $\mathcal{F}'$ on $\mathcal{R}$. We say that M is **$\mathcal{F}$-centric** if $I_Q^Q\cdot M=0$ for every $Q\in\mathcal{F}'\backslash\left(\mathcal{F}^c\cap\mathcal{F}'\right)$. The **category of $\mathcal{F}$-centric Mackey functors over $\mathcal{F}'$ on $\mathcal{R}$** (denoted by $\text{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}')$) is the full subcategory of $\text{Mack}_{\mathcal{R}}(\mathcal{F}')$ whose objects are $\mathcal{F}$-centric Mackey functors over $\mathcal{F}'$.

If $P=S$ and $\mathcal{F}'=\mathcal{F}$ we simply say that M is **centric** and denote by $\text{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ the **category of centric Mackey functors over $\mathcal{F}$ on $\mathcal{R}$.**

Let $\mathcal{F}'$ be a fusion subsystem of $\mathcal{F}$ and let M be an $\mathcal{F}'$-centric Mackey functor over $\mathcal{F}'$. The induced Mackey functor $M\uparrow_{\mathcal{F}'}^{\mathcal{F}}$ over $\mathcal{F}$ might not be $\mathcal{F}$-centric since we don't necessarily have $\mathcal{F}'^c\subseteq\mathcal{F}^c$. However, we have the following result.

Proposition 2.2.30. *Let P and Q be subgroups of S such that $P\leq Q$ and let $\mathcal{F}'$ and $\mathcal{F}''$ be fusion systems over P and Q respectively such that $\mathcal{F}'\subseteq\mathcal{F}''\subseteq\mathcal{F}$. Then we have that:*

(1) The functor $\downarrow_{\mathcal{F}'}^{\mathcal{F}''}$ maps $\text{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}'')$ to $\text{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}')$. In particular $\downarrow_{\mathcal{F}'}^{\mathcal{F}}$ maps $\text{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ to $\text{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}')$.

(2) The functor $\uparrow_{\mathcal{F}'}^{\mathcal{F}''}$ maps $\text{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}')$ to $\text{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}'')$. In particular $\uparrow_{\mathcal{F}'}^{\mathcal{F}}$ maps $\text{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}')$ to $\text{Mack}_{\mathcal{R}}(\mathcal{F}^c)$.

(3) *For every isomorphism $\varphi\colon P \to \varphi(P)$ in $\mathcal{F}$ the functor ${}^{\varphi}\cdot$ maps $Mack_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$ to $Mack_{\mathcal{R}}^{\mathcal{F}^c}\left(\mathcal{F}_{\varphi(P)}\right)$.*

Proof.

(1) Let $M \in \mathrm{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}'')$. For every $J \in \mathcal{F}' \backslash (\mathcal{F}' \cap \mathcal{F}^c)$ we have that $J \notin \mathcal{F}^c$ and, therefore, $I_J^J\left(M \downarrow_{\mathcal{F}'}^{\mathcal{F}''}\right) = I_J^J M = 0$. This proves that $M \downarrow_{\mathcal{F}'}^{\mathcal{F}''} \in \mathrm{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}')$.

(2) Let $M \in \mathrm{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}')$ and let $J \in \mathcal{F}'' \backslash (\mathcal{F}'' \cap \mathcal{F}^c)$. From Proposition 2.2.24 and Definition 2.2.28 we know that any element in $M \uparrow_{\mathcal{F}'}^{\mathcal{F}"}$ can be written as an $\mathcal{R}$-linear combination of elements of the form

$$y := I_{\varphi(C)}^{B} c_{\varphi} R_C^A 1_{\mu_{\mathcal{R}}(\mathcal{F}')} \otimes x.$$

Where $x \in M$ and $I_{\varphi(C)}^{B} c_{\varphi} R_C^A \in \mu_{\mathcal{R}}(\mathcal{F}'')$. Thus, it suffices to prove that $I_J^J y = 0$ for every such $y \in M \uparrow_{\mathcal{F}'}^{\mathcal{F}''}$. From Lemma 2.2.22 (5) we can assume without loss of generality that $A \leq P$ and $B = J$. With this setup we have that

$$\begin{aligned} I_J^J y &= I_J^J I_{\varphi(C)}^{J} c_{\varphi} R_C^A 1_{\mu_{\mathcal{R}}(\mathcal{F}')} \otimes x, \\ &= I_{\varphi(C)}^{J} c_{\varphi} R_C^A \otimes x, \\ &= I_{\varphi(C)}^{J} c_{\varphi} \otimes R_C^A x. \end{aligned}$$

Where, in the last identity, we are using the fact that the tensor product is over $\mu_{\mathcal{R}}(\mathcal{F}_P)$ and $R_C^A \in \mu_{\mathcal{R}}(\mathcal{F}_P)$ (see Corollary 2.2.25). Since $J \notin \mathcal{F}^c$ and $\varphi(C) \leq J$ then we can deduce from [Li07, Proposition 4.4] that $\varphi(C) \notin \mathcal{F}^c$. From definition of $\mathcal{F}$-centric subgroup we can deduce that $C \notin \mathcal{F}^c$. Since $C \leq P$ then we can conclude that $C \in \mathcal{F}' \backslash (\mathcal{F}' \cap \mathcal{F}^c)$. Since $M \in \mathrm{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}')$ and $x \in M$ this implies that $R_C^A x \in I_C^C M = 0$. Therefore we can conclude once again that $I_J^J y = 0$ thus proving that $M \uparrow_{\mathcal{F}'}^{\mathcal{F}''} \in \mathrm{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}'')$.

(3) Let $M \in \mathrm{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$ and let $J \in \mathcal{F}_{\varphi(P)} \backslash \left(\mathcal{F}_{\varphi(P)} \cap \mathcal{F}^c\right)$ with φ as in the statement. By definition of $\mathcal{F}$-centric subgroup we know that $\varphi^{-1}(J) \in \mathcal{F}_P \backslash (\mathcal{F}_P \cap \mathcal{F}^c)$. Then, by definition of the functor ${}^{\varphi}\cdot$ (see Definition 2.2.28), we have that $I_J^J\, {}^{\varphi}M = I_{\varphi^{-1}(J)}^{\varphi^{-1}(J)} M = 0$ thus proving that ${}^{\varphi}M \in \mathrm{Mack}_{\mathcal{R}}^{\mathcal{F}^c}\left(\mathcal{F}_{\varphi(P)}\right)$.

□

Proposition 2.2.30 motivates the introduction of the following.

Notation 2.2.31. Let $\mathcal{F}', \mathcal{F}'', \mathcal{F}$ and $\varphi\colon P \to \varphi(P)$ be as in Proposition 2.2.30. We use the same notation to refer to the functors $\uparrow_{\mathcal{F}'}^{\mathcal{F}''}$, $\downarrow_{\mathcal{F}'}^{\mathcal{F}''}$ and ${}^{\varphi}\cdot$ of Definition 2.2.28 and their restrictions given by Proposition 2.2.30.

With this setup we are now just one Lemma away from providing a result analogue to Equation (2.4) in the context of centric Mackey functors over fusion systems.

Lemma 2.2.32. *Let $P, Q \in \mathcal{F}^c$, let $M \in \mathit{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$, let $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]$, let $y \in I_A^A \mu_{\mathcal{R}}(\mathcal{F}_P) I_A^A$, let $\mathcal{I}$ be the two sided ideal of $\mu_{\mathcal{R}}(\mathcal{F})$ generated by elements of the form I_J^J with $J \in \mathcal{F}_P \backslash (\mathcal{F}_P \cap \mathcal{F}^c)$ and let $\pi\colon \mu_{\mathcal{R}}(\mathcal{F}) \to \mu_{\mathcal{R}}(\mathcal{F})/\mathcal{I}$ be the natural projection. If $\pi\left(I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} y\right) = \pi\left(I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}}\right)$ (see Corollary 2.2.25) then $\pi(y) = \pi\left(I_A^A\right)$. In particular, viewing the subset $I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} \mu_{\mathcal{R}}(\mathcal{F}_P)$ of $\mu_{\mathcal{R}}(\mathcal{F})$ as a right $\mu_{\mathcal{R}}(\mathcal{F}_P)$-module, and defining*

$$I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M := I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} \mu_{\mathcal{R}}(\mathcal{F}_P) \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M,$$

we have an isomorphism of $\mathcal{R}$-modules from $I_A^A M$ to $I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M$ that sends x to $I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} \otimes x$.

Proof. From Lemma 2.2.22, Proposition 2.2.24 and [Li07, Proposition 4.4] we know that the ideal $\mathcal{I}$ is spanned as an $\mathcal{R}$-module by elements of the form $I_{\psi(J)}^C c_\psi R_J^B$ such that exists $J' \in \mathcal{F}_P \backslash (\mathcal{F}_P \cap \mathcal{F}^c)$ satisfying $J =_{\mathcal{F}} J'$. Define now $\mathcal{J} := \mathcal{I} \cap \mu_{\mathcal{R}}(\mathcal{F}_P)$. From the above we can conclude that $\mathcal{J}$ is spanned as an $\mathcal{R}$-module by elements of the form $I_{{}^xJ}^C c_{c_x} R_J^B$ with $J \in \mathcal{F}_P \backslash (\mathcal{F}_P \cap \mathcal{F}^c)$ and $x \in P$. Since M is $\mathcal{F}$-centric, we have that $\mathcal{J} M = 0$ (by definition). On the other hand, from the above description of $\mathcal{I}$ and $\mathcal{J}$, we know that $\pi\left(I_{\overline{\varphi}(A)}^J c_{\overline{\varphi}} \mu_{\mathcal{R}}(\mathcal{F}_P)\right)$ is equivalent, as a right $\mu_{\mathcal{R}}(\mathcal{F}_P)$ module, to $\left(I_{\overline{\varphi}(A)}^J c_{\overline{\varphi}} \mu_{\mathcal{R}}(\mathcal{F}_P)\right) / \left(I_{\overline{\varphi}(A)}^J c_{\overline{\varphi}} \mathcal{J}\right)$. We can therefore conclude that

$$\pi\left(I_{\overline{\varphi}(A)}^J c_{\overline{\varphi}}\right) \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M := \pi\left(I_{\overline{\varphi}(A)}^J c_{\overline{\varphi}} \mu_{\mathcal{R}}(\mathcal{F}_P)\right) \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M \cong I_{\overline{\varphi}(A)}^J c_{\overline{\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M.$$

With this setup we obtain a surjective morphism of $\mathcal{R}$-modules Γ from $I_A^A M$ to $\pi\left(I_{\overline{\varphi}(A)}^J c_{\overline{\varphi}}\right) \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M$ that sends any $x \in I_A^A M$ to $\pi\left(I_{\overline{\varphi}(A)}^J c_{\overline{\varphi}}\right) \otimes x$. Assume that $\pi\left(I_{\overline{\varphi}(A)}^J c_{\overline{\varphi}}\right) \otimes x = 0$. Then there exists $y \in I_A^A \mu_{\mathcal{R}}(\mathcal{F}_P) I_A^A$ such that $\pi\left(I_{\overline{\varphi}(A)}^J c_{\overline{\varphi}} y\right) =$

$\pi\left(I^J_{\overline{\varphi}(A)}c_{\overline{\varphi}}\right)$ and $yx=0$. Since $\mathcal{J}M=0$, if the first part of the statement were true, we would have that $yx=I^A_Ax=x$. This would prove that $x=0$ and, therefore, that Γ is an isomorphism of $\mathcal{R}$-modules. In other words the second part of the statement follows from the first.

Let's now prove the first part of the statement. For $i=1,2$ let $x_i\in P$ and $B_i\leq A^{x_i}\cap A$ such that $B_i\in\mathcal{F}_P\cap\mathcal{F}^c$ and that

$$\pi\left(I^Q_{\overline{\varphi\iota c_{x_1}}(B_1)}c_{\overline{\varphi\iota c_{x_1}}}R^A_{B_1}\right)=\pi\left(I^Q_{\overline{\varphi\iota c_{x_2}}(B_2)}c_{\overline{\varphi\iota c_{x_2}}}R^A_{B_2}\right).$$

Since $B_i\notin\mathcal{F}^c$ then we can deduce from the description of $\mathcal{I}$ given at the start of the proof, the above identity and Proposition 2.2.24 that

$$I^Q_{\overline{\varphi\iota c_{x_1}}(B_1)}c_{\overline{\varphi\iota c_{x_1}}}R^A_{B_1}=I^Q_{\overline{\varphi\iota c_{x_2}}(B_2)}c_{\overline{\varphi\iota c_{x_2}}}R^A_{B_2}.$$

From Items (1) and (3) of Lemma 2.2.22 and Proposition 2.2.24 we can conclude that there exists $a\in A$ such that $B:=B_1=B_2^a$ and $\overline{\varphi}\,\overline{\iota c_{x_1a}}=\overline{\varphi}\,\overline{\iota c_{x_2}}$. Since $x_1a,x_2\in P$ we also have that $\overline{\iota^P_A}\,\overline{\iota c_{x_1a}}=\overline{\iota^P_A}=\overline{\iota^P_A}\,\overline{\iota c_{x_2}}$. From the universal properties of product we can therefore conclude that $\overline{\iota^A_{^{x_1}B}c_{x_1a}}=\overline{\iota^A_{^{x_2}B}c_{x_2}}$. From definition of $\mathcal{O}\left(\mathcal{F}^c\right)$ this implies that there exists $b\in A$ such that $c_{bx_1a}=c_{x_2}$ as an isomorphism from B to ^{x_2}B. Therefore, there exists $z\in C_P(B)$ such that $bx_1az=x_2$. Since $B\in\mathcal{F}^c$ we can conclude that $z\in B\leq A$ and, therefore, $x_2\in Ax_1A$. Now let y be as in the statement. From Proposition 2.2.24 we can write

$$y:=\sum_{x\in[A\backslash P/A]}\sum_{\substack{B\leq A^x\cap A\\ \text{up to }A\text{-conj.}}}\lambda_{x,B}I^A_{^xB}c_{c_x}R^A_B,$$

for some $\lambda_{x,B}\in\mathcal{R}$. Since we are only interested in the projection $\pi(y)$ then we can assume without loss of generality that $\lambda_{x,B}=0$ whenever $B\in\mathcal{F}_P\backslash\left(\mathcal{F}_P\cap\mathcal{F}^c\right)$. From the above and Proposition 2.2.24 we can conclude that if y satisfies $\pi\left(I^J_{\overline{\varphi}(A)}c_{\overline{\varphi}}y\right)=\pi\left(I^J_{\overline{\varphi}(A)}c_{\overline{\varphi}}\right)$ then $\lambda_{x,B}=0$ unless $B=A$ and $x\in A$ in which case $\lambda_{x,B}=1_{\mathcal{R}}$. In other words we have that $\pi(y)=\pi\left(I^A_A\right)$ just as we wanted to prove. $\square$

We can now prove an analogue to Equation (2.4) in the context of centric Mackey functors over fusion systems.

Proposition 2.2.33. *Let $P \in \mathcal{F}^c$, let $\mathcal{I}$ be the two sided ideal of $\mu_{\mathcal{R}}(\mathcal{F})$ generated by elements of the form I_Q^Q with $Q \in \mathcal{F}\backslash\mathcal{F}^c$ and let $\pi\colon \mu_{\mathcal{R}}(\mathcal{F}) \to \mu_{\mathcal{R}}(\mathcal{F})/\mathcal{I}$ be the natural projection map. Then the set $\pi\left(\mu_{\mathcal{R}}(\mathcal{F})1_{\mu_{\mathcal{R}}(\mathcal{F}_P)}\right)$ inherits from $\mu_{\mathcal{R}}(\mathcal{F})1_{\mu_{\mathcal{R}}(\mathcal{F}_P)}$ a right $\mu_{\mathcal{R}}(\mathcal{F}_P)$-module structure and the following is a $\mu_{\mathcal{R}}(\mathcal{F}_P)$ basis of $\pi\left(\mu_{\mathcal{R}}(\mathcal{F})1_{\mu_{\mathcal{R}}(\mathcal{F}_P)}\right)$*

$$\mathcal{B} := \bigsqcup_{Q\in\mathcal{F}^c}\ \bigsqcup_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} \left\{\pi\left(I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}}\right)\right\}.$$

In particular, for any $M \in \mathit{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$, we have the following equivalence of $\mathcal{R}$-modules

$$M\uparrow_{\mathcal{F}_P}^{\mathcal{F}} \cong_{\mathcal{R}} \bigoplus_{Q\in\mathcal{F}^c}\ \bigoplus_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} \otimes M \cong_{\mathcal{R}} \bigoplus_{Q\in\mathcal{F}^c}\ \bigoplus_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} I_A^A M.$$

Where each $I_{\overline{\varphi}(A)}^K c_{\overline{\varphi}} \otimes M$ is seen as an $\mathcal{R}$-submodule of $M\uparrow_{\mathcal{F}_P}^{\mathcal{F}}$.

Proof. Throughout this proof we denote the right $\mu_{\mathcal{R}}(\mathcal{F}_P)$-module $\pi\left(\mu_{\mathcal{R}}(\mathcal{F})1_{\mu_{\mathcal{R}}(\mathcal{F}_P)}\right)$ simply by $\overline{\mu_{\mathcal{R}}(\mathcal{F})_P}$.

From Lemma 2.2.22, Proposition 2.2.24 and [Li07, Proposition 4.4] we know that the ideal $\mathcal{I}$ is spanned as an $\mathcal{R}$-module by elements of the form $I_{\varphi(C)}^B c_\varphi R_C^A$ with $C \in \mathcal{F}\backslash\mathcal{F}^c$. If $A \not\leq P$ we have that $R_C^A 1_{\mu_{\mathcal{R}}(\mathcal{F}_P)} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M = 0$. On the other hand, if $A \leq P$, we have that $C \leq P$ and, therefore, $C \in \mathcal{F}_P\backslash(\mathcal{F}_P \cap \mathcal{F}^c)$. Since $M \in \mathrm{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$ this implies that $R_C^A 1_{\mu_{\mathcal{R}}(\mathcal{F}_P)} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M = I_C^C \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} R_C^A M = 0$. In either case we have that $\mathcal{I}1_{\mu_{\mathcal{R}}(\mathcal{F}_P)} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M = 0$. Using right exactness of the tensor product functor we can conclude from the above and definition of $\uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ (see Definition 2.2.28) that the following isomorphism of $\mu_{\mathcal{R}}(\mathcal{F})$-modules holds

$$M\uparrow_{\mathcal{F}_P}^{\mathcal{F}} \cong \overline{\mu_{\mathcal{R}}(\mathcal{F})_P} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M.$$

Assume now that $\mathcal{B}$ is a $\mu_{\mathcal{R}}(\mathcal{F}_P)$ basis of the right $\mu_{\mathcal{R}}(\mathcal{F}_P)$-module $\overline{\mu_{\mathcal{R}}(\mathcal{F})_P}$. Since tensor product preserves direct sums, we obtain from the previous equivalence the following equivalences of $\mathcal{R}$-modules

$$\begin{aligned} M\uparrow_{\mathcal{F}_P}^{\mathcal{F}} &\cong_{\mathcal{R}} \bigoplus_{Q\in\mathcal{F}^c}\ \bigoplus_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} \pi\left(I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} \mu_{\mathcal{R}}(\mathcal{F}_P)\right) \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M, \\ &\cong_{\mathcal{R}} \bigoplus_{Q\in\mathcal{F}^c}\ \bigoplus_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} \mu_{\mathcal{R}}(\mathcal{F}_P) \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M. \end{aligned}$$

Where, for the second identity, we are using that $\mathcal{I}1_{\mu_{\mathcal{R}}(\mathcal{F}_P)} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M = 0$ and right exactness of tensor product. The second part of the statement follows from Lemma 2.2.32 and the above by viewing each $I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}} \mu_{\mathcal{R}}(\mathcal{F}_P) \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M$ as the $\mathcal{R}$-submodule $I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M$ of $M \uparrow^{\mathcal{F}}_{\mathcal{F}_P}$. This proves that the second part of the statement follows from the first.

Let's now prove the first part of the statement. From Proposition 2.2.24 and the previous description of $\mathcal{I}$ we obtain the following equivalence of right $\mu_{\mathcal{R}}(\mathcal{F}_P)$-modules

$$\overline{\mu_{\mathcal{R}}(\mathcal{F})_P} \cong \bigoplus_{Q\in\mathcal{F}^c} \pi\left(I^Q_Q\right) \overline{\mu_{\mathcal{R}}(\mathcal{F})_P}.$$

For every $Q \in \mathcal{F}^c$ we can now define

$$\mathcal{B}^Q := \bigsqcup_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} \left\{\pi\left(I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}}\right)\right\}.$$

In order to prove the statement it suffices to prove that $\mathcal{B}^Q$ is a right $\mu_{\mathcal{R}}(\mathcal{F}_P)$-basis of $\pi\left(I^Q_Q\right) \overline{\mu_{\mathcal{R}}(\mathcal{F})_P}$. In other words we need to prove that for every $Q \in \mathcal{F}^c$ there exists a direct sum decomposition of right $\mu_{\mathcal{R}}(\mathcal{F}_P)$-modules of the form

$$\pi\left(I^Q_Q\right) \overline{\mu_{\mathcal{R}}(\mathcal{F})_P} = \bigoplus_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} \pi\left(I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}}\right) \overline{\mu_{\mathcal{R}}(\mathcal{F})_P}, \tag{2.6}$$

Where the summands on the right hand side are seen as right $\mu_{\mathcal{R}}(\mathcal{F}_P)$-submodules of $\pi\left(I^Q_Q\right) \overline{\mu_{\mathcal{R}}(\mathcal{F})_P}$.

Fix $Q \in \mathcal{F}^c$. From Proposition 2.2.24 and the above description of $\mathcal{I}$ we know that $\pi\left(I^Q_Q\right) \overline{\mu_{\mathcal{R}}(\mathcal{F})_P}$ has an $\mathcal{R}$-basis of the form

$$\mathcal{B}^Q_{\mathcal{R}} := \bigsqcup_{J\in\mathcal{F}_P\cap\mathcal{F}^c} \bigsqcup_{\substack{B\in\mathcal{F}_J\cap\mathcal{F}^c \\ \text{up to } J\text{-conj.}}} \bigsqcup_{\psi\in\left[\operatorname{Aut}_Q(Q)\backslash \operatorname{Hom}_{\mathcal{F}_P}(B,Q)/\operatorname{Aut}_J(B)\right]} \left\{\pi\left(I^Q_{\overline{\psi}(B)} c_{\overline{\psi}} R^J_B\right)\right\},$$

For each $\pi\left(I^Q_{\overline{\psi}(B)} c_{\overline{\psi}} R^J_B\right) \in \mathcal{B}^Q_{\mathcal{R}}$ we get a map $\overline{\psi} : B \to Q$ and a map $\overline{\iota^P_B} : B \to P$. From the universal properties of product we can then conclude that there exists a unique $\left(B^{P,Q}, \overline{\psi^{P,Q}}\right) \in [P \times_{\mathcal{F}} Q]$ and a unique $\overline{\gamma^{P,Q}_{\left(B,\overline{\psi}\right)}} \in \operatorname{Hom}_{\mathcal{O}(\mathcal{F}^c)}\left(B, B^{P,Q}\right)$ such that $\overline{\iota^P_{B^{P,Q}}}\,\overline{\gamma^{P,Q}_{\left(B,\overline{\psi}\right)}} = \overline{\iota^P_B}$ and that $\overline{\psi^{P,Q}}\,\overline{\gamma^{P,Q}_{\left(B,\overline{\psi}\right)}} = \overline{\psi}$. From the first identity and definition of

$\mathcal{O}(\mathcal{F})$ we can conclude that $\overline{\gamma^{P,Q}_{(B,\overline{\psi})}} \in \mathcal{O}(\mathcal{F}_P)$. From the second identity and Corollary 2.2.23 we can deduce that

$$I^Q_{\overline{\psi}(B)} c_{\overline{\psi}} = I^Q_{\overline{\psi^{P,Q}}(B^{P,Q})} c_{\overline{\psi^{P,Q}}} I^{B^{P,Q}}_{\overline{\gamma^{P,Q}_{(B,\overline{\psi})}}(B)} c_{\overline{\gamma^{P,Q}_{(B,\overline{\psi})}}}. \tag{2.7}$$

This allows us to write $\mathcal{B}^Q_{\mathcal{R}} = \bigsqcup_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} \mathcal{B}^{Q,(A,\overline{\varphi})}_{\mathcal{R}}$ where

$$\mathcal{B}^{Q,(A,\overline{\varphi})}_{\mathcal{R}} := \bigsqcup_{J\in\mathcal{F}_P\cap\mathcal{F}^c} \bigsqcup_{\substack{B\in\mathcal{F}_J\cap\mathcal{F}^c \\ \text{up to } J\text{-conj.}}} \bigsqcup_{\substack{\psi\in\left[\operatorname{Aut}_Q(Q)\backslash \operatorname{Hom}_{\mathcal{F}_P}(B,Q)/\operatorname{Aut}_J(B)\right] \\ \left(B^{P,Q},\overline{\psi^{P,Q}}\right)=(A,\overline{\varphi})}} \left\{\pi\left(I^Q_{\overline{\psi}(B)} c_{\overline{\psi}} R^J_B\right)\right\}.$$

Fix $(A,\overline{\varphi}) \in [P\times_{\mathcal{F}} Q]$. From Equation (2.7) we know that $\mathcal{B}^{Q,(A,\overline{\varphi})}_{\mathcal{R}}$ is contained in $\pi\left(I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}}\right)\overline{\mu_{\mathcal{R}}(\mathcal{F})_P}$. If we now prove that $\mathcal{B}^{Q,(A,\overline{\varphi})}_{\mathcal{R}}$ is in fact a generating set of $\pi\left(I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}}\right)\overline{\mu_{\mathcal{R}}(\mathcal{F})_P}$ (seen as an $\mathcal{R}$-module) then, since $\pi\left(I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}}\right)\overline{\mu_{\mathcal{R}}(\mathcal{F})_P}$ is a right $\mu_{\mathcal{R}}(\mathcal{F}_P)$-submodule of $\pi\left(I^Q_Q\right)\overline{\mu_{\mathcal{R}}(\mathcal{F})_P}$ and $\mathcal{B}^Q_{\mathcal{R}}$ is an $\mathcal{R}$-basis of $\pi\left(I^Q_Q\right)\overline{\mu_{\mathcal{R}}(\mathcal{F})_P}$, we would obtain Equation (2.6) and the result would follow. From Proposition 2.2.24 and the above description of $\mathcal{I}$ it suffices to prove that for every $J\in\mathcal{F}_P\cap\mathcal{F}^c$, every $C\in\mathcal{F}_J\cap\mathcal{F}^c$ and every $\overline{\theta}\in\operatorname{Hom}_{\mathcal{O}(\mathcal{F}_P)}(C,A)$ there exists $\pi\left(I^Q_{\overline{\psi}(B)} c_{\overline{\psi}} R^J_B\right)\in\mathcal{B}^{Q,(A,\overline{\varphi})}_{\mathcal{R}}$ such that $\pi\left(I^Q_{\overline{\psi}(B)} c_{\overline{\psi}} R^J_B\right) = \pi\left(I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}} I^A_{\overline{\theta}(C)} c_{\overline{\theta}} R^J_C\right)$. From the description of $\mathcal{B}^Q_{\mathcal{R}}$ there exist $j\in J$ and $\pi\left(I^Q_{\overline{\psi}(B)} c_{\overline{\psi}} R^J_B\right)\in\mathcal{B}^Q_{\mathcal{R}}$ such that ${}^jB = C$ and $\overline{\psi} = \overline{\varphi}\overline{\theta}\overline{c_j}$. Here we are viewing c_j as an isomorphism from B to C. Therefore, by definition, we have that $\overline{\gamma^{P,Q}_{(B,\overline{\psi})}} = \overline{\theta}\overline{c_j}$ and $\left(B^{P,Q},\overline{\psi^{P,Q}}\right) = (A,\overline{\varphi})$. In other words $\pi\left(I^Q_{\overline{\psi}(B)} c_{\overline{\psi}} R^J_B\right)\in\mathcal{B}^{Q,(A,\overline{\varphi})}_{\mathcal{R}}$. From Lemma 2.2.22 (3) we know that $c_j R^J_B = R^J_C$ and, therefore, from the identities above and Corollary 2.2.23 we can conclude that $I^Q_{\overline{\psi}(B)} c_{\overline{\psi}} R^J_B = I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}} I^A_{\overline{\theta}(C)} c_{\overline{\theta}} R^J_C$ thus completing the proof. □

Before proceeding it is worth introducing the following result motivated by the notation of Proposition 2.2.33.

Lemma 2.2.34. *Let $P,Q\in\mathcal{F}^c$. Then we have that:*

(1) For every $(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]$, every $J\in\mathcal{F}^c$ and every $\theta\in\operatorname{Hom}_{\mathcal{F}^c}(Q,J)$ there exist a unique $\left(A^\theta,\overline{\varphi^\theta}\right)\in[P\times_{\mathcal{F}}J]$ and a unique $\overline{\gamma^\theta_{(A,\overline{\varphi})}}\in\operatorname{Hom}_{\mathcal{O}(\mathcal{F}^c)}\left(A,A^\theta\right)$ such that $\overline{\varphi^\theta}\overline{\gamma^\theta_{(A,\overline{\varphi})}} = \overline{\theta}\overline{\varphi}$ and $\overline{\iota^P_{A^\theta}}\overline{\gamma^\theta_{(A,\overline{\varphi})}} = \overline{\iota^P_A}$. Moreover $\overline{\gamma^\theta_{(A,\overline{\varphi})}}\in\mathcal{O}(\mathcal{F}_P)$ and, given $J'\in\mathcal{F}^c$ and $\delta\in\operatorname{Hom}_{\mathcal{F}^c}(J,J')$ we have that $A^{\delta\theta} = \left(A^\theta\right)^\delta$, that $\overline{\varphi^{\delta\theta}} = \overline{\left(\varphi^\theta\right)^\delta}$

and that $\overline{\gamma^{\delta\theta}_{(A,\overline{\varphi})}} = \overline{\gamma^{\delta}_{\left(A^{\theta},\overline{\varphi^{\theta}}\right)}\gamma^{\theta}_{(A,\overline{\varphi})}}$. *If* $\theta = \iota_Q^J$ *we write* $\left(A^J, \overline{\varphi^J}\right) := \left(A^{\theta}, \overline{\varphi^{\theta}}\right)$ *and* $\overline{\gamma^{J}_{(A,\overline{\varphi})}} := \overline{\gamma^{\theta}_{(A,\overline{\varphi})}}$.

(2) *Let* $J \in \mathcal{F}^c$ *such that* $J \geq Q$ *and let* $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]$. *The following identities are satisfied*

$$I_Q^J I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}} = I^J_{\overline{\varphi^J}(A^J)} c_{\overline{\varphi^J}} I^{A^J}_{\overline{\gamma^J_{(A,\overline{\varphi})}}(A)} c_{\overline{\gamma^J_{(A,\overline{\varphi})}}},$$

$$c_{\overline{\varphi^{-1}}} R^Q_{\overline{\varphi}(A)} R_Q^J = c_{\overline{\left(\gamma^J_{(A,\overline{\varphi})}\right)^{-1}}} R^{A^J}_{\overline{\gamma^J_{(A,\overline{\varphi})}}(A)} c_{\overline{(\varphi^J)^{-1}}} R^J_{\overline{\varphi^J}(A^J)}.$$

(3) *Let* $J \in \mathcal{F}^c$ *such that* $J \geq Q$ *and let* $\mathcal{I}$ *be the two sided ideal of* $\mu_{\mathcal{R}}(\mathcal{F})$ *generated by elements of the form* I_C^C *such that* $C \in \mathcal{F}\backslash\mathcal{F}^c$. *The following equivalences are satisfied*

$$\sum_{\left(B,\overline{\psi}\right)\in[P\times_{\mathcal{F}}J]} R_Q^J I^J_{\overline{\psi}(B)} c_{\overline{\psi}} \equiv \sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}} c_{\overline{\left(\gamma^J_{(A,\overline{\varphi})}\right)^{-1}}} R^{A^J}_{\overline{\gamma^J_{(A,\overline{\varphi})}}(A)}, \qquad \mod \mathcal{I}$$

$$\sum_{\left(B,\overline{\psi}\right)\in[P\times_{\mathcal{F}}J]} c_{\overline{\psi^{-1}}} R^J_{\overline{\psi}(B)} I_Q^J \equiv \sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} I^{A^J}_{\overline{\gamma^J_{(A,\overline{\varphi})}}(A)} c_{\overline{\gamma^J_{(A,\overline{\varphi})}}} c_{\overline{\varphi^{-1}}} R^Q_{\overline{\varphi}(A)}. \qquad \mod \mathcal{I}$$

More precisely, for every $\left(B, \overline{\psi}\right) \in [P \times_{\mathcal{F}} J]$ *we have that*

$$R_Q^J I^J_{\overline{\psi}(B)} c_{\overline{\psi}} \equiv \sum_{\substack{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q] \\ \left(A^J,\overline{\varphi^J}\right)=\left(B,\overline{\psi}\right)}} I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}} c_{\overline{\left(\gamma^J_{(A,\overline{\varphi})}\right)^{-1}}} R^{A^J}_{\overline{\gamma^J_{(A,\overline{\varphi})}}(A)}, \qquad \mod \mathcal{I}$$

$$c_{\overline{\psi^{-1}}} R^J_{\overline{\psi}(B)} I_Q^J \equiv \sum_{\substack{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q] \\ \left(A^J,\overline{\varphi^J}\right)=\left(B,\overline{\psi}\right)}} I^{A^J}_{\overline{\gamma^J_{(A,\overline{\varphi})}}(A)} c_{\overline{\gamma^J_{(A,\overline{\varphi})}}} c_{\overline{\varphi^{-1}}} R^Q_{\overline{\varphi}(A)}. \qquad \mod \mathcal{I}$$

(4) *Let* $\rho\colon Q \to \rho(Q)$ *be an isomorphism in* $\mathcal{F}$, *for every* $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]$, *the morphism* $\overline{\gamma^{\rho}_{(A,\overline{\varphi})}}$ *is an isomorphism and we have*

$$c_{\rho} I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}} = I^{\rho(Q)}_{\overline{\varphi^{\rho}}(A^{\rho})} c_{\overline{\varphi^{\rho}}} c_{\gamma^{\rho}_{(A,\overline{\varphi})}}, \qquad c_{\overline{\varphi^{-1}}} R^Q_{\overline{\varphi}(A)} c_{\rho^{-1}} = c_{\left(\gamma^{\rho}_{(A,\overline{\varphi})}\right)^{-1}} c_{\overline{(\varphi^{\rho})^{-1}}} R^{\rho(Q)}_{\overline{\varphi^{\rho}}(A^{\rho})}.$$

For any representative $\gamma^{\rho}_{(A,\overline{\varphi})} \in \overline{\gamma^{\rho}_{(A,\overline{\varphi})}}$. *In particular, from Proposition 2.2.18* (3)

$$\sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} c_{\rho} I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}} = \sum_{\left(B,\overline{\psi}\right)\in[P\times_{\mathcal{F}}\rho(Q)]} I^{\rho(Q)}_{\overline{\psi}(B)} c_{\overline{\psi}} c_{\gamma^{\rho^{-1}}_{(B,\overline{\psi})}},$$

$$\sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} c_{\overline{\varphi^{-1}}} R^Q_{\overline{\varphi}(A)} c_{\rho^{-1}} = \sum_{\left(B,\overline{\psi}\right)\in[P\times_{\mathcal{F}}\rho(Q)]} c_{\gamma^{\rho}_{(B,\overline{\psi})}} c_{\overline{\psi^{-1}}} R^{\rho(Q)}_{\overline{\psi}(B)}.$$

Proof. We only prove the first equation of each item since the proof of the second one is analogous.

(1) Item (1) is an immediate consequence of the universal properties of products. The fact that $\overline{\gamma^{\theta}_{(A,\overline{\varphi})}} \in \mathcal{O}(\mathcal{F}_P)$ follows from definition of $\mathcal{O}(\mathcal{F})$ and the identity $\overline{\iota^{P}_{A^{\theta}}}\,\overline{\gamma^{\theta}_{(A,\overline{\varphi})}} = \overline{\iota^{P}_{A}}$.

(2) Item (2) follows from the identity $\overline{\varphi^{J}}\,\overline{\gamma^{J}_{(A,\overline{\varphi})}} = \overline{\iota^{J}_{Q}}\overline{\varphi}$ and Corollary 2.2.23.

(3) Let $(B,\overline{\psi}) \in [P \times_{\mathcal{F}} J]$, fix a representative $\psi \in \overline{\psi}$ and view it as an isomorphism between the appropriate restrictions. Item (3) now follows from the identities below

$$
\begin{aligned}
R^{J}_{Q} I^{J}_{\overline{\psi}(B)} c_{\overline{\psi}} &= \sum_{x\in[Q\backslash J/\psi(B)]} I^{Q}_{Q\cap{}^{x}(\psi(B))} c_{c_x\psi} R^{B}_{\psi^{-1}(Q^{x}\cap\psi(B))}, && \text{Lemma 2.2.22 (3) and (4)}\\
&\equiv \sum_{\substack{x\in[Q\backslash J/\psi(B)]\\ Q^{x}\cap\psi(B)\in\mathcal{F}^{c}}} I^{Q}_{Q\cap{}^{x}(\psi(B))} c_{c_x\psi} R^{B}_{\psi^{-1}(Q^{x}\cap\psi(B))}, && \mod \mathcal{I}\\
&= \sum_{\substack{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]\\ \left(A^{J},\overline{\varphi^{J}}\right)=\left(B,\overline{\psi}\right)}} I^{Q}_{\overline{\varphi}(A)} c_{\overline{\varphi}} c_{\overline{\left(\gamma^{J}_{(A,\overline{\varphi})}\right)^{-1}}} R^{A^{J}}_{\overline{\gamma^{J}_{(A,\overline{\varphi})}}(A)}. && \text{Proposition 2.2.18 (5)}
\end{aligned}
$$

Where, in the last identity, we are using the fact that the bijection of Proposition 2.2.18 (5), which sends every $(B,\overline{\psi}) \in [P \times_{\mathcal{F}} J]$ and every $x \in [Q\backslash J/\psi(B)]$ to $(A,\overline{\varphi}) = \left((\psi^{-1}(Q^{x}\cap\psi(B)))^{y}, \overline{\iota c_x \psi c_y}\right)$ for some $y \in P$ (which from Proposition 2.2.18 (5) we can assume to be 1_S), satisfies $\left(A^{J},\overline{\varphi^{J}}\right) = \left(B,\overline{\psi}\right)$ and $\overline{\gamma^{J}_{(A,\overline{\varphi})}} = \overline{\iota^{B}_{{}^{y}A} c_y}$.

(4) From uniqueness of the map $\overline{\gamma^{\mathsf{Id}_Q}_{(A,\overline{\varphi})}}$ we know that $\overline{\gamma^{\mathsf{Id}_Q}_{(A,\overline{\varphi})}} = \overline{\mathsf{Id}_A}$. From Item (1), we can therefore deduce that $\overline{\gamma^{\rho}_{(A,\overline{\varphi})}}$ is an isomorphism with inverse $\overline{\gamma^{\rho^{-1}}_{\left(A^{\rho},\overline{\varphi^{\rho}}\right)}}$. Item (4) now follows from the identity $\overline{\varphi^{\rho}}\,\overline{\gamma^{\rho}_{(A,\overline{\varphi})}} = \overline{\rho}\,\overline{\varphi}$ and Corollary 2.2.23.

□

As a consequence of Proposition 2.2.33 we can recover a result that appears in Mackey functors over groups and which is, in general, not true for Mackey functors over fusion systems. We do not prove it in detail since it falls outside the scope of this paper but it's worth sketching a proof.

Remark 2.2.35. Let $P \in \mathcal{F}^c$ and view the functors $\uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ and $\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ as functors between the categories $\mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$ and $\mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ (see Proposition 2.2.30). Then $\uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ is both right and left adjoint to $\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$.

To prove this start by defining the coinduction Mackey functor $\Uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ as the functor that sends any $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}_P)$ to

$$M \Uparrow_{\mathcal{F}_P}^{\mathcal{F}} := \mathrm{Hom}_{\mu_{\mathcal{R}}(\mathcal{F}_P)}\left(\mu_{\mathcal{R}}(\mathcal{F}) \downarrow_{\mathcal{F}_P}^{\mathcal{F}}, M\right) \in \mu_{\mathcal{R}}(\mathcal{F})\text{-mod}\,.$$

Here we are viewing $M \Uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ as a $\mu_{\mathcal{R}}(\mathcal{F})$-module by setting for every $f \in M \Uparrow_{\mathcal{F}_P}^{\mathcal{F}}$, every $y \in \mu_{\mathcal{R}}(\mathcal{F})$ and every $x \in \mu_{\mathcal{R}}(\mathcal{F}) \downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ the image $(y \cdot f)(x) := f(xy)$. It is well known that $\Uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ is the right adjoint of the restriction functor $\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ while $\uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ is its left adjoint. Therefore, proving that $\uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ and $\Uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ coincide on $\mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$ would prove the statement. The broad steps to prove this are as follows. First use the fact that M is $\mathcal{F}$-centric in order to obtain the isomorphism

$$\mathrm{Hom}_{\mu_{\mathcal{R}}(\mathcal{F}_P)}\left(\mu_{\mathcal{R}}(\mathcal{F}) \downarrow_{\mathcal{F}_P}^{\mathcal{F}}, M\right) \cong \mathrm{Hom}_{\mu_{\mathcal{R}}(\mathcal{F}_P)}\left(\mu_{\mathcal{R}}(\mathcal{F}) \downarrow_{\mathcal{F}_P}^{\mathcal{F}} / \mathcal{I} \downarrow_{\mathcal{F}_P}^{\mathcal{F}}, M\right).$$

With $\mathcal{I}$ as in Proposition 2.2.33. Using again Proposition 2.2.33 and the anti involution $\cdot^*$ of $\mu_{\mathcal{R}}(\mathcal{F})$ which sends every $I_{\varphi(C)}^B c_\varphi R_C^A \in \mu_{\mathcal{R}}(\mathcal{F})$ to $\left(I_{\varphi(C)}^B c_\varphi R_C^A\right)^* := I_C^A c_{\varphi^{-1}} R_{\varphi(C)}^B$ it can now be proven, using arguments dual to those of Proposition 2.2.33, that the following is a $\mu_{\mathcal{R}}(\mathcal{F}_P)$ basis of $\mu_{\mathcal{R}}(\mathcal{F}) \downarrow_{\mathcal{F}_P}^{\mathcal{F}} / \mathcal{I} \downarrow_{\mathcal{F}_P}^{\mathcal{F}}$

$$\mathcal{B} := \bigsqcup_{Q \in \mathcal{F}^c} \bigsqcup_{(A,\overline{\varphi}) \in [P \times_{\mathcal{F}} Q]} \left\{\pi\left(c_{\overline{\varphi^{-1}}} R_{\overline{\varphi}(A)}^Q\right)\right\}.$$

Where $\pi\colon \mu_{\mathcal{R}}(\mathcal{F}) \downarrow_{\mathcal{F}_P}^{\mathcal{F}} \to \mu_{\mathcal{R}}(\mathcal{F}) \downarrow_{\mathcal{F}_P}^{\mathcal{F}} / \mathcal{I} \downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ denotes the natural projection. Using this we can now define for every $Q \in \mathcal{F}^c$, every $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]$ and every $x \in I_A^A M$ the $\mu_{\mathcal{R}}(\mathcal{F}_P)$-module morphism $f_{(A,\overline{\varphi})}^x \in M \Uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ that sends every element in $\mathcal{B}$ to 0 except for $\pi\left(c_{\overline{\varphi^{-1}}} R_{\overline{\varphi}(A)}^Q\right)$ which is sent to x. With this notation it can be proven that the $f_{(A,\overline{\varphi})}^x$ form an $\mathcal{R}$-basis of $M \Uparrow_{\mathcal{F}_P}^{\mathcal{F}}$. Finally an isomorphism from $M \Uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ to $M \uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ can be obtained from Proposition 2.2.33 by sending any morphism of the form $f_{(A,\overline{\varphi})}^x$ to $I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} \otimes x \in M \uparrow_{\mathcal{F}_P}^{\mathcal{F}}$. Some care is needed in this last step to prove that this morphism is in fact a morphism of $\mu_{\mathcal{R}}(\mathcal{F})$-modules but Proposition 2.2.18 and Lemma 2.2.34 can be used to this end.

As we show in Subsection 2.4.2 there are at least 2 ways of translating Equation (2.5) to the context of Mackey functors over fusion systems. We are now ready to give the first one.

Lemma 2.2.36. *Let $P, Q \in \mathcal{F}^c$, let $\mathcal{G}$ be a fusion system containing $\mathcal{F}$ and let $M \in$ Mack$^{\mathcal{G}^c}_{\mathcal{R}}(\mathcal{F}_P)$, for every $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]$ fix a representative φ of $\overline{\varphi}$ viewed as an isomorphism onto its image and define $M_{(A,\overline{\varphi})} := {}^{\varphi}\left(M \downarrow^{\mathcal{F}_P}_{\mathcal{F}_A}\right)$. Each $M_{(A,\overline{\varphi})}$ is $\mathcal{G}$-centric and there exists an isomorphism*

$$\begin{array}{ccc} \bigoplus_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} M_{(A,\overline{\varphi})}\uparrow^{\mathcal{F}_Q}_{\mathcal{F}_{\varphi(A)}} & \overset{\Gamma}{\hookrightarrow\!\!\!\!\rightarrow} & M\uparrow^{\mathcal{F}}_{\mathcal{F}_P}\downarrow^{\mathcal{F}}_{\mathcal{F}_Q}, \\ I^J_{\overline{\theta}(C)} c_{\overline{\theta}} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} x & \to & I^J_{\overline{\theta}(C)} c_{\overline{\theta\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} x \end{array} \tag{2.8}$$

where we are viewing φ as an isomorphism between the appropriate restrictions and we are using Proposition 2.2.33 and the fact that $M_{(A,\overline{\varphi})} \in$ Mack$^{\mathcal{G}^c}_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)$ to define Γ via $\mathcal{R}$ linearity by setting its image on elements of the form $I^J_{\overline{\theta}(C)} c_{\overline{\theta}} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} x \in M_{(A,\overline{\varphi})}\uparrow^{\mathcal{F}_Q}_{\mathcal{F}_{\varphi(A)}}$ with $J \in \mathcal{F}_Q \cap \mathcal{F}^c$, $\left(C, \overline{\theta}\right) \in \left[\varphi(A) \times_{\mathcal{F}_Q} J\right]$ such that $C \in \mathcal{F}_Q \cap \mathcal{F}^c$ and $x \in I^{\varphi^{-1}(C)}_{\varphi^{-1}(C)} M = I^C_C M_{(A,\overline{\varphi})}$.

Proof. The fact that each $M_{(A,\overline{\varphi})}$ is $\mathcal{G}$-centric follows from their definition and Proposition 2.2.30.

From Propositions 2.2.18 and 2.2.33 we have the following isomorphism of $\mathcal{R}$-modules

$$\begin{aligned} M\uparrow^{\mathcal{F}}_{\mathcal{F}_P}\downarrow^{\mathcal{F}}_{\mathcal{F}_Q} &\cong_{\mathcal{R}} \bigoplus_{J\in\mathcal{F}_Q\cap\mathcal{F}^c} \bigoplus_{\left(B,\overline{\psi}\right)\in[P\times_{\mathcal{F}}J]} I^J_{\overline{\psi}(B)} c_{\overline{\psi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M, && \text{Proposition 2.2.33} \\ &\cong_{\mathcal{R}} \bigoplus_{\substack{J\in\mathcal{F}_Q\cap\mathcal{F}^c \\ (A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]}} \bigoplus_{\substack{x\in[J\backslash Q/\varphi(A)] \\ J^x\cap\varphi(A)\in\mathcal{F}^c}} I^J_{J\cap^x(\varphi(A))} c_{c_x\varphi} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M, && \text{Proposition 2.2.18 (5)} \\ &\cong_{\mathcal{R}} \bigoplus_{\substack{J\in\mathcal{F}_Q\cap\mathcal{F}^c \\ (A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]}} \bigoplus_{\substack{\left(C,\overline{\theta}\right)\in\left[\varphi(A)\times_{\mathcal{F}_Q}J\right] \\ C\in\mathcal{F}_Q\cap\mathcal{F}^c}} I^J_{\overline{\theta}(C)} c_{\overline{\theta\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M. && \text{Proposition 2.2.18 (2)} \end{aligned}$$

Where each φ is as in the statement. From Proposition 2.2.30 we know that $M_{(A,\overline{\varphi})} \uparrow^{\mathcal{F}}_{\mathcal{F}_{\varphi(A)}}$ is $\mathcal{F}$-centric and, therefore, $I^J_J M_{(A,\overline{\varphi})} \uparrow^{\mathcal{F}_Q}_{\mathcal{F}_{\varphi(A)}} = 0$ for every $J \in \mathcal{F}_Q \backslash (\mathcal{F}_Q \cap \mathcal{F}^c)$. The same argument also tells us that $I^C_C M_{(A,\overline{\varphi})} = 0$ for every $C \in \mathcal{F}_{\varphi(A)} \backslash \left(\mathcal{F}_{\varphi(A)} \cap \mathcal{F}^c\right)$. We can therefore use Proposition 2.2.33 in order to conclude that

$$M_{(A,\overline{\varphi})}\uparrow_{\mathcal{F}_{\varphi(A)}}^{\mathcal{F}_Q} \cong_{\mathcal{R}} \bigoplus_{J\in\mathcal{F}_Q\cap\mathcal{F}^c} \bigoplus_{\substack{(C,\overline{\theta})\in[\varphi(A)\times_{\mathcal{F}_Q} J] \\ C\in\mathcal{F}_Q\cap\mathcal{F}^c}} I_{\overline{\theta}(C)}^{J} c_{\overline{\theta}} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} M_{(A,\overline{\varphi})}.$$

By definition of the functor ${}^{\varphi}\cdot$ (see Definition 2.2.28) we now have that for every $J \in \mathcal{F}_Q \cap \mathcal{F}^c$ and every $(C,\overline{\theta}) \in [\varphi(A) \times_{\mathcal{F}_Q} J]$ such that $C \in \mathcal{F}_Q \cap \mathcal{F}^c$ there is an equivalence of $\mathcal{R}$-modules $I_C^C M_{(A,\overline{\varphi})} \cong_{\mathcal{R}} I_{\varphi^{-1}(C)}^{\varphi^{-1}(C)} M$ realized by sending every $x \in I_C^C M_{(A,\overline{\varphi})}$ to x seen as an element in $I_{\varphi^{-1}(C)}^{\varphi^{-1}(C)} M$. This leads in turn to an equivalence of $\mathcal{R}$-modules $I_{\overline{\theta}(C)}^{J} c_{\overline{\theta}} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} M_{(A,\overline{\varphi})} \cong_{\mathcal{R}} I_{\overline{\theta}(C)}^{J} c_{\overline{\theta\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M$ realized by sending every $I_{\overline{\theta}(C)}^{J} c_{\overline{\theta}} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} x \in I_{\overline{\theta}(C)}^{J} c_{\overline{\theta}} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} M_{(A,\overline{\varphi})}$ to $I_{\overline{\theta}(C)}^{J} c_{\overline{\theta\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} x \in I_{\overline{\theta}(C)}^{J} c_{\overline{\theta\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M$. Therefore, viewing each $I_{\overline{\theta}(C)}^{J} c_{\overline{\theta}} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} M_{(A,\overline{\varphi})}$ as an $\mathcal{R}$-submodule of $M_{(A,\overline{\varphi})}\uparrow_{\mathcal{F}_{\varphi(A)}}^{\mathcal{F}_Q}$ and each $I_{\overline{\theta}(C)}^{J} c_{\overline{\theta\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} M$ as an $\mathcal{R}$-submodule of $M \uparrow_{\mathcal{F}_P}^{\mathcal{F}}\downarrow_{\mathcal{F}_Q}^{\mathcal{F}}$, we can conclude that the morphism Γ of the statement is a bijective $\mathcal{R}$-module morphism. We are now just left with proving that Γ is a morphism of $\mu_{\mathcal{R}}(\mathcal{F}_Q)$-modules. Take $(A,\overline{\varphi}), J, (C,\overline{\theta})$ and x as in the statement and let $J' \in \mathcal{F}_Q \cap \mathcal{F}^c$ such that $J' \leq J$. Then we have that

$$\begin{aligned}
R_{J'}^{J}\Gamma\left(I_{\overline{\theta}(C)}^{J} c_{\overline{\theta}} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} x\right) &= \sum_{(B,\overline{\psi})} I_{\overline{\psi}(B)}^{J} c_{\overline{\psi}} c_{\overline{\left(\gamma_{(B,\overline{\psi})}^{J}\right)^{-1}}} R_{\gamma_{(B,\overline{\psi})}^{J}(B)}^{C} c_{\varphi} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} x, \\
&= \sum_{(B,\overline{\psi})} I_{\overline{\psi}(B)}^{J} c_{\overline{\psi}} c_{\overline{\left(\gamma_{(B,\overline{\psi})}^{J}\right)^{-1}}} R_{\gamma_{(B,\overline{\psi})}^{J}(B)}^{C} c_{\varphi} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} x, \\
&= \sum_{(B,\overline{\psi})} I_{\overline{\psi}(B)}^{J} c_{\overline{\psi}} c_{\varphi} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} c_{\varphi^{-1}} c_{\overline{\left(\gamma_{(B,\overline{\psi})}^{J}\right)^{-1}}} R_{\gamma_{(B,\overline{\psi})}^{J}(B)}^{C} c_{\varphi} \cdot x, \\
&= \sum_{(B,\overline{\psi})} \Gamma\left(I_{\overline{\psi}(B)}^{J} c_{\overline{\psi}} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} c_{\overline{\left(\gamma_{(B,\overline{\psi})}^{J}\right)^{-1}}} R_{\gamma_{(B,\overline{\psi})}^{J}(B)}^{C} \cdot x\right), \\
&= \Gamma\left(R_{J'}^{J} I_{\overline{\theta}(C)}^{J} c_{\overline{\theta}} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} x\right).
\end{aligned}$$

Where the $(B,\overline{\psi})$ are iterating over the elements in $[P \times_{\mathcal{F}_Q} J]$ such that $\left(B^J, \overline{\psi^J}\right) = (C,\overline{\theta})$, we are taking φ as in the statement, in the first and second identities we are using Items (3) and (1) of Lemma 2.2.34 respectively, in the third identity we are using the definition of $M_{(A,\overline{\varphi})}$ and in the last identity we are repeating the same operations backwards.

Let now $J' \in \mathcal{F}_Q \cap \mathcal{F}^c$ such that $J' \geq J$ and let $\rho\colon J \to \rho(J)$ be an isomorphism in $\mathcal{F}_Q$. The same arguments used above but now replacing Item (3) of Lemma 2.2.34 with Items (2) and (4) (which remove the sum thus making the operations simpler to carry) we obtain the identities below

$$I_J^{J'}\Gamma\left(I_{\overline{\theta}(C)}^{J}c_{\overline{\theta}}\otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)}x\right)=\Gamma\left(I_J^{J'}I_{\overline{\theta}(C)}^{J}c_{\overline{\theta}}\otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)}x\right),$$
$$c_\rho\Gamma\left(I_{\overline{\theta}(C)}^{J}c_{\overline{\theta}}\otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)}x\right)=\Gamma\left(c_\rho I_{\overline{\theta}(C)}^{J}c_{\overline{\theta}}\otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)}x\right).$$

This proves that Γ is indeed an $\mu_{\mathcal{R}}(\mathcal{F}_Q)$-module morphism thus concluding the proof. □

Using Proposition 2.2.33 we can now define a morphism θ^P from a centric Mackey functor M over $\mathcal{F}$ to the centric Mackey functor $M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ by setting for every $Q \in \mathcal{F}^c$ and every $x \in I_Q^Q M$

$$\theta_M^P(x) := \sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} I_{\overline{\varphi}(A)}^{Q}c_{\overline{\varphi}}\otimes c_{\overline{\varphi^{-1}}}R_{\overline{\varphi}(A)}^{Q}x.$$

Since the tensor product is over $\mu_{\mathcal{R}}(\mathcal{F}_P)$ we know that θ^P does not depend on the choice of $[P\times_{\mathcal{F}}Q]$. Thus we can conclude that it is well defined and an $\mathcal{R}$-module morphism. Let $Q \in \mathcal{F}^c$, let $x \in I_Q^Q M$ and let $\rho\colon Q \to \rho(Q)$ be an isomorphism in $\mathcal{F}$. Applying Items (1) and (4) of Lemma 2.2.34 we have that

$$\begin{aligned}
c_\rho\theta_M^P(x) &= \sum_{\left(B,\overline{\psi}\right)\in[P\times_{\mathcal{F}}Q]} I_{\overline{\psi^\rho}(B)}^{Q}c_{\overline{\psi^\rho}}c_{\gamma^\rho_{(B,\overline{\psi})}}\otimes c_{\overline{\psi^{-1}}}R_{\overline{\psi}(B)}^{Q}x,\\
&= \sum_{\left(B,\overline{\psi}\right)\in[P\times_{\mathcal{F}}Q]} I_{\overline{\psi^\rho}(B)}^{Q}c_{\overline{\psi^\rho}}\otimes c_{\left(\gamma^{\rho^{-1}}_{(B,\overline{\psi})}\right)^{-1}}c_{\overline{\psi^{-1}}}R_{\overline{\psi}(B)}^{Q}x,\\
&= \sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}\rho(Q)]} I_{\overline{\varphi}(A)}^{\rho(Q)}c_{\overline{\varphi}}\otimes c_{\overline{\varphi^{-1}}}R_{\overline{\varphi}(A)}^{Q}c_\rho x=\theta_M^P(c_\rho x).
\end{aligned}$$

With the same notation as above let $J \in \mathcal{F}^c$ such that $J \geq Q$ then we have that

$$\begin{aligned}
\theta_M^P\left(I_Q^J x\right) &= \sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} I_{\overline{\varphi^J}(A^J)}^{J}c_{\overline{\varphi^J}}\otimes I_{\gamma^J_{(A,\overline{\varphi})}(A)}^{A^J}c_{\gamma^J_{(A,\overline{\varphi})}}c_{\overline{\varphi^{-1}}}R_{\overline{\varphi}(A)}^{K}x,\\
&= \sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} I_Q^J I_{\overline{\varphi}(A)}^{Q}c_{\overline{\varphi}}\otimes c_{\overline{\varphi^{-1}}}R_{\overline{\varphi}(A)}^{Q}x=I_Q^J\theta_M^P(x).
\end{aligned}$$

Where, in the first identity, we are using Lemma 2.2.34 (3) together with the fact that M is $\mathcal{G}$-centric and, therefore, annihilated by the ideal $\mathcal{I}$ of Lemma 2.2.34 and, in the second identity, we are using Lemma 2.2.34 (1) to move things from one side of the tensor product to the other and Lemma 2.2.34 (2) to simplify the equation. If $J \in \mathcal{F}^c$ is such that $J \leq Q$ then the exact same arguments (but starting with $R_J^Q \theta_M^P(x)$ instead of $\theta_M^P\left(R_J^Q x\right)$) prove that θ_M^P also commutes with restriction. We can therefore conclude that θ_M^P is a morphism of $\mu_{\mathcal{R}}(\mathcal{F})$-modules for every $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$. This allows us to give the following definition with which we conclude this subsection.

