Resonant Separatrix Activation in Weakly-Dissipative Systems

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***Abstract*—We report a new phenomenon, calling it resonant separatrix activation (RSA): a significant decrease of the activation barrier due to a moderately weak periodic drive of a proper frequency. The observed decrease greatly exceeds the maximum decrease caused with the adiabatic drive of the same amplitude. The RSA is observed in weakly-dissipative systems possessing two or more saddles. The mechanism is a strong resonant quasi-regular variation of energy within the area of transient chaos related to the saddles-associated separatrix(ces) of the undriven system. The most pronounced RSA occurs when the saddles have different energies, and this difference ΔE is comparable to the original activation barrier: the resonant drive provides the inter-separatrix transport, thus reducing the activation barrier for the value ΔE. The RSA can be observed in natural systems, e.g. in fluidic, atmospheric and planetary transport, and can be used in engineered systems with a large quality factor.**

***Index Terms*—weakly-dissipative systems, separatrix chaos, zero-dispersion resonance, resonance, noise-induced transitions**

I. Introduction

Weakly dissipative systems are abundant. Examples include particle transport in fluid, planetary and atmospheric transport, NEMS/MEMS, ferromagnetic and optomechanical systems. Studies of noise-induced transitions in them form a subject of a big scientific and practical interest (see, e.g. [1], [2] and references therein). An important case is that of weak noise: a multistable system stays most of time near one of the stable states while rarely transiting to the vicinity of a different state. The transition rate r characterises the state's stability and depends on the effective "temperature" T in an activation-like fashion [1], [2]:

$$r \propto \exp\left(-\frac{S}{T}\right) \ , \quad T \ll S \ , \tag{1}$$

where S is often called an activation barrier. The activation barrier depends both on the properties of noise and on noise-free dynamics. In the archetypical case of Gaussian white noise in a potential system subjected to linear friction, the activation barrier merely coincides with the potential barrier. For the activation control, one can apply an external periodic drive. There were many works on various types of such a control. In particular, such control is exploited in stochastic resonance (see, e.g. [3] and references therein). However, a weak drive only causes there a weak change in the activation barrier. Some resonant mechanism of the activation barrier control is suggested in [4], but if the drive is weak, the reduction of the barrier is still much smaller than the original barrier. The reduction of the activation barrier in an underdamped double-well potential system by means of a non-adiabatic periodic force due to the transient chaos was reported in [5]. However the magnitude of the reduction was of the order of the amplitude of the periodic force, i.e. it was again small if the force was weak.

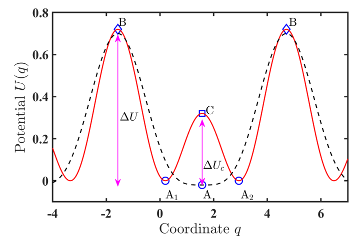

Fig. 1. Potential $U(q) = a(b - \sin(q))^2$ for two sets of parameters. The dashed black line shows $U(q)$ for $a = 0.18, b = 1$: it has one barrier per period. The solid red line shows $U(q)$ for $a = 0.5, b = 0.2$: it possesses an additional moderately small barrier Δ_c while the larger barrier is the same.

Here we report an observation of a *dramatic* decrease of the activation barrier by means of a *weak* drive, achieved if the drive frequency is properly tuned. We have uncovered the physical mechanism of the effect and partially developed its theory. The effect involves transient chaos, resulting in the fractal-like mixture of basins of various attractors. The transient chaos occurs around the destroyed separatrices of the undriven system. As it was shown before [6]–[9], the separatrix chaos in the periodically driven dissipationless systems may occupy a large area in the phase space. We show here that the great expansion of chaotic area still holds in the presence of a weak dissipation, with the only difference being that the transient chaos replaces the dissipationless chaos. It is, therefore, sufficient for the noise to bring the system to the lower boundary of the separatrix chaotic layer while the further transition beyond the separatrix occurs noise-free, which is equivalent to the decrease of the activation barrier. Altogether this gave us grounds to call the phenomenon *resonant separatrix activation* (RSA). The RSA can be used for controlling activation in engineered systems (e.g. memory cells), sensing a weak force and can be relevant for explaining transport in natural systems. Also, the effect is characterised by a novel type of escape process.

