Enhancing Betting Against Beta with Stochastic Dominance

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Abstract

The performance of the widely used betting-against-beta (BAB) investment strategy is improved by controlling for the stochastic dominance (SD) relation between individual stocks and the market portfolio. Dominating stocks, preferred by all risk-averse and prudent investors, are excluded from the short leg of the BAB strategy. Stocks that are dominated by the market are excluded from the long leg of the strategy. This prefiltering significantly enhances a wide range of performance and risk measures including abnormal returns relative to various factor models. The improvements are especially pronounced for the third-order SD, are robust to transaction costs and different market conditions.

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1 Introduction

Stocks with low market beta often generate superior excess returns. This puzzle that seemingly contradicts financial theory has been discussed for decades in the academic literature and it has given rise to multiple investment products, including the betting-against-beta (BAB) strategy. In this paper we show that the BAB strategy may not always be consistent with the notion of stochastic dominance (SD) – a more general framework encompassing a wide variety of individual preferences. We suggest a methodology of prefiltering individual stocks based on their SD-relation with the market index and adjusting the BAB strategy such that the long leg excludes dominated stocks, and the short leg excludes dominating stocks. The resulting SD-enhanced BAB portfolio exhibits performance improvement across multiple dimensions, from mean and variance to Omega ratio, lower partial moments, certainty equivalents, and abnormal returns relative to various factors models. The outperformance is especially pronounced for third-order SD (TSD) prefiltering. Allocation adjustments in the short leg contribute the most to performance improvement of the overall portfolio.

The underpinning mechanism of low beta strategies is related to the observation that the security market line is too flat, as emphasized by Fama and French (1992). Black (1993) suggests that "Beta is a valuable investment tool if the line is as steep as the CAPM predicts. It is even more valuable if the line is flat." He further develops the intuition of beta factor, which is the precursor of the BAB strategy of Frazzini and Pedersen (2014). Findings that the security market line is "essentially flat" or "completely flat" are discussed and debated by, for example, Black (1993), Jagannathan et al. (1995), Jagannathan and Wang (1996), Fama and French (1996), Campbell and Vuolteenaho (2004), and Bai et al. (2019), among others. Table 1 summarizes the key findings in the academic literature related to the low beta anomaly, its potential explanations, and implications.

[Table 1 around here]

The beta anomaly is often utilized as an investment strategy. Invesco Russell 1000 Low Beta Equal Weight ETF (USLB), for example, directly targets low beta stocks within Russell 1000 index. Frazzini and Pedersen (2014) point out that correlation moves slowly compared to volatility, thus, low volatility investing is practically close to low beta investing. Various ETFs track low volatility stocks. For example, Invesco S&P 500 Low Volatility ETF (SPLV) targets the quintile of S&P 500 index components with the lowest volatilities. The assets under management of this fund are about $10.3 billion as of May 27, 2022. Financial index providers such as S&P and Dow Jones develop multiple low beta indices covering the US, the UK, Japan, Developed, and Global markets. AQR – one of the leading investment firms – directly tracks a long-short portfolio replicating the BAB factor.[1]

Conceptually, beta is inherent to the mean-variance framework. It is a valuable measure characterizing risk-return trade-off if returns on assets are normally distributed or investors have a quadratic utility function. Stock returns, however, often do not follow a normal distribution (Cont 2001), and investors are likely to have heterogeneous preferences. Accommodating non-normal/non-quadratic-utility world of heterogenous investors calls for a more general framework of financial decision making. Here, the SD approach is the most suitable. First, it ranks complete return distributions. Hence, it accounts for various risks (including, e.g., tail risk),

[1] The AUM of SPLV can be found via https://www.invesco.com/us/financial-products/etfs/product-detail?productId=ETF-SPLV, and the AQR's BAB can be accessed via https://www.aqr.com/Insights/Datasets/Betting-Against-Beta-Equity-Factors-Monthly.

and it does not restrict comparisons to a limited number of moments (such as only mean/variance/skewness). Second, the general preferences captured by SD are particularly suitable for collective investments, such as mutual or hedge funds, where individual investors are likely to have different risk tolerance.

In this paper we focus on the second order SD (SSD) which is related to the preference of the risk-averse investors and the third order SD (TSD), which also encompasses investors' prudence. If return distribution of one asset dominates the return distribution of another asset by SSD (TSD), all risk-averse (and prudent) investors, irrespective of their actual risk aversion or the exact shape of the utility function, prefer the former to the latter. Post (2003) and Kuosmanen (2004) find that SD efficient strategies enhance portfolio performance. Hodder et al. (2015) show that investment strategies based on SSD significantly outperform strategies based on the mean-variance approach. Post and Kopa (2017), Kolokolova et al. (2022), and Fang and Post (2022) further discuss the implication of higher-order SD for investment decisions and portfolio selection.

Our paper merges the insights from these strands of research and proposes a methodology that allows improving performance of the BAB strategy across a spectrum of performance-risk measures and during different market conditions. Stocks are evaluated in terms of their SSD and TSD relations with the market index, and dominating stocks are excluded from the short leg of the BAB strategy while dominated stocks are excluded from the long leg of the strategy. The TSD prefiltering performs especially well, significantly increasing the mean return (by over 50 basis points per year), the Sharpe, Sortino, Omega and Upside potential ratios, as well as certainty equivalents for different levels of risk aversion. It also results in a significant increase in the alpha relative to various factor models, such as the Fama-French 6 factor model and the q5 model of Hou et al. (2021) among others. The positive and significant alpha of the SD-enhanced BAB is not subsumed by the FMAX factor of Bali et al. (2017), contrary to that of the original BAB strategy. The outperformance remains pronounced after controlling for realistic transaction costs. Our results are of interest not only to academics but also practitioners. We clearly show how following the SD-enhanced strategy significantly improves the properties of portfolio returns over a classical investment strategy that merely considers the stock beta.

The rest of the paper is organized as follows: Section 2 introduces the methodology for augmenting the BAB portfolio and describes various performance evaluation metrics. Section 3 discusses the data and presents the empirical results. Several robustness checks are examined in Section 4, and concluding comments are provided in Section 5.

2 Methodology

2.1 SD - theoretical preliminaries

The concept of stochastic dominance (SD) allows ranking complete distributions of outcomes, for example, asset returns, without making any parametric assumptions about the exact utility function of an investor. SD of different orders captures different types of preferences. Specifically, first order SD (FSD) characterizes monotonic preferences, second order SD (SSD) characterizes risk-averse preferences, while third order SD (TSD) is related to risk-averse and prudent preferences. No investors with increasing utilities (that is, investors preferring higher returns to lower ones) will choose FSD dominated investment strategy. No investors with increasing and concave utilities (that is, risk-averse investors) will choose SSD dominated strategy. And no investors with increasing and concave utilities, that have also a positive third derivative (that is, prudent investors making precautionary savings) will choose TSD dominated strategy.[2]

Formally, SD relationship between two distributions can be defined as follows. Let $F = F^{[1]}$ be a cumulative distribution function defined on an interval $[a, b]$. Define $F^{[n]}$ recursively as:

$$F^{[n]}(r) = \int_a^r F^{[n-1]}(k)dk. \tag{1}$$

A distribution F is said to dominate distribution G at order S in the stochastic sense if $F^{[S]}(r) \leq G^{[S]}(r)$ for all $r \in [a, b]$ and $F^{[m]}(b) \leq G^{[m]}(b)$ for all $m = 2, \ldots, S-1$.

In this paper, we focus on SSD and TSD because of their rich implications for financial decision making. As mentioned earlier, if a return distribution F dominates another return distribution G by SSD (TSD), no risk-averse (and prudent) investor prefers G over F. SSD relation can also be viewed through the prism of expected losses relative to a threshold. If F dominates G by SSD, then expected losses relative to all admissible thresholds of F are smaller in absolute values that those of G. This makes SSD closely related to advanced risk measures, such as *Omega*, which is the ratio of the expected gains over expected losses relative to a specified return level. If F dominates G by SSD, then *Omega* of F is always larger than that of G, for any required return levels. Hence, risk-averse investors should try to avoid SSD dominated

[2]For a comprehensive discussion of the SD framework see Levy (2016), and for the application to portfolio choice see Hodder et al. (2015), among others.

assets in their portfolios. Similarly, TSD is closely related to preferences for skewness by the investors. Indeed, Fang and Post (2017) and Fang and Post (2022) have shown benefits of higher-order SD for asset pricing and portfolio optimization.

