Contests on Networks

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Abstract

We develop a model of contests on networks. Each player is connected to a set of contests and exerts a single effort to increase the probability of winning each contest to which she is connected. We explore how behavior is shaped by the pattern of interactions and characterize the networks that tend to induce greater effort; in particular, we show that the complete bipartite network is the unique structure that maximizes aggregate player effort. We also obtain a new exclusion result – akin to the Exclusion Principle of Baye et al. (1993) – which holds under the lottery CSF, and contrasts prior work in contests. Finally, new insight into uniqueness of equilibrium for network contest games is provided. Our framework has a broad range of applications, including research and development, advertising, and research funding.

Keywords: Network Games, Contests, Bipartite Graph, Exclusion Principle, Tullock Contest

JEL Classifications: C72, D70, D85

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1 Introduction

In recent years, economists have recognized the importance of understanding how the structure of interactions affects economic behavior, which has fueled research on networks. The importance of this field is unquestionable, given the broad applicability of these models in many economically relevant settings. Jackson and Zenou (2014) provide a comprehensive overview of the literature on network games and emphasize three classes of games on which researchers have focused: (1) games with linear best-replies; (2) games of strategic complements and substitutes; and (3) games with an uncertain pattern of interactions.[1,2] In games with more complicated best-replies – contests, for example – general conclusions prove difficult to ascertain due to the inherent complexity of network models.

In this paper, we develop and study a contest network game.[3] Our model consists of a set of players and a set of contests, which form a bipartite graph.[4] Each player competes in contests to which she is connected by exerting a single effort; the contest success function (CSF) takes the form of the logit CSF.[5] Similar to the setup in Xu et al. (2022), our general framework allows for an arbitrary network structure, arbitrary player-specific convex cost functions as well as contest-specific prize values and arbitrary concave impact functions. Within this general framework, it is shown that there exists a unique pure-strategy Nash equilibrium. Our uniqueness result extends that of Xu et al. to accommodate the fact that an equilibrium in our model need not be contained in the restricted strategy space over which Xu et al.'s result applies. As our

[1] See also Jackson (2008) and Bramoullé and Kranton (2016).

[2] Examples of models in the first two categories will be discussed later in the Introduction. Examples of models in the third category include Jackson and Yariv (2007), Galeotti et al. (2010).

[3] For relevant surveys of the contest literature see Nitzan (1994), Congleton et al. (2008), and Konrad (2009).

[4] A bipartite graph is a graph in which the vertices may be partitioned into two disjoint subsets; within each subset, no two vertices are connected. The bipartite structure of the network in our model is similar to the oligopoly framework of Bimpikis et al. (2019) (see also, Bulow et al., 1985).

[5] The logit-form CSF is used extensively in the contest literature. For a recent contribution, see, e.g., Rosokha et al. (2024)

primary focus is to understand how patterns of interaction affect behavior in contests, we defer the detailed discussion of this point to Section 3.1.

In order to obtain meaningful insights on how behavior is shaped by the network, we then introduce more structure on the model. Under the lottery CSF (Friedman, 1958; Tullock, 1980), we provide necessary and sufficient conditions characterizing equilibrium. We provide sharp upper-bounds on individual and aggregate efforts, given in terms of network characteristics. We then compare equilibrium behavior on different network structures, with an emphasis on characterizing the networks that induce greater aggregate player effort. We show, in particular, that for a given set of players and contests/prizes, the complete bipartite network is the *unique* network structure that maximizes aggregate player effort.

To obtain further comparative statics and study the impact of player entry/exit, we then introduce some additional structure on the network. We describe a class of "quasiregular" networks, which includes as special cases, complete bipartite networks, biregular networks, star networks, and the class of bilateral complete bipartite networks studied by Franke and Öztürk (2015); henceforth FÖ. Within this class, we show that the entry [exit] of a particularly well-connected player may result in a decrease [increase] in aggregate equilibrium effort. This finding contrasts a large body of work in contests on the "Exclusion Principle" (Baye et al., 1993). The Exclusion Principle states that aggregate equilibrium effort in a contest may increase if the most competitive (the highest value) player is excluded. In single-prize contests, it is well-known that the Exclusion Principle holds under the all-pay auction CSF but does *not* apply to the Tullock CSF (see, for example, Fang, 2002; Matros, 2006; Menicucci, 2006).[6] In this paper, we derive a new exclusion result, given in terms of network characteristics, which also applies under the lottery CSF.

Our model has a number of applications including, for example, centralized R&D decisions by multinational firms (MNFs). Prizes are commonly used

[6] These papers assume a linear or convex cost of effort. If costs are concave, then even in a symmetric contest under the lottery CSF aggregate effort may be decreasing in the number of players (see Gama and Rietzke, 2019).

tools to encourage R&D activity,[7] and contests can be used to model both explicit R&D contests or patent races (e.g. Che and Gale, 2003; Baye and Hoppe, 2003). To take advantage of economies of scale and scope, historically, R&D activity within MNFs has tended to be centralized, and undertaken at the corporate level (Gassmann and Von Zedtwitz, 1999). In this interpretation, the firm chooses a single level of R&D effort, the benefits of which are then realized by each branch of the firm. Our model could also be interpreted in the context of a national advertising campaign by a geographically dispersed franchised firm. In this context, each firm chooses a level of expenditure on a national advertising campaign, which increases the share of the market each franchise expects to capture.[8] Finally, one might also interpret our model in the context of project funding – e.g., researchers applying for grant funding or start-up firms applying for venture capital funding. Many research funding bodies offer matching grants, which cover only a fraction of the cost of a given project; the other funds must be raised from other sources. In this interpretation, researchers exert effort on a single project proposal, which they submit to various funding bodies. Similarly, start-ups may seek out funding from several different venture capitalists using the same funding proposal.[9]

The main contributions of this research are twofold. First, we contribute to the literature on networks by analyzing a new class of network games. We adapt the uniqueness argument of Xu et al. to accommodate our environment and, in so doing, extend their finding. We then provide necessary and sufficient conditions characterizing equilibrium and study how network characteristics influence individual and aggregate behavior. We provide insights into the network structures that tend to induce greater aggregate effort and we characterize the unique network structure that maximizes aggregate network activity. Second, we contribute to the literature on contests by studying

[7]Innovation inducement prizes were a central feature of the Obama Administration's efforts to stimulate American innovation as part of the Recovery Act of 2009. Since 2010, more than 800 inducement prizes have been offered by federal agencies in areas ranging from national defense to education. See, `http://challenge.gov/about`.

[8]For a recent contribution on networks and advertising see Bimpikis et al. (2016).

[9]We thank the anonymous referee who suggested the example of start-ups applying for venture capital.

how behavior in contests is influenced by the structure of interactions between players. Our new exclusion result illustrates how changing the pattern of interactions can significantly alter existing conclusions from the literature.

There is a large literature on network games, with a broad range of applications – to name a few: job search and employment dynamics (Calvó-Armengol and Jackson, 2004; Calvó-Armengol, 2004), the provision of public goods (Bramoullé and Kranton, 2007; Bramoullé et al., 2014), collaboration/research and development (Goyal and Moraga-Gonzalez, 2001; Goyal and Joshi, 2003), and criminal activity (Calvó-Armengol and Zenou, 2004; Ballester et al., 2006). But as already noted, the bulk of this literature has focused on games with linear or monotonic best-replies; contests, inherently, do not fall into either of these categories.

There is a growing literature that combines contests and networks. FÖ's analysis relates closely[10] and it is worth pointing out the main differences between our studies. First, FÖ examine bilateral conflicts, while we allow for multilateral conflicts. Second, the players in FÖ's model choose a vector of efforts – one effort for each contest in which the player competes. As will be discussed in Section 4.1, our model can be interpreted as one in which each player has increasing returns-to-scale over contests, while FÖ's model can be interpreted as one in which players have decreasing returns-to-scale over contests. This difference has significant consequences for behavior; for instance, for a given quasiregular network in our model, individual efforts and payoffs are greater for players with higher degrees, which contrasts the findings in FÖ. Finally, to gain tractability, FÖ emphasize a class of complete bipartite networks, which are a special case of our quasiregular networks.

Xu et al. (2022) extends the analysis of FÖ in several interesting dimensions. The authors adopt tools from the study of Variational Inequalities to address issues such as existence and uniqueness of equilibrium; we rely greatly on their approach and results to address these points in our model. Nevertheless, the focus of our studies is quite different. In particular, their comparative

[10]See also, Jiao et al. (2019), Doğan et al. (2020), and Huremovic (2020). See Corrales and Arjona (2022) for a related experimental study.

statics analysis focuses on changes in parameters in the game (prize values for instance), rather than on changes in the network structure, which is the focus of this study. König et al. (2017) studies a setting in which players, competing for a single prize, are linked as enemies or allies. Several studies –including Goyal and Vigier (2014) and Kovenock and Roberson (2018) – examine the attack and defense of networked targets. Marinucci and Vergote (2011) and Grandjean et al. (2016) study a model of network formation in an all-pay auction, and a Tullock contest, respectively. In these models, players compete for a single prize, but the value of the prize to each player depends on the number of links she forms. Jackson and Nei (2015) study the interaction between networks of alliances, international trade, and conflict. Finally, Dziubiński et al. (2021) study a dynamic model in which "kingdoms" can attack their neighbors and expand their empire. In contrast to our model in which players choose a continuous level of effort, in Dziubiński et al. (2021), players are resource constrained and choose only whether to attack their neighbor(s) or not.

The remainder of the paper is organized as follows: In Section 2, we introduce the model. In Section 3, we analyze equilibrium behavior for an arbitrary network. In Section 4, we introduce and study the class of quasiregular networks. In Section 5, we discuss an alternative modelling approach in which the cost of effort depends on a player's degree. Concluding remarks are given in Section 6.

2 The Model

Let $\mathcal{N} = \{1, \ldots, N\}$ denote the set of players, and $\mathcal{M} = \{1, \ldots, M\}$ denote the set of contests. The structure of interactions can be represented by a bipartite graph and summarized by the $N \times M$ biadjacency matrix, $\mathbf{G} = (g_{im})$, where $g_{im} = 1$ if player i competes in contest m and $g_{im} = 0$, otherwise. Figure 1 illustrates the bipartite structure of the network. We let $\mathcal{N}_m = \{i \in \mathcal{N} | g_{im} = 1\}$ denote the set of players competing in contest m and $\mathcal{M}_i = \{m \in \mathcal{M} | g_{im} = 1\}$ denote the set of contests in which player i competes. We let $n_m = |\mathcal{N}_m|$ denote the degree of contest m and $v_i = |\mathcal{M}_i|$ denote the degree of player i.

We denote by, $\mathbf{v} \in \mathbb{N}^N$, the vector of player degrees and, $\mathbf{n} \in \mathbb{N}^M$, the vector of contest degrees. We assume throughout that for each m, $n_m \geq 2$ and for each i, $v_i \geq 1$. This ensures there is at least some competition in each contest and that each player competes in at least 1 contest. Throughout this analysis, we express vectors in bold and should be understood to be column vectors.