II. Model

To illustrate the RSA, we use the archetypal model of a multistable potential system [10] subjected to weak linear friction, weak white noise and weak periodic drive:

$$\begin{aligned} &\ddot{q}+\gamma\dot{q}+dU/dq=\sqrt{\gamma T}\xi(t)+h\cos(\omega_f t), \qquad (2)\\ &U(q)=a(b-\sin(q))^2, \qquad \gamma\ll\Omega\equiv\sqrt{a\max(1,b)},\\ &\langle\xi\rangle=0, \quad \langle\xi(t)\xi(0)\rangle=2\delta(t), \quad T\ll\Omega^2, \quad h\ll\Omega^2. \end{aligned}$$

Fig. 1 shows two potential profiles. If $h=0$, a transition from any of stable states (A or A1 and A2, respectively) beyond state B has the same activation barrier $S=\Delta U$.

The RSA is possible in both profiles because the necessary condition for the strong widening of the separatrix chaotic areas - a presence of at least two saddles [8], [9] - is satisfied in both cases. However, the RSA in the case with an additional barrier of a smaller height is more pronounced. It is due to the fact that the eigenfrequency ω vs energy $E\equiv U(q)+\dot{q}^2/2$ necessarily possesses a local maximum ω_m in the inter-separatrix range of energy [6], [8], [10] (Fig. 2). If $\omega_f\approx\omega_m$, the zero-dispersion nonlinear resonances [6], [8], [10], [11], which are wide in energy, arise, and they significantly facilitate the onset of the inter-separatrix transport. We consider below only such a case, i.e. $U(q)$ shown in Fig. 1 with the solid red line (see also Fig. 2(a)).

III. Results

We first discuss the case when only periodic driving is present while dissipation and noise are absent: $h\neq 0$, $\gamma=0$, $T=0$. Let us define an *inter-separatrix transport* (IST) as a possibility for the system to reach a given separatrix in a finite time while starting from any state on another separatrix. It can be shown that, for the system (2), the latter condition is equivalent to a possibility for the system starting from the inner saddle ($q=\pi/2, p\equiv\dot{q}=0$) to reach a coordinate of any of the two outer saddles, i.e. $-\pi/2$ or $3\pi/2$. Fig. 3 shows the bifurcation diagram in the plane of the driving parameters which is obtained by means of numerical integration of the equation of motion with the initial state in the inner saddle for a very long time: shaded and white areas correspond to the presence and absence of the IST respectively. Let us fix a given value of ω_f, and denote the lowest value of h at which the IST occurs as h_{IST}. As function of ω_f, h_{IST} has a very deep minimum at $\omega_f\approx 0.401$ (which is a little smaller than the maximum eigenfrequency $\omega_m\approx 0.425$): $h_{IST}(\omega_f=0.401)\approx 0.0048$ [9]. It is almost 30 times smaller than the minimal value of the *adiabatic* driving which provides for the IST and almost 40 times smaller than that for ω_f close to the eigenfrequency in the minima of $U(q)$ (≈ 0.98). The origin of such a great facilitation of the IST onset when ω_f is close to ω_m is the wide in energy quasi-regular resonant transport which drastically widens the chaotic layers in the

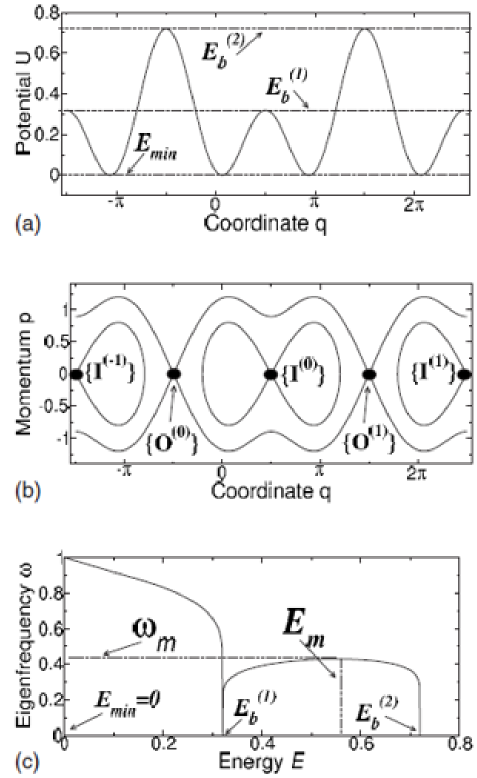

Fig. 2. Potential $U(q)$ (2) with $a=0.5$ and $b=0.2$ (a), separatrices in the phase plane ($p\equiv\dot{q},q$) (b), and frequency of eigenoscillation ω vs. energy $E\equiv U(q)+\dot{q}^2/2$ (c) for the system (2) at the absence of dissipation ($\gamma=0$), noise ($T=0$) and periodic driving ($h=0$). Note that, in such a case, energy E is conserved along a trajectory (in particular along a separatrix).

area of the destroyed separatrices and ultimately connects them with each other (Fig. 4).