2.2 SD and low beta investing

Market beta relates to the first two moments of the return distribution, while SD accounts for the full spectrum of moments. This may cause inconsistencies between beta and SD rankings and make the beta-based trading strategies less desirable. A stylized example in Table 2 illustrates a potential mismatch between SD and low-beta investments.[3]

Suppose we have five assets (A, B, C, D, and E) across three periods (t=1, 2, 3), as well as their respective initial market capitalizations at the beginning of period 1. We compute the equal-weighted (M_{EQ}) and value-weighted (M_{VW}) market index returns based on these five assets. The weights for the value weighted market returns are adjusted each period to account for the previous growth of the assets. For example, the asset A has a payoff of 1.1% for the first period, accounting for 20% of M_{EQ} and 25% of M_{VW}. The value-weighed scheme adjusts for the size effect – likely outperformance of small stocks relative to large stocks.[4] This is exactly the case in our example with the smallest stock (D) having the highest mean return.

Given the payoff profiles of these assets, we know that D dominates assets A, B, and C by FSD, as ranked payoffs of D are always higher than those of these assets. The returns of asset A at any time are smaller than those of C, D and E, thus,

[3] In this example our asset E has the same properties as the one used in the example of Post and Kopa (2017).

[4] See Fama and French (1992) among many others.

A is dominated by these assets by state-wise dominance. State-wise dominance is related to explicit relationship between probability density functions and is the most fundamental form of FSD. Asset B dominates C by SSD, as the cumulative sum of the ranked returns of B is always higher than that of C. Note this case, the mean return of asset B (1.10) is larger than that of asset C (1.05), but the standard deviation of B (0.17) is also higher than that of C (0.16). Hence, there is no clear dominance relation between these two assets in the mean-variance sense. However, a slightly larger variance of B is a result of "good" returns, which can be assessed by comparing the empirical cumulative distribution functions of the returns of these two assets. The relation of the cumulative distribution functions can also be captured by the partial sums of the ranked returns (Kuosmanen 2004). The cumulative sums of returns for asset B of [0.9; 1.99; 3.3] are always higher than the sums of C of [0.83; 1.93; 3.14], implying dominance of B over C in the SSD sense. One could look at this ranking also from the point of view of the expected losses relative to all possible thresholds for these two distributions. The expected losses are always smaller in absolute values for asset B than asset C. For example, consider a threshold 1.1. Relative to this threshold, with a probability of $1/3$ asset B loses $1.1 - 0.9 = 0.2$, and with a probably of $1/3$, it loses $1.1 - 1.09 = 0.01$. Hence, the expected loss is $0.2/3 + 0.01/3 = 0.07$. For asset C, the expected loss is larger, $(1.1 - 0.83)/3 = 0.09$, and the relation holds for all other thresholds as well.

Comparing the asset return with the market return distributions in the SD sense, we can see that the equal-weighted market index dominates A and C by FSD, it dominates B by SSD, and it is dominated by D in terms of FSD and by E in terms of TSD. Asset E has the same mean return as the equal-weighted market index (1.10) and a slightly higher variance (0.17 vs 0.16), which rules out the SSD relation between the two. However, asset E exhibits much higher skewness (0.69 vs 0), which results in its dominance over the market in the TSD sense.

The SD relations remain the same for A and D relative to the value-weighted market index. This is because A (D) has a lower or equal (higher or equal) return at each state than the two market indices, hence, the weighting scheme of the market index does not affect pairwise dominance. However, the dominance relations change for B, C, and E. The value-weighted market index and B lie in the same dominance class, meaning that none of them dominates each other by either FSD or SSD. Asset C is dominated by equal-weighted market index by FSD and by value-weighted index by SSD. In fact, C is dominated by D in terms of FSD, and D contributes substantially to the performance of the equal-weighted market index. On the contrary, C dominates A by FSD, but it is dominated by B by SSD, and the performance of the value-weighted market index is substantially driven by the larger stocks.

We next calculate two betas for each asset, using the equally weighted and value weighted market indices. Asset A has the lowest β_{EQ} of 0.50 as well as the lowest β_{VW} of 0.56, while D has the highest beta estimates of 2.08 and 2.72 respectively. Low beta investing suggests taking a long position in A and a short position in D, which is against the fact that D dominates A by FSD. Low beta investing in this case is equivalent to purchasing a dominated asset and selling a dominating asset, which results in utility losses for all investors.

Hence, a beta-based asset choice can provide misleading indications when it is going against the SD relationship, and this issue persists when different weighting schemes are used to compute the market return.

[Table 2 around here]

2.3 SD-enhanced BAB strategy

To enhance performance of the BAB strategy and avoid potential losses due to inconsistencies between SD- and market-beta-based rankings, we suggest implementing SD-prefiltering of stocks before constructing the BAB strategy. All dominated stocks relative to the market are to be removed from the long leg, and all dominating stocks are to be removed from the short leg.

To assess the contribution of such pre-filtering to performance, we collect monthly and daily return data for the CRSP stock universe from July 1926 to December 2020. Following Frazzini and Pedersen (2014), we use 5-year rolling window for beta estimation. Specifically, at the beginning of each month from July 1931 to December 2020, we compute beta estimate for stock i as follows:

$$\hat{\beta}_i^{FP} = 0.6 \times \hat{\rho}\frac{\hat{\sigma}_i}{\hat{\sigma}_m} + 0.4, \tag{2}$$

where $\hat{\sigma}_i$ is the stock return volatility calculated using daily log returns from previous one-year period, while $\hat{\rho}$ is the correlation between stock returns and market returns calculated using overlapping three-day log returns from previous 5-year period. We require at least 120 daily returns to estimate the volatility and at least 750 daily returns to estimate the correlation.

In parallel, for each stock we conduct the non-dominance test of Davidson (2009) to determine if the stock dominates the market portfolio or is dominated by it based on one-year period of past returns. We use the CRSP US total market index as the market portfolio and a one-year estimation window to be consistent with volatility estimation for the beta. The Davidson (2009) test is chosen due to the convenient formulation of the null hypothesis in terms of non-dominance.[5] Rejection of the null hypothesis leaves only one conclusive alternative of dominance.[6]

The test procedure works as follows. We are interested in the potential SD relations at order S for a pair of assets A and B with T observations each. The test works on the joint support of the empirical return distributions, trimming 5% extreme observations, which forms the set of thresholds z. For each asset, its dominance function with respect to a certain threshold z can be computed as:

$$D^S_{Asset}(z) = \frac{1}{(S-1)!T}\sum_{i=1}^{T}(\max(z - y_i, 0))^{S-1} \tag{3}$$

where y_i is the ith ranked return of the Asset, which is either A or B in this example.

The test statistics of the null hypothesis that asset A does not dominate asset B at order S (t^*) is the minimum of the standardized differences in the dominance functions across all thresholds:

$$t^* = \min_z \frac{D^S_B(z) - D^S_A(z)}{\sqrt{var(D^S_A(z)) + var(D^S_B(z)) - 2cov(D^S_A(z), D^S_B(z))}} \tag{4}$$

where $var(\cdot)$ and $cov(\cdot)$ are the estimated variance and covariance of the corresponding terms. t^* is asymptotically normally distributed. See Davidson (2009) and Hodder et al. (2015) for more technical details.

[5]There exist other potential SD-related tests that one could use for stock prefiltering, including Post (2003), Kuosmanen (2004), Fang and Post (2017), Linton et al. (2005, 2010). The chosen Davidson (2009) test is characterised by computational advantage compared to the other tests, as we discuss in detail in Online Appendix.

[6]The MATLAB code for the SD test is available from the authors upon request.

When implementing this procedure, for each stock i, we first test the null hypothesis that this stock does not dominate the market portfolio by SSD or TSD. Rejection of it implies that stock i dominates the market portfolio. Economically, such dominance means that all risk-averse (for SSD) and prudent (for TSD) investors would be better off investing in stock i as opposed to the index.