We now extend the classical logit contest to our environment. If player i wins contest m, she receives a prize, $\beta_m > 0$. We let $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_M)$ be the vector of all prize values. Player i chooses a single effort, $x_i \geq 0$, to increase her probability of winning each contest in which she competes. If i chooses effort x_i, she incurs a cost of $C_i(x_i)$, where $C_i : \mathbb{R}_+ \to \mathbb{R}_+$ is twice differentiable with, $C_i(0) = 0$, $C_i' > 0$ and $C_i'' \geq 0$. Let $\mathbf{x}_{-i} \in \mathbb{R}_+^{N-1}$ denote the profile of efforts chosen by players other than player i and let $\mathbf{x} \in \mathbb{R}_+^N$ denote the profile of efforts chosen by all players. If the players' efforts are $\mathbf{x}$, the probability that player i wins contest $m \in \mathcal{M}$ is

$$p_{im}(x_i, \mathbf{x}_{-i}) = \begin{cases} \frac{g_{im}\phi_m(x_i)}{\sum_{j \in \mathcal{N}_m} \phi_m(x_j)}, & \max_{j \in \mathcal{N}_m}\{x_j\} > 0, \\ \frac{g_{im}}{n_m}, & \max_{j \in \mathcal{N}_m}\{x_j\} = 0. \end{cases}$$

For each m, assume ϕ_m is twice continuously differentiable with $\phi_m(0) = 0$, $\phi_m' > 0$, and $\phi_m'' \leq 0$. The expected payoff to player i is

$$\pi_i(x_i, \mathbf{x}_{-i}) = \sum_{m \in \mathcal{M}} p_{im}(x_i, \mathbf{x}_{-i})\beta_m - C_i(x_i). \tag{1}$$

Note that for any $\mathbf{x}_{-i}$, $\lim_{x_i \to \infty} \pi_i(x_i, \mathbf{x}_{-i}) < 0$. So we may, without loss of generality, restrict each player's strategy space to $\Delta_i = [0, \overline{x}_i]$ for some arbitrarily large $\overline{x}_i$. We let $\mathcal{S} = \times_{i \in \mathcal{N}} \Delta_i \subset \mathbb{R}_+^N$ and $\mathcal{S}_{-i} = \times_{j \neq i} \Delta_j \subset \mathbb{R}_+^{N-1}$. Player i takes $\mathbf{x}_{-i}$ as given and solves, $\max_{x_i \in \Delta_i} \pi_i(x_i, \mathbf{x}_{-i})$.

Let $s_m = \sum_{i \in \mathcal{N}_m} x_i$ denote the total effort allocated to contest m and $\mathbf{s} \in \mathbb{R}_+^M$ denote the vector of total contest efforts. We also introduce two notions of aggregate effort in the network: let $z_p = \sum_{i \in \mathcal{N}} x_i$ denote the aggregate player effort and $z_c = \sum_{m \in \mathcal{M}} s_m$ denote the aggregate contest effort.[11] We

[11] In a single-battle N-player contest, there is only one notion of aggregate effort; i.e., the

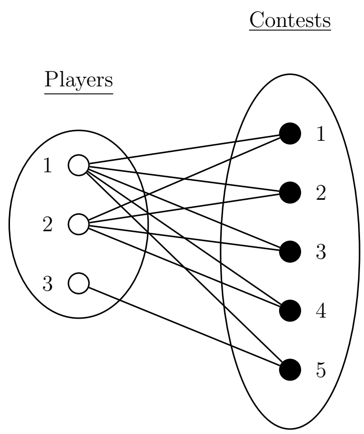

Figure 1: The contest network as a bipartite graph.

let x_i^* denote the equilibrium effort of player, i, and let $\mathbf{x}^* \in \mathcal{S}$ denote the profile of individual equilibrium efforts. We analogously define s_m^*, $\mathbf{s}^*$, z_p^* and z_c^*. Finally, we let $\overline{\beta}_i = \sum_{m \in \mathcal{M}_i} \beta_m$ denote the total value for which player i competes and let $B = \sum_{m \in \mathcal{M}} \beta_m$ denote the total value of all prizes.

3 Results

3.1 Existence and Uniqueness of Equilibrium

As is well-known, the logit-form contest success function gives rise to a discontinuity in payoffs. "Off-the-shelf" existence and uniqueness results by Rosen (1965) and Goodman (1980) therefore do not directly apply. Xu et al. (2022) study a closely-related multi-battle contest model and apply results from Reny (1999) to establish existence. The same arguments used by Xu et al. can be applied in our setting.

Xu et al. also show that there is at most one equilibrium within a particular subset of the strategy space. Following their notation, let $\mathcal{S}^2$ denote the subset

aggregate effort of the players is equal to the aggregate effort devoted to the contest. In our environment, there are two distinct notions of aggregate effort, depending on whether one aggregates over the players or over contests. As aggregating over contests necessarily leads to double-counting the effort of any player connected to more than one contest, in general, $z_c \geq z_p$. We are grateful to the anonymous referee who suggested aggregating over contests.

of the strategy space corresponding to profiles in which at least two players are active (exert strictly positive effort) in every contest. Their Proposition 5, which can be directly applied in our setting, shows that there is at most one equilibrium in $\mathcal{S}^2$. To establish this result, Xu et al. first show that the problem of characterizing equilibrium can be expressed as an equivalent variational inequality problem. They then demonstrate that the operator associated with that variational inequality possesses a strict monotonicity property on $\mathcal{S}^2$ (see Proposition 3 in Xu et al.). Uniqueness of equilibrium (in $\mathcal{S}^2$) follows almost immediately from this monotonicity property (see, Theorem 2 and Proposition 5 in Xu et al.).

In Xu et al.'s baseline model, each player chooses a vector of efforts – one effort for each contest in which the player competes. If the cost function is strictly increasing in each effort, Xu et al. show that any equilibrium must be in $\mathcal{S}^2$. In this case, uniqueness of equilibrium is guaranteed. In our model, players choose only a single effort; as a consequence, there may be equilibria outside of $\mathcal{S}^2$. To see this, consider the conflict structure depicted in Figure 1. In this example, Player 1 competes in all 5 contests, Player 2 competes in contests 1-4, and Player 3 competes only in contest 5. Suppose each contest utilizes the lottery CSF (i.e., $\phi_m(x_i) = x_i$), each prize is equal to 1 (i.e., $\beta_m = 1$ for each m), and $C_i(x_i) = x_i$. It is straightforward to show that there is an equilibrium in which Players 1 and 2 each choose $x_1^* = x_2^* = 1$, while player 3 is inactive: $x_3^* = 0$. In this equilibrium, Player 1 is the only active player in contest 5 and, hence, $\mathbf{x}^* \notin \mathcal{S}^2$.

The example above demonstrates why Xu et al.'s uniqueness result does not imply uniqueness of equilibrium in our framework. Nevertheless, in this example, the equilibrium is unique. This begs the question of whether a similar uniqueness result obtains over some larger subset containing $\mathcal{S}^2$. Indeed, we show this is the case. Note that in equilibrium in our model (1) each contest must have at least one active player; and (2) each active player, i, faces at least one active opponent in some contest to which i is connected. Point (1) follows since, if no player is active in some contest m, then any player connected to that contest could increase her effort slightly and strictly increase her payoff.

To understand point (2), note that if player i is active, but faces no active opponents, then i could raise her payoff by slightly reducing her effort. Let $\tilde{\mathcal{S}}^2$ denote the set of effort profiles, which satisfy (1) and (2). As just argued, any equilibrium must be in $\tilde{\mathcal{S}}^2$; moreover, see that $\mathcal{S}^2 \subset \tilde{\mathcal{S}}^2$. Our first result establishes existence and uniqueness of equilibrium. The arguments used to establish uniqueness are similar to those made by Xu et al., but reveal that there is at most one equilibrium in $\tilde{\mathcal{S}}^2$.

Proposition 1. *For any network,* $\mathbf{G}$*, there exists a unique pure-strategy equilibrium of the logit contest game.*

3.2 Equilibrium Behavior

To gain some traction in the analysis, let us now suppose, for each m, the CSF is the lottery CSF in which $\phi_m(x) = x$; further, for each i suppose $C_i(x_i) = x_i$. Let $b_i : \mathbb{R}_+^{N-1} \to \mathbb{R}_+$ denote player i's best-response: $b_i(\mathbf{x}_{-i}) = \arg\max_{x_i \in \Delta_i} \pi_i(x_i, \mathbf{x}_{-i})$. As π_i is strictly concave in x_i, b_i is a single-valued function. Moreover, when $b_i(\mathbf{x}_{-i}) > 0$, the first-order condition is necessary and sufficient for characterizing the best-response. In particular, $b_i(\mathbf{x}_{-i})$ is the unique solution to:

$$\frac{\partial \pi_i(b_i(\mathbf{x}_{-i}), \mathbf{x}_{-i})}{\partial x_i} = \sum_{m \in \mathcal{M}} g_{im} \frac{\sum_{j \neq i} g_{jm} x_j}{\left(b_i(\mathbf{x}_{-i}) + \sum_{j \neq i} g_{jm} x_j\right)^2} \beta_m - 1 = 0. \qquad (2)$$

For each m let $y_{-im} = \sum_{j \neq i} g_{jm} x_j$ denote the total effort of players other than i to contest m and let $\mathbf{y}_{-i} = (y_{-i1}, \cdots, y_{-iM})$. Examining Equation (2), it is clear that b_i depends on $\mathbf{x}_{-i}$ insofar as it depends on $\mathbf{y}_{-i}$. Let $\tilde{b}_i(\mathbf{y}_{-i})$ be i's best reply when the vector of others' total efforts is $\mathbf{y}_{-i}$. By (2), when interior, $\tilde{b}(\mathbf{y}_{-i})$ is defined by,

$$\sum_{m \in \mathcal{M}} g_{im} \frac{y_{-im}}{\left(\tilde{b}_i(\mathbf{y}_{-i}) + y_{-im}\right)^2} \beta_m = 1. \qquad (3)$$

Now, although our game is not an aggregative game, we will use Equation

(3) to construct an object analogous to the backwards reply mapping (see, e.g., Novshek, 1984; Cornes and Hartley, 2005; Jensen, 2010). Specifically, let $r_i(\mathbf{s})$ be the effort of player i that is consistent with i choosing a best-response when the vector of total contest efforts is $\mathbf{s} \gg 0$.[12] That is, $r_i(\mathbf{s}) = \tilde{b}_i(s_1 - g_{i1}r_i(\mathbf{s}), \cdots, s_M - g_{iM}r_i(\mathbf{s}))$. From (3), it follows that if $r_i(\mathbf{s}) > 0$ then,

$$\sum_{m \in \mathcal{M}} g_{im} \frac{s_m - r_i(\mathbf{s})}{s_m^2} \beta_m = 1. \tag{4}$$

Thus,

$$r_i(\mathbf{s}) = \max \left\{ \frac{\sum_m \frac{g_{im}\beta_m}{s_m} - 1}{\sum_m \frac{g_{im}\beta_m}{s_m^2}}, 0 \right\}. \tag{5}$$

As is typical in contest models, r_i is non-monotonic. In particular, if $r_i(\mathbf{s}) > 0$, it is straightforward to show that for any $m \in \mathcal{M}_i$, $\frac{\partial r_i(\mathbf{s})}{\partial s_m} > [<] 0$ if and only if $r_i(\mathbf{s}) > [<] \frac{s_m}{2}$. We let $\mathbf{r}(\mathbf{s}) = (r_1(\mathbf{s}), \cdots r_N(\mathbf{s}))^T$. Our next result gives necessary and sufficient conditions characterizing equilibrium.