Definition 2.2.37. Let $\mathcal{G}$ be a fusion system (not necessarily over S) containing $\mathcal{F}$, let $M \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{G}^c}(\mathcal{F})$ and let $P \in \mathcal{F}^c$. From Proposition 2.2.30 we know that the following is a $\mathcal{G}$-centric Mackey functor over $\mathcal{F}$

$$M_P := M \downarrow_{\mathcal{F}_P}^{\mathcal{F}} \uparrow_{\mathcal{F}_P}^{\mathcal{F}} .$$

Thus the above discussion allows us to define the Mackey functor morphisms

$$\theta_P^M : M_P \to M, \qquad \theta_M^P : M \to M_P,$$

by setting for every $y \otimes x \in M_P$, every $Q \in \mathcal{F}^c$ and every $z \in I_Q^Q M$

$$\theta_P^M(y \otimes x) := y \cdot x, \qquad \theta_M^P(z) := \sum_{(A,\overline{\varphi}) \in [P \times_{\mathcal{F}} Q]} I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} \otimes c_{\overline{\varphi^{-1}}} R_{\overline{\varphi}(A)}^Q z.$$

If there is no possible confusion regarding M we write $\theta_P := \theta_P^M$ and $\theta^P := \theta_M^P$.

2.2.3 The centric Burnside ring over a fusion system

Through this subsection we will be using Notations 2.1.1, 2.2.3, 2.2.9, 2.2.21 and 2.2.31.

Let G be a finite group. It is known (see [TW95, Proposition 9.2]) that the Burnside ring of G can be embedded in the center of the Mackey algebra of G. In this subsection we prove that there exists a similar embedding of the centric Burnside ring of $\mathcal{F}$ into the center of a certain quotient of $\mu_{\mathcal{R}}(\mathcal{F})$ (see Proposition 2.2.43).

Let us start by recalling the definition of centric Burnside ring of a fusion system.

Definition 2.2.38 ([DL09, Definition 2.11]). The **centric Burnside ring of $\mathcal{F}$** (denoted by $B^{\mathcal{F}^c}$) is the Grothendieck group of the semigroup whose elements are isomorphism classes of $\mathcal{O}(\mathcal{F}^c)_{\sqcup}$ and addition is given by taking the isomorphism class of the coproduct of two representatives. This is doted with a ring structure by taking multiplication of two isomorphism classes to be the isomorphism class of the product of two of their representatives and extending by linearity. Given a commutative ring $\mathcal{R}$ we also define the **centric Burnside ring of $\mathcal{F}$ on $\mathcal{R}$** as

$$B_{\mathcal{R}}^{\mathcal{F}^c} := \mathcal{R} \otimes_{\mathbb{Z}} B^{\mathcal{F}^c}.$$

An important distinction between the ring $B_{\mathcal{R}}^{\mathcal{F}^c}$ and the Burnside ring of a group is that, in general, the isomorphism class $\overline{S}$ of S is not the identity in $B_{\mathcal{R}}^{\mathcal{F}^c}$. However, we have the following result due to Sune Reeh.

Proposition 2.2.39. *If every integer prime other than p is invertible in $\mathcal{R}$ then the isomorphism class $\overline{S}$ of S is invertible in $B_{\mathcal{R}}^{\mathcal{F}^c}$.*

Proof. See [Pr16, Proposition 4.13]. □

This result motivates the following definition.

Definition 2.2.40. We say that a ring $\mathcal{R}$ is **p-local** if all integer primes other than p are invertible in $\mathcal{R}$.

Remark 2.2.41. The definition of p-local ring does not specify if p is invertible or not. This distinction is not relevant towards the results shown in this paper. It is however worth noting that, if $\mathcal{R}$ is a field of characteristic 0, then arguments analogous to those of [TW90, Theorem 9.1] can be used in order to prove that $\mu_{\mathcal{R}}(\mathcal{F})$ is a semisimple $\mathcal{R}$-algebra. The exact condition is in fact for $\mathcal{R}$ to be a field where $|\mathrm{Aut}_{\mathcal{F}}(P)|$ is invertible for every $P \leq S$.

Before proceeding let us recall precisely how the Burnside ring of a finite group G embeds into the center of the Mackey algebra. Let G be a finite group and let $\mathcal{R}$ be a commutative ring, [TW95, Proposition 9.2] describes the above mentioned embedding as the map that, for every $H \leq G$, sends the isomorphism class $\overline{G/H}$ of the transitive G-set G/H to

$$\overline{G/H} \to \sum_{K \leq G} \sum_{x \in [K \backslash G/H]} I^K_{K \cap {}^x H} R^K_{K \cap {}^x H} \in Z\left(\mu_{\mathcal{R}}\left(G\right)\right).$$

This embedding leads to an action of the Burnside ring of G on any Mackey functor over G. When trying to obtain a similar result for the case of Mackey functors over fusion systems many difficulties arise. These can, once again, be traced back to the fact that the category $\mathcal{O}\left(\mathcal{F}\right)_{\sqcup}$ does not in general admit products. However, we have the following results with which we conclude this section.

Lemma 2.2.42. *Let $\mathcal{I}$ be the two sided ideal of $\mu_{\mathcal{R}}\left(\mathcal{F}\right)$ defined in Proposition 2.2.33, define $\mu_{\mathcal{R}}\left(\mathcal{F}^c\right) := \mu_{\mathcal{R}}\left(\mathcal{F}\right)/\mathcal{I}$ and denote by $\pi\colon \mu_{\mathcal{R}}\left(\mathcal{F}\right) \twoheadrightarrow \mu_{\mathcal{R}}\left(\mathcal{F}^c\right)$ the natural projection. The $\mathcal{R}$-algebra $\mu_{\mathcal{R}}\left(\mathcal{F}^c\right)$ is naturally equipped with a $\mu_{\mathcal{R}}\left(\mathcal{F}\right)$-module structure given by setting $y \cdot \pi\left(x\right) = \pi\left(y\right)\pi\left(x\right)$ for every $x, y \in \mu_{\mathcal{R}}\left(\mathcal{F}\right)$. Moreover, for every $P \in \mathcal{F}\backslash\mathcal{F}^c$ we have that $I^P_P \in \mathcal{I}$ and, therefore, $I^P_P \mu_{\mathcal{R}}\left(\mathcal{F}^c\right) = 0$. Thus we can view $\mu_{\mathcal{R}}\left(\mathcal{F}^c\right)$ as a centric Mackey functor over $\mathcal{F}$ and, using Definition 2.2.37, we can define*

$$\Gamma\left(P\right) := \theta_P^{\mu_{\mathcal{R}}\left(\mathcal{F}^c\right)}\left(\theta^P_{\mu_{\mathcal{R}}\left(\mathcal{F}^c\right)}\left(1_{\mu_{\mathcal{R}}\left(\mathcal{F}^c\right)}\right)\right) \in \mu_{\mathcal{R}}\left(\mathcal{F}^c\right)$$

for every $P \in \mathcal{F}^c$. With this setup we have that:

(1) $\Gamma\left(P\right)$ is in the center of the $\mathcal{R}$-algebra $\mu_{\mathcal{R}}\left(\mathcal{F}^c\right)$ and

$$\Gamma\left(P\right) = \sum_{J \in \mathcal{F}^c} \sum_{(A,\overline{\varphi}) \in [J \times P]} \pi\left(I^J_A R^J_A\right).$$

(2) For every $P' =_{\mathcal{F}} P$ (see Notation 2.2.3) we have that $\Gamma\left(P'\right) = \Gamma\left(P\right)$.

(3) For every $Q \in \mathcal{F}^c$ we have that

$$\Gamma\left(Q\right)\Gamma\left(P\right) = \sum_{(A,\overline{\varphi}) \in [Q \times P]} \Gamma\left(A\right).$$

Proof.

(1) For every $P \in \mathcal{F}^c$, we have that

$$\Gamma\left(P\right) = \sum_{Q \in \mathcal{F}^c} \sum_{(A,\overline{\varphi}) \in [P \times Q]} \pi\left(I^Q_{\varphi(A)} c_{\varphi} c_{\varphi^{-1}} R^Q_{\varphi(A)}\right) = \sum_{Q \in \mathcal{F}^c} \sum_{\left(B,\overline{\psi}\right) \in [Q \times P]} \pi\left(I^Q_B R^Q_B\right).$$

Where, for the first identity, we are using the fact that $1_{\mu_{\mathcal{R}}\left(\mathcal{F}^c\right)} = \sum_{Q \in \mathcal{F}^c} \pi\left(I^Q_Q\right)$ and Corollary 2.2.23 in order to take φ to be any representative of $\overline{\varphi}$ and view it as an isomorphism onto its image. For the second identity we are using Items (1) and (2) of Lemma 2.2.22 in order to remove $c_{\varphi}c_{\varphi^{-1}}$ and Proposition 2.2.18 (1) in order to rewrite the sum. This proves the second part of Item (1). For the first part recall from Definition 2.2.37 that both $\theta^{P}_{\mu_{\mathcal{R}}(\mathcal{F}^c)}$ and $\theta_{P}^{\mu_{\mathcal{R}}(\mathcal{F}^c)}$ are morphisms of Mackey functors and, since the projection π is an $\mathcal{R}$-algebra morphism, we also know that $\pi\left(1_{\mu_{\mathcal{R}}(\mathcal{F})}\right) = 1_{\mu_{\mathcal{R}}(\mathcal{F}^c)}$. We can therefore conclude that, for every $x \in \mu_{\mathcal{R}}(\mathcal{F})$, we have

$$\pi(x)\,\Gamma(P) = x \cdot \theta_{P}^{\mu_{\mathcal{R}}(\mathcal{F}^c)}\left(\theta^{P}_{\mu_{\mathcal{R}}(\mathcal{F}^c)}\left(\pi\left(1_{\mu_{\mathcal{R}}(\mathcal{F})}\right)\right)\right) = \theta_{P}^{\mu_{\mathcal{R}}(\mathcal{F}^c)}\left(\theta^{P}_{\mu_{\mathcal{R}}(\mathcal{F}^c)}\left(\pi(x)\right)\right).$$

The fact that $\Gamma(P)$ is in the center of $\mu_{\mathcal{R}}(\mathcal{F}^c)$ now follows from definition of $\theta^{P}_{\mu_{\mathcal{R}}(\mathcal{F}^c)}$ and $\theta_{P}^{\mu_{\mathcal{R}}(\mathcal{F}^c)}$ via the identities below

$$\begin{aligned}
\Gamma(P)\,\pi(x) &= \left(\sum_{Q\in\mathcal{F}^c}\sum_{(A,\overline{\varphi})\in[P\times Q]} I^{Q}_{\varphi(A)}c_{\varphi}c_{\varphi^{-1}}R^{Q}_{\varphi(A)}\cdot\pi\left(I^{Q}_{Q}\right)\right)\pi(x),\\
&= \left(\sum_{Q\in\mathcal{F}^c}\sum_{(A,\overline{\varphi})\in[P\times Q]} I^{Q}_{\varphi(A)}c_{\varphi}c_{\varphi^{-1}}R^{Q}_{\varphi(A)}\cdot\pi\left(I^{Q}_{Q}x\right)\right),\\
&= \theta_{P}^{\mu_{\mathcal{R}}(\mathcal{F}^c)}\left(\theta^{H}_{\mu_{\mathcal{R}}(\mathcal{F}^c)}\left(\pi(x)\right)\right) = \pi(x)\,\Gamma(P).
\end{aligned}$$

(2) Let $\psi\colon P \hookrightarrow\!\!\!\!\rightarrow HP$ be an isomorphism in $\mathcal{F}$. Item (2) follows from Item (1) and Proposition 2.2.18 (5) via the identities below

$$\Gamma(P) = \sum_{Q\in\mathcal{F}^c}\sum_{(A,\overline{\varphi})\in[Q\times P]}\pi\left(I^{Q}_{A}R^{Q}_{A}\right) = \sum_{Q\in\mathcal{F}^c}\sum_{\left(A,\overline{\psi\varphi}\right)\in[Q\times P']}\pi\left(I^{Q}_{A}R^{Q}_{A}\right) = \Gamma(P').$$

(3) Item (3) follows from the identities below

$$\begin{aligned}
\Gamma(Q)\,\Gamma(P) &= \sum_{J\in\mathcal{F}^c}\sum_{\substack{(A,\overline{\varphi})\in[J\times P]\\ \left(B,\overline{\psi}\right)\in[J\times Q]}}\pi\left(I^{J}_{B}R^{J}_{B}I^{J}_{A}R^{J}_{A}\right),\\
&= \sum_{J\in\mathcal{F}^c}\sum_{\substack{(A,\overline{\varphi})\in[J\times P]\\ \left(B,\overline{\psi}\right)\in[J\times Q]}}\sum_{\substack{x\in[B\backslash J/A]\\ B^x\cap A\in\mathcal{F}^c}}\pi\left(I^{J}_{B^x\cap A}R^{J}_{B^x\cap A}\right),\\
&= \sum_{\left(C,\overline{\theta}\right)\in[P\times Q]}\sum_{J\in\mathcal{F}^c}\sum_{(D,\overline{\gamma})\in[J\times C]}\pi\left(I^{J}_{D}R^{J}_{D}\right) = \sum_{\left(C,\overline{\theta}\right)\in[P\times Q]}\Gamma(C).
\end{aligned}$$

Where, for the first identity, we are using the fact that π is a morphism of $\mathcal{R}$-algebras and Lemma 2.2.22 (5), for the second identity, we are using Lemma 2.2.22 (4) and definition of $\mathcal{I}$ and, for the third identity, we are using Proposition 2.2.18 (7).

□

Proposition 2.2.43. *Let p be a prime, let S be a finite p-group, let $\mathcal{F}$ be a saturated fusion system over S, let $\mathcal{R}$ be a commutative ring with unit and let $\mathcal{I}, \mu_{\mathcal{R}}(\mathcal{F}^c)$ and Γ be as in Lemma 2.2.42. For every $X \in \mathcal{O}(\mathcal{F}^c)_{\sqcup}$ (see Definition 2.2.12) denote by $\overline{X} \in B_{\mathcal{R}}^{\mathcal{F}^c}$ (see Definition 2.2.38) its isomorphism class and define the (non necessarily unit preserving) $\mathcal{R}$-algebra morphism $\overline{\Gamma} : B_{\mathcal{R}}^{\mathcal{F}^c} \to \mu_{\mathcal{R}}(\mathcal{F}^c)$ by setting $\overline{\Gamma}\left(\overline{P}\right) := \Gamma(P)$ for every $P \in \mathcal{F}^c$ and extending by $\mathcal{R}$-linearity. If $B_{\mathcal{R}}^{\mathcal{F}^c}$ contains a non-zero divisor then $\overline{\Gamma}$ is injective and, if $\mathcal{R}$ is p-local (see Definition 2.2.40), then $B_{\mathcal{R}}^{\mathcal{F}^c}$ contains a unit (see Proposition 2.2.39) and $\overline{\Gamma}\left(1_{B_{\mathcal{R}}^{\mathcal{F}^c}}\right) = 1_{\mu_{\mathcal{R}}(\mathcal{F}^c)}$. Moreover, if $\mathcal{R}$ is p-local, then, for every fusion system $\mathcal{G}$ containing $\mathcal{F}$ and every $M \in Mack_{\mathcal{R}}^{\mathcal{G}^c}(\mathcal{F}) \subseteq Mack_{\mathcal{R}}(\mathcal{F}^c)$ (see Definition 2.2.29), the ring $B_{\mathcal{R}}^{\mathcal{F}^c}$ acts on M by defining for every $P \in \mathcal{F}^c$*

$$\overline{P}\cdot := \theta_P^M \theta_M^P \in \operatorname{End}(M).$$

Where we are using the notation of Definition 2.2.37.

Proof. From Items (2) and (3) of Lemma 2.2.42 we know that $\overline{\Gamma}$ is a well defined (non necessarily unit preserving) $\mathcal{R}$-algebra morphism.

Viewing $\mu_{\mathcal{R}}(\mathcal{F}_S)$ as a subset of $\mu_{\mathcal{R}}(\mathcal{F})$ (see Corollary 2.2.25) we can define the $\mathcal{R}$-algebra morphism $\varUpsilon$ from $\pi\left(I_S^S \mu_{\mathcal{R}}(\mathcal{F}_S) I_S^S\right)$ to $\operatorname{End}\left(B_{\mathcal{R}}^{\mathcal{F}^c}\right)$ by setting for every $\pi\left(I_Q^S R_Q^S\right) \in \pi\left(I_S^S \mu_{\mathcal{R}}(\mathcal{F}_S) I_S^S\right)$ and every $P \in \mathcal{F}^c$

$$\varUpsilon\left(\pi\left(I_Q^S R_Q^S\right)\right)\left(\overline{P}\right) := \sum_{\substack{x \in [Q \backslash S / P] \\ Q^x \cap P \in \mathcal{F}^c}} \overline{Q^x \cap P},$$

From Proposition 2.2.24 and Items (1) and (3) of Lemma 2.2.22 we know that this is sufficient to define $\varUpsilon$ via $\mathcal{R}$-linearity. From Lemma 2.2.42 (1) and definition of $\overline{\Gamma}$ we also know that $\overline{\Gamma}\left(B_{\mathcal{R}}^{\mathcal{F}^c}\right) \subseteq \pi\left(\mu_{\mathcal{R}}(\mathcal{F}_S)\right)$. Therefore we can define $\varUpsilon' : \overline{\Gamma}\left(B_{\mathcal{R}}^{\mathcal{F}^c}\right) \to \operatorname{End}\left(B_{\mathcal{R}}^{\mathcal{F}^c}\right)$ by setting $\varUpsilon'(x) = \varUpsilon\left(\pi\left(I_S^S\right) x \pi\left(I_S^S\right)\right)$ for every $x \in B_{\mathcal{R}}^{\mathcal{F}^c}$. With this setup we can conclude from Proposition 2.2.18 (5) and Lemma 2.2.42 (1) that $\Upsilon'\left(\Gamma\left(\overline{P}\right)\right)\left(\overline{Q}\right) = \overline{P\times Q}$ for every $P, Q\in\mathcal{F}^c$.

Assume now that $B_{\mathcal{R}}^{\mathcal{F}^c}$ admits a non zero divisor $\overline{\Omega}$. Then, for every $\overline{\Psi},\overline{\Phi}\in B_{\mathcal{R}}^{\mathcal{F}^c}$, we have that

$$\Upsilon'\left(\Gamma\left(\overline{\Psi}\right)\right)\left(\overline{\Omega}\right) = \Upsilon'\left(\Gamma\left(\overline{\Phi}\right)\right)\left(\overline{\Omega}\right) \Rightarrow \overline{\Psi\times\Omega} = \overline{\Phi\times\Omega} \Rightarrow \overline{\Psi} = \overline{\Phi}.$$

This proves that the composition $\Upsilon'\Gamma$ is injective and, in particular, that Γ is injective.

Assume now that $\mathcal{R}$ is p-local. By Proposition 2.2.39 we know that $B_{\mathcal{R}}^{\mathcal{F}^c}$ admits a unit. Let us denote by $1_{B_{\mathcal{R}}^{\mathcal{F}^c}} := \sum_{Q\in\mathcal{F}^c}\lambda_Q\overline{Q}$ this unit. From Lemma 2.2.42 (1) and definition of product in $B_{\mathcal{R}}^{\mathcal{F}^c}$ we have that

$$\overline{\Gamma}\left(1_{B_{\mathcal{R}}^{\mathcal{F}^c}}\right) = \sum_{P,Q\in\mathcal{F}^c}\sum_{(A,\overline{\varphi})\in[P\times Q]}\lambda_Q\pi\left(I_A^PR_A^P\right) = \sum_{P\in\mathcal{F}^c}\pi\left(I_P^P\right) = 1_{\mu_{\mathcal{R}}(\mathcal{F}^c)}.$$

Finally, for every $M\in\text{Mack}_{\mathcal{R}}^{\mathcal{G}^c}(\mathcal{F})$, we have that $M\in\text{Mack}_{\mathcal{R}}(\mathcal{F}^c)$. Therefore, by definition of $\mathcal{I}$, we have that $\mathcal{I}M = 0$. In particular M acquires a $\mu_{\mathcal{R}}(\mathcal{F}^c)$-module structure by setting $\pi(y)\cdot x = y\cdot x$ for every $y\in\mu_{\mathcal{R}}(\mathcal{F})$ and every $x\in M$. This leads us to the equivalence of $\mathcal{R}$-algebras $\text{End}(M) := \text{End}_{\mu_{\mathcal{R}}(\mathcal{F})}(M)\cong\text{End}_{\mu_{\mathcal{R}}(\mathcal{F}^c)}(M)$. Notice now that there exists a natural map $\Theta\colon Z(\mu_{\mathcal{R}}(\mathcal{F}^c))\to\text{End}(M)$ defined by setting $\Theta(y)(x) = y\cdot x$ for every $y\in Z(\mu_{\mathcal{R}}(\mathcal{F}^c))$ and every $x\in M$. With this notation we can define $\overline{\Omega}\cdot := \Theta\left(\overline{\Gamma}\left(\overline{\Omega}\right)\right)\in\text{End}(M)$ for every $\overline{\Omega}\in B_{\mathcal{R}}^{\mathcal{F}^c}$. Then, for every $P\in\mathcal{F}^c$ and every $x\in M$, we have that

$$\begin{aligned}\overline{P}\cdot x &= \theta_P^{\mu_{\mathcal{R}}(\mathcal{F}^c)}\left(\theta_{\mu_{\mathcal{R}}(\mathcal{F}^c)}^P\left(1_{\mu_{\mathcal{R}}(\mathcal{F}^c)}\right)\right)\cdot x,\\ &= \sum_{Q\in\mathcal{F}^c}\sum_{(A,\overline{\varphi})\in[P\times Q]}\pi\left(I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}}c_{\overline{\varphi^{-1}}}R_{\overline{\varphi}(A)}^Q\right)\cdot x,\\ &= \sum_{Q\in\mathcal{F}^c}\sum_{(A,\overline{\varphi})\in[P\times Q]} I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}}c_{\overline{\varphi^{-1}}}R_{\overline{\varphi}(A)}^Q\cdot x = \theta_P^M\left(\theta_M^P(x)\right).\end{aligned}$$

Where, in the last identity, we are using the fact that $M\in\text{Mack}_{\mathcal{R}}^{\mathcal{G}^c}(\mathcal{F})\subseteq\text{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ and, in particular $\sum_{Q\in\mathcal{F}^c}I_Q^Q\cdot x = x$. This concludes the proof. □

2.3 Relative projectivity and Higman's criterion

Through this section we will be using Notations 2.1.1, 2.2.3, 2.2.9, 2.2.21 and 2.2.31.

Let G be a finite group and let M be a Mackey functor over G on $\mathcal{R}$. It is known (see [We00, Section 3]) that there exists a minimal family $\mathcal{X}_M$ of subgroups of G closed under G-subconjugacy such that M is a direct summand of $\bigoplus_{H\in\mathcal{X}_M} M\downarrow_H^G\uparrow_H^G$. If $\mathcal{R}$ is a complete local PID then the Krull-Schmidt-Azumaya theorem (see [CR81, Theorem 6.12 (ii)]) allows us to use this fact in order to obtain a decomposition of M of the form $M\cong\bigoplus_{H\in\mathcal{X}_M} N^H$ where each N^H is a (possibly 0) direct summand of $M\downarrow_H^G\uparrow_H^G$. From this decomposition and minimality of $\mathcal{X}_M$ it follows that, whenever M is indecomposable, then $\mathcal{X}_M$ is generated by a single element that we call vertex. This fact is essential in order to describe the Green correspondence and, during this section, we prove that a similar process can be applied to centric Mackey functors over fusion systems. Moreover we prove that Higman's criterion (see [NT89, Theorem 2.2]) can be translated to the context of centric Mackey functors over fusion systems (see Theorem 2.3.17). This provides us with a link between the vertex of an indecomposable $M\in\mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ and certain ideals of $\mathrm{End}(M)$. Such link turns out to be essential towards proving the Green correspondence for centric Mackey functors.

2.3.1 The defect set

During this subsection we translate the notion of relative projectivity (see [We00, Section 3]) to the context of centric Mackey functors over a fusion system (see Definition 2.3.1). We also prove that, if $\mathcal{R}$ is p-local, the notions of defect set and vertex (see [We00, Section 3]) can also be translated to the context of centric Mackey functors over fusion systems (see Definition 2.3.7).

Definition 2.3.1. Let $\mathcal{G}$ be a fusion system containing $\mathcal{F}$, let $M\in\mathsf{Mack}_{\mathcal{R}}^{\mathcal{G}^c}(\mathcal{F})$ and let $\mathcal{X}$ be a family of $\mathcal{F}$-centric subgroups of S. With notation as in Definition 2.2.37 we define

$$M_{\mathcal{X}}:=\bigoplus_{P\in\mathcal{X}} M_P,\qquad \theta_{\mathcal{X}}^M:=\sum_{P\in\mathcal{X}}\theta_P^M: M_{\mathcal{X}}\to M,\qquad \theta_M^{\mathcal{X}}:=\sum_{P\in\mathcal{X}}\theta_M^P: M\to M_{\mathcal{X}}.$$

If there is no possible confusion regarding M we write $\theta_{\mathcal{X}} := \theta_{\mathcal{X}}^{M}$ and $\theta^{\mathcal{X}} := \theta_{M}^{\mathcal{X}}$. We say that M is **projective relative to** $\mathcal{X}$ (or $\mathcal{X}$-**projective**) if $\theta_{\mathcal{X}}$ is split surjective. If $\mathcal{X} = \{P\}$ for some $P \in \mathcal{F}^c$ we simply say that M is **projective relative to** P (or P-**projective**).

There is a key difference between the above definition of relative projectivity and the one given in the case of Mackey functors over finite groups (see [We00, Section 3]). Let G be a finite group and let M be a Mackey functor over G. In this case we have that $M_G := M \downarrow_G^G \uparrow_G^G \cong M$ and that $\theta_G = \mathrm{Id}_M$. In particular θ_G splits and, therefore, any Mackey functor over G is projective relative to G. This result is however lost in the case of Mackey functors over fusion systems since, given $N \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$, we do not, in general, have $N_S \cong N$ (unless $\mathcal{F} = \mathcal{F}_S$). We do however have the following.

Lemma 2.3.2. *Let $\mathcal{G}$ be a fusion system containing $\mathcal{F}$, let $\mathcal{R}$ be p-local and let $M \in$ $\mathsf{Mack}_{\mathcal{R}}^{\mathcal{G}^c}(\mathcal{F})$. Then M is S-projective.*

Proof. Since $\mathcal{F} \subseteq \mathcal{G}$ then all $\mathcal{G}$-centric subgroups of S are also $\mathcal{F}$-centric. In particular we have that $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$. Since $\mathcal{R}$ is p-local, from Proposition 2.2.39, we know that the centric Burnside ring $B_{\mathcal{R}}^{\mathcal{F}^c}$ contains an inverse of $\overline{S}$. Then, with notation as in Proposition 2.2.43 we have that

$$\theta_S \theta^S \overline{S}^{-1} \cdot = \left(\overline{S} \cdot\right) \left(\overline{S}^{-1} \cdot\right) = 1_{B_{\mathcal{R}}^{\mathcal{F}^c}} \cdot = \mathrm{Id}_M .$$

This proves that θ_S is split surjective or, equivalently, that M is S-projective thus concluding the proof. ☐

This last result tells us that, whenever $\mathcal{R}$ is p-local, any centric Mackey functor is projective relative to some family of $\mathcal{F}$-centric subgroups of S (namely $\{S\}$). We would now like for this family to be unique under certain minimality conditions and use this uniqueness to define the defect set. In the case of Mackey functors over finite groups this uniqueness follows from [We00, Lemma 3.2 and Proposition 3.3]. In order to translate these results to the context of centric Mackey functors over fusion systems we first need the following.

Lemma 2.3.3. *Let $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$, let $\mathcal{X}$ and $\mathcal{Y}$ be families of objects in $\mathcal{F}^c$, let $\sigma \colon \mathcal{X} \to \mathcal{Y}$ be a map between sets and let $\Phi = \{\overline{\varphi_P} : P \to \sigma(P)\}_{P \in \mathcal{X}}$ be a family* *of morphisms in $\mathcal{O}(\mathcal{F}^c)$. There exists a (non necessarily unique) morphism of $\mu_{\mathcal{R}}(\mathcal{F})$-modules $\theta_\Phi : M_{\mathcal{X}} \to M_{\mathcal{Y}}$ such that $\theta_{\mathcal{X}} = \theta_{\mathcal{Y}}\theta_\Phi$. In particular, if M is $\mathcal{X}$-projective, then it is also $\mathcal{Y}$-projective.*

Proof. Because of the direct sum decomposition of $M_{\mathcal{X}}$ and $M_{\mathcal{Y}}$ given in Definition 2.3.1 it suffices to prove the claim in the case where $\mathcal{X} := \{P\}$, $\mathcal{Y} := \{Q\}$ and $\Phi := \{\overline{\varphi} : P \to Q\}$ for some $P, Q \in \mathcal{F}^c$ and some $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F}^c)}(P, Q)$.

Fix a representative φ of $\overline{\varphi}$ and view it as an isomorphism onto its image. Then, for every $x \in P$ we have that $\varphi c_x = c_{\varphi(x)}\varphi$ as isomorphisms from P to $\varphi(P)$. With this in mind Items (2) and (3) of Lemma 2.2.22 tell us that, for every $I^B_{{}^xC} c_{c_x} R^A_C \in \mathcal{F}_P$, we have

$$c_{\varphi,B} I^B_{{}^xC} c_{c_x} R^A_C = I^{\varphi(B)}_{{}^{\varphi(x)}(\varphi(C))} c_{c_{\varphi(x)}} R^{\varphi(A)}_{\varphi(C)} c_{\varphi,A} \in \mathcal{F}_{\varphi(P)} c_{\varphi,A}.$$

Where we are viewing φ as an isomorphism between the appropriate restrictions and we are viewing $\mathcal{F}_{\varphi(P)} c_{\varphi,A}$ as a subset of $\mu_{\mathcal{R}}(\mathcal{F})$. Because of Proposition 2.2.24 this allows us to define the $\mu_{\mathcal{R}}(\mathcal{F})$-module morphism $\theta_\varphi \colon M_P \hookrightarrow\!\!\!\!\rightarrow M_{\varphi(P)}$ that, for every $y \in \mu_{\mathcal{R}}(\mathcal{F})$, every $J \in \mathcal{F}_P$ and every $x \in I^J_J M \downarrow^{\mathcal{F}}_{\mathcal{F}_P}$, sends $y \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} x$ to $y c_{\varphi^{-1},\varphi(J)} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(P)}\right)} c_{\varphi,J} x$. Notice now that $\mathcal{F}_{\varphi(P)} \subseteq \mathcal{F}_Q$. Because of Corollary 2.2.25 this inclusion allows us to define $\theta_{\iota^Q_{\varphi(P)}} : M_{\varphi(P)} \twoheadrightarrow M_Q$ as the natural $\mu_{\mathcal{R}}(\mathcal{F})$-module morphism that, for every $y' \in \mu_{\mathcal{R}}(\mathcal{F}) 1_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(P)}\right)}$ and every $x' \in M \downarrow^{\mathcal{F}}_{\mathcal{F}_{\varphi(P)}}$, sends $y' \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(P)}\right)} x'$ to $y' \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_Q\right)} x'$. With this setup we can finally define the $\mu_{\mathcal{R}}(\mathcal{F})$-module morphism $\theta_{\overline{\varphi}} \colon M_P \twoheadrightarrow M_Q$ as $\theta_{\overline{\varphi}} := \theta_{\iota^Q_{\varphi(P)}} \theta_\varphi$. Then, with x, y and J as above, we have that

$$\theta_P\left(y \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} x\right) = yx = y c_{\varphi^{-1},\varphi(J)} c_{\varphi,J} x = \theta_Q\left(\theta_{\overline{\varphi}}\left(y \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} x\right)\right).$$

Where we are viewing φ as an isomorphism between the appropriate restrictions and, for the second identity, we are using Items (1) and (2) of Lemma 2.2.22 in order to add the terms $c_{\varphi^{-1},\varphi(J)} c_{\varphi,J}$. This proves that $\theta_P = \theta_Q \theta_{\overline{\varphi}}$ thus concluding the proof. □

Using Lemma 2.3.3 we can now translate [We00, Lemma 3.2] to the context of centric Mackey functors over fusion systems.

Corollary 2.3.4. *Let $M \in Mack_{\mathcal{R}}(\mathcal{F}^c)$, let $\mathcal{X}$ and $\mathcal{Y}$ be families of $\mathcal{F}$-centric subgroups of S and denote by $\mathcal{X}^{max} \subseteq \mathcal{X}$ any family of maximal elements of $\mathcal{X}$ (under the preorder $\leq_{\mathcal{F}}$ of Notation 2.2.3) taken up to $\mathcal{F}$-isomorphism. We have that:*

(1) If M is $\mathcal{X}$-projective and $\mathcal{X} \subseteq \mathcal{Y}$ then M is $\mathcal{Y}$-projective.

(2) If M is $\mathcal{X}$-projective then it is $\mathcal{X}^{\max}$-projective.

Proof. From definition of $\mathcal{X}^{\max}$ for every $P \in \mathcal{X}$ exists $J_P \in \mathcal{X}^{\max}$ such that $P \leq_{\mathcal{F}} J_P$ or, equivalently, such that $\mathrm{Hom}_{\mathcal{O}(\mathcal{F}^c)}(P, J_P) \neq \emptyset$. On the other hand, for every $P \in \mathcal{X}$ we can take $Q_P := P \in \mathcal{Y}$ and we have $\mathrm{Id}_P \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F}^c)}(P, Q_P) \neq \emptyset$. The result now follows from Lemma 2.3.3. □

Finally we can translate [We00, Proposition 3.3] to the context of centric Mackey functors over fusion systems.

Proposition 2.3.5. *Let $M \in \mathit{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ and let $\mathcal{X}$ and $\mathcal{Y}$ be families of $\mathcal{F}$-centric subgroups of S closed under $\mathcal{F}$-subconjugacy (i.e. $Q \in \mathcal{X}$ and $P \leq_{\mathcal{F}} Q$ imply $P \in \mathcal{X}$ and analogously with $\mathcal{Y}$). If M is both $\mathcal{X}$-projective and $\mathcal{Y}$-projective then:*

(1) M is $\mathcal{X} \times \mathcal{Y}$-projective where

$$\mathcal{X} \times \mathcal{Y} := \{A \in \mathcal{F}^c \,|\, \exists P \in \mathcal{X}, Q \in \mathcal{Y} \text{ and } \overline{\varphi}\colon A \to Q \text{ s.t. } (A, \overline{\varphi}) \in [P \times Q]\}.$$

(2) M is $\mathcal{X} \cap \mathcal{Y}$-projective.

Proof. For every $A \in \mathcal{X} \times \mathcal{Y}$, there exist, by definition, $P \in \mathcal{X}$ and $Q \in \mathcal{Y}$ such that $A \leq_{\mathcal{F}} P, Q$. Since both $\mathcal{X}$ and $\mathcal{Y}$ are closed under $\mathcal{F}$-subconjugacy this implies that $A \in \mathcal{X} \cap \mathcal{Y}$. In other words we have that $\mathcal{X} \times \mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{Y}$. From Corollary 2.3.4 (1) we can now deduce that Item (2) follows from Item (1).

Let's prove Item (1). For every $P \in \mathcal{X}$, every $Q \in \mathcal{Y}$ and every $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]$ let us fix a representative φ of $\overline{\varphi}$ and view it as an isomorphism onto its image. Using the notation of Lemma 2.2.36 we have that

$$M' := \bigoplus_{P \in \mathcal{X}, Q \in \mathcal{Y}} \bigoplus_{(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]} M_{(A, \overline{\varphi})} \uparrow_{\mathcal{F}_{\varphi(A)}}^{\mathcal{F}} \cong (M_{\mathcal{X}})_{\mathcal{Y}} := \bigoplus_{P \in \mathcal{X}, Q \in \mathcal{Y}} M \downarrow_{\mathcal{F}_P}^{\mathcal{F}} \uparrow_{\mathcal{F}_P}^{\mathcal{F}} \downarrow_{\mathcal{F}_Q}^{\mathcal{F}} \uparrow_{\mathcal{F}_Q}^{\mathcal{F}},$$

We can now define $\Gamma^{-1}\colon (M_{\mathcal{X}})_{\mathcal{Y}} \hookrightarrow\!\!\!\!\!\rightarrow M'$ to be the inverse of the isomorphism described in Lemma 2.2.36 and define $\Upsilon\colon M' \to M_{\mathcal{X} \times \mathcal{Y}}$ by setting for every $P \in \mathcal{X}$, every

$Q \in \mathcal{Y}$, every $(A, \overline{\varphi}) \in [P \times Q]$, every $J \leq A$, every $x \in I_J^J M \downarrow_{\mathcal{F}_A}^{\mathcal{F}}$ and every $y \in \mu_{\mathcal{R}}(\mathcal{F}) 1_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)}$

$$\Upsilon\left(y \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} x\right) := y c_{\varphi, J} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_A)} x.$$

where we are viewing φ as an isomorphism between the appropriate restrictions and, on the left hand side, we are viewing x as an element of $M_{(A,\overline{\varphi})}$ while, on the right hand side, we are viewing x as an element of $M \downarrow_{\mathcal{F}_A}^{\mathcal{F}}$. Notice that, for every $x \in P$, we have that $c_{\varphi(P)}\varphi = c_h \varphi$ is an isomorphisms from P to $\varphi(P)$. With this in mind Proposition 2.2.24 and Items (2) and (3) of Lemma 2.2.22 ensure us that the definition of Υ does not depend on the choice of representatives of $y \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} x$. Moreover it is immediate from definition that Υ commutes with the action of $\mu_{\mathcal{R}}(\mathcal{F})$ and, therefore, it's a $\mu_{\mathcal{R}}(\mathcal{F})$-module morphism.

Finally, since M is both $\mathcal{X}$-projective and $\mathcal{Y}$-projective, there exist Mackey functor morphisms $u_{\mathcal{X}} \colon M \to M_{\mathcal{X}}$ and $u_{\mathcal{Y}} \colon M \to M_{\mathcal{Y}}$ such that $\theta_{\mathcal{X}}^M u_{\mathcal{X}} = \theta_{\mathcal{Y}}^M u_{\mathcal{Y}} = \mathrm{Id}_M$. Applying restriction and induction functors to the morphisms $u_{\mathcal{X}}$ and $\theta_{\mathcal{X}}$ we can define

$$\begin{aligned} u_{\mathcal{X},\mathcal{Y}} &:= \sum_{Q \in \mathcal{Y}} \uparrow_{\mathcal{F}_Q}^{\mathcal{F}} \left(\downarrow_{\mathcal{F}_Q}^{\mathcal{F}} (u_{\mathcal{X}})\right) : M_{\mathcal{Y}} \to (M_{\mathcal{X}})_{\mathcal{Y}}, \\ \theta_{\mathcal{X},\mathcal{Y}} &:= \sum_{Q \in \mathcal{Y}} \uparrow_{\mathcal{F}_Q}^{\mathcal{F}} \left(\downarrow_{\mathcal{F}_Q}^{\mathcal{F}} (\theta_{\mathcal{X}}^M)\right) : (M_{\mathcal{X}})_{\mathcal{Y}} \to M_{\mathcal{Y}}. \end{aligned}$$

From functoriality of induction and restriction, we have that $\theta_{\mathcal{X},\mathcal{Y}} u_{\mathcal{X},\mathcal{Y}} = \mathrm{Id}_{M_{\mathcal{Y}}}$.

Let $P \in \mathcal{X}$, let $Q, J \in \mathcal{Y}$ such that $Q \leq J$, let $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]$, let φ be the previously fixed representative of $\overline{\varphi}$ viewed as an isomorphism onto its image, let $\left(C, \overline{\theta}\right) \in \left[\varphi(A) \times_{\mathcal{F}_Q} J\right]$, let $x \in I_C^C M$ and let $y \in 1_{\mu_{\mathcal{R}}(\mathcal{F}_Q)} \mu_{\mathcal{R}}(\mathcal{F}) 1_{\mu_{\mathcal{R}}(\mathcal{F}_P)}$. Using the notation of Corollary 2.2.23 we have that

$$\theta_{\mathcal{Y}}^M \left(\theta_{\mathcal{X},\mathcal{Y}}\left(y \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_Q)} I_{\overline{\theta}(C)}^J c_{\overline{\theta\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} x\right)\right) = \theta_{\mathcal{Y}}^M \left(y \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_Q)} I_{\overline{\theta}(C)}^J c_{\overline{\theta\varphi}} x\right) = y I_{\overline{\theta}(C)}^J c_{\overline{\theta\varphi}} x.$$

and that

$$\begin{aligned} y I_{\overline{\theta}(C)}^J c_{\overline{\theta}\varphi} x &= \theta_{\mathcal{X}\times\mathcal{Y}}^M \left(y I_{\overline{\theta}(C)}^J c_{\overline{\theta}\varphi} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_A)} x\right), \\ &= \theta_{\mathcal{X}\times\mathcal{Y}}^M \left(\Upsilon\left(y I_{\overline{\theta}(C)}^J c_{\overline{\theta}} \otimes_{\mu_{\mathcal{R}}\left(\mathcal{F}_{\varphi(A)}\right)} x\right)\right), \\ &= \theta_{\mathcal{X}\times\mathcal{Y}}^M \left(\Upsilon\left(\Gamma\left(y \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_Q)} I_{\overline{\theta}(C)}^J c_{\overline{\theta\varphi}} \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} x\right)\right)\right). \end{aligned}$$

Where, in the second identity, we are viewing x as an element of $M_{(A,\overline{\varphi})}$. From Lemma 2.2.36 we know that every element in $M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}\uparrow^{\mathcal{F}}_{\mathcal{F}_P}\downarrow^{\mathcal{F}}_{\mathcal{F}_Q}\uparrow^{\mathcal{F}}_{\mathcal{F}_Q}$ can be written as a finite sum of elements of the form $y\otimes_{\mu_{\mathcal{R}}(\mathcal{F}_Q)} I^J_{\overline{\theta}(C)}c_{\overline{\theta\varphi}}\otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} x$. Therefore the previous identities prove that $\theta^M_{\mathcal{X}\times\mathcal{Y}}\Upsilon\Gamma^{-1} = \theta^M_{\mathcal{Y}}\theta_{\mathcal{X},\mathcal{Y}}$. With this in mind we obtain

$$\theta^M_{\mathcal{X}\times\mathcal{Y}}\Upsilon\Gamma^{-1}u_{\mathcal{X},\mathcal{Y}}u_{\mathcal{Y}} = \theta^M_{\mathcal{Y}}\theta_{\mathcal{X},\mathcal{Y}}u_{\mathcal{X},\mathcal{Y}}u_{\mathcal{Y}} = \theta^M_{\mathcal{Y}}u_{\mathcal{Y}} = \mathrm{Id}_M .$$

This proves that $\theta^M_{\mathcal{X}\times\mathcal{Y}}$ is split surjective or, equivalently, that M is $\mathcal{X}\times\mathcal{Y}$-projective thus concluding the proof. □

We can now finally define the defect set of a centric Mackey functor over a fusion system.

Corollary 2.3.6. *Let $\mathcal{R}$ be p-local and let $M\in$ Mack$_{\mathcal{R}}(\mathcal{F}^c)$. There exists a unique minimal family of $\mathcal{F}$-centric subgroups of S that is closed under $\mathcal{F}$-subconjugacy and such that M is projective relative to it.*

Proof. This is an immediate consequence of Lemma 2.3.2, Corollary 2.3.4 (1) and Proposition 2.3.5 (2). □

Definition 2.3.7. Let $\mathcal{R}$ be p-local and let $M\in \mathrm{Mack}_{\mathcal{R}}(\mathcal{F}^c)$. We call the minimal family of elements in $\mathcal{F}^c$ given in Corollary 2.3.6 the **defect set of** M (denoted as $\mathcal{X}_M$). Using the notation of Corollary 2.3.4 we call **defect group** of M any element in $\mathcal{X}_M^{\max}$ (for any choice of $\mathcal{X}_M^{\max}$. If $|\mathcal{X}_M^{\max}| = 1$ we say that M **admits a vertex** and we call **vertex of** M (and denote it by V_M) any fully $\mathcal{F}$-normalized defect group of M.

2.3.2 Trace maps and Higman's criterion

The main goal of this subsection is translating Higman's criterion (see [NT89, Theorem 2.2]) to centric Mackey functors over fusion systems (see Theorem 2.3.17). This allows us to relate the concept of relative projectivity of an indecomposable centric Mackey functor $M\in \mathrm{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ to the images of certain trace maps (see Definitions 2.3.8 and 2.3.12). In order to understand this relation we need to start by introducing some notation.

Definition 2.3.8. Let $\mathcal{G}$ be a fusion system containing $\mathcal{F}$, let $M\in \mathrm{Mack}^{\mathcal{G}^c}_{\mathcal{R}}(\mathcal{F})$, let $P\in\mathcal{F}^c$ and let $\varphi\colon P\to\varphi(P)$ be an isomorphism in $\mathcal{F}$. We define the **conjugation**

map from $\mathcal{F}_P$ to $\mathcal{F}_{\varphi(P)}$ on M as the $\mathcal{R}$-algebra morphism ${}^{M,\varphi}\cdot : \operatorname{End}\left(M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right) \to \operatorname{End}\left(M\downarrow^{\mathcal{F}}_{\mathcal{F}_{\varphi(P)}}\right)$, obtained by setting for every $f \in \operatorname{End}\left(M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right)$, every $Q \in \mathcal{F}_{\varphi(P)} \cap \mathcal{F}^c$ and every $x \in I_Q^Q M\downarrow^{\mathcal{F}}_{\mathcal{F}_{\varphi(P)}}$

$$ {}^{M,\varphi}f(x) := c_{\varphi,\varphi^{-1}(Q)}\left(f\left(c_{\varphi^{-1},Q}\,x\right)\right). $$

Where we are viewing φ as an isomorphism between the appropriate restrictions and we are viewing $M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}$ and $M\downarrow^{\mathcal{F}}_{\mathcal{F}_{\varphi(P)}}$ as subsets of M.

We define the **trace map from $\mathcal{F}_P$ to $\mathcal{F}$ on** M as the $\mathcal{R}$-module morphism

$$ {}^{M}\operatorname{tr}^{\mathcal{F}}_{\mathcal{F}_P} : \underset{f}{\operatorname{End}\left(M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right)} \underset{\longrightarrow}{\longrightarrow} \underset{\theta_P^M f\uparrow^{\mathcal{F}}_{\mathcal{F}_P}\theta_M^P}{\operatorname{End}(M)}. $$

where $f\uparrow^{\mathcal{F}}_{\mathcal{F}_P}$ denotes the image of f via the induction functor $\uparrow^{\mathcal{F}}_{\mathcal{F}_P}$: $\mathsf{Mack}_{\mathcal{R}}(\mathcal{F}_P) \to \mathsf{Mack}_{\mathcal{R}}(\mathcal{F})$ (see Definition 2.2.28). More precisely, for every $Q \in \mathcal{F}^c$, every $x \in I_Q^Q M$ and every $f \in \operatorname{End}\left(M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right)$ we have that

$$ {}^{M}\operatorname{tr}^{\mathcal{F}}_{\mathcal{F}_P}(f)(x) = \sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}} f\left(c_{\overline{\varphi^{-1}}} R^Q_{\overline{\varphi}(A)} x\right). $$

Finally, given any fusion subsystem $\mathcal{H} \subseteq \mathcal{F}$, we define the **restriction map from $\mathcal{F}$ to $\mathcal{H}$ on** M as the $\mathcal{R}$-algebra morphism

$$ {}^{M}\mathrm{r}^{\mathcal{F}}_{\mathcal{H}} : \underset{f}{\operatorname{End}(M)} \underset{\longrightarrow}{\longrightarrow} \underset{f\downarrow^{\mathcal{F}}_{\mathcal{H}}}{\operatorname{End}\left(M\downarrow^{\mathcal{F}}_{\mathcal{H}}\right)}. $$

where $f\downarrow^{\mathcal{F}}_{\mathcal{H}}$ denotes the image of f via the restriction functor $\downarrow^{\mathcal{F}}_{\mathcal{H}}$: $\mathsf{Mack}_{\mathcal{R}}(\mathcal{F}) \to \mathsf{Mack}_{\mathcal{R}}(\mathcal{H})$ (see Definition 2.2.28).

Whenever there is no doubt regarding M we simply write

$$ \operatorname{tr}^{\mathcal{F}}_{\mathcal{F}_P} := {}^{M}\operatorname{tr}^{\mathcal{F}}_{\mathcal{F}_P}, \qquad \mathrm{r}^{\mathcal{F}}_{\mathcal{H}} := {}^{M}\mathrm{r}^{\mathcal{F}}_{\mathcal{H}}, \qquad {}^{\varphi}\cdot := {}^{M,\varphi}\cdot. $$

Trace, restriction and conjugation maps satisfy the following properties which are analogous to those satisfied in the case of Mackey functors over groups (see [Sa82, Definition 2.7]).

Proposition 2.3.9. *Let $M \in Mack_{\mathcal{R}}(\mathcal{F}^c)$ then:*

(1) For every $P \in \mathcal{F}^c$ and every $x \in P$ we have that $\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}_P} = \mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}_P} = {}^{c_x}\cdot = \mathrm{Id}_{\mathrm{End}\left(M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)}$.

(2) For every fusion subsystems $\mathcal{H} \subseteq \mathcal{K} \subseteq \mathcal{F}$ we have that $\mathrm{r}_{\mathcal{H}}^{\mathcal{K}} \mathrm{r}_{\mathcal{K}}^{\mathcal{F}} = \mathrm{r}_{\mathcal{H}}^{\mathcal{F}}$.

(3) For every $P \leq Q \in \mathcal{F}^c$ we have that $\mathrm{tr}_{\mathcal{F}_Q}^{\mathcal{F}} \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}_Q} = \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}$.

(4) For every isomorphisms φ, ψ in $\mathcal{F}^c$ such that $\varphi\psi$ is defined we have that ${}^{\psi}\cdot\, {}^{\varphi}\cdot = {}^{\psi\varphi}\cdot$.

(5) For every $P \leq Q \in \mathcal{F}^c$ and every isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, \varphi(Q))$ we have that ${}^{\varphi}\cdot\, \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}_Q} = \mathrm{tr}_{\mathcal{F}_{\varphi(P)}}^{\mathcal{F}_{\varphi(Q)}}\, {}^{\varphi}\cdot$.

(6) For every $P \leq Q \in \mathcal{F}^c$ and every isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, \varphi(Q))$ we have that ${}^{\varphi}\cdot\, \mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}_Q} = \mathrm{r}_{\mathcal{F}_{\varphi(P)}}^{\mathcal{F}_{\varphi(Q)}}\, {}^{\varphi}\cdot$.

(7) For every $P \in \mathcal{F}^c$ and every isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, \varphi(P))$ we have that $\mathrm{tr}_{\mathcal{F}_{\varphi(P)}}^{\mathcal{F}}\, {}^{\varphi}\cdot = \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}$.

(8) For every $P \in \mathcal{F}^c$ and every isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, \varphi(P))$ we have that ${}^{\varphi}\cdot\, \mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}} = \mathrm{r}_{\mathcal{F}_{\varphi(P)}}^{\mathcal{F}}$.

(9) For every $P, Q \in \mathcal{F}^c$ we have

$$\mathrm{r}_{\mathcal{F}_Q}^{\mathcal{F}} \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}} = \sum_{(A,\overline{\varphi}) \in [P \times_{\mathcal{F}} Q]} \mathrm{tr}_{\mathcal{F}_{\varphi(A)}}^{\mathcal{F}_Q}\, {}^{\varphi}\cdot\, \mathrm{r}_{\mathcal{F}_A}^{\mathcal{F}_P}.$$

Here φ is any representative of $\overline{\varphi}$ seen as an isomorphism onto its image.

(10) For every $P \in \mathcal{F}^c$, every $f \in \mathrm{End}(M)$ and every $g \in \mathrm{End}\left(M \downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)$ we have that

$$f \,\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(g) = \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(\mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}}(f)\, g\right), \qquad \text{and that} \quad \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(g)\, f = \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(g\, \mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}}(f)\right).$$

(11) Let $P \in \mathcal{F}^c$. Using Notation 2.1.1 and the notation of Proposition 2.2.43 we have that $\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}} \mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}} = \left(\overline{P}\cdot\right)_*$.

Proof.

(1) Let $Q \in \mathcal{F}_P \cap \mathcal{F}^c$, let $y \in I_Q^Q M \downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ and let $f \in \mathrm{End}\left(M \downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)$. By definition of restriction we have that $\mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}}(f)(y) = f(y)$. Since f is a $\mu_{\mathcal{R}}(\mathcal{F}_P)$-module

morphism we have that

$$^{c_x}f\left(y\right)=c_{c_x}f\left(c_{c_{x^{-1}}}y\right)=c_{c_x}c_{c_{x^{-1}}}f\left(y\right)=f\left(y\right).$$

Where we are viewing c_x as an isomorhism from Q^x to Q. Finally, from Proposition 2.2.18 (2), we have that $[P\times_{\mathcal{F}_P}Q]=\left\{\left(Q,\overline{\mathrm{Id}_Q}\right)\right\}$ and, therefore, from Lemma 2.2.22 (1), we can conclude that

$$\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(f\right)\left(y\right)=I_Q^Q c_{\overline{\mathrm{Id}_Q}}\left(f\left(c_{\overline{\mathrm{Id}_Q}}R_Q^Q y\right)\right)=f\left(y\right).$$

(2) Since the restriction functor satisfies $\downarrow_{\mathcal{H}}^{\mathcal{K}}\downarrow_{\mathcal{K}}^{\mathcal{F}}=\downarrow_{\mathcal{H}}^{\mathcal{F}}$, then Item (2) follows.

(3) Let $J\in\mathcal{F}^c$, let $x\in I_J^J M$ and let $f\in\mathrm{End}\left(M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)$. From Proposition 2.2.18 (6) we have that

$$\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(f\right)\left(x\right)=\sum_{(A,\overline{\varphi})\in[Q\times_{\mathcal{F}}J]}\sum_{\substack{y\in[A\backslash Q/P]\\ A^y\cap P\in\mathcal{F}^c}}I_{\overline{\varphi c_y}(A^y\cap H)}^J c_{\overline{\varphi c_y}}\left(f\left(c_{\overline{(\varphi c_y)^{-1}}}R_{\overline{\varphi c_y}(A^y\cap P)}^J x\right)\right).$$

Since $M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\in\mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}\left(\mathcal{F}_P\right)$ we know that $c_{\overline{(\varphi c_y)^{-1}}}R_{\overline{\varphi c_y}(A^y\cap P)}^J\cdot x=0$ for every $(A,\overline{\varphi})\in[Q\times_{\mathcal{F}}J]$ and every $y\in[A\backslash Q/P]$ such that $A^y\cap P\in\mathcal{F}_P^c\backslash\left(\mathcal{F}_P^c\cap\mathcal{F}^c\right)$. Thus, we can replace the second sum of the above equation as a sum over $y\in[A\backslash Q/P]$ such that $A^y\cap P\in\mathcal{F}_P^c$. Using Proposition 2.2.18 (2) we can now rewrite.

$$\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(f\right)\left(x\right)=\sum_{(A,\overline{\varphi})\in[Q\times_{\mathcal{F}}J]}\sum_{\left(B,\overline{\psi}\right)\in\left[P\times_{\mathcal{F}_Q}A\right]}I_{\overline{\varphi\psi}(B)}^J c_{\overline{\varphi\psi}}\left(f\left(c_{\overline{(\varphi\psi)^{-1}}}R_{\overline{\varphi\psi}(B)}^J x\right)\right).$$

From Corollary 2.2.23 we know that the above is equal to $\mathrm{tr}_{\mathcal{F}_Q}^{\mathcal{F}}\left(\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}_Q}\left(f\right)\right)\left(x\right)$ thus proving Item (3).