Next, we use the non-dominance test with the complementary null hypothesis that the market portfolio does not dominate stock i by SSD or TSD. Rejecting this null implies that stock i is dominated by the market portfolio, hence, all risk averse or prudent investors are better off investing in the market. For both cases, we choose a conservative approach to reject the null hypothesis only when the p-value is smaller than 1%. If none of the hypotheses can be rejected, stock i and the market portfolio lie in the same dominance class. This means that for some risk-averse or prudent investors stock i is the preferred option, while for other investors investing in the market would be preferred.

As a result, at the beginning of each month each stock is characterized by its estimated beta and indicators of whether it is dominated by the market, it dominates the market, or it lies in the same dominance class with the market portfolio according to SSD and TSD criteria. We next create long-short SSD- and TSD-enhanced BAB portfolios consistent with Frazzini and Pedersen (2014). The long leg of the portfolios includes half of stocks with the market betas below the median, excluding those stocks that are dominated by the market by SSD or TSD respectively. The short leg of the portfolios includes half of stocks with the market betas above the median, excluding those stocks that dominate the market by SSD or TSD. The portfolios are held for one month and then rebalanced. We label the resulting SD-enhanced BAB portfolios as SSDBAB and TSDBAB for SSD and TSD prefiltering respectively.

For comparison purposes, we also construct portfolios purely based on SSD and TSD. Such choices are not necessarily straightforward and are often subject to standalone research.[7] To assure computational feasibility and conceptual consistency with the previous procedure of enhancing the BAB portfolio, we construct SSD and TSD long-short portfolios based on a pairwise comparison of each stock with the market index. Each month t using the past year of daily data, we test for the SSD and TSD relations between each stock and the market, similar to the first step of our prefiltering approach. The long leg of the SSD or TSD portfolio is then constructed as an equally weighted portfolio of all stocks that dominate the market at the required order, with the null hypotheses of non-dominance of the stock over the market rejected at the 1% level. The short leg is an equally weighted portfolio of all stocks that are dominated by the market, with the null hypotheses of non-dominance of the market over the stock rejected at the 1% level. The SSD and TSD based long-short portfolios are constructed separately.

2.4 Performance evaluation

Barroso et al. (2021) argue that the BAB outperformance can be explained by several

[7] Often, research focuses on SD-efficient portfolios, which cannot be dominated by any combination of the same underlying assets. The corresponding linear programming algorithms are computationally intensive and work only with a limited number of assets. Hodder et al. (2015), for example, use 49 industry portfolios and construct SSD efficient portfolios based on Kuosmanen (2004) and Kopa and Post (2015) linear programming approaches. Post and Kopa (2017) use TSD and again 49 industry portfolios as base assets. Kolokolova et al. (2022) use a pairwise comparison between industry indices and the market and show that past dominance of an industry portfolio over the market index predicts future dominance. A notable exception is Clark and Kassimatis (2014), where the authors use individual stocks in the UK, and their investment rule is based on pairwise comparisons of all stocks with each other. This renders $N \times (N-1)$ comparisons every evaluation period, which is not computationally feasible for the CRSP universe of stocks.

risk factors. Specifically, they find that the Fama-French 6-factor model subsumes the BAB abnormal return. Hence, in order to comprehensively evaluate the performance of the SD-enhanced BAB portfolios and compare it with the original BAB, we estimate a set of abnormal returns (alphas) relative to the following benchmark models:

(1) FF4 – the Fama-French 3-factor model, that includes the excess return of the market over the risk-free rate (MKT_RF), the size (SMB) and value (HML) factors, plus the Carhart momentum factor (MOM);

(2) FF4+PS – the FF4 model augmented by the Pastor and Stambaugh (2003) traded liquidity factor;

(3) FF5 – the Fama-French 5-factor model, that includes the profitability (RMW), and investment (CMA) factors;

(4) FF6 – the Fama-French 6-factor model, that adds MOM to FF5;

(5) q5 – the q-5 factor model, that includes MKT_RF, SMB, as well as the investment (IA), return on equity (ROE), and expected growth (EG) factors (Hou et al. 2021).

Bali et al. (2017) show that the lottery-demand related factor FMAX can explain the outperformance of the BAB strategy. Thus, we further augment the FF4 model with the FMAX factor (FF4+FMAX). Last but not least, to assess if the SD prefiltering expands the investment opportunity set relative to the original BAB, we regress the performance of our alternative portofplios on the BAB factor only.

In addition to the risk-adjusted performance relative to the factor models, we estimate a wide range of performance and risk measures that are often used in portfolio management and risk reporting. These include:

(1) classical symmetric measures: mean return, return standard deviation, and Sharpe ratio;

(2) measures focusing on downside risk and upside potential: Omega, Sortino, and Upside potential ratios;

(3) measures focusing on tail returns: Value-at-Risk (VaR) at 5%, expected shortfall (ES) at 5%, maximum drawdown (MDD), minimum return, maximum return, skewness, and kurtosis;

(4) lower partial moments (LPM): zero-, first-, and second-order lower partial moments which capture the loss frequency, average shortfall, and semi-variance, respectively;

(5) utility-based measures: mean-variance utility functions with the market level of risk aversion m, as well as risk aversions of 1, 3, and 5 (Um, U1, U3, U5), and certainty equivalents relative to the power utility function with three risk aversion specifications of 3, 5, and 10 (CEV3, CEV5, and CEV10);

(6) the manipulation-proof performance measures (MPPM) of Ingersoll et al. (2007);

Table 3 summarizes all the performance measures and provides their exact definitions.

[Table 3 around here]

2.5 Transaction cost

Monthly portfolio rebalancing often requires substantial changes in the portfolio composition, and the associated transaction costs may erode performance. To control for these effects, we estimate turnover and transaction cost following Barroso and Santa-Clara (2015) and Barroso and Detzel (2021). Specifically, the turnover (TO) of the long leg is computed as

$$TO_{long,t} = \frac{1}{2} \sum_{i=1}^{N_t} |w_{i,t} - \widetilde{w}_{i,t-1}|, \tag{5}$$

where N_t is the number of valid stocks in month t, $w_{i,t}$ is the weight of stock i in the long leg of month t after rebalancing, and $\widetilde{w}_{i,t-1}$ is its weight before rebalancing defined as $\widetilde{w}_{i,t-1} = \frac{w_{i,t-1}(1+r_{i,t})}{\sum_{j=1}^{N_t} w_{j,t-1}(1+r_{j,t})}$ with $r_{i,t}$ denoting the raw return of stock i. The turnover of the short leg is similarly defined, and the turnover of the overall portfolio is the sum of the estimated turnovers of the long and short legs.

The transaction cost (TC) associated with the long leg of trades is computed as

$$TC_{long,t} = \sum_{i=1}^{N_t} |w_{i,t} - \widetilde{w}_{i,t-1}| c_{i,t}, \tag{6}$$

where $c_{i,t}$ is one-way transaction cost. The transaction cost of the short leg is computed similarly, and then the total transaction cost of the portfolio is the sum of those for the short and the long legs. Following Moreira and Muir (2017, Section II.B), we consider three scenarios of constant trading cost: 1bp, 10 bps, and 14 bps.

We adjust the performance of all the portoflios for the transaction costs and repeat the analysis.

The transaction cost adjustment as discussed above captures predominantly the trading costs arising from rebalancing portfolios due to a bid-ask spread and a price impact. In practice, additional substantial costs can be associated with short selling especially for hard-to-borrow stocks, that tend to be smaller and less liquid. For example, using proprietary lending data, Cohen et al. (2007) report that between 1999 and 2003 the average stock borrowing fee for large stocks is 0.4% per year, while it is around 4% for small stocks. Bekjarovski (2018) provides a more detailed description of borrowing costs, that range from as little as 0.35% per year for the largest 10% of stocks to 21.85% for the smallest 10% of stocks.