Proposition 2. $\mathbf{x}^*$ *is an equilibrium profile of efforts if and only if* $\mathbf{x}^* = \mathbf{r}(\mathbf{s}^*)$ *where* $\mathbf{s}^*$ *satisfies* $\mathbf{G}^T\mathbf{r}(\mathbf{s}^*) = \mathbf{s}^*$ *and* $\mathbf{s}^* \gg 0$. *Moreover, in equilibrium:*

$$x_i^* = \sum_{m \in \mathcal{M}} p_{im}(\mathbf{x}^*)(1 - p_{im}(\mathbf{x}^*))\beta_m.$$

The condition $\mathbf{x}^* = \mathbf{r}(\mathbf{s}^*)$ ensures that each player i chooses an optimal action that is consistent with the vector of total contest efforts, $\mathbf{s}^*$. The condition, $\mathbf{G}^T\mathbf{r}(\mathbf{s}^*) = \mathbf{s}^*$ ensures that the vector of total contest efforts, $\mathbf{s}^*$, is consistent with players' individual efforts. That is, $\mathbf{G}^T\mathbf{r}(\mathbf{s}^*) = \mathbf{s}^*$ ensures that in each contest, m, $\sum_i g_{im}x_i^* = s_m^*$. Finally, it should be clear that, in equilibrium, it must be that for each m, $s_m^* > 0$; if not, any player competing in contest m could strictly increase their payoff by increasing effort slightly.

As far as we are aware, there is no way to express equilibrium efforts in closed-form for a general network structure. Given the non-monotonicity

[12] For $\mathbf{t} \in \mathbb{R}^K$, we write $\mathbf{t} \gg 0$ if $t_k > 0$ for each $k = 1, \cdots, K$.

of best-replies, performing comparative statics in contest models without a closed-form representation of equilibrium efforts often proves challenging. Yet the second part of Proposition 2 provides useful insights into equilibrium behavior and comparisons between behavior across different patters of interaction. Note, for instance, that for any $p \in [0,1]$, $p(1-p) \leq \frac{1}{4}$. It follows immediately that $x_i^* = \sum_{m \in \mathcal{M}} p_{im}(\mathbf{x}^*)(1 - p_{im}(\mathbf{x}^*))\beta_m \leq \frac{\overline{\beta}_i}{4}$. We state this observation as a corollary.

Corollary 1. *Individual equilibrium effort satisfies,* $x_i^* \leq \frac{\overline{\beta}_i}{4}$.

Our next result characterizes an upper (lower) bound on aggregate player (contest) effort. In what follows, for $\mathbf{t} \in \mathbb{R}^K$, we let $h(\mathbf{t})$ denote the harmonic mean of the elements of $\mathbf{t}$: $h(\mathbf{t}) = \frac{K}{\sum_k t_k^{-1}}$; we also let $\boldsymbol{\eta} = \left(\frac{n_1}{\beta_1}, \cdots, \frac{n_M}{\beta_M}\right)$.

Proposition 3. *In equilibrium, aggregate player and contest efforts satisfy,*

$$z_p^* \leq B - \frac{M}{h(\boldsymbol{\eta})} \leq z_c^*.$$

For a fixed prize vector, note that the upper (lower) bound on aggregate player (contest) effort is increasing in $h(\boldsymbol{\eta})$. As will be discussed in the next section, the term, $h(\boldsymbol{\eta})$, is a measure of both the overall number of competitors in contests across the network, and the extent to which higher-value contests are also those with more competitors. In the next section, we draw comparisons between equilibrium behavior on different networks; we place a particular emphasis on characterizing the network structures that induce the greatest aggregate activity.

3.3 Comparing Networks

As we will now be comparing equilibrium behavior for different networks, we make the dependence of equilibrium behavior on the network explicit and write $x^*(\mathbf{G})$ to denote equilibrium individual effort when the network is $\mathbf{G}$. We similarly use $z_p^*(\mathbf{G})$ and $z_c^*(\mathbf{G})$ to denote aggregate player and contest efforts (respectively). Before proceeding it will be useful to define some particular network structures of interest.

Definition 1. *The network* $\mathbf{G}$ *is biregular if* $n_m = n \geq 2$ *for each* $m \in \mathcal{M}$ *and* $v_i = v \geq 1$ *for each* $i \in \mathcal{N}$. $\mathbf{G}$ *is a player-symmetric biregular network if* $\mathbf{G}$ *is biregular and* $\overline{\beta}_i = \overline{\beta}_j$ *for each* $i, j \in \mathcal{N}$. $\mathbf{G}$ *is a complete bipartite network if* $\mathbf{G}$ *is biregular and* $n = N$ *(and hence,* $v = M$*).*

In a biregular network, each player has the same degree (v) and each contest has the same degree, (n) (although it need not be the case that $v = n$); such a network is completely summarized by the parameters, $[n, v, N, M]$. In a biregular player-symmetric network, each player competes for the same total value. In a complete bipartite network, each player competes in every contest and thus $n = N$ and $v = M$. Note that a complete bipartite network is also a biregular player-symmetric network in which, for each player i, $\overline{\beta}_i = B$. The next definition describes a particular type of equilibrium that may be induced by certain network structures.

Definition 2. *The network* $\mathbf{G}$ *induces a competitively-balanced equilibrium if, for each* $m \in \mathcal{M}$ *and* $i \in \mathcal{N}_m$, $p_{im}(\mathbf{x}^*) = \frac{1}{n_m}$.

If $\mathbf{G}$ induces a competitively-balanced equilibrium then, in equilibrium, each player competing in contest, m, has an equal chance of winning that contest. Our next result shows that networks that induce balanced competition tend to generate greater aggregate player effort.

Proposition 4. *Fix* N, M, *and* $\boldsymbol{\beta}$. *Let* $\mathbf{G}^0$ *and* $\mathbf{G}^1$ *be two networks. Let* $\mathbf{n}^k$ *denote the vector of contest degrees in network* $\mathbf{G}^k$, *and similarly define* $\boldsymbol{\eta}^k$, *where* $k \in \{0, 1\}$. *If* $\mathbf{G}^1$ *induces a competitively-balanced equilibrium and* $h(\boldsymbol{\eta}^0) \leq h(\boldsymbol{\eta}^1)$, *then* $z_p^*(\mathbf{G}^0) \leq z_p^*(\mathbf{G}^1)$. *If, in addition,* $\mathbf{G}^0$ *does not induce a competitively balanced-equilibrium or* $h(\boldsymbol{\eta}^0) < h(\boldsymbol{\eta}^1)$, *then* $z_p^*(\mathbf{G}^0) < z_p^*(\mathbf{G}^1)$.

Given two networks, $\mathbf{G}^0$ and $\mathbf{G}^1$, Proposition 4 reveals that aggregate player effort will be higher in $\mathbf{G}^1$ if (i) $\mathbf{G}^1$ induces a competitively-balanced equilibrium and (ii) $h(\boldsymbol{\eta}^0) \leq h(\boldsymbol{\eta}^1)$. It is well-understood in single-prize contests that a level playing field tends to induce greater aggregate effort. Condition (i) extends this idea to an environment with more complex interactions. To provide some intuition for condition (ii), note that the term, $h(\boldsymbol{\eta})$, is greater if the average number of competitors across contests is greater and/or contests with greater prizes are also those with more competitors. To see these points, first see that condition (ii) obviously holds if there are more competitors in each contest in the network $\mathbf{G}^1$ (i.e., if $n_m^0 \leq n_m^1$ for each m). Next, see that if $\beta_m = \beta$ for each m, then $h(\boldsymbol{\eta}) = \frac{h(\mathbf{n})}{\beta}$. In this case, condition (ii) holds if and only if the average (in the harmonic sense) number of competitors across contests is higher in network $\mathbf{G}^1$. Next, suppose the prize values differ across contests and, without loss of generality, suppose $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_M$. If $\mathbf{n}^1$ is such that $n_1^1 \geq n_2^1 \geq \cdots \geq n_M^1$ and $\mathbf{n}^0$ is any permutation of $\mathbf{n}^1$ then, $h(\boldsymbol{\eta}^0) \leq h(\boldsymbol{\eta}^1)$.[13]

Of course, the statement of Proposition 4 is only useful if one can identify classes of networks that induce competitively balanced equilibria. The next observation does just that.

Observation 1. *If* $\mathbf{G}$ *is a biregular player-symmetric network, then* $\mathbf{G}$ *induces a competitively-balanced equilibrium. In particular, the complete bipartite network induces a competitively-balanced equilibrium.*

Bipartite player-symmetric networks are not the only network structures that will induce a competitively-balanced equilibrium. For instance, if $\mathbf{G}$ is a collection of disconnected bipartite player-symmetric subnetworks, then $\mathbf{G}$ will also induce a competitively-balanced equilibrium. In general, to give rise to such an equilibrium, the network must induce a strong symmetry between the players in each contest (but the players in different contests, need not share this symmetry). As a consequence of Proposition 4 and Observation 1, we can characterize the network that induces the greatest aggregate effort, for a given N, M, and $\boldsymbol{\beta}$.

Proposition 5. *Fix* N*,* M*, and* $\boldsymbol{\beta}$*. Let* $\mathbf{G}^1$ *denote the complete bipartite network and let* $\mathbf{G}^0$ *be any network such that* $\mathbf{G}^0 \neq \mathbf{G}^1$*. Then* $z_p^*(\mathbf{G}^0) < z_p^*(\mathbf{G}^1)$*,* $z_c^*(\mathbf{G}^0) < z_c^*(\mathbf{G}^1)$ *and, for each* m*,* $s_m^*(\mathbf{G}^0) < s_m^*(\mathbf{G}^1)$*.*

[13] In general, $h(\boldsymbol{\eta}) = \frac{M}{\sum_{m \in \mathcal{M}} F(\beta_m, n_m)}$, where $F(\beta, n) = \frac{\beta}{n}$. Since F is strictly submodular in (β, n), it follows from well-known results from the literature on matching (e.g., Becker, 1973) that $\sum_m F(\beta_m, n_m^1) \leq \sum_m F(\beta_m, n_m^0)$, and hence, $h(\boldsymbol{\eta}^0) \leq h(\boldsymbol{\eta}^1)$.