When a *weak dissipation* is added, elliptic points transform into attractors (limit cycles) while the dissipationless chaotic layer transforms into the layer of the fractal-like mixture of basins of attraction of different attractors (Fig. 5). The system wanders random-like through the layer for a long time before residing on one of the attractors. The random-like wandering and the corresponding area are quite similar to those in the dissipationless case (cf. Fig. 4). The notion IST means in the weakly dissipative case that the basins of attraction of the limit cycles associated with A_1 and A_2 (shown in Fig. 5 by red and blue colors respectively) are mixed with those of the attractors situated beyond the coordinate range $]-\pi/2,3\pi/2[$ (shown in Fig. 5 by cyan and green colors).

Finally, we add weak *noise*. Then it may be of scientific and practical interest to know rates of direct inter-attractor transitions or of a direct escape through a given boundary in the Poincare section (the term “direct”means that only those paths are considered which do not enter a close vicinity of any attractor except the initial one). Such rates depend on the effective temperature T (see (2)) in the activation manner (1).

We show here that, if the given transition or escape necessarily requires for the system to intersect the outer separatrix, then

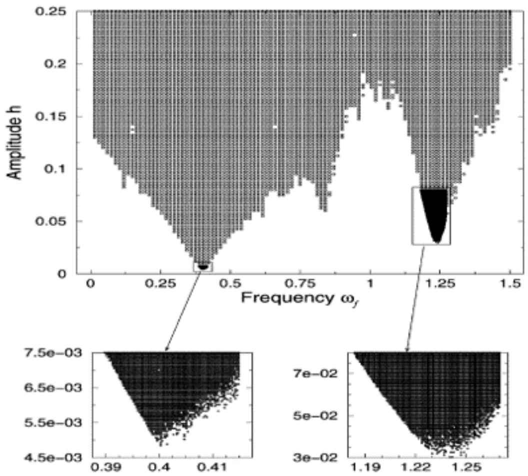

Fig. 3. The diagram indicating (by shading) areas of the driving parameters for which the inter-separatrix transport exists. The integration time for each point of the grid is 12000π.

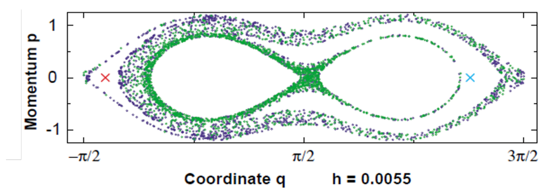

Fig. 4. The stroboscopic Poincare section (for instants $t_i = 2\pi i/\omega_f$ with $i = 0, 1, 2, \ldots$) of two trajectories of the system (2) with $U(q)$ shown in Fig.2 at the absence of the dissipation and noise ($\gamma = 0$ and $T = 0$) whilst $h = 0.0055$ and $\omega_f = 0.401$. The trajectories starting from the inner saddle ($q = \pi/2, p \equiv \dot{q} = 0$) and from the left outer saddle ($q = -\pi/2, p \equiv \dot{q} = 0$) are drawn in green and blue respectively. The number of points in each trajectory is equal to 2000. The stable stationary points of the section which are associated with the nonlinear resonances are indicated by the red and cyan crosses. It can be seen that the green and blue trajectories merge with each other.

even a rather weak perioic driving is sufficient for decreasing the activation barrier S by the factor ~ 2, provided the driving frequency is properly fitted: this phenomenon is closely related to the strong broadening of the resonance energy range near sepatratrices discussed above in cases without noise.