We use the average borrowing costs for stocks in different size deciles as reported in Bekjarovski (2018), and further adjust the resulting returns of our long-short portfolios. Each month prior to investing, all stocks are sorted into deciles from smallest to largest, according to NYSE size break points.[8] The annual borrowing costs $c_{i,t}^{BOR}$ for stock i are determined according to the size decile it belongs to. For the size deciles (from smallest to largest stocks) the costs used are [21.85%, 10.90%, 3.40%, 2.37%, 0.92%, 0.58%, 0.44%, 0.38%, 0.37%, 0.35%] per year, respectively. Then, monthly borrowing costs for the short leg of our portfolios are computed as:

$$TC_{short,t}^{BOR} = \sum_{i=1}^{N_t} w_{i,t} c_{i,t}^{BOR} / 12, \tag{7}$$

and we subtract these additional borrowing costs from the returns of the long-short portfolios.

[8] See http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html#Breakpoints.

3 Data and Empirical results

3.1 Sample construction and BAB replication

We collect monthly and daily return data from CRSP database. Our sample includes all common stocks listed on the NYSE, AMEX, or NASDAQ that have a CRSP share code of 10 or 11 in CRSP universe from July 1926 and December 2020. We also assemble daily and monthly data of common risk factors such as the market (MKT), size (SMB), and value (HML) factors from Fama and French (1993), the momentum (MOM) factor from Carhart (1997), the profitability (RMW) and investment (CMA) factors from Fama and French (2016), the investment (IA) and return on equity (ROE) factors from Hou et al. (2015), and the expected growth (EG) factor from Hou et al. (2021). The risk-free rate is the one-month US Treasury bill rate. The MKT, SMB, HML, MOM, RMW, and CMA factors are from Kenneth French data library. The IA, ROE, and EG factors are from Lu Zhang global-q data library.[9] Table 4 reports the descriptive statistics of the stock returns and the factors.

Next, we replicate the performance of the BAB portfolio using our data and compare it with the one maintained by AQR.[10] The descriptive statistics reported in Table 5 indicate that our replication is very close to that of AQR. The mean returns are statistically indistinguishable from one another, and we do not detect significant differences in the compete return distributions using the Kolmogorov-Smirnov test.

[Tables 4 and 5 around here]

[9]The corresponding websites are https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html and http://global-q.org/index.html.

[10]The data is from AQR data library: https://www.aqr.com/Insights/Datasets/.

3.2 Performance evaluation of SD-enhanced BAB portfolios

We compare high-beta stocks with the market portfolio in terms of SSD and TSD and find that the original BAB portfolio contains a substantial fraction of SD-dominating stocks in its short leg. These stocks are expected to improve utility of all risk-averse investors, hence, shorting these stocks is suboptimal. Our prefiltering approach effectively removes these stocks from the short leg of the BAB portfolio. The average fraction of SSD dominating stocks is about 15% across the whole sample and about 18% during the past 30 years, but it varies substantially over time as depicted in Figure 1.[11] As for the low-beta stocks in the long leg of the BAB portfolio, we find very few instances in which these stocks are dominated by the market. In fact, the fraction of such stocks is, on average, below 1%. TSD suggests even larger exclusions with the average fraction of stocks excluded from the short leg being 35%, while that for stocks excluded from the long leg remaining below 1%.

[Figure 1 around here]

We now compare the performance of the portfolios in terms of their alphas relative to different benchmark models. Barroso et al. (2021) argue that the outperformance of the BAB portfolio can be explained by its exposure to different risk factors beyond the CAPM. Specifically, they find that the Fama-French 6-factor model which combines

[11] The fraction of excluded stocks based on SSD is negatively related to the market performance over the estimation window and the FMAX factor performance with correlation coefficients of -0.44 and -0.43, respectively. It is positively correlated with the performance of the Pastor and Stambaugh (2003) traded liquidity factor with a correlation coefficient of 0.19. Hence, we perform a subsample analysis of the performance of our trading strategy based on market conditions, lottery demand, and liquidity in the later sections. The fraction of stocks excluded from the short leg based on TSD has virtually zero correlation with these factors, highlighting the fact that TSD prefiltering provides a unique perspective on stock allocation, which is not directly captured by the major commonly used risk factors.

the momentum factor with Fama-French 5 factors subsumes the BAB abnormal return. Our results in Table 6 provide consistent evidence. The abnormal return of the BAB portfolio declines from 6.49% per year relative to the FF4 model to just 3.76% per year relative to the Fama-French 6-factor model, with much reduced economical and statistical size. The abnormal return is even smaller and weakly statistically significant relative to the q-5 model. On the contrary, including the additional systematic risk factors does not erode the performance of SSDBAB and TSDBAB to the same extent. The minimum abnormal return for SSDBAB of 5.26% relative to the Fama-French 6-factor model is still statistically significant at the 1% and it is some 40% higher than its BAB counterpart. The difference is even higher when the q-5 factor model is used as a benchmark, with the abnormal returns being 5.45% for SSDBAB and 3.64% for BAB. These differences are not only economically large, but also statistically significant at the 1% level, with SSDBAB and TSDBAB significantly outperforming BAB in terms of the alphas.

The alpha of the BAB strategy loses statistical significance after inclusion of the FMAX into the FF4 model, consistent with the findings in Bali et al. (2017). The alphas of SSDBAB and TSDBAB still remain statistically significant in the presence of the FMAX factor. Conceptually, third-order SD captures preferences for skewness. That is, when the first two moments of distributions remain unchanged, the third-order SD ranking will favour the distribution with higher skewness. Hence, TSDBAB restricts BAB from shorting high-skewness stocks, everything else being equal. Second-order SD, however, allows for skewness averse preferences. As a result, the lottery factor FMAX captures more of the extra return of TSDBAB as compared to SSDBAB. The alpha of SSDBAB relative to the Fama-French 4 factors plus FMAX is higher (3.07% per year) than that of TSDBAB (2.06% per year). At the same time, TSDBAB seems to expand the investment set relative to BAB, as the direct regression of the performance of TSDBAB on BAB results in a highly significant alpha of 0.84% per year. SSDBAB remains relatively close to BAB and its corresponding alpha of 0.65% per year is not statistically significant.[12]

As for pure SD-based portfolios, their alphas relative to Fama-French 4- and 5-factor models are not statistically different from those of BAB. These portfolios, however, seem to be loading on different factors. For example, once the FMAX factor (capturing returns on the lottery-type stocks) is included in the regression, the alpha of BAB turns not statistically significant, while the alphas of SSD and TSD portfolios become high and statistically significant, reaching over 10% per year for SSD. These portfolios also have very high alphas relative to BAB itself, up to almost 18% per year for SSD. However, these results should be interpreted with caution. Pure SSD and TSD portfolios are substantially under-diversified compared to BAB. On average, they include 462 (for SSD) and 1064 (for TSD) stocks in the long leg and only one or two stocks in the short leg, compared to 1709 stocks for BAB in each of the legs. These makes these portfolios a much riskier investment option, as can be seen from Table 7.

[Table 6 around here]

Table 7 reports the other performance and risk measures of the portfolios, as well as their differences. To access the statistical significance of the differences in the

[12]We further decompose the strategy performance into that of the long and the short leg separately. Consistent with the observation that SD-prefiltering impacts the short leg the most, we see that the improvement of the overall portfolio alphas is due to the "better" (more negative) performance of the short leg. The detailed results are reported in Online Appendix.

measures, we use a bootstrap with replacement from the original dates of portfolio returns and 1,000 replications. Overall, SD-enhanced strategies exhibit better performance across all dimensions, with the difference being especially pronounced and often statistically significant for TSDBAB. Compared to BAB, TSDBAB exhibits significantly higher mean return, Sharpe ratio, Sortino ratio, Omega, and the Upside potential ratio, higher MPPM, as well as higher certainty equivalents. As for SSDBAB, it significantly increases portfolio skewness and converts large losses (smaller LPM2) to small losses (higher LPM0), which highlights the SD essence to smooth payoffs across the states of nature. As mentioned above, pure SSD and TSD portfolios are much riskier. They exhibit significantly higher mean than BAB, which is more than offset by higher variance, resulting in significantly lower Sharpe ratios and underperformance based on many other measures from VaR to MPPM and CEV. We, hence, concentrate on BAB and SD-enhanced BAB portfolios in the following analysis.