(4) Let $P\in\mathcal{F}^c$, let $\varphi\colon P\to\varphi\left(P\right)$ and $\psi\colon\varphi\left(P\right)\to\psi\varphi\left(P\right)$ be isomorphisms in $\mathcal{F}$, let $J\in\mathcal{F}_{\psi\varphi(P)}\cap\mathcal{F}^c$, let $x\in I_J^J M\downarrow_{\mathcal{F}_{\psi\varphi(P)}}^{\mathcal{F}}$ and let $f\in\mathrm{End}\left(M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)$. Item (4). follows from Lemma 2.2.22 (2) via the identities below

$$^{\psi\varphi}f\left(x\right)=c_{\psi\varphi}\left(f\left(c_{\varphi^{-1}\psi^{-1}}x\right)\right)=c_{\psi}\left(c_{\varphi}f\left(c_{\varphi^{-1}}c_{\psi^{-1}}x\right)\right)={}^{\psi}\left({}^{\varphi}f\right)\left(x\right).$$

(5) Let $J \in \mathcal{F}_Q \cap \mathcal{F}^c$. Viewing $[\varphi(J)\backslash\varphi(Q)/\varphi(P)]$ as a subset of $\varphi(Q)$ we can take $[J\backslash Q/P] = \varphi^{-1}([\varphi(J)\backslash\varphi(Q)/\varphi(P)])$. Moreover, for every $\varphi(A) \leq \varphi(Q)$ we have that $\varphi(A) \in \mathcal{F}^c_{\varphi(Q)}$ if and only if $A \in \mathcal{F}^c_Q$ and, for every $\varphi(y) \in \varphi(Q)$ we have that $\varphi^{-1}\left(\varphi(J)^{\varphi(y)} \cap \varphi(P)\right) = J^y \cap P$. From Proposition 2.2.18 (2) we can therefore conclude that

$$\left[P \times_{\mathcal{F}_Q} J\right] = \bigsqcup_{\substack{y\in[J\backslash Q/P]\\ J^y\cap P\in\mathcal{F}^c_Q}} \{(J^y \cap P, \overline{\iota c_y})\} = \bigsqcup_{\left(B,\overline{\psi}\right)\in\left[\varphi(P)\times_{\mathcal{F}_{\varphi(Q)}}\varphi(J)\right]} \left\{\left(\varphi^{-1}(B), \overline{\iota\varphi^{-1}\psi\varphi}\right)\right\}.$$

Where, for the second identity, we are using that c_y and $\varphi^{-1}c_{\varphi(y)}\varphi$ are equal as automorphisms of Q and viewing ψ as an isomorphism onto its image. Let $x \in I^{\varphi(J)}_{\varphi(J)} M \downarrow^{\mathcal{F}}_{\mathcal{F}_{\varphi(Q)}}$. Using the above identity we have that

$$\begin{aligned}\operatorname{tr}^{\mathcal{F}_{\varphi(Q)}}_{\mathcal{F}_{\varphi(P)}}\left({}^{\varphi}f\right)(x) &= \sum_{\left(B,\overline{\psi}\right)\in\left[\varphi(P)\times_{\mathcal{F}_{\varphi(Q)}}\varphi(J)\right]} I^{\varphi(J)}_{\overline{\psi}(B)} c_{\overline{\psi\varphi}}\left(f\left(c_{\overline{(\psi\varphi)^{-1}}} R^{\varphi(J)}_{\overline{\psi}(B)} x\right)\right),\\ &= \sum_{\left(C,\overline{\theta}\right)\in\left[P\times_{\mathcal{F}_Q}J\right]} c_\varphi I^J_{\overline{\theta}(C)} c_{\overline{\theta}}\left(f\left(c_{\overline{\theta}} R^J_{\overline{\theta}(C)} c_{\varphi^{-1}} x\right)\right) = {}^{\varphi}\left(\operatorname{tr}^{\mathcal{F}_Q}_{\mathcal{F}_P}(f)\right)(x).\end{aligned}$$

Where, for the second identity, we are using Items (1) and (3) of Lemma 2.2.22 in order to obtain the identities $I^{\varphi(J)}_{\psi(B)} = c_\varphi I^J_{\varphi^{-1}\psi(B)} c_{\varphi^{-1}}$ and $R^{\varphi(J)}_{\psi(B)} = c_\varphi R^J_{\varphi^{-1}\psi(B)} c_{\varphi^{-1}}$ for any representative ψ of $\overline{\psi}$. This proves Item (5).

(6) Let $J \in \mathcal{F}_{\varphi(P)} \cap \mathcal{F}^c$ and let $x \in I^J_J M$. Item (6) follows from the identities below

$${}^{\varphi}\left(\operatorname{r}^{\mathcal{F}_Q}_{\mathcal{F}_P}(f)\right)(x) = c_\varphi\left(f\left(c_{\varphi^{-1}}x\right)\right) = {}^{\varphi}f(x) = \operatorname{r}^{\mathcal{F}_Q}_{\mathcal{F}_{\varphi(P)}}\left({}^{\varphi}f\right)(x).$$

(7) Let $Q \in \mathcal{F}^c$, let $x \in I^Q_Q M$ and let $f \in \operatorname{End}\left(M \downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right)$. Using Proposition 2.2.18 (4) we have that.

$$\begin{aligned}\operatorname{tr}^{\mathcal{F}}_{\mathcal{F}_{\varphi(P)}}\left({}^{\varphi}f\right)(x) &= \sum_{\left(B,\overline{\psi}\right)\in[\varphi(P)\times_{\mathcal{F}}Q]} I^Q_{\overline{\theta\varphi}(\varphi^{-1}(B))} c_{\overline{\theta\varphi}}\left(f\left(c_{\overline{(\theta\varphi)^{-1}}} R^Q_{\overline{\theta\varphi}(\varphi^{-1}(B))} x\right)\right),\\ &= \sum_{\left(C,\overline{\theta}\right)\in[P\times_{\mathcal{F}}Q]} I^Q_{\overline{\theta}(C)} c_{\overline{\theta}}\left(f\left(c_{\overline{\theta^{-1}}} R^Q_{\overline{\theta}(C)} x\right)\right) = \operatorname{tr}^{\mathcal{F}}_{\mathcal{F}_P}(f)(x).\end{aligned}$$

Where we are viewing φ as an isomorphism between the appropriate restrictions.

This proves Item (7).

(8) Let $Q \in \mathcal{F}_{\varphi(P)} \cap \mathcal{F}^c$, let $x \in I_Q^Q M$ and let $f \in \mathrm{End}\,(M)$. Since f is a morphism of $\mu_{\mathcal{R}}(\mathcal{F})$-modules we have that

$$^{\varphi}\left(\mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}}(f)\right)(x) = c_{\varphi} f\left(c_{\varphi^{-1}} x\right) = c_{\varphi} c_{\varphi^{-1}} f(x) = f(x).$$

Where we are viewing φ as an isomorphism between the appropriate restrictions. This proves Item (8).

(9) Let $J \in \mathcal{F}_Q \cap \mathcal{F}^c$, let $x \in I_J^J M$ and let $f \in \mathrm{End}\left(M \downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)$. From Proposition 2.2.18 (5) we have that

$$\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(f)(x) = \sum_{(A,\overline{\varphi}) \in [P \times_{\mathcal{F}} Q]} \sum_{\substack{y \in [J \backslash Q/\varphi(A)] \\ J^y \cap \varphi(A) \in \mathcal{F}^c}} I_{J \cap^y \varphi(A)}^J c_{c_y \varphi} \left(f\left(c_{(c_y\varphi)^{-1}} R_{J \cap^y \varphi(A)}^J x\right)\right).$$

Where we are fixing a representative φ of $\overline{\varphi}$ and viewing it as an isomorphism onto its image. The same arguments employed to prove Item (3) allow us to replace the second sum of the previous equation with a sum over $\left[\varphi(A) \times_{\mathcal{F}_Q} J\right]$. This leads us to the identities

$$\begin{aligned} \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(f)(x) &= \sum_{(A,\overline{\varphi}) \in [P \times_{\mathcal{F}} Q]} \sum_{\left(B,\overline{\psi}\right) \in \left[J \times_{\mathcal{F}_Q} \varphi(A)\right]} I_{\overline{\psi}(B)}^J c_{\overline{\psi\varphi}} \left(f\left(c_{\overline{(\psi\varphi)^{-1}}} R_{\overline{\psi}(B)}^J x\right)\right), \\ &= \sum_{(A,\overline{\varphi}) \in [P \times_{\mathcal{F}} Q]} \mathrm{tr}_{\mathcal{F}_{\varphi(A)}}^{\mathcal{F}_Q} \left({}^{\varphi}\left(\mathrm{r}_{\mathcal{F}_A}^{\mathcal{F}_P}(f)\right)\right)(x). \end{aligned}$$

Here we are viewing $M \downarrow_{\mathcal{F}_Q}^{\mathcal{F}}$ as a subset of M. With this inclusion in mind we also have that $\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(f)(x) = \mathrm{r}_{\mathcal{F}_Q}^{\mathcal{F}}\left(\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(f)\right)(x)$ and, therefore, the above identities prove Item (9).

(10) We prove just the first identity since the second is proved similarly. Let $Q \in \mathcal{F}^c$. Since f is a morphism of $\mu_{\mathcal{R}}(\mathcal{F})$-modules, for every $y \in M \downarrow_{\mathcal{F}_P}^{\mathcal{F}} \subset M$ and every $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]$ we have that

$$f\left(I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} y\right) = I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} f(y) = I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}}\, \mathrm{r} \downarrow_{\mathcal{F}_P}^{\mathcal{F}}(f)(y).$$

Let $x \in I_Q^Q M$. Item (10) follows from the above via the identities below

$$\begin{aligned} f\left(\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(g)(x)\right) &= \sum_{(A,\overline{\varphi}) \in [P \times_{\mathcal{F}} Q]} I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} \cdot \left(\left(\mathrm{r} \downarrow_{\mathcal{F}_P}^{\mathcal{F}}(f)\, g\right)\left(c_{\overline{\varphi^{-1}}} R_{\overline{\varphi}(A)}^Q \cdot x\right)\right), \\ &= \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(\mathrm{r} \downarrow_{\mathcal{F}_P}^{\mathcal{F}}(f)\, g\right)(x). \end{aligned}$$

(11) Let $Q \in \mathcal{F}^c$, let $x \in I_K^K M$ and let $f \in \mathrm{End}(M)$. Since f is a $\mu_{\mathcal{R}}(\mathcal{F})$-module morphism, for every $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]$, we have that $f\left(c_{\overline{\varphi^{-1}}} R^Q_{\overline{\varphi}(A)} x\right) = c_{\overline{\varphi^{-1}}} R^Q_{\overline{\varphi}(A)} f(x)$. Item (11) follows from this identity and Proposition 2.2.43 via the identities below

$$\begin{aligned}\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(\mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}}(f)\right)(x) &= \sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}Q]} I^Q_{\overline{\varphi}(A)} c_{\overline{\varphi}} c_{\overline{\varphi^{-1}}} R^Q_{\overline{\varphi}(A)}\left(f(x)\right),\\ &= \theta_P^M\left(\theta_M^P\left(f(x)\right)\right) = \left(\overline{P}\cdot\right)_*(f)(x).\end{aligned}$$

□

Remark 2.3.10. Given a fusion system $\mathcal{K}$ contained in $\mathcal{F}$ the trace $\mathrm{tr}_{\mathcal{K}}^{\mathcal{F}}$ is in general not defined. However, as we show in Subsection 2.4.5, something similar can be defined when $\mathcal{K} = N_{\mathcal{F}}(P)$ for some fully $\mathcal{F}$-normalized $P \in \mathcal{F}^c$. In this situation we obtain a result similar to Proposition 2.3.9 (3) but replacing $\mathcal{F}_Q$ with $N_{\mathcal{F}}(P)$ (see Lemma 2.4.29).

Corollary 2.3.11. *Let $M \in$ Mack$_{\mathcal{R}}(\mathcal{F}^c)$, let $P \in \mathcal{F}^c$ and let $\varphi\colon P \to \varphi(P)$ be an isomorphism in $\mathcal{F}$ then ${}^{\mathrm{Id}_P}\cdot = \mathrm{Id}_{\mathrm{End}\left(M\downarrow_{\mathcal{F}_P}^{\mathcal{F}^c}\right)}$ and ${}^{\varphi}\cdot$ is an isomorphism.*

Proof. Let $Q \in \mathcal{F}_P \cap \mathcal{F}^c$, let $x \in I_Q^Q M$ and let $f \in \mathrm{End}\left(M\downarrow_{\mathcal{F}_P}^{\mathcal{F}^c}\right)$. From definition of conjugation map and Lemma 2.2.22 (1) we have that

$${}^{\mathrm{Id}_P} f(x) = c_{\mathrm{Id}_P} f\left(c_{\mathrm{Id}_P} x\right) = f(x).$$

Thus we have that ${}^{\mathrm{Id}_P}\cdot = \mathrm{Id}_{\mathrm{End}\left(M\downarrow_{\mathcal{F}_P}^{\mathcal{F}^c}\right)}$. Using Proposition 2.3.9 (4) we can now deduce that

$${}^{\varphi}\cdot\,{}^{\varphi^{-1}}\cdot = {}^{\varphi^{-1}}\cdot\,{}^{\varphi}\cdot = {}^{\mathrm{Id}_P}\cdot = \mathrm{Id}_{\mathrm{End}\left(M\downarrow_{\mathcal{F}_P}^{\mathcal{F}^c}\right)}.$$

This proves that ${}^{\varphi}\cdot$ has an inverse and, therefore, is an isomorphism. □

Definition 2.3.12. Let $M \in$ Mack$_{\mathcal{R}}(\mathcal{F}^c)$, let $P \in \mathcal{F}^c$ and let $\mathcal{X}$ be a family of objects in $\mathcal{F}^c$. We define the **trace image from P to $\mathcal{F}$ on M** and the **trace image from $\mathcal{X}$ to $\mathcal{F}$ on M** respectively as

$${}^{M}\mathrm{Tr}_P^{\mathcal{F}} := \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(\mathrm{End}\left(M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)\right), \qquad \text{and} \qquad {}^{M}\mathrm{Tr}_{\mathcal{X}}^{\mathcal{F}} := \sum_{P\in\mathcal{X}} {}^{M}\mathrm{Tr}_P^{\mathcal{F}}.$$

If there is no possible confusion we simply write $\mathrm{Tr}_P^{\mathcal{F}} := {}^M\mathrm{Tr}_P^{\mathcal{F}}$ and $\mathrm{Tr}_{\mathcal{X}}^{\mathcal{F}} := {}^M\mathrm{Tr}_{\mathcal{X}}^{\mathcal{F}}$.

Lemma 2.3.13. *With notation as in Definition 2.3.12, both* $\mathrm{Tr}_P^{\mathcal{F}}$ *and* $\mathrm{Tr}_{\mathcal{X}}^{\mathcal{F}}$ *are two sided ideals of* $\mathrm{End}\,(M)$.

Proof. This is an immediate consequence of Proposition 2.3.9 (10). □

We now have the following result reminiscent of Lemma 2.3.3.

Lemma 2.3.14. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be families of objects in* $\mathcal{F}^c$, *let* $\sigma\colon \mathcal{X} \to \mathcal{Y}$ *be a map between sets and let* $\Phi = \{\varphi_P : P \to \sigma(P)\}_{P\in\mathcal{X}}$ *be a family of morphisms in* $\mathcal{F}^c$. *Then, we have that* $\mathrm{Tr}_{\mathcal{X}}^{\mathcal{F}} \subseteq \mathrm{Tr}_{\mathcal{Y}}^{\mathcal{F}}$ *regardless of the associated centric Mackey functor.*

Proof. From definition of $\mathrm{Tr}_{\mathcal{X}}^{\mathcal{F}}$ and $\mathrm{Tr}_{\mathcal{Y}}^{\mathcal{F}}$ it suffices to prove the statement in the case where $\mathcal{X} := \{P\}$, $\mathcal{Y} := \{Q\}$ and $\Phi := \{\varphi\colon P \to Q\}$ for some objects $P, Q \in \mathcal{F}^c$ and some morphism $\varphi \in \mathcal{F}$. In what follows we view φ as an isomorphism onto its image. From Proposition 2.3.9 (7) we have that $\mathrm{Tr}_P^{\mathcal{F}} = \mathrm{tr}_{\mathcal{F}_{\varphi(A)}}^{\mathcal{F}}\left({}^{\varphi}\left(\mathrm{End}\left(M \downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)\right)\right)$. From Corollary 2.3.11 we can conclude that $\mathrm{Tr}_P^{\mathcal{F}} = \mathrm{Tr}_{\varphi(P)}^{\mathcal{F}}$. Finally, using Proposition 2.3.9 (3) on the groups $\varphi(P) \leq Q$ we can conclude that $\mathrm{Tr}_P^{\mathcal{F}} \subseteq \mathrm{Tr}_Q^{\mathcal{F}}$ just as we wanted to prove. □

We can now provide the following definition which, as we show in Theorem 2.3.17, is closely related to Definition 2.3.1.

Definition 2.3.15. Let $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$, let $f \in \mathrm{End}\,(M)$ and let $\mathcal{X}$ be a family of objects in $\mathcal{F}^c$. We say that f is **projective relative to** $\mathcal{X}$ (or $\mathcal{X}$-**projective**) if $f \in \mathrm{Tr}_{\mathcal{X}}^{\mathcal{F}}$. If $\mathcal{X} = \{P\}$ for some $P \in \mathcal{F}^c$ we simply say that f is **projective relative to** P (or P-**projective**).

Let G be a finite group, let $H \leq G$ and let M be a Mackey functor over G. Using Equation (2.5) we can define π_M as the natural projection of $M \uparrow_H^G\downarrow_H^G$ onto the summand $\left({}^{1_G}\left(M \downarrow_H^H\right)\right) \uparrow_H^H \cong M$. By composing it with the natural inclusion, the morphism π_M can be seen as an endomorphism of $M \downarrow_H^G\uparrow_H^G$. In order to prove Higman's criterion for Mackey functors over finite groups (see [NT89, Theorem 2.2]) Hirosi and Tsushima use the identity $\mathrm{tr}_H^G(\pi_M) = \mathrm{Id}_{M\uparrow_H^G}$ where tr_H^G denotes the trace map for Mackey functors over finite groups (see [Sa82, Definition 2.7]). In order to prove Higman's criterion for centric Mackey functors over fusion systems (and thus relate Definitions 2.3.1 and 2.3.15) we need a similar result.

Lemma 2.3.16. *Let $P \in \mathcal{F}^c$, let $M \in \mathit{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$ and let $\pi_M \in \operatorname{End}\left(M \uparrow_{\mathcal{F}_P}^{\mathcal{F}} \downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)$ be the composition of the projection onto the summand $\left({}^{\mathrm{Id}_P}\left(M \downarrow_{\mathcal{F}_P}^{\mathcal{F}_P}\right)\right) \uparrow_{\mathcal{F}_P}^{\mathcal{F}_P} \cong M$ (see Lemma 2.2.36) and the natural inclusion. Then we have that $\operatorname{tr}_{\mathcal{F}_P}^{\mathcal{F}}(\pi_M) = \mathrm{Id}_{M \uparrow_{\mathcal{F}_P}^{\mathcal{F}}}$.*

Proof. From Definition 2.2.28 we know that every element in $M \uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ is of the form $y \otimes x$ for some $y \in \mu_{\mathcal{R}}(\mathcal{F})$ and some $x \in M$. Therefore, since $\operatorname{tr}_{\mathcal{F}_P}^{\mathcal{F}}(\pi_M)$ is a morphism of $\mu_{\mathcal{R}}(\mathcal{F})$-modules, it suffices to prove that $\operatorname{tr}_{\mathcal{F}_P}^{\mathcal{F}}(\pi_M)\left(I_Q^Q \otimes x\right) = I_Q^Q \otimes x$ for every $Q \in \mathcal{F}_P \cap \mathcal{F}^c$ and every $x \in I_Q^Q M$. Fix x and Q as described. From definition of π_M we have that

$$\operatorname{tr}_{\mathcal{F}_P}^{\mathcal{F}}(\pi_M)\left(I_Q^Q \otimes x\right) = \sum_{\substack{(A,\overline{\varphi}) \in [P \times_{\mathcal{F}} Q] \\ c_{\overline{\varphi^{-1}}} R_{\overline{\varphi}(A)}^Q \in \mu_{\mathcal{R}}(\mathcal{F}_P)}} I_{\overline{\varphi}(A)}^Q c_{\overline{\varphi}} c_{\overline{\varphi^{-1}}} R_{\overline{\varphi}(A)}^Q \otimes x.$$

Since $Q \leq P$ by assumption, then we have that $c_{\overline{\varphi^{-1}}} R_{\overline{\varphi}(A)}^Q \in \mu_{\mathcal{R}}(\mathcal{F}_P)$ if and only if $\overline{\varphi} \in \mathcal{O}(\mathcal{F}_P)$. For every $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]$ satisfying $\overline{\varphi} \in \mathcal{O}(\mathcal{F}_P)$ we can assume without loss of generality that $A \leq Q$ and that $\overline{\varphi} = \overline{\iota_A^Q}$ (see Definition 2.2.17). From maximality of the pair $(A, \overline{\varphi})$ (see again Definition 2.2.17) the previous description implies that $A = Q$. We can therefore conclude that there exists a unique $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} Q]$ such that $c_{\overline{\varphi^{-1}}} R_{\overline{\varphi}(A)}^Q \in \mu_{\mathcal{R}}(\mathcal{F}_P)$. Moreover $[P \times_{\mathcal{F}} Q]$ can be taken so that this element satisfies $c_{\overline{\varphi^{-1}}} R_{\overline{\varphi}(A)}^Q = I_Q^Q$. The result now follows from the equation above. □

We are now finally ready to translate Higman's criterion to the context of centric Mackey functors over fusion systems.

Theorem 2.3.17 (Higman's criterion). *Let $\mathcal{G}$ be a fusion system containing $\mathcal{F}$, let $M \in \mathit{Mack}_{\mathcal{R}}^{\mathcal{G}^c}(\mathcal{F}) \subseteq \mathit{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ (see Definition 2.2.29) be an indecomposable Mackey functor and let $P \in \mathcal{F}^c$. The following are equivalent:*

(1) There exists $N \in \mathit{Mack}_{\mathcal{R}}^{\mathcal{G}^c}(\mathcal{F}_P)$ such that M is a summand of $N \uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ (see Definition 2.2.28).

(2) There exists $N \in \mathit{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$ such that M is a summand of $N \uparrow_{\mathcal{F}_P}^{\mathcal{F}}$.

(3) Id_M *is P-projective (see Definition 2.3.15).*

(4) $\mathrm{End}\,(M) = \mathrm{Tr}_P^{\mathcal{F}}$ (see Definition 2.3.12).

(5) θ_P *(see Definition 2.2.37) is an epimorphism and, given $N, L \in$ Mack$_{\mathcal{R}}(\mathcal{F}^c)$ and Mackey functor morphisms $\varphi\colon N \twoheadrightarrow L$ and $\psi\colon M \to L$ with φ surjective, if there exists a Mackey functor morphism $\gamma\colon M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\to N\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ such that $\varphi\downarrow_{\mathcal{F}_P}^{\mathcal{F}} \gamma = \psi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ then there exists a Mackey functor morphism $\hat{\gamma} : M \to N$ such that $\varphi\hat{\gamma} = \psi$.*

(6) θ^P *(see Definition 2.2.37) is a monomorphism and, given $N, L \in$ Mack$_{\mathcal{R}}(\mathcal{F}^c)$ and Mackey functor morphisms $\varphi\colon L \hookrightarrow N$ and $\psi\colon L \to M$ with φ injective, if there exists a Mackey functor morphism $\gamma\colon N\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\to M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ such that $\gamma\varphi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}= \psi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ then there exists a Mackey functor morphism $\hat{\gamma}\colon N \to M$ such that $\hat{\gamma}\varphi = \psi$.*

(7) θ_P *is an epimorphism and, given $N \in$ Mack$_{\mathcal{R}}(\mathcal{F}^c)$ and an epimorphism of Mackey functors $\varphi\colon N \twoheadrightarrow M$, if $\varphi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ splits then φ splits.*

(8) θ^P *is a monomorphism and, given $N \in$ Mack$_{\mathcal{R}}(\mathcal{F}^c)$ and a monomorphism of Mackey functors $\varphi\colon M \hookrightarrow N$, if $\varphi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ splits then φ splits.*

(9) θ_P *is split surjective (equivalently M is P-projective see Definition 2.3.1).*

(10) θ^P *is split injective.*

(11) M *is a direct summand of M_P (see Definition 2.2.37).*

Proof. The proof is analogous to that of [NT89, Theorem 2.2] except for some details in the proof of implications (2)⇒(3), (7)⇒(9) and (8)⇒(10) for which we need to use Lemmas 2.2.36 and 2.3.16 in order to replace analogous results for Mackey functors over finite groups.

(1)⇒(2).

Since $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{F}_P \cap \mathcal{G}^c \subseteq \mathcal{F}_P \cap \mathcal{F}^c$ and, therefore, $\mathrm{Mack}_{\mathcal{R}}^{\mathcal{G}^c}(\mathcal{F}_P) \subseteq \mathrm{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$. The implication follows.

(2)⇒(3).

Let $N \in \mathrm{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$ such that there exists $L \in \mathrm{Mack}_{\mathcal{R}}(\mathcal{F})$ satisfying $N\uparrow_{\mathcal{F}_P}^{\mathcal{F}}= M \oplus L$, Let π_M be the endomorphism of $N\uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ given by the natural projection onto M followed by the natural inclusion and let $\pi_N \in \operatorname{End}\left(N \uparrow_{\mathcal{F}_P}^{\mathcal{F}} \downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)$ be the endomorphism of Lemma 2.3.16 satisfying ${}^{N}\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(\pi_N) = \mathrm{Id}_{N\uparrow_{\mathcal{F}_P}^{\mathcal{F}}}$. Since restriction preserves direct sums then we have that $N \uparrow_{\mathcal{F}_P}^{\mathcal{F}} \downarrow_{\mathcal{F}_P}^{\mathcal{F}} = M \downarrow_{\mathcal{F}_P}^{\mathcal{F}} \oplus L \downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ and that the endomorphism ${}^{N}\mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}}(\pi_M)$ of $N \uparrow_{\mathcal{F}_P}^{\mathcal{F}} \downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ is the projection onto $M \downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ followed by the natural inclusion. We can now define $f \in \operatorname{End}\left(M \downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)$ by setting for every $x \in M \downarrow_{\mathcal{F}_P}^{\mathcal{F}}$

$$f(x) := {}^{N}\mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}}(\pi_M)(\pi_N(x)).$$

Here we are seeing $M \downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ as a subset of $N \uparrow_{\mathcal{F}_P}^{\mathcal{F}} \downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ in order to apply π_N. With this setup, for every $Q \in \mathcal{F}^c$ and every $x \in I_Q^Q M \subseteq I_Q^Q N \uparrow_{\mathcal{F}_P}^{\mathcal{F}}$, we have that.

$${}^{M}\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(f)(x) = {}^{N}\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left({}^{N}\mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}}(\pi_M)\pi_N\right)(x) = \pi_M\, {}^{N}\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(\pi_N)(x) = \pi_M(x) = x.$$

where the first identity follows from definition, for the second identity we are using Proposition 2.3.9 (10), for the third we are using Lemma 2.3.16 and for the last we are using the fact that $x \in M$ and definition of π_M. From the above we can conclude that ${}^{M}\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(f) = \mathrm{Id}_M$ which implies that Id_M is H-projective thus proving the implication.

(3)⇔(4).

By definition we have that Id_M is P-projective if and only if $\mathrm{Id}_M \in \mathrm{Tr}_P^{\mathcal{F}}$. From Lemma 2.3.13 we know that $\mathrm{Tr}_P^{\mathcal{F}}$ is an ideal of $\operatorname{End}(M)$. Therefore $\mathrm{Tr}_P^{\mathcal{F}} = \operatorname{End}(M)$ if and only if $\mathrm{Id}_M \in \mathrm{Tr}_P^{\mathcal{F}}$. This proves that Items (3) and (4) are equivalent.

(3)⇒(5).

If Item (3) is satisfied then there exists $f \in \operatorname{End}\left(M \downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)$ such that $\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(f) := \theta_P^M f \uparrow_{\mathcal{F}_P}^{\mathcal{F}} \theta_M^P = \mathrm{Id}_M$ (see Definition 2.3.8). Therefore θ_M^P is a split injective and θ_P^M is split surjective. In particular θ_P^M is surjective. Let N, L, φ, ψ and γ be as in the statement of item (5) and define $\hat{\gamma} := \theta_P^N(\gamma f) \uparrow_{\mathcal{F}_P}^{\mathcal{F}} \theta_M^P$. Then, for every $x \in N$ and every $y \in \mu_{\mathcal{R}}(\mathcal{F})$, we have that

$$\varphi\left(\theta_P^N\left(\gamma \uparrow_{\mathcal{F}_P}^{\mathcal{F}}(y \otimes x)\right)\right) = y\varphi \downarrow_{\mathcal{F}_P}^{\mathcal{F}}(\gamma(x)) = y\psi \downarrow_{\mathcal{F}_P}^{\mathcal{F}}(x) = \theta_P^M\left(\psi \downarrow_{\mathcal{F}_P}^{\mathcal{F}} \uparrow_{\mathcal{F}_P}^{\mathcal{F}}(y \otimes x)\right).$$

Where, for the first identity, we are using the fact that φ is a $\mu_{\mathcal{R}}(\mathcal{F})$-module morphism in order to get $\varphi(y\gamma(x)) = y\varphi(\gamma(x)) = y\varphi \downarrow_{\mathcal{F}_P}^{\mathcal{F}}(\gamma(x))$. The above equation proves that $\varphi\theta_P^N\gamma\uparrow_{\mathcal{F}_P}^{\mathcal{F}}=\theta_P^M\psi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\uparrow_{\mathcal{F}_P}^{\mathcal{F}}$. The implication now follows from the identities below

$$\varphi\hat{\gamma}=\varphi\theta_P^N(\gamma f)\uparrow_{\mathcal{F}_P}^{\mathcal{F}}\theta_M^P=\theta_P^M\big(\psi\downarrow_{\mathcal{F}_P}^{\mathcal{F}} f\big)\uparrow_{\mathcal{F}_P}^{\mathcal{F}}\theta_M^P=\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(\mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}}(\psi)\, f\right)=\psi\,\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(f)=\psi.$$

Where, for the third identity, we are using Definition 2.3.8 while, for the fourth identity, we are using Proposition 2.3.9 (10).

(3)⇒(6).

Let f be as in the previous implication. As before we have that θ_M^P is split injective and, in particular, it is injective. Let N, L, φ, ψ and γ be as in the statement of Item (6) and define $\hat{\gamma}:=\theta_P^M(f\gamma)\uparrow_{\mathcal{F}_P}^{\mathcal{F}}\theta_N^P$. Then, for every $Q\in\mathcal{F}^c$ and every $x\in I_Q^Q M$, we have that

$$\left(\gamma\uparrow_{\mathcal{F}_P}^{\mathcal{F}}\theta_N^P\varphi\right)(x)=\sum_{(A,\overline{\alpha})\in[P\times_{\mathcal{F}}Q]} I_{\overline{\alpha}(A)}^Q c_{\overline{\alpha}}\otimes\gamma\varphi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\left(c_{\overline{\alpha^{-1}}}R_{\overline{\alpha}(A)}^Q x\right)=\left(\psi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\uparrow_{\mathcal{F}_P}^{\mathcal{F}}\theta_M^P\right)(x).$$

Where, for the second identity, we are using the identity $\gamma\varphi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}=\psi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ while, for the first identity, we are using that φ is a morphism of $\mu_{\mathcal{R}}(\mathcal{F})$-modules in order to get $c_{\overline{\alpha^{-1}}}R_{\overline{\alpha}(A)}^Q\varphi(x)=\varphi\left(c_{\overline{\alpha^{-1}}}R_{\overline{\alpha}(A)}^Q x\right)$ and we are using that $c_{\overline{\alpha^{-1}}}R_{\overline{\alpha}(A)}^Q x\in M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ in order to write $\varphi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ instead of φ. The above equation proves that $\gamma\uparrow_{\mathcal{F}_P}^{\mathcal{F}}\theta_N^P\varphi=\psi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\uparrow_{\mathcal{F}_P}^{\mathcal{F}}\theta_M^P$. The implication now follows from the identities below

$$\hat{\gamma}\varphi=\theta_P^M(f\gamma)\uparrow_{\mathcal{F}_P}^{\mathcal{F}}\theta_N^P\varphi=\theta_P^M\left(f\psi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)\uparrow_{\mathcal{F}_P}^{\mathcal{F}}\theta_M^P=\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(f\,\mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}}(\psi)\right)=\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(f)\,\psi=\psi.$$

Where, for the third identity, we are using Definition 2.3.8 while, for the fourth identity, we are using Proposition 2.3.9 (10).

(5)⇒(7).

With the notation of Item (5) let $L:=M$ and $\psi:=\mathrm{Id}_M$. Since $\varphi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ splits then there exists $\gamma\colon M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\to N\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$ such that $\varphi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\gamma=\mathrm{Id}_{M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}}=\psi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$. Therefore, by hypothesis, there exists $\hat{\gamma}:M\to N$ such that $\varphi\hat{\gamma}=\psi=\mathrm{Id}_M$. In other words φ splits.

(6)⇒(8).

With notation as in Item (6) let $L:=M$, $\psi:=\mathrm{Id}_M$ and $\gamma\colon N\to M$ such that $\gamma\varphi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}=\mathrm{Id}_{M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}}=\psi\downarrow_{\mathcal{F}_P}^{\mathcal{F}}$. Then, by hypothesis, there exists $\hat{\gamma}\colon N\to M$ such that $\hat{\gamma}\varphi=\mathrm{Id}_M$. In other words φ splits.

(7)⇒(9).

Let $f\colon M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}\hookrightarrow M_P\downarrow^{\mathcal{F}}_{\mathcal{F}_P}$ be the $\mu_{\mathcal{R}}(\mathcal{F}_P)$-module morphism given by Lemma 2.2.36 and that sends $M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}$ isomorphically into the summand $\left({}^{\mathrm{Id}_P}\left(M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right)\right)\uparrow^{\mathcal{F}_P}_{\mathcal{F}_P}$ of $M_P\downarrow^{\mathcal{F}}_{\mathcal{F}_P}$. With this setup we have that $\theta_P\downarrow^{\mathcal{F}}_{\mathcal{F}_P} f=\mathrm{Id}_M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}$. Item (9) now follows from Item (7) by taking $N:=M_P$ and $\varphi=\theta_P$.

(8)⇒(10).

From Lemma 2.2.36 we can take $\pi\colon M_P\downarrow^{\mathcal{F}}_{\mathcal{F}_P}\twoheadrightarrow M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}$ to be the natural projection onto the summand $M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}\cong\left({}^{\mathrm{Id}_P}\left(M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right)\right)\uparrow^{\mathcal{F}_P}_{\mathcal{F}_P}$. Dually to the previous implication we have that $\pi\theta^P\downarrow^{\mathcal{F}}_{\mathcal{F}_P}=\mathrm{Id}_M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}$. Item (10) now follows from Item (8) by taking $N:=M_P$ and $\varphi=\theta^P$.

(9)⇒(11) and (10)⇒(11).

The fact that M is a summand of M_P is an immediate consequence of either θ_P being split surjective (Item (9)) or θ^P being split injective (Item (10)).

(11)⇒(1).

From Proposition 2.2.30 we know that $N:=M\downarrow^{\mathcal{F}}_{\mathcal{F}_P}$ is $\mathcal{G}$-centric and, from Item (11) we have that M is a summand of $N\uparrow^{\mathcal{F}}_{\mathcal{F}_P}=M_P$. □

Remark 2.3.18. The equivalence (2)⇔(1) of Theorem 2.3.17 can be proven independently from the rest.

We conclude this section with the following result which allows us to always talk about the vertex of an indecomposable centric Mackey functor over a fusion system.

Corollary 2.3.19. *Let $\mathcal{R}$ be a complete local and p-local PID, let $\mathcal{G}$ be a fusion system containing $\mathcal{F}$ and let $M\in \mathit{Mack}^{\mathcal{G}^c}_{\mathcal{R}}(\mathcal{F})$ be an indecomposable Mackey functor. Then M admits a vertex (see Definition 2.3.7). Moreover $V_M\in\mathcal{F}\cap\mathcal{G}^c$ and, for any $N\in \mathit{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ such that M is a summand of N, we have that $V_M\in\mathcal{X}_N$.*

Proof. By definition of defect set we know that the map $\theta^M_{\mathcal{X}_M}\colon\bigoplus_{P\in\mathcal{X}_M}M_P\to M$ is split surjective, in particular M is a summand of $\bigoplus_{P\in\mathcal{X}_M}M_P$. Since $\mathcal{R}$ is a complete local PID and M is indecomposable, then we can apply the Krull-Schmidt-Azumaya theorem (see [CR81, Theorem 6.12 (ii)]) in order to deduce that there exists $P\in\mathcal{X}_M$ such that M is a summand of M_P. Because of Theorem 2.3.17 this implies that M is

P-projective. Since M is $\mathcal{G}$-centric then $M_P = 0$ for every $P \in \mathcal{F} \backslash (\mathcal{F} \cap \mathcal{G}^c)$. Therefore we necessarily have $P \in \mathcal{F} \cap \mathcal{G}^c$. Define $\mathcal{X}_P := \{Q \in \mathcal{F}^c : Q \leq_{\mathcal{F}} P\}$. Since M is P-projective we can deduce from Corollary 2.3.4 (1) that M is also $\mathcal{X}_P$-projective. From minimality of $\mathcal{X}_M$ (see Definition 2.3.7) this implies that $\mathcal{X}_M \subseteq \mathcal{X}_P$. Since $\mathcal{X}_M$ is closed under $\mathcal{F}$-subconjugacy and $P \in \mathcal{X}_M$ we also have that $\mathcal{X}_P \subseteq \mathcal{X}_M$ and, therefore, $\mathcal{X}_P = \mathcal{X}_M$. By construction of $\mathcal{X}_P$ this is equivalent to saying that M admits a vertex (namely any fully $\mathcal{F}$-normalized $Q =_{\mathcal{F}} P$).

Let N be as in the statement and let $L \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ such that $N = M \oplus L$. Since induction and restriction preserve direct sum decomposition we have that $N_{\mathcal{X}_N} = M_{\mathcal{X}_N} \oplus L_{\mathcal{X}_N}$. Immediately from the definition of $\theta^N_{\mathcal{X}_N}$ we also have that $\theta^N_{\mathcal{X}_N}(M_{\mathcal{X}_N}) \subseteq M$ and that $\theta^N_{\mathcal{X}_N}(L_{\mathcal{X}_N}) \subseteq L$. Moreover, the restrictions of $\theta^N_{\mathcal{X}_N}$ to $M_{\mathcal{X}_N}$ and $L_{\mathcal{X}_N}$ coincide with $\theta^M_{\mathcal{X}_N}$ and $\theta^L_{\mathcal{X}_N}$ respectively. In other words we have that $\theta^N_{\mathcal{X}_N} = \theta^M_{\mathcal{X}_N}\pi_M + \theta^L_{\mathcal{X}_N}\pi_L$ where π_M and π_L denote the natural projections onto $M_{\mathcal{X}_N}$ and $L_{\mathcal{X}_N}$ respectively. On the other hand, from definition of defect set, we know that there exists a Mackey functor morphism $u \colon N \to N_{\mathcal{X}_N}$ such that $\theta^N_{\mathcal{X}_N} u = \mathrm{Id}_N$. Denote by $u_{|M} : M \to N_{\mathcal{X}_N}$ the restriction of u to M followed by the natural inclusion into $N_{\mathcal{X}_N}$. Since $\theta^L_{\mathcal{X}_N}$ maps to L and $L \cap M = \{0\}$ then we can conclude that $\mathrm{Id}_M = \left(\theta^M_{\mathcal{X}_N}\pi_M + \theta^L_{\mathcal{X}_N}\pi_L\right) u_{|M} = \theta^M_{\mathcal{X}_N}\pi_M u_{|M}$. In particular $\theta^M_{\mathcal{X}_N}$ is split surjective or, equivalently, M is $\mathcal{X}_N$-projective. From minimality of the defect set we can then conclude that $V_M \in \mathcal{X}_N$. □

2.4 Green correspondence

Through this section we will be using Notations 2.1.1, 2.2.3, 2.2.9, 2.2.21 and 2.2.31.

In this section we prove the main result of this paper. More precisely we prove that a Green correspondence holds for centric Mackey functors over fusion systems (see Theorem 2.4.38).

We start in Subsection 2.4.1 by proving Proposition 2.4.7 which gives us a list of sufficient conditions to prove a Green correspondence like result in the context of endomorphisms. Subsections 2.4.2 to 2.4.5 are dedicated to building the tools needed in order to prove that Proposition 2.4.7 can be applied to endomorphism rings of $\mathcal{F}$-centric Mackey functors.

Finally we conclude with Subsection 2.4.6 where we use Proposition 2.4.7 together with

Theorem 2.3.17 in order to translate Green correspondence to the context of centric Mackey functors over fusion systems (see Theorem 2.4.38).

2.4.1 Correspondence of endomorphisms

The goal of this subsection is that of stating and proving Proposition 2.4.7. This result is one of the cornerstones for proving Theorem 2.4.38.

Let's start with some notation.

Definition 2.4.1. Let A and B be rings (not necessarily having a unit) and let $f\colon A \twoheadrightarrow B$ be a surjective ring morphism. We say that f is a **near isomorphism** if $A\ker(f) = \ker(f)A = 0$.

The following Lemmas are useful in later sections and provide examples of near isomorphisms.

Lemma 2.4.2. *Let A and B be rings (not necessarily having a unit) and let $f\colon A \to B$ be a ring morphism. If f is an isomorphism then it is a near isomorphism and if f is a near isomorphism and A has a unit then f is an isomorphism.*

Proof. If f is an isomorphism it is surjective and $\ker(f) = 0$. In particular $A\ker(f) = \ker(f)A = 0$ and, therefore, f is a near isomorphism. Assume now that f is a near isomorphism and A has a unit. Then, for every $x \in \ker(f)$, we have that $x1_A = 0$ and, therefore, $\ker(f) = 0$. Thus f is injective. Since f is also surjective by definition of near isomorphism then it is an isomorphism thus concluding the proof. □

Lemma 2.4.3. *Let A be a ring (not necessarily having a unit) and let I and J be two sided ideals of A such that $I \subseteq J$ and $JA, AJ \subseteq I$. Then the natural surjective ring morphism $f\colon A/I \twoheadrightarrow A/J$ is a near isomorphism.*

Proof. For every $C \subseteq A$ denote by $\overline{C}$ the image of C under the natural projection onto A/I. Then, by construction, we have that $\ker(f) = \overline{J}$. Since $AJ, JA \subseteq I$ we have that $\overline{A}\,\overline{J} = \overline{J}\,\overline{A} = \overline{I} = \overline{0}$ thus concluding the proof. □

Lemma 2.4.4. *Let A, B and C be rings (not necessarily having a unit) and let $f\colon A \to B$ and $g\colon B \to C$ be ring homomorphisms. If gf is a near isomorphism and f is surjective then both f and g are near isomorphisms.*

Proof. First of all notice that $\ker(f) \subseteq \ker(gf)$. Since gf is a near isomorphims then we have that $A\ker(f) \subseteq A\ker(gf) = 0$ and that $\ker(f)A \subseteq \ker(gf)A = 0$. Since f is surjective by hypothesis then we can conclude that f is a near isomorphism.

On the other hand, since gf is a near isomorphism, then it is surjective and, therefore, g is also surjective. Since f is surjective, then we have that $\ker(g) = f(\ker(gf))$ and $B = f(A)$. Therefore we can conclude that $B\ker(g) = f(A\ker(gf)) = 0$ and that $\ker(g)B = f(\ker(gf)A) = 0$ thus concluding the proof. ☐

Lemma 2.4.5. *Let A, B and C be rings (not necessarily having a unit), let $f\colon A \twoheadrightarrow B$ be a near isomorphism and let $g\colon B \hookrightarrow\!\!\!\!\rightarrow C$ be an isomorphism then gf is a near isomorphism.*

Proof. Since both f and g are surjective, then $h := gf$ is also surjective. The result follows from applying Lemma 2.4.4 to $f = g^{-1}h$. ☐

The importance of near isomorphisms comes from the following well known lemma due to Green which we state without proving.

Lemma 2.4.6 ([Gr71, Lemma 4.22])**.** *Let A and B be $\mathcal{R}$-algebras and let $f\colon A \twoheadrightarrow B$ be a near isomorphism. Denote by $E(A)$ and $E(B)$ the sets of idempotents of A and B respectively. Then the following are satisfied*

(1) f induces a bijection from $E(A)$ to $E(B)$.

(2) Let $x \in E(A)$ be a local idempotent. Then $f(x) \in E(B)$ is also a local idempotent.

(3) Let $x, y \in E(A)$ be idempotents. Then x and y are conjugate in A if and only if $f(x)$ and $f(y)$ are conjugate in B.

With this in mind we can now prove the main result of this subsection.

Proposition 2.4.7. *Let A and B be $\mathcal{R}$-algebras, let C, J be two sided ideals of A, let I and K be two sided ideals of C and B respectively (C seen as a ring with potentially no unit) and let $f\colon C \to B$ and $g\colon B \to C + J$ be $\mathcal{R}$-linear maps. Assume that the following are satisfied:*

(1) $(C \cap J)\, C, C\, (C \cap J) \subseteq I \subseteq C \cap J$,

(2) $g\,(K) \subseteq J$,

(3) $f\,(I) \subseteq K$,

(4) f *is surjective.*

(5) g *sends idempotents to idempotents.*

(6) The $\mathcal{R}$*-linear maps* $\overline{f}\colon C/I \to B/K$ *and* $\overline{g}\colon B/K \to (C+J)/J$ *induced by* f *and* g *respectively commute with multiplication (i.e.* $\overline{f}\,(xy) = \overline{f}\,(x)\,\overline{f}\,(y)$ *and* $\overline{g}\,(vw) = \overline{f}\,(v)\,\overline{f}\,(w)$ *for every* $x, y \in C/I$ *and every* $v, w \in B/K$*).*

(7) The natural isomorphism $s\colon C/(C \cap J) \to (C+J)/J$ *and the natural projection* $q\colon C/I \to C/(C \cap J)$ *satisfy* $sq = \overline{g}\overline{f}$.

(8) For every idempotent $x \in A$ *there exists a unique (up to conjugation) decomposition of* x *as a finite sum of orthogonal local idempotents.*

Let $b \in B$ *be a local idempotent such that* $b \notin K$. *Then* $g\,(b) \in C + J \subseteq A$ *and, from Conditions* (5) *and* (8)*, there exists a unique* $n \in \mathbb{N}$ *and a unique (up to conjugation) set of orthogonal local idempotents* $\{a_0, \ldots, a_n\} \subseteq A$ *such that*

$$g\,(b) = \sum_{i=0}^{n} a_i.$$

There exists exactly one value $j \in \{0, \ldots, n\}$ *such that* $a_j \in C \setminus (C \cap J)$. *Moreover, if we define* $a := a_j$*, we have that*

$$g(b) \equiv a \mod J, \qquad\qquad f\,(a) \equiv b \mod K.$$

Proof. Since both C and J are two sided ideals of A then $C + J$ is also a two sided ideal of A. With notation as in the statement, since all the a_i are pairwise orthogonal, for every $i = 0, \ldots, n$, we have that $a_i = a_i g\,(b)$ and, since $g\,(b) \in C + J$, we can conclude that $a_i \in C + J$. Since $C + J$ is a two sided ideal of A we can conclude that $a_i A a_i \subseteq a_i\,(C + J)\,a_i$. Since $C + J \subseteq A$ we obtain the other inclusion and, therefore, we obtain the identity

$$a_i\,(C + J)\,a_i = a_i A a_i.$$

In particular, since each a_i is a local idempotent of A, we have that $a_i (C+J) a_i$ is a local ring and, since $a_i \in C+J$, we can conclude that each a_i is a local idempotent of $C+J$ (and not just of A).

Since, by hypothesis, $b \notin K$ then the projection $\bar{b}$ of b onto B/K is non zero. Since, by hypothesis, b is a local idempotent then we can conclude that $\bar{b}$ is also a local idempotent (because quotients of local rings are still local rings). Likewise, for every $i = 0, \ldots, n$, we have that the projection $\overline{a_i}$ of a_i onto $(C+J)/J$ is either 0 or a local idempotent of $(C+J)/J$.

From Lemma 2.4.3 and Condition (1) we know that the natural projection q of Condition (7) is a near isomorphism. From Lemma 2.4.2 we know that s is also a near isomorphism. From Lemma 2.4.5 and Condition (7) we can conclude that $\overline{g}\overline{f}$ is also a near isomorphism. Finally, from Lemma 2.4.4 and Condition (4), we can conclude that $\overline{f}$ and $\overline{g}$ are near isomorphisms. Since $\bar{b}$ is a local idempotent then we can conclude from Lemma 2.4.6 (2) that $\overline{g}\left(\bar{b}\right) = \sum_{i=0}^{n} \overline{a_i}$ is also a local idempotent. Since local idempotents are primitive we can conclude that there exists exactly one $j \in \{0, \ldots, n\}$ such that $\overline{a_j} \neq 0$. We can assume without loss of generality that $j = 0$ and define $a := a_0$. In other words we have that $\overline{g}\left(\bar{b}\right) = \overline{a}$ (equivalently $g(b) \equiv a \mod (J)$) while for every $i = 1, \ldots, n$ we have that $\overline{a_i} = 0$ (equivalently $a_i \in J$). This proves the first equivalence in the statement. Since $a \notin J$ (because $\overline{a} \neq 0$), in order to complete the proof, we just need to prove that $a \in C$ and that the second equivalence of the statement is satisfied.

Since both C and J are two sided ideals of A then we can deduce that aCa and aJa are two sided ideals of $a(C+J)a$. Since a is a local idempotent of $C+J$, by definition, we have that $a(C+J)a$ is a local ring. Notice also that, from the distributive property of the product, we have that $aCa + aJa = a(C+J)a$. From definition of local ring we can conclude that either

$$a(C+J)a = aCa \subseteq C, \qquad \text{or} \qquad a(C+J)a = aJa \subseteq J.$$

Since a is an idempotent and $a \notin J$ then we can conclude that the identity on the right in the above equation is not possible. Therefore the identity on the left must be satisfied and we can conclude that $a \in C \backslash (C \cap J)$.

In order to complete the proof we are just left with proving that $f(a)$ is equivalent to

b modulo K. Denote by $\overline{\overline{a}}$ the projection of a on C/I. Since a is an idempotent then $\overline{\overline{a}}$ must also be an idempotent and, from Condition (6) we can deduce that $\overline{f}\left(\overline{\overline{a}}\right)$ is an idempotent. On the other hand, from the first part of the proof, we know that $\overline{a} = \overline{g}\left(\overline{b}\right)$. Thus, from Condition (7), we can deduce that

$$\overline{g}\left(\overline{f}\left(\overline{\overline{a}}\right)\right) = s\left(q\left(\overline{\overline{a}}\right)\right) = \overline{a} = \overline{g}\left(\overline{b}\right).$$

Since $\overline{g}$ is a near isomorphism (as already proven), from the above identities and Lemma 2.4.6 (1), we can conclude that $\overline{f}\left(\overline{\overline{a}}\right) = \overline{b}$. From Condition (3) and definition of $\overline{f}$ this is equivalent to saying that $f(a)$ is equivalent to b modulo K. This concludes the proof. □

Let's conclude this subsection by giving an example where Proposition 2.4.7 is used in order to prove that Green correspondence holds for Green functors (see [Gr71, Proposition 4.34]).

Example 2.4.8. Let $\mathcal{R}$ be a complete local PID, let G be a finite group, let $D, H \leq G$ be subgroups such that $N_G(D) \leq H$ and let M be a Green functor over G on $\mathcal{R}$ (see the first definition of [Gr71, Subsection 1.3]). With the notation of Proposition 2.4.7 we can define

$$\begin{aligned} A &:= \mathrm{End}\left(M\downarrow^G_H\right), & B &:= \mathrm{tr}^G_D\left(\mathrm{End}\left(M\downarrow^G_D\right)\right), \\ C &:= \mathrm{tr}^H_D\left(\mathrm{End}\left(M\downarrow^G_D\right)\right), & K &:= \sum_{x\in G-H} \mathrm{tr}^G_{D^x\cap D}\left(\mathrm{End}\left(M\downarrow^G_{D^x\cap D}\right)\right), \\ I &:= \sum_{x\in G-H} \mathrm{tr}^H_{D^x\cap D}\left(\mathrm{End}\left(M\downarrow^G_{D^x\cap D}\right)\right), & J &:= \sum_{x\in G-H} \mathrm{tr}^H_{D^x\cap H}\left(\mathrm{End}\left(M\downarrow^G_{D^x\cap H}\right)\right), \\ f &:= \mathrm{tr}^G_H, & g &:= \mathrm{r}^G_H. \end{aligned}$$

With this setup the Green correspondence for Green functors (see [Gr71, Proposition 4.34]) follows from Proposition 2.4.7 and the first remark after [Gr71, Hypothesis 4.31].

2.4.2 Composing induction and restriction

We have seen in Subsection 2.2.2 that, when working with Mackey functors over finite groups, there exists a way of rewriting the composition of induction and restriction functors (see Equation (2.5)). In that same subsection we have proven that a similar result holds for centric Mackey functors over fusion systems when composing induction functors of the form $\uparrow_{\mathcal{F}_P}^{\mathcal{F}}$ with restriction functors of the form $\downarrow_{\mathcal{F}_Q}^{\mathcal{F}}$ for some $P, Q \in \mathcal{F}^c$ (see Lemma 2.2.36). However, we haven't shown any result regarding compositions of induction and restriction functors when the fusion systems $\mathcal{F}_P$ and $\mathcal{F}_Q$ of Lemma 2.2.36 are replaced with other fusion subsystems of $\mathcal{F}$. That is precisely the goal of this subsection. More precisely, let $P \in \mathcal{F}^c$ be fully $\mathcal{F}$-normalized, let $M \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$ and let $N \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(N_{\mathcal{F}}(P))$ (see Example 2.2.8), in this subsection we study the $\mathcal{F}$-centric Mackey functors of the form $M \uparrow_{\mathcal{F}_P}^{\mathcal{F}}\downarrow_{N_{\mathcal{F}}(P)}^{\mathcal{F}}$ (see Lemma 2.4.11) and $N \uparrow_{N_{\mathcal{F}}(P)}^{\mathcal{F}}\downarrow_{N_{\mathcal{F}}(P)}^{\mathcal{F}}$ (see Lemma 2.4.12).

Before proceeding let us introduce some notation that is used throughout the rest of this document.

Notation 2.4.9. From now on and unless otherwise specified P **denotes a fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric subgroup of** S, we denote **the normalizer of** P **in** S (i.e. $N_S(P)$) simply **as** N_S, we denote **the normalizer fusion system** $N_{\mathcal{F}}(P)$ (see Example 2.2.8) simply **as** $N_{\mathcal{F}}$ and $\mathcal{X}$ and $\mathcal{Y}$ denote the following sets

$$\mathcal{Y} := \{Q \leq_{\mathcal{F}} P \,:\, Q \leq N_S,\ Q \in \mathcal{F}^c \text{ and } Q \neq P\},$$
$$\mathcal{X} := \{Q \lneq P \,:\, Q \in \mathcal{F}^c\} = \{Q \in \mathcal{Y} \,:\, Q \leq P\}.$$

Lemma 2.4.10. *Let $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} N_S]$, fix a representative φ of $\overline{\varphi}$, let $Q \in \mathcal{F}^c \cap N_{\mathcal{F}}$, let $(B, \overline{\psi}) \in [\varphi(A) \times_{N_{\mathcal{F}}} Q]$ such that $B \in \mathcal{F}_{\varphi(A)} \cap \mathcal{F}^c$ and denote by $\tilde{\varphi}\colon \tilde{\varphi}^{-1}(B) \to B$ the morphism φ seen as an isomorphism between the given subgroups (i.e. the unique morphism such that $\varphi\iota_{\tilde{\varphi}^{-1}(B)}^{A} = \iota_B^{N_S}\tilde{\varphi}$). From the universal properties of products we know that there exist a unique $\left(B^{\mathcal{F},\varphi}, \overline{\psi^{\mathcal{F},\varphi}}\right) \in [P \times_{\mathcal{F}} Q]$ and a unique morphism $\overline{\gamma_{(B,\overline{\psi})}^{\mathcal{F},\varphi}}\colon \tilde{\varphi}^{-1}(B) \to B^{\mathcal{F},\varphi}$ such that $\overline{\iota_{B^{\mathcal{F},\varphi}}^{P}}\,\overline{\gamma_{(B,\overline{\psi})}^{\mathcal{F},\varphi}} = \overline{\iota_{\varphi^{-1}(B)}^{P}}$ and $\overline{\psi^{\mathcal{F},\varphi}}\,\overline{\gamma_{(B,\overline{\psi})}^{\mathcal{F},\varphi}} = \overline{\psi\tilde{\varphi}}$. With this setup the morphism $\overline{\gamma_{(B,\overline{\psi})}^{\mathcal{F},\varphi}}$ belongs to $\mathcal{O}(\mathcal{F}_P)$ and the morphism $\overline{\psi^{\mathcal{F},\varphi}}$ belongs to $\mathcal{O}(N_{\mathcal{F}})$ if and only if $\overline{\varphi}$ belongs to $\mathcal{O}(N_{\mathcal{F}})$.*

Proof. Throughout this proof, contrary to Notation 2.1.1, we write $\overline{\alpha} \in \mathcal{O}(N_{\mathcal{F}})$ to denote that $\overline{\alpha}$ is a morphism in $\mathcal{O}(N_{\mathcal{F}})$ instead of an object in $\mathcal{O}(N_{\mathcal{F}})$.

The fact that $\overline{\gamma_{(B,\overline{\psi})}^{\mathcal{F},\varphi}}$ is a morphism in $\mathcal{O}(\mathcal{F}_P)$ follows immediately from the identity

$\overline{\iota^P_{B^{\mathcal{F},\varphi}}}\overline{\gamma^{\mathcal{F},\varphi}_{(B,\overline{\psi})}} = \overline{\iota^P_{\tilde{\varphi}^{-1}(B)}}$ and definition of the orbit category (see Definition 2.2.10).