Proposition 5 shows that, for a given N, M and $\boldsymbol{\beta}$, the complete bipartite network, in which all players compete for every prize, is the *unique* network structure that maximizes aggregate effort. Obviously, the complete bipartite network must also be the unique network that maximizes total effort within each contest. There are two driving forces for this result: First, the complete bipartite network maximizes the total value for which each player competes. Second, the complete bipartite network induces symmetry between all players, which, as discussed following Proposition 5, promotes greater player activity.

Our final result in this section describes the network structure that induces the greatest effort of some individual player, holding fixed the total value for which that player competes.

Proposition 6. *Let $\mathbf{G}^1$ be a biregular player-symmetric network with contest degree $n^1 = 2$ and total player prize value $\overline{\beta}_i^1 = \overline{\beta}^1$ for each i. Let $\mathbf{G}^0$ be any network and let $\overline{\beta}_i^0$ be the total prize value of some player i in this network. If $\overline{\beta}_i^0 \leq [<]\overline{\beta}^1$ then $x_i^*(\mathbf{G}^0) \leq [<] x_i^*(\mathbf{G}^1)$.*

Holding the total prize value of some player i fixed, Proposition 6 reveals that a biregular player-symmetric network in which two players compete in each contest induces the greatest individual effort from player i.

Our characterization of equilibrium in Section 3.1 has proved fruitful for performing some comparative statics and characterizing the network structures that induce the greatest aggregate/individual effort. However, in order to perform some additional comparative statics and study the effects of entry/exit from the network, a more complete characterization of equilibrium is necessary. In the next section, we introduce some additional structure on the network that enables such an analysis.

4 Quasiregular Networks

In this section, we analyze behavior in a class of networks, which we refer to as quasiregular networks. In order to focus exclusively on how equilibrium behavior is affected by patterns of interactions, in this section we assume that

$\beta_m = \beta_{m'} = \beta$ for each $m, m' \in \mathcal{M}$. Without loss of generality, we normalize $\beta = 1$; hence, for each i, $\overline{\beta}_i = v_i$. The following definition describes the class of quasiregular networks.

Definition 3. *Let $K_m(\gamma)$ denote the number of players with degree γ who compete in contest m : $K_m(\gamma) = \sum_{i \in \mathcal{N}} g_{im} \mathbb{1}(v_i = \gamma)$, where $\mathbb{1}(\cdot)$ is an indicator function. We say that the network $\mathbf{G}$ is quasiregular if for each $m, m' \in \mathcal{M}$ and $\gamma \in \{1, \ldots, M\}$, $n_m \geq 2$ and $K_m(\gamma) = K_{m'}(\gamma)$.*

In a quasiregular network, the number of players with degree $\gamma \in \{1, \ldots, M\}$ competing in a contest is equal across contests. Obviously, this means that each contest has the same degree, which we denote by n. Player degrees need not be equal but the following link property is satisfied:

$$\sum_{i \in \mathcal{N}} v_i = Mn. \tag{6}$$

The left-hand side of (6) is the number of links from players to prizes; the right-hand side is the number of links from prizes to players; obviously, these two numbers must be equal. The class of quasiregular networks includes some important sub-classes, such as biregular networks (defined in Section 3.3), complete biregular networks, star networks,[14] and the class of bilateral complete bipartite networks, studied in FÖ.[15]

As each contest in a quasiregular network has the same total number of competitors and the same number of competitors with each degree, $\gamma \in \{1, \ldots, M\}$, it should be clear that there is a symmetry between each contest in the network. Players, on the other hand, need not be symmetric and may have different degrees. But given the symmetric nature of the contests, the

[14] In a star network, one central player has degree, M; each of M periphery players has degree 1; and the degree of each contest is 2. See the left panel of Figure 4 for an illustration.

[15] In a bilateral complete bipartite network players are partitioned into two coalitions, say C_1 and C_2. Each member of coalition C_j competes with each member in the opposing coalition, C_k, in one bilateral contest. No two members of the same coalition compete with one another. Note that each player in coalition C_j has degree $|C_k|$. This structure is a quasiregular network: Each contest has one player with degree, $|C_1|$ and one player with degree, $|C_2|$.

only meaningful source of asymmetry between two players is their respective degree; i.e., the identities of the contests in which a player competes do not matter. Indeed, it will be shown that, in a quasiregular network, two players choose the same equilibrium effort if and only if those two players have the same degree. Therefore, in what follows, when we refer to the asymmetry of a quasiregular network, this should be understood to be a reference to the level of dispersion of player degrees.

Given the discussion above, it will be useful to have a measure summarizing the overall dispersion in player degrees. In this spirit, for $\mathbf{t} \in \mathbb{R}^K$, we let $a(\mathbf{t}) = \frac{\frac{1}{K}\sum_k t_k}{h(\mathbf{t})} \geq 1$ denote the ratio of the arithmetic mean of entries of $\mathbf{t}$ to the harmonic mean of elements of $\mathbf{t}$. For a quasiregular network, $a(\mathbf{v})$ can be interpreted as a measure of overall asymmetry between players. Note, for instance, that if $\mathbf{v}$ is such that $v_i = v_j$ for each i and j, while $\mathbf{v}'$ is such that $v'_i \neq v'_j$ for some i and j then $a(\mathbf{v}) = 1 < a(\mathbf{v}')$. More generally, given two quasiregular networks, $\mathbf{G}^1$ and $\mathbf{G}^0$ with M contests and N players, if player degrees in $\mathbf{G}^1$ are a mean-preserving spread of the player degrees in network $\mathbf{G}^0$ then $a(\mathbf{v}^0) < a(\mathbf{v}^1)$ (see Mitchell, 2004). Our first result provides an explicit characterization of equilibrium efforts for quasiregular networks.

Proposition 7. *Let* $\mathbf{G}$ *be a quasiregular network with* N *players,* M *contests, and contest degree,* n*. Let* $s^* = \frac{M(n-1)}{N}$ *and suppose, for each* i*,* $v_i > s^*$*. In the unique equilibrium,* $x_i^* = s^* - \frac{s^{*2}}{v_i} > 0$*. Moreover, for each contest,* m*,* $s_m^* = s^*$ *and hence,* $z_c^* = Ms^*$*. Aggregate player effort can be expressed,* $z_p^* = M(n-1)\left(1 - \frac{n-1}{n}a(\mathbf{v})\right)$*.*

The assumption, $v_i > \frac{M(n-1)}{N}$, ensures that the equilibrium is interior (i.e., all players choose a strictly positive effort in equilibrium); it is satisfied by, for example, any biregular network, star network, or bilateral complete bipartite network. This assumption is made for ease of exposition, but a complete characterization of equilibrium behavior without the interiority assumption, may be found in our working paper (Matros and Rietzke, 2018).

In the next subsection, we provide several comparative statics results for quasiregular networks. We then examine the effects of player entry/exit from the network. Throughout this next section, we will maintain the interiority assumption, $v_i > \frac{M(n-1)}{N}$ for all i; however, some of our illustrative examples do not satisfy this condition. Analogous comparative statics results without the interiority assumption can be found in our aforementioned working paper.

4.1 Comparative Statics on Quasiregular Networks

Our first result shows that, for a given quasiregular network, players with higher degrees exert greater effort and receive higher equilibrium payoffs.

Proposition 8. *Let* $\mathbf{G}$ *be a quasiregular network. For any two players, i and j, the following statements are equivalent: (i) $x_i^* > x_j^*$; (ii) $\pi_i(\mathbf{x}^*) > \pi_j(\mathbf{x}^*)$; and (iii) $v_i > v_j$.*

It is worth comparing Proposition 8 with results from FÖ. In their model, for regular or complete bipartite networks, FÖ find that players with higher degrees exert *less* effort in each contest and receive lower equilibrium payoffs. This point of contrast arises because of a fundamental difference in the externalities that exist across contests in the two models. In FÖ's model, the network effects are captured through a linkage between each of i's efforts in the quadratic cost function.[16] Thus, greater effort by player i in contest m raises the marginal cost of effort in all of i's other contests, yielding negative externalities between contests. In our model, there are positive externalities across contests – greater effort in contest m reduces the marginal cost of effort (to zero) in all of i's other contests. As a consequence, there is a difference in the returns-to-scale over contests for each player. If some player, i, with degree, v_i, wants to allocate the same effort, say $\overline{x}$, to each of her contests, then in FÖ's model, the total cost of doing so is $v_i^2\overline{x}^2$. The average cost per-contest is $v_i\overline{x}^2$, which is clearly increasing in v_i. In this way, the setup in FÖ can be interpreted as one in which players have decreasing returns-to-scale over contests. In our model, the average cost per-contest of delivering $\overline{x}$ units of

[16] Specifically, in FÖ's model, if $\mathbf{x}_i \in \mathbb{R}_+^{v_i}$ is the vector of efforts chosen by player i then the cost of effort is $C(\mathbf{x}_i) = \left(\sum_{m\in\mathcal{M}_i} x_{im}\right)^2$.

effort to each of i's contests is $\frac{\overline{x}}{v_i}$, which is clearly decreasing in v_i and can be interpreted as a form of increasing returns-to-scale over contests.[17] We revisit this issue in Section 5 when we introduce a model with degree-dependent costs.

We now explore how the structure of a quasiregular network affects equilibrium behavior. Our first result shows the impact of increasing the number of competitors in each contest (i.e., increasing the prize degree, n) on aggregate efforts.

Proposition 9. *Fix N and M. Let $\mathbf{G}^0$ and $\mathbf{G}^1$ be two quasiregular networks; let n^k denote the contest degree in network $\mathbf{G^k}$ and suppose $n^0 < n^1$. Then,*

(i) For each m, $s^*_m(\mathbf{G}^0) < s^*_m(\mathbf{G}^1)$ and, hence, $z^*_c(\mathbf{G}^0) < z^*_c(\mathbf{G}^1)$.

(ii) If $a(\mathbf{v^1}) \leq [<] \min\left\{a(\mathbf{v^0}), \frac{n^0 n^1}{n^0 n^1 - 1}\right\}$, then $z^*_p(\mathbf{G}^0) \leq [<] z^*_p(\mathbf{G}^1)$.