[Table 7 around here]

After the transaction costs are taken into account, the outperformance pattern of SSDBAB and TSDBAB remains pronounced. For example, without transaction costs, SSDBAB exhibits a positive alpha relative to BAB of 0.65%, while TSDBAB exhibits a positive and significant alpha of 0.84%. Proportional transaction costs of 14bps reduce the alphas to 0.56% and 0.68% respectively (Panel A of Table 8). Such relative outperformance of SD-enhanced portfolios persists even after incorporation of the borrowing costs (Panel B of Table 8), with the alphas relative to BAB being 0.87% highly significant for SSDBAB and 0.41% significant at the 10% level for TSDBAB.

Remarkably, SSD-based pre-filtering seems to exclude more hard to sell stocks, hence, reducing the return losses due to the associated costs.

It is worth mentioning that after adjusting portfolio returns for stock borrowing costs, the alphas relative to all classical factor models become negative. This, however, could not be directly interpreted as underperformance relative to the factors, since the factors themselves are long-short portfolios which are not adjusted for the borrowing costs of their short legs. Still, SD-enhanced BAB portfolios exhibit significantly higher alphas relative to the factor models than BAB after controlling for transaction costs.

The differences in other measures are also preserved, even though they become smaller in absolute values after the inclusion of transaction costs. For instance, the difference in Sortino ratios between TSDBAB and BAB reduces from 0.12 without transaction costs to 0.05 with 14bps proportional transaction costs and borrowing costs, remaining significant at the 5% level. Similar to the results related to the alphas, SSDBAB displays the most robust performace when the stock borrowing costs are accounted for. Remarkably, the introduction of borrowing costs leads to a negative mean return of BAB, while it remains positive for SSDBAB.

[Tables 8 and 9 around here]

Figure 2 plots the cumulative performance of the BAB, SSDBAB, and TSDBAB portfolios from July 1931 until December 2020. During early years the contribution of the SD-prefiltering seems to be minor: the cumulative performance of the portfolios is almost indistinguishable up until mid 1990s. The SD-enhanced portfolios perform considerably better than the simple BAB in the last 20 years, with TSDBAB exhibiting the largest improvement. The last decades have seen the most dramatic episodes for financial markets and general economic development, including the Internet bubble and its burst in 2000, the housing bubble and the following financial crisis in 2007-2009. The bottom sub-figure of Figure 2 zooms into the cumulative performance of the strategies from January 1990 up until December 2020. An initial investment of $1 in 1990 in BAB turns into $13.88 at the end of 2020, the investment in SSDBAB results in a higher reward of $17.03 while TSDBAB delivers $20.20 as of the end of 2020.

[Figure 2 around here]

4 Robustness: Different Market Conditions

Financial assets are likely to exhibit different dynamics depending on the state of the economy. In particular, during turbulent or crisis period many assets become more heavily interdependent, especially in the tails of the distributions (Chabi-Yo et al. 2018), and the marginal benefit of any changes in portfolio composition reduces. Hence, we now evaluate the relative performance of the BAB and SD-enhanced BAB strategies during different market conditions, using a sub-sample analysis.

First, we report sub-sample results for periods of economic recessions and expansions. We use the recession indicator as provided by the Federal Reserve Bank of St. Louis (FRED)[13] to determine recession months (those with the recession indicator=1) and not-recession months (the recession indicator=0). Second, we use the traded liquidity factor of Pastor and Stambaugh (2003) and split the months according to their liquidity based on the median value of the factor. Last but not least, we

[13] https://fred.stlouisfed.org/series/USREC

split the sample according to the lottery demand FMAX factor of Bali et al. (2017) being above or below the median.

Tables 10 to 15 report the results for the estimated alphas and other risk measures. Figure 3 plots the alphas relative to the q5 model, to illustrate the general patterns in performance. During non-recession periods, both SSDBAB and TSDBAB significantly outperform BAB along most of the dimensions, including alphas relative to the factor models (Table 10) and other risk measures (Table 11). During recession, as expected, we can see less differences in the performance. During recession period, SSDBAB performs best, still delivering significantly higher alphas than those of BAB (although all the alphas are negative during this period), and having significantly lower return standard deviation, higher Upside potential ratio, and higher return skewness.

Looking at the differences with respect to the liquidity conditions, TSDBAB performs significantly better than BAB in terms of alphas and many other risk measures in both sub-periods, but the gains are larger in absolute values during periods with high values of the traded liquidity factor.

An interesting pattern arises with respect to lottery demand. TSDBAB exhibits significantly higher alphas during periods with low lottery demand, while no significant differences in alphas can be seen during high lottery demand periods. Still, in both periods TSDBAB has a positive and significant alpha relative to BAB only. As for all the other risk measures, TSDBAB outperforms BAB in both sub-periods to the same extent.

Overall, SD prefiltering improves performance of BAB across periods with different market conditions. TSD-based enhancement usually delivers the strongest improvement, while during periods of economic recession SSD-based enhancement seems the most valuable.

5 Conclusion

Low-beta investing has attracted substantial interest from both academics and practitioners due to its superior performance, which may seem to contradict expectations drawn from the classical finance theories. In this paper we argue that ranking stocks based on their market beta to determine the components of the long and short legs of a zero-cost investment strategy may not always be consistent with a more powerful notion of stochastic dominance.

Stochastic dominance considers complete distributions and not only the first two moments (which are the implicit drivers of the beta). More importantly, stochastic dominance is closely linked to the decision making under uncertainty. If one distribution stochastically dominates the other one by second order, all risk-averse investors will prefer the former to the latter.

We propose an intuitive SD prefiltering rule to improve the beta-based investment strategy. It excludes stocks that have been dominated by the market index during the previous year from the long leg of the strategy, and stocks that dominated the market from the short leg of the strategy. Empirically, this pre-filtering suggests very few exclusions of low-beta stocks from the long leg, while a substantial fraction of high-beta stocks should be excluded from the short leg, averaging 15% to 35% for SSD and TSD prefiltering.

Compared to the conventional BAB portfolio, SD-enhanced BAB exhibits superior performance in terms of a wide range of performance and risk measures, including abnormal returns relative to various risk-factor models. These benefits are pronounced across different economic conditions and are especially relevant during the recent decades.

On a broader note, the SD prefiltering is likely to improve performance of other diversified long-short trading strategies, such as strategies based on various market anomalies. Excluding dominated stocks from long legs and dominating stocks from short legs of such strategies is likely to increase utilities of all risk-averse and prudent investors while potentially reducing the implementation costs of the strategies.

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Table 1: Key low-beta anomaly research papers

This table lists the key research contributions related to the low-beta anomaly in a chronological order.

Main theme	Key findings	References
Classic CAPM	Expected return is significantly positively related to market beta	Sharpe (1964), Lintner (1965), Mossin (1966)
Flat security market line (SML)	SML is too flat and low beta stocks have a premium	Friend and Blume (1970), Jensen et al. (1972), Miller and Scholes (1972), Haugen and Heins (1975), Reinganum (1981), Fama and French (1992), and Rouwenhorst (1999)
The intuition of beta arbitrage	A portfolio that longs low beta stocks and shorts high beta stocks should generate excess return	Black (1972, 1993)
Persistence of the beta anomaly	A low-beta portfolio outperforms even after controlling for liquidity, volume, and momentum effects	Ang et al. (2006)
	The low-risk anomaly can be a result of irrationality and institutional constraints preventing arbitrage	Baker et al. (2011)
	Leverage constraints are related to the low-beta anomaly, and betting-against-beta (BAB) exhibits excellent performance	Frazzini and Pedersen (2014)
	BAB performance is driven by non-standard procedures used in its construction that effectively, but non-transparently, equal weights stocks returns	Novy-Marx and Velikov (2021)
Explanations of the beta anomaly	The poor performance of the CAPM since 1963 can be explained by the fact that growth stocks and high-past-beta stocks exhibit predominantly high betas with low price risks	Campbell and Vuolteenaho (2004)
	Trading friction anomalies, including strategies related to a low beta and Amihud (2002) illiquidity, are not significant with the q 4-factor model which consists of the market, size, investment, and return on equity factors.	Hou et al. (2015)
	Positive exposures to RMW (Robust Minus Weak) and CMA (Conservative Minus Aggressive) capture the low market premium of high-beta stocks.	Fama and French (2016)
	The demand for the lottery-type stocks explains the beta anomaly	Bali et al. (2017)

Table 2: Low-beta vs SD – An example

This table presents an example of potential inconsistency between beta-based ranking and SD relation between the assets. A, B, C, D, and E stands for different assets. M_{EQ} and M_{VW} stands for the equal-weighted and value-weighted market index respectively, based on the four assets. μ is the mean return and σ is the return standard deviation. β_{EQ} and β_{VW} are market betas calculated relative to the equal-weighted and value-weighted market indices, respectively. $\prec^n$ indicates that the asset is dominated by the market at order n, $\succ^n$ indicates that the asset dominates the market at order n, while $\approx$ stands for no dominance relation between two assets.