Assume that $\overline{\varphi} \notin \mathcal{O}(N_{\mathcal{F}})$. Since $\varphi^{-1}(B) \leq P$, by definition of $N_{\mathcal{F}}$, If $\overline{\tilde{\varphi}} \in \mathcal{O}(N_{\mathcal{F}})$ there would exist $\overline{\hat{\varphi}} \in \mathrm{Hom}_{\mathcal{O}(N_{\mathcal{F}})}(P, N_S)$ such that $\overline{\hat{\varphi}}\overline{\iota^P_{\tilde{\varphi}^{-1}(B)}} = \overline{\iota^{N_S}_B}\overline{\tilde{\varphi}}$. By definition of $\tilde{\varphi}$ this would imply that $\overline{\hat{\varphi}}\overline{\iota^P_A\iota^A_{\tilde{\varphi}^{-1}(B)}} = \overline{\varphi}\overline{\iota^A_{\tilde{\varphi}^{-1}(B)}}$. From [Li07, Theorem 4.9] we would deduce that $\overline{\iota^A_{\tilde{\varphi}^{-1}(B)}}$ is an epimorphism and, therefore, we would conclude that $\overline{\hat{\varphi}}\overline{\iota^P_A} = \overline{\varphi}$. In particular we would have that $\overline{\varphi} \in \mathcal{O}(N_{\mathcal{F}})$ which contradicts our assumption. We can therefore deduce that $\overline{\tilde{\varphi}} \notin \mathcal{O}(N_{\mathcal{F}})$. Since $\overline{\psi} \in \mathcal{O}(N_{\mathcal{F}})$ this implies that $\overline{\psi\tilde{\varphi}} \notin \mathcal{O}(N_{\mathcal{F}})$. On the other hand, since $\mathcal{O}(\mathcal{F}_P) \subseteq \mathcal{O}(N_{\mathcal{F}})$, then we have that $\overline{\gamma^{\mathcal{F},\varphi}_{(B,\overline{\psi})}} \in \mathcal{O}(N_{\mathcal{F}})$. Thus, from the identity $\overline{\psi^{\mathcal{F},\varphi}}\overline{\gamma^{\mathcal{F},\varphi}_{(B,\overline{\psi})}} = \overline{\psi\tilde{\varphi}}$, we can conclude that $\overline{\psi^{\mathcal{F},\varphi}} \notin \mathcal{O}(N_{\mathcal{F}})$.

Assume now that $\overline{\varphi} \in \mathcal{O}(N_{\mathcal{F}})$. In this situation we have that $\overline{\tilde{\varphi}} \in \mathcal{O}(N_{\mathcal{F}})$ and, therefore, $\overline{\psi\tilde{\varphi}} = \overline{\psi^{\mathcal{F},\varphi}}\overline{\gamma^{\mathcal{F},\varphi}_{(B,\overline{\psi})}} \in \mathcal{O}(N_{\mathcal{F}})$. Since $\tilde{\varphi}^{-1}(B) \leq P$, by definition of $N_{\mathcal{F}}$, there exists a morphism $\overline{\theta}\colon P \to N_S$ in $\mathcal{O}(N_{\mathcal{F}})$ such that $\overline{\theta}\overline{\iota^P_{\tilde{\varphi}^{-1}(B)}} = \overline{\iota^{N_S}_Q}\overline{\psi^{\mathcal{F},\varphi}}\overline{\gamma^{\mathcal{F},\varphi}_{(B,\overline{\psi})}}$. Since $\overline{\gamma^{\mathcal{F},\varphi}_{(B,\overline{\psi})}} \in \mathcal{O}(\mathcal{F}_P)$ then there exists $x \in P$ such that $\tilde{\varphi}^{-1}(B)^x \leq B^{\mathcal{F},\varphi}$ and $\overline{\gamma^{\mathcal{F},\varphi}_{(B,\overline{\psi})}} = \overline{\iota^{B^{\mathcal{F},\varphi}}_{\tilde{\varphi}^{-1}(B)^x}c_{x^{-1}}}$. Using both these identities and the definition of the orbit category we can deduce that $\overline{\theta}\overline{\iota^P_{B^{\mathcal{F},\varphi}}}\overline{\iota^{B^{\mathcal{F},\varphi}}_{\tilde{\varphi}^{-1}(B)^x}} = \overline{\iota^{N_S}_Q}\overline{\psi^{\mathcal{F},\varphi}}\overline{\iota^{B^{\mathcal{F},\varphi}}_{\tilde{\varphi}^{-1}(B)^x}}$. From [Li07, Theorem 4.9] we know that $\overline{\iota^{B^{\mathcal{F},\varphi}}_{\tilde{\varphi}^{-1}(B)^x}}$ is an epimorphism and, therefore, we can conclude from the previous identity that $\overline{\theta}\overline{\iota^P_{B^{\mathcal{F},\varphi}}} = \overline{\iota^{N_S}_Q}\overline{\psi^{\mathcal{F},\varphi}} \in \mathcal{O}(N_{\mathcal{F}})$. In particular $\overline{\psi^{\mathcal{F},\varphi}} \in \mathcal{O}(N_{\mathcal{F}})$ thus concluding the proof. □

Using Lemma 2.4.10 we can now give the first of the two main results of this section.

Lemma 2.4.11. *Let $\mathcal{R}$ be a complete local and p-local PID, let $\mathcal{G}$ be a fusion system containing $\mathcal{F}$ and let $M \in Mack^{\mathcal{G}^c}_{\mathcal{R}}(\mathcal{F}_P)$. Then*

$$M \uparrow^{N_{\mathcal{F}}}_{\mathcal{F}_P} \oplus \bigoplus_{Q\in\mathcal{Y}} M^Q \cong M \uparrow^{\mathcal{F}}_{\mathcal{F}_P}\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}$$

where, for every $Q \in \mathcal{Y}$, we have that $M^Q \in Mack^{\mathcal{G}^c}_{\mathcal{R}}(N_{\mathcal{F}})$ is Q-projective. Moreover the isomorphism realizing the above equivalence can be taken so that the summand $M \uparrow^{N_{\mathcal{F}}}_{\mathcal{F}_P}$ on the left hand side is mapped isomorphically to the $\mu_{\mathcal{R}}(N_{\mathcal{F}})$-submodule $\mu_{\mathcal{R}}(N_{\mathcal{F}}) \otimes M$ of $M \uparrow^{\mathcal{F}}_{\mathcal{F}_P}\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}$. Here we are using Corollary 2.2.25 in order to view $\mu_{\mathcal{R}}(N_{\mathcal{F}}) \otimes M$ as a submodule of $M \uparrow^{\mathcal{F}}_{\mathcal{F}_P}\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}$.

Proof. In order to simplify notation we define $M^{N_{\mathcal{F}}} := M \uparrow^{\mathcal{F}}_{\mathcal{F}_P}\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}$. From Proposition

2.2.30 we know that $M^{N_\mathcal{F}}$ is $\mathcal{G}$-centric and, therefore, every M^K (if exists) must necessarily be $\mathcal{G}$-centric.

For every $(A,\overline{\varphi}) \in [P \times_\mathcal{F} N_S]$ fix a representative φ of $\overline{\varphi}$ and view it as an isomorphism onto its image. Since $M^{N_\mathcal{F}} \downarrow^{N_\mathcal{F}}_{\mathcal{F}_{N_S}} = M \uparrow^{\mathcal{F}}_{\mathcal{F}_P} \downarrow^{\mathcal{F}}_{\mathcal{F}_{N_S}}$ we can use Lemma 2.2.36 in order to obtain a decomposition of $M^{N_\mathcal{F}} \downarrow^{N_\mathcal{F}}_{\mathcal{F}_{N_S}}$ as a direct sum of $\mu_\mathcal{R}(\mathcal{F}_{N_S})$-modules. Applying the additive functor $\uparrow^{N_\mathcal{F}}_{\mathcal{F}_{N_S}}$ to the resulting decomposition we can conclude that $\left(M^{N_\mathcal{F}}\right)_{N_S} = M^P \oplus M^\mathcal{Y}$ (see Definition 2.2.37) where, using the notation of Lemma 2.2.36, we define

$$M^P := \bigoplus_{\substack{(A,\overline{\varphi}) \in [P \times_\mathcal{F} N_S] \\ \overline{\varphi} \in \mathcal{O}(N_\mathcal{F})}} M_{(A,\overline{\varphi})} \uparrow^{N_\mathcal{F}}_{\mathcal{F}_{\varphi(A)}}, \qquad M^\mathcal{Y} := \bigoplus_{\substack{(A,\overline{\varphi}) \in [P \times_\mathcal{F} N_S] \\ \overline{\varphi} \notin \mathcal{O}(N_\mathcal{F})}} M_{(A,\overline{\varphi})} \uparrow^{N_\mathcal{F}}_{\mathcal{F}_{\varphi(A)}}.$$

Here we are viewing the right hand sides of the above definitions as submodules of $\left(M^{N_\mathcal{F}}\right)_{N_S}$ via the isomorphism described in Lemma 2.2.36. From Proposition 2.2.33 we know that all the elements in $\theta^{M^{N_S}}_{N_S}\left(M_{(A,\overline{\varphi})} \uparrow^{N_\mathcal{F}}_{\mathcal{F}_{\varphi(A)}}\right)$ (see Definition 2.2.37) can be written as finite sums of elements of the form $I^Q_{\overline{\psi}(B)} c_{\overline{\psi\tilde{\varphi}}} \otimes x$ for some $Q \in N_\mathcal{F} \cap \mathcal{G}^c$, some $\left(B,\overline{\psi}\right) \in [\varphi(A) \times_{N_\mathcal{F}} Q]$ such that $B \in \mathcal{F}_{\varphi(A)} \cap \mathcal{G}^c$ and some $x \in I^{\varphi^{-1}(B)}_{\varphi^{-1}(B)} M$. Here $\tilde{\varphi}\colon \varphi^{-1}(B) \to B$ denotes the morphism φ seen as an isomorphism between the given subgroups. From Lemma 2.4.10 we can now conclude that, for every $Q \in N_\mathcal{F} \cap \mathcal{G}^c$, the elements of $I^Q_Q \theta^{M^{N_S}}_{N_S}\left(M^P\right)$ can be written as finite sums of elements of the form $I^Q_{\overline{\theta}(C)} c_{\overline{\theta}} \otimes x$ for some $x \in I^C_C M$ and some $\left(C,\overline{\theta}\right) \in [P \times_\mathcal{F} Q]$ such that $\overline{\theta} \in \mathcal{O}(N_\mathcal{F})$. Notice that the tensor product is over $\mu_\mathcal{R}(\mathcal{F}_P)$ and not $\mu_\mathcal{R}(\mathcal{F}_{N_S})$ since $I^Q_{\overline{\theta}(C)} c_{\overline{\theta}} \otimes x$ is an element of $M^{N_\mathcal{F}} := M \uparrow^{\mathcal{F}}_{\mathcal{F}_P} \downarrow^{\mathcal{F}}_{N_\mathcal{F}}$. Likewise, the elements of $I^Q_Q \theta^{M^{N_\mathcal{F}}}_{N_S}\left(M^\mathcal{Y}\right)$ can be written as finite sums of elements of the form $I^Q_{\overline{\theta}(C)} c_{\overline{\theta}} \otimes x$ for some $x \in I^C_C M$ and some $\left(C,\overline{\theta}\right) \in [P \times_\mathcal{F} Q]$ such that $\overline{\theta} \notin \mathcal{O}(N_\mathcal{F})$. Applying again Proposition 2.2.33 we can conclude that $\theta^{M^{N_\mathcal{F}}}_{N_S}\left(M^P\right) \cap \theta^{M^{N_\mathcal{F}}}_{N_S}\left(M^\mathcal{Y}\right) = \{0\}$. On the other hand, since $\mathcal{R}$ is p-local, we have from Lemma 2.3.2 that $\theta^{M^{N_\mathcal{F}}}_{N_S}$ is split surjective and, in particular, surjective. Since $\left(M^{N_\mathcal{F}}\right)_{N_S} = M^P \oplus M^\mathcal{Y}$, from the previous result, we can conclude that

$$M^{N_\mathcal{F}} = \theta^{M^{N_\mathcal{F}}}_{N_S}\left(M^P\right) \oplus \theta^{M^{N_\mathcal{F}}}_{N_S}\left(M^\mathcal{Y}\right). \tag{2.9}$$

By definition of $\mathcal{O}(N_\mathcal{F})$ (see Example 2.2.8) we have that for every $A \leq P$ and every $\overline{\varphi}\colon A \to N_S$ in $\mathcal{O}(N_\mathcal{F})$ there exists a morphism $\overline{\hat{\varphi}}\colon P \to N_S$ in $\mathcal{O}(N_\mathcal{F})$ such that

$\overline{\widehat{\varphi}\iota_A^P} = \overline{\varphi}$. We also have that $\mathcal{O}(N_\mathcal{F}) \subseteq \mathcal{O}(\mathcal{F})$. Therefore, for every $(A, \overline{\varphi}) \in [P \times_\mathcal{F} N_S]$ such that $\overline{\varphi} \in \mathcal{O}(N_\mathcal{F})$, we can deduce from maximality (see Definition 2.2.17) that $A = P$ and $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{O}(\mathcal{F})}(P)$. From Proposition 2.2.18 (4) and the above description of elements in $\theta_{N_S}^{M^{N_\mathcal{F}}}\left(M_{(A,\overline{\varphi})}\uparrow_{\mathcal{F}_P}^{N_\mathcal{F}}\right)$ we can then conclude that, for every $Q \in N_\mathcal{F} \cap \mathcal{G}^c$, the elements in $I_Q^Q \theta_{N_S}^{M^{N_S}}\left(M^P\right)$ are finite sums of elements of the form $I_{\overline{\psi}(B)}^Q c_{\overline{\psi}} \otimes x$ for some $\left(B, \overline{\psi}\right) \in [P \times_{N_\mathcal{F}} Q]$ and some $x \in I_B^B M$. From Proposition 2.2.33 we can then conclude that $\theta_{N_S}^{M^{N_\mathcal{F}}}\left(M_{(A,\overline{\varphi})}\uparrow_{\mathcal{F}_P}^{N_\mathcal{F}}\right)$ is precisely the submodule $\mu_\mathcal{R}(N_\mathcal{F}) \otimes M$ of $M\uparrow_{\mathcal{F}_P}^{\mathcal{F}}\downarrow_{N_\mathcal{F}}^{\mathcal{F}}$ which is, by definition, isomorphic to $M\uparrow_{\mathcal{F}_P}^{N_\mathcal{F}}$.

From Equation (2.9) and the fact that $\theta_{N_S}^{M^{N_\mathcal{F}}}$ is split surjective, we conclude that the restriction of $\theta_{N_S}^{M^{N_\mathcal{F}}}$ as a map from $M^\mathcal{Y}$ to $\theta_{N_S}^{M^{N_\mathcal{F}}}\left(M^\mathcal{Y}\right)$ is also split surjective. In particular we have that $\theta_{N_S}^{M^{N_\mathcal{F}}}\left(M^\mathcal{Y}\right)$ is isomorphic to a summand of $M^\mathcal{Y}$. Notice now that, for every $(A, \overline{\varphi}) \in [P \times_\mathcal{F} N_S]$, we have that $\varphi(A) \leq_\mathcal{F} P$ and, if $\varphi(A) = P$, then we necessarily have that $A = P$ and $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{O}(\mathcal{F})}(P) = \mathrm{Aut}_{\mathcal{O}(N_\mathcal{F})}(P)$. We can therefore conclude that

$$M^\mathcal{Y} = \bigoplus_{Q \in \mathcal{Y}} M'^Q \qquad \text{where} \qquad M'^Q := \bigoplus_{\substack{(A,\overline{\varphi}) \in [P \times_\mathcal{F} N_S] \\ \varphi(A) = Q}} M_{(A,\overline{\varphi})}\uparrow_{\mathcal{F}_Q}^{N_\mathcal{F}}.$$

Since $\mathcal{R}$ is a complete local PID we can now apply the Krull-Schmidt-Azumaya theorem (see [CR81, Theorem 6.12 (ii)]) in order to write $\theta_{N_S}^{M^{N_\mathcal{F}}}\left(M^\mathcal{Y}\right) = \bigoplus_{Q \in \mathcal{Y}} M^Q$ where each M^Q is a summand of M'^Q. From Theorem 2.3.17 we know that each M'^Q is Q-projective. Therefore since each M^Q is a summand of M'^Q we can conclude, again from Theorem 2.3.17, that M^Q is Q-projective thus concluding the proof. □

Using Lemma 2.4.11 we can now obtain the following result with which we conclude this subsection.

Lemma 2.4.12. *Let $\mathcal{R}$ be a complete local and p-local PID, let $\mathcal{G}$ be a fusion system containing $\mathcal{F}$ and let $M \in$ Mack$_\mathcal{R}^{\mathcal{G}^c}(N_\mathcal{F})$ be P-projective. Then, there exists an $\mathcal{Y}$-projective $M' \in$ Mack$_\mathcal{R}^{\mathcal{G}^c}(N_\mathcal{F})$ such that*

$$M\uparrow_{N_\mathcal{F}}^{\mathcal{F}}\downarrow_{N_\mathcal{F}}^{\mathcal{F}} \cong M \oplus M'.$$

Proof. From Proposition 2.2.30 we know that if such a direct sum decomposition exists then M' is necessarily $\mathcal{G}$-centric. From Theorem 2.3.17 we know that there exist $N \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{G}^c}(\mathcal{F}_P)$ and $U \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{G}^c}(N_{\mathcal{F}})$ such that $M \oplus U \cong N \uparrow_{\mathcal{F}_P}^{N_{\mathcal{F}}}$. Since induction and restriction preserve direct sum decomposition, from Lemma 2.4.11, we obtain an isomorphism

$$f : M \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} \oplus U \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} \hookrightarrow\!\!\!\!\rightarrow N \uparrow_{\mathcal{F}_P}^{N_{\mathcal{F}}} \oplus \bigoplus_{Q \in \mathcal{Y}} N^Q.$$

Where each N^Q is Q-projective. Lemma 2.4.11 also tells us that f sends the sub-module $M \oplus U$ of $M \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} \oplus U \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ isomorphically onto the summand $N \uparrow_{\mathcal{F}_P}^{N_{\mathcal{F}}}$ of the right hand side. Using this we obtain the following equivalence of $\mu_{\mathcal{R}}(N_{\mathcal{F}})$-modules

$$\begin{aligned} M \oplus \bigoplus_{Q \in \mathcal{Y}} N^Q &\cong \left(M \oplus U \oplus \bigoplus_{Q \in \mathcal{Y}} N^Q \right) /U, \\ &\cong \left(M \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} \oplus U \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} \right) /U, \\ &\cong M \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} \oplus \left(U \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} /U \right). \end{aligned}$$

In particular we can conclude that $M \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ is a summand of $M \oplus \bigoplus_{Q \in \mathcal{Y}} N^Q$. Moreover, again from the description of f, we have that $M \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ contains the summand M. Since $\mathcal{R}$ is complete local and p-local then we can use this and the Krull-Schmidt-Azumaya theorem in order to conclude that there exists a summand M' of $\bigoplus_{Q \in \mathcal{Y}} N^Q$ (which is necessarily $\mathcal{Y}$-projective from Theorem 2.3.17) such that $M \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} \cong M \oplus M'$. This concludes the proof. □

2.4.3 Composing trace and restriction

Through this subsection we will be using Notations 2.1.1, 2.2.3, 2.2.9, 2.2.21, 2.2.31 and 2.4.9.

Let G be a finite group, let H, K and J be subgroups of G such that $J \leq K$ and for every $x \in [K \backslash G/H]$ define

$$[J \backslash K/ (K \cap {}^x H)]\, x := \{ yx \in G \,:\, y \in [J \backslash K/ (K \cap {}^x H)] \}.$$

It is well known that the following decomposition of double cosets representatives holds

$$[J \backslash G/H] = \bigsqcup_{x \in [K \backslash G/H]} [J \backslash K/ (K \cap {}^x H)]\, x. \tag{2.10}$$

Denoting by tr_J^G and r_H^G the trace and restriction maps of the Endomorphism Mackey functor $\mathrm{End}(M)$ (see [Sa82, Definition 2.7]) Equation (2.10) can be used in order to prove that for any Mackey functor M over G

$$\mathrm{r}_K^G\,\mathrm{tr}_H^G = \sum_{x\in[K\backslash G/H]} \mathrm{tr}_{K\cap {}^xH}^K\, c_{c_x}\, \mathrm{r}_{K^x\cap H}^H\,. \tag{2.11}$$

We know from Proposition 2.3.9 (9) that a similar result holds in the case of the trace and restriction maps of Definition 2.3.8. However, Proposition 2.3.9 (9) only involves composition of trace and restriction maps of the form $\mathrm{r}_{\mathcal{F}_Q}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}$ for some $P, Q \in \mathcal{F}^c$ and tells us nothing regarding compositions of trace and restriction of the form $\mathrm{r}_{\mathcal{G}}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}$ for other fusion system $\mathcal{G}$ contained in $\mathcal{F}$. Attempting to obtain a decomposition similar to that of Proposition 2.3.9 (9) in this situation leads to several complications. These can be traced back to the lack of a result analogous to Proposition 2.2.18 (6) in the case where P is replaced with $\mathcal{G}$ and $[A\backslash P/J]$ is replaced with $[A\times_{\mathcal{G}} J]$. Some experimentation leads us to believe that such a result is possible when $P=Q$ and $\mathcal{G}=N_{\mathcal{F}}$ (i.e. a result dual to Theorem 2.4.27), however we were unable to prove it. Nonetheless we were able to obtain a result analogous to Equation (2.11) for the composition $\mathrm{r}_{N_{\mathcal{F}}}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}$ (see Proposition 2.4.16) and this subsection is dedicated to proving it. In order to do so we first need to develop some tools.

Lemma 2.4.13. *Let $\mathcal{R}$ be a p-local ring, let $M \in \mathit{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ and let $\overline{N_S} \in B_{\mathcal{R}}^{(N_{\mathcal{F}})^c}$ be the isomorphism class of N_S. From Proposition 2.2.39 we know that $\overline{N_S}$ admits an inverse in $B_{\mathcal{R}}^{(N_{\mathcal{F}})^c}$. With this setup we have that*

$$\mathrm{r}_{N_{\mathcal{F}}}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}} = \sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}N_S]} \left(\overline{N_S}^{-1}\cdot\right)_*\mathrm{tr}_{\mathcal{F}_{\varphi(A)}}^{N_{\mathcal{F}}}\,{}^{\varphi}\cdot\,\mathrm{r}_{\mathcal{F}_A}^{\mathcal{F}_P}\,.$$

where we are using Notation 2.1.1 as well as the notation of Proposition 2.2.43 and Definition 2.3.8 and we are viewing the representative φ of $\overline{\varphi}$ as an isomorphism onto its image. Equivalently, using the same notation, we have that

$$\left(\overline{N_S}\cdot\right)_*\mathrm{r}_{N_{\mathcal{F}}}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}} = \sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}N_S]} \mathrm{tr}_{\mathcal{F}_{\varphi(A)}}^{N_{\mathcal{F}}}\,{}^{\varphi}\cdot\,\mathrm{r}_{\mathcal{F}_A}^{\mathcal{F}_P}\,.$$

Proof. Since the first and second identities of the statement are equivalent we just prove the second identity. Notice that we can rewrite

$$\left(\overline{N_S}\cdot\right)_*\mathrm{r}_{N_{\mathcal{F}}}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}} = \mathrm{tr}_{\mathcal{F}_{N_S}}^{N_{\mathcal{F}}}\,\mathrm{r}_{\mathcal{F}_{N_S}}^{N_{\mathcal{F}}}\,\mathrm{r}_{N_{\mathcal{F}}}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}} = \mathrm{tr}_{\mathcal{F}_{N_S}}^{N_{\mathcal{F}}} \sum_{(A,\overline{\varphi})\in[P\times_{\mathcal{F}}N_S]} \mathrm{tr}_{\mathcal{F}_{\varphi(A)}}^{\mathcal{F}_{N_S}}\,{}^{\varphi}\cdot\,\mathrm{r}_{\mathcal{F}_A}^{\mathcal{F}_P}\,.$$

Here we are using Item (11) of Proposition 2.3.9 for the first identity and Items (2) and (9) of the same proposition for the second identity. The Lemma follows after applying Proposition 2.3.9 (3) to the identity above. □

Lemma 2.4.14. *Let $P \in \mathcal{F}^c$ be such that $\mathcal{F} = N_{\mathcal{F}}(P)$ and let $Q \in \mathcal{F}_P \cap \mathcal{F}^c$. Then we have that*

$$[Q \times_{\mathcal{F}} S] = \left\{ (Q, \overline{\varphi}) \mid \overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(Q, S) \right\}, \quad \textit{and} \quad \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(Q, S) \cong \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P, S).$$

In particular we have the following bijection of finite sets

$$[Q \times_{\mathcal{F}} S] \cong [P \times_{\mathcal{F}} S] \cong \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(Q, S).$$

Proof. Since $Q \leq P$, for any subgroup $A \leq Q$, we have that $PA = P$. Analogously we also have that $PS = S$. Since $\mathcal{F} = N_{\mathcal{F}}(P)$, by definition (see Example 2.2.8), we can conclude that for every $A \leq Q$ and every morphism $\overline{\varphi}\colon A \to S$ in $\mathcal{O}(\mathcal{F})$ there exists a morphism $\overline{\hat{\varphi}}\colon P \to S$ in $\mathcal{O}(\mathcal{F})$ such that $\overline{\hat{\varphi}}\,\overline{\iota_Q^P}\,\overline{\iota_A^Q} = \overline{\hat{\varphi}}\,\overline{\iota_A^P} = \overline{\varphi}$. From maximality of the pairs $(A, \overline{\varphi}) \in [Q \times_{\mathcal{F}} S]$ (see Definition 2.2.17) we can conclude that $A = Q$ and, therefore, we have that

$$[Q \times_{\mathcal{F}} S] = \left\{ (Q, \overline{\varphi}) \mid \overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(Q, S) \right\} \cong \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(Q, S).$$

Thus we are only left with proving that $\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(Q, S) \cong \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P, S)$.

It suffices to prove that the map $\left(\overline{\iota_Q^P}\right)^*$ from $\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P, S)$ to $\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(Q, S)$ (see Notation 2.1.1) is bijective. From [Li07, Theorem 4.9] we know that $\overline{\iota_Q^P}$ is surjective. On the other hand it is well known that the contravariant Hom functor $\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(-, S)$ is left exact and, in particular, sends surjective morphisms to injective morphisms. Joining both these facts we can conclude that $\left(\overline{\iota_Q^P}\right)^*$ is injective.

On the other hand, as mentioned at the beginning of the proof, for every morphism $\overline{\varphi} : Q \to S$ in $\mathcal{O}(\mathcal{F})$ there exists a morphism $\overline{\hat{\varphi}} : P \to S$ in $\mathcal{O}(\mathcal{F})$ such that $\overline{\varphi} = \overline{\hat{\varphi}}\,\overline{\iota_Q^P}$. This proves that $\left(\overline{\iota_Q^P}\right)^*$ is also surjective thus concluding the proof. □

We can now finally obtain the last ingredient needed in order to prove Proposition 2.4.16.

Lemma 2.4.15. *Let $P \in \mathcal{F}^c$ be such that $\mathcal{F} = N_{\mathcal{F}}(P)$, let $M \in Mack_{\mathcal{R}}(\mathcal{F}^c)$, let $\mathcal{R}$ be a p-local ring and let $\overline{S} \in B_{\mathcal{R}}^{\mathcal{F}^c}$ be the isomorphism class of S. From Proposition 2.2.39 we know that $\overline{S}$ has an inverse in $B_{\mathcal{R}}^{\mathcal{F}^c}$. With this setup the following equivalent identities are satisfied*

$$\sum_{\overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P,S)} \left(\overline{S}^{-1} \cdot\right)_* \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}} {}^{\varphi}\cdot = \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}, \qquad \sum_{\overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P,S)} \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}} {}^{\varphi}\cdot = \left(\overline{S}\cdot\right)_* \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}.$$

Where we are viewing the representative φ of $\overline{\varphi}$ as an isomorphism onto its image and we are dropping the left superindex M in order to keep notation simple.

Proof. We only prove the second identity since both identities are equivalent. From Proposition 2.3.9 (11) we know that $\left(\overline{S}\cdot\right)_* = \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}} \mathrm{r}_{\mathcal{F}_P}^{\mathcal{F}}$. Combining this with Proposition 2.3.9 (9) we obtain the identity $\left(\overline{S}\cdot\right)_* \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}} = \mathrm{tr}_{\mathcal{F}_S}^{\mathcal{F}} \sum_{(A,\overline{\varphi}) \in [P \times_{\mathcal{F}} S]} \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}_S} {}^{\varphi}\cdot$. The result now follows from Proposition 2.3.9 (3) and Lemma 2.4.14. □

We are now finally able to give a result for centric Mackey functors over fusion systems analogous to that of Equation (2.11) in a case not covered by Proposition 2.3.9 (9).

Proposition 2.4.16. *Let $\mathcal{R}$ be a p-local ring, let $M \in Mack_{\mathcal{R}}(\mathcal{F}^c)$ and for every $(A, \overline{\varphi}) \in [P \times_{\mathcal{F}} N_S]$ fix a representative φ of $\overline{\varphi}$ seen as an isomorphism onto its image. From Proposition 2.2.39 we know that the $N_{\mathcal{F}}$-conjugacy class $\overline{N_S} \in B_{\mathcal{R}}^{(N_{\mathcal{F}})^c}$ of N_S has an inverse in $B_{\mathcal{R}}^{(N_{\mathcal{F}})^c}$ and, using the notation of Proposition 2.2.43, we have that $\overline{N_S}^{-1} \cdot \in \mathrm{End}\left(M \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}\right)$. For every $f \in \mathrm{End}\left(M \downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)$ and every $Q \in \mathcal{Y}$ (see Notation 2.4.9) we can now define*

$$f_Q := \sum_{\substack{(A,\overline{\varphi}) \in [P \times_{\mathcal{F}} N_S] \\ \varphi(A) = Q}} \left(\mathrm{r}_{\mathcal{F}_{\varphi(A)}}^{N_{\mathcal{F}}} \left(\overline{N_S}^{-1} \cdot\right)\right)^{\varphi} \left(\mathrm{r}_{\mathcal{F}_A}^{\mathcal{F}_P}(f)\right) \in \mathrm{End}\left(M \downarrow_{\mathcal{F}_Q}^{\mathcal{F}}\right).$$

and the following identity holds

$$\mathrm{r}_{N_{\mathcal{F}}}^{\mathcal{F}} \left(\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(f)\right) = \mathrm{tr}_{\mathcal{F}_P}^{N_{\mathcal{F}}}(f) + \sum_{Q \in \mathcal{Y}} \mathrm{tr}_{\mathcal{F}_Q}^{N_{\mathcal{F}}}(f_Q).$$

Different choices of $[P \times_{\mathcal{F}} N_S]$ and representative $\varphi \in \overline{\varphi}$ can lead to different definitions of each individual f_Q but the result holds for any such choice.

Proof. Applying Proposition 2.3.9 (10) to $\mathrm{tr}_{\mathcal{F}_Q}^{N_\mathcal{F}}(f_Q)$ for every $Q \in \mathcal{Y}$ we obtain

$$\sum_{Q\in\mathcal{Y}} \mathrm{tr}_{\mathcal{F}_Q}^{N_\mathcal{F}}(f_Q) = \sum_{Q\in\mathcal{Y}} \sum_{\substack{(A,\overline{\varphi})\in[P\times_\mathcal{F} N_S]\\ \varphi(A)=Q}} \left(\overline{N_S}^{-1}\cdot\right)_* \left(\mathrm{tr}_{\mathcal{F}_Q}^{N_\mathcal{F}}\left({}^{\varphi}\left(\mathrm{r}_{\mathcal{F}_A}^{\mathcal{F}_P}(f)\right)\right)\right),$$

$$= \sum_{\substack{(A,\overline{\varphi})\in[P\times_\mathcal{F} N_S]\\ \varphi(A)\in\mathcal{Y}}} \left(\overline{N_S}^{-1}\cdot\right)_* \left(\mathrm{tr}_{\mathcal{F}_Q}^{N_\mathcal{F}}\left({}^{\varphi}\left(\mathrm{r}_{\mathcal{F}_A}^{\mathcal{F}_P}(f)\right)\right)\right).$$

Subtracting the above identity to the one in the statement and applying Lemma 2.4.13 to $\mathrm{r}_{N_\mathcal{F}}^{\mathcal{F}}\left(\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}(f)\right)$ we obtain that the following identity is equivalent to the one in the statement

$$\sum_{\substack{(A,\overline{\varphi})\in[P\times_\mathcal{F} N_S]\\ \varphi(A)\notin\mathcal{Y}}} \left(\overline{N_S}^{-1}\cdot\right)_* \mathrm{tr}_{\mathcal{F}_Q}^{N_\mathcal{F}}\, {}^{\varphi}\cdot\, \mathrm{r}_{\mathcal{F}_A}^{\mathcal{F}_P} = \mathrm{tr}_{\mathcal{F}_P}^{N_\mathcal{F}}.$$

Because of Lemma 2.4.15 it now suffices to prove the identity

$$\{(A,\overline{\varphi}) \in [P\times_\mathcal{F} N_S] \mid \varphi(A) \notin \mathcal{Y}\} = \left\{(P,\overline{\varphi}) \mid \overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(N_\mathcal{F})}(P, N_S)\right\}.$$

For every $(A,\overline{\varphi}) \in [P\times_\mathcal{F} N_S]$ we have that $\varphi(A) \leq_\mathcal{F} N_S$ and, therefore, by definition of $\mathcal{Y}$ (see Notation 2.4.9), we have that $\varphi(A) \notin \mathcal{Y}$ if and only if $\varphi(A) = P$. Since φ is an isomorphism, $A \leq P$ and the groups A and P are finite then the identity $\varphi(A) = P$ implies that $A = P$ and, therefore, $\varphi \in \mathrm{Aut}_\mathcal{F}(P) = \mathrm{Aut}_{N_\mathcal{F}}(P)$. Equivalently $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(N_\mathcal{F})}(P, N_S)$. This proves one inclusion. On the other hand, for every $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(N_\mathcal{F})}(P, N_S)$, we know from the universal properties of product that there exist a unique $\left(B,\overline{\psi}\right) \in [P\times_\mathcal{F} N_S]$ and a unique $\overline{\gamma}\colon P \to B$ such that $\overline{\psi}\overline{\gamma} = \overline{\varphi}$ and $\overline{\iota_B^P}\overline{\gamma} = \overline{\iota_P^P} = \overline{\mathrm{Id}_P}$. From these identities we can conclude that $\left(B,\overline{\psi}\right) = (P,\overline{\varphi})$. This proves the second inclusion thus completing the proof. □

2.4.4 Decomposing the product in $\mathcal{O}(\mathcal{F}^c)_\sqcup$

Through this subsection we will be using Notations 2.1.1, 2.2.3, 2.2.9, 2.2.21, 2.2.31 and 2.4.9.

Let G be a finite group, let H, K and J be subgroups of G such that $J \leq K$ and for every $x \in [H\backslash G/K]$ define

$$x\left[(H^x \cap K)\backslash K/J\right] := \{xy \in G \,:\, y \in \left[(H^x \cap K)\backslash K/J\right]\}.$$

It is well known that the following identity, dual to Equation (2.10), holds

$$[H\backslash G/J] = \bigsqcup_{x\in[H\backslash G/K]} x\left[(H^x \cap K)\backslash K/J\right]. \tag{2.12}$$

In the case of Mackey functors over finite groups, this can be used to prove that $\mathrm{tr}_K^G \mathrm{tr}_J^K = \mathrm{tr}_J^G$ where tr_A^B denotes the trace map of the endomorphism Mackey functor $\mathrm{End}(M)$ for some Mackey functor M over G (see [Sa82, Definition 2.7]). Proposition 2.3.9 (3) proves that a similar result holds for fusion systems. However, in the case of Mackey functors over fusion systems, given $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ and a fusion system $\mathcal{H}$, such that $\mathcal{F}_P \subseteq \mathcal{H} \subseteq \mathcal{F}$ the trace $\mathrm{tr}_{\mathcal{H}}^{\mathcal{F}}\colon \mathrm{End}\left(M\downarrow_{\mathcal{H}}^{\mathcal{F}}\right) \to \mathrm{End}(M)$ is in general not defined. We show with Definition 2.4.28 and Lemma 2.4.29 that the trace $\mathrm{tr}_{\mathcal{H}}^{\mathcal{F}}$ can be defined when $\mathcal{H} = N_{\mathcal{F}}$ (see Notation 2.4.9) and that, in this case, we have $\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}} \mathrm{tr}_{\mathcal{F}_P}^{N_{\mathcal{F}}} = \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}$. However, in order to prove such a result, we first need to translate Equation (2.12) to the context of fusion systems. More precisely, we need to prove that, for every $Q \in \mathcal{F}^c$, we can write $[P \times_{\mathcal{F}} Q]$ in terms of sets of the form $[P \times_{N_{\mathcal{F}}} A]$ with $A \in N_{\mathcal{F}} \cap \mathcal{F}^c$. This section is dedicated to proving exactly that (see Theorem 2.4.27).

Let us start by finding what can replace the groups $H^x \cap K$ of Equation (2.12) in the context of fusion systems.

Lemma 2.4.17. *Let $A, Q \in \mathcal{F}^c$ with $A \leq N_S$ and let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(A, Q)$. Using Notation 2.4.9 we define the* ***normalizer after*** φ ***in*** $N_{\mathcal{F}}$ *as*

$${}^{N_{\mathcal{F}}}_{\varphi}N := \left\{x \in N_Q(\varphi(A)) \,:\, \varphi^{-1} c_x \varphi \in \mathrm{Aut}_{N_{\mathcal{F}}}(A)\right\},$$

where, on the right hand side, we are viewing φ as an isomorphism onto its image. Then ${}^{N_{\mathcal{F}}}_{\varphi}N$ is the unique maximal subgroup of $N_Q(\varphi(A))$ such that

$$\mathrm{Aut}_{{}^{N_{\mathcal{F}}}_{\varphi}N}(\varphi(A)) \leq {}^{\varphi}\mathrm{Aut}_{N_{\mathcal{F}}}(A).$$

Moreover there exist a fully $N_{\mathcal{F}}$-normalized subgroup $A' \leq N_S$, an isomorphism $\theta \in \mathrm{Hom}_{N_{\mathcal{F}}}(A', A)$ and a subgroup $N_{\varphi\theta}^{N_{\mathcal{F}}}$ of $N_{N_S}(A')$ containing A' such that

$$\mathrm{Aut}_{{}^{N_{\mathcal{F}}}_{\varphi}N}(\varphi(A)) = {}^{\varphi\theta}\mathrm{Aut}_{N_{\varphi\theta}^{N_{\mathcal{F}}}}(A').$$

More precisely we can take θ such that

$$N_{\varphi\theta}^{N_{\mathcal{F}}} = \left\{x \in N_{N_S}(A') \,:\, c_x \in \mathrm{Aut}_{{}^{N_{\mathcal{F}}}_{\varphi}N}(\varphi(A))^{\varphi\theta}\right\}.$$

*We call any morphism of the form $\varphi\theta$ with θ as before $N_{\mathcal{F}}$-**top of** φ and denote it by*

$\varphi^{N_\mathcal{F}}$. *We also call* ***normalizer before*** $\varphi^{N_\mathcal{F}}$ ***in*** $N_\mathcal{F}$ *any group of the form* $N^{N_\mathcal{F}}_{\varphi^{N_\mathcal{F}}}$.

Proof. First of all notice that $\varphi^{-1}c_{1_{N_K(\varphi(A))}}\varphi = \mathrm{Id}_A$, that for any $x \in {}^{N_\mathcal{F}}_{\varphi}N$ we have $\varphi^{-1}c_{x^{-1}}\varphi = (\varphi^{-1}c_x\varphi)^{-1}$ and that for any other $y \in {}^{N_\mathcal{F}}_{\varphi}N$ we have $\varphi^{-1}c_{xy}\varphi = (\varphi^{-1}c_x\varphi)(\varphi^{-1}c_y\varphi)$. Since $\mathrm{Aut}_{N_\mathcal{F}}(A)$ is a subgroup of $\mathrm{Aut}(A)$ the previous equations prove that $1_{N_Q(\varphi(A))} \in {}^{N_\mathcal{F}}_{\varphi}N$, that $x^{-1} \in {}^{N_\mathcal{F}}_{\varphi}N$ and that $xy \in {}^{N_\mathcal{F}}_{\varphi}N$ respectively. We can therefore conclude that ${}^{N_\mathcal{F}}_{\varphi}N$ is indeed a subgroup of $N_Q(\varphi(A))$. Moreover, from definition of ${}^{N_\mathcal{F}}_{\varphi}N$, we have that $\mathrm{Aut}_{{}^{N_\mathcal{F}}_{\varphi}N}(\varphi(A))^{\varphi}$ is a subgroup of $\mathrm{Aut}_{N_\mathcal{F}}(A)$. Equivalently, $\mathrm{Aut}_{{}^{N_\mathcal{F}}_{\varphi}N}(\varphi(A))$ is a subgroup of ${}^{\varphi}\mathrm{Aut}_{N_\mathcal{F}}(A)$. On the other hand, for every $x \in N_Q(\varphi(A))$ such that $c_x \in {}^{\varphi}\mathrm{Aut}_{N_\mathcal{F}}(A)$ we have by definition that $\varphi^{-1}c_x\varphi \in \mathrm{Aut}_{N_\mathcal{F}}(A)$ and, therefore, that $x \in {}^{N_\mathcal{F}}_{\varphi}N$. This proves that ${}^{N_\mathcal{F}}_{\varphi}N$ is indeed the unique maximal subgroup of $N_Q(\varphi(A))$ with the desired properties.

Let's now prove the second half of the statement. Let $A' =_{N_\mathcal{F}} A$ be fully $N_\mathcal{F}$-normalized and let $\alpha : A' \to A$ be an isomorphism in $N_\mathcal{F}$. Since ${}^{N_\mathcal{F}}_{\varphi}N \leq S$ and S is a p-group then ${}^{N_\mathcal{F}}_{\varphi}N$ is also a p-group. It follows that $\mathrm{Aut}_{{}^{N_\mathcal{F}}_{\varphi}N}(\varphi(A))^{\varphi\alpha}$ is also a p-group. On the other hand, from construction of A' and α have that

$$\mathrm{Aut}_{{}^{N_\mathcal{F}}_{\varphi}N}(\varphi(A))^{\varphi\alpha} \leq \mathrm{Aut}_{N_\mathcal{F}}(A').$$

Since $\mathrm{Aut}_{N_S}(A')$ is a Sylow p-subgroup of $\mathrm{Aut}_{N_\mathcal{F}}(A')$ (see [St03, Proposition 2.5]) we can apply second Sylow theorem in order to obtain $\beta \in \mathrm{Aut}_{N_\mathcal{F}}(A')$ satisfying

$$\mathrm{Aut}_{{}^{N_\mathcal{F}}_{\varphi}N}(\varphi(A))^{\varphi\alpha\beta} \leq \mathrm{Aut}_{N_S}(A'). \tag{2.13}$$

We can now define $\theta := \alpha\beta$ and let $N^{N_\mathcal{F}}_{\varphi\theta}$ be as in the statement. The same arguments used to prove that ${}^{N_\mathcal{F}}_{\varphi}N$ is a subgroup of $N_Q(\varphi(A))$ can be used to prove that $N^{N_\mathcal{F}}_{\varphi\theta}$ is a subgroup of $N_{N_S}(A')$. For every $x \in A'$ we have $\varphi\theta c_x(\varphi\theta)^{-1} = c_{\varphi\theta(x)} \in \mathrm{Aut}_{{}^{N_\mathcal{F}}_{\varphi}N}(\varphi(A))$ and, therefore, the inclusion $A' \leq N^{N_\mathcal{F}}_{\varphi\theta}$ follows. It is also immediate from definition that ${}^{\varphi\theta}\mathrm{Aut}_{N^{N_\mathcal{F}}_{\varphi\theta}}(A')$ is contained in $\mathrm{Aut}_{{}^{N_\mathcal{F}}_{\varphi}N}(\varphi(A))$. The converse inclusion follows from Equation (2.13) and definition of $N^{N_\mathcal{F}}_{\varphi\theta}$. This concludes the proof. □

Corollary 2.4.18. *With the notation of Lemma 2.4.17 assume that* $\varphi = \varphi^{N_\mathcal{F}}$. *If there exists* $\hat{\varphi} \in \mathrm{Hom}_\mathcal{F}\left(N^{N_\mathcal{F}}_{\varphi}, S\right)$ *such that* $\iota^S_Q\varphi = \hat{\varphi}\iota^{N^{N_\mathcal{F}}_{\varphi}}_A$ *then* $\hat{\varphi}\left(N^{N_\mathcal{F}}_{\varphi}\right) = {}^{N_\mathcal{F}}_{\varphi}N$.

Proof. By definition, we have that $N_\varphi^{N_\mathcal{F}} \leq N_{N_S}(A)$ and that $\hat{\varphi}(A) = \varphi(A)$. Therefore we can deduce that $\hat{\varphi}\left(N_\varphi^{N_\mathcal{F}}\right) \leq N_{N_S}(\varphi(A))$. Moreover, from Lemma 2.4.17 we have that

$$\mathrm{Aut}_{{}^{N_\mathcal{F}}_{\varphi}N}(\varphi(A)) = {}^{\varphi}\mathrm{Aut}_{N_\varphi^{N_\mathcal{F}}}(A) = \mathrm{Aut}_{\hat{\varphi}\left(N_\varphi^{N_\mathcal{F}}\right)}(\varphi(A)).$$

From these identities we can conclude that

$${}^{N_\mathcal{F}}_{\varphi}NC_S(\varphi(A)) = \hat{\varphi}\left(N_\varphi^{N_\mathcal{F}}\right)C_S(\varphi(A)).$$

Recall now that, by hypothesis, we have $A \in \mathcal{F}^c$. Therefore we also have $\varphi(A) \in \mathcal{F}^c$. In particular $C_S(\varphi(A)) \leq \varphi(A)$. Finally, from Lemma 2.4.17 we have that $\varphi(A) \leq {}^{N_\mathcal{F}}_{\varphi}N$ and that $A \leq N_\varphi^{N_\mathcal{F}}$. Putting all this together we obtain the following identities from which the result follows.

$${}^{N_\mathcal{F}}_{\varphi}N = {}^{N_\mathcal{F}}_{\varphi}NC_S(\varphi(A)) = \hat{\varphi}\left(N_\varphi^{N_\mathcal{F}}\right)C_S(\varphi(A)) = \hat{\varphi}\left(N_\varphi^{N_\mathcal{F}}\right).$$

□

Corollary 2.4.19. *With notation as in Lemma 2.4.17, for every $A' \in \mathcal{F}^c$ and isomorphism $\theta \in \mathrm{Hom}_{N_\mathcal{F}}(A', A)$ we have that ${}^{N_\mathcal{F}}_{\varphi}N = {}^{N_\mathcal{F}}_{\varphi\theta}N$.*

Proof. Since θ is an isomorphism in $N_\mathcal{F}$ then we have that $\mathrm{Aut}_{N_\mathcal{F}}(A) = {}^{\theta}\mathrm{Aut}_{N_\mathcal{F}}(A')$. With this in mind the result follows from the identities below.

$$\begin{aligned}
{}^{N_\mathcal{F}}_{\varphi\theta}N &= \left\{x \in N_Q(\varphi\theta(A')) \,:\, (\varphi\theta)^{-1}c_x\varphi\theta \in \mathrm{Aut}_{N_\mathcal{F}}(A')\right\}, \\
&= \left\{x \in N_Q(\varphi(A)) \,:\, \varphi^{-1}c_x\varphi \in {}^{\theta}(\mathrm{Aut}_{N_\mathcal{F}}(A'))\right\}, \\
&= \left\{x \in N_Q(\varphi(A)) \,:\, \varphi^{-1}c_x\varphi \in \mathrm{Aut}_{N_\mathcal{F}}(A)\right\} = {}^{N_\mathcal{F}}_{\varphi}N.
\end{aligned}$$

□

Corollary 2.4.20. *With the notation of Lemma 2.4.17:*

(1) We can always take $\left(\varphi^{N_\mathcal{F}}\right)^{N_\mathcal{F}} = \varphi^{N_\mathcal{F}}$.

(2) If $\varphi = \varphi^{N_\mathcal{F}}$ then for every $x \in P$ we have that $\varphi c_x = (\varphi c_x)^{N_\mathcal{F}}$ where $c_x : A^x \to A$ is seen as an isomorphism in $N_\mathcal{F}$.

Proof.

(1) The result follows from definition of $\left(\varphi^{N_{\mathcal{F}}}\right)^{N_{\mathcal{F}}}$ and the identities below

$$^{\varphi^{N_{\mathcal{F}}}}\mathrm{Aut}_{N^{N_{\mathcal{F}}}_{\varphi^{N_{\mathcal{F}}}}}(A') = \mathrm{Aut}_{^{N_{\mathcal{F}}}_{\varphi}N}(\varphi(A)) = \mathrm{Aut}_{^{N_{\mathcal{F}}}_{\varphi^{N_{\mathcal{F}}}}N}\left(\varphi^{N_{\mathcal{F}}}(A')\right).$$

Here we are using Corollary 2.4.19 for the second identity.

(2) With the notation of Item (2) we have that

$$\mathrm{Aut}_{^{N_{\mathcal{F}}}_{\varphi c_x}N}(\varphi c_x(A^x)) = \mathrm{Aut}_{^{N_{\mathcal{F}}}_{\varphi}N}(\varphi(A)) = {^{\varphi}\mathrm{Aut}_{N_{\varphi}^{N_{\mathcal{F}}}}}(A) = {^{\varphi c_x}\mathrm{Aut}_{\left(N_{\varphi}^{N_{\mathcal{F}}}\right)^x}}(A^x).$$

Where, for the first identity, we are using Corollary 2.4.19, while, for the second identity, we are using the fact that $\varphi = \varphi^{N_{\mathcal{F}}}$. Using the above and the description of $N_{\varphi}^{N_{\mathcal{F}}}$ given in Lemma 2.4.17 we obtain

$$\begin{aligned}\left(N_{\varphi}^{N_{\mathcal{F}}}\right)^x &= \left\{y \in N_{N_S}(A) \,:\, c_y \in \mathrm{Aut}_{^{N_{\mathcal{F}}}_{\varphi}N}(\varphi(A))^{\varphi}\right\}^x,\\ &= \left\{z \in N_{N_S}(A^x) \,:\, c_{^xz} \in \mathrm{Aut}_{^{N_{\mathcal{F}}}_{\varphi}N}(\varphi c_x(A^x))^{\varphi}\right\}^x,\\ &= \left\{z \in N_{N_S}(A^x) \,:\, c_z \in \mathrm{Aut}_{^{N_{\mathcal{F}}}_{\varphi}N}(\varphi c_x(A^x))^{\varphi c_x}\right\}^x,\\ &= \left\{z \in N_{N_S}(A^x) \,:\, c_z \in \mathrm{Aut}_{^{N_{\mathcal{F}}}_{\varphi c_x}N}(\varphi c_x(A^x))^{\varphi c_x}\right\}^x.\end{aligned}$$

Where we are using Corollary 2.4.19 for the last identity. The result follows by defining $N_{\varphi c_x}^{N_{\mathcal{F}}} := \left(N_{\varphi}^{N_{\mathcal{F}}}\right)^x$.

Finally we obtain the following □

Lemma 2.4.21. *With the notation of Lemma 2.4.17 for every $x \in Q$ we have that ${^x\left({^{N_{\mathcal{F}}}_{\varphi}N}\right)} = N^{N_{\mathcal{F}}}_{c_x\varphi}$. Moreover, if $\varphi = \varphi^{N_{\mathcal{F}}}$, we also have that $c_x\varphi = (c_x\varphi)^{N_{\mathcal{F}}}$ and that $N_{\varphi}^{N_{\mathcal{F}}} = N_{c_x\varphi}^{N_{\mathcal{F}}}$.*

Proof. First of all notice that

$$\begin{aligned}{^{N_{\mathcal{F}}}_{c_x\varphi}N} &= \left\{y \in N_Q\left({^x(\varphi(A))}\right) \,:\, (c_x\varphi)^{-1}c_yc_x\varphi \in \mathrm{Aut}_{N_{\mathcal{F}}}(A)\right\},\\ &= \left\{y \in {^x(N_Q(\varphi(A)))} \,:\, \varphi^{-1}c_{y^x}\varphi \in \mathrm{Aut}_{N_{\mathcal{F}}}(A)\right\},\\ &= \left\{{^xz} \,:\, z \in N_Q(\varphi(A)) \text{ and } \varphi^{-1}c_z\varphi \in \mathrm{Aut}_{N_{\mathcal{F}}}(A)\right\} = {^x\left({^{N_{\mathcal{F}}}_{\varphi}N}\right)}.\end{aligned}$$

This proves the first half of the lemma. For the second part we can use the above and the identity $\varphi = \varphi^{N_{\mathcal{F}}}$ to obtain the identities below

$$^{c_x\varphi}\mathrm{Aut}_{N_{\varphi}^{N_{\mathcal{F}}}}(A) = {}^{c_x}\left(\mathrm{Aut}_{{}^{N_{\mathcal{F}}}_{\varphi}N}(\varphi(A))\right) = \mathrm{Aut}_{{}^{x}\left({}^{N_{\mathcal{F}}}_{\varphi}N\right)}(c_x\varphi(A)) = \mathrm{Aut}_{{}^{N_{\mathcal{F}}}_{c_x\varphi}N}(c_x\varphi(A)).$$

This proves both that $c_x\varphi = (c_x\varphi)^{N_{\mathcal{F}}}$ and that $N_{\varphi}^{N_{\mathcal{F}}} = N_{c_x\varphi}^{N_{\mathcal{F}}}$. □

Lemma 2.4.21 allows us to introduce the following definition.

Definition 2.4.22. Let $A, Q \in \mathcal{F}^c$ with $A \leq N_S$, let $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(A, Q)$ if there exists a representative φ of $\overline{\varphi}$ such that $\varphi = \varphi^{N_{\mathcal{F}}}$ then, from Lemma 2.4.21, this happens for every representative of $\overline{\varphi}$. If this is the case we write $\overline{\varphi} = \overline{\varphi}^{N_{\mathcal{F}}}$ and define the **normalizer before $\overline{\varphi}$ in $N_{\mathcal{F}}$** as $N_{\overline{\varphi}}^{N_{\mathcal{F}}} := N_{\varphi}^{N_{\mathcal{F}}}$. Because of Lemma 2.4.21 we know that $N_{\overline{\varphi}}^{N_{\mathcal{F}}}$ does not depend on the choice of representative φ of $\overline{\varphi}$.

As we show in Theorem 2.4.27, for every $P, Q \in \mathcal{F}^c$ and every $(A, \overline{\varphi}) \in [P \times Q]$ such that $\overline{\varphi} = \overline{\varphi}^{N_{\mathcal{F}}}$ the groups $N_{\overline{\varphi}}^{N_{\mathcal{F}}}$ play, in the context of fusion systems, a role analogous to the one that the groups $H^x \cap K$ play in Equation (2.12).

Let's now look into what objects play, in the context of fusion systems, a role analogous to that of the biset representatives $x \in [H\backslash G/K]$ of Equation (2.12).

Definition 2.4.23. Let $Q \in \mathcal{F}^c$. We define an equivalence relation $\sim$ in $[P \times_{\mathcal{F}} Q]$ by setting $(A, \overline{\varphi}) \sim (B, \overline{\psi})$ if and only if there exists an isomorphism $\overline{\theta} \in \mathrm{Hom}_{\mathcal{O}(N_{\mathcal{F}})}(A, B)$ such that $\overline{\varphi} = \overline{\psi\theta}$. Lemma 2.4.17 ensures us that for each equivalence class in $[P \times Q]/\sim$ we can choose one representative $(A, \overline{\varphi})$ such that A is fully $N_{\mathcal{F}}$-normalized and $\overline{\varphi} = \overline{\varphi}^{N_{\mathcal{F}}}$. We define the **product of $N_{\mathcal{F}}$ and Q in $\mathcal{F}$** to be any subset $[N_{\mathcal{F}} \times Q] \subseteq [P \times_{\mathcal{F}} Q]$ formed by such representatives.

We want the elements $[N_{\mathcal{F}} \times Q]$ and $N_{\overline{\varphi}}^{N_{\mathcal{F}}}$ to play, in the context of fusion systems, the same role that the elements $[H\backslash G/K]$ and $H^x \cap K$ play in Equation (2.12). In order to do so we need to be able to define something analogous to the set $x[H^x \cap K\backslash K/J]$ of Equation (2.12). In other words, for every $(A, \overline{\varphi}) \in [N_{\mathcal{F}} \times Q]$ we need to be able to lift the morphism $\overline{\varphi}: A \to Q$ in a unique way to a morphism $\overline{\hat{\varphi}}: N_{\overline{\varphi}}^{N_{\mathcal{F}}} \to Q$.

Proposition 2.4.24. *Let $Q \in \mathcal{F}^c$, let $(A, \overline{\varphi}) \in [N_{\mathcal{F}} \times Q]$ and let φ be a representative of $\overline{\varphi}$. There exists a morphism $\hat{\varphi}: N_{\varphi}^{N_{\mathcal{F}}} \to Q$ in $\mathcal{F}$ such that $\varphi = \hat{\varphi}\iota_A^{N_{\varphi}^{N_{\mathcal{F}}}}$. In particular,* *from Corollary 2.4.18, we have that* $\hat{\varphi}\left(N_{\varphi}^{N_{\mathcal{F}}}\right) = {}^{N_{\mathcal{F}}}_{\varphi}N \leq N_Q(\varphi(A))$*. Moreover there exists a unique morphism* $\overline{\hat{\varphi}} : N_{\overline{\varphi}}^{N_{\mathcal{F}}} \to Q$ *in* $\mathcal{O}(\mathcal{F}^c)$ *such that* $\overline{\hat{\varphi}}\overline{\iota_A^{N_{\overline{\varphi}}^{N_{\mathcal{F}}}}} = \overline{\varphi}$ *and* $\hat{\varphi}$ *is necessarily a representative of* $\overline{\hat{\varphi}}$*.*

Proof. If the first part of the statement is satisfied then the morphism $\overline{\hat{\varphi}}$ in $\mathcal{O}(\mathcal{F}^c)$ having representative $\hat{\varphi}$ satisfies $\overline{\hat{\varphi}}\overline{\iota_A^{N_{\overline{\varphi}}^{N_{\mathcal{F}}}}} = \overline{\varphi}$. From [Li07, Theorem 4.9] we know that $\overline{\iota_A^{N_{\overline{\varphi}}^{N_{\mathcal{F}}}}}$ is an epimorphism. Therefore, for any morphism $\overline{\psi}\colon N_{\varphi}^{N_{\mathcal{F}}} \to Q$ in $\mathcal{O}(\mathcal{F}^c)$ satisfying $\overline{\psi}\overline{\iota_A^{N_{\varphi}^{N_{\mathcal{F}}}}} = \overline{\varphi} = \overline{\hat{\varphi}}\overline{\iota_A^{N_{\varphi}^{N_{\mathcal{F}}}}}$, we must necessarily have $\overline{\psi} = \overline{\hat{\varphi}}$. This proves uniqueness of $\overline{\hat{\varphi}}$.