Fix N and M. Consider two contest networks, $\mathbf{G}^0$ and $\mathbf{G}^1$, and suppose each contest in $\mathbf{G}^k$ has $n^k \geq 2$ competitors. If $n^0 < n^1$, under what conditions will it be true that the total effort in each contest is higher in $\mathbf{G}^1$? Proposition 9(i) provides one set of sufficient conditions addressing this question. In particular, if $\mathbf{G}^0$ and $\mathbf{G}^1$ are both quasiregular then, indeed, total effort in each contest is higher in $\mathbf{G}^1$. To understand why quasiregularity is important for this finding, it is instructive to first examine an example where $\mathbf{G}^0$ and $\mathbf{G}^1$ are *not* quasiregular:

Example 1. *Consider the two networks illustrated in Figure 2. Let $\mathbf{G}^0$ denote the network on the left and let $\mathbf{G}^1$ denote the network on the right. In $\mathbf{G}^0$, each contest has degree $n^0 = 2$; in $\mathbf{G}^1$, each contest has degree, $n^1 = 3$. Neither network is quasiregular. In $\mathbf{G}^0$, the total equilibrium effort in contest c_1 is approximately 1.476. In $\mathbf{G}^1$, the total equilibrium effort in contest c_1 is approximately 1.111.*

[17] Note that it is not the linearity of the cost function in our model that drives the difference between our result and the result of FÖ. Rather, it is the fact that there are positive externalities between contests, combined with the fact that the total cost of effort to player i is independent of her degree. For instance, if each player has cost function, $C(x_i)$, then the average cost per contest of i delivering $\overline{x}$ units of effort to each contest is $\frac{C(\overline{x})}{v_i}$, which is decreasing in v_i. For the cost function, $C(x_i) = x_i^2$ (which is analogous to FÖ) it is straightforward to show that the statement of Proposition 8 remains true.

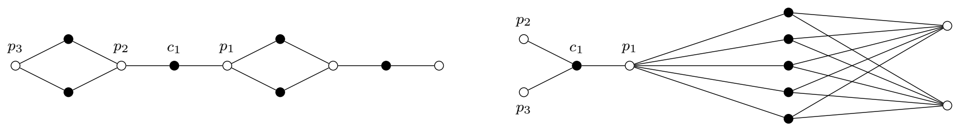

Figure 2: The two network structures discussed in Example 1. Players are represented by hollow nodes; contests are represented by solid nodes.

Example 1 illustrates that, in general, if the degree of each contest increases, total effort in any particular contest may decrease. Intuitively, although there are fewer competitors in each contest in network $\mathbf{G}^0$, the competition is relatively balanced: the two competitors in c_1 – players 1 and 2 – both compete in a total of 3 contests; in equilibrium, player 1 wins c_1 with probability of about .506, while player 2 wins with probability .494. In network $\mathbf{G}^1$, by contrast, competition in c_1 is relatively imbalanced. Players 2 and 3 compete for only a single prize, while player 1 competes for a total of six. The presence of the high-value player discourages effort from players 2 and 3; in equilibrium, player 1 is the only active player in c_1. The comparatively high effort from player 1 is not enough to offset the loss of effort from the inactive players and total effort in c_1 decreases when the network changes from $\mathbf{G}^0$ to $\mathbf{G}^1$.

Within the class of quasiregular networks, strong discouragement effects are also possible. Indeed, an increase in the degree of each contest may result in a reduction in aggregate player effort. At the same time, the structure of these networks is such that, in any particular contest, some level of competition is preserved. Our next example illustrates.

Example 2. *Consider the two networks in Figure 3. Let $\mathbf{G}^0$ denote the network on the left and let $\mathbf{G}^1$ denote the network illustrated on the right. In $\mathbf{G}^0$, the contest degree is $n^0 = 3$. In $\mathbf{G}^1$, the contest degree is $n^1 = 4$. Both networks are quasiregular. In equilibrium in $\mathbf{G}^0$ all players are active and $s_1^*(\mathbf{G}^0) = s_2^*(\mathbf{G}^0) = \frac{2}{3}$, and $z_p^*(\mathbf{G}^0) = \frac{4}{3}$. In equilibrium in $\mathbf{G}^1$ only players 3 and 4 are active and $s_1^*(\mathbf{G}^1) = s_2^*(\mathbf{G}^1) = z_p^*(\mathbf{G}^1) = 1$.*

Similar to Example 1, when the network changes from $\mathbf{G}^0$ to $\mathbf{G}^1$ in Example 2, competition in each contest becomes relatively imbalanced. In $\mathbf{G}^0$, each player competes for a total value of 1, and this network induces a competitively-balanced equilibrium. In $\mathbf{G}^1$, players 3 and 4 compete for a total value of 2, while players 1,2, 5, and 6 compete in a single contest. The presence of the two high-value players discourages effort from all other players; in equilibrium only players 3 and 4 are active. But, contrasting Example 2, *within each contest*, the increased competition between players 3 and 4 offsets the loss of effort from the inactive players; total effort in each contest increases.

Example 2 illustrates a general feature of quasiregular networks: If $\mathbf{G}^0$ and $\mathbf{G}^1$ are quasiregular networks with contest degrees, $n^0 < n^1$, then, within each contest, the degree of at least two players must be higher in $\mathbf{G}^1$.[18] As a consequence, in any particular contest, discouragement effects are offset by the increased competition between the players of higher degree.

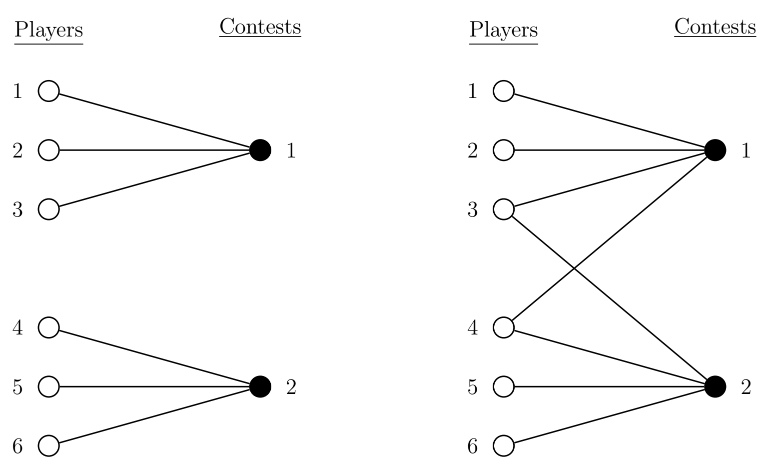

Figure 3: The two network structures described in Example 2.

Example 2 also demonstrates that an increase in the contest degree may result in a reduction in aggregate player effort, even while the total effort in each contest increases. Intuitively, adding additional links may have two competing

[18]For instance, suppose $n^1 = n^0 + 1$. Then, $\sum_i v_i^1 - \sum_i v_i^0 = Mn^1 - Mn^0 = M$. This means that in network $\mathbf{G}^1$, M new links must be created between players and contests. It cannot be the case that the degree of 1 player increases by M (otherwise, that player must have been competing in 0 contests in network $\mathbf{G}^0$). Hence, the degree of at least two players must increase. But since each contest contains an equal number of players of every degree, this means that the degree of at least two players in each contest must increase.

effects on aggregate network effort. For those players whose degree increases, effort tends to increase since these players now compete for a greater number of prizes. However, additional links may give rise to a greater overall asymmetry between players, as measured by $a(\mathbf{v})$. Using the characterization of equilibrium in Proposition 7, it is clear that aggregate player effort is decreasing in $a(\mathbf{v})$. Of course, additional links may also *reduce* the level of player asymmetry, giving rise to a more competitive contest network. Proposition 9(ii) shows that if additional links are added to a network in such a way that the resulting structure is "symmetric enough" then aggregate equilibrium effort in the network increases. The next example illustrates; Figure 4 depicts the two networks described in the example.

Example 3. *Let $\mathbf{G}^0$ denote the star network in the left panel of Figure 4 and $\mathbf{G}^1$ denote the augmented star depicted in the right panel. For the star network, $n^0 = 2$ and $a(\mathbf{v^0}) \approx 1.444$. In the augmented star, $n^1 = 3$, and $a(\mathbf{v^1}) = 1.125$. It is straightforward to check that the hypotheses of Proposition 9 are satisfied in this example. And indeed it holds, $z_p^*(\mathbf{G}^0) \approx 1.389 < z_p^*(\mathbf{G}^1) = 2.5$.*

Example 3 illustrates a scenario in which a higher contest degree results in a reduction in overall player asymmetry. In $\mathbf{G}^0$ there is a single player with degree 5 and 5 players with degree 1. In $\mathbf{G}^1$, the discrepancy between the high and low-value players is reduced – there is one player of degree 5 and 5 players of degree 2. When the network changes from $\mathbf{G}^0$ to $\mathbf{G}^1$, overall player asymmetry, as measured by $a(\cdot)$, decreases and aggregate player effort increases.

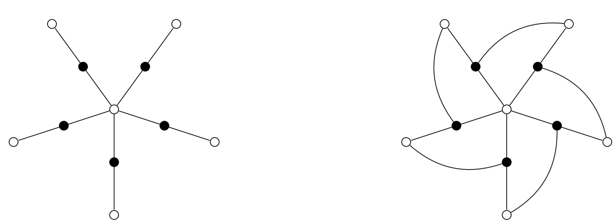

Figure 4: The two network structures described in Example 3. Players are represented by hollow nodes and contests are represented by solid nodes.

4.2 Effects of Player Entry/Exit

In this section, we study the effects of player entry/exit. We focus on the impact of the entry/exit of a particularly "well-connected" player on the aggregate player effort. Specifically, our next result examines a scenario in which a player with degree M is added to a biregular network.

Proposition 10. *Let* $\mathbf{G}$ *be a biregular network summarized by* $[n, v, N, M]$. *Let* $\mathbf{G}^+$ *be the quasiregular network formed by adding an additional player with degree* M *to the network, and let* $\mathbf{v}^+$ *denote the vector of player degrees in network* $\mathbf{G}^+$. *Then* $z_p^*(\mathbf{G}^+) < z_p^*(\mathbf{G})$ *if and only if*

$$1 + \frac{1}{n^3} < a(\mathbf{v}^+). \tag{7}$$

Proposition 10 demonstrates that the entry [exit] of an additional player may result in lower [greater] aggregate player effort. Condition (7) can be interpreted as saying that the new entrant induces a significant amount of asymmetry in the network. Our result closely relates to the Exclusion Principle (Baye et al., 1993). The driving force behind the Exclusion Principle is well-understood in single-prize contests: a player with a high prize value discourages competition from players with lower prize values. Removing the high-value player, levels the playing field and results in a more competitive contest. As discussed in the introduction, prior work has found that the Exclusion Principle holds under the all-pay auction CSF but does not apply under the lottery CSF with linear/convex costs (see, e.g., Stein, 2002; Fang, 2002; Matros, 2006). The reason is that the lottery CSF introduces a significant amount of noise in determining the winner, which dampens the discouragement effect.[19]

But previous work in the contest literature focuses on one particular pattern of interactions (one in which all players compete for a single prize); the entry/exit of one player does not affect this structure.[20] In contrast, the entry/exit of a player in our model, has indirect effects on other players, which

[19]For more recent work on the discouragement effect see also Chowdhury et al. (2023), Drugov and Ryvkin (2022) and Cortes-Corrales and Gorny (2022).