	Assets					Market	
	A	B	C	D	E	M_{EQ}	M_{VW}
Time t			Returns				
1	1.1	0.9	1.1	1.4	1	1.10	1.04
2	0.7	1.09	0.83	0.91	0.97	0.90	0.91
3	0.9	1.31	1.21	1.74	1.34	1.30	1.21
		Weighting schemes for market returns					
Wgt_{EQ}	0.2	0.2	0.2	0.2	0.2		
Initial Market Cap	25	35	25	5	10		
$Wgt_{VW}(1)$	0.25	0.35	0.25	0.05	0.1		
$Wgt_{VW}(2)$	0.2502	0.3495	0.2502	0.0502	0.1000		
$Wgt_{VW}(3)$	0.2496	0.3502	0.2500	0.0502	0.1000		
		Return distribution characteristics					
μ	0.90	1.10	1.05	1.35	1.10	1.10	1.05
σ	0.16	0.17	0.16	0.34	0.17	0.16	0.12
skewness	0.00	0.09	-0.46	-0.22	0.69	0.00	0.18
β_{EQ}	0.50	0.55	0.95	2.08	0.93	1	
β_{VW}	0.56	0.82	1.23	2.72	1.27		1
		SD relation: asset to market					
M_{EQ}	$\prec^1$	$\prec^2$	$\prec^1$	$\succ^1$	$\succ^3$		
M_{VW}	$\prec^1$	$\approx$	$\prec^2$	$\succ^1$	$\succ^2$		

Table 3: Performance and risk measures: Definitions

This table lists the equations and definitions for the performance and risk measures used in the paper. r_i denotes the ith return observation and n denotes the total number of observations.

Variable	Formula	Definition
μ	$\sum_{i=1}^{n} r_i/n$	Mean return
σ	$\sqrt{\sum_{i=1}^{n}(r_i-\mu)^2/(n-1)}$	Return standard deviation measures the variation of return observations relative to the mean
SR	SR=μ/σ	Sharpe ratio measures the return per unit of risk. We keep the risk-free rate at zero in this specification
Omega	$\frac{\sum_{i=1}^{n} r_i I_{(r_i>0)}/n}{LPM1}$	Omega measures the ratio of the expected gains to expected losses.
Sortino	$\frac{\mu}{\sqrt{LPM2}}$	Sortino ratio measures excess return per unit of downside deviation
Upside	$\frac{\sum_{i=1}^{n} r_i I_{(r_i>0)}/n}{\sqrt{LPM2}}$	Upside potential ratio measures the ratio of gains per unit of downside deviation
VaR(5%)	$P(r \leq VaR(5\%))$=5%	Value-at-Risk (5%) is the threshold for which the probability of a loss worse than VaR is 5%
ES(5%)	$\frac{\sum_{i=1}^{n} r_i I_{(r_i \leq VaR(5\%))}}{\sum_{i=1}^{n} I_{(r_i \leq VaR(5\%))}}$	Expected shortfall (5%) measures the expected loss if it is worse than VaR(5%)
MDD	$max_{\tau \leq n}(max_{t \leq \tau}(X_t)-X_\tau)$	Maximum drawdown is the maximum loss of the value of investment from a peak to a trough
Min	$min(r_i)$	Minimum return is the minimum value of all return observations
Max	$max(r_i)$	Maximum return is the maximum value of all return observations
Skew	$\frac{\sum_{i=1}^{n}(r_i-\mu)^3/n}{\sigma^3}$	Skewness measures the asymmetry of return distribution
Kurt	$\frac{\sum_{i=1}^{n}(r_i-\mu)^4/n}{\sigma^4}$	Kurtosis measures how heavy-tailed the return distributing is
LPMp	$\sum_{i=1}^{n}(\mid r_i \mid)^p I_{(r_i<0)}/n$	The p-order lower partial moment, p=0, 1, and 2. LPM0 measures the frequency of losses; LPM1 measures the average loss; LPM2 measures the variation of losses.
U$_\gamma$	$\mu - \frac{\gamma}{2}\sigma^2$	Mean-variance utility relative to risk aversion γ m=2.6 is the market-implied risk aversion and we also consider γ of 1, 3, and 5.
CEV$_\gamma$	$u^{-1}(\sum_{i=1}^{n} u(r_i+1))/n)$ $u(r_i+1)=(1+r_i)^{1-\gamma}/(1-\gamma)$	Certainty equivalent is a risk-free return that an investor with a power utility function and a risk aversion of γ regards as equivalent to the risky investment. We consider γ of 3, 5, and 10.
MPPM	$\frac{1}{1-m}ln(\frac{1}{T}\sum_{t=1}^{T}(\frac{1+r_t}{1+r_{ft}})^{1-m})$	Manipulation-proof performance measure proposed by Goetzmann et al. (2007)

Table 4: Descriptive statistics of stock and factor returns

This table summarizes the descriptive statistics of the returns of the individual stocks in our sample and various risk factors. The returns are in percent per month. For individual stocks, we first compute their individual descriptive statistics across their lifespan, and then report cross-sectional averages, medians, and standard deviations of the statistics.

	Mean	StD	Min	Max	Skewness	Kurtosis
Panel A: Individual stocks						
Average measure	0.61	18.7	-40.06	78.63	1.05	8.38
Median measure	1.22	16.41	-39.2	58.82	0.81	5.38
Measures st. dev.	4.68	12.11	18.1	78.33	1.31	11.43
Panel B: Factors						
MKT	0.68	5.35	-29.13	38.85	0.17	10.57
SMB	0.20	3.18	-16.82	36.70	1.88	21.97
HML	0.32	3.50	-13.96	35.46	2.09	21.61
MOM	0.65	4.71	-52.27	18.36	-2.98	30.00
RMW	0.25	2.17	-18.48	13.38	-0.33	15.26
CMA	0.26	1.99	-6.86	9.56	0.31	4.61
IA	0.34	1.89	-7.16	9.24	0.15	4.24
ROE	0.51	2.56	-14.46	10.38	-0.90	8.70
EG	0.81	1.98	-9.72	11.51	0.10	6.94
PS	0.41	3.47	-13.33	11.76	-0.12	4.28
FMAX	-0.48	4.81	-27.28	33.33	0.14	10.54

Table 5: BAB replication

This table summarizes the descriptive statistics of the BAB monthly return series constructed by AQR (BAB-AQR) and our replicated BAB monthly return series. Panel A reports the mean return, standard deviation, minimum return, maximum return, skewness, and kurtosis. Panel B reports the test statistics and p-values of the paired-sample t-test for the null hypothesis of mean return equality, and of the two-sample Kolmogorov-Smirnov test for the null hypothesis of the return distributions being the same.