We are now only left with proving that there exists a morphism $\hat{\varphi}$ as in the statement.

We know from definition of $[N_{\mathcal{F}} \times Q]$ that $A \in \mathcal{F}^c$. Therefore we must also have $\varphi(A) \in \mathcal{F}^c$ and, in particular, $\varphi(A)$ is fully $\mathcal{F}$-centrialized. From [St03, Proposition 4.4] (see also [Li07, Proposition 2.7]) we can now deduce that there exists a morphism $\psi\colon N_{\iota_Q^S\varphi} \to S$ (see Definition 2.2.5) such that $\psi\iota_A^{N_{\iota_Q^S\varphi}} = \iota_Q^S\varphi$. By definition of $N_{\varphi}^{N_{\mathcal{F}}}$ (see Lemma 2.4.17) we have that $N_{\varphi}^{N_{\mathcal{F}}} \leq N_{\iota_Q^S\varphi}$. Therefore we can apply Corollary 2.4.18 (taking $\hat{\varphi} := \psi\iota_{N_{\varphi}^{N_{\mathcal{F}}}}^{N_{\iota_Q^S\varphi}}$) to deduce that $\psi\left(N_{\varphi}^{N_{\mathcal{F}}}\right) = {}^{N_{\mathcal{F}}}_{\varphi}N$. In particular we have that $\psi\left(N_{\varphi}^{N_{\mathcal{F}}}\right) \leq Q$. This allows us to define the morphism $\hat{\varphi}\colon N_{\varphi}^{N_{\mathcal{F}}} \to Q$ in $\mathcal{F}$ by setting $\hat{\varphi}(x) := \psi(x)$ for every $x \in N_{\varphi}^{N_{\mathcal{F}}}$. Since $\psi\iota_A^{N_{\iota_Q^S\varphi}} = \iota_Q^S\varphi$ then we have that $\hat{\varphi} = \varphi\iota_A^{N_{\varphi}^{N_{\mathcal{F}}}}$ thus completing the proof. □

Notice that maximality of the pairs $(A, \overline{\varphi}) \in [N_{\mathcal{F}} \times Q]$ does not imply that the pair $\left(N_{\overline{\varphi}}^{N_{\mathcal{F}}}, \overline{\hat{\varphi}}\right)$ given by Proposition 2.4.24 satisfies $\left(N_{\overline{\varphi}}^{N_{\mathcal{F}}}, \overline{\hat{\varphi}}\right) = (A, \overline{\varphi})$ since we might have that $N_{\overline{\varphi}}^{N_{\mathcal{F}}} \not\leq P$. However, we have the following corollary.

Corollary 2.4.25. *Let* $Q \in \mathcal{F}^c$ *and let* $(A, \overline{\varphi}) \in [N_{\mathcal{F}} \times Q]$ *then* $A = N_{\overline{\varphi}}^{N_{\mathcal{F}}} \cap P$*.*

Proof. From Lemma 2.4.17 we know that $A \leq N_{\overline{\varphi}}^{N_{\mathcal{F}}}$ and from Definitions 2.2.17 and 2.4.23 we know that $A \leq P$. Therefore we can deduce that $A \leq N_{\overline{\varphi}}^{N_{\mathcal{F}}} \cap P$. Then, using the notation of Proposition 2.4.24, we obtain the identity $\overline{\hat{\varphi}\iota_{N_{\overline{\varphi}}^{N_{\mathcal{F}}} \cap P}^{N_{\overline{\varphi}}^{N_{\mathcal{F}}}}}\,\overline{\iota_A^{N_{\overline{\varphi}}^{N_{\mathcal{F}}} \cap P}} = \overline{\varphi}$. Since $N_{\varphi}^{N_{\mathcal{F}}} \cap P \leq P$ then we can deduce from maximality of the pair $(A, \overline{\varphi})$ that $A = N_{\overline{\varphi}}^{N_{\mathcal{F}}} \cap P$ thus concluding the proof. □

From Corollary 2.4.25 we now obtain the following.

Lemma 2.4.26. *Let $Q \in N_{\mathcal{F}} \cap \mathcal{F}^c$ and let $(A, \overline{\varphi}) \in [N_{\mathcal{F}} \times Q]$. For every $\left(B, \overline{\psi}\right) \in \left[P \times_{N_{\mathcal{F}}} N_{\overline{\varphi}}^{N_{\mathcal{F}}}\right]$ and every representative ψ of $\overline{\psi}$ we have that $\psi(B) = N_{\overline{\varphi}}^{N_{\mathcal{F}}} \cap P = A$.*

Proof. Because of Corollary 2.4.25 we just need to prove that $\psi(B) = N_{\overline{\varphi}}^{N_{\mathcal{F}}} \cap P$ for any representative ψ of $\overline{\psi}$. Since ψ is a morphism in $N_{\mathcal{F}}$ and, by definition, every morphism on $N_{\mathcal{F}}$ sends subgroups of P to subgroups of P, then $\psi(B) \leq P$. Therefore we must have $\psi(B) \leq N_{\overline{\varphi}}^{N_{\mathcal{F}}} \cap P$. From definition of $N_{\mathcal{F}}$ this implies that there exists $\hat{\psi} \in \mathrm{Aut}_{N_{\mathcal{F}}}(P)$ such that $\hat{\psi}(x) = \psi(x)$ for every $x \in B$. Define now $\theta\colon C := \hat{\psi}^{-1}\left(N_{\overline{\varphi}}^{N_{\mathcal{F}}} \cap P\right) \to N_{\overline{\varphi}}^{N_{\mathcal{F}}}$ by setting $\theta(x) := \hat{\psi}(x)$ for every $x \in C$. Then we have that $B \leq C$ and $\overline{\theta \iota_B^C} = \overline{\psi}$. Since θ is a morphism in $N_{\mathcal{F}}$ and $C \leq P$, from maximality of the pair $\left(B, \overline{\psi}\right)$ (see Definition 2.2.17), we can conclude that $\left(C, \overline{\theta}\right) = \left(B, \overline{\psi}\right)$. In particular we have that $\psi(B) = \theta(C)$ and since $\theta(C) = \hat{\psi}\left(\hat{\psi}^{-1}\left(N_{\overline{\varphi}}^{N_{\mathcal{F}}} \cap P\right)\right) = N_{\overline{\varphi}}^{N_{\mathcal{F}}} \cap P$ the result follows. □

We have now gathered all the ingredients needed to prove Theorem 2.4.27 with which we conclude this subsection.

Theorem 2.4.27. *Using Notation 2.4.9 let $Q \in \mathcal{F}^c$ (see Definition 2.2.11) and for every $(A, \overline{\varphi}) \in [N_{\mathcal{F}} \times Q]$ (see Example 2.2.8 and Definition 2.4.23) let $\overline{\hat{\varphi}}$ be as in Proposition 2.4.24. Then, for every $(A, \overline{\varphi}) \in [N_{\mathcal{F}} \times Q]$ we can take $\left[P \times_{N_{\mathcal{F}}} N_{\overline{\varphi}}^{N_{\mathcal{F}}}\right]$ (see Definition 2.2.17 and Lemma 2.4.21) so that*

$$[P \times_{\mathcal{F}} Q] = \bigsqcup_{(A, \overline{\varphi}) \in [N_{\mathcal{F}} \times Q]} \overline{\hat{\varphi}}\left[P \times_{N_{\mathcal{F}}} N_{\overline{\varphi}}^{N_{\mathcal{F}}}\right],$$

where

$$\overline{\hat{\varphi}}\left[P \times_{N_{\mathcal{F}}} N_{\overline{\varphi}}^{N_{\mathcal{F}}}\right] := \bigsqcup_{\left(B, \overline{\psi}\right) \in \left[P \times_{N_{\mathcal{F}}} N_{\overline{\varphi}}^{N_{\mathcal{F}}}\right]} \left\{\left(B, \overline{\hat{\varphi}\psi}\right)\right\}.$$

Proof. Let $(A, \overline{\varphi}) \in [N_{\mathcal{F}} \times Q]$, let $\left(B, \overline{\psi}\right) \in \left[P \times_{N_{\mathcal{F}}} N_{\overline{\varphi}}^{N_{\mathcal{F}}}\right]$, let ψ be a representative of $\overline{\psi}$ and let $\tilde{\psi} : B \to \psi(B)$ be the morphism ψ seen as an isomorphism onto its image. From Lemma 2.4.26 we know that $\tilde{\psi}$ is in fact an isomorphism from B to A. From definition of $N_{\mathcal{F}}$ (see Example 2.2.8) we can now choose an automorphism $\hat{\psi} \in \mathrm{Aut}_{N_{\mathcal{F}}}(P)$ satisfying $\hat{\psi}\iota_B^P = \iota_A^P \psi$. For every $D \leq P$ we denote by $\hat{\psi}_D\colon D \to \hat{\psi}(D)$ the isomorphism in $N_{\mathcal{F}}$ obtained by restricting $\hat{\psi}$ to D.

From the universal property of products, we know that there exist a unique $\left(C, \overline{\theta}\right) \in [P \times_{\mathcal{F}} Q]$ and a unique $\overline{\gamma}\colon B \to C$ such that $\overline{\theta}\,\overline{\gamma} = \overline{\hat{\varphi}}\,\overline{\psi}$ and $\overline{\iota_C^P}\overline{\gamma} = \overline{\iota_B^P}$. From the second identity we can deduce that $\overline{\gamma} \in \mathcal{O}(\mathcal{F}_P)$ (see Example 2.2.2 and Definition 2.2.10). Therefore we can choose $\left[P \times_{N_\mathcal{F}} N_{\overline{\varphi}}^{N_\mathcal{F}}\right]$ so that $B \leq C$ and $\overline{\gamma} = \overline{\iota_B^C}$. With this setup the first identity can be rewritten as $\overline{\theta}\,\overline{\iota_B^C} = \overline{\hat{\varphi}}\,\overline{\psi} = \overline{\varphi}\,\overline{\tilde{\psi}}$. Using this and the notation introduced at the start we obtain the identity $\overline{\theta}\overline{\hat{\psi}_C^{-1}}\overline{\iota_A^{\hat{\psi}(C)}} = \overline{\varphi}$. Since $\left(C, \overline{\theta}\right) \in [P \times_\mathcal{F} Q]$ we know from Proposition 2.2.18 (4) that there exists $x \in P$ satisfying $\left(\hat{\psi}(C)^x, \overline{\theta\hat{\psi}_C^{-1} c_x}\right) \in [P \times_\mathcal{F} Q]$. With this setup we obtain the identities $\overline{\varphi} = \overline{\theta\hat{\psi}_C^{-1} c_x}\,\overline{c_{x^{-1}}\iota_A^{\hat{\psi}(C)}}$ and $\overline{\iota_A^P} = \overline{\iota_{\hat{\psi}(C)^h}^P}\,\overline{c_{h^{-1}}\iota_A^{\hat{\psi}(C)}}$. Since $(A, \overline{\varphi}) \in [N_\mathcal{F} \times Q] \subseteq [P \times_\mathcal{F} Q]$ we can conclude from the previous identities and the universal properties of product that $(A, \overline{\varphi}) = \left(\hat{\psi}(C)^x, \overline{\theta\hat{\psi}_C^{-1} c_x}\right)$ and $\overline{c_{x^{-1}}\iota_A^{\hat{\psi}(C)}} = \overline{\mathrm{Id}_A}$. In particular we have that $A = \hat{\psi}(C)$ and, from Lemma 2.4.26, we can conclude that $B = C$. This implies that $\overline{\iota_B^C} = \overline{\mathrm{Id}_B}$ which allows us to deduce from the identity $\overline{\theta}\,\overline{\iota_B^C} = \overline{\hat{\varphi}}\,\overline{\psi}$ that $\left(B, \overline{\hat{\varphi}}\,\overline{\psi}\right) = \left(C, \overline{\theta}\right) \in [P \times_\mathcal{F} Q]$ thus proving that

$$\bigcup_{(A,\overline{\varphi}) \in [N_\mathcal{F} \times_\mathcal{F} Q]} \; \bigcup_{\left(B,\overline{\psi}\right) \in \left[P \times_{N_\mathcal{F}} N_{\overline{\varphi}}^{N_\mathcal{F}}\right]} \left\{\left(B, \overline{\hat{\varphi}}\,\overline{\psi}\right)\right\} \subseteq [P \times_\mathcal{F} Q]. \tag{2.14}$$

Let's now prove the other inclusion. By definition of $[N_\mathcal{F} \times Q]$ (see Definition 2.4.23) we know that for every $\left(C, \overline{\theta}\right) \in [P \times_\mathcal{F} Q]$ there exist a unique $(A, \overline{\varphi}) \in [N_\mathcal{F} \times Q]$ and an isomorphism $\overline{\gamma} : C \to A$ in $\mathcal{O}(N_\mathcal{F})$ such that $\overline{\theta} = \overline{\varphi}\,\overline{\gamma} = \overline{\hat{\varphi}}\,\overline{\iota_A^{N_{\overline{\varphi}}^{N_\mathcal{F}}}}\,\overline{\gamma}$. From the universal properties of products we know that there exist a unique $\left(B, \overline{\psi}\right) \in \left[P \times_{N_\mathcal{F}} N_\varphi^{N_\mathcal{F}}\right]$ and a unique map $\overline{\delta} : C \to B$ such that $\overline{\iota_B^P}\,\overline{\delta} = \overline{\iota_C^P}$ and that $\overline{\psi}\,\overline{\delta} = \overline{\iota_A^{N_{\overline{\varphi}}^{N_\mathcal{F}}}}\,\overline{\gamma}$. Joining the second identity with the previous one we obtain the identity $\overline{\theta} = \overline{\hat{\varphi}}\,\overline{\psi}\,\overline{\delta}$. Since $\left(B, \overline{\hat{\varphi}}\,\overline{\psi}\right) \in [P \times_\mathcal{F} Q]$ as shown in the first part of the proof we can conclude from the universal properties of products that $\overline{\delta} = \overline{\mathrm{Id}_B}$ and that $\left(B, \overline{\hat{\varphi}}\,\overline{\psi}\right) = \left(C, \overline{\theta}\right)$. This gives us the inclusion converse to that of Equation (2.14) thus leading us to the identity

$$[P \times_\mathcal{F} Q] = \bigcup_{(A,\overline{\varphi}) \in [N_\mathcal{F} \times_\mathcal{F} Q]} \; \bigcup_{\left(B,\overline{\psi}\right) \in \left[P \times_{N_\mathcal{F}} N_{\overline{\varphi}}^{N_\mathcal{F}}\right]} \left\{\left(B, \overline{\hat{\varphi}}\,\overline{\psi}\right)\right\}.$$

We are now only left with proving that the above unions are disjoint.

Let $(A, \overline{\varphi}), \left(A', \overline{\varphi'}\right) \in [N_\mathcal{F} \times Q]$, let $\left(B, \overline{\psi}\right) \in \left[P \times_{N_\mathcal{F}} N_{\overline{\varphi}}^{N_\mathcal{F}}\right]$ and let $\left(B', \overline{\psi'}\right) \in \left[P \times_{N_\mathcal{F}} N_{\overline{\varphi'}}^{N_\mathcal{F}}\right]$ such that $\left(B', \overline{\hat{\varphi}'}\,\overline{\psi'}\right) = \left(B, \overline{\hat{\varphi}}\,\overline{\psi}\right)$. Fix representatives, ψ and ψ' of $\overline{\psi}$ and $\overline{\psi'}$ respectively, let $\tilde{\psi}$ and $\tilde{\psi}'$ be the isomorphisms obtained by viewing ψ and ψ

respectively as isomorphisms onto their images and let $\overline{\hat{\varphi}}$ and $\overline{\hat{\varphi}'}$ be as in Proposition 2.4.24. With this setup we have that $\overline{\hat{\varphi}}\,\overline{\psi} = \overline{\varphi}\,\overline{\tilde{\psi}}$ and that $\overline{\hat{\varphi}'}\,\overline{\psi'} = \overline{\varphi'}\,\overline{\tilde{\psi}'}$. Thus, from the identity $\overline{\hat{\varphi}}\,\overline{\psi} = \overline{\hat{\varphi}'}\,\overline{\psi'}$ we can conclude that $\overline{\varphi} = \overline{\varphi'}\,\overline{\tilde{\psi}'}\,\overline{\tilde{\psi}^{-1}}$. Since $\overline{\tilde{\psi}'}\,\overline{\tilde{\psi}^{-1}}$ is an isomorphism in $\mathcal{O}(N_{\mathcal{F}})$, by definition of $[N_{\mathcal{F}} \times Q]$, we can conclude that $(A, \overline{\varphi}) = \left(A', \overline{\varphi'}\right)$.

Take now a representative φ of $\overline{\varphi}$ and let $\tilde{\varphi}\colon A \to \varphi(A)$ be the isomorphism obtained by viewing φ as an isomorphism onto its image. With this setup we have that $\overline{\varphi}\overline{\tilde{\psi}} = \overline{\iota_{\varphi(A)}^{Q}}\overline{\tilde{\varphi}}\overline{\tilde{\psi}}$ and that $\overline{\varphi}\overline{\tilde{\psi}'} = \overline{\iota_{\varphi(A)}^{Q}}\overline{\tilde{\varphi}}\overline{\tilde{\psi}'}$. Thus we obtain the identity $\overline{\iota_{\varphi(A)}^{Q}}\overline{\tilde{\varphi}}\overline{\tilde{\psi}} = \overline{\iota_{\varphi(A)}^{Q}}\overline{\tilde{\varphi}}\overline{\tilde{\psi}'}$ and we can deduce that there exists $x \in Q$ such that $c_x\tilde{\varphi}\tilde{\psi} = \tilde{\varphi}\tilde{\psi}'$ as isomorphisms from B to $\varphi(A)$. Since $\tilde{\varphi}\tilde{\psi}$ is an isomorphism from B to A, from the previous identity, we can conclude that $x \in N_Q(\varphi(A))$. Always from the previous identity we obtain the identity $\tilde{\varphi}^{-1}c_x\tilde{\varphi} = \tilde{\psi}'\tilde{\psi}^{-1}$. Since both $\tilde{\psi}$ and $\tilde{\psi}'$ are isomorphism in $N_{\mathcal{F}}$ we can deduce from Lemma 2.4.26 that $\tilde{\psi}'\tilde{\psi}^{-1} \in \mathrm{Aut}_{N_{\mathcal{F}}}(A)$. Thus, from Lemma 2.4.17, we have that $x \in {}^{N_{\mathcal{F}}}_{\varphi}N$. Now let $\hat{\varphi}$ be a representative of $\overline{\hat{\varphi}}$ as in Proposition 2.4.24. From Corollary 2.4.18 we know that there exists a unique $y \in N_{\overline{\varphi}}^{N_{\mathcal{F}}}$ such that $x = \hat{\varphi}(y)$. With this setup we can deduce that $\tilde{\varphi}^{-1}c_x\tilde{\varphi} = \tilde{\varphi}^{-1}\tilde{\varphi}c_y = c_y$. Thus we can conclude that $c_y\tilde{\psi} = \tilde{\psi}'$ and, since $y \in N_{\overline{\varphi}}^{N_{\mathcal{F}}}$ we can conclude by definition of the orbit category (see Definition 2.2.10) that $\overline{\psi} = \overline{\iota_A^Q c_y\tilde{\psi}} = \overline{\iota_A^Q\tilde{\psi}'} = \overline{\psi'}$ thus concluding the proof. □

2.4.5 The trace from $N_{\mathcal{F}}$ to $\mathcal{F}$

Through this subsection we will be using Notations 2.1.1, 2.2.3, 2.2.9, 2.2.21, 2.2.31 and 2.4.9.

As we already explained at the beginning of Subsection 2.4.4 given a fusion system $\mathcal{H} \subseteq \mathcal{F}$ and a Mackey functor $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ the trace $\mathrm{tr}_{\mathcal{H}}^{\mathcal{F}}\colon \mathrm{End}\left(M\downarrow_{\mathcal{H}}^{\mathcal{F}}\right) \to \mathrm{End}(M)$ is, in general, not defined. However, just like Equation (2.12) can be used in the case of Mackey functors over finite groups in order to prove that trace maps compose nicely, Theorem 2.4.27 can be used in the case of centric Mackey functors over fusion systems in order to define $\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}$ and prove that $\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}\mathrm{tr}_{\mathcal{F}_P}^{N_{\mathcal{F}}} = \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}$ (see Examples 2.2.2 and 2.2.8 for notation). More precisely we have the following.

Definition 2.4.28. Let $\mathcal{R}$ be a p-local ring and let $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$. From Proposition 2.2.39 we know that the isomorphism class $\overline{S} \in B_{\mathcal{R}}^{\mathcal{F}^c}$ of S has an inverse in $B_{\mathcal{R}}^{\mathcal{F}^c}$. Thus, using the notation of Examples 2.2.2 and 2.2.8 and of Proposition 2.2.43, we define the

trace from $N_{\mathcal{F}}$ to $\mathcal{F}$ as the $\mathcal{R}$-module morphism

$$ {}^{M}\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}} := \left(\overline{S}^{-1}\cdot\right)_{*} \sum_{(A,\overline{\varphi})\in[N_{\mathcal{F}}\times S]} {}^{M}\mathrm{tr}_{\mathcal{F}_{N_{\overline{\varphi}}^{N_{\mathcal{F}}}}}^{\mathcal{F}} \; {}^{M\downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}}\mathrm{r}_{\mathcal{F}_{N_{\overline{\varphi}}^{N_{\mathcal{F}}}}}^{N_{\mathcal{F}}} : \mathrm{End}\left(M\downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}\right) \to \mathrm{End}\left(M\right). $$

From Items (7) and (8) of Proposition 2.3.9 we know that ${}^{M}\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}$ does not depend on the choice of the set $[N_{\mathcal{F}}\times S]$ (see Definition 2.4.23). Whenever there is no confusion regarding M we simply write $\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}} := {}^{M}\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}$.

Lemma 2.4.29. *Let $\mathcal{R}$ be a p-local ring. Regardless of the centric Mackey functor involved we have that $\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{N_{\mathcal{F}}} = \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}$. In particular $\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}$ sends $\mathrm{Tr}_P^{N_{\mathcal{F}}}$ surjectively onto $\mathrm{Tr}_P^{\mathcal{F}}$ (see Definition 2.3.12).*

Proof. For every $(A,\overline{\varphi}) \in [N_{\mathcal{F}}\times S]$ fix representatives φ of $\overline{\varphi}$ and $\hat{\varphi}$ of $\overline{\hat{\varphi}}$ as in Proposition 2.4.24 and for every $\left(B,\overline{\psi}\right) \in \left[P\times_{N_{\mathcal{F}}} N_{\overline{\varphi}}^{N_{\mathcal{F}}}\right]$ fix a representative ψ of $\overline{\psi}$. Let $\tilde{\varphi},\tilde{\hat{\varphi}}$ and $\tilde{\psi}$ denote the isomorphisms obtained by viewing φ, $\hat{\varphi}$ and ψ respectively as isomorphisms onto their images. With this notation and the notation of Proposition 2.2.43 we have that

$$ \begin{aligned} \left(\overline{S}\cdot\right)_{*}\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{N_{\mathcal{F}}} &= \sum_{(A,\overline{\varphi})\in[N_{\mathcal{F}}\times S]} \mathrm{tr}_{\mathcal{F}_{\tilde{\hat{\varphi}}\left(N_{\overline{\varphi}}^{N_{\mathcal{F}}}\right)}}^{\mathcal{F}} \,{}^{\tilde{\hat{\varphi}}}\cdot\, \mathrm{r}_{\mathcal{F}_{N_{\overline{\varphi}}^{N_{\mathcal{F}}}}}^{N_{\mathcal{F}}}\,\mathrm{tr}_{\mathcal{F}_P}^{N_{\mathcal{F}}}, \\ &= \sum_{(A,\overline{\varphi})\in[N_{\mathcal{F}}\times S]} \; \sum_{\left(B,\overline{\psi}\right)\in\left[P\times_{N_{\mathcal{F}}} N_{\overline{\varphi}}^{N_{\mathcal{F}}}\right]} \mathrm{tr}_{\mathcal{F}_{\tilde{\hat{\varphi}}\left(N_{\overline{\varphi}}^{N_{\mathcal{F}}}\right)}}^{\mathcal{F}} \,{}^{\tilde{\hat{\varphi}}}\cdot\, \mathrm{tr}_{\mathcal{F}_{\tilde{\psi}(B)}}^{\mathcal{F}_{N_{\overline{\varphi}}^{N_{\mathcal{F}}}}} \,{}^{\tilde{\psi}}\cdot\, \mathrm{r}_{\mathcal{F}_B}^{\mathcal{F}_P}, \\ &= \sum_{(A,\overline{\varphi})\in[N_{\mathcal{F}}\times S]} \; \sum_{\left(B,\overline{\psi}\right)\in\left[P\times_{N_{\mathcal{F}}} N_{\overline{\varphi}}^{N_{\mathcal{F}}}\right]} \mathrm{tr}_{\mathcal{F}_{\tilde{\varphi}(\tilde{\psi}(B))}}^{\mathcal{F}} \,{}^{\tilde{\varphi}\tilde{\psi}}\cdot\, \mathrm{r}_{\mathcal{F}_B}^{\mathcal{F}_P}. \end{aligned} $$

Where we are using Item (7) of Proposition 2.3.9 for the first identity, Item (9) for the second identity and Items (3) and (5) for the third identity. From Theorem 2.4.27 we can now replace the two sums in the last line of the previous equation with a sum over the pairs $\left(C,\overline{\theta}\right) \in [P\times_{\mathcal{F}} S]$ and replace the isomorphisms $\tilde{\varphi}\tilde{\psi}$ with the isomorphisms $\tilde{\theta}$ where θ is a representative of $\overline{\theta}$ and $\tilde{\theta}$ is the isomorphism obtained by viewing θ as an isomorphism onto its image. With this change in mind we can apply Items (3) and (9) of Proposition 2.3.9 in order to obtain the identity $\left(\overline{S}\cdot\right)_{*}\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{N_{\mathcal{F}}} = \mathrm{tr}_{\mathcal{F}_S}^{\mathcal{F}}\,\mathrm{r}_{\mathcal{F}_S}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}$. Applying now Item (11) we obtain from here the identity $\left(\overline{S}\cdot\right)_{*}\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}\,\mathrm{tr}_{\mathcal{F}_P}^{N_{\mathcal{F}}} = \left(\overline{S}\cdot\right)_{*}\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}$. Finally, from Propositions 2.2.39 and 2.2.43, we know that $\left(\overline{S}\cdot\right)_{*}$ is invertible and, therefore, we can deduce from the previous identity that $\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}\mathrm{tr}_{\mathcal{F}_P}^{N_{\mathcal{F}}} = \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}$ thus concluding the proof. □

Corollary 2.4.30. *Let $\mathcal{R}$ be a p-local ring and let $M \in \mathrm{Mack}_{\mathcal{R}}(\mathcal{F}^c)$. For every family $\mathfrak{X}$ of elements in $\mathcal{F}_P \cap \mathcal{F}^c$ we have that $\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}\left(\mathrm{Tr}_{\mathfrak{X}}^{N_{\mathcal{F}}}\right) = \mathrm{Tr}_{\mathfrak{X}}^{\mathcal{F}}$.*

Proof. Because of linearity of $\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}$ it suffices to prove the statement when $\mathfrak{X} = \{Q\}$ for some $Q \in \mathcal{F}_P \cap \mathcal{F}^c$. From Proposition 2.3.9 (3) we know that $\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\mathrm{tr}_{\mathcal{F}_Q}^{\mathcal{F}_P} = \mathrm{tr}_{\mathcal{F}_Q}^{\mathcal{F}}$ and that $\mathrm{tr}_{\mathcal{F}_P}^{N_{\mathcal{F}}}\mathrm{tr}_{\mathcal{F}_Q}^{\mathcal{F}_P} = \mathrm{tr}_{\mathcal{F}_Q}^{N_{\mathcal{F}}}$. Thus, from definition of $\mathrm{Tr}_Q^{\mathcal{F}}$ and $\mathrm{Tr}_Q^{N_{\mathcal{F}}}$ (see Definition 2.3.12), we have that $\mathrm{Tr}_Q^{\mathcal{F}} = \mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(\mathrm{Tr}_Q^{\mathcal{F}_P}\right)$ and that $\mathrm{Tr}_Q^{N_{\mathcal{F}}} = \mathrm{tr}_{\mathcal{F}_P}^{N_{\mathcal{F}}}\left(\mathrm{Tr}_Q^{\mathcal{F}_P}\right)$. Since $\mathrm{Tr}_Q^{N_{\mathcal{F}}} \subseteq \mathrm{End}\left(M\downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}\right)$ the result now follows from the above and Lemma 2.4.29 after applying $\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}$ to $\mathrm{Tr}_K^{N_{\mathcal{F}}}$. □

Corollary 2.4.30 allows us to give the following definition.

Definition 2.4.31. Let $\mathcal{R}$ be a p-local ring, let $M \in \mathrm{Mack}_{\mathcal{R}}(\mathcal{F}^c)$, let $\mathcal{X}$ be as in Notation 2.4.9 and let $E_{\mathcal{X}}^{N_{\mathcal{F}}} := \mathrm{End}\left(M\downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}\right)/\mathrm{Tr}_{\mathcal{X}}^{N_{\mathcal{F}}}$ and $E_{\mathcal{X}}^{\mathcal{F}} := \mathrm{End}(M)/\mathrm{Tr}_{\mathcal{X}}^{\mathcal{F}}$. We define the **quotient trace from $N_{\mathcal{F}}$ to $\mathcal{F}$** as the $\mathcal{R}$-module morphism $\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}} : E_{\mathcal{X}}^{N_{\mathcal{F}}} \to E_{\mathcal{X}}^{\mathcal{F}}$ obtained by setting $\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}\left(\overline{f}\right) := \overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}(f)}$ for every $f \in \mathrm{End}\left(M\downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}\right)$. Here we are using the overline $(\bar{\cdot})$ to denote the projections onto the appropriate quotients. Corollary 2.4.30 assures us that $\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}\left(\overline{f}\right)$ does not depend of the chosen preimage f of $\overline{f}$.

An important property of the $\mathcal{R}$-module morphism $\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}$ of Definition 2.4.31 is the following.

Lemma 2.4.32. *With notation as in Definition 2.4.31 view $\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}$ as a morphism from $\overline{\mathrm{Tr}_P^{N_{\mathcal{F}}}} := \mathrm{Tr}_P^{N_{\mathcal{F}}}/\mathrm{Tr}_{\mathcal{X}}^{N_{\mathcal{F}}}$ to $\overline{\mathrm{Tr}_P^{\mathcal{F}}} := \mathrm{Tr}_P^{\mathcal{F}}/\mathrm{Tr}_{\mathcal{X}}^{\mathcal{F}}$. Then $\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}$ is surjective and commutes with multiplication (i.e. $\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}\left(\overline{fg}\right) = \overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}\left(\overline{f}\right)\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}\left(\overline{g}\right)$ for every $\overline{f}, \overline{g} \in \overline{\mathrm{Tr}_P^{N_{\mathcal{F}}}}$). In particular, if $\mathrm{Tr}_P^{N_{\mathcal{F}}}$ has a multiplicative unit, then $\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}$ is a surjective $\mathcal{R}$-algebra morphism.*

Proof. During this proof we use the overline symbol $(\bar{\cdot})$ in order to represent the projection of an endomorphism on the appropriate quotient ring. From Corollary 2.4.30 we know that the map $\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}$ viewed as in the statement is surjective. If $\mathrm{Tr}_P^{N_{\mathcal{F}}}$ has a multiplicative unit (denoted $1_{\mathrm{Tr}_P^{N_{\mathcal{F}}}}$) and commutes with multiplication then, from surjectivenes, we necessarily have that $\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}\left(1_{\mathrm{Tr}_P^{N_{\mathcal{F}}}}\right) = 1_{\mathrm{Tr}_P^{\mathcal{F}}}$. Thus we only need to prove that $\overline{\mathrm{tr}_{N_{\mathcal{F}}}^{\mathcal{F}}}$ commutes with multiplication.

Let $\mathcal{H} \in \{N_{\mathcal{F}}, \mathcal{F}\}$, let $(A, \overline{\varphi}) \in [P \times_{\mathcal{H}} P]$ and let φ be a representative of $\overline{\varphi}$. Since $A \leq P$, φ is injective and both A and P are finite groups then we have that $\varphi(A) \lneq P$ unless $A = P$. From definition of $\mathcal{X}$ (see Notation 2.4.9), this is equivalent to saying that $\varphi(A) \in \mathcal{X}$ unless $\varphi \in \mathrm{Aut}_{N_{\mathcal{F}}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)$. On the other hand, from maximality of the pairs in $[P \times_{\mathcal{H}} P]$ (see Definition 2.2.17), we have that $(P, \overline{\varphi}) \in [P \times_{\mathcal{H}} P]$ for every $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{O}(N_{\mathcal{F}})}(P) = \mathrm{Aut}_{\mathcal{O}(\mathcal{F})}(P)$. Let $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ and let $f \in \mathrm{End}\left(M \downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right)$. From the above discussion and Proposition 2.3.9 (9) we can conclude that

$$\mathrm{r}^{\mathcal{H}}_{\mathcal{F}_P}\left(\mathrm{tr}^{\mathcal{H}}_{\mathcal{F}_P}(f)\right) \in {}^{\mathcal{F},P}f + \mathrm{Tr}^{\mathcal{F}_P}_{\mathcal{X}}. \qquad \text{where} \qquad {}^{\mathcal{F},P}f := \sum_{\overline{\varphi} \in \mathrm{Aut}_{\mathcal{O}(\mathcal{F})}(P)} {}^{\varphi}f.$$

From Proposition 2.3.9 (3) we also have that $\mathrm{tr}^{\mathcal{H}}_{\mathcal{F}_P}\left(\mathrm{Tr}^{\mathcal{F}_P}_{\mathcal{X}}\right) = \mathrm{Tr}^{\mathcal{H}}_{\mathcal{X}}$. Using the above and Proposition 2.3.9 (10) we can conclude that

$$\overline{\mathrm{tr}^{\mathcal{H}}_{\mathcal{F}_P}(f)\,\mathrm{tr}^{\mathcal{H}}_{\mathcal{F}_P}(g)} = \overline{\mathrm{tr}^{\mathcal{H}}_{\mathcal{F}_P}\left(f\,\mathrm{r}^{\mathcal{H}}_{\mathcal{F}_P}\left(\mathrm{tr}^{\mathcal{H}}_{\mathcal{F}_P}(g)\right)\right)} = \overline{\mathrm{tr}^{\mathcal{H}}_{\mathcal{F}_P}\left(f\ {}^{\mathcal{F},P}g\right)}$$

for every $f, g \in \mathrm{End}\left(M \downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right)$. On the other hand we know from Lemma 2.4.29 that $\overline{\mathrm{tr}^{\mathcal{F}}_{N_{\mathcal{F}}}}\left(\overline{\mathrm{tr}^{N_{\mathcal{F}}}_{\mathcal{F}_P}(\alpha)}\right) = \overline{\mathrm{tr}^{\mathcal{F}}_{\mathcal{F}_P}}(\overline{\alpha})$ for $\alpha \in \left\{f, g, f\ {}^{\mathcal{F},P}g\right\}$ and, therefore, we can conclude that

$$\overline{\mathrm{tr}^{\mathcal{F}}_{N_{\mathcal{F}}}}\left(\overline{\mathrm{tr}^{N_{\mathcal{F}}}_{\mathcal{F}_P}(f)}\right)\overline{\mathrm{tr}^{\mathcal{F}}_{N_{\mathcal{F}}}}\left(\overline{\mathrm{tr}^{N_{\mathcal{F}}}_{\mathcal{F}_P}(g)}\right) = \overline{\mathrm{tr}^{\mathcal{F}}_{N_{\mathcal{F}}}}\left(\overline{\mathrm{tr}^{N_{\mathcal{F}}}_{\mathcal{F}_P}\left(f\ {}^{\mathcal{F},P}g\right)}\right) = \overline{\mathrm{tr}^{\mathcal{F}}_{N_{\mathcal{F}}}}\left(\overline{\mathrm{tr}^{N_{\mathcal{F}}}_{\mathcal{F}_P}(f)}\,\overline{\mathrm{tr}^{N_{\mathcal{F}}}_{\mathcal{F}_P}(g)}\right).$$

Since all elements in $\overline{\mathrm{Tr}^{N_{\mathcal{F}}}_{P}}$ are, by definition, of the form $\overline{\mathrm{tr}^{N_{\mathcal{F}}}_{\mathcal{F}_P}(f)}$ for some $f \in \mathrm{End}\left(M \downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right)$ the result follows. $\square$

2.4.6 Green correspondence for centric Mackey functors

Through this subsection we will be using Notations 2.1.1, 2.2.3, 2.2.9, 2.2.21, 2.2.31 and 2.4.9.

In this subsection we follow similar ideas as those Sasaki uses in [Sa82, Proposition 3.1] in order to prove that the Green correspondence holds for centric Mackey functors over fusion systems (Theorem 2.4.38). To do so we need to replace some results valid for Mackey functors over finite groups with the analogue results developed in Subsections 2.4.2 to 2.4.5. First however we need to prove that Proposition 2.4.7 can be applied to centric Mackey functors over fusion systems just like it can be applied to Green functors (see Example 2.4.8).

Let $\mathcal{R}$ be a complete local and p-local PID and let $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ be indecomposable with vertex P (see Corollary 2.3.19). Using Notation 2.4.9 we can define

$$\begin{aligned} A &:= \mathrm{End}\left(M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right), & B &:= \mathrm{Tr}^{\mathcal{F}}_{P} = \mathrm{End}\left(M\right), \\ C &:= \mathrm{Tr}^{N_{\mathcal{F}}}_{P}, & K &:= \mathrm{Tr}^{\mathcal{F}}_{\mathcal{X}}, \\ I &:= \mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{X}}, & J &:= \mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{Y}}, \\ f &:= \mathrm{tr}^{\mathcal{F}}_{N_{\mathcal{F}}}, & g &:= \mathrm{r}^{\mathcal{F}}_{N_{\mathcal{F}}}. \end{aligned}$$

Here we are using Theorem 2.3.17 and the fact that M has vertex P to define B and we are viewing f as a morphism from $\mathrm{Tr}^{N_{\mathcal{F}}}_{P}$ to $\mathrm{Tr}^{\mathcal{F}}_{P}$ (see Lemma 2.4.29) and $\mathrm{r}^{\mathcal{F}}_{N_{\mathcal{F}}}$ as a morphism from $\mathrm{Tr}^{\mathcal{F}}_{P}$ to $\mathrm{Tr}^{N_{\mathcal{F}}}_{P} + \mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{Y}}$ (see Proposition 2.4.16). With this setup we can now provide the following.

Lemma 2.4.33. *With notation as above the conditions needed to apply Proposition 2.4.7 are met. Moreover* Id_M *is a local idempotent of* $\mathrm{End}(M)$ *satisfying* $\mathrm{r}^{\mathcal{F}}_{N_{\mathcal{F}}}(\mathrm{Id}_M) = \mathrm{Id}_{M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}}$.

Proof. Since $\mathcal{R}$ is a complete local PID and M is indecomposable we can apply [CR81, Proposition 6.10 (ii)] to deduce that $\mathrm{End}(M)$ is a local ring. In particular Id_M is a local idempotent of $\mathrm{End}(M)$. The identity $\mathrm{r}^{\mathcal{F}}_{N_{\mathcal{F}}}(\mathrm{Id}_M) = \mathrm{Id}_{M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}}$ follows immediately from Definition 2.3.8.

Therefore, if we prove the first part of the statement, the second part follows.

Let us start by proving that A, B, C, K, I, J, f and g are as defined in Proposition 2.4.7. First of all notice that $\mathrm{End}\left(M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right)$ and $\mathrm{End}(M)$ are both $\mathcal{R}$-algebras. From Lemma 2.3.13 we also know that $\mathrm{Tr}^{\mathcal{F}}_{\mathcal{X}}$ is a two sided ideal of $\mathrm{End}(M)$ and that $\mathrm{Tr}^{N_{\mathcal{F}}}_{P}, \mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{X}}$ and $\mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{Y}}$ are two sided ideals of $\mathrm{End}\left(M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right)$. By definition of $\mathcal{X}$ (see Notation 2.4.9) we know that for every $Q \in \mathcal{X}$ then $\mathrm{Hom}_{\mathcal{F}}(Q, P) \neq \emptyset$. Therefore, from Lemma 2.3.14 we can conclude that $\mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{X}} \subseteq \mathrm{Tr}^{N_{\mathcal{F}}}_{P}$ and since $\mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{X}}$ is a two sided ideal of $\mathrm{End}\left(M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right)$ then we can view $\mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{X}}$ as a two sided ideal of $\mathrm{Tr}^{N_{\mathcal{F}}}_{P}$ (seen as a ring with potentially no unit). As mentioned in the statement we can use Lemma 2.4.29 to view $\mathrm{tr}^{\mathcal{F}}_{N_{\mathcal{F}}}$ as a morphism of $\mathcal{R}$-modules from $\mathrm{Tr}_P^{N_\mathcal{F}}$ to $\mathrm{Tr}_P^{\mathcal{F}}$. Finally, writing $E_Q := \mathrm{End}\left(M\downarrow_{\mathcal{F}_Q}^{\mathcal{F}}\right)$ for any $Q \in \mathcal{F}^c$, we have from Proposition 2.4.16 that

$$\mathrm{r}_{N_\mathcal{F}}^{\mathcal{F}}\left(\mathrm{Tr}_P^{\mathcal{F}}\right) = \mathrm{r}_{N_\mathcal{F}}^{\mathcal{F}}\left(\mathrm{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(E_P\right)\right) \subseteq \mathrm{tr}_{\mathcal{F}_P}^{N_\mathcal{F}}\left(E_P\right) + \sum_{Q\in\mathcal{Y}} \mathrm{tr}_{\mathcal{F}_Q}^{N_\mathcal{F}}\left(E_Q\right) = \mathrm{Tr}_P^{N_\mathcal{F}} + \mathrm{Tr}_\mathcal{Y}^{N_\mathcal{F}}.$$

Thus, we can view $\mathrm{r}_{N_\mathcal{F}}^{\mathcal{F}}$ as an $\mathcal{R}$-module morphism from $\mathrm{Tr}_P^{\mathcal{F}}$ to $\mathrm{Tr}_P^{N_\mathcal{F}} + \mathrm{Tr}_\mathcal{Y}^{N_\mathcal{F}}$.

With this setup we just need to prove that the Conditions (1)-(8) of Proposition 2.4.7 are met for our choices of A, B, C, K, I, J, f and g.

(1) *For Condition* (1) *we need to check that the following inclusions are satisfied*

$$\left(\mathrm{Tr}_P^{N_\mathcal{F}} \cap \mathrm{Tr}_\mathcal{Y}^{N_\mathcal{F}}\right)\mathrm{Tr}_P^{N_\mathcal{F}} \subseteq \mathrm{Tr}_\mathcal{X}^{N_\mathcal{F}}, \qquad \mathrm{Tr}_P^{N_\mathcal{F}}\left(\mathrm{Tr}_P^{N_\mathcal{F}} \cap \mathrm{Tr}_\mathcal{Y}^{N_\mathcal{F}}\right) \subseteq \mathrm{Tr}_\mathcal{X}^{N_\mathcal{F}},$$
$$\mathrm{Tr}_\mathcal{X}^{N_\mathcal{F}} \subseteq \mathrm{Tr}_P^{N_\mathcal{F}} \cap \mathrm{Tr}_\mathcal{Y}^{N_\mathcal{F}}.$$

From definition of $\mathcal{X}$ and $\mathcal{Y}$ (see Notation 2.4.9) we know that $\mathcal{X} \subseteq \mathcal{Y}$ and that every element in $\mathcal{X}$ is a subgroup of P. Therefore, from Lemma 2.3.14 we can conclude that $\mathrm{Tr}_\mathcal{X}^{N_\mathcal{F}} \subseteq \mathrm{Tr}_P^{N_\mathcal{F}}$ and $\mathrm{Tr}_\mathcal{X}^{N_\mathcal{F}} \subseteq \mathrm{Tr}_\mathcal{Y}^{N_\mathcal{F}}$. This proves the bottom inclusion. Let's now prove the top right inclusion (the top left inclusion follows similarly). Let $f \in \mathrm{End}\left(M\downarrow_{\mathcal{F}_P}^{\mathcal{F}}\right)$ and for every $Q \in \mathcal{Y}$ let $g_Q \in \mathrm{End}\left(M\downarrow_{\mathcal{F}_Q}^{\mathcal{F}}\right)$ such that $\sum_{Q\in\mathcal{Y}} \mathrm{tr}_{\mathcal{F}_Q}^{N_\mathcal{F}}\left(g_Q\right) \in \mathrm{Tr}_P^{N_\mathcal{F}} \cap \mathrm{Tr}_\mathcal{Y}^{N_\mathcal{F}}$. Then, from Items (9) and (10) of Proposition 2.3.9, we have that

$$\begin{aligned}\mathrm{tr}_{\mathcal{F}_P}^{N_\mathcal{F}}(f)\left(\sum_{Q\in\mathcal{Y}} \mathrm{tr}_{\mathcal{F}_Q}^{N_\mathcal{F}}\left(g_Q\right)\right) &= \sum_{Q\in\mathcal{Y}} \mathrm{tr}_{\mathcal{F}_P}^{N_\mathcal{F}}\left(f\,\mathrm{r}_{\mathcal{F}_P}^{N_\mathcal{F}}\left(\mathrm{tr}_{\mathcal{F}_Q}^{N_\mathcal{F}}\left(g_Q\right)\right)\right), \\ &= \sum_{Q\in\mathcal{Y}} \sum_{(A,\overline{\varphi})\in\left[Q\times_{N_\mathcal{F}}P\right]} \mathrm{tr}_{\mathcal{F}_P}^{N_\mathcal{F}}\left(f\,\mathrm{tr}_{\mathcal{F}_{\varphi(A)}}^{\mathcal{F}_P}\left({}^{\varphi}\left(\mathrm{r}_{\mathcal{F}_A}^{\mathcal{F}_Q}\left(g_Q\right)\right)\right)\right).\end{aligned}$$

Where φ is a representative of $\overline{\varphi}$ seen as an isomorphism onto its image. Fix now $Q \in \mathcal{Y}$ and $(A,\overline{\varphi}) \in \left[Q\times_{N_\mathcal{F}}P\right]$ and take φ as before. Since M is $\mathcal{F}$-centric, if $A \notin \mathcal{F}^c$ we have that $M\downarrow_{\mathcal{F}_A}^{\mathcal{F}} = 0$ and, in particular, $\mathrm{r}_{\mathcal{F}_A}^{\mathcal{F}_Q}\left(g_Q\right) = 0$. We can therefore assume without loss of generality that $A \in \mathcal{F}^c$. Since $A \in \mathcal{F}_Q \cap \mathcal{F}^c$ and $Q \in \mathcal{Y}$ then we can conclude from definition of $\mathcal{Y}$ (see Notation 2.4.9) that $A \in \mathcal{Y}$. From this we can deduce that $\varphi(A) \in \mathcal{Y}$ and since $\varphi(A) \leq P$ we can conclude that $\varphi(A) \in \mathcal{X}$. With this in mind, applying Items (3) and (10) of

Proposition 2.3.9, we obtain that

$$\operatorname{tr}_{\mathcal{F}_P}^{N_\mathcal{F}}\left(f\operatorname{tr}_{\mathcal{F}_{\varphi(A)}}^{\mathcal{F}_P}\left({}^{\varphi}\left(\operatorname{r}_{\mathcal{F}_A}^{\mathcal{F}_Q}(g_Q)\right)\right)\right)=\operatorname{tr}_{\mathcal{F}_{\varphi(A)}}^{\mathcal{F}}\left(\operatorname{r}_{\mathcal{F}_{\varphi(A)}}^{\mathcal{F}_P}(f)^{\varphi}\left(\operatorname{r}_{\mathcal{F}_A}^{\mathcal{F}_Q}(g_Q)\right)\right)\in\operatorname{Tr}_{\varphi(A)}^{N_\mathcal{F}}\subseteq\operatorname{Tr}_{\mathcal{X}}^{N_\mathcal{F}}.$$

Therefore $\operatorname{Tr}_P^{N_\mathcal{F}}\left(\operatorname{Tr}_P^{N_\mathcal{F}}\cap\operatorname{Tr}_\mathcal{Y}^{N_\mathcal{F}}\right)\subseteq\operatorname{Tr}_\mathcal{X}^{N_\mathcal{F}}$ thus proving that Condition (1) is satisfied.

(2) *For Condition* (2) *we need to check that* $\operatorname{r}_{N_\mathcal{F}}^{\mathcal{F}}\left(\operatorname{Tr}_\mathcal{X}^{\mathcal{F}}\right)\subseteq\operatorname{Tr}_\mathcal{Y}^{N_\mathcal{F}}$. For every $Q\in\mathcal{X}$ we have that $Q\lneq P$ and, therefore, from Proposition 2.3.9 (3), we have that $\operatorname{Tr}_Q^{\mathcal{F}}=\operatorname{tr}_{\mathcal{F}_P}^{\mathcal{F}}\left(\operatorname{Tr}_Q^{\mathcal{F}_P}\right)$. From Proposition 2.4.16 we can then deduce that $\operatorname{r}_{N_\mathcal{F}}^{\mathcal{F}}\left(\operatorname{Tr}_Q^{\mathcal{F}}\right)\subseteq\operatorname{tr}_{\mathcal{F}_P}^{N_\mathcal{F}}\left(\operatorname{Tr}_Q^{\mathcal{F}_P}\right)+\operatorname{Tr}_\mathcal{Y}^{N_\mathcal{F}}$. Applying Proposition 2.3.9 (3) once again we obtain that $\operatorname{tr}_{\mathcal{F}_P}^{N_\mathcal{F}}\left(\operatorname{Tr}_Q^{\mathcal{F}_P}\right)=\operatorname{Tr}_Q^{N_\mathcal{F}}$ and, since $Q\in\mathcal{X}\subseteq\mathcal{Y}$, we can deduce that $\operatorname{Tr}_Q^{N_\mathcal{F}}\subseteq\operatorname{Tr}_\mathcal{Y}^{N_\mathcal{F}}$. Thus we can conclude that $\operatorname{r}_{N_\mathcal{F}}^{\mathcal{F}}\left(\operatorname{Tr}_Q^{\mathcal{F}}\right)\subseteq\operatorname{Tr}_\mathcal{Y}^{N_\mathcal{F}}$. Since this works for every $Q\in\mathcal{X}$ the result follows.

(3) *For Condition* (3) *we need to check that* $\operatorname{tr}_{N_\mathcal{F}}^{\mathcal{F}}\left(\operatorname{Tr}_\mathcal{X}^{N_\mathcal{F}}\right)\subseteq\operatorname{Tr}_\mathcal{X}^{\mathcal{F}}$. This follows from Corollary 2.4.30.

(4) *For Condition* (4) *we need to check that* $\operatorname{tr}_{N_\mathcal{F}}^{\mathcal{F}}$*, seen as a morphism from* $\operatorname{Tr}_P^{N_\mathcal{F}}$ *to* $\operatorname{Tr}_P^{\mathcal{F}}$*, is surjective.* This is given by Lemma 2.4.29.

(5) *For Condition* (5) *we need to check that* $\operatorname{r}_{N_\mathcal{F}}^{\mathcal{F}}$ *sends idempotents to idempotents.* This follows immediately from definition of $\operatorname{r}_{N_\mathcal{F}}^{\mathcal{F}}$.

(6) *For Condition* (6) *we need to check that the* $\mathcal{R}$*-linear maps* $\overline{\operatorname{tr}_{N_\mathcal{F}}^{\mathcal{F}}}:\operatorname{Tr}_P^{N_\mathcal{F}}/\operatorname{Tr}_\mathcal{X}^{N_\mathcal{F}}\to\operatorname{Tr}_P^{\mathcal{F}}/\operatorname{Tr}_\mathcal{X}^{\mathcal{F}}$ *and* $\overline{\operatorname{r}_{N_\mathcal{F}}^{\mathcal{F}}}:\operatorname{Tr}_P^{\mathcal{F}}/\operatorname{Tr}_\mathcal{X}^{\mathcal{F}}\to\left(\operatorname{Tr}_P^{N_\mathcal{F}}+\operatorname{Tr}_\mathcal{Y}^{N_\mathcal{F}}\right)/\operatorname{Tr}_\mathcal{Y}^{N_\mathcal{F}}$ *commute with multiplication.* From Lemma 2.4.32 we know that $\overline{\operatorname{tr}_{N_\mathcal{F}}^{\mathcal{F}}}$ commutes with multiplication. On the other hand it is immediate from definition that $\operatorname{r}_{N_\mathcal{F}}^{\mathcal{F}}$ commutes with multiplication and, therefore, so does $\overline{\operatorname{r}_{N_\mathcal{F}}^{\mathcal{F}}}$.

(7) *For Condition* (7) *we need to check that the natural isomorphism*

$$s:\operatorname{Tr}_P^{N_\mathcal{F}}/\left(\operatorname{Tr}_P^{N_\mathcal{F}}\cap\operatorname{Tr}_\mathcal{Y}^{N_\mathcal{F}}\right)\hookrightarrow\!\!\!\!\rightarrow\left(\operatorname{Tr}_P^{N_\mathcal{F}}+\operatorname{Tr}_\mathcal{Y}^{N_\mathcal{F}}\right)/\operatorname{Tr}_\mathcal{Y}^{N_\mathcal{F}}$$

and the natural projection

$$q:\operatorname{Tr}_P^{N_\mathcal{F}}/\operatorname{Tr}_\mathcal{X}^{N_\mathcal{F}}\twoheadrightarrow\operatorname{Tr}_P^{N_\mathcal{F}}/\left(\operatorname{Tr}_P^{N_\mathcal{F}}\cap\operatorname{Tr}_\mathcal{Y}^{N_\mathcal{F}}\right)$$

satisfy $sq=\overline{\operatorname{r}_{N_\mathcal{F}}^{\mathcal{F}}}\,\overline{\operatorname{tr}_{N_\mathcal{F}}^{\mathcal{F}}}$. Abusing a bit of notation we denote with an overline $(\bar{\cdot})$ the projection of an endomorphism on the appropriate quotient. With this notation, for every $f \in \mathrm{End}\left(M \downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right)$, we have that

$$\overline{\mathrm{r}^{\mathcal{F}}_{N_{\mathcal{F}}}}\left(\overline{\mathrm{tr}^{\mathcal{F}}_{N_{\mathcal{F}}}}\left(\overline{\mathrm{tr}^{N_{\mathcal{F}}}_{\mathcal{F}_P}(f)}\right)\right) = \overline{\mathrm{r}^{\mathcal{F}}_{N_{\mathcal{F}}}\left(\mathrm{tr}^{\mathcal{F}}_{\mathcal{F}_P}(f)\right)} = \overline{\mathrm{tr}^{N_{\mathcal{F}}}_{\mathcal{F}_P}(f)} = s\left(q\left(\overline{\mathrm{tr}^{N_{\mathcal{F}}}_{\mathcal{F}_P}(f)}\right)\right),$$

Where we are using Lemma 2.4.29 for the first identity, we are using Proposition 2.4.16 and the fact that $\overline{\mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{Y}}} = \overline{0}$ for the second identity and we are using the definitions of q and s for the third identity. Since every element in $\mathrm{Tr}^{N_{\mathcal{F}}}_{P} / \mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{X}}$ is of the form $\overline{\mathrm{tr}^{N_{\mathcal{F}}}_{\mathcal{F}_P}(f)}$ for some $f \in \mathrm{End}\left(M \downarrow^{\mathcal{F}}_{\mathcal{F}_P}\right)$ the result follows.

(8) *For Condition* (8) *we need to prove that for every idempotent* $f \in \mathrm{End}\left(M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right)$ *there exists a unique (up to conjugation) decomposition of* f *as a finite sum of orthogonal local idempotents.* From Proposition 2.2.24 we know that the $\mathcal{R}$-algebra $\mu_{\mathcal{R}}(N_{\mathcal{F}})$ is finitely generated as an $\mathcal{R}$-module. Therefore we can apply the Krull-Schmidt-Azumaya theorem (see [CR81, Theorem 6.12 (ii)]) together with [CR81, Proposition 6.10 (ii)] to conclude that Condition (8) is satisfied.