For the sake of simplicity, we consider in simulation only the escape problem while the relevant boundary consists merely of two walls: at $q = -3$ and $q = 5$ (note that the range $[-3, 5]$ fully includes the most relevant range $[-\pi/2, 3\pi/2]$). The qualitative analysis is generic however. Fig. 6 shows that, if ω_f corresponds to the minimum of the function $h_{IST}(\omega_f)$ (Fig. 3), then, as h increases, the activation barrier S quickly decreases from the value of the upper potential barrier $E_b^{(2)} = 0.72$ to that of the lower potential barrier $E_b^{(1)} = 0.32$. Thus, S decreases by the factor 2.25 at as tiny h as $h_{min} \approx 0.007$. The value of h of the adiabatic drive which would be required for the same barrier reduction, would be about ≈ 0.08, i.e. ≈ 11 times larger! Even for rather close frequencies 0.3 and 0.5, the required value of h is about 6 times larger (Fig. 3). For small temperatures such

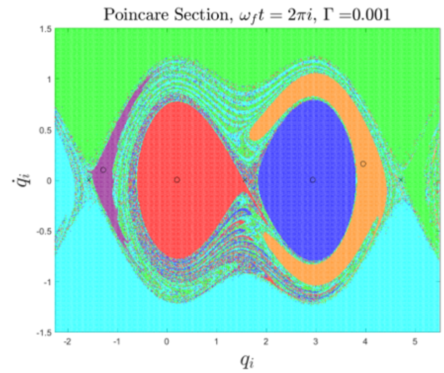

Fig. 5. Stationary points within the coordinate range $]-\pi/2, 3\pi/2[$ and basins of attraction in the stroboscopic Poincare section for the system (2) with $U(q)$ shown in Fig. 2(a) in the presence of dissipation ($\gamma = 0.001$) and periodic driving ($h = 0.01$ and $\omega_f = 0.401$) whilst noise is absent ($T = 0$). Limit cycles corresponding to the linear response to the periodic driving for the case when the systems stays close to the states A_1 and A_2 and saddle cycles near the states B and C are shown by the same markers as the corresponding states in Fig. 1. Limit cycles corresponding to the large-aimplitude oscillations induced by the periodic force are sometimes called as nonlinear resonances, and they are shown with the marker '+'. Basins of attraction of the limit cycles associated with the states A_1 and A_2 are shown with red and blue colors respectively. Basins of attraction of the larger-amplitude and smaller-amplitude nonlinear resonances lying within the coordinate range $]-\pi/2, 3\pi/2[$ are shown with pink and golden colors respectively. Basins of attraction of all limit cycles lying to the left from $q = -\pi/2$ are shown with cyan color. Basins of attraction of all limit cycles lying to the right from $q = 3\pi/2$ are shown with green color. The figure demonstrates a presence of a broad layer where all basins mentioned above mix with each other in a fractal manner. Such a layer may be called as the transient chaos layer.

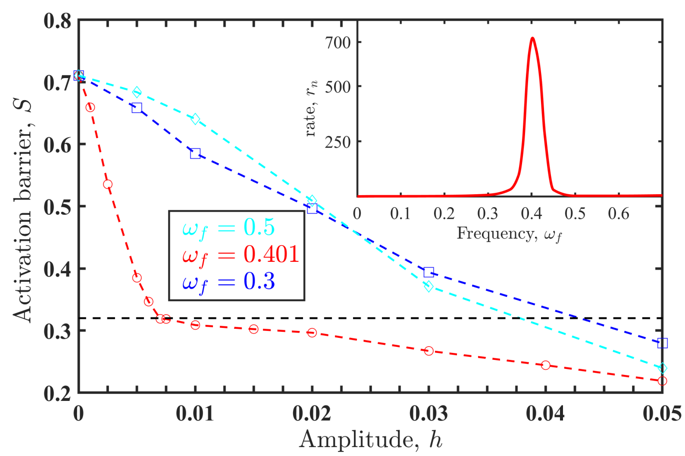

Fig. 6. The function $S(h)$ exctracted from the simulations for three values of ω_f ($U(q)$ and γ are the same as in Fig. 5). The dashed black line marks the lower barrier level ΔU_c. The inset shows the dependence on ω_f for the escape rate for $T = 0.055$ and $h = 0.01$ normalized with its value for $h = 0$.

a substantial decrease of the activation barrier leads to a huge increase of the escape rate: see the inset in Fig. 3. The latter inset also explicitly demonstrates the resonant nature of the phenomenon.