Panel A: Descriptive statistics of the return distributions						
	Mean	StD	Min	Max	Skewness	Kurtosis
BAB-AQR	0.68	3.25	-21.95	18.65	-0.72	10.1
BAB	0.72	3.06	-19.16	21.18	-0.35	9.38

Panel B: Statistical tests for the differences		
	Test statistics	p-value
Difference in means (t-test)	-0.31	0.76
Difference in distributions (KS-test)	0.02	0.92

Table 6: Abnormal returns of BAB vs SSDBAB and TSDBAB

This table reports the abnormal return of the BAB, SSD, TSD, and SD-enhanced BAB (SSDBAB and TSDBAB) strategies based on the Fama-French 4-factor model (market, size, value, and momentum), Fama-French 4-factor model augmented with PS liquidity, Fama-French 4-factor model augmented with lottery demand FMAX, Fama-French 4-factor model augmented with BAB, BAB, Fama-French 5-factor model (market, size, value, robust operating profitability, and conservative investment), Fama-French 6-factor model (market, size, value, momentum, robust operating profitability, and conservative investment), and q-5 factor model (market, size, investment, return on equity, and expected growth). Values are annualized and t-statistics are robust to heteroscedasticity and autocorrelation. *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

						Difference with BAB (Portfolio − BAB)			
	BAB	SSD	TSD	SSDBAB	TSDBAB	SSD	TSD	SSDBAB	TSDBAB
FF4	6.49***	5.8***	4.38***	7.59***	6.94***	-0.69	-2.11	1.10***	0.46**
	[4.67]	[3.28]	[3.66]	[5.38]	[4.93]	[-0.28]	[-1.06]	[3.24]	[2.53]
FF4+PS	6.23***	8.71***	5.64***	7.55***	7.00***	2.47	-0.60	1.32***	0.77***
	[3.37]	[3.79]	[3.49]	[4.11]	[3.79]	[0.81]	[-0.23]	[3.02]	[3.63]
FF4+FMAX	1.54	10.83***	7.42***	3.07**	2.60*	9.29***	5.88**	1.54***	1.06***
	[1.02]	[4.72]	[4.26]	[1.99]	[1.70]	[3.37]	[2.41]	[3.41]	[4.91]
FF4+BAB		6.51***	5.23***	1.19***	0.63***	6.51***	5.23***	1.19***	0.63***
		[3.46]	[3.92]	[3.07]	[2.68]	[3.46]	[3.92]	[3.07]	[2.68]
BAB		17.77***	16.33***	0.65	0.84***	17.77***	16.33***	0.65	0.84***
		[3.46]	[4.11]	[1.51]	[3.06]	[3.46]	[4.11]	[1.51]	[3.06]
FF5	5.14***	4.84*	4.83***	5.30***	5.99***	-0.30	-0.31	0.16	0.85***
	[2.99]	[1.69]	[2.78]	[3.06]	[3.53]	[-0.09]	[-0.12]	[0.28]	[4.28]
FF6	3.76**	10.82***	7.18***	5.26***	4.78***	7.06**	3.42	1.50***	1.02***
	[2.17]	[4.28]	[4.07]	[3.06]	[2.78]	[2.22]	[1.26]	[2.96]	[4.71]
q5	3.64*	11.8***	8.91***	5.45***	4.62**	8.16*	5.27	1.81***	0.98***
	[1.86]	[3.43]	[3.95]	[2.86]	[2.41]	[1.89]	[1.53]	[2.66]	[3.93]

Table 7: Performance and risk measures of BAB vs SSDBAB and TSDBAB

This table reports the performance and risk measures of the BAB, SSD, TSD, and SD-enhanced BAB (SSDBAB and TSDBAB) strategies. Values are annualized and one-sided p-values for the differences are obtained using 1,000 bootstrap samples. *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

						Difference with BAB (Portfolio − BAB)			
	BAB	SSD	TSD	SSDBAB	TSDBAB	SSD	TSD	SSDBAB	TSDBAB
Mean	8.47	15.13	14.51	8.69	9.01	6.66*	6.04**	0.22	0.54***
Std	10.77	37.28	27.89	10.62	10.57	26.51***	17.12***	-0.16	-0.20
SR	0.79	0.41	0.52	0.82	0.85	-0.38***	-0.27**	0.03	0.07***
Omega	6.66	5.05	5.44	6.81	6.95	-1.61**	-1.22*	0.15	0.29***
Sortino	1.21	0.77	0.90	1.31	1.33	-0.45*	-0.32	0.10*	0.12**
Upside	2.53	2.44	2.46	2.66	2.66	-0.09	-0.06	0.14**	0.13**
VaR	-4.29	-12.93	-9.61	-4.08	-4.40	-8.64***	-5.32***	0.21	-0.11
ES	-7.15	-19.14	-15.85	-6.75	-7.01	-12.00***	-8.70***	0.40*	0.14
MDD	0.47	0.76	0.74	0.47	0.44	0.29***	0.27***	0.00	-0.03
Min	-22.04	-37.50	-35.41	-19.16	-17.57	-15.46***	-13.37***	2.88*	4.47
Max	20.77	108.76	76.05	21.18	14.66	87.99***	55.28***	0.41	-6.11***
Skewness	-0.62	2.45	1.41	-0.35	-0.65	3.07***	2.03***	0.27**	-0.03
Kurtosis	10.47	21.37	16.24	9.38	7.90	10.9***	5.77	-1.09*	-2.57**
LPM0	0.34	0.46	0.41	0.36	0.34	0.12***	0.07***	0.02***	0.00
LPM1	0.76	2.75	2.12	0.75	0.75	1.99***	1.35***	-0.02	-0.02*
LPM2	4.06	32.44	21.86	3.66	3.80	28.38***	17.8***	-0.40**	-0.26
Um	6.96	-2.94	4.40	7.22	7.56	-9.90***	-2.56	0.26	0.60***
U1	7.89	8.18	10.62	8.12	8.45	0.29	2.73	0.24	0.57***
U3	6.73	-5.72	2.84	7.00	7.33	-12.44***	-3.88*	0.27	0.61***
U5	5.57	-19.61	-4.94	5.87	6.22	-25.18***	-10.50***	0.30	0.65***
MPPM	3.65	-3.54	1.67	3.93	4.25	-7.19**	-1.98	0.28	0.60***
CEV3	0.07	-0.03	0.03	0.07	0.07	-0.09***	-0.03	0.00	0.01***
CEV5	0.05	-0.14	-0.04	0.06	0.06	-0.19***	-0.10***	0.00	0.01***
CEV10	0.02	-0.50	-0.30	0.02	0.03	-0.52***	-0.32***	0.01**	0.01***

Table 8: Abnormal returns of BAB vs SSDBAB and TSDBAB: transaction cost

This tables reports the abnormal return of the BAB and SDBAB strategies relative to three assumptions of trading cost: 1bp, 10bps, and 14bps, following Moreira and Muir (2017) in Panel A, and adding short selling borrowing costs in Panel B. The alphas are computed relative to the factor models described in Section 2.4. All values are annualized, and t-statistics are adjusted for heteroscedasticity and autocorrelation. *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