Since all conditions are verified we can conclude the proof. □

Corollary 2.4.34. *With notation as in Lemma 2.4.33 there exists a unique way (up to conjugation) of writing*

$$\mathrm{Id}_{M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}} = \sum_{i=0}^{n} \varepsilon_i$$

where each ε_i *is a local idempotent in* $\mathrm{Tr}^{N_{\mathcal{F}}}_{P}$ *and they are all mutually orthogonal. Moreover there exists a unique* $j \in \{0,\dots,n\}$ *such that* $\varepsilon_j \in \mathrm{Tr}^{N_{\mathcal{F}}}_{P} - \mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{Y}}$ *and, defining* $(\mathrm{Id}_M)_{N_{\mathcal{F}}} := \varepsilon_j$ *the following hold*

$$\mathrm{tr}^{\mathcal{F}}_{N_{\mathcal{F}}}\left((\mathrm{Id}_M)_{N_{\mathcal{F}}}\right) \equiv \mathrm{Id}_M \mod \mathrm{Tr}^{\mathcal{F}}_{\mathcal{X}}, \qquad \mathrm{r}^{\mathcal{F}}_{N_{\mathcal{F}}}(\mathrm{Id}_M) \equiv (\mathrm{Id}_M)_{N_{\mathcal{F}}} \mod \mathrm{Tr}^{N_{\mathcal{F}}}_{\mathcal{Y}}.$$

Proof. This is an immediate consequence of applying Proposition 2.4.7 to the setup of Lemma 2.4.33. □

Corollary 2.4.35. *Let* $\mathcal{R}$ *be a complete local and* p*-local* PID *and let* $M \in \mathit{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ *be indecomposable with vertex* P *(see Corollary 2.3.19). There exists a unique (up to isomorphism) decomposition of* $M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}$ *as a direct sum of indecomposable* $\mathcal{F}$*-centric Mackey functors*

$$M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}} = \bigoplus_{i=0}^{n} M_i.$$

*With this notation, there exists exactly one $j \in \{0, \dots, n\}$ such that M_j has vertex P while, for every other $i \in \{0, \dots, n\} - \{j\}$, we have that M_i has vertex in $\mathcal{Y}$. We call M_j the **Green correspondent** of M and denote it by $M_{N_{\mathcal{F}}}$.*

Proof. Applying Corollary 2.4.34 we know that there exists a unique (up to conjugation) decomposition of $\mathrm{Id}_{M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}}$ of the form

$$\mathrm{Id}_{M\downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}} = (\mathrm{Id}_M)_{N_{\mathcal{F}}} + \sum_{i=1}^{n} \varepsilon_i$$

where the ε_i and $(\mathrm{Id}_M)_{N_{\mathcal{F}}}$ are mutually orthogonal local idempotents satisfying $(\mathrm{Id}_M)_{N_{\mathcal{F}}} \in \mathrm{Tr}_P^{N_{\mathcal{F}}} - \mathrm{Tr}_{\mathcal{Y}}^{N_{\mathcal{F}}}$ and $\varepsilon_i \in \mathrm{Tr}_{\mathcal{Y}}^{N_{\mathcal{F}}}$ for every $i = 1, \dots, n$. From this decomposition and [CR81, Proposition 6.10 (ii)] we can deduce that there exists a unique (up to isomorphism) decomposition of $M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}$ as a direct sum of indecomposable Mackey functors and it is given by

$$M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}} = (\mathrm{Id}_M)_{N_{\mathcal{F}}} \left(M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right) \oplus \bigoplus_{i=1}^{n} \varepsilon_i \left(M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right).$$

Since $(\mathrm{Id}_M)_{N_{\mathcal{F}}} \in \mathrm{Tr}_P^{N_{\mathcal{F}}} - \mathrm{Tr}_{\mathcal{Y}}^{N_{\mathcal{F}}}$, using Theorem 2.3.17, we can conclude that $(\mathrm{Id}_M)_{N_{\mathcal{F}}} \left(M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right)$ has vertex P. On the other hand, since $\varepsilon_i \in \mathrm{Tr}_{\mathcal{Y}}^{N_{\mathcal{F}}}$ and $\mathrm{Tr}_{\mathcal{Y}}^{N_{\mathcal{F}}}$ is an ideal (see Lemma 2.3.13) we have that $\varepsilon_i \in \varepsilon_i \mathrm{Tr}_{\mathcal{Y}}^{N_{\mathcal{F}}} \varepsilon_i = \varepsilon_i \mathrm{End}\left(M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right) \varepsilon_i$. Since ε_i is a local idempotent of $\mathrm{End}\left(M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right)$ then we can conclude that $\varepsilon_i \mathrm{Tr}_{\mathcal{Y}}^{N_{\mathcal{F}}} \varepsilon_i$ is a local ring and, therefore, there exists $Q \in \mathcal{Y}$ such that $\varepsilon_i \mathrm{Tr}_Q^{N_{\mathcal{F}}} \varepsilon_i = \varepsilon_i \mathrm{End}\left(M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right) \varepsilon_i$. In particular ε_i is Q-projective and, from Theorem 2.3.17, we can conclude that $\varepsilon_i \left(M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right)$ is also Q-projective. Since $Q \in \mathcal{Y}$ we can conclude from minimality of the defect set that $\varepsilon_i \left(M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right)$ has vertex in $\mathcal{Y}$. The result follows by setting $M_{N_{\mathcal{F}}} := (\mathrm{Id}_M)_{N_{\mathcal{F}}} \left(M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right)$ and $M_i := \varepsilon_i \left(M \downarrow^{\mathcal{F}}_{N_{\mathcal{F}}}\right)$. □

Corollary 2.4.35 gives us the first half of the Green correspondence. Let's now get the other half.

Lemma 2.4.36. *Let $\mathcal{R}$ be a complete local and p-local PID, let $N \in \mathit{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(N_{\mathcal{F}})$ be indecomposable with vertex P. Using the notation of Corollary 2.4.35 there exists an indecomposable $M \in \mathit{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ with vertex P such that $M_{N_{\mathcal{F}}} \cong N$. Moreover M is a summand of $N \uparrow^{\mathcal{F}}_{N_{\mathcal{F}}} \cong M_{N_{\mathcal{F}}} \uparrow^{\mathcal{F}}_{N_{\mathcal{F}}}$.*

Proof. From Proposition 2.2.24 we know that $\mu_{\mathcal{R}}(\mathcal{F})$ is finitely generated as an $\mathcal{R}$-module. Therefore we can apply the Krull-Schmidt-Azumaya theorem (see [CR81, Theorem 6.12 (ii)]) in order to write $N \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \cong \bigoplus_{i=0}^{n} M_i$. Where each $M_i \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ (see Proposition 2.2.30) is indecomposable. From Lemma 2.4.12 we know that N is a summand of $N \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$. Since the restriction functor is additive we can now use the fact that N is indecomposable and uniqueness of the Krull-Schmidt-Azumaya theorem (now applied to $N \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$) in order to chose $j \in \{0, \ldots, n\}$ such that N is a summand of $M_j \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$. To simplify notation let us define $M := M_j$. We are now only left with proving that M has vertex P. Since N is P-projective and M is a summand of $N \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ then we can deduce from Theorem 2.3.17 that M is P-projective. From minimality of the defect set (see Corollary 2.3.6) we can now conclude that the vertex V_M of M satisfies $V_M \leq_{\mathcal{F}} P$ (see Notation 2.2.3). Assume that $V_M \lneq_{\mathcal{F}} P$. From Corollary 2.3.4 (2) we can deduce that there exists $Q \lneq P$ such that M is Q-projective. From Theorem 2.3.17 we can then deduce that there exists $L \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_Q)$ such that M is a summand of $L' \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ where $L' := L \uparrow_{\mathcal{F}_Q}^{N_{\mathcal{F}}}$. Since N is a summand of $M \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ we can deduce that N is a summand of $L' \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$. From Lemma 2.4.12 we can now deduce that there exists an $\mathcal{Y}$-projective $K \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(N_{\mathcal{F}})$ such that $L' \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} = L' \oplus K = L \uparrow_{\mathcal{F}_Q}^{N_{\mathcal{F}}} \oplus K$. Since $Q \lneq P$ by hypothesis then we can conclude that $Q \in \mathcal{X} \subseteq \mathcal{Y}$ and, therefore, that $L \uparrow_{\mathcal{F}_Q}^{N_{\mathcal{F}}} \oplus K$ is $\mathcal{Y}$-projective. Since N is an indecomposable summand of $L \uparrow_{\mathcal{F}_Q}^{N_{\mathcal{F}}} \oplus K$ we can then conclude from Corollary 2.3.19 that the vertex of N lies in $\mathcal{Y}$. Since $P \notin \mathcal{Y}$ this contradicts the hypothesis that N has vertex P. Therefore we cannot have $V_M \lneq_{\mathcal{F}} P$. Since we have proven that $V_M \leq_{\mathcal{F}} P$ we can therefore conclude that $V_M =_{\mathcal{F}} P$. Since P is fully $\mathcal{F}$-normalized (see Notation 2.4.9) then we can conclude that M has vertex P. We can therefore apply Corollary 2.4.35 to M in order to conclude that there exists a unique (up to isomorphism) indecomposable summand $M_{N_{\mathcal{F}}}$ of $M \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ with vertex P. Since N is a summand of $M \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ and has vertex P the result follows. $\square$

Lemma 2.4.37. *Let $\mathcal{R}$ be a complete local and p-local PID, let $M \in$ Mack$_{\mathcal{R}}(\mathcal{F}^c)$ be indecomposable with vertex P and let $M_{N_{\mathcal{F}}}$ be as in Corollary 2.4.35. Since $\mu_{\mathcal{R}}(\mathcal{F})$ is finitely generated as an $\mathcal{R}$-module (see Proposition 2.2.24) we can apply the Krull-Schmidt-Azumaya theorem (see [CR81, Theorem 6.12 (ii)]) together with Lemma 2.4.36 in order to write*

$$M_{N_{\mathcal{F}}} \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \cong M \oplus \bigoplus_{i=1}^{n} M_i.$$

Where each M_i is indecomposable. With this notation we have that each M_i is $\mathcal{F}$-centric and has vertex $\mathcal{F}$-isomorphic to an element in $\mathcal{X}$ (see Notation 2.4.9).

Proof. From Proposition 2.2.30 we know that $M_i \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ for every $i = 1, \dots, n$. From Corollary 2.4.35 we know $M_{N_\mathcal{F}}$ has vertex P. Therefore, from Theorem 2.3.17, we know that there exists $N \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_P)$ such that $M_{N_\mathcal{F}}$ is a summand of $N \uparrow_{\mathcal{F}_P}^{N_\mathcal{F}}$. Since induction preserves direct sum decomposition then we can conclude that each M_i is a summand of $N \uparrow_{\mathcal{F}_P}^{\mathcal{F}}$. From Theorem 2.3.17 this implies that each M_i is P-projective. Assume now that there exists $j \in \{1, \dots, n\}$ such that M_j has vertex P. Since restriction preserves direct sum decomposition we can conclude, using Corollary 2.4.35 that, $M_{N_\mathcal{F}} \oplus (M_j)_{N_\mathcal{F}}$ is a summand of $M_{N_\mathcal{F}} \uparrow_{N_\mathcal{F}}^{\mathcal{F}} \downarrow_{N_\mathcal{F}}^{\mathcal{F}}$. However, from Lemma 2.4.12, we know that there exists an $\mathcal{Y}$-projective $L \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(N_\mathcal{F})$ such that $M_{N_\mathcal{F}} \uparrow_{N_\mathcal{F}}^{\mathcal{F}} \downarrow_{N_\mathcal{F}}^{\mathcal{F}} \cong M_{N_\mathcal{F}} \oplus L$. From uniqueness of the Krull-Schmidt-Axumaya theorem we can then deduce that $(M_j)_{N_\mathcal{F}}$ is a summand of L. Thus, from Corollary 2.3.19, we can conclude that $(M_j)_{N_\mathcal{F}}$ has vertex in $\mathcal{Y}$. This contradicts Corollary 2.4.35. Thus we can deduce that none of the M_i has vertex in P. Since they are all P-projective then we can conclude from minimality of the defect set and Corollary 2.3.4 that they are all $\mathcal{X}$-projective. From Corollary 2.3.19 this implies that each M_i has vertex $\mathcal{F}$-isomorphic to an element in $\mathcal{X}$ thus concluding the proof. □

Putting the previous results together we can prove that the Green correspondence holds for centric Mackey functors over fusion systems.

Theorem 2.4.38 (Green correspondence). *Let $\mathcal{R}$ be a complete local and p-local PID (see Definition 2.2.40), let $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ (see Definition 2.2.29) be indecomposable with vertex P (see Definition 2.3.7 and Notation 2.4.9) and let $N \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(N_\mathcal{F})$ (see Example 2.2.8) be indecomposable with vertex P. There exist unique (up to isomorphism) decompositions of $M \downarrow_{N_\mathcal{F}}^{\mathcal{F}}$ and $N \uparrow_{N_\mathcal{F}}^{\mathcal{F}}$ (see Definition 2.2.28) into direct sums of indecomposable Mackey functors. Moreover, writing these decompositions as*

$$M \downarrow_{N_\mathcal{F}}^{\mathcal{F}} := \bigoplus_{i=0}^{n} M_i, \qquad N \uparrow_{N_\mathcal{F}}^{\mathcal{F}} := \bigoplus_{j=0}^{m} N_j,$$

*there exist unique $i \in \{0, \dots, n\}$ and $j \in \{0, \dots, m\}$ such that both M_i and N_j have vertex P. We call these summands the **Green correspondents** of M and N and denote them as $M_{N_\mathcal{F}}$ and $N^\mathcal{F}$ respectively. Moreover every indecomposable summand of*

$M\downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ other than $M_{N_{\mathcal{F}}}$ has vertex in $\mathcal{Y}$ (see Notation 2.4.9) while every indecomposable summand of $N\uparrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ other than $N^{\mathcal{F}}$ has vertex $\mathcal{F}$-isomorphic to an element in $\mathcal{X}$ (see Notation 2.4.9). Finally we have that $(M_{N_{\mathcal{F}}})^{\mathcal{F}}\cong M$ and that $\left(N^{\mathcal{F}}\right)_{N_{\mathcal{F}}}\cong N$.

Proof. From Lemma 2.4.36 we know that there exists an indecomposable $L\in\mathrm{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ with vertex P such that $N=L_{N_{\mathcal{F}}}$. It follows from Lemma 2.4.37 that there exists a unique (up to isomorphism) decomposition of $N\uparrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ as the one in the statement and that $L\cong N^{\mathcal{F}}$. In particular $\left(N^{\mathcal{F}}\right)_{N_{\mathcal{F}}}\cong N$.

From Corollary 2.4.35 we know that there exists a unique (up to isomorphism) decomposition of $M\downarrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ as the one in the statement. From Lemma 2.4.37 and the first part of the statement we have that $(M_{N_{\mathcal{F}}})^{\mathcal{F}}\cong M$ which concludes the proof. □

Before concluding this paper let us see an example where Theorem 2.4.38 can be applied.

Example 2.4.39. Let $\mathcal{R}$ be a complete local and p-local PID and let $\mathcal{F}$ be a fusion system. For example, we can take $\mathcal{R}=\mathbb{Z}_p$ and, using the notation of Example 2.2.2, we can take the fusion system $\mathcal{F}:=\mathcal{F}_1:=\mathcal{F}_{D_8}(GL_2(3))$, or the Ruiz-Viruel exotic fusion system $\mathcal{F}:=\mathcal{F}_2$ on 7_+^{1+2} having two $\mathcal{F}_2$-orbits of elementary abelian subgroups of rank 2 the first of which has 6 elements while the second has 2 elements (see [RV04, Theorem 1.1]).

Choose now $P\in\mathcal{F}^c$ fully $\mathcal{F}$-normalized and minimal under the preorder $\leq_{\mathcal{F}}$ (see Notation 2.2.3). For $\mathcal{F}_1$ we can take $P:=P_1$ to be any one of the two characteristic elementary abelian subgroups of rank 2 of D_8. For $\mathcal{F}_2$ we can take $P:=P_2$ to be one of the two elementary abelian subgroups of rank two whose $\mathcal{F}_2$-orbit contains only 2 elements (make sure to take one that is fully $\mathcal{F}_2$-normalized).

In order to visualize this example it might help to notice the following identities

$$N_{\mathcal{F}_1}(P_1)=\mathcal{F}_{D_8}(S_4),\qquad\qquad N_{\mathcal{F}_2}(P_2)=\mathcal{F}_{7_+^{1+2}}(L_3(7).3).$$

The first one follows after a straightforward calculation while the second one follows from [RV04, Theorem 1.1] and [Br05, Section 4].

Let $\mathcal{I}$ be as in Proposition 2.2.33 and for every $x\in\mu_{\mathcal{R}}(\mathcal{F})$ denote by $\overline{x}\in\mu_{\mathcal{R}}(\mathcal{F})/\mathcal{I}$ its image via the natural projection. From Proposition 2.2.24 we know that $\mu_{\mathcal{R}}(\mathcal{F})$ is finitely generated as an $\mathcal{R}$-module. As a consequence $\mu_{\mathcal{R}}(\mathcal{F})/\mathcal{I}$ is also finitely generated as an $\mathcal{R}$-module. Therefore we can apply the Krull-Schmidt-Azumaya theorem (see [CR81, Theorem 6.12 (ii)]) together with [CR81, Proposition 6.10 (ii)] in order to conclude that, for every $P \in \mathcal{F}^c$, there exists a unique (up to conjugation) decomposition of $\overline{I_P^P}$ in $\mu_{\mathcal{R}}(\mathcal{F})/\mathcal{I}$ as a sum of orthogonal local idempotents. Let $\overline{I_P^P} = \sum_{i=0}^{n} \overline{x_i}$ be such decomposition. Define now $\overline{x} := \overline{x_0}$. For example, for $\mathcal{F}_1$ we have that $\mathrm{Aut}_{\mathcal{F}_1}(P_1) \cong S_3$ and, therefore, we can take $\varphi \in \mathrm{Aut}_{\mathcal{F}_1}(P_1)$ to be one of the two elements of order 3 and $\overline{x} := \frac{2}{3}\overline{I_{P_1}^{P_1}} - \frac{1}{3}\overline{c_\varphi} - \frac{1}{3}\overline{c_{\varphi^2}}$ is a local idempotent in the decomposition of $\overline{I_{P_1}^{P_1}}$.

Since $\overline{I_P^P x I_P^P} = \overline{x}$ by construction, from Proposition 2.2.24, we know that

$$\overline{x} = \sum_{j=0}^{m} \lambda_j \overline{I_{\varphi_j(A_j)}^P c_{\varphi_j} R_{A_j}^P}.$$

for some $\lambda_j \in \mathcal{R}$, some $A_j \leq P$ and some isomorphisms $\varphi_j \colon A_j \to \varphi_j(A_j)$ in $\mathcal{F}$ such that $\varphi_j(A_j) \leq P$. Since P is minimal $\mathcal{F}$-centric, by definition of $\mathcal{I}$, we can conclude that $\overline{I_{\varphi_j(A_j)}^P c_{\varphi_j} R_{A_j}^P} = \overline{0}$ unless $A_j = P$. When this is the case then we necessarily have that $\varphi_j \in \mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{N_{\mathcal{F}}}(P)$. Viewing $\mu_{\mathcal{R}}(N_{\mathcal{F}})$ as a subset of $\mu_{\mathcal{R}}(\mathcal{F})$ (see Corollary 2.2.25), we have in particular that $\overline{x} \in \overline{\mu_{\mathcal{R}}(N_{\mathcal{F}})}$. Define now the two sided ideal $\mathcal{J}$ of $\mu_{\mathcal{R}}(N_{\mathcal{F}})$ as $\mathcal{J} := \mathcal{I} \cap \mu_{\mathcal{R}}(N_{\mathcal{F}})$. We know that $\mu_{\mathcal{R}}(N_{\mathcal{F}})/\mathcal{J} \cong (\mu_{\mathcal{R}}(N_{\mathcal{F}}) + \mathcal{I})/\mathcal{I}$ and, therefore, we can view $\mu_{\mathcal{R}}(N_{\mathcal{F}})/\mathcal{J}$ as a subset of $\mu_{\mathcal{R}}(\mathcal{F})\mathcal{I}$ and $\overline{x}$ as an idempotent in $\mu_{\mathcal{R}}(N_{\mathcal{F}})/\mathcal{J}$. Since $\overline{x}$ is a primitive idempotent of $\mu_{\mathcal{R}}(\mathcal{F})/\mathcal{I}$ (recall that every local idempotent is primitive), it is also a primitive idempotent of $\mu_{\mathcal{R}}(N_{\mathcal{F}})/\mathcal{J}$. In particular we have that $M := (\mu_{\mathcal{R}}(\mathcal{F})/\mathcal{I})\overline{x}$ and $N := (\mu_{\mathcal{R}}(N_{\mathcal{F}})/\mathcal{J})\overline{x}$ are indecomposable as left $\mu_{\mathcal{R}}(\mathcal{F})/\mathcal{I}$ and $\mu_{\mathcal{R}}(N_{\mathcal{F}})/\mathcal{J}$-modules respectively. In particular they are indecomposable as $\mu_{\mathcal{R}}(\mathcal{F})$ and $\mu_{\mathcal{R}}(N_{\mathcal{F}})$ modules respectively (i.e. as Mackey functors over $\mathcal{F}$ and $N_{\mathcal{F}}$ respectively). From definition of $\mathcal{I}$ and $\mathcal{J}$ we can also conclude that $M \in \mathsf{Mack}_{\mathcal{R}}(\mathcal{F}^c)$ and $N \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(N_{\mathcal{F}})$.

From Lemma 2.2.22, Proposition 2.2.24 and [Li07, Proposition 4.4] we know that $\mathcal{I}$ is spanned as an $\mathcal{R}$-module by elements of the form $I_{\varphi(C)}^B c_\varphi R_C^A$ with $C \in \mathcal{F}_A \setminus (\mathcal{F}_A \cap \mathcal{F}^c)$. In particular $R_C^A \in \mu_{\mathcal{R}}(\mathcal{F}_A) \cap \mathcal{I}$ and we can write any element in $\mathcal{I} I_P^P$ (resp. $\mathcal{J} I_P^P$) as a finite sum of elements of the form bc with $b \in \mu_{\mathcal{R}}(\mathcal{F})$ (resp. $\mu_{\mathcal{R}}(N_{\mathcal{F}})$) and $c \in \mu_{\mathcal{R}}(\mathcal{F}_P) \cap \mathcal{I}$. Therefore, for every $y \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} \overline{x} \in M_P$ (resp. $y \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} \overline{x} \in N_P$) such that $y \in \mathcal{I}$ (resp. $y \in \mathcal{J}$) we have that $y \otimes_{\mu_{\mathcal{R}}(\mathcal{F}_P)} \overline{x} = 0$. This allows us to define the morphisms of Mackey functors $u_P^M : M \to M_P$ and $u_P^N : N \to N_P$ by setting

$u_P^M(\overline{a}) = aI_P^P \otimes \overline{x}$ and $u_P^N(\overline{b}) = bI_P^P \otimes \overline{x}$ for any representative $a \in \mu_{\mathcal{R}}(\mathcal{F})$ of $\overline{a} \in M$ and any representative $b \in \mu_{\mathcal{R}}(N_{\mathcal{F}})$ of $\overline{b} \in N$. Since $\overline{x}$ is an idempotent and $\overline{I_P^P}\overline{x} = \overline{x}$ by construction, using the previous notation we have that $\overline{aI_P^P}\overline{x} = \overline{a}$ and that $\overline{bI_P^P}\overline{x} = \overline{b}$. In other words we have that $\theta_P^M u_P^M = \mathrm{Id}_M$ and that $\theta_P^N u_P^N = \mathrm{Id}_N$. Equivalently both M and N are P-projective. Since P is minimal $\mathcal{F}$-centric and fully $\mathcal{F}$-normalized we can conclude from minimality of the defect set and Corollary 2.3.19 that P is in fact the vertex of both M and N. We now have by construction that $M \cong N \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}}$ which proves that $M \cong N^{N_{\mathcal{F}}}$. From Theorem 2.4.38 we can then conclude that $N = M_{N_{\mathcal{F}}}$ and, therefore, that there exists an $\mathcal{Y}$-projective $N' \in \mathsf{Mack}_{\mathcal{R}}^{\mathcal{F}^c}(N_{\mathcal{F}})$ such that $M \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} \cong N \oplus N'$. In the case $\mathcal{F} := \mathcal{F}_1$, since P_1 is characteristic and P_1 is minimal $\mathcal{F}$-centric, then we have that $\mathcal{Y} = \emptyset$ and, therefore, $N' = 0$ and $M \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} \cong N$. On the other hand, in the case of $\mathcal{F} := \mathcal{F}_2$, we have that $\mathcal{Y} = \{Q\}$ where Q is the only other group in the $\mathcal{F}_2$ orbit of P_2. Thus N' is Q-projective. Since P_2 is minimal $\mathcal{F}$-centric then so is Q and since N' is $\mathcal{F}$-centric then $N'_J = 0$ for every $J \lneq Q$ and, therefore, we can conclude from Theorem 2.3.17 that N' has vertex Q.

Bibliography of chapter 2

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Chapter 3

Sharpness for the Benson-Solomon fusion systems

Abstract

We develop tools to prove Díaz and Park's sharpness conjecture (see [DP15]) for fusion systems admitting tame families of fusion subsystems (see Theorem 3.A). We use such tools to prove the conjecture for all Benson-Solomon fusion systems (see Theorem 3.B) thus completing the work started in [HLL23, Theorems 1.1 and 1.4].

3.1 Introduction

Let G be a finite group, let p be a prime number, let $\boldsymbol{C}$ be a small category and let $F : \boldsymbol{C} \to \mathrm{Top}$ be a functor sending every object in $\boldsymbol{C}$ to the classifying space BH of a subgroup H of G. A (mod p) homology decomposition of BG is a mod-p homology equivalence of the form

$$\mathrm{hocolim}_{\boldsymbol{C}}(F) \underset{p}{\tilde{\to}} BG.$$

Due to a result of Bousfield and Kan (see [BK87, Chapter XII Section 4.5]) we know that any such a homology decomposition leads to a first quadrant cohomology spectral sequence

$$\lim^i_{\boldsymbol{C}} H^j(F(-), \mathbb{F}_p) \Rightarrow H^{i+j}(G, \mathbb{F}_p). \tag{3.1}$$

In [Dw97, Theorem 1.6 and Example 1.17] Dwyer proves that an homology decomposition of G can be obtained by taking $\boldsymbol{C}$ to be the p-centric orbit category $\mathcal{O}^c_p(G)$ of G and defining $F(H) := BH$ for every subgroup $H \leq G$ in $\mathcal{O}^c_p(G)$. Building on this Dwyer proves in [Dw98, Theorem 10.3] that the spectral sequence of Equation (3.1) deriving from such homology decomposition is in fact sharp (i.e. $\lim^i_{\mathcal{O}^c_p(G)}(H^j(-, \mathbb{F}_p)) = 0$ for every $i \geq 1$). It follows that for every $n \geq 0$ the isomorphism of abelian groups $\lim_{\mathcal{O}^c_p(G)}(H^n(-, \mathbb{F}_p)) \cong H^n(G, \mathbb{F}_p)$ holds.

On the other hand, work of Broto, Levi and Oliver (see [BLO03]) and of Chermak (see [Ch13]) leads to the description, existence and uniqueness of classifying space of a fusion system $\mathcal{F}$

$$B\mathcal{F} \simeq \mathrm{hocolim}_{\mathcal{O}(\mathcal{F}^c)}(B(-)). \tag{3.2}$$

Moreover, from [BLO03, Lemma 5.3] we know that, analogously to the case of finite groups, the isomorphism $\lim_{\mathcal{O}(\mathcal{F}^c)}(H^n(-, \mathbb{F}_p)) \cong H^n(B\mathcal{F}, \mathbb{F}_p)$ holds. This parallelism naturally leads to the following question which was first formulated by Aschbacher, Kessar and Oliver in [AKO11, Section III.7] and later generalized by Díaz and Park in [DP15].

Conjecture 3.1.1 (Sharpness for fusion system). *Let S be a finite p-group, let $\mathcal{F}$ be a fusion system over S and let $M = (M_*, M^*)$ be a Mackey functor over $\mathcal{F}$ on $\mathbb{F}_p$ (see Definition 2.2.26). Then $\lim^n_{\mathcal{O}(\mathcal{F}^c)}\left(M^* \downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}(\mathcal{F}^c)}\right) = 0$ for every $n \geq 1$.*

Although still unresolved there have recently been several developments regarding this conjecture (see [GL23; GM22; HLL23; Ya22]) which add to the results in [DP15]. In this paper we aim to add to these efforts by developing some tools that can be used to approach Conjecture 3.1.1. More precisely we prove the following.

Theorem 3.A. *Let S be a finite p-group, let $\mathcal{F}$ be a fusion system over S, let I be a finite set, let $\boldsymbol{F} := \{\mathcal{F}_i\}_{i\in I}$ be a collection of fusion subsystems of $\mathcal{F}$, for each $i \in I$ let $S_i \leq S$ be the finite p-group such that $\mathcal{F}_i$ is a fusion system over S_i, let $\underline{\mathbb{F}_p}^{\mathcal{O}(\mathcal{F}^c)}$ be the right $\mathcal{RO}(\mathcal{F}^c)$-module associated to the constant contravariant functor sending every object of $\mathcal{O}(\mathcal{F}^c)$ to the $\mathbb{F}_p$-module $\mathbb{F}_p$ (see [We07, Proposition 2.1] and Definition 3.2.18), for every $i \in I$ let $\mathcal{O}_{\mathcal{C}}(\mathcal{F}_i)$ be the full subcategory of $\mathcal{O}(\mathcal{F}_i)$ with objects all $\mathcal{F}$-centric subgroups of S_i, define $\underline{\mathbb{F}_p}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)} := \bigoplus_{i\in I} \underline{\mathbb{F}_p}^{\mathcal{O}(\mathcal{F}^c)} \downarrow^{\mathbb{F}_p\mathcal{O}(\mathcal{F}^c)}_{\mathbb{F}_p\mathcal{O}_{\mathcal{C}}(\mathcal{F}_i)} \uparrow^{\mathbb{F}_p\mathcal{O}(\mathcal{F}^c)}_{\mathbb{F}_p\mathcal{O}_{\mathcal{C}}(\mathcal{F}_i)}$ and let $\theta_{\boldsymbol{F}} : \underline{\mathbb{F}_p}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)} \to \underline{\mathbb{F}_p}^{\mathcal{O}(\mathcal{F}^c)}$ be the natural map (see Definition 3.4.2).*

If the following 4 conditions are satisfied then Conjecture 3.1.1 is satisfied for every Mackey functor over $\mathcal{F}$ on $\mathbb{F}_p$:

1. *(1) $\theta_{\boldsymbol{F}}$ is an epimorphism.*
2. *(2) For every $i \in I$ all $\mathcal{F}_i$-centric-radical subgroups of S_i are $\mathcal{F}$-centric (see Definition 3.2.11 (3)).*
3. *(3) For every with $i \in I$ Conjecture 3.1.1 is satisfied for $\mathcal{F}_i$.*
4. *(4) For every $\mathcal{RO}(\mathcal{F}^c)$-module M and every morphism $f : \ker(\theta_{\boldsymbol{F}}) \to M$ there exists a morphism $\hat{f} : \underline{\mathbb{F}_p}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)} \to M$ lifting f. In other words the family $\boldsymbol{F}$ satisfies the lifting property (see Definition 3.4.3).*

The conditions needed to apply Theorem 3.A might seem too restrictive. However, as we show in Section 3.4, for any given fusion system there exist several families of fusion subsystems satisfying Conditions (1)-(3) and, therefore, the only real problem is given by Condition (4) (see Conjecture 3.4.6).

As an application of Theorem 3.A we complete the work started in [HLL23, Theorems 1.1 and 1.4] by proving

Theorem 3.B. *Conjecture 3.1.1 is satisfied for all Benson-Solomon fusion systems (see [LO05, Definition 1.6]).*

More generally we prove that

Theorem 3.C. *Let S be a finite p-group, let $\mathcal{F}, \mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems over S such that:*

1. *(1) $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{F}$ and $\mathcal{F}$ is generated by $\mathcal{F}_1$ and $\mathcal{F}_2$ (i.e. all morphisms of $\mathcal{F}$ can be written as a composition of finitely many morphisms in $\mathcal{F}_1$ and $\mathcal{F}_2$).*

2. *(2) For every $i = 1, 2$ all $\mathcal{F}_i$-centric-radical subgroups of S are $\mathcal{F}$-centric.*

3. *(3) For every $i = 1, 2$ Conjecture 3.1.1 is satisfied for $\mathcal{F}_i$.*

Then Conjecture 3.1.1 is satisfied for $\mathcal{F}$.

We conclude this section with a list of common notation that we use throughout this chapter.

Notation 3.1.2.

- The letter p denotes a prime number.
- Every module is understood to be a right module. All arguments can however be adapted to left modules.
- All rings are understood to have a unit.
- For any ring $\mathcal{R}$ we denote by $\mathcal{R}\text{-Mod}$ the category of right $\mathcal{R}$-modules and by $\mathcal{R}\text{-mod}$ the category of finitely generated right $\mathcal{R}$-modules.
- Let $\boldsymbol{C}$ be a small category, let A, B and M be objects in $\boldsymbol{C}$ and let $f \in \operatorname{Hom}_{\boldsymbol{C}}(A, B)$. We denote by $f^* : \operatorname{Hom}_{\boldsymbol{C}}(B, M) \to \operatorname{Hom}_{\boldsymbol{C}}(A, M)$ the set map (or morphism of abelian groups if $\boldsymbol{C}$ is abelian) sending every morphism $g \in \operatorname{Hom}_{\boldsymbol{C}}(B, M)$ to the composition $gf \in \operatorname{Hom}_{\boldsymbol{C}}(A, M)$.
- With notation as above we denote by $f_* : \operatorname{Hom}_{\boldsymbol{C}}(M, A) \to \operatorname{Hom}_{\boldsymbol{C}}(M, B)$ the set map (or morphism of abelian groups if $\boldsymbol{C}$ is abelian) sending every morphism $g \in \operatorname{Hom}_{\boldsymbol{C}}(M, A)$ to the composition $fg \in \operatorname{Hom}_{\boldsymbol{C}}(M, B)$.

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3.2 Preliminaries

During this section we briefly recall the definitions of fusion systems (Definition 3.2.6), Mackey functors (Definition 2.2.26), Cartan-Eilenberg resolutions (Definition 3.2.23) and other related concepts. We also state, without a proof, some known results that are relevant for later sections. For a more detailed introduction we refer the interested reader to [Li07] for fusion systems, to [We00] for Mackey functors and to [We94, Chapter 5.7] for Cartan-Eilenberg resolutions.

3.2.1 Fusion systems

Fusion systems were first devised by Puig in [Pu06] as a common framework between p-fusion of finite groups and p-blocks of finite groups. Intuitively they can be thought of as categories that collect the p-local structure of a finite group. In this subsection we recall the definitions of fusion system and orbit category (Definitions 3.2.6 and 3.2.10), provide an example of particular notice (Example 3.2.9) and highlight certain full subcategories of the orbit category (see Definition 3.2.12).

Definition 3.2.1. Let S be a finite p-group. An S-**category** is a category $\mathcal{F}$ having as objects all subgroups of S and such that the following conditions are satisfied:

(1) Every morphism in $\mathcal{F}$ is an injective group morphism.

(2) For every $P, Q \leq S$ we have that $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q)$. Where $\mathrm{Hom}_S(P, Q)$ denotes the set of morphisms from P to Q obtained by conjugating with an element in S.

(3) The composition of morphisms in $\mathcal{F}$ coincides with the composition of morphisms in the category of groups. In particular if two morphisms in $\mathcal{F}$ can be composed then their composition is also a morphism in $\mathcal{F}$.

(4) For every $P, Q \leq S$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ both the isomorphism $\tilde{\varphi} : P \to \varphi(P)$, obtained by viewing φ as an isomorphism onto is image, and its inverse $\tilde{\varphi}^{-1} : \varphi(P) \to P$ are morphisms in $\mathcal{F}$.

Example 3.2.2. Let G be a group and let S be a finite p-subgroup of G. The category $\mathcal{F}_S(G)$ having as objects subgroups of S and as morphisms

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) := \mathrm{Hom}_G(P,Q),$$

is an S-category.

Example 3.2.3. Let S be a finite p-group, the category having objects all subgroups of S and as morphisms all injective group morphisms is an S-category.

Fusion systems are particular types of S categories. In order to describe these we first need to introduce some further notation.

Definition 3.2.4. Let S be a finite p-group and let $\mathcal{F}$ be an S-category. We say that $P \leq S$ is **fully $\mathcal{F}$-normalized** if for every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ we have $|N_S(P)| \geq |N_S(\varphi(P))|$.

Definition 3.2.5. Let S be a finite p-group, let $\mathcal{F}$ be an S-category, let $P,Q \leq S$ and let $\varphi \in \mathrm{Hom}(P,Q)$. We define the **normalizer of φ in S** to be

$$N_\varphi := \{y \in N_S(P) \,:\, \exists z \in N_S(\varphi(P)) \text{ s.t. } \varphi({}^y x) = {}^z\varphi(x) \ \ \forall x \in P\}.$$

We are now ready to state the following.

Definition 3.2.6. Let S be a finite p-group, a **fusion system over S** is an S-category $\mathcal{F}$ satisfying:

(1) $\mathrm{Aut}_S(S) \in \mathsf{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S))$.

(2) For every $P \leq S$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ such that $\varphi(P)$ is fully $\mathcal{F}$-normalized there exists $\hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\varphi(x) = \hat{\varphi}(x)$ for every $x \in P$.

Remark 3.2.7. It is not uncommon in the literature to use the term fusion system to refer to S-categories and the term *saturated* fusion systems to refer to fusion systems. Our use of the term fusion system is however equally common and simplifies the notation of this paper.

Example 3.2.8. With notation as in Example 3.2.2, if G is finite and $S \in \mathrm{Syl}_p(G)$ then $\mathcal{F}_S(G)$ is a fusion system over S.

Example 3.2.9. Let S be a finite p-group, let $\mathcal{F}$ be a fusion system over S and let $P \leq S$. We define $N_{\mathcal{F}}(P)$ as the $N_S(P)$-category whose morphism sets are given by

$$\mathrm{Hom}_{N_{\mathcal{F}}(P)}(A,B) := \big\{\varphi \in \mathrm{Hom}_{\mathcal{F}}(A,B) \,:\, \exists\hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(AP,BP) \text{ s.t } \hat{\varphi}(a) = \varphi(a)\ \forall a \in A \text{ and } \hat{\varphi}(P) = P\big\}.$$

It is a result of Puig (see also [Li07, Theorem 3.6] for a proof) that, whenever P is fully $\mathcal{F}$-normalized, then $N_{\mathcal{F}}(P)$ is a fusion system over $N_S(P)$.

Throughout this paper we don't often work with fusion systems directly but rather with their orbit categories.

Definition 3.2.10. Let S be a finite p-group and let $\mathcal{F}$ be a fusion over S. The **orbit category of** $\mathcal{F}$ is the category $\mathcal{O}(\mathcal{F})$ whose objects are subgroups of S and whose morphism sets are given by

$$\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P,Q) := \mathrm{Aut}_Q(Q) \setminus \mathrm{Hom}_{\mathcal{F}}(P,Q)$$

where $\mathrm{Aut}_Q(Q)$ acts on $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ by left composition.

In this paper we often use certain subcategories of $\mathcal{O}(\mathcal{F})$ which we now introduce.

Definition 3.2.11. Let S be a finite p-group and let $\mathcal{F}$ be a fusion system over S. We say that $P \leq S$ is:

(1) $\mathcal{F}$-**centric**: If for every $Q \leq S$ isomorphic to P in $\mathcal{F}$ we have that $C_S(Q) \leq Q$.

(2) $\mathcal{F}$-**radical**: If the p-core of $\mathrm{Aut}_{\mathcal{F}}(P)$ satisfies $O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Aut}_P(P)$.

(3) $\mathcal{F}$-**centric-radical**: If P is both $\mathcal{F}$-centric and $\mathcal{F}$-radical

Definition 3.2.12. Let S be a finite p-group, let $\mathcal{F}$ be a fusion system over S and let $\mathcal{C}$ be a family of subgroups of S closed under $\mathcal{F}$-overconjugation (i.e. for every $P, Q \leq S$ if $P \in \mathcal{C}$ and $\mathrm{Hom}_{\mathcal{F}}(P,Q) \neq \emptyset$ then $Q \in \mathcal{C}$). We denote by $\mathcal{O}_{\mathcal{C}}(\mathcal{F})$ the full subcategory of $\mathcal{O}(\mathcal{F})$ having as objects the elements of $\mathcal{C}$. Whenever $\mathcal{C}$ is the family of $\mathcal{F}$-centric subgroups of S (which we know from [Li07, Proposition 4.4] to be closed under $\mathcal{F}$-overconjucation) we write $\mathcal{O}(\mathcal{F}^c) := \mathcal{O}_{\mathcal{C}}(\mathcal{F})$ instead.

3.2.2 Mackey functors

A Mackey functor is an algebraic structure with operations that resemble the induction, restriction and conjugation maps in representation theory. In this subsection we recall the definition of Mackey functor over a fusion system (Definition 2.2.26) as well as some known results that will allow us to view them as pairs of left and right modules over a certain ring (Proposition 3.2.17).

Definition 3.2.13. Let S be a finite p-group, let $\mathcal{F}$ be a fusion system over S and let $\mathcal{R}$ be a commutative ring. A **Mackey functor over** $\mathcal{F}$ **on** $\mathcal{R}$ is a pair $M = (M_*, M^*)$ of a covariant functor $M_* : \mathcal{O}(\mathcal{F}) \to \mathcal{R}\text{-mod}$ and a contravariant functor $M^* : \mathcal{O}(\mathcal{F})^{\mathrm{op}} \to \mathcal{R}\text{-mod}$ such that:

(1) $M(P) := M_*(P) = M^*(P)$ for every $P \leq S$.

(2) $M_*(\overline{\varphi}) = M^*(\overline{\varphi}^{-1})$ for every isomorphism $\overline{\varphi}$ in $\mathcal{O}(\mathcal{F})$.

(3) For every $A, B \leq C \leq S$ then

$$M^*\left(\overline{\iota_B^C}\right) M_*\left(\overline{\iota_A^C}\right) = \sum_{x \in [B \backslash C / A]} M_*\left(\overline{\iota_{B \cap {}^x A}^B c_x}\right) M^*\left(\overline{\iota_{B^x \cap A}^A}\right)$$

where, for every morphism φ in $\mathcal{F}$, we denote by $\overline{\varphi}$ the image of φ in $\mathcal{O}(\mathcal{F})$(see Definition 3.2.10), for every $x \in C$ the map $c_x : B^x \cap A \to B \cap {}^x A$ is the morphism which sends every $a \in B^x \cap A$ to xax^{-1} and ι_X^Y denotes the natural inclusion.

Remark 3.2.14. Definition 3.2.13 might at first glance seem different from the definition of Mackey functor given in Chapter 2 (see Definition 2.2.26) but they are in fact equivalent.

More precisely, for every Mackey functor $M = (M_*, M^*)$ as in Definition 3.2.13, we can define a left $\mu_{\mathcal{R}}(\mathcal{F})$-module $\boldsymbol{M}$ by setting $\boldsymbol{M} := \bigoplus_{P \leq S} M(P)$. Here we define the action of $\mu_{\mathcal{R}}(\mathcal{F})$ on $\boldsymbol{M}$ by setting for every $P \leq Q \leq S$, every isomorphism $\varphi : P \to \varphi(P)$ in $\mathcal{F}$ and every $x \in \boldsymbol{M}$

$$I_P^Q \cdot x := M_*\left(\overline{\iota_P^Q}\right)(\pi_P(x)), \quad R_P^Q \cdot x := M^*\left(\overline{\iota_P^Q}\right)(\pi_Q(x)), \quad c_{\varphi,P} \cdot x := M_*(\overline{\varphi})(\pi_P(x)),$$

where $\pi_A : \boldsymbol{M} \to M(A)$ denotes the natural projection and $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P, \varphi(P))$ is the isomorphism in $\mathcal{O}(\mathcal{F})$ with representative φ.

On the other hand for every left $\mu_{\mathcal{R}}(\mathcal{F})$-module $\boldsymbol{M}$ we can define a Mackey functor $M = (M_*, M^*)$ by setting $M(P) = M_*(P) = M^*(P) := I_P^P \boldsymbol{M}$ for every $P \leq S$ and defining for every $P \leq Q \leq S$ and every isomorphism $\varphi : P' \hookrightarrow\!\!\!\!\rightarrow P$

$$M_*\left(\overline{\iota_P^Q \varphi}\right)(x) := I_P^Q c_{\varphi,P'} \cdot x, \qquad M^*\left(\overline{\iota_P^Q \varphi}\right)(y) := c_{\varphi^{-1},P} R_P^Q \cdot y,$$

for every $x \in M(P')$ and every $y \in M(Q)$. Once again, for every morphism ψ in $\mathcal{F}$ we denote by $\overline{\psi}$ the morphism in $\mathcal{O}(\mathcal{F})$ with representative ψ.

It is easy to check that the above described maps between left $\mu_{\mathcal{R}}(\mathcal{F})$-modules and Mackey functors over $\mathcal{F}$ on $\mathcal{R}$ is in fact natural.

Example 3.2.15. For any finite p-group S, any fusion system $\mathcal{F}$ over S and any commutative ring $\mathcal{R}$:

- The cohomology functor $H^n(-;\mathcal{R})$ restricted to $\mathcal{O}(\mathcal{F})$ is the contravariant part of a Mackey functor over $\mathcal{F}$ on $\mathcal{R}$.
- The homology functor $H_n(-;\mathcal{R})$ restricted to $\mathcal{O}(\mathcal{F})$ is the covariant part of a Mackey functor over $\mathcal{F}$ on $\mathcal{R}$.
- The Burnside functor $B(-;\mathcal{R})$ that sends every subgroup of S to the underlying $\mathcal{R}$-module of its Burnside ring with coefficients on $\mathcal{R}$ defines a Mackey functor over $\mathcal{F}$ on $\mathcal{R}$.

It is often useful to view the covariant and contravariant parts of a Mackey functor as left and right modules over a certain ring. To this end we introduce the following.

Definition 3.2.16. Let $\mathcal{R}$ be a commutative ring and let $\boldsymbol{C}$ be a small category. The **category algebra of $\boldsymbol{C}$ over $\mathcal{R}$** is the $\mathcal{R}$-algebra $\mathcal{R}\boldsymbol{C}$ that is freely generated as an

$\mathcal{R}$-module by the set of morphisms in $\boldsymbol{C}$ and with product given by

$$\psi \cdot \varphi = \begin{cases} \psi\varphi & \text{if the composition of } \varphi \text{ and } \psi \text{ is defined} \\ 0 & \text{otherwise} \end{cases}.$$

The relevance of this Definition in our context is given by the following result which we report without a proof. The interested reader can refer to [We07, Proposition 2.1] for a proof.

Proposition 3.2.17 ([We07, Proposition 2.1])**.** *Let $\boldsymbol{C}$ be a small category with finitely many objects and let $\mathcal{R}$ be a commutative ring. The category of right $\mathcal{R}\boldsymbol{C}$-modules is isomorphic to the category of contravariant functors from $\boldsymbol{C}$ to $\mathcal{R}\text{-}\mathrm{Mod}$. Equivalently the category of left $\mathcal{R}\boldsymbol{C}$-modules is isomorphic to the category of covariant functors from $\boldsymbol{C}$ to $\mathcal{R}\text{-}\mathrm{Mod}$.*

The following is a particularly relevant $\mathcal{R}\boldsymbol{C}$-module that we use repeatedly in Section 3.4.

Definition 3.2.18. Let $\mathcal{R}$ and $\boldsymbol{C}$ be as in Proposition 3.2.17. The **constant $\mathcal{R}\boldsymbol{C}$-module** (denoted $\underline{\mathcal{R}}^{\boldsymbol{C}}$) is the $\mathcal{R}\boldsymbol{C}$-module corresponding, via Proposition 3.2.17, to the constant contravariant functor sending every object to the trivial $\mathcal{R}$-module $\mathcal{R}$ and every morphism to the identity. If there is no confusion regarding the category $\boldsymbol{C}$ we write $\underline{\mathcal{R}} := \underline{\mathcal{R}}^{\boldsymbol{C}}$.

Let S be a finite p-group, let $\mathcal{F}$ be a fusion system and let $\mathcal{R}$ be a commutative ring. Since S is finite then it has finitely many subgroups. Equivalently $\mathcal{O}(\mathcal{F})$ has finitely many objects. Therefore, Proposition 3.2.17, allows us to view any Mackey functor $M = (M_*, M^*)$ over $\mathcal{F}$ on $\mathcal{R}$ as a pair of a left and right $\mathcal{R}\mathcal{O}(\mathcal{F})$-modules respectively. This viewpoint has two main advantages. First, for every family $\mathcal{C}$ of subgroups of S closed under $\mathcal{F}$-overconjugation, if we denote by $\iota : \mathcal{O}_{\mathcal{C}}(\mathcal{F}) \to \mathcal{O}(\mathcal{F})$ the natural inclusion of categories we obtain the isomorphism of abelian groups $\lim^n_{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}(M^*\iota) \cong \mathrm{Ext}^n_{\mathcal{R}\mathcal{O}_{\mathcal{C}}(\mathcal{F})}\left(\underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}, M^*\iota\right)$ (see [We07, Corollary 5.2]). Another advantage is that, for any small category $\boldsymbol{C}$ with finitely many objects and any subcategory $\boldsymbol{D} \subseteq \boldsymbol{C}$ we obtain a pair of adjoint functors between the diagram categories $\mathcal{R}\text{-}\mathrm{Mod}^{\boldsymbol{C}}$ and $\mathcal{R}\text{-}\mathrm{Mod}^{\boldsymbol{D}}$. More precisely we have the following.

Lemma 3.2.19. *Let $\mathcal{R}$ be a commutative ring, let $\boldsymbol{C}$ be a small category with finitely many objects and let $\boldsymbol{D} \subseteq \boldsymbol{C}$ be a subcategory. There is a natural inclusion of rings $\mathcal{R}\boldsymbol{D} \subseteq \mathcal{R}\boldsymbol{C}$. In particular the change of ring operations between the rings $\mathcal{R}\boldsymbol{D}$ and $\mathcal{R}\boldsymbol{C}$ lead to an* ***induction*** $\uparrow_{\boldsymbol{D}}^{\boldsymbol{C}}\colon \mathcal{R}\boldsymbol{D}\text{-Mod} \to \mathcal{R}\boldsymbol{C}\text{-Mod}$ *and a* ***restriction*** $\downarrow_{\boldsymbol{D}}^{\boldsymbol{C}}\colon \mathcal{R}\boldsymbol{C}\text{-Mod} \to \mathcal{R}\boldsymbol{D}\text{-Mod}$ *functors such that $\uparrow_{\boldsymbol{D}}^{\boldsymbol{C}}$ is left adjoint to $\downarrow_{\boldsymbol{D}}^{\boldsymbol{C}}$.*

Proof. Since every morphism in $\boldsymbol{D}$ is also a morphism in $\boldsymbol{C}$ then there exists a natural inclusion of the $\mathcal{R}$-basis of $\mathcal{R}\boldsymbol{D}$ into the $\mathcal{R}$-basis of $\mathcal{R}\boldsymbol{C}$ (see Definition 3.2.16). This leads to a natural inclusion of $\mathcal{R}\boldsymbol{D}$ into $\mathcal{R}\boldsymbol{C}$. The result follows. □

We conclude this subsection with the following trivial but useful insight on the restriction functor.

Lemma 3.2.20. *Let $\mathcal{R}, \boldsymbol{C}$ and $\boldsymbol{D}$ be as in Lemma 3.2.19. Then $\underline{\mathcal{R}}^{\boldsymbol{C}} \downarrow_{\boldsymbol{D}}^{\boldsymbol{C}} = \underline{\mathcal{R}}^{\boldsymbol{D}}$.*

Proof. This is immediate from the definition. □

3.2.3 Cartan-Eilenberg resolution

Cartan-Eilenberg resolutions are, intuitively speaking, projective (resp. injective) resolutions for chain complexes (resp. cochain complexes). During this subsection we recall the definition of Cartan-Eilenberg resolution (Definition 3.2.23) as well as some known sufficient conditions for their existence (Proposition 3.2.25).

Definition 3.2.21. Let $\mathcal{A}$ be an abelian category. A **double chain complex in $\mathcal{A}$** is a family of triples $\{(C_{i,j}, d^v_{i,j}, d^h_{i,j})\}_{i,j\in\mathbb{Z}}$ (often abbreviated to $C_{*,*}$) where, for every $i,j \in \mathbb{Z}$, $C_{i,j}$ is an object in $\mathcal{A}$ and $d^v_{i,j} : C_{i,j} \to C_{i,j-1}$ and $d^h_{i,j} : C_{i,j} \to C_{i-1,j}$ are morphisms in $\mathcal{A}$ satisfying

$$d^v_{i,j-1}d^v_{i,j} = d^h_{i-1,j}d^h_{i,j} = d^v_{i-1,j}d^h_{i,j} + d^h_{i,j-1}d^v_{i,j} = 0.$$

A **double cochain complex in $\mathcal{A}$** can be defined in an analogous way just by inverting the directions of arrows. Double cochain complexes are denoted as $\{(C^{i,j}, d_v^{i,j}, d_h^{i,j})\}_{i,j\in\mathbb{Z}}$ (often abbreviated to $C^{*,*}$) with $C^{i,j}$ objects in $\mathcal{A}$ and $d_v^{i,j} : C^{i,j} \to C^{i,j+1}$ and $d_h^{i,j} : C^{i,j} \to C^{i+1,j}$ morphisms in $\mathcal{A}$.

For each double chain complex there exist two (potentially isomorphic) chain complexes. These complexes are related (see [We94, Section 5.6]) and play a fundamental role in Section 3.3.

Definition 3.2.22. Let $\mathcal{A}$ be an abelian category and let $C_{*,*}$ be a double chain complex. We define the **total sum chain complex** ($\mathrm{Tot}^{\oplus}$) and **total product chain complex** (Tot^{Π}) **of** $C_{*,*}$ by setting for every $n \in \mathbb{Z}$

$$\mathrm{Tot}_n^{\oplus}(C_{*,*}) := \bigoplus_{i\in\mathbb{Z}} C_{i,n-i}, \qquad \mathrm{Tot}_n^{\Pi}(C_{*,*}) := \prod_{i\in\mathbb{Z}} C_{i,n-i},$$

and taking defining the differentials $d_n^{\oplus} : \mathrm{Tot}_n^{\oplus}(C_{*,*}) \to \mathrm{Tot}_{n-1}^{\oplus}(C_{*,*})$ and $d_n^{\Pi} : \mathrm{Tot}_n^{\Pi}(C_{*,*}) \to \mathrm{Tot}_{n-1}^{\Pi}(C_{*,*})$ component-wise by setting

$$d_n^{\oplus} = d_n^{\Pi} := d_{i,n-i}^{h} + d_{i,n-i}^{v},$$

for each component $C_{i,n-i} \subseteq \mathrm{Tot}_n^{\oplus}(C_{*,*}) \subseteq \mathrm{Tot}_n^{\Pi}(C_{*,*})$. If $\mathrm{Tot}_n^{\oplus}(C_{*,*}) \cong \mathrm{Tot}_n^{\Pi}(C_{*,*})$ for every $n \in \mathbb{Z}$ then we define the **total chain complex of** $C_{*,*}$ as

$$\mathrm{Tot}(C_{*,*}) := \mathrm{Tot}^{\oplus}(C_{*,*}) \cong \mathrm{Tot}^{\Pi}(C_{*,*}).$$

For a double cochain complex $C^{*,*}$ the **total sum cochain complex** ($\mathrm{Tot}_{\oplus}(C^{*,*})$), **total product cochain complex** ($\mathrm{Tot}_{\Pi}(C^{*,*})$) and **total cochain complex** ($\mathrm{Tot}(C^{*,*})$) **of** $C^{*,*}$ are defined analogously.

A Cartan-Eilenberg resolution of a chain complex is a particular type of double chain complex.

Definition 3.2.23. Let $\mathcal{A}$ be an abelian category and let C_* be a chain complex in $\mathcal{A}$ with differentials $d_i : C_i \to C_{i-1}$. A **Cartan-Eilenberg projective resolution of** C_* is a pair $\left(P_{*,*}, \{\varepsilon_*\}_{i\in\mathbb{Z}}\right)$ (often abbreviated as $P_{*,*} \overset{\varepsilon_*}{\twoheadrightarrow} C_*$) where $P_{*,*}$ is a double chain complex in $\mathcal{A}$, for every $i \in \mathbb{Z}$, the **augmentation map** $\varepsilon_i : P_{0,i} \twoheadrightarrow C_i$ is an epimorphism in $\mathcal{A}$ and the following conditions are satisfied:

(1) $\varepsilon_* : P_{0,*} \twoheadrightarrow C_*$ is a chain map.

(2) For every $i, j \in \mathbb{Z}$ the object $P_{i,j}$ is projective.

(3) If $C_j = 0$ for some $j \in \mathbb{Z}$ then $P_{i,j} = 0$ for every $i \in \mathbb{Z}$.

(4) $P_{*,i} \overset{\varepsilon_i}{\twoheadrightarrow} C_i$ is a projective resolution of C_i for every $i \in \mathbb{Z}$.

(5) The projective resolution of Condition (4) induces for every $i \in \mathbb{Z}$ a projective resolution $\mathrm{im}\left(d^v_{*,i}\right) \twoheadrightarrow \mathrm{im}\left(d_i\right)$.

(6) The projective resolution of Condition (4) induces for every $i \in \mathbb{Z}$ a projective resolution $\ker\left(d^v_{*,i}\right) \twoheadrightarrow \ker\left(d_i\right)$.

(7) The projective resolution of Condition (4) induces for every $i \in \mathbb{Z}$ a projective resolution $\ker\left(d^v_{*,i}\right) / \mathrm{im}\left(d^v_{*,i+1}\right) \twoheadrightarrow H_i\left(C_*\right)$.

For any cochain complex C^* a **Cartan-Eilenberg injective resolution of C^*** is defined analogously by inverting arrows and replacing homology and projective objects with cohomology and injective objects respectively.

Remark 3.2.24. Conditions (4) and (5) are usually omitted from the definition of Cartan-Eilenberg resolution as they can be derived from the rest (see [We94, Exercise 5.7.1]).

Despite the numerous properties that Cartan-Eilenberg resolutions possess there are quite lax sufficient conditions for their existence. More precisely we have the following.

Proposition 3.2.25 ([We94, Lemma 5.7.2 and Paragraph 5.7.9]). *Let $\mathcal{A}$ be an abelian category and let C_* be a chain complex in $\mathcal{A}$. If $\mathcal{A}$ has enough projectives then there exists a Cartan-Eilenberg projective resolution of C_*. The same result holds for Cartan-Eilenberg injective resolutions and cochain complexes if $\mathcal{A}$ has enough injectives.*

We conclude this section with three examples of categories admitting Cartan-Eilenberg projective resolutions.