Let us give a more explicit qualitative explanation of the phenomenon. The system tends to escape via paths which provide the minimum of action: it means that the system moves along noise-free trajectories or close to them whenever possible. The analysis of typical escape paths shows (cf. Fig. 7) that the system first slowly diffuses towards the chaotic layer near the lower barrier and then escapes purely *noise-free*. The structure of the noise-free part of the path is as follows. The system first performs a fast chaotic wandering within the thin layer about the destroyed inner separatrix. Then it follows a *quasi-regular* resonant path until the latter reaches the thin layer about the destroyed outer separatrix and then reaches one of the bounadary walls in one or few steps. Thus, the activation barrier decreases from $\Delta U = 0.72$ at $h = 0$ to $\Delta U_C = 0.32$ at as tiny h as $h_{min} \approx 0.007$ (Fig. 6).

We note also that the statistical distribution of the escape paths has important differences from all those known before. It includes a part which neither concentrates near a single 1D line, often called an optimal trajectory (cf. [4], [12] in case of a regular motion in a noise-free system and [13] where the noise-free system possesses the chaotic saddle), nor represents a smooth distribution of strongly chaotic trajectories within some 2D area [5]. Rather it represents a smooth distribution of the quasi-regular trajectories within certain 2D areaof the Poincare section (cf. Fig. 7).

Finally, we comment that, if h is smaller than h_{min} (i.e. the minimal h which provides the IST) while still being comparable with it, then the resonant driving still provides a strong increase of energy, thus largely freeing noise from this work, which is equivalent to a significant decrease of the activation barrier.(Fig. 6)

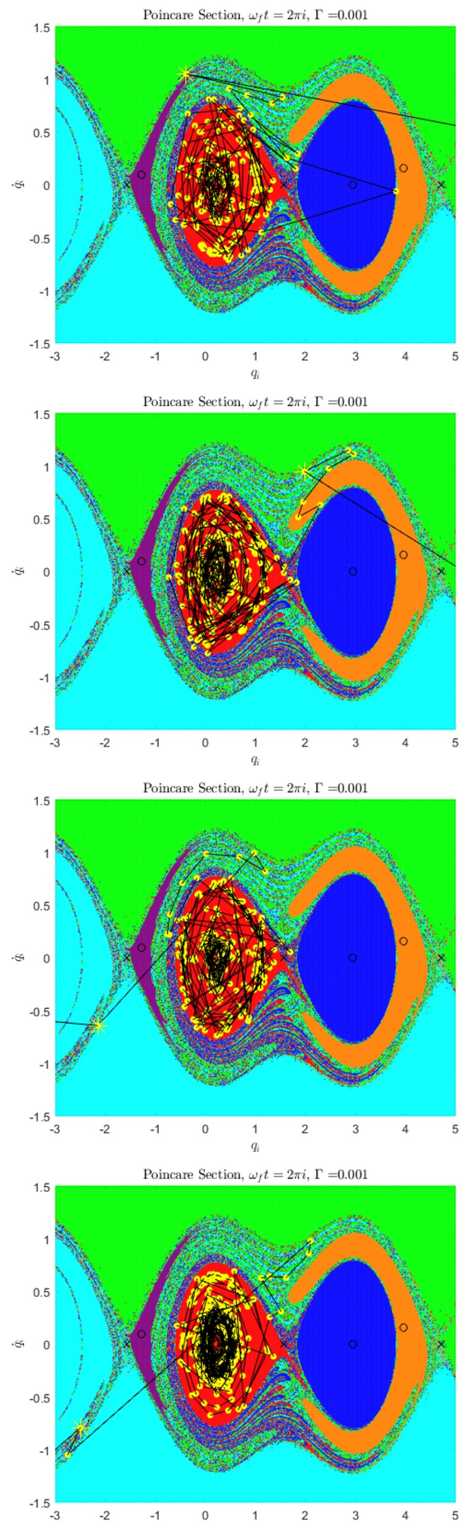

Fig. 7. An example of four subsequently generated noise-induced trajectories in the Poincare section escaping from the limit cycle associated with A_1 directly to one of the walls $q_i = -2$ or $q_i = 5$ (the term "direct"means that only those trajectories are considered which do not follow a close vicinity of any other attractor). The trajectory is shown by yellow points connected with dashed lines for visualisation of the escape path. $U(q)$, γ, h, ω_f are the same as in Fig. 5 and $T = 0.02$.

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