[20]Two exceptions are Dahm and Esteve-Gonzalez (2018) and Dahm (2018). Both studies

stem from the player's influence on the structure of interactions.[21] When indirect network effects are taken into account, our Proposition 10 shows that the phenomenon behind the Exclusion Principle can also apply under the noisy lottery CSF. The following example illustrates our finding; Figure 5 depicts the networks described in the example.

Example 4. *Let* $\mathbf{G}$ *denote the biregular network consisting of only players 1-6 in Figure 5. Let* $\mathbf{G}^+$ *denote the network formed when player 7 is added. It holds that* $a(\mathbf{v}^+) = \frac{52}{49} \approx 1.06 > 1 + \frac{1}{n^3} = 1 + \frac{1}{27} \approx 1.04$. *Consistent with Proposition 10, the addition of player 7 reduces aggregate player effort. Specifically,* $z_p^*(\mathbf{G}^+) = \frac{120}{49} \approx 2.45 < z_p^*(\mathbf{G}) = \frac{8}{3} \approx 2.67$.

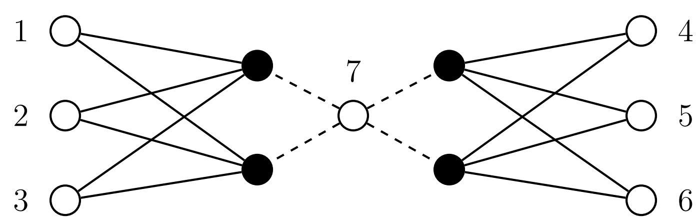

Figure 5: The networks described in Example 4. Players are represented by hollow nodes; contests are represented by solid nodes.

5 Degree-Dependent Costs

In Sections 3.2–4, we assumed that the cost of effort to player i is independent of the number of contests in which i competes. In some situations of interest, this assumption may be overly restrictive. In the context of R&D, for example, a multinational firm may undertake R&D centrally and distribute that technology to each market in which the firm operates. But the cost of

explore a particular network structure in which all players compete for a main prize, while a subset of disadvantaged players also compete for a secondary prize. Dahm (2018) shows that excluding an advantaged player altogether may increase total effort under the all-pay auction CSF, but even greater effort can be generated by only excluding the advantaged player from competing for the secondary prize.

[21] A similar effect is described in Ballester et al. (2006)

distributing that technology may be non-trivial and may impact her initial investment decision. In this section we explore this issue.

Suppose if player i chooses effort, x, her total cost is, $C_i(x) = c(v_i)x$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is weakly increasing, twice continuously differentiable, and satisfies $c(0) = 0$. The payoff to player i is, $\pi_i(\mathbf{x}) = \sum_m \frac{g_{im}x_i}{\sum_j g_{jm}x_j} - c(v_i)x_i$.

In this alternative model, the qualitative nature of our results from Section 3 continue to hold, but with regards to equilibrium *expenditures*. For instance, the characterization of equilibrium individual behavior given in Proposition 2 remains true if one replaces x_i^* with $e_i^* = c(v_i)x_i^*$; that is, it can be shown that in equilibrium, $e_i^* = \sum_m p_{im}(\mathbf{x}^*)(1 - p_{im}(\mathbf{x}^*))\beta_m$. The upper-bound on aggregate player effort given in Proposition 3 also holds if one replaces z_p^* by aggregate equilibrium expenditures, z_e^*, where, $z_e^* = \sum_i e_i^*$. Similarly, the comparisons between equilibrium aggregate player efforts for different network structures given in Section 3.3 also hold if one replaces aggregate player effort, z_p^*, by aggregate player expenditures, z_e^*. In particular, fixing N and M, the complete network is the unique network structure that maximizes aggregate player expenditures.

Note, however, that in the degree-dependent cost model, the comparative statics of equilibrium efforts are highly dependent upon the shape of the cost function, $c(\cdot)$. For instance, for any quasiregular network, Proposition 8 shows that $x_i^* > x_j^*$ if and only if $v_i > v_j$. Moreover, it is implied by Proposition 9 that, for biregular networks, an increase in the contest degree, n, yields greater aggregate player effort. Our next result summarizes how these comparative statics may change in the degree-dependent cost model.

Proposition 11. *Consider the degree-dependent cost model with equal prize values across contests. Let* $f(v) = \frac{v}{c(v)}$ *and suppose* $f(v_i) > \frac{M(n-1)}{\sum_i c(v_i)}$[22] *for each* i*. Then,*

(i) If $\mathbf{G}$ *is a quasiregular network, then for any two players,* i *and* j*, the*

[22]This condition is analogous to the condition given in Proposition 7 that $v_i > s^*$. It ensures that, for quasiregular networks, the equilibrium is interior (i.e., all players choose strictly positive effort). It is always satisfied for biregular networks. A statement analogous to (i) holds without this interiority assumption for any two active players.

following statements are equivalent: (i) $x_i^* > x_j^*$ *and (ii)* $f(v_i) > f(v_j)$*.*

(ii) Let $\mathbf{G}^0$ *and* $\mathbf{G}^1$ *are two biregular networks with* N *players,* M *contests, and contest degrees,* n^0 *and* n^1*, respectively, and suppose* $n^1 > n^0$*. If* $f' < 0$*, then* $z_p^*(\mathbf{G}^0) > z_p^*(\mathbf{G}^1)$*.*

Proposition 11(i) reveals which players choose the greatest effort in the model with degree-dependent costs. Note that $c'' < 0$ implies f is strictly increasing; in this case, the average cost per contest, $\frac{c(v_i)x_i}{v_i}$ is decreasing in v_i. Hence, $c'' < 0$ can be interpreted as a form of increasing-returns-to-scale over contests. Analogously, $c'' > 0$ can be interpreted as a form of decreasing-returns-to-scale over contests. Thus, it is implied by Proposition 11(i) that, for quasiregular networks, if there are increasing returns-to-scale over contests then the players with the highest degrees choose the greatest efforts. Similarly, with decreasing returns-to-scale, the players with the lowest degrees choose the highest efforts. Following the same interpretation, Proposition 11(ii) reveals that, for biregular networks, if there are decreasing returns-to-scale over contests then aggregate player effort is decreasing in the contest degree.

6 Conclusion

In this paper we propose a new framework for studying contests on networks. We provided necessary and sufficient conditions characterizing equilibrium for an arbitrary network and gave insights into how behavior depends on the structure of interactions. For a given set of players and contests, we showed that the complete network is the *unique* network structure that maximizes aggregate effort. We also introduced a class of quasiregular networks, which enabled further comparative statics results. Finally, we derived a new exclusion result, akin to the Exclusion Principle of Baye et al. (1993), given in terms of network characteristics, which is relevant under the lottery CSF. This finding shows that prior conclusions from the contest literature may be significantly affected by changing the pattern of interactions.

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Appendix

A Proofs

Proof of Proposition 1

The proof of existence follows nearly identical arguments as those made by Xu et al., so we omit this here. The proof of uniqueness also follows along similar lines as the proofs in Xu et al. For this reason, we will provide only an outline of most of the arguments but we will be more explicit when appropriate. Let $\tilde{\mathcal{S}}^2$ be as defined in the discussion prior to Proposition 1. As argued in the main body, if $\mathbf{x}^*$ is an equilibrium profile then $\mathbf{x}^* \in \tilde{\mathcal{S}}^2$. We will show that there cannot exist two distinct equilibria in $\tilde{\mathcal{S}}^2$.

Let $\mathcal{S}^1$ denote the set of all effort profiles in which at least one player is active in each contest; i.e., $\mathcal{S}^1 = \{\mathbf{x} \in \mathbb{R}_+ | \forall m \in \mathcal{M}, \max_{i \in \mathcal{N}_m}\{x_i\} > 0\}$. It follows by definition of $\tilde{\mathcal{S}}^2$ that $\tilde{\mathcal{S}}^2 \subset \mathcal{S}^1$. Define the operator $\mathbf{F} : \mathcal{S}^1 \to \mathbb{R}^N$:

$$\mathbf{F}(\mathbf{x}) = -\begin{bmatrix} \frac{\partial \pi_1(\mathbf{x})}{\partial x_1} \\ \vdots \\ \frac{\partial \pi_N(\mathbf{x})}{\partial x_N} \end{bmatrix}.$$

Following similar arguments made by Xu et al. in the proof of their Proposition 1, it can be shown that $\mathbf{x}^*$ is a Nash equilibrium if and only if,

$$\mathbf{x}^* \in \mathcal{S}^1 \text{ and } \langle \mathbf{F}(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle \geq 0, \forall \mathbf{x} \in \mathcal{S}^1, \tag{8}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. For $\mathbf{x} \in \mathcal{S}^1$, let $\mathbf{M}(\mathbf{x})$ denote the Jacobian matrix of $\mathbf{F}$: $\mathbf{M}(\mathbf{x}) = \nabla_{\mathbf{x}} \mathbf{F}(\mathbf{x})$, $\mathbf{x} \in \mathcal{S}^1$.

A critical step in the proof of Xu et al.'s uniqueness result is showing that $\frac{1}{2}(\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^T)$ is positive definite for $\mathbf{x} \in \mathcal{S}^2$; here, we will show that is true for any $\mathbf{x} \in \tilde{\mathcal{S}}^2$. To do so, we follow Xu et al. and apply a result from Goodman (1980), which ensures that $\frac{1}{2}(\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^T)$ is positive definite for $\mathbf{x} \in \tilde{\mathcal{S}}^2$ if on $\tilde{\mathcal{S}}^2$ for each i, (1) $\frac{\partial^2 \pi_i(\mathbf{x})}{\partial x_i^2} < 0$, (2) π_i is convex in $\mathbf{x}_{-i}$ for each x_i, and (3) $\sum_i \pi_i$ is concave in $\mathbf{x}$. Points (2) and (3) follow from similar arguments made in Xu et al. Here, we show point (1). Take any $i \in \mathcal{N}$ and let $m \in \mathcal{M}_i$. At any $x \in \mathcal{S}^2$, p_{im} is twice differentiable in x_i and

$$\frac{\partial^2 p_{im}(\mathbf{x})}{\partial x_i^2} = \frac{\left(\sum_{j\in\mathcal{N}_m\setminus\{i\}} \phi_m(x_j)\right)\left[\phi_m''(x_j)\sum_{j\in\mathcal{N}_m}\phi_m(x_j) - 2\phi_m'(x_i)^2\right]}{\left(\sum_{j\in\mathcal{N}_m}\phi_m(x_j)\right)^3}.$$