Corollary 3.2.26. *Let $\mathcal{R}$ be a ring. The category $\mathcal{R}\text{-}\mathrm{Mod}$ admits both Cartan-Eilenberg projective and injective resolutions.*

Proof. We know from [CR88, Theorem 57.8] that $\mathcal{R}\text{-}\mathrm{Mod}$ has enough injectives and it is well known that it has enough projectives. The rest of the statement follows from Proposition 3.2.25. □

Corollary 3.2.27. *Let $\mathcal{R}$ be a finite ring. The category $\mathcal{R}\text{-mod}$ is abelian and has enough injectives and projectives. In particular $\mathcal{R}\text{-mod}$ admits both Cartan-Eilenberg projective and injective resolutions.*

Proof. Since $\mathcal{R}$ is finite then it is Artinian. Since all Artinian rings are Noetherian we can conclude from [We94, Example 1.6.3 (1)] that $\mathcal{R}\text{-mod}$ is an abelian category.

Since $\mathcal{R}$ is an Artinian ring we know from [ARS97, Theorem I.4.2] that $\mathcal{R}\text{-mod}$ has enough projectives.

Finally let $k \leq \mathcal{R}$ denote the subring of $\mathcal{R}$ that is generated by the multiplicative identity. That is $k \cong \mathbb{Z}/\text{char}(\mathcal{R})\,\mathbb{Z}$. Then k is a finite (and therefore Artinian) commutative ring. Moreover we can view $\mathcal{R}$ as a finite k-algebra. We can therefore apply [ARS97, Corollary II.3.4] to conclude that $\mathcal{R}\text{-mod}$ has enough injectives. The rest of the statement follows from Proposition 3.2.25. □

Remark 3.2.28. Let k be a field. If $\mathcal{R}$ is a finite dimensional k-algebra then it is Artinian and the same arguments used to prove Corollary 3.2.27 still hold.

3.3 $\mathrm{Ext}^n_{\mathcal{R}}$ groups of cokernels

In this section we develop the main techniques used to prove Theorem 3.A. Namely we prove Proposition 3.3.7 and Corollary 3.3.8. To do so we start by taking a ring $\mathcal{R}$, $\mathcal{R}$-modules A, B and M and an $\mathcal{R}$-module morphism $f : A \to B$. We then construct a 2-term chain complex C_* having A and B as elements and $\mathrm{coker}\,(f)$ as an homology $\mathcal{R}$-module (see Lemma 3.3.1). Thereafter we take a Cartan-Eilenberg projective resolution of C_* and use spectral sequences associated to this resolution (see [We94, Proposition 5.7.6 and Paragraph 5.7.9]) in order to obtain a short exact sequence involving $\mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}\,(f), M)$ (see Corollary 3.3.4). Under certain conditions we can then use this short exact sequence in order to prove that $\mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}\,(f), M) = 0$ (see Proposition 3.3.7). With this in mind we are finally able to prove that, under stronger conditions, $\mathrm{Ext}^n_{\mathcal{R}}(\mathrm{coker}\,(f), M) = 0$ for every $n \geq 1$ (see Corollary 3.3.8).

Let us start by defining the above mentioned chain complex and studying how its elements relate to those of its related spectral sequence.

Lemma 3.3.1. *Let $\mathcal{R}$ be a ring, let M, A, B be $\mathcal{R}$-modules, let $f : A \to B$ be an $\mathcal{R}$-module morphism and let $\iota : \ker(f) \hookrightarrow A$ be the natural inclusion. We define the chain complex C_* in $\mathcal{R}\text{-Mod}$ by setting $C_i := 0$ for every $i \neq 0, 1$, $C_1 := A$ and $C_0 := B$ and by taking differentials $d_i = 0$ if $i \neq 1$ and $d_1 := f : C_1 \to C_0$. Fix $P_{*,*} \overset{\varepsilon_*}{\twoheadrightarrow} C_*$ a Cartan-Eilenberg projective resolution of C_* (see Corollary 3.2.26). Finally define $D^{*,*}$ as the first quadrant double cochain complex obtained by applying the contravariant functor $\operatorname{Hom}_{\mathcal{R}}(-, M)$ to $P_{*,*}$. From [We94, Proposition 5.7.6 and Paragraph 5.7.9] we know that there exists a first quadrant cohomology spectral sequence*

$$E_2^{i,j} := \operatorname{Ext}^i_{\mathcal{R}}(H_j(C_*), M) \Rightarrow H^{i+j}(\operatorname{Tot}(D^{*,*})). \tag{3.3}$$

which is obtained from the double cochain complex $D^{*,*}$ by first computing cohomology with respect to the vertical differentials and then with respect to the horizontal differentials (i.e. the spectral sequence corresponding to the filtration $F^r \operatorname{Tot}^n := \bigoplus_{i+j=n, j \geq r} D^{i,j}$ of $\operatorname{Tot}(D^{*,*})$).

If $\operatorname{Ext}^1_{\mathcal{R}}(B, M) = 0$ then there exists an isomorphism of $\mathcal{R}$-modules

$$\operatorname{im}(\iota^*) \cong \ker\left(d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}\right).$$

Proof. In order to simplify notation throughout this proof we write for every $i, j \in \mathbb{Z}$

$$\operatorname{Ho}_{i,j} := \operatorname{Hom}_{\mathcal{R}}(P_{i,j}, M) \qquad \text{and} \qquad \mathrm{K} := \ker(f_1^*) \cap \ker\left(d_{1,0}^*\right).$$

Where $f_i : P_{i,1} \to P_{i,0}$ are the non zero vertical differentials of $P_{*,*}$ and $d_{i,j} : P_{i+1,j} \to P_{i,j}$ be its horizontal differentials. From Proposition 3.A.3 we have that

$$\begin{aligned} E_2^{0,1} &\cong \frac{\left\{(\varphi, \psi) \in \operatorname{Ho}_{0,1} \oplus \operatorname{Ho}_{1,0} \ : \ d_{0,1}^*(\varphi) + f_1^*(\psi) = 0\right\}}{\left\{\left(f_0^*(\alpha), d_{0,0}^*(\alpha) + \beta\right) \ : \ (\alpha, \beta) \in \operatorname{Ho}_{0,0} \times \ker(f_1^*)\right\}}, \\ E_2^{2,0} &\cong \frac{\left\{\theta \in \operatorname{Ho}_{2,0} \ : \ f_2^*(\theta) = d_{2,0}^*(\theta) = 0\right\}}{\left\{d_{1,0}^*(\gamma) \ : \ \gamma \in \ker(f_1^*)\right\}}, \\ &= \left(\ker(f_2^*) \cap \ker\left(d_{2,0}^*\right)\right) / d_{1,0}^*(\ker(f_1^*)). \end{aligned}$$

Denote by $\overline{(\varphi, \psi)}$ the element in $E_2^{0,1}$ with representative $(\varphi, \psi) \in \operatorname{Ho}_{0,1} \oplus \operatorname{Ho}_{1,0}$ and by $\overline{\theta}$ the element in $E_2^{2,0}$ with representative $\theta \in \ker(f_2^*) \cap \ker\left(d_{2,0}^*\right)$. Proposition 3.A.3

tells us that the differential $d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}$ of the spectral sequence in Equation (3.3) is given by setting $d_2^{0,1}\left(\overline{(\varphi, \psi)}\right) := \overline{d_{1,0}^*(\psi)}$. We can therefore conclude that

$$\ker\left(d_2^{0,1}\right) = \frac{\left\{(\varphi,\psi) \in \mathrm{Ho} \,:\, \exists\theta \in \ker\left(f_1^*\right) \text{ s.t. } d_{0,1}^*(\varphi) + f_1^*(\psi) = d_{1,0}^*(\theta+\psi) = 0\right\}}{\left\{\left(f_0^*(\alpha), d_{0,0}^*(\alpha)+\beta\right) \,:\, (\alpha,\beta) \in \mathrm{Ho}_{0,0} \times \ker\left(f_1^*\right)\right\}}.$$

Where $\mathrm{Ho} := \mathrm{Ho}_{0,1} \oplus \mathrm{Ho}_{1,0}$. Since $\beta, \theta \in \ker(f_1^*)$ we can redefine $\psi = \theta + \psi$ and use the third isomorphism theorem in order to obtain

$$\ker\left(d_2^{0,1}\right) \cong \frac{\left\{(\varphi,\psi) \in \mathrm{Ho}_{0,1} \oplus \mathrm{Ho}_{1,0} \,:\, d_{0,1}^*(\varphi) + f_1^*(\psi) = d_{1,0}^*(\psi) = 0\right\}}{\left\{\left(f_0^*(\alpha), d_{0,0}^*(\alpha)+\beta\right) \,:\, (\alpha,\beta) \in \mathrm{Ho}_{0,0} \times \mathrm{K}\right\}}.$$

From Property (4) of Definition 3.2.23 we know that $P_{*,0} \overset{\varepsilon_0}{\twoheadrightarrow} B$ is a projective resolution of B. Since $\mathrm{Ext}^1_{\mathcal{R}}(B, M) = 0$ we can conclude that for every $\psi \in \ker\left(d_{1,0}^*\right)$ there exists $\alpha \in \mathrm{Ho}_{0,0}$ such that $\psi = d_{0,0}^*(\alpha)$. This allows us to redefine $\psi = \psi - d_{0,0}^*(\alpha) = 0$. Then, using the third isomorphism theorem again, we obtain

$$\begin{aligned}
\ker\left(d_2^{0,1}\right) &\cong \frac{\left\{\varphi \in \mathrm{Ho}_{0,1} \,:\, d_{0,1}^*(\varphi) = 0\right\}}{\left\{f_0^*(\alpha) \,:\, \alpha \in \mathrm{Ho}_{0,0} \text{ s.t. } \exists\beta \in \mathrm{K} \text{ satisfying } d_{0,0}^*(\alpha)+\beta = 0\right\}}, \\
&\cong \frac{\ker\left(d_{0,1}^*\right)}{f_0^*\left(\left\{\alpha \in \mathrm{Ho}_{0,0} \,:\, d_{0,0}^*(\alpha) \in \mathrm{K}\right\}\right)}, \\
&\cong \ker\left(d_{0,1}^*\right) / f_0^*\left(\ker\left(\left(d_{0,0} f_1\right)^*\right)\right), \qquad (3.4) \\
&\cong \ker\left(d_{0,1}^*\right) / f_0^*\left(\ker\left(\left(f_0 d_{0,1}\right)^*\right)\right), \\
&\cong \ker\left(d_{0,1}^*\right) / \left(\mathrm{im}\left(f_0^*\right) \cap \ker\left(d_{0,1}^*\right)\right).
\end{aligned}$$

Where, for the third isomorphism, we use the fact that $d_{0,0}^*(\alpha) \in \ker\left(d_{1,0}^*\right)$ for every $\alpha \in \mathrm{Ho}_{0,0}$ while, for the fourth isomorphism, we use anti-commutativity of double chain complexes (see Definition 3.2.21).

From Properties (3) and (7) of Definition 3.2.23 we know that $\mathrm{coker}(f_0) = P_{0,0}/\,\mathrm{im}(f_0)$ is projective. Therefore the short exact sequence $0 \to \mathrm{im}(f_0) \to P_{0,0} \to \mathrm{coker}(f_0) \to 0$ splits. In particular $\mathrm{im}(f_0)$ is a summand of $P_{0,0}$. Let $\iota' : \mathrm{im}(f_0) \to P_{0,0}$ be the inclusion in the previous short exact sequence, let $\pi : P_{0,0} \to \mathrm{im}(f_0)$ be the natural projection and let $\tilde{f}_0 : P_{0,1} \to \mathrm{im}(f_0)$ be the unique morphism such that $\iota'\tilde{f}_0 = f_0$. Then we have that $\pi f_0 = \pi\iota'\tilde{f}_0 = \tilde{f}_0$ and, therefore

$$\mathrm{im}\left(\tilde{f}_0^*\right) = \mathrm{im}\left(f_0^*\pi^*\right) \subseteq \mathrm{im}\left(f_0^*\right), \qquad \mathrm{im}\left(f_0^*\right) = \mathrm{im}\left(\tilde{f}_0^*\left(\iota'\right)^*\right) \subseteq \mathrm{im}\left(\tilde{f}_0^*\right).$$

Therefore we obtain the identity $\mathrm{im}\left(f_0^*\right) = \mathrm{im}\left(\tilde{f}_0^*\right)$. This allows us to rewrite the isomorphism in Equation (3.4) as

$$\ker\left(d_2^{0,1}\right) \cong \ker\left(d_{0,1}^*\right) / \left(\operatorname{im}\left(\tilde{f_0}^*\right) \cap \ker\left(d_{0,1}^*\right)\right). \tag{3.5}$$

On the other hand, denoting by $\iota_0 : \ker(f_0) \to P_{0,1}$ the natural inclusion, we obtain the short exact sequence $0 \to \ker(f_0) \stackrel{\iota_0}{\to} P_{0,1} \stackrel{\tilde{f_0}}{\to} \operatorname{im}(f_0) \to 0$. Since the $\operatorname{Hom}(-, M)$ contravariant functor is left exact the above leads to the short exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{R}}(\operatorname{im}(f_0), M) \xrightarrow{\tilde{f_0}^*} \operatorname{Ho}_{0,1} \xrightarrow{\iota_0^*} \operatorname{Hom}_{\mathcal{R}}(\ker(f_0), M)$$

In particular we have that $\operatorname{im}\left(\tilde{f_0}^*\right) \cong \ker(\iota_0^*)$. Applying the first isomorphism theorem we can deduce that for every $K' \subseteq \operatorname{Ho}_{0,1}$

$$\iota_0^*(K') \cong K'/(\ker(\iota_0^*) \cap K') = K'/\left(\operatorname{im}\left(\tilde{f_0}^*\right) \cap K'\right).$$

In particular, taking $K' := \ker\left(d_{0,1}^*\right)$ we can deduce from the above and Equation (3.5) that

$$\ker\left(d_2^{0,1}\right) \cong \iota_0^*\left(\ker\left(d_{0,1}^*\right)\right).$$

On the other hand, from Property (4) of Definition 3.2.23 we know that the sequence $P_{1,1} \stackrel{d_{0,1}}{\to} P_{0,1} \stackrel{\varepsilon_1}{\twoheadrightarrow} A \to 0$ is exact. Applying the left exact contravariant $\operatorname{Hom}_{\mathcal{R}}(-, M)$ to this sequence we can deduce that $\ker\left(d_{0,1}^*\right) = \operatorname{im}(\varepsilon_1^*)$. Therefore we can rewrite the previous equation as

$$\ker\left(d_2^{0,1}\right) \cong \iota_0^*(\operatorname{im}(\varepsilon_1^*)) = \operatorname{im}((\varepsilon_1\iota_0)^*).$$

Let $\hat{\varepsilon}_1 : \ker(f_0) \to \ker(f)$ be the appropriate restriction of ε_1 and let $\iota : \ker(f) \to A$ be as in the statement. By construction we have that $\varepsilon_1\iota_0 = \iota\hat{\varepsilon}_1$ and, therefore, we obtain from the previous equation that

$$\ker\left(d_2^{0,1}\right) \cong \operatorname{im}((\iota\hat{\varepsilon}_1)^*) = \operatorname{im}(\hat{\varepsilon}_1^*\iota^*) = \hat{\varepsilon}_1^*(\operatorname{im}(\iota^*)).$$

Finally, from Property (6) of Definition 3.2.23 we know that $\ker(f_0) \stackrel{\hat{\varepsilon}_1}{\twoheadrightarrow} \ker(f)$ is an epimorphism. Applying once again left exactness of the contravariant $\operatorname{Hom}_{\mathcal{R}}(-, M)$

functor we can then deduce that $\hat{\varepsilon}_1^*$ is injective. From the first isomorphism theorem we can conclude that $\hat{\varepsilon}_1^* (\mathrm{im}(\iota^*)) \cong \mathrm{im}(\iota^*)$. The result follows. □

As a consequence of Lemma 3.3.1 we obtain a short exact sequence involving $\mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}(f), M)$.

Lemma 3.3.2. *Let $\mathcal{R}, M, A, B, D^{*,*}, f$ and ι be as in Lemma 3.3.1. If $\mathrm{Ext}^1_{\mathcal{R}}(B, M) = 0$ then there exists a short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}(f), M) \to H^1(\mathrm{Tot}(D^{*,*})) \to \mathrm{im}(\iota^*) \to 0.$$

Proof. With notation as in Lemma 3.3.1 we have that $E_2^{i,j} \Rightarrow H^{i+j}(\mathrm{Tot}(D^{*,*}))$ is a first quadrant cohomology spectral sequence. In particular we have that $E_2^{3,-1} = E_2^{-1,1} = E_2^{-2,2} = 0$. It follows from Definition 3.A.1

$$E_3^{1,0} \cong \ker\left(E_2^{1,0} \to E_2^{3,-1}\right) / \mathrm{im}\left(E_2^{-1,1} \to E_2^{1,0}\right) \cong E_2^{1,0} := \mathrm{Ext}^1_{\mathcal{R}}(H_0(C_*), M),$$
$$E_3^{0,1} \cong \ker\left(E_2^{0,1} \to E_2^{2,0}\right) / \mathrm{im}\left(E_2^{-2,2} \to E_2^{0,1}\right) \cong \ker\left(d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}\right).$$

From Lemma 3.3.1 and the second equation we obtain that $E_3^{0,1} \cong \mathrm{im}(\iota^*)$. On the other hand, by construction of C_* we have that $C_i = 0$ for every $i \neq 0, 1$. We can therefore conclude that $H_0(C_*) = \mathrm{coker}(f)$ and that $H_i(C_*) = 0$ for every $i \neq 0, 1$. Combining this with the first equation we obtain

$$E_3^{1,0} \cong \mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}(f), M) \qquad \text{and} \qquad E_2^{i,j} = 0 \;\forall\, j \neq 0, 1.$$

The result now follows from the short exact sequence of Proposition 3.A.4 taken with $n = 1$. □

The following allows us to describe the term $H^1(\mathrm{Tot}(D^{*,*}))$ appearing in the short exact sequence of Lemma 3.3.2.

Lemma 3.3.3. *Let $\mathcal{R}, M, A, B, D^{*,*}$ and f be as in Lemma 3.3.1. If $\mathrm{Ext}^1_{\mathcal{R}}(B, M) = 0$ then*

$$H^1(\mathrm{Tot}(D^{*,*})) \cong \mathrm{Hom}_{\mathcal{R}}(A, M) / \mathrm{im}(f^*).$$

Proof. Let C_* be as in Lemma 3.3.1. Since $\mathrm{Hom}_{\mathcal{R}}(-, M)$ is a left exact contravariant functor we know from [We94, Proposition 5.7.6 and Paragraph 5.7.9] that there exists a first quadrant cohomology spectral sequence

$$E_2^{i,j} := H^i\left(\mathrm{Ext}^j_{\mathcal{R}}(C_*, M)\right) \Rightarrow H^{i+j}\left(\mathrm{Tot}\left(D^{*,*}\right)\right).$$

Where $H^i\left(\mathrm{Ext}^j_{\mathcal{R}}(C_*, M)\right)$ denotes the i^{th} cohomology group of the cochain complex $\mathrm{Ext}^j_{\mathcal{R}}(C_*, M)$ obtained by applying the contravariant functor $\mathrm{Ext}^j_{\mathcal{R}}(-, M)$ to the chain complex C_*.

Since $C_i = 0$ for every $i \neq 0, 1$ then we can conclude that $\mathrm{Ext}^j_{\mathcal{R}}(C_i, M) = E_2^{i,j} = 0$ for every $i \neq 0, 1$. We can therefore apply Proposition 3.A.5 with $n = 1$ in order to obtain the short exact sequence

$$0 \to E_2^{1,0} \to H^1\left(\mathrm{Tot}\left(D^{*,*}\right)\right) \to E_2^{0,1} \to 0.$$

On the other hand, since $B = C_0$ and $\mathrm{Ext}^1_{\mathcal{R}}(B, M) = 0$ by hypothesis, we have that

$$E_2^{0,1} := H^0\left(\mathrm{Ext}^1_{\mathcal{R}}(C_*, M)\right) = \mathrm{Ext}^1_{\mathcal{R}}(B, M) = 0.$$

Combining this with the previous short exact sequence we obtain the isomorphism

$$H^1\left(\mathrm{Tot}\left(D^{*,*}\right)\right) \cong E_2^{1,0}.$$

Recall that for every $\mathcal{R}$-module N there exists a natural isomorphism of abelian groups $\mathrm{Ext}^0_{\mathcal{R}}(N, M) \cong \mathrm{Hom}_{\mathcal{R}}(N, M)$. Since the only non zero terms in the chain complex C_* are $C_0 := B$ and $C_1 := A$ with differential $f : A \to B$ we can then conclude that

$$E_2^{1,0} := H^1\left(\mathrm{Ext}^0_{\mathcal{R}}(C_*, M)\right) \cong H^1\left(\mathrm{Hom}_{\mathcal{R}}(C_*, M)\right) = \mathrm{Hom}_{\mathcal{R}}(A, M) / \mathrm{im}\left(f^*\right).$$

The result follows. □

We can now rewrite the short exact sequence of Lemma 3.3.2.

Corollary 3.3.4. *Let* $\mathcal{R}, M, A, B, D^{*,*}, f$ *and* ι *be as in Lemma 3.3.1. If* $\mathrm{Ext}^1_{\mathcal{R}}(B, M) =$

0 *then there exists a short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}(f), M) \to \mathrm{Hom}_{\mathcal{R}}(A, M) / \mathrm{im}(f^*) \to \mathrm{im}(\iota^*) \to 0.$$

Proof. This is an immediate consequence of Lemmas 3.3.2 and 3.3.3. □

We know from [We07, Corollary 5.2] that there exists an equivalence between certain $\mathrm{Ext}^n_{\mathcal{R}}$ groups and higher limits. It therefore stands to reason that, in order to study Conjecture 3.1.1, we should study the conditions under which $\mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}(f), M) = 0$. Corollary 3.3.4 tells us that this happens precisely when the epimorphism from $\mathrm{Hom}(A, M) / \mathrm{im}(f^*)$ to $\mathrm{im}(\iota^*)$ is in fact an isomorphism. The following provides us with some insight regarding sufficient conditions for this to happen.

Lemma 3.3.5. *Let $\mathcal{R}, M, A, B$ and ι be as in Lemma 3.3.1 and let $\tilde{f} : A \to Im(f)$ be the $\mathcal{R}$-module morphism obtained by viewing f as an epimorphism onto its image. If $\mathrm{im}\left(\tilde{f}^*\right) = \mathrm{im}(f^*)$ then the morphism $\iota^* : \mathrm{Hom}_{\mathcal{R}}(A, M) \to \mathrm{Hom}_{\mathcal{R}}(\ker(f), M)$ induces an isomorphism of $\mathcal{R}$-modules*

$$\mathrm{Hom}_{\mathcal{R}}(A, M) / \mathrm{im}(f^*) \xrightarrow{\cong} \mathrm{im}(\iota^*) \subseteq \mathrm{Hom}_{\mathcal{R}}(\ker(f), M).$$

Proof. Applying the left exact contravariant $\mathrm{Hom}_{\mathcal{R}}(-, M)$ functor to the short exact sequence $0 \to \ker(f) \xrightarrow{\iota} A \xrightarrow{\tilde{f}} \mathrm{im}(f) \to 0$, we obtain the short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{R}}(\mathrm{im}(f), M) \xrightarrow{\tilde{f}^*} \mathrm{Hom}_{\mathcal{R}}(A, M) \xrightarrow{\iota^*} \mathrm{Hom}_{\mathcal{R}}(\ker(f), M).$$

This leads to an injective morphism $\mathrm{Hom}_{\mathcal{R}}(A, M) / \mathrm{im}\left(\tilde{f}^*\right) \hookrightarrow \mathrm{Hom}_{\mathcal{R}}(\ker(f), M)$ which factors through $\mathrm{im}(\iota^*)$. Since $\mathrm{im}\left(\tilde{f}^*\right) = \mathrm{im}(f^*)$ by hypothesis then the result follows. □

From now on we concentrate on finitely generated modules. It is therefore useful before proceeding to recall the following well known result which prevents confusions.

Proposition 3.3.6. *Let $\mathcal{R}$ be a Noetherian ring, let M and N be finitely generated $\mathcal{R}$-modules, let $n \geq 0$ and denote by $\mathrm{Ext}^n_{\mathcal{R}\text{-mod}}(M, N)$ and $\mathrm{Ext}^n_{\mathcal{R}\text{-Mod}}(M, N)$ the Ext^n groups in the categories $\mathcal{R}$-mod and $\mathcal{R}$-Mod respectively. There exists an isomorphism* *of abelian groups*

$$\mathrm{Ext}^n_{\mathcal{R}}(M,N) := \mathrm{Ext}^n_{\mathcal{R}\text{-Mod}}(M,N) \cong \mathrm{Ext}^n_{\mathcal{R}\text{-mod}}(M,N).$$

Proof. Since $\mathcal{R}$ is Noetherian then every finitely generated module is finitely presented. We can therefore take a projective resolution $P_* \overset{\varepsilon}{\twoheadrightarrow} M$ in $\mathcal{R}\text{-Mod}$ where each P_* is free and finitely generated. In particular we have that $P_* \overset{\varepsilon}{\twoheadrightarrow} M$ is also a projective resolution in $\mathcal{R}\text{-mod}$. The result follows from computing $\mathrm{Ext}^n_{\mathcal{R}\text{-mod}}(M,N)$ and $\mathrm{Ext}^n_{\mathcal{R}\text{-Mod}}(M,N)$ using this resolution. □

We are now ready to provide sufficient conditions for the group $\mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}(f), M)$ of Lemma 3.3.2 to vanish.

Proposition 3.3.7. *Let $\mathcal{R}$ be a finite ring, let M, A and B be finitely generated $\mathcal{R}$-modules, let $f : A \to B$ be an $\mathcal{R}$-module morphism, and let $\tilde{f} : A \to \mathrm{im}(f)$ be as in Lemma 3.3.5. If $\mathrm{im}\left(\tilde{f}^*\right) = \mathrm{im}(f^*)$ and $\mathrm{Ext}^1_{\mathcal{R}}(B, M) = 0$ then $\mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}(f), M) = 0$.*

Proof. Since M, A and B are all finitely generated modules over a finite ring then they are all finite. In particular we have that $\mathrm{Hom}_{\mathcal{R}}(\ker(f), M)$, $\mathrm{Hom}_{\mathcal{R}}(A, M)$ and $\mathrm{Hom}_{\mathcal{R}}(B, M)$ are finite and, therefore, $\mathrm{im}(\iota^*)$ and $\mathrm{Hom}_{\mathcal{R}}(A, M)/\mathrm{im}(f^*)$ are also finite.

We know from Lemma 3.3.5 that there exists an isomorphism $\mathrm{Hom}_{\mathcal{R}}(A, M)/\mathrm{im}(f^*) \cong \mathrm{im}(\iota^*)$. In particular we have that $|\mathrm{Hom}_{\mathcal{R}}(A, M)/\mathrm{im}(f^*)| = |\mathrm{im}(\iota^*)|$.

Let $0 \to \mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}(f), M) \to \mathrm{Hom}_{\mathcal{R}}(A, M)/\mathrm{im}(f^*) \overset{g}{\to} \mathrm{im}(\iota^*) \to 0$ be the short exact sequence of Corollary 3.3.4. Since g is surjective, $\mathrm{Hom}_{\mathcal{R}}(A, M)/\mathrm{im}(f^*)$ and $\mathrm{im}(\iota^*)$ are finite and $|\mathrm{Hom}_{\mathcal{R}}(A, M)/\mathrm{im}(f^*)| = |\mathrm{im}(\iota^*)|$ we can conclude that g is also injective. From exactness of the sequence in Corollary 3.3.4 this implies that $\mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}(f), M) = 0$ just as we wanted to prove. □

We conclude this section with an application of Proposition 3.3.7 which provides us with sufficient conditions to obtain $\mathrm{Ext}^n_{\mathcal{R}}(\mathrm{coker}(f), M) = 0$ for every $n \geq 1$.

Corollary 3.3.8. *Let $\mathcal{R}, M, A, B, f$ and $\tilde{f}$ be as in Proposition 3.3.7. If* $\mathrm{Ext}^n_{\mathcal{R}}(B, M) = 0$ *for every* $n \geq 1$ *and* $\tilde{f}^*(\mathrm{Hom}_{\mathcal{R}}(\mathrm{im}(f), N)) = f^*(\mathrm{Hom}_{\mathcal{R}}(B, N))$ *for every finitely generated $\mathcal{R}$-module N then* $\mathrm{Ext}^n_{\mathcal{R}}(\mathrm{coker}(f), M) = 0$ *for every* $n \geq 1$.

Proof. From Corollary 3.2.27 there exists an injective resolution $0 \to M \to I_0 \to \cdots$ of M in $\mathcal{R}$-mod. Let $d_n : I_n \to I_{n+1}$ be the differentials of this resolution and let $d_{-1} : M \to I_0$ be the augmentation map. For every $n \geq 0$ we have an injective resolution $0 \to \mathrm{coker}(d_{n-1}) \to I_{n+1} \to \cdots$ of $\mathrm{coker}(d_{n-1})$ in $\mathcal{R}$-mod. We conclude that for every $\mathcal{R}$-module N and every $n \geq 0$ then

$$\mathrm{Ext}^1_{\mathcal{R}}(N, \mathrm{coker}(d_{n-1})) \cong \ker((d_{n+2})_*) / \mathrm{im}((d_{n+1})_*) \cong \mathrm{Ext}^{n+2}_{\mathcal{R}}(N, M).$$

In particular, for every $n \geq 0$ we obtain

$$\mathrm{Ext}^1_{\mathcal{R}}(B, \mathrm{coker}(d_{n-1})) \cong \mathrm{Ext}^{n+2}_{\mathcal{R}}(B, M) = 0.$$

We can now apply Proposition 3.3.7 with $M = \mathrm{coker}(d_{n-1})$ to deduce that for every $n \geq 0$ then

$$\mathrm{Ext}^{n+2}_{\mathcal{R}}(\mathrm{coker}(f), M) \cong \mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}(f), \mathrm{coker}(d_{n-1})) = 0.$$

From Proposition 3.3.7 we also know that $\mathrm{Ext}^1_{\mathcal{R}}(\mathrm{coker}(f), M) = 0$ thus concluding the proof. □

3.4 Sharpness from fusion subsystems

In this section we use the tools developed in Section 3.3 in order to prove Theorem 3.A. Let us start by introducing some notation appearing in the statement of Theorem 3.A.

Definition 3.4.1. Let $\boldsymbol{D} \subseteq \boldsymbol{C}$ be small categories with finitely many objects and let $\mathcal{R}$ be a commutative ring. With notation as in Definition 3.2.18 and Lemma 3.2.19 we define the **constant module induced from $\mathcal{R}\boldsymbol{D}$ to $\mathcal{R}\boldsymbol{C}$** as the $\mathcal{R}\boldsymbol{C}$-module $\underline{\mathcal{R}}^{\boldsymbol{C}}_{\boldsymbol{D}} := \underline{\mathcal{R}}^{\boldsymbol{D}} \uparrow^{\boldsymbol{C}}_{\boldsymbol{D}}$. Moreover we define the **identity morphism induced from $\mathcal{R}\boldsymbol{D}$ to $\mathcal{R}\boldsymbol{C}$** as the morphism $\theta^{\boldsymbol{C}}_{\mathcal{R},\boldsymbol{D}} : \underline{\mathcal{R}}^{\boldsymbol{C}}_{\boldsymbol{D}} \to \underline{\mathcal{R}}^{\boldsymbol{C}}$ that, for every $x \in \underline{\mathcal{R}}^{\boldsymbol{D}}$ and every $y \in \mathcal{R}\boldsymbol{C}$

sends $x \otimes y$ to $x \cdot y$. If there are no doubts regarding the ring $\mathcal{R}$ and the category $\boldsymbol{C}$ we simply write $\theta_{\boldsymbol{D}} := \theta^{\boldsymbol{C}}_{\mathcal{R},\boldsymbol{D}}$.

We are mostly interested in the above definitions in the case where $\boldsymbol{C}$ is of the form $\mathcal{O}(\mathcal{F}^c)$ (see Definition 3.2.12). It is therefore useful to introduce the following notation.

Definition 3.4.2. Let S be a p-group, let $S' \leq S$, let $\mathcal{F}$ be a fusion system over S, let $\mathcal{F}' \subseteq \mathcal{F}$ be a fusion system over S', let $\mathcal{C}$ be a family of subgroups of S closed under $\mathcal{F}$-overconjugation (see Definition 3.2.12), define $\mathcal{C}' := \{P \leq S' \,:\, P \in \mathcal{C}\}$ and let $\mathcal{R}$ be a commutative ring. With notation as in Definition 3.4.1 we define $\underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}_{\mathcal{F}'} := \underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}_{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}$ and $\theta^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}_{\mathcal{R},\mathcal{F}'} := \theta^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}_{\mathcal{R},\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}$. If $\mathcal{O}_{\mathcal{C}}(\mathcal{F})$ and $\mathcal{R}$ are clear we simply write $\theta_{\mathcal{F}'} := \theta^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}_{\mathcal{R},\mathcal{F}'}$. Moreover, given a family $\boldsymbol{F} := \{\mathcal{F}_i\}_{i \in I}$ of fusion subsystems of $\mathcal{F}$ we define $\underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}_{\boldsymbol{F}} := \bigoplus_{i \in I} \underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}_{\mathcal{F}_i}$ and $\theta^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}_{\mathcal{R},\boldsymbol{F}} = \bigoplus_{i \in I} \theta^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}_{\mathcal{R},\mathcal{F}_i} : \underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}_{\boldsymbol{F}} \to \underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}$. If $\mathcal{R}$ and $\mathcal{O}_{\mathcal{C}}(\mathcal{F})$ are clear we simply write $\theta_{\boldsymbol{F}} := \theta^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}_{\mathcal{R},\boldsymbol{F}}$.

Finally we introduce the remaining notation stated in Theorem 3.A.

Definition 3.4.3. Let $\mathcal{F}$ be a fusion system, let $\boldsymbol{F}$ be a family of fusion subsystems of $\mathcal{F}$, let $\mathcal{R}$ be a commutative ring and let $\iota : \ker(\theta_{\boldsymbol{F}}) \hookrightarrow \underline{\mathcal{R}}^{\mathcal{O}(\mathcal{F}^c)}_{\boldsymbol{F}}$ be the natural inclusion. We say that $\boldsymbol{F}$ **satisfies the $\mathcal{R}$-lifting property** (or simply **lifting property** if $\mathcal{R}$ is clear) if for every $\mathcal{R}\mathcal{O}(\mathcal{F}^c)$-module M the equality $\iota^*\left(\mathrm{Hom}_{\mathcal{R}\mathcal{O}(\mathcal{F}^c)}\left(\underline{\mathcal{R}}^{\mathcal{O}(\mathcal{F}^c)}_{\boldsymbol{F}}, M\right)\right) = \mathrm{Hom}_{\mathcal{R}\mathcal{O}(\mathcal{F}^c)}(\ker(\theta_{\boldsymbol{F}}), M)$ holds. This is equivalent to saying that for every morphism $f : \ker(\theta_{\boldsymbol{F}}) \to M$ there exists $\hat{f} : \underline{\mathcal{R}}^{\mathcal{O}(\mathcal{F}^c)}_{\boldsymbol{F}} \to M$ such that $f = \hat{f}\iota$.

Let S be a finite p-group, let $\mathcal{F}$ be a fusion system over S, let $\boldsymbol{F}$ be a family of fusion subsystems of $\mathcal{F}$ satisfying the $\mathbb{F}_p$-lifting property and let $\boldsymbol{M} = (\boldsymbol{M}_*, \boldsymbol{M}^*)$ be a Mackey functor over $\mathcal{F}$ on $\mathbb{F}_p$. The main idea behind the proof of Theorem 3.A is to apply Corollary 3.3.8 with $\mathcal{R} := \mathbb{F}_p\mathcal{O}(\mathcal{F}^c)$, $M := \boldsymbol{M}^* \downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}(\mathcal{F}^c)}$ (see Proposition 3.2.17), $B := \underline{\mathbb{F}_p}^{\mathcal{O}(\mathcal{F}^c)}_{\boldsymbol{F}}$, $A := \ker(\theta_{\boldsymbol{F}})$ and $f : \ker(\theta_{\boldsymbol{F}}) \hookrightarrow \underline{\mathbb{F}_p}^{\mathcal{O}(\mathcal{F}^c)}_{\boldsymbol{F}}$ the natural inclusion. However, in order to interpret the results thus obtained with Theorem 3.A we need to relate the $\mathrm{Ext}^n_{\mathcal{R}}\left(\mathrm{coker}(f), \boldsymbol{M}^* \downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}(\mathcal{F}^c)}\right)$ groups with the higher limits $\lim^n_{\mathcal{O}(\mathcal{F}^c)}\left(\boldsymbol{M}^* \downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}(\mathcal{F}^c)}\right)$.

To do so the following adaptation of [Ya22, Proposition 4.5] is necessary.

Lemma 3.4.4 ([Ya22, Proposition 4.5])**.** *Let $\mathcal{R}$ be a commutative ring, let S be a finite p-group, let $\mathcal{F}$ be an S-category, let S' be a subgroup of S, let $\mathcal{F}' \subseteq \mathcal{F}$ be an*

S'-category, let $\mathcal{C}$ be a family of subgroups of S closed under $\mathcal{F}$-overconjugation (see Definition 3.2.12) and let $\mathcal{C}' := \{P \in \mathcal{C} : P \leq S'\}$. With notation as in Definitions 3.2.12 and 3.2.16 the functors $\uparrow_{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}$ and $\downarrow_{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}$ of Lemma 3.2.19 are both exact. Moreover, for every $P \leq S$ and every $\mathcal{R}\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')$-module M we have the following isomorphism of $\mathcal{R}$-modules

$$M \uparrow_{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})} \cdot \mathrm{Id}_P \cong \bigoplus_{\left(\varphi: P \xrightarrow{\cong} Q_\varphi\right) \in I_{\mathcal{F}',P}^{\mathcal{F}}} M \cdot \mathrm{Id}_{Q_\varphi},$$

where $I_{\mathcal{F}',P}^{\mathcal{F}}$ is a (necessarily finite) set of representatives of the isomorphisms in $\mathcal{F}$ of the form $\varphi : P \xrightarrow{\cong} Q_\varphi$ (for some $Q_\varphi \leq S$) modulo the equivalence relation

$$\varphi \sim \varphi' \Leftrightarrow \varphi' = \psi\varphi \text{ for some isomorphism } \psi \text{ in } \mathcal{F}'.$$

Proof. Exactness of the restriction functor is immediate. The rest of the statement follows from [Ya22, Proposition 4.5] after taking (non necessarily finite) groups G and H realizing $\mathcal{F}$ and $\mathcal{F}'$ respectively (see [LS07]) and applying the equivalence of categories described in [Ya22, Lemma 2.5]. □

As a consequence of the above we obtain the following particular case of [Ya22, Proposition 4.8].

Lemma 3.4.5. *Let $\mathcal{R}, \mathcal{F}, \mathcal{F}', \mathcal{C}$ and $\mathcal{C}'$ be as in Lemma 3.4.4, let $M : \mathcal{O}_{\mathcal{C}}(\mathcal{F})^{op} \to \mathcal{R}\text{-Mod}$ be a contravariant functor and let $n \geq 0$ be an integer. With notation as in Definition 3.2.18 the following isomorphism of abelian groups holds*

$$\mathrm{Ext}^n_{\mathcal{R}\mathcal{O}_{\mathcal{C}}(\mathcal{F})}\left(\underline{\mathcal{R}}_{\mathcal{F}'}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}, M\right) \cong \lim{}^n_{\mathcal{R}}\left(M \downarrow_{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}\right).$$

Proof. From Lemma 3.4.4 and [Ya22, Proposition 3.7] we have that

$$\mathrm{Ext}^n_{\mathcal{R}\mathcal{O}_{\mathcal{C}}(\mathcal{F})}\left(\underline{\mathcal{R}}_{\mathcal{F}'}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}, M\right) \cong \mathrm{Ext}^n_{\mathcal{R}\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}\left(\underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}, M \downarrow_{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}\right).$$

The result follows from [We07, Corollary 5.2]. □

We are now finally ready to prove Theorem 3.A.

Proof. (of Theorem 3.A). Since the functor $\mathrm{Ext}^n_{\mathbb{F}_p\mathcal{O}(\mathcal{F}^c)}(-,-)$ is additive on both variables for every $n \geq 0$ we can apply Lemma 3.4.5 in order to obtain the isomorphism of abelian groups

$$\mathrm{Ext}^n_{\mathbb{F}_p\mathcal{O}_{\mathcal{C}}(\mathcal{F})}\left(\underline{\mathbb{F}_p}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)}, N\downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}(\mathcal{F}^c)}\right)\cong\bigoplus_{i\in I}\lim{}^n_{\mathcal{O}_{\mathcal{C}_i}(\mathcal{F}_i)}\left(N\downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}_{\mathcal{C}_i}(\mathcal{F}_i)}\right). \tag{3.6}$$

Where $\mathcal{O}(\mathcal{F}^c)$ is as in Definition 3.2.17, N is any $\mathbb{F}_p\mathcal{O}(\mathcal{F})$-module and for every $i\in I$ we define $\mathcal{C}_i$ to be the set of $\mathcal{F}$-centric subgroups of S_i.

Let $\iota:\mathbb{F}_p\text{-Mod}\to\mathbb{Z}_p\text{-Mod}$ be the natural inclusion of categories. From Corollary 3.B.4 we know that the following equivalences of abelian groups hold for every $i\in I$ and every n and N as above

$$\lim{}^n_{\mathcal{O}_{\mathcal{C}_i}(\mathcal{F}_i)}\left(N\downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}_{\mathcal{C}_i}(\mathcal{F}_i)}\right)\cong\lim{}^n_{\mathcal{O}_{\mathcal{C}_i}(\mathcal{F}_i)}\left(\iota N\downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}_{\mathcal{C}_i}(\mathcal{F}_i)}\right),$$
$$\lim{}^n_{\mathcal{O}\left(\mathcal{F}_i^c\right)}\left(N\downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}\left(\mathcal{F}_i^c\right)}\right)\cong\lim{}^n_{\mathcal{O}\left(\mathcal{F}_i^c\right)}\left(\iota N\downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}\left(\mathcal{F}_i^c\right)}\right).$$

From Condition (2) and [Ya22, Proposition 10.5] we can then conclude that for every i,n and N as above there exists an isomorphism of abelian groups

$$\lim{}^n_{\mathcal{O}_{\mathcal{C}_i}(\mathcal{F}_i)}\left(N\downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}_{\mathcal{C}_i}(\mathcal{F}_i)}\right)\cong\lim{}^n_{\mathcal{O}\left(\mathcal{F}_i^c\right)}\left(N\downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}\left(\mathcal{F}_i^c\right)}\right).$$

Therefore, using Condition (3) we obtain that for every Mackey functor $M=(M_*,M^*)$ over $\mathcal{F}$ on $\mathbb{F}_p$, every $i\in I$ and every $n\geq 1$ then

$$\lim{}^n_{\mathcal{O}_{\mathcal{C}_i}(\mathcal{F}_i)}\left(M^*\downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}_{\mathcal{C}_i}(\mathcal{F}_i)}\right)=0.$$

We conclude from Equation (3.6) that $\mathrm{Ext}^n_{\mathbb{F}_p\mathcal{O}_{\mathcal{C}}(\mathcal{F})}\left(\underline{\mathbb{F}_p}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)}, M^*\downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}(\mathcal{F}^c)}\right)=0$ for every $n\geq 1$ and M as before. Using this result and Condition (4) we can now apply Corollary 3.3.8 with $A:=\ker(\theta_{\boldsymbol{F}})$, $B:=\underline{\mathbb{F}_p}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)}$ and $f:A\hookrightarrow B$ the natural inclusion in order to deduce that for every such n and M then

$$\mathrm{Ext}^n_{\mathbb{F}_p\mathcal{O}(\mathcal{F}^c)}\left(\underline{\mathbb{F}_p}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)}/\ker(\theta_{\boldsymbol{F}}), M^*\downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}(\mathcal{F}^c)}\right)=0. \tag{3.7}$$

From Condition (1) and the first isomorphism theorem we know that $\underline{\mathbb{F}_p}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)}/\ker(\theta_{\boldsymbol{F}})\cong\underline{\mathbb{F}_p}^{\mathcal{O}(\mathcal{F}^c)}$. The result now follows from Equation (3.7) and [We07, Corollary 5.2]. $\square$

The four conditions required to apply Theorem 3.A might seem too restrictive. However, there are several families of fusion subsystems satisfying Conditions (1)-(3). Let S be a finite p-group, let $\mathcal{F}$ be a fusion system over S, let I be a finite indexing set, let $S\in\{P_i\}_{i\in I}$ be a family of fully $\mathcal{F}$-normalized and $\mathcal{F}$-centric subgroups of S, for every $i\in I$ define $\mathcal{F}_i:=N_{\mathcal{F}}(P_i)$ (see Example 3.2.9) and let $\boldsymbol{F}$ be as in Theorem 3.A. From

Definition 3.4.2 and Lemma 3.4.4 we know that $\theta_{\boldsymbol{F}}\left(\underline{\mathbb{F}_p}_{N_{\mathcal{F}}(S)}^{\mathcal{O}(\mathcal{F}^c)}\right) = \underline{\mathbb{F}_p}^{\mathcal{O}(\mathcal{F}^c)}$ (see Definition 3.2.18) and, therefore, Condition (1) is satisfied. From [Ya22, Lemma 10.4] we can deduce that $\boldsymbol{F}$ satisfies Condition (2). Finally, from [Br05, Proposition C] and [DP15, Theorem B], we know $\boldsymbol{F}$ satisfies Condition (3). Thus the only condition of Theorem 3.A that we need to show is Condition (4). We therefore formulate the following.

Conjecture 3.4.6. *Let $\mathcal{F}$ be a fusion system over a p-group S, let $\mathcal{F}^e$ be the set of fully $\mathcal{F}$-normalized and $\mathcal{F}$-essential subgroups of S (see [Li07, Definition 5.1 (ii)]), and define*

$$\boldsymbol{F} := \{N_{\mathcal{F}}(S)\} \cup \{N_{\mathcal{F}}(P) \, : \, P \in \mathcal{F}^e\}.$$

Then $\boldsymbol{F}$ satisfies the lifting property.

Since all $\mathcal{F}$-essential subgroups of S are $\mathcal{F}$-centric we know from the above discussion that Conditions (1)-(3) of Theorem 3.A are satisfied when taking $\boldsymbol{F} = \{N_{\mathcal{F}}(P)\}_{P \in \mathcal{F}^e \cup \{S\}}$. Conjecture 3.4.6 claims that $\boldsymbol{F}$ also satisfies Condition (2) of 3.A. It follows, by applying Theorem 3.A that Conjecture 3.4.6 implies Conjecture 3.1.1.

3.5 Sharpness for the Benson-Solomon fusion systems

In [HLL23, Theorems 1.1 and 1.4] Henke, Libman and Lynd prove that Conjecture 3.1.1 is satisfied for all Benson-Solomon fusion systems $\mathcal{F}_{\mathrm{Sol}}(q^n)$ such that $q^n \equiv \pm 3 \pmod 8$. In this section we aim to extend this result to all Benson-Solomon fusion systems.

Let us start by recalling a few key facts regarding the construction of the Benson-Solomon fusion systems. Let q be an odd prime, let n be a positive integer and let $S(q^n) \in \mathrm{Syl}_2(\mathrm{Spin}_7(q^n))$. From [LO02, Definition 2.2 and Proposition 2.5] we know that there exists an elementary abelian group U of rank 2 such that $U \trianglelefteq S(q^n)$. In [LO02, Definition 2.6] Levi and Oliver introduce a group $S_0(q^n)$ which satisfies $S_0(q^n) \leq S(q^n)$ and $S_0(q^n) \in \mathrm{Syl}_2\left(C_{\mathrm{Spin}_7(q^n)}(U)\right)$ (see [LO02, Proposition 2.5 and Lemma 2.7]). Since $S(q^n) \in \mathrm{Syl}_2(\mathrm{Spin}_7(q^n))$ we can deduce from these facts that $S_0(q^n) = C_{S(q^n)}(U)$. Finally in [LO05, Definition 1.6] Levi and Oliver define the Benson-Solomon fusion system over $S(q^n)$ as the fusion system $\mathcal{F}_{\mathrm{Sol}}(q^n)$ generated by a group of automorphisms $\Gamma(q^n) \leq \mathrm{Aut}\left(C_{S(q^n)}(U)\right)$ and the fusion system $\mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n))$.

In order to keep notation simple we introduce the following.

Definition 3.5.1. Let $q, n, U, S(q^n)$ and $\mathcal{F}_{\mathrm{Sol}}(q^n)$ be as above. With notation as in Example 3.2.9 we define the following fusion subsystems of $\mathcal{F}_{\mathrm{Sol}}(q^n)$

$$\mathcal{F}_1(q^n) := \mathcal{F}_{S(q^n)}(\mathrm{Spin}_7(q^n)), \qquad \mathcal{F}_2(q^n) := N_{\mathcal{F}_{\mathrm{Sol}}(q^n)}\left(C_{S(q^n)}(U)\right).$$

Since $\Gamma(q^n) \leq \mathrm{Aut}_{\mathcal{F}_{\mathrm{Sol}}(q^n)}\left(C_{S(q^n)}(U)\right)$ then $\mathcal{F}_2(q^n)$ contains the automorphisms of $\Gamma(q^n)$. Thus, from [LO05, Definition 1.6] we immediately obtain the following.

Lemma 3.5.2. *Let q be an odd prime and let n be a positive integer. The Benson-Solomon fusion system $\mathcal{F}_{\mathrm{Sol}}(q^n)$ is generated by its fusion subsystems $\mathcal{F}_1(q^n)$ and $\mathcal{F}_2(q^n)$.*

In this section we aim to show that Theorem 3.A applies to $\mathcal{F} := \mathcal{F}_{\mathrm{Sol}}(q^n)$, $I := \{1, 2\}$ and $\mathcal{F}_i := \mathcal{F}_i(q^n)$ for every $i \in I$. To do so we first need to prove that both $\mathcal{F}_1(q^n)$ and $\mathcal{F}_2(q^n)$ satisfy Conditions (2) and (3) of Theorem 3.A. Let us start by proving that this is true for $\mathcal{F}_1(q^n)$.

Lemma 3.5.3. *With notation as in Definition 3.5.1 the following hold:*

(1) $\mathcal{F}_1(q^n)$ is a fusion system over $S(q^n)$.

(2) Every $\mathcal{F}_1(q^n)$-centric-radical subgroup of $S(q^n)$ is $\mathcal{F}_{\mathrm{Sol}}(q^n)$-centric.

(3) Conjecture 3.1.1 is satisfied for $\mathcal{F}_1(q^n)$.

Proof. Part (1) follows from Definition 3.5.1. Part (2) holds since [LO02, Proposition 3.3 (a)] shows that any $\mathcal{F}_1(q^n)$-centric subgroup of $S(q^n)$ is $\mathcal{F}_{\mathrm{Sol}}(q^n)$-centric. Finally Part (3) follows from [DP15, Theorem B]. □

In order to obtain a result analogous to Lemma 3.5.3 for the fusion system $\mathcal{F}_2(q^n)$ we first need some further work. The following is a well known result in group theory.

Lemma 3.5.4. *Let S be a 2-group, let $P \trianglelefteq S$ be an elementary abelian subgroup of S of rank 2. Then $[S : C_S(P)] \in \{1, 2\}$. In particular, since S is a 2-group we have that $N_S(C_S(P)) = S$.*

Proof. By definition we know that $P \cong C_2 \times C_2$ and, therefore, $\mathrm{Aut}(P) = S_3$. In particular, since $N_S(P) = S$ we have that $S/C_S(P) \cong \mathrm{Aut}_S(P) \leq S_3$. Since $S/C_S(P)$ is a 2-group we can then conclude that it is isomorphic to either C_2 or the trivial group. It follows that $[S : C_S(P)] = |S/C_S(P)| \in \{2, 1\}$ thus concluding the proof. □

As a consequence of the above we obtain the following.

Lemma 3.5.5. *With notation as in Definition 3.5.1 we have that $C_{S(q^n)}(U)$ is $\mathcal{F}_{\mathrm{Sol}}(q^n)$-centric.*

Proof. Throughout this proof we write $\mathcal{F} := \mathcal{F}_{\mathrm{Sol}}(q^n)$, $\mathcal{F}_1 := \mathcal{F}_1(q^n)$ and $S_0 := C_{S(q^n)}(U)$. From Alperin's fusion theorem we know that for every $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S_0)$ there exist a set $P_1, \ldots, P_n$ of $\mathcal{F}$-centric subgroups of S and automorphisms $\varphi_i \in \mathrm{Aut}_{\mathcal{F}}(P_i)$ such that $\varphi(x) = \varphi_n(\cdots \varphi_1(x))$ for every $x \in S_0$. Notice that each P_i contains a subgroup $S_i := \varphi_i(\cdots \varphi_1(S_0))$ isomorphic to S_0 in $\mathcal{F}$.

If S_0 isn't $\mathcal{F}$-centric then none of the S_i is $\mathcal{F}$-centric either. Since each P_i is $\mathcal{F}$-centric we can conclude that $P_i \gneq S_i$ for every $i = 1, \ldots, n$. From [LO05, Definition 1.6] we know that $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}_1}(S)$. we can therefore conclude that $\varphi \in \mathrm{Aut}_{\mathcal{F}_1}(S_0)$. Again from [LO05, Definition 1.6] this implies that $\mathcal{F} = \mathcal{F}_1$ which contradicts the fact that $\mathcal{F}$ is an exotic fusion system. We can therefore conclude that S_0 is $\mathcal{F}$-centric just as we wanted to prove. □

We can now provide a result analogous to Lemma 3.5.3 for the fusion system $\mathcal{F}_2(q^n)$.

Lemma 3.5.6. *With notation as in Definition 3.5.1 we have that:*

(1) $\mathcal{F}_2(q^n)$ is a fusion system over $S(q^n)$.

(2) Every $\mathcal{F}_2(q^n)$-centric-radical subgroup of $S(q^n)$ is $\mathcal{F}_{\mathrm{Sol}}(q^n)$-centric.

(3) Conjecture 3.1.1 is satisfied for $\mathcal{F}_2(q^n)$.

Proof. From Definition 3.5.1 and Example 3.2.9 we know that $\mathcal{F}_2(q^n)$ is a fusion system over $N_{S(q^n)}\left(C_{S(q^n)}(U)\right)$. Part (1) now follows from Lemma 3.5.4. From Lemma 3.5.5 we know that $C_{S(q^n)}(U)$ is $\mathcal{F}_{\mathrm{Sol}}(q^n)$-centric while from Lemma 3.5.4 we know that it is fully $\mathcal{F}_{\mathrm{Sol}}(q^n)$-normalized. Part (2) now follows from [Ya22, Lemma 10.4]. Finally Part (3) follows from [Br05, Proposition C] and [DP15, Theorem B]. □

Lemmas 3.5.3 and 3.5.6 tell us that, taking $\mathcal{F} := \mathcal{F}_{\mathrm{Sol}}(q^n)$, $I = \{1, 2\}$, $\mathcal{F}_1 := \mathcal{F}_1(q^n)$ and $\mathcal{F}_2 := \mathcal{F}_2(q^n)$ then Conditions (2) and (3) of Theorem 3.A are satisfied. Condition (1) then follows from the following and either Lemma 3.5.3 (1) or Lemma 3.5.6 (1).

Lemma 3.5.7. *Let $\mathcal{R}$ be a commutative ring, let $\boldsymbol{C}$ and $\boldsymbol{D}$ be small categories with finitely many objects such that $\boldsymbol{D} \subseteq \boldsymbol{C}$ and let $\theta_{\boldsymbol{D}} : \underline{\mathcal{R}}_{\boldsymbol{D}}^{\boldsymbol{C}} \to \underline{\mathcal{R}}^{\boldsymbol{C}}$ be as in Definition 3.4.1. If $\boldsymbol{C}$ and $\boldsymbol{D}$ have the same objects then $\theta_{\boldsymbol{D}}$ is an epimorphism.*

Proof. It follows from Definitions 3.2.16 and 3.2.18 that the following are direct sum decompositions of $\underline{\mathcal{R}}^{\boldsymbol{C}}$ and $\underline{\mathcal{R}}^{\boldsymbol{D}}$ as $\mathcal{R}$-modules

$$\underline{\mathcal{R}}^{\boldsymbol{C}} = \bigoplus_{X \in \mathsf{Ob}(\boldsymbol{C})} \underline{\mathcal{R}}^{\boldsymbol{C}} \cdot \mathrm{Id}_X \cong \bigoplus_{X \in \mathsf{Ob}(\boldsymbol{C})} \mathcal{R}, \quad \underline{\mathcal{R}}^{\boldsymbol{D}} = \bigoplus_{Y \in \mathsf{Ob}(\boldsymbol{D})} \underline{\mathcal{R}}^{\boldsymbol{D}} \cdot \mathrm{Id}_Y \cong \bigoplus_{Y \in \mathsf{Ob}(\boldsymbol{D})} \mathcal{R}.$$

By definition, for each object $Y \in \mathsf{Ob}(\boldsymbol{D}) \subseteq \mathsf{Ob}(\boldsymbol{C})$ the morphism $\theta_{\boldsymbol{D}}$ sends the $\mathcal{R}$-submodule $\underline{\mathcal{R}}^{\boldsymbol{D}} \cdot \mathrm{Id}_Y \otimes \mathrm{Id}_Y$ of $\underline{\mathcal{R}}_{\boldsymbol{D}}^{\boldsymbol{C}}$ isomorphically onto the component $\underline{\mathcal{R}}^{\boldsymbol{C}} \cdot \mathrm{Id}_Y$ of $\underline{\mathcal{R}}^{\boldsymbol{C}}$. The result follows. □

Now it only remains to prove that Condition (4) of Theorem 3.A holds. This follows from the following three results.