See that $\phi_m'' \leq 0 < \phi_m' \implies \frac{\partial^2 p_{im}(\mathbf{x})}{\partial x_i^2} \leq 0$, holding with strict inequality if $\sum_{j\in\mathcal{N}_m\setminus\{i\}} \phi_m(x_j) > 0$. Moreover, $\mathbf{x} \in \tilde{\mathcal{S}}^2$ implies that for at least one $m^\dagger \in \mathcal{M}_i$, $\sum_{j\in\mathcal{N}_{m^\dagger}\setminus\{i\}} \phi_{m^\dagger}(x_j) > 0$ and hence, $\frac{\partial^2 p_{im^\dagger}(\mathbf{x})}{\partial x_i^2} < 0$. These facts along with the convexity of C_i imply,

$$\frac{\partial^2 \pi_i(\mathbf{x})}{\partial x_i^2} = \sum_{m\in\mathcal{M}_i} \frac{\partial^2 p_{im}(\mathbf{x})}{\partial x_i^2}\beta_m - C_i''(x_i) \leq \frac{\partial^2 p_{im^\dagger}(\mathbf{x})}{\partial x_i^2}\beta_{m^\dagger} < 0,$$

which establishes that $\frac{1}{2}(\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^T)$ is positive definite for $\mathbf{x} \in \tilde{\mathcal{S}}^2$. From here, the remaining arguments are similar to those made by Xu et al. in the proofs of their Proposition 3 and Theorem 2. Namely, for $\mathbf{x} \in \tilde{\mathcal{S}}^2$ $\frac{1}{2}(\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^T)$ positive definite implies $\mathbf{F}$ is a strictly monotone operator on $\tilde{\mathcal{S}}^2$:

$$\mathbf{x}', \mathbf{x}'' \in \tilde{\mathcal{S}}^2, \mathbf{x}' \neq \mathbf{x}'' \implies \langle \mathbf{F}(\mathbf{x}') - \mathbf{F}(\mathbf{x}''), \mathbf{x}' - \mathbf{x}''\rangle > 0. \tag{9}$$

Then suppose $\mathbf{x}^{**}$ and $\mathbf{x}^*$ are two distinct equilibria. By (8),

$$\langle \mathbf{F}(\mathbf{x}^*), \mathbf{x}^{**} - \mathbf{x}^*\rangle \geq 0 \text{ and } \langle \mathbf{F}(\mathbf{x}^{**}), \mathbf{x}^* - \mathbf{x}^{**}\rangle \geq 0,$$

which implies

$$\langle \mathbf{F}(\mathbf{x}^*) - \mathbf{F}(\mathbf{x}^{**}), \mathbf{x}^* - \mathbf{x}^{**}\rangle \leq 0.$$

But since $\mathbf{x}^*, \mathbf{x}^{**} \in \tilde{\mathcal{S}}^2$ the inequality above contradicts (9), which means that any equilibrium must be unique.

□

Proof of Proposition 2

First we show necessity. Let $\mathbf{x}^*$ be a PSNE. For each $i \in \mathcal{N}$ and $m \in \mathcal{M}$, let $s_m^* = \sum_j g_{jm} x_j^*$ and let $y_{-im}^* = s_m^* - x_i^*$. The fact that $\mathbf{s}^* \gg 0$ follows from arguments made in the proof of Proposition 1 (recall that any equilibrium must be in $\tilde{\mathcal{S}}^2$). Now choose any player i. By definition of equilibrium, $x_i^* = \tilde{b}_i(y_{-i1}^*, \cdots, y_{-iM}^*) = \tilde{b}_i(s_1^* - g_{i1}x_i^*, \cdots, s_M^* - g_{iM}x_i^*)$. By definition of r_i, it follows that $x_i^* = r_i(\mathbf{s}^*)$, where $\mathbf{s}^* = (s_1^*, \cdots, s_M^*)$. Thus, for each $i \in \mathcal{N}$, $x_i^* = r_i(\mathbf{s}^*)$. So, for each $m \in \mathcal{M}$, $s_m^* = \sum_j g_{jm} x_j^* = \sum_j g_{jm} r_j(\mathbf{s}^*)$. This means, $\mathbf{G}^T \mathbf{r}(\mathbf{s}^*) = \mathbf{s}^*$.

Now let $\mathbf{x}^* = \mathbf{r}(\mathbf{s}^*)$, where $\mathbf{G}^T\mathbf{r}(\mathbf{s}^*) = \mathbf{s}^* \gg 0$. We will show that $\mathbf{x}^*$ is an equilibrium profile. To this end, it suffices to show that, for each player, i, $r_i(\mathbf{s}^*)$ is a best-response to $\mathbf{r}_{-i}(\mathbf{s}^*) = (r_1(\mathbf{s}^*), \cdots, r_{i-1}(\mathbf{s}^*), r_{i+1}(\mathbf{s}^*), \cdots, r_N(\mathbf{s}^*))$. Consider any player, i. Suppose $r_i(\mathbf{s}^*) > 0$. See that

$$
\begin{aligned}
\frac{\partial \pi_i}{\partial x_i}\Big|_{(r_i(\mathbf{s}^*), \mathbf{r}_{-i}(\mathbf{s}^*))} &= \sum_{m \in \mathcal{M}} g_{im} \frac{\sum_{j \neq i} g_{jm} r_j(\mathbf{s}^*)}{\left(\sum_{j \in \mathcal{N}} g_{jm} r_j(\mathbf{s}^*)\right)^2} \beta_m - 1 \\
&= \sum_{m \in \mathcal{M}} g_{im} \frac{s_m^* - r_i(\mathbf{s}^*)}{s_m^{*2}} \beta_m - 1 \\
&= 0.
\end{aligned}
$$

The second equality follows from the first since $\mathbf{G}^T\mathbf{r}(\mathbf{s}^*) = \mathbf{s}^*$. The final equality follows by definition of $r_i(\mathbf{s}^*)$ (see Equation (4)). By strict concavity of π_i in x_i, $r_i(\mathbf{s}^*)$ is the best-response for player i.

Next, suppose $r_i(\mathbf{s}^*) = 0$. This means $\sum_{m \in \mathcal{M}_i} \frac{\beta_m}{s_m^*} \leq 1$. Then, for any $x_i > 0$, $\frac{\partial \pi_i}{\partial x_i}|_{(x_i, \mathbf{r}_{-i}(\mathbf{s}^*))} = \sum_{m \in \mathcal{M}_i} \frac{s_m^*}{(x_i + s_m^*)^2} \beta_m - 1 < \sum_{m \in \mathcal{M}_i} \frac{\beta_m}{s_m^*} - 1 \leq 0$. Hence, for any $x_i > 0$, $\pi_i(x_i, \mathbf{r}_{-i}(\mathbf{s}^*)) < \pi_i(r_i(\mathbf{s}^*), \mathbf{r}_{-i}(\mathbf{s}^*))$. This establishes that $\mathbf{x}^*$ is an equilibrium profile.

Finally, let $\mathbf{x}^*$ be a PSNE; we will show $x_i^* = \sum_{m \in \mathcal{M}} p_{im}(\mathbf{x}^*)(1 - p_{im}(\mathbf{x}^*))\beta_m$. This is trivially true if $x_i^* = 0$; so suppose $x_i^* > 0$. Using equation (4) along with the fact that $x_i^* = r_i(\mathbf{s}^*)$, in equilibrium it holds, $x_i^* = \sum_{m \in \mathcal{M}} g_{im} \frac{x_i^* s_m^* - x_i^{*2}}{s_m^{*2}} \beta_m$; or $x_i^* = \sum_{m \in \mathcal{M}} p_{im}(\mathbf{x}^*)(1 - p_{im}(\mathbf{x}^*))\beta_m$.

□

Proof of Proposition 3

We first show $s_m^* \geq \frac{n_m-1}{n_m}\beta_m$. We proceed by contradiction. Contrary to the proposition, suppose there exists a contest $\tilde{m}$ such that, in equilibrium, $s_{\tilde{m}} < \frac{n_{\tilde{m}}-1}{n_{\tilde{m}}}\beta_{\tilde{m}}$. Now, for each active player, $i \in \mathcal{N}_{\tilde{m}}$, $\frac{\partial \pi_i(\mathbf{x}^*)}{\partial x_i} = \frac{s_{\tilde{m}}-x_i^*}{s_{\tilde{m}}^2}\beta_{\tilde{m}} + \sum_{m\neq\tilde{m}} \frac{s_m-x_i^*}{s_m^2}\beta_m - 1 \geq \frac{s_{\tilde{m}}-x_i^*}{s_{\tilde{m}}^2}\beta_{\tilde{m}} - 1$. Thus, $\sum_{i\in\mathcal{N}_{\tilde{m}}} \frac{\partial \pi_i(x^*)}{\partial x_i} \geq \sum_{i\in\mathcal{N}_{\tilde{m}}} \left(\frac{s_{\tilde{m}}-x_i^*}{s_{\tilde{m}}^2}\beta_{\tilde{m}} - 1\right) = \frac{n_{\tilde{m}}\beta_{\tilde{m}}-\beta_{\tilde{m}}-n_{\tilde{m}}s_{\tilde{m}}}{s_{\tilde{m}}} > 0$, where the final inequality holds since $s_{\tilde{m}} < \frac{n_{\tilde{m}}-1}{n_{\tilde{m}}}\beta_{\tilde{m}}$. But this means that, for at least one player $i \in \mathcal{N}_{\tilde{m}}$, $\frac{\partial \pi_i(\mathbf{x}^*)}{\partial x_i} > 0$, which contradicts (4). Thus, for each m, $s_m^* \geq \frac{n_m-1}{n_m}\beta_m$. By definition of z_c^*, $z_c^* \geq \sum_{m\in\mathcal{M}} \frac{n_m-1}{n_m}\beta_m$

We now show $z_p^* \leq \sum_{m\in\mathcal{M}} \frac{n_m-1}{n_m}\beta_m$. Using Proposition 2,

$$\begin{aligned} z_p^* &= \sum_{i\in\mathcal{N}} \sum_{m\in\mathcal{M}} p_{im}(\mathbf{x}^*)(1-p_{im}(\mathbf{x}^*))\beta_m \\ &= \sum_{m\in\mathcal{M}} \beta_m \sum_{i\in\mathcal{N}_m} p_{im}(\mathbf{x}^*)(1-p_{im}(\mathbf{x}^*)) \\ &= \sum_{m\in\mathcal{M}} \beta_m - \sum_{m\in\mathcal{M}} \beta_m \sum_{i\in\mathcal{N}_m} p_{im}(\mathbf{x}^*)^2, \end{aligned}$$

where the final equality holds since $\sum_{i\in\mathcal{N}_m} p_{im}(\mathbf{x}) = 1$ for all m and $\mathbf{x}$. For any contest, m, consider the following constrained minimization problem: $\min_{\mathbf{p}_m\in[0,1]^{n_m}} \beta_m \sum_{i\in\mathcal{N}_m} p_{im}^2$ subject to the constraint $\sum_{i\in\mathcal{N}_m} p_{im} = 1$. There is a unique solution to this problem, which is to choose $p_{im} = \frac{1}{n_m}$ for each $i \in \mathcal{N}_m$. Therefore,

$$\sum_{m\in\mathcal{M}} \beta_m - \sum_{m\in\mathcal{M}} \beta_m \sum_{i\in\mathcal{N}_m} p_{im}(\mathbf{x}^*)^2 \leq \sum_{m\in\mathcal{M}} \beta_m - \sum_{m\in\mathcal{M}} \beta_m \sum_{i\in\mathcal{N}_m} \frac{1}{n_m^2} = \sum_{m\in\mathcal{M}} \frac{n_m-1}{n_m}\beta_m.$$