Lemma 3.5.8. *Let $\mathcal{R}$ be a ring, let A, B and C be $\mathcal{R}$-modules, let $f : A \to B$ be an $\mathcal{R}$-module morphism, let $\iota : \mathrm{im}(f) \to B$ be the natural inclusion and let $\tilde{f} : A \to \mathrm{im}(f)$ be the unique epimorpism satisfying $f = \iota \tilde{f}$. Then $f^*(\mathrm{Hom}_{\mathcal{R}}(B, C)) = \tilde{f}^*(\mathrm{Hom}_{\mathcal{R}}(\mathrm{im}(f), C))$ if and only if $\iota^*(\mathrm{Hom}(B, C)) = \mathrm{Hom}(\mathrm{im}(f), C)$.*

Proof. Since $f = \iota \tilde{f}$ then we have that

$$f^*(\mathrm{Hom}_{\mathcal{R}}(B, C)) = \tilde{f}^*(\iota^*(\mathrm{Hom}_{\mathcal{R}}(B, C))). \tag{3.8}$$

Therefore, whenever $\iota^*(\mathrm{Hom}_{\mathcal{R}}(B, C)) = \mathrm{Hom}_{\mathcal{R}}(\mathrm{im}(f), C)$, then

$$f^*(\mathrm{Hom}_{\mathcal{R}}(B, C)) = \tilde{f}^*(\mathrm{Hom}_{\mathcal{R}}(\mathrm{im}(f), C)).$$

On the other hand, if $f^*(\mathrm{Hom}_{\mathcal{R}}(B, C)) = \tilde{f}^*(\mathrm{Hom}_{\mathcal{R}}(\mathrm{im}(f), C))$ then we can once again use Equation (3.8) to obtain the identity

$$\tilde{f}^*(\mathrm{Hom}_{\mathcal{R}}(\mathrm{im}(f), C)) = \tilde{f}^*(\iota^*(\mathrm{Hom}_{\mathcal{R}}(B, C))).$$

Since $\tilde{f}$ is an epimorphism and the functor $\mathrm{Hom}_{\mathcal{R}}(-, C)$ sends epimorphisms to monomorphisms we conclude that $\tilde{f}^*$ is a monomorphism. The result follows. □

Lemma 3.5.8 provides us with a condition equivalent to Condition (4) of Theorem 3.A. The next result is useful to prove this equivalent condition.

Lemma 3.5.9. *Let $\mathcal{R}, \mathcal{F}, \mathcal{F}', \mathcal{C}$ and $\mathcal{C}'$ be as in Lemma 3.4.4. For each $P \in \mathcal{C}'$ we know that there exists a natural isomorphism of $\mathcal{R}$-modules $\underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')} \cdot \mathrm{Id}_P \xrightarrow{\cong} \mathcal{R}$. Let $1_P \in \underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')} \cdot \mathrm{Id}_P$ denote the image of the identity in $\mathcal{R}$ via this isomorphism. Then every element $\lambda \in \underline{\mathcal{R}}_{\mathcal{F}'}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}$ (see Definition 3.4.2) can be written as $\lambda = 1_{S'} \otimes_{\mathcal{R}\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')} x_\lambda$ for some $x_\lambda \in \mathcal{R}\mathcal{O}_{\mathcal{C}}(\mathcal{F})$.*

Proof. By definition of $\underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}$ (see Definition 3.2.18) we know that $1_{S'} \cdot \overline{\iota_P^{S'}} = 1_P$ where $\overline{\iota_P^{S'}} : \mathrm{Hom}_{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}(P, S')$ denotes the equivalence class of the natural inclusion seen as an element in $\mathcal{R}\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')$ (see Definition 3.2.12). Moreover every element in $\underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}$ is of the form $\mu = \sum_{P \in \mathcal{C}'} 1_P \cdot \mu_P$ for some $\mu_P \in \mathcal{R}$. Define $y_\mu := \sum_{P \in \mathcal{C}'} \mu_P \overline{\iota_P^{S'}} \in \mathcal{R}\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')$. Then we have that

$$\mu = \sum_{P \in \mathcal{C}'} 1_{S'} \cdot \overline{\iota_P^{S'}} \mu_P = 1_{S'} \cdot y_\mu.$$

On the other hand, we know that every element in $\lambda \in \underline{\mathcal{R}}_{\mathcal{F}'}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}$ can be written as a finite sum of the form $\lambda = \sum_{i=1}^n \lambda_i \otimes z_i$ for some $\lambda_i \in \underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}'}(\mathcal{F}')}$ and some $z_i \in \mathcal{R}\mathcal{O}_{\mathcal{C}}(\mathcal{F})$. Therefore, with the above notation, we have that

$$\lambda = \sum_{i=1}^n 1_{S'} \cdot y_{\lambda_i} \otimes z_i = 1_{S'} \otimes \left(\sum_{i=1}^n y_{\lambda_i} z_i \right).$$

The result follows by setting $x_\lambda := \sum_{i=1}^n y_{\lambda_i} z_i$. □

The fact that Condition (4) of Theorem 3.A is satisfied for the Benson-Solomon fusion system is now just a special case of the following.

Proposition 3.5.10. *Let $\mathcal{R}$ be a commutative ring, let S be a p-group, let $\mathcal{F}$ be a fusion system over S, let $\mathcal{F}_1, \mathcal{F}_2$ be fusion subsystems of $\mathcal{F}$ over S, let $\boldsymbol{F} := \{\mathcal{F}_1, \mathcal{F}_2\}$, let $\mathcal{C}$ be the collection of all $\mathcal{F}$-centric subgroups of S and, with notation as in Definition 3.4.2, define $f : \underline{\mathcal{R}}_{\mathcal{F}_S(S)}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})} \to \underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}$ by setting*

$$f\left(x \otimes_{\mathcal{R}\mathcal{O}_{\mathcal{C}}(\mathcal{F}_S(S))} y\right) := \sum_{i=1}^2 (-1)^i x \otimes_{\mathcal{R}\mathcal{O}_{\mathcal{C}}(\mathcal{F}_i)} y,$$

for every $x \in \underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F}_S(S))} \subseteq \underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F}_1)}, \underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F}_2)}$ *and every* $y \in \mathcal{RO}_{\mathcal{C}}(\mathcal{F})$*. The following are satisfied*

(1) For every $\mathcal{RO}(\mathcal{F}^c)$*-module* N *the image* $f^*\left(\mathrm{Hom}_{\mathcal{RO}_{\mathcal{C}}(\mathcal{F})}\left(\underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)}, N\right)\right)$ *equals the image* $\tilde{f}^*\left(\mathrm{Hom}_{\mathcal{RO}_{\mathcal{C}}(\mathcal{F})}(\mathrm{im}(f), N)\right)$ *where* $\tilde{f} : \underline{\mathcal{R}}_{\mathcal{F}_S(S)}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})} \twoheadrightarrow \mathrm{im}(f)$ *is the morphism obtained by viewing* f *as an epimorphism onto its image.*

(2) If $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *generate* $\mathcal{F}$ *then* $\mathrm{im}(f) = \ker(\theta_{\boldsymbol{F}})$ *(see Definition 3.4.2). In particular, from Lemma 3.5.8 and Part (1) we have that* $\boldsymbol{F}$ *satisfies the lifting property (see Definition 3.4.3).*

Proof. In order to simplify notation throughout this proof we will denote $\mathrm{Hom}_{\mathcal{RO}_{\mathcal{C}}(\mathcal{F})}$ simply as Hom.

(1) Let $\iota : \mathrm{im}(f) \hookrightarrow \underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)}$ denote the natural inclusion. By definition of $\tilde{f}$ we have that $f = \iota\tilde{f}$ and, therefore

$$f^*\left(\mathrm{Hom}\left(\underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)}, N\right)\right) = \tilde{f}^*\left(\iota^*\left(\mathrm{Hom}\left(\underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)}, N\right)\right)\right) \subseteq \tilde{f}^*\left(\mathrm{Hom}(\mathrm{im}(f), N)\right).$$

In order to prove the converse inclusion we need to prove that for every morphism $\varphi \in \mathrm{Hom}(\mathrm{im}(f), N)$ there exists a morphism $\hat{\varphi} \in \mathrm{Hom}\left(\underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)}, N\right)$ such that $\hat{\varphi}f = \varphi\tilde{f}$. With notation as in Lemma 3.5.9 let $1_{\mathcal{H}} := 1_S \in \mathcal{RO}_{\mathcal{C}}(\mathcal{H})$ for every $\mathcal{H} \in \{\mathcal{F}_S(S), \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}\}$ and define $\hat{\varphi} : \underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)} \to N$ by setting

$$\hat{\varphi}\left(1_{\mathcal{F}_1} \otimes_{\mathcal{RO}_{\mathcal{C}}(\mathcal{F}_1)} \mathrm{Id}_S\right) := \varphi\left(\tilde{f}\left(-1_{\mathcal{F}_S(S)} \otimes_{\mathcal{RO}_{\mathcal{C}}(\mathcal{F}_S(S))} \mathrm{Id}_S\right)\right), \qquad \text{and}$$

$$\hat{\varphi}\left(1_{\mathcal{F}_2} \otimes_{\mathcal{RO}_{\mathcal{C}}(\mathcal{F}_2)} \mathrm{Id}_{S_2}\right) := 0.$$

From Lemma 3.5.9 this defines a morphism $\hat{\varphi} \in \mathrm{Hom}\left(\underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}(\mathcal{F}^c)}, N\right)$. If we now view $\mathcal{RO}_{\mathcal{C}}(\mathcal{F}_S(S))$, $\mathcal{RO}_{\mathcal{C}}(\mathcal{F}_1)$ and $\mathcal{RO}_{\mathcal{C}}(\mathcal{F}_2)$ as subrings of $\mathcal{RO}_{\mathcal{C}}(\mathcal{F})$ we have that $1_{\mathcal{F}_S(S)} = 1_{\mathcal{F}_1} = 1_{\mathcal{F}_2} = 1_{\mathcal{F}}$ and therefore

$$\begin{aligned}
\hat{\varphi}\left(f\left(1_{\mathcal{F}_S(S)} \otimes_{\mathcal{RO}_{\mathcal{C}}(\mathcal{F}_S(S))} \mathrm{Id}_S\right)\right) &= \hat{\varphi}\left(\sum_{i=1}^{2}(-1)^i 1_{\mathcal{F}_i} \otimes_{\mathcal{RO}_{\mathcal{C}}(\mathcal{F}_i)} \mathrm{Id}_S\right), \\
&= \hat{\varphi}\left(-1_{\mathcal{F}_1} \otimes_{\mathcal{RO}_{\mathcal{C}}(\mathcal{F}_1)} \mathrm{Id}_S\right), \\
&= \varphi\left(\tilde{f}\left(1_{\mathcal{F}_S(S)} \otimes_{\mathcal{RO}_{\mathcal{C}}(\mathcal{F}_S(S))} \mathrm{Id}_S\right)\right).
\end{aligned}$$

From Lemma 3.5.9 the above proves that $\hat{\varphi}f = \varphi\tilde{f}$. Part (1) follows.

(2) By definition of f and $\theta_{\boldsymbol{F}}$ we have that

$$\theta_{\boldsymbol{F}}\left(f\left(1_{\mathcal{F}_S(S)}\otimes_{\mathcal{R}\mathcal{O}_{\mathcal{C}}(\mathcal{F}_S(S))}\mathrm{Id}_S\right)\right)=\sum_{i=1}^{2}(-1)^i\,1_{\mathcal{F}_i}\cdot\mathrm{Id}_S=-1_{\mathcal{F}}+1_{\mathcal{F}}=0.$$

Because of Lemma 3.5.9 we can deduce from the above that $\mathrm{im}\,(f)\subseteq\ker\,(\theta_{\boldsymbol{F}})$.

In order to prove the converse inclusion observe that $\ker\,(\theta_{\boldsymbol{F}})$ and $\mathrm{im}\,(f)$ decompose as direct sums of $\mathcal{R}$-submodules

$$\ker\,(\theta_{\boldsymbol{F}})=\bigoplus_{P\in\mathcal{C}}\ker\,(\theta_{\boldsymbol{F}})\,\mathrm{Id}_P\qquad\text{and}\qquad\mathrm{im}\,(f)=\bigoplus_{P\in\mathcal{C}}\mathrm{im}\,(f)\,\mathrm{Id}_P\,.$$

From this decomposition we deduce that, in order to prove the inclusion $\ker\,(\theta_{\boldsymbol{F}})\subseteq\mathrm{im}\,(f)$, it suffices to prove the inclusion $\ker\,(\theta_{\boldsymbol{F}})\,\mathrm{Id}_P\subseteq\mathrm{im}\,(f)\,\mathrm{Id}_P$ for every $P\in\mathcal{C}$.

Fix $P\in\mathcal{C}$ and for every $i=1,2$ let $I_{\mathcal{F}_i}:=I^{\mathcal{F}}_{\mathcal{F}_i,P}$ be as in Lemma 3.4.4. Then we have the following decomposition of $\underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}\,\mathrm{Id}_P$ as a direct sum of $\mathcal{R}$-submodules

$$\underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}\,\mathrm{Id}_P\cong\bigoplus_{i=1}^{2}\bigoplus_{\varphi\in I_{\mathcal{F}_i}}\underline{\mathcal{R}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F}_i)}\,\mathrm{Id}_{\varphi(P)}\,. \tag{3.9}$$

In other words, with notation as in Lemma 3.5.9 we can write every $x\in\underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}\,\mathrm{Id}_P$ in a unique way as

$$x=\sum_{i=1}^{2}\sum_{\varphi\in I_{\mathcal{F}_i}}1_{\varphi,\mathcal{F}_i}\,x_{\mathcal{F}_i,\varphi},$$

where for every $i=1,2$ and every $\varphi\in I_{\mathcal{F}_i}$ we are taking $x_{\mathcal{F}_i,\varphi}\in\mathcal{R}$ and we define $1_{\varphi,\mathcal{F}_i}:=1_{\varphi(P)}\otimes\overline{\varphi}$ where $\overline{\varphi}\in\mathcal{R}\mathcal{O}_{\mathcal{C}}\,(\mathcal{F})$ is the $\mathcal{R}$-basis element corresponding to the morphism in $\mathcal{O}_{\mathcal{C}}\,(\mathcal{F})$ with representative φ. It is now immediate from definition of $\theta_{\boldsymbol{F}}$ (see Definition 3.4.2) that

$$\ker\,(\theta_{\boldsymbol{F}})\,\mathrm{Id}_P=\left\{x\in\underline{\mathcal{R}}_{\boldsymbol{F}}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}\,\mathrm{Id}_P\;:\;\sum_{i=1}^{2}\sum_{\varphi\in I_{\mathcal{F}_i}}x_{\mathcal{F}_i,\varphi}=0\right\}.$$

Let us assume without loss of generality that $\mathrm{Id}_P\in I_{\mathcal{F}_1}\cap I_{\mathcal{F}_2}$. From the above we conclude that, in order to prove the inclusion $\ker\,(\theta)\,\mathrm{Id}_P\subseteq\mathrm{im}\,(f)\,\mathrm{Id}_P$, it suffices to prove that $1_{\varphi,\mathcal{F}_i}\lambda\in1_{\mathrm{Id}_P,\mathcal{F}_1}\lambda+\mathrm{im}\,(f)$ for every $\lambda\in\mathcal{R}$, every $i=1,2$ and every $\varphi\in I_{\mathcal{F}_i}$.

Using again the notation of Lemma 3.4.4 fix $I_{\mathcal{F}_S(S)}:=I^{\mathcal{F}}_{\mathcal{F}_S(S),P}$ and for every $\varphi\in I_{\mathcal{F}_S(S)}$ define $1_{\varphi,\mathcal{F}_S(S)}\in\underline{\mathcal{R}}_{\mathcal{F}_S(S)}^{\mathcal{O}_{\mathcal{C}}(\mathcal{F})}$ as before. We can assume without loss of generality that $\mathrm{Id}_P\in I_{\mathcal{F}_S(S)}$. Since both $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems over S then $\mathcal{F}_S\,(S)\subseteq\mathcal{F}_1,\mathcal{F}_2$. Therefore, for every $i=1,2$ and every $\varphi\in I_{\mathcal{F}_S(S)}$ there exist a unique $\varphi^{(i)}\in I_{\mathcal{F}_i}$ and a unique isomorphism ψ in $\mathcal{F}_i$ satisfying $\varphi^{(i)} = \psi\varphi$. We conclude that for every $\lambda \in \mathcal{R}$ and every $\varphi \in I_{\mathcal{F}_S(S)}$ the following holds

$$f\left(1_{\varphi,\mathcal{F}_S(S)}\lambda\right) = \sum_{i=1}^{2} (-1)^i \, 1_{\varphi^{(i)},\mathcal{F}_i}\lambda.$$

Fix $\lambda \in \mathcal{R}$ and $\varphi' \in I_{\mathcal{F}_2}$. Since $\mathcal{F}_S(S) \subseteq \mathcal{F}_2$ then there exists $\varphi \in I_{\mathcal{F}_S(S)}$ such that $\varphi^{(2)} = \varphi'$. Moreover, since $\mathcal{F}_1$ and $\mathcal{F}_2$ generate $\mathcal{F}$ we know that there exist $n \in \mathbb{N}$ and isomorphisms $\varphi_{j,i}$ in $\mathcal{F}_j$ for $j = 1,2$ and $i = 1, \ldots, n$ such that

$$\varphi = \varphi_{2,n}\varphi_{1,n} \cdots \varphi_{2,1}\varphi_{1,1}.$$

For every $i = 1, \ldots, n$ define

$$\varphi^{2,i} := \varphi_{2,i}\varphi_{1,i} \cdots \varphi_{2,1}\varphi_{1,1}, \qquad \varphi^{1,i} := \varphi_{1,i} \cdots \varphi_{2,1}\varphi_{1,1}.$$

By composing with isomorphisms in $\mathcal{F}_S(S)$ we can assume without loss of generality that $\varphi^{1,i}, \varphi^{2,i} \in I_{\mathcal{F}_S(S)}$ for every $i = 1, \ldots, n$. Since $\varphi_{2,i}$ is an isomorphism in $\mathcal{F}_2$ and $\varphi_{1,i}$ is an isomorphism in $\mathcal{F}_1$ for every $i = 1, \ldots, n$, by definition, we have that

$$\left(\varphi^{2,i}\right)^{(2)} = \left(\varphi^{1,i}\right)^{(2)}, \qquad \left(\varphi^{1,i}\right)^{(1)} = \left(\varphi^{2,i-1}\right)^{(1)}.$$

Here we take $\varphi^{2,i-1} = \mathrm{Id}_P$. With this setup we can conclude that

$$\begin{aligned}
1_{\varphi',\mathcal{F}_2}\lambda = 1_{\varphi^{(2)},\mathcal{F}_2}\lambda &\in 1_{\varphi^{(2)},\mathcal{F}_2}\lambda + \mathrm{im}(f), \\
&= 1_{(\varphi^{2,n})^{(2)},\mathcal{F}_2}\lambda + f\left(-1_{\varphi^{1,n},\mathcal{F}_S(S)}\lambda\right) + \mathrm{im}(f), \\
&= 1_{(\varphi^{1,n})^{(2)},\mathcal{F}_2}\lambda + 1_{(\varphi^{1,n})^{(1)},\mathcal{F}_1}\lambda - 1_{(\varphi^{1,n})^{(2)},\mathcal{F}_2}\lambda + \mathrm{im}(f), \\
&= 1_{(\varphi^{1,n})^{(1)},\mathcal{F}_1}\lambda + \mathrm{im}(f), \\
&= 1_{(\varphi^{1,n})^{(1)},\mathcal{F}_1}\lambda + f\left(1_{\varphi^{2,n-1},\mathcal{F}_S(S)}\lambda\right) + \mathrm{im}(f), \\
&= 1_{(\varphi^{2,n-1})^{(2)},\mathcal{F}_2}\lambda + \mathrm{im}(f), \\
&\;\;\vdots \\
&= 1_{(\varphi_{1,1})^{(1)},\mathcal{F}_1}\lambda + f\left(1_{\mathrm{Id}_P,\mathcal{F}_S(S)}\lambda\right) + \mathrm{im}(f), \\
&= 1_{\mathrm{Id}_P,\mathcal{F}_2}\lambda + \mathrm{im}(f), \\
&= 1_{\mathrm{Id}_P,\mathcal{F}_2}\lambda + f\left(-1_{\mathrm{Id}_P,\mathcal{F}_S(S)}\lambda\right) + \mathrm{im}(f), \\
&= 1_{\mathrm{Id}_P,\mathcal{F}_1}\lambda + \mathrm{im}(f).
\end{aligned}$$

The same arguments can be used in order to prove that $1_{\psi,\mathcal{F}_1}\lambda \in 1_{\mathrm{Id}_P,\mathcal{F}_1}\lambda + \mathrm{im}\,(f)$ for every $\psi \in I_{\mathcal{F}_1}$ and every $\lambda \in \mathcal{R}$. Part (2) follows from the previous arguments. □

As a corollary of Proposition 3.5.10 and Theorem 3.A we can now prove Theorem 3.C.

Proof. (of Theorem 3.C). Assume the notation and hypothesis of Theorem 3.C. Let $I = \{1, 2\}$. Since each $\mathcal{F}_i$ is a fusion system over S then Condition (1) of Theorem 3.A is satisfied by Lemma 3.5.7. Conditions (2) and (3) of Theorem 3.A are satisfied because of Conditions (2) and (3) of Theorem 3.C. Finally Condition (4) of Theorem 3.A is satisfied because of Condition (1) of Theorem 3.C and Proposition 3.5.10 (2). Hence we can apply Theorem 3.A and the result follows. □

We conclude this section with the proof of Theorem 3.B.

Proof. (of Theorem 3.B). Assume the notation of Definition 3.5.1. It suffices to prove that we can apply Theorem 3.C with $\mathcal{F} := \mathcal{F}_{\mathrm{Sol}}\,(q^n)$, $\mathcal{F}_1 := \mathcal{F}_1\,(q^n)$ and $\mathcal{F}_2 := \mathcal{F}_2\,(q^n)$. From Lemmas 3.5.3 (1) and 3.5.6 (1) we know that both $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems over S. From Lemma 3.5.2 we know that Condition (1) of Theorem 3.C is satisfied. Finally from Lemmas 3.5.3 and 3.5.6 we know that Conditions (2) and (3) of Theorem 3.C are satisfied. The result follows by applying Theorem 3.C with the above setup. □

3.A Cohomology spectral sequences

In this appendix we recall the definition of a spectral sequence and prove some of the results that are left as an exercise in [We94, Sections 5.1 and 5.2]. These will help us provide a description of the page 2 differentials of a spectral sequence (see [We94, Exercise 5.1.2]) and obtain short and long exact sequences relating page 2 and page 3 terms of certain spectral sequences (see [We94, Exercises 5.2.1 and 5.2.2]).

Let us start by recalling the definition of a spectral sequence.

Definition 3.A.1. Let $\mathcal{A}$ be an abelian category and let a be a non negative integer. A **cohomology spectral sequence in $\mathcal{A}$ starting at a** is a family of objects $\{E_k^{i,j}\}_{\substack{i,j,k\in\mathbb{Z}\\ k\geq a}}$ together with maps $d_k^{i,j} : E_k^{i,j} \to E_k^{i+k,j-k+1}$ for every i, j and k such that:

(1) For every i, j and k the composition $d_k^{i+k,j-k+1} d_k^{i,j}$ is zero.

(2) For every i, j and k there is an isomorphism

$$E_{k+1}^{i,j} \cong \ker\left(d_k^{i,j}\right) / \operatorname{im}\left(d_k^{i-k,j+k-1}\right).$$

For every $k \geq a$ we call **page k elements** the elements $E_k^{i,j}$ and **page k differentials** the maps $d_k^{i,j}$. We usually denote a cohomology spectral sequence in $\mathcal{A}$ starting at a with elements $E_k^{i,j}$ and differentials $d_k^{i,j} : E_k^{i,j} \to E_k^{i+k,j-k+1}$ simply by its page a elements (i.e. $E_a^{i,j}$).

We are mostly interested in spectral sequences that converge in the following sense.

Definition 3.A.2. Let $\mathcal{A}$ be an abelian category, let a be a non negative integer, let $E_a^{i,j}$ be a cohomology spectral sequence in $\mathcal{A}$ starting at a and let $H^* := \{H^n\}_{n\in\mathbb{Z}}$ be a family of objects in $\mathcal{A}$. We say that $E_a^{i,j}$ **converges to** H^* (denoted as $E_a^{i,j} \Rightarrow H^{i+j}$) if:

- For every $i, j \in \mathbb{Z}$ there exists $N^{i,j} \in \mathbb{Z}$ and $E_\infty^{i,j} \in \mathcal{A}$ such that for every $n \geq N^{i,j}$ then $E_\infty^{i,j} \cong E_k^{i,j}$.
- For every $n \in \mathbb{Z}$ there exists a filtration of H^n

  $$\cdots \subseteq F_n^u H^n \subseteq F_n^{u-1} H^n \subseteq \cdots,$$

  such that

  $$\varprojlim\left(F_n^* H^n\right) = 0, \qquad \varinjlim\left(F_n^* H^n\right) = H^n,$$

  and for every $i, j \in \mathbb{Z}$ there exists an isomorphism

  $$E_\infty^{i,j} \cong F_{i+j}^i H^{i+j} / F_{i+j}^{i+1} H^{i+j}.$$

It would be quite impractical if, in order to describe any spectral sequence, it was necessary to define the differentials at each individual page. The following Proposition tells us that, when a spectral sequence arises from a double cochain (resp. chain) complex, it is in fact only necessary to describe the differentials at the first two pages.

Proposition 3.A.3 ([We94, Exercise 5.1.2]). *Let $\mathcal{R}$ be a ring and let $\left\{\left(C^{i,j}, d_v^{i,j}, d_h^{i,j}\right)\right\}_{i,j\in\mathbb{Z}}$ be a double cochain complex in $\mathcal{R}$-Mod (see Definition 3.2.21). Define $E_0^{i,j}$ as the cohomology spectral sequence in $\mathcal{A}$ starting at 0 and such that $E_0^{i,j} := C^{i,j}$, $d_0^{i,j} := d_v^{i,j}$, $E_1^{i,j} := \ker\left(d_v^{i,j}\right)/\operatorname{im}\left(d_v^{i,j-1}\right)$ and $d_1^{i,j} : E_1^{i,j} \to E_1^{i,j+1}$ is the map induced by $d_h^{i,j}$ (anti-commutativity of the vertical and horizontal maps ensures that $d_1^{i,j}$ is well defined). Then we have an isomorphism of $\mathcal{R}$-modules*

$$E_2^{i,j} \cong \frac{\left\{(x,y)\in E_0^{i,j}\times E_0^{i+1,j-1} \,:\, d_v^{i,j}(x) = d_h^{i,j}(x) + d_v^{i+1,j-1}(y) = 0\right\}}{\left\{\left(d_v^{i,j-1}(a) + d_h^{i-1,j}(c), d_h^{i,j-1}(a) + b\right) \,:\, (a,b,c)\in \mathrm{K}\right\}},$$

where

$$\mathrm{K} := E_0^{i,j-1}\times \ker\left(d_v^{i+1,j-1}\right)\times \ker\left(d_v^{i-1,j}\right),$$

and we can define $d_2^{i,j} : E_2^{i,j} \to E_2^{i+2,j-1}$ by setting $d_2^{i,j}\left(\overline{(x,y)}\right) = \overline{\left(d_h^{i+1,j-1}(y), 0\right)}$. For all spectral sequences in $\mathcal{R}$-Mod arising in this way from a double cochain complex we take this to be the differential at page 2.

Proof. From Definition 3.A.1 we have that

$$\begin{aligned} E_1^{i,j} &\cong \ker\left(d_0^{i,j}\right)/\operatorname{im}\left(d_0^{i,j-1}\right), \\ &= \left\{x\in E_0^{i,j} \,:\, d_v^{i,j}(x) = 0\right\} / \left\{d_v^{i,j-1}(a) \,:\, a\in E_0^{i,j-1}\right\}, \\ &\cong \frac{\left\{(x,y)\in E_0^{i,j}\times E_0^{i+1,j-1} \,:\, d_v^{i,j}(x)=0\right\}}{\left\{\left(d_v^{i,j-1}(a), b\right) \,:\, (a,b)\in E_0^{i,j-1}\times E_0^{i+1,j-1}\right\}}, \end{aligned}$$

Using this isomorphism we can redefine $d_1^{i,j}$ by setting $d_1^{i,j}\left(\overline{(x,y)}\right) = \overline{\left(d_h^{i,j}(x), 0\right)}$. With this in mind we can now conclude that

$$\ker\left(d_1^{i,j}\right) = \frac{\left\{(x,y)\in E_0^{i,j}\times E_0^{i+1,j-1} \,:\, d_v^{i,j}(x) = d_h^{i,j}(x) + d_v^{i+1,j-1}(y) = 0\right\}}{\left\{\left(d_v^{i,j-1}(a), b\right) : (a,b)\in E_0^{i,j-1}\times E_0^{i+1,j-1} \text{ s.t. } d_h^{i,j}\left(d_v^{i,j-1}(a)\right) + d_v^{i+1,j-1}(b) = 0\right\}}.$$

From anti-commutativity of the cochain complex differentials we now have that

$$\begin{aligned} d_h^{i,j}\left(d_v^{i,j-1}(a)\right) + d_v^{i+1,j-1}\left(b + d_h^{i,j-1}(a)\right) &= d_v^{i+1,j-1}(b) + d_h^{i,j}\left(d_v^{i,j-1}(a)\right) + \\ &\quad + d_v^{i+1,j-1}\left(d_h^{i,j-1}(a)\right), \\ &= d_v^{i+1,j-1}(b). \end{aligned}$$

Therefore, by redefining $b := b + d_h^{i,j-1}(a)$, we can rewrite

$$\ker\left(d_1^{i,j}\right) = \frac{\left\{(x,y) \in E_0^{i,j} \times E_0^{i+1,j-1} \,:\, d_v^{i,j}(x) = d_h^{i,j}(x) + d_v^{i+1,j-1}(y) = 0\right\}}{\left\{\left(d_v^{i,j-1}(a), d_h^{i,j-1}(a) + b\right) \,:\, (a,b) \in E_0^{i,j-1} \times \ker\left(d_v^{i+1,j-1}\right)\right\}}.$$

The first part of the statement follows from the above, the third isomorphism theorem and the isomorphism $E_2^{i,j} \cong \ker\left(d_1^{i,j}\right) / \operatorname{im}\left(d_1^{i-1,j}\right)$ given in Definition 3.A.1.

To prove the rest of the statement first notice that, whenever the differentials $d_2^{i,j}$ are well defined, we have that $d_2^{i+2,j-1} d_2^{i,j} = 0$. Therefore we are only left with proving that the differential $d_2^{i,j}$ is well defined for every $i,j \in \mathbb{Z}$. For every $x \in E_0^{i,j}$ and $y \in E_0^{i+1,j-1}$ such that $d_v^{i,j}(x) = d_h^{i,j}(x) + d_v^{i+1,j-1}(y) = 0$ we can use anti-commutativity of the cochain complex differentials to deduce that

$$\begin{aligned} d_v^{i+2,j-1}\left(d_h^{i+1,j-1}(y)\right) = -d_h^{i+1,j}\left(d_v^{i+1,j-1}(y)\right) = d_h^{i+1,j}\left(d_h^{i,j}(x)\right) = 0, \\ d_h^{i+2,j-1}\left(d_h^{i+1,j-1}(y)\right) + d_v(0) = 0. \end{aligned}$$

Therefore, denoting with an overline $(\bar{\cdot})$ the equivalence class of the appropriate elements, we have that $d_2^{i,j}\left(\overline{(x,y)}\right) \in E_2^{i+2,j-1}$ for every $\overline{(x,y)} \in E_2^{i,j}$. We are now only left with proving that $d_2^{i,j}\left(\overline{(x,y)}\right)$ does not depend on the choice of representative (x,y) of $\overline{(x,y)}$. For every $(a,b,c) \in E_0^{i,j-1} \times \ker\left(d_v^{i+1,j-1}\right) \times \ker\left(d_v^{i-1,j}\right)$ let $a' := b' := 0$ and $c' := b \in \ker\left(d_v^{(i+2)-1,j-1}\right)$. Then we have that

$$\begin{aligned} \left(d_h^{i+1,j-1}\left(d_h^{i,j-1}(a) + b\right), 0\right) &= \left(d_h^{i+1,j-1}(b), 0\right), \\ &= \left(d_h^{i+1,j-1}(c'), 0\right), \\ &= \left(d_v^{i+2,j-2}(a') + d_h^{i+1,j-1}(c'), d_h^{i+2,j-2}(a') + b'\right). \end{aligned}$$

This proves that $d_2^{i,j}$ does not depend on the choice of representative thus concluding the proof. □

If a convergent cohomology spectral sequence starting at 2 has only two non zero rows then there exist a short exact sequence relating its page 3 elements and a long exact sequence relating its page 2 elements. More precisely we have the following.

Proposition 3.A.4 ([We94, Exercise 5.2.2]). *Let $\mathcal{A}$ be an abelian category, let $E_2^{i,j}$ be a cohomology spectral sequence in $\mathcal{A}$ starting at 2 and let $H^* = \{H^n\}_{n\in\mathbb{Z}}$ a family of* *elements in $\mathcal{A}$ such that $E_2^{i,j} \Rightarrow H^{i+j}$. If $E_k^{i,j} = 0$ whenever $j \notin \{0,1\}$ then there exists a long exact sequence of the form*

$$\cdots \longrightarrow H^{n-1} \longrightarrow E_2^{n-2,1} \xrightarrow{d_2^{n-2,1}} E_2^{n,0} \longrightarrow H^n \longrightarrow E_2^{n-1,1} \xrightarrow{d_2^{n-1,1}} E_2^{n+1,0} \longrightarrow H^{n+1} \longrightarrow \cdots .$$

Moreover, for every $n \in \mathbb{Z}$, there exists a short exact sequence of the form

$$0 \to E_3^{n,0} \to H^n \to E_3^{n-1,1} \to 0.$$

Proof. We know by definition that for every $i, j, k \in \mathbb{Z}$ with $k \geq 2$ the differential $d_k^{i,j}$ maps the page k element $E_k^{i,j}$ to the page k element $E_k^{i+k,j-k+1}$. Since $E_k^{i,j} = 0$ whenever $j \notin \{0,1\}$ this implies that the differential $d_k^{i,j}$ is the zero map for every $k \geq 3$. In particular we have that $\ker\left(d_k^{i,j}\right) = E_k^{i,j}$ and that $\operatorname{im}\left(d_k^{i-k,j+k-1}\right) = 0$ for every $k \geq 3$ and, therefore

$$E_{k+1}^{i,j} \cong \ker\left(d_k^{i,j}\right) / \operatorname{im}\left(d_k^{i-k,j+k-1}\right) = E_k^{i,j}/0 \cong E_k^{i,j}.$$

We can therefore conclude that, with notation as in Definition 3.A.2, we have $E_\infty^{i,j} = E_3^{i,j}$ for every $i, j \in \mathbb{Z}$. In particular, for every $n \in \mathbb{Z}$ there exists a filtration

$$\cdots \hookrightarrow F_n^{n-u} H^n \hookrightarrow F_n^{n-(u+1)} H^n \hookrightarrow \cdots , \tag{3.10}$$

such that

$$\varprojlim \left(F_n^* H^n\right) = 0, \qquad \varinjlim \left(F_n^* H^n\right) = H^n.$$

and for every $i, j \in \mathbb{Z}$ there exists an isomorphism

$$E_3^{i,j} \cong F_{i+j}^{(i+j)-j} H^{i+j} / F_{i+j}^{(i+j)-(j-1)} H^{i+j}. \tag{3.11}$$

We can conclude from the above that for every $j \notin \{0,1\}$ and every $n \in \mathbb{Z}$ then

$$0 = E_3^{n-j,j} \cong F_n^{n-j} H^n / F_n^{n-(j-1)} H^n.$$

Equivalently, for every $j \notin \{0,1\}$ we have that

$$F_n^{n-j}H^n \cong F_n^{n-(j-1)}H^n.$$

Since $\varprojlim (F_n^* H^n) = 0$ and $\varinjlim (F_n^* H^n) = H^n$ then we can rewrite the filtration of Equation (3.10) as

$$\cdots \hookrightarrow 0 \hookrightarrow 0 = F_n^{n+1}H^n \hookrightarrow F_n^n H^n \hookrightarrow F_n^{n-1}H^n = H^n \hookrightarrow H^n \hookrightarrow \cdots . \tag{3.12}$$

We can therefore conclude from Equation (3.11) that for every $n \in \mathbb{Z}$ we have

$$E_3^{n,0} \cong F_n^n H^n / F_n^{n+1} H^n \cong F_n^n H^n / 0 \cong F_n^n H^n.$$

Applying again Equation (3.11) with the filtration of Equation (3.12) we obtain the isomorphism

$$E_3^{n-1,1} \cong F_n^{n-1}H^n / F_n^n H^n \cong H^n / E_3^{n,0}.$$

The short exact sequence of the statement follows

On the other hand we know by definition that

$$E_3^{n,0} \cong \ker\left(d_2^{n,0}\right) / \operatorname{im}\left(d_2^{n-2,1}\right), \qquad E_3^{n-1,1} \cong \ker\left(d_2^{n-1,1}\right) / \operatorname{im}\left(d_2^{n-3,2}\right).$$

Since $E_2^{n+2,-1} = E_2^{n-3,2} = 0$ then we can conclude that $\ker\left(d_2^{n,0}\right) = E_2^{n,0}$ and $\operatorname{im}\left(d_2^{n-3,2}\right) = 0$. Therefore, we can rewrite the above equivalences as

$$E_3^{n,0} \cong \mathsf{coker}\left(d_2^{n-2,1}\right), \qquad E_3^{n-1,1} \cong \ker\left(d_2^{n-1,1}\right).$$

We can therefore rewrite the short exact sequences of the statement as

$$0 \to \mathsf{coker}\left(d_2^{n-2,1}\right) \to H^n \to \ker\left(d_2^{n-1,1}\right) \to 0.$$

The long exact sequence of the statement follows from joining all such short exact sequences. □

Analogously to Proposition 3.A.4, whenever a cohomology spectral sequence has only two non zero columns, there exists a short exact sequence relating its page 2 elements. More precisely we have the following results with which we conclude this appendix.

Proposition 3.A.5 ([We94, Exercise 5.2.1]). *Let $\mathcal{A}, E_2^{i,j}$ and H^* be as in Proposition 3.A.4. If $E_k^{i,j} = 0$ for every $i \notin \{0,1\}$ then, for every $n \in \mathbb{Z}$ there exists a short exact sequence of the form*

$$0 \to E_2^{1,n-1} \to H^n \to E_2^{0,n} \to 0.$$

Proof. With arguments analogue to those in proof of Proposition 3.A.4 we can deduce that $d_k^{i,j} = 0$ whenever $k \geq 2$ and, therefore, $E_\infty^{i,j} \cong E_2^{i,j}$ for every $i, j \in \mathbb{Z}$. In particular, for every $n \in \mathbb{Z}$ there exists a filtration

$$\cdots \hookrightarrow F_n^u H^n \hookrightarrow F_n^{u-1} H^n \hookrightarrow \cdots,$$

such that

$$\varprojlim (F_n^* H^n) = 0, \qquad \varinjlim (F_n^* H^n) = H^n.$$

and for every $i, j \in \mathbb{Z}$ there exists an isomorphism

$$E_2^{i,j} \cong F_{i+j}^i H^{i+j} / F_{i+j}^{i+1} H^{i+j}. \tag{3.13}$$

We can conclude from the above that for every $i \notin \{0,1\}$ and every $n \in \mathbb{Z}$ then $F_n^i H^n = F_n^{i+1} H^n$. We can therefore rewrite the filtration of Equation (3.13) as

$$\cdots \hookrightarrow 0 = F_n^2 H^n \hookrightarrow F_n^1 H^n \hookrightarrow F_n^0 H^n = H^n \hookrightarrow \cdots. \tag{3.14}$$

We can therefore conclude from Equation (3.13) that for every $n \in \mathbb{Z}$ then

$$E_2^{1,n-1} \cong F_n^1 H^n / F_n^2 H^n \cong F_n^1 H^n / 0 \cong F_n^1 H^n.$$

Applying once again Equation (3.13) we obtain from the above and Equation (3.14) that

$$E_2^{0,n} \cong F_n^0 H^n / F_n^1 H^n \cong H^n / E_2^{1,n-1}.$$

The short exact sequence of the statement follows. □

3.B Higher limits over $\mathbb{F}_p$-modules and $\mathbb{Z}_p$-modules

Let p be a prime, let $\boldsymbol{C}$ be a small category with finitely many objects, let $F : \boldsymbol{C} \to \mathbb{F}_p$-Mod be a functor and let $\iota : \mathbb{F}_p$-Mod $\to \mathbb{Z}_p$-Mod be the natural inclusion of categories. In this appendix we prove that $\lim^n_{\boldsymbol{C}} (F) \cong \lim^n_{\boldsymbol{C}} (\iota \circ F)$ as abelian groups for every non negative integer n. This seems to be widely used in the literature, but we were unable to find any reference proving it. Therefore we include a proof in this paper.

Let us start by relating free resolutions of $\mathbb{Z}_p$-modules with free resolutions of $\mathbb{F}_p$-modules.

Lemma 3.B.1. *Let $\boldsymbol{C}$ be a small category with finitely many objects, let $\mathcal{R}$ be a commutative ring, let $x \in \mathcal{R}$ be a non zero divisor, let M be an $\mathcal{R}\boldsymbol{C}$-module (see Definition 3.2.16) such that $m \cdot x \neq 0$ for every $m \in M$, let $F_* \xrightarrow{\varepsilon} M \to 0$ be a free resolution of M in $\mathcal{R}\boldsymbol{C}$ and define the quotient ring $\mathcal{S} := \mathcal{R}/x\mathcal{R}$. By viewing $\mathcal{S}\boldsymbol{C}$ as an $(\mathcal{R}\boldsymbol{C}, \mathcal{S}\boldsymbol{C})$-bimodule in the natural way we have that $F_* \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C} \xrightarrow{\varepsilon \otimes \mathrm{Id}_{\mathcal{S}\boldsymbol{C}}} M \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C} \to 0$ is a free resolution of the $\mathcal{S}\boldsymbol{C}$-module $M \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C}$.*

Proof. For every free $\mathcal{R}\boldsymbol{C}$-module $F \cong (\mathcal{R}\boldsymbol{C})^n$ we know that $F \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C} \cong (\mathcal{S}\boldsymbol{C})^n$ is a free $\mathcal{S}\boldsymbol{C}$-module. Therefore we only need to prove that the sequence in the statement is exact.

For every $\mathcal{R}\boldsymbol{C}$-module N let $x\cdot : N \to N$ denote the endomorphism of $\mathcal{R}\boldsymbol{C}$-modules corresponding to multiplication by x (seen as an element in $\mathcal{R}\boldsymbol{C}$). Since x is not a zero divisor of $\mathcal{R}$, is not a zero divisor of $\mathcal{R}\boldsymbol{C}$ either (when seen as an element in $\mathcal{R}\boldsymbol{C}$). Therefore, since each F_i is free we can deduce that the endomorphisms $x\cdot : F_i \to F_i$ are injective for every i. By definition of M we also know that the endomorphism $x\cdot : M \to M$ is injective. Take now the chain complex C_* defined by setting $C_{-1} = M$, $C_i = F_i$ for every $i \geq 0$ and $C_j = 0$ for every $j \leq -2$. Since $\mathcal{S}\boldsymbol{C} \cong \mathcal{R}\boldsymbol{C}/(\mathcal{R}\boldsymbol{C}x)$ by construction, viewing $\mathcal{S}\boldsymbol{C}$ as an $(\mathcal{R}\boldsymbol{C}, \mathcal{R}\boldsymbol{C})$-bimodule in the natural way, we obtain the exact sequence of chain complexes

$$0 \to C_* \xrightarrow{x\cdot} C_* \to (C_* \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C}) \to 0.$$

This leads (see [We94, Theorem 1.3.1]) to the long exact sequence in homology

$$\cdots \to H_1 (C_* \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C}) \to H_0 (C_*) \to H_0 (C_*) \to H_0 (C_* \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C}) \to 0.$$

Since $F_* \xrightarrow{\varepsilon} M \to 0$ is a free resolution of M, we have that it is a long exact sequence. Equivalently we have that $H_n (C_*) = 0$ for every $n \in \mathbb{Z}$. From the above long exact sequence we can therefore conclude that $H_n (C_* \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C}) = 0$ for every $n \in \mathbb{Z}$. Equivalently the sequence of $\mathcal{R}\boldsymbol{C}$-modules $F_* \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C} \xrightarrow{\varepsilon \otimes \mathsf{Id}_{\mathcal{S}\boldsymbol{C}}} M \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C}$ is exact. The result follows from viewing this sequence as a sequence of $\mathcal{S}\boldsymbol{C}$-modules. □

The following result, together with Lemma 3.B.1 allows us to switch from free resolution in $\mathbb{Z}_p\text{-Mod}$ to free resolutions in $\mathbb{F}_p\text{-Mod}$ when applying a certain contravariant Hom functor.

Lemma 3.B.2. *Let $\mathcal{R}$ be a ring, let I be a two sided ideal of $\mathcal{R}$, let $\mathcal{S} := \mathcal{R}/I$, let $\iota : \mathcal{S}\text{-Mod} \hookrightarrow \mathcal{R}\text{-Mod}$ be the natural inclusion of categories, let M be an $\mathcal{S}$-module and let N be an $\mathcal{R}$-module. Viewing $\mathcal{S}$ as an $(\mathcal{R}, \mathcal{S})$-bimodule in the natural way there exists an isomorphism of abelian groups $\mathrm{Hom}_{\mathcal{R}} (N, \iota (M)) \cong \mathrm{Hom}_{\mathcal{S}} (N \otimes_{\mathcal{R}} \mathcal{S}, M)$ which is natural in both N and M.*

Proof. The functor ι is in fact the restriction functor resulting from the projection $\pi : \mathcal{R} \to \mathcal{S}$. It is well known that such restriction functor is right adjoint to the functor $- \otimes_{\mathcal{R}} \mathcal{S} : \mathcal{R}\text{-Mod} \to \mathcal{S}\text{-Mod}$. More precisely there exists a natural bijection of sets $\Gamma : \mathrm{Hom}_{\mathcal{R}} (N, \iota (M)) \hookrightarrow\!\!\!\!\to \mathrm{Hom}_{\mathcal{S}} (N \otimes_{\mathcal{R}} \mathcal{S}, M)$ which sends every $f \in \mathrm{Hom}_{\mathcal{R}} (N, \iota (M))$ to the morphism $\Gamma (f) \in \mathrm{Hom}_{\mathcal{S}} (N \otimes_{\mathcal{R}} \mathcal{S}, M)$ defined by setting

$$\Gamma (f) (n \otimes \pi (x)) = f (n) \cdot \pi (x),$$

for every $n \in N$ and every $x \in \mathcal{R}$.

For every $f, g \in \mathrm{Hom}_{\mathcal{R}} (N, \iota (M))$, every $x \in \mathcal{R}$ and every $n \in N$ we have that

$$\begin{aligned} \Gamma (f + g) (n \otimes \pi (x)) &= (f + g) (n) \cdot \pi (x), \\ &= f (n) \cdot \pi (x) + g (n) \cdot \pi (x), \\ &= \Gamma (f) (n \otimes \pi (x)) + \Gamma (g) (n \otimes \pi (x)). \end{aligned}$$

We can therefore deduce that $\Gamma(f+g) = \Gamma(f) + \Gamma(g)$. Since Γ sends the zero morphism to the zero morphism this implies that Γ is in fact a morphism of abelian groups thus concluding the proof. □

As a consequence of Lemmas 3.B.1 and 3.B.2 we have the following.

Proposition 3.B.3. *Let $\mathcal{R}, \boldsymbol{C}, x$ and $\mathcal{S}$ be as in Lemma 3.B.1 and let $\iota : \mathcal{S}\text{-Mod} \to \mathcal{R}\text{-Mod}$ be the natural inclusion of categories. Then, for every non negative integer n and every contravariant functor $M : \boldsymbol{C}^{\mathsf{op}} \to \mathcal{S}\text{-Mod}$ there exists an isomorphism of abelian groups $\lim^n_{\boldsymbol{C}}(M) \cong \lim^n_{\boldsymbol{C}}(\iota \circ M)$ which is natural in M.*

Proof. Let $\underline{\mathcal{R}}^{\boldsymbol{C}}$ be as in Definition 3.2.18, let $F_* \xrightarrow{\varepsilon} \underline{\mathcal{R}}^{\boldsymbol{C}} \to 0$ be a free resolution of $\underline{\mathcal{R}}^{\boldsymbol{C}}$ in $\mathcal{R}\boldsymbol{C}\text{-Mod}$, denote by $d_n : F_{n+1} \to F_n$ its differentials and view $\iota \circ M : \boldsymbol{C}^{\mathsf{op}} \to \mathcal{R}\text{-Mod}$ as an $\mathcal{R}\boldsymbol{C}$-module (see Proposition 3.2.17). By definition of the $\mathrm{Ext}^n_{\mathcal{R}\boldsymbol{C}}$ groups, for every integer $n \geq 0$ we have the isomorphism of abelian groups

$$\mathrm{Ext}^n_{\mathcal{R}\boldsymbol{C}}\left(\underline{\mathcal{R}}^{\boldsymbol{C}}, \iota \circ M\right) \cong \frac{\ker\left(d^*_n : \mathrm{Hom}_{\mathcal{R}}(F_n, \iota \circ M) \to \mathrm{Hom}_{\mathcal{R}}(F_{n+1}, \iota \circ M)\right)}{\mathrm{im}\left(d^*_{n-1} : \mathrm{Hom}_{\mathcal{R}}(F_{n-1}, \iota \circ M) \to \mathrm{Hom}_{\mathcal{R}}(F_n, \iota \circ M)\right)},$$

where we take $d_{-1} := 0$ and $F_{-1} = 0$. Because of Lemma 3.B.2 we can obtain from the above the following isomorphism of abelian groups

$$\mathrm{Ext}^n_{\mathcal{R}\boldsymbol{C}}\left(\underline{\mathcal{R}}^{\boldsymbol{C}}, \iota \circ M\right) \cong \ker\left((d_n \otimes \mathrm{Id}_{\mathcal{S}\boldsymbol{C}})^*\right) / \,\mathrm{im}\left((d_{n-1} \otimes \mathrm{Id}_{\mathcal{S}\boldsymbol{C}})^*\right), \tag{3.15}$$

where

$$(d_n \otimes \mathrm{Id}_{\mathcal{S}\boldsymbol{C}})^* : \mathrm{Hom}_{\mathcal{S}}(F_n \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C}, M) \to \mathrm{Hom}_{\mathcal{S}}(F_{n+1} \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C}, M), \text{ and}$$
$$(d_{n-1} \otimes \mathrm{Id}_{\mathcal{S}\boldsymbol{C}})^* : \mathrm{Hom}_{\mathcal{S}}(F_{n-1} \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C}, M) \to \mathrm{Hom}_{\mathcal{S}}(F_n \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C}, M).$$

Since $\underline{\mathcal{R}}^{\boldsymbol{C}} \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C} \cong \underline{\mathcal{S}}^{\boldsymbol{C}}$ as $\mathcal{S}\boldsymbol{C}$-modules, from Lemma 3.B.1 we obtain the free resolution $F_* \otimes_{\mathcal{R}\boldsymbol{C}} \mathcal{S}\boldsymbol{C} \xrightarrow{\varepsilon \otimes \mathrm{Id}_{\mathcal{S}\boldsymbol{C}}} \underline{\mathcal{S}}^{\boldsymbol{C}} \to 0$ of $\underline{\mathcal{S}}^{\boldsymbol{C}}$ in $\mathcal{S}\boldsymbol{C}\text{-Mod}$. Therefore, from definition of the $\mathrm{Ext}^n_{\mathcal{S}\boldsymbol{C}}$ groups, we can rewrite Equation (3.15) as

$$\mathrm{Ext}^n_{\mathcal{R}\boldsymbol{C}}\left(\underline{\mathcal{R}}^{\boldsymbol{C}}, \iota \circ M\right) \cong \mathrm{Ext}^n_{\mathcal{S}\boldsymbol{C}}\left(\underline{\mathcal{S}}^{\boldsymbol{C}}, M\right).$$

The result follows from the above equivalence of abelian groups and [We07, Corollary

5.2]. ☐

As a corollary of Proposition 3.B.3 we obtain the result that motivates the introduction of this appendix.

Corollary 3.B.4. *Let $\boldsymbol{C}$ be a small category with finitely many objects, let p be a prime and let $\iota : \mathbb{F}_p\text{-Mod} \to \mathbb{Z}_p\text{-Mod}$ be the natural inclusion. Then, for every non negative integer n and every contravariant functor $M : \boldsymbol{C}^{op} \to \mathbb{F}_p\text{-Mod}$ there exists an isomorphism of abelian groups $\lim^n_{\boldsymbol{C}}(M) \cong \lim^n_{\boldsymbol{C}}(\iota \circ M)$ which is natural in M.*

Proof. This is just a particular case of Proposition 3.B.3 taken with $\mathcal{R} := \mathbb{Z}_p$ and $x := p$. ☐

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Chapter 4

Conclusions and further work

Let $\mathcal{F}$ be a fusion system. In this thesis we defined Mackey functors over $\mathcal{F}$ and $\mathcal{F}$-centric Mackey functors (also known as $\mathcal{F}^c$-restricted Mackey functors) (see Definitions 2.2.26 and 2.2.29). These definitions are in fact equivalent to those given in [DP15]. We then explored the properties of $\mathcal{F}$-centric Mackey functors proving that results like Higman's criterion and the Green correspondence, which are known to be satisfied for Mackey functors over groups (see [Sa82]), can be translated to $\mathcal{F}$-centric Mackey functors (see Theorems 2.3.17 and 2.4.38).

Moreover we proved results which contribute to proving the sharpness conjecture for fusion systems (see [DP15]) by both providing some tools to approach it (see Theorems 3.A and 3.C) and using such tools to prove that the conjecture is satisfied for the only known family of exotic fusion systems over 2-groups (see Theorem 3.B).

We would like to conclude this thesis by outlining a research project that could be pursued as a continuation of the results exposed during Chapter 3.

We believe there exist at least two potential methods of using Theorems 3.A and 3.C in order to make further progress towards proving the sharpness conjecture for fusion systems.

The first method was explained in Section 3.4. It is based on the observation that, because of [Br05, Proposition C], [DP15, Theorem B] and [Ya22, Lemma 10.4], then for any fusion system $\mathcal{F}$ over a p-group S and any family $\boldsymbol{P} := \{P_i\}_{i\in I}$ of fully $\mathcal{F}$-normalized and $\mathcal{F}$-centric subgroups of S the set $\boldsymbol{F} := \{N_{\mathcal{F}}(S)\} \cup \{N_{\mathcal{F}}(P_i) : P \in \boldsymbol{P}\}$ satisfies

Conditions (1)-(3) of Theorem 3.A. We suspect that, under some minimality conditions on $\boldsymbol{F}$, Condition (4) of Theorem 3.A is also satisfied. More precisely we conjecture that, if $\boldsymbol{P}$ is taken to be the family of all fully $\mathcal{F}$-normalized and $\mathcal{F}$-essential subgroups of S (see [Li07, Definition 5.1 (ii)]), then we can apply Theorem 3.A in order to prove that the sharpness conjecture holds for $\mathcal{F}$ (see Conjecture 3.4.6). We think that the main difficulty to overcome in attempting to prove the above is the varying nature and number of $\mathcal{F}$-essential subgroups depending on the fusion system $\mathcal{F}$. This variety might in fact lead to complications when attempting to obtain general results.

The second method is based on Theorem 3.C. With notation as before we know that Conditions (2) and (3) of Theorem 3.C are satisfied for any $\mathcal{F}_1 \in \boldsymbol{F}$ and $\mathcal{F}_2 := N_{\mathcal{F}}(S)$. We conjecture that, with some extra work, we can apply Theorem 3.C with $\mathcal{F}_1$ and $\mathcal{F}_2$ as above in order to prove the sharpness conjecture for the (non necessarily saturated) fusion system $\mathcal{F}'$ over S generated by $\mathcal{F}_1$ and $\mathcal{F}_2$. Some additional work might then allow us to repeat this process replacing $\mathcal{F}_2$ with $\mathcal{F}'$ and $\mathcal{F}_1$ with a different fusion system in $\boldsymbol{F}$. We know from Alperin's fusion theorem that the fusion systems in $\boldsymbol{F}$ generate $\mathcal{F}$. Therefore, repeating this process a finite number of times, we would prove that the sharpness conjecture is satisfied for $\mathcal{F}$. We believe that the main difficulties to overcome when trying to prove the above are:

- Unlike $N_{\mathcal{F}}(S)$ the fusion system $\mathcal{F}_1$ might not be a fusion system over S. This prevents us from applying Theorem 3.C with $\mathcal{F}_2 := N_{\mathcal{F}}(S)$. In order to obtain a similar result, major adaptations of the proof of Theorem 3.C might be necessary.
- The fusion system $\mathcal{F}'$ defined above might not be saturated. It is therefore necessary to retrace the steps leading to Theorem 3.C and adapt them to non saturated fusion systems.
- The fusion system $\mathcal{F}'$ defined above might not satisfy Condition (2) of Theorem 3.C. To solve this problem it might be necessary to replace Conditions (2) and (3) by the weaker condition "$\lim^n_{\mathcal{O}_{\mathcal{F}^c}(\mathcal{F}_i)}\left(M^* \downarrow^{\mathcal{O}(\mathcal{F})}_{\mathcal{O}_{\mathcal{F}^c}(\mathcal{F}_i)}\right) = 0$ for every $n \geq 1$, every $i = 1, 2$ and every Mackey functor $M = (M_*, M^*)$ over $\mathcal{F}$". This relaxation of conditions might lead to a less powerful result.

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