Finally, using the definitions of $h(\cdot)$ and B, straightforward algebraic manipulation reveals that $\sum_{m\in\mathcal{M}} \frac{n_m-1}{n_m}\beta_m = B - \frac{M}{h(\boldsymbol{\eta})}$. □

Proof of Proposition 4

Let $\mathbf{G}^0$ and $\mathbf{G}^1$ be as given in the proposition. From the proof of Proposition 3 it is clear that the upper-bound given in the proposition is achieved if and only if the network induces a competitively-balanced equilibrium. So, if $\mathbf{G}^1$ induces a competitively-balanced equilibrium then, $z_p^*(\mathbf{G}^1) = B - \frac{M}{h(\boldsymbol{\eta}^1)}$. As $h(\boldsymbol{\eta}^0) \leq h(\boldsymbol{\eta}^1)$, $z_p^*(\mathbf{G}^0) \leq B - \frac{M}{h(\boldsymbol{\eta}^0)} \leq B - \frac{M}{h(\boldsymbol{\eta}^1)}$. Note that the first inequality, which holds by Propositon 3, is strict if $\mathbf{G}^0$ does not induce a competitively balanced equilibrium; and the second inequality is strict if $h(\boldsymbol{\eta}^0) < h(\boldsymbol{\eta}^1)$. □

Proof of Observation 1

Let $\mathbf{G}$ be a biregular player-symmetric network with contest degree, n, and common total prize value, $\overline{\beta}_i = \overline{\beta}$. Following arguments analogous to those presented in the proof of Proposition 7, it is straightforward to confirm that in the unique equilibrium $x_i^* = x_j^* = \overline{\beta}\left(\frac{n-1}{n^2}\right)$ for each $i, j \in \mathcal{N}$. Thus, for each m and $i \in \mathcal{N}_m$, $p_{im}(\mathbf{x}^*) = \frac{1}{n}$. □

Proof of Proposition 5

Fix M, N, and $\boldsymbol{\beta}$. Let $\mathbf{G}^1$ denote the complete bipartite network and let $\mathbf{G}^0 \neq \mathbf{G}^1$. Let n_m^k denote the degree of contest m in $\mathbf{G}^k$ and let $\mathbf{n}^k$ denote the vector of contest degrees. By Observation 1, $\mathbf{G}^1$ induces a competitively-balanced equilibrium. Moreover, $n_m^1 = N$ for each m. Obviously, it must be that $n_m^0 \leq N$ for each m; moreover, since $\mathbf{G}^0 \neq \mathbf{G}^1$, $n_{m'}^0 < N$ for at least one m'. It follows that $h(\boldsymbol{\eta}^0) < h(\boldsymbol{\eta}^1)$. By Proposition 4, $z_p^*(\mathbf{G}^0) < z_p^*(\mathbf{G}^1)$. Moreover, for each m, $s_m^*(\mathbf{G}^0) \leq z_p^*(\mathbf{G}^0) < z_p^*(\mathbf{G}^1) = s_m^*(\mathbf{G}^1)$. By definition of z_c^*, it follows that $z_c^*(\mathbf{G}^0) < z_c^*(\mathbf{G}^1)$. □

Proof of Proposition 6

Let $\mathbf{G}^1$ be a biregular player-symmetric network with total prize value $\overline{\beta}^1$ and contest degree n^1. As shown in the proof of Observation 1, $x_i^*(\mathbf{G}^1) = \frac{\overline{\beta}^1}{4}$. The result follows immediately by Corollary 1. □

Proof of Proposition 7

Let $\mathbf{s}^*$ be the $M \times 1$ column vector, whose entries are equal to $s^* = \frac{M(n-1)}{N}$. Let $\mathbf{x}^* = (x_1^*, \cdots, x_N^*)^T$, where $x_i^* = s^* - \frac{s^{*2}}{v_i}$. To show that $\mathbf{x}^*$ is an equilibrium profile, Proposition 2 implies that it is sufficient to show $x_i^* = r_i(\mathbf{s}^*)$ for each i and that $\sum_i g_{im} r_i(s^*) = s^*$ for each contest, m.

First, see that $r_i(\mathbf{s}^*) = \max\left\{\frac{\sum_m \frac{g_{im}}{s_m^*} - 1}{\sum_m \frac{g_{im}}{s_m^{*2}}}, 0\right\} = \max\left\{\frac{v_i s^* - s^{*2}}{v_i}, 0\right\} = s^* - \frac{s^{*2}}{v_i} = x_i^*$, where the second-to-last equality holds since $v_i > s_i^*$, by assumption. Thus, $x_i^* = r(\mathbf{s}^*)$.

Before completing the proof, we provide a simplification of the sum, $t_m = \sum_i \frac{g_{im}}{v_i}$. Note that t_m can be written as $t_m = \sum_{\gamma=1}^{M} \frac{K_m(\gamma)}{\gamma}$, where $K_m(\gamma)$ is as defined in Definition 3. But, by definition of a quasiregular network, for each $m, m' \in \mathcal{M}$ $K_m(\gamma) = K_{m'}(\gamma)$. Thus, for each $m, m' \in \mathcal{M}$, $t_m = \sum_i \frac{K_m(\gamma)}{\gamma} = \sum_i \frac{K_{m'}(\gamma)}{\gamma} = t_{m'} = t$. It follows that $\sum_m t_m = Mt = \sum_m \sum_i \frac{g_{im}}{v_i} = N$. Thus, for each $m \in M$, $t_m = \frac{N}{M}$.

Now let m be given. We will show $\sum_i g_{im} r_i(\mathbf{s}^*) = s^*$. As already shown, $x_i^* = r_i(\mathbf{s}^*)$. So, $\sum_i g_{im} r_i(\mathbf{s}^*) = \sum_i g_{im}(s^* - \frac{s^{*2}}{v_i}) = ns^* - s^{*2} t_m = ns^* - s^{*2}\frac{N}{M} = \frac{M(n-1)}{N} = s^*$, where the second-to-last inequality holds by plugging in $s^* = \frac{M(n-1)}{N}$ to the expression prior. This establishes that $\mathbf{x}^*$ is an equilibrium profile of efforts.

Now see that $z_p^* = \sum_i x_i^* = Ns^* - s^{*2} \sum_i v_i^{-1} = M(n-1)\left(1 - \frac{M(n-1)}{N}\frac{1}{h(\mathbf{v})}\right) = M(n-1)\left(1 - \frac{n-1}{n} a(\mathbf{v})\right)$, where the first equality holds by plugging in the expression for s^*; the second equality holds since $\frac{1}{N}\sum_i v_i = \frac{Mn}{N}$ by (6). □

Proof of Proposition 8

The fact that $x_i^* > x_j^*$ if and only if $v_i > v_j$ follows immediately from the characterization of equilibrium efforts in Proposition 7. Now see that $\pi_i(\mathbf{x}^*) = v_i \frac{x_i^*}{s^*} - x_i^* = \frac{s^{*2}}{v_i} + v_i - 2s^*$; under the interiority assumption $v_i > s^*$ and it is straightforward to show that $\pi_i(\mathbf{x}^*)$ is strictly increasing in v_i. □

Proof of Proposition 9

Part (i) is immediate from the characterization in Proposition 7. Here, we show Part (ii). Using the characterization provided in Proposition 7,

$$\begin{aligned}\frac{1}{M}(z_p^*(\mathbf{G}^1) - z_p^*(\mathbf{G}^0)) &= n^1 - n^0 - \frac{(n^1-1)^2}{n^1}a(\mathbf{v^1}) + \frac{(n^0-1)^2}{n^0}a(\mathbf{v^0}) \\ &\geq [>] n^1 - n^0 - a(\mathbf{v^1})\left[\frac{(n^1-1)^2}{n^1} - \frac{(n^0-1)^2}{n^0}\right] \\ &= n^1 - n^0 - a(\mathbf{v^1})\left(\frac{n^1-n^0}{n^0 n^1}\right)(n^0 n^1 - 1) \\ &\geq [>] 0.\end{aligned}$$

The first inequality follows since $a(\mathbf{v^1}) \leq [<] a(\mathbf{v^0})$. The final inequality holds since $n^1 > n^0$ and $a(\mathbf{v^1}) \leq [<] \frac{n^0 n^1}{n^0 n^1 - 1}$. □

Proof of Proposition 10

Let $\mathbf{G}$ and $\mathbf{G}^+$ be as described in the proposition. The network, $\mathbf{G}^+$ is a quasiregular network with contest degree, $n+1$. Using Proposition 7, equilibrium aggregate player effort in $\mathbf{G}^+$ is, $z_p^*(\mathbf{G}^+) = Mn[1 - \frac{n}{n+1}a(\mathbf{v}^+)]$; aggregate player effort in the biregular network simplifies to, $z_p^*(\mathbf{G}) = \frac{M(n-1)}{n}$. Using these two expressions, it holds that $z_p^*(\mathbf{G}^+) < z_p^*(\mathbf{G})$ if and only if expression (7) holds. □

Proof of Proposition 11

Following analogous steps as in the proof of Proposition 7, it may be verified that if $\mathbf{G}$ is a quasiregular network with N players, M contests and contest degree, n then, in the unique equilibrium, $x_i^* = s^* - \frac{s^{*2}}{f(v_i)}$, where $s^* = \frac{M(n-1)}{\sum_i c(v_i)}$. Moreover, in each contest, $s_m^* = s^*$. The statement of item (i) follows immediately from this characterization.

In a biregular network, $v_i = v$ for each i. In this case, the characterization above simplifies to yield, $s_m^* = s^* = \frac{M(n-1)}{Nc(v)}$ and $x_i^* = x^* = s^* - \frac{c(v)}{v}s^{*2}$. Using the fact that in a biregular network, $Mn \equiv Nv$, aggregate player effort can be written $z_p^* = \frac{n-1}{n} \frac{M}{c\left(\frac{Mn}{N}\right)}$. Momentarily viewing n as a continuous variable, note that $\text{sgn}\left(\frac{\partial z_p^*}{\partial n}\right) = \text{sgn}\left(\frac{1}{n} - \frac{(n-1)M}{N} \frac{c'(v)}{c(v)}\right)$. Then, $f'(v) < 0$ implies $\frac{c'(v)}{c(v)} > \frac{1}{v}$ for all $v > 0$. It follows that $\frac{1}{n} - \frac{(n-1)M}{N} \frac{c'(v)}{c(v)} < \frac{1}{n} - \frac{M(n-1)}{Nv} = \frac{2-n}{n} \leq 0$, where the final equality holds by using the identity $Mn \equiv Nv$. Item (*ii*) follows immediately. □