	I. 1bp					II. 10bps					III. 14bps				
	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB
Panel A: Proportional trading costs															
FF4	6.47***	7.56***	6.91***	1.10***	0.44**	6.29***	7.34***	6.64***	1.05***	0.35*	6.21***	7.23***	6.52***	1.02***	0.31*
	[4.65]	[5.36]	[4.91]	[3.23]	[2.47]	[4.52]	[5.19]	[4.72]	[3.08]	[1.94]	[4.46]	[5.12]	[4.63]	[3.01]	[1.70]
FF4+PS	6.21***	7.52***	6.97***	1.32***	0.76***	6.00***	7.26***	6.66***	1.25***	0.66***	5.91***	7.14***	6.53***	1.23***	0.62***
	[3.35]	[4.09]	[3.77]	[3.01]	[3.59]	[3.24]	[3.94]	[3.61]	[2.87]	[3.14]	[3.19]	[3.88]	[3.53]	[2.81]	[2.94]
FF4+FMAX	1.51	3.05**	2.56*	1.53***	1.05***	1.31	2.78*	2.26	1.47***	0.95***	1.22	2.67*	2.13	1.45***	0.91***
	[1.01]	[1.97]	[1.68]	[3.40]	[4.86]	[0.87]	[1.80]	[1.48]	[3.27]	[4.42]	[0.81]	[1.73]	[1.39]	[3.22]	[4.22]
FF4+BAB		1.19***	0.62***	1.19***	0.62***		1.13***	0.52**	1.13***	0.52**		1.11***	0.47**	1.11***	0.47**
		[3.06]	[2.63]	[3.06]	[2.63]		[2.93]	[2.22]	[2.93]	[2.22]		[2.87]	[2.03]	[2.87]	[2.03]
BAB		0.65	0.83***	0.65	0.83***		0.59	0.73***	0.59	0.73***		0.56	0.68**	0.56	0.68**
		[1.50]	[3.02]	[1.50]	[3.02]		[1.37]	[2.67]	[1.37]	[2.67]		[1.31]	[2.51]	[1.31]	[2.51]
FF5	5.12***	5.27***	5.96***	0.15	0.84***	4.92***	5.01***	5.66***	0.09	0.74***	4.83***	4.89***	5.52***	0.07	0.70***
	[2.97]	[3.04]	[3.51]	[0.27]	[4.22]	[2.85]	[2.89]	[3.33]	[0.17]	[3.74]	[2.80]	[2.82]	[3.25]	[0.12]	[3.53]
FF6	3.74**	5.23***	4.75***	1.50***	1.01***	3.53**	4.97***	4.45***	1.44***	0.91***	3.44**	4.86***	4.31**	1.41***	0.87***
	[2.16]	[3.04]	[2.76]	[2.95]	[4.66]	[2.04]	[2.89]	[2.58]	[2.83]	[4.22]	[1.99]	[2.82]	[2.50]	[2.78]	[4.02]
q5	3.62*	5.42***	4.59**	1.80***	0.97***	3.42*	5.16***	4.29**	1.74**	0.87***	3.33*	5.04***	4.15**	1.72**	0.83***
	[1.85]	[2.85]	[2.39]	[2.65]	[3.89]	[1.74]	[2.70]	[2.24]	[2.57]	[3.51]	[1.70]	[2.64]	[2.17]	[2.53]	[3.34]
Panel B: Proportional trading costs and borrowing costs															
FF4	-2.42*	-0.54	-1.93	1.88***	0.49**	-2.60*	-0.76	-2.20	1.83***	0.39**	-2.67*	-0.87	-2.32	1.81***	0.35*
	[-1.70]	[-0.38]	[-1.35]	[5.21]	[2.45]	[-1.82]	[-0.53]	[-1.54]	[5.07]	[1.97]	[-1.88]	[-0.60]	[-1.63]	[5.01]	[1.75]
FF4+PS	-4.58**	-2.06	-3.67**	2.52***	0.91***	-4.79**	-2.33	-3.97**	2.45***	0.81***	-4.88***	-2.45	-4.11**	2.43***	0.77***
	[-2.45]	[-1.11]	[-1.97]	[5.46]	[3.71]	[-2.55]	[-1.26]	[-2.13]	[5.34]	[3.32]	[-2.60]	[-1.32]	[-2.21]	[5.28]	[3.15]
FF4+FMAX	-9.20***	-6.54***	-8.03***	2.66***	1.17***	-9.40***	-6.80***	-8.33***	2.60***	1.07***	-9.49***	-6.91***	-8.46***	2.58***	1.03***
	[-6.05]	[-4.21]	[-5.20]	[5.65]	[4.86]	[-6.18]	[-4.37]	[-5.39]	[5.53]	[4.46]	[-6.23]	[-4.44]	[-5.47]	[5.48]	[4.28]
FF4+BAB		1.85***	0.42**	1.85***	0.42**		1.79***	0.32	1.79***	0.32		1.77***	0.27	1.77***	0.27
		[5.33]	[2.12]	[5.33]	[2.12]		[5.20]	[1.61]	[5.20]	[1.61]		[5.14]	[1.39]	[5.14]	[1.39]
BAB		0.96***	0.56**	0.96***	0.56**		0.90***	0.46*	0.90***	0.46*		0.87***	0.41*	0.87***	0.41*
		[3.10]	[2.32]	[3.10]	[2.32]		[2.93]	[1.91]	[2.93]	[1.91]		[2.85]	[1.72]	[2.85]	[1.72]
FF5	-5.60***	-4.34**	-4.65***	1.26**	0.96***	-5.81***	-4.60***	-4.95***	1.20**	0.86***	-5.89***	-4.72***	-5.08***	1.18**	0.81***
	[-3.24]	[-2.51]	[-2.73]	[2.19]	[4.19]	[-3.35]	[-2.65]	[-2.90]	[2.09]	[3.77]	[-3.40]	[-2.72]	[-2.98]	[2.05]	[3.58]
FF6	-6.98***	-4.35**	-5.85***	2.63***	1.12***	-7.18***	-4.61***	-6.15***	2.57***	1.02***	-7.27***	-4.73***	-6.29***	2.54***	0.98***
	[-3.98]	[-2.52]	[-3.38]	[4.94]	[4.54]	[-4.09]	[-2.67]	[-3.55]	[4.84]	[4.15]	[-4.14]	[-2.73]	[-3.62]	[4.80]	[3.97]
q5	-7.11***	-4.15**	-6.01***	2.97***	1.10***	-7.32***	-4.41**	-6.31***	2.91***	1.00***	-7.41***	-4.52**	-6.44***	2.88***	0.96***
	[-3.60]	[-2.18]	[-3.13]	[4.22]	[4.02]	[-3.70]	[-2.31]	[-3.28]	[4.14]	[3.68]	[-3.74]	[-2.37]	[-3.35]	[4.11]	[3.52]

Table 9: Performance and risk measures of BAB vs SSDBAB and TSDBAB: transaction cost

This tables reports the performance and risk measures of the BAB and SDBAB strategies relative to three assumptions of trading cost: 1bp, 10bps, and 14bps, following Moreira and Muir (2017). All values are annualized, and one-sided p-values for the differences are obtained using 1,000 bootstrap samples. *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

	I. 1bp					II. 10bps					III. 14bps				
	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB
	Panel A: Proportional trading costs														
Mean	8.45	8.66	8.98	0.22	0.53***	8.27	8.44	8.71	0.16	0.44***	8.20	8.34	8.59	0.14	0.40**
Std	10.77	10.62	10.57	-0.16	-0.20	10.78	10.62	10.57	-0.15	-0.20	10.78	10.62	10.58	-0.15	-0.20
SR	0.78	0.82	0.85	0.03	0.07***	0.77	0.79	0.82	0.03	0.06***	0.76	0.78	0.81	0.02	0.05***
Omega	6.65	6.80	6.93	0.15	0.28***	6.56	6.68	6.79	0.12	0.23**	6.52	6.63	6.73	0.11	0.21**
Sortino	1.21	1.31	1.33	0.10	0.12***	1.18	1.27	1.28	0.08	0.10**	1.17	1.25	1.26	0.08	0.09**
Upside	2.53	2.66	2.66	0.14*	0.13**	2.50	2.63	2.62	0.13**	0.12**	2.49	2.61	2.60	0.12*	0.11**
VaR	-4.29	-4.08	-4.40	0.21	-0.11	-4.31	-4.11	-4.44	0.21	-0.12	-4.32	-4.12	-4.45	0.21	-0.13
ES	-7.15	-6.76	-7.01	0.40*	0.14	-7.17	-6.78	-7.04	0.39**	0.13	-7.18	-6.79	-7.06	0.39*	0.12
MDD	0.47	0.47	0.44	0.00	-0.03	0.47	0.48	0.44	0.00	-0.03	0.47	0.48	0.44	0.00	-0.03
Min	-22.04	-19.16	-17.58	2.88*	4.47	-22.07	-19.19	-17.65	2.88*	4.41*	-22.08	-19.21	-17.69	2.87*	4.39*
Max	20.77	21.18	14.65	0.41	-6.11***	20.74	21.15	14.62	0.41	-6.12***	20.73	21.14	14.61	0.41	-6.12***
Skewness	-0.62	-0.35	-0.65	0.27**	-0.03	-0.62	-0.35	-0.65	0.27***	-0.03	-0.62	-0.35	-0.66	0.27**	-0.03
Kurtosis	10.47	9.38	7.90	-1.09*	-2.57**	10.47	9.38	7.91	-1.09*	-2.56**	10.47	9.38	7.91	-1.09*	-2.56**
LPM0	0.34	0.36	0.34	0.02***	0.00	0.34	0.36	0.34	0.02***	0.00	0.34	0.36	0.35	0.02***	0.00
LPM1	0.77	0.75	0.75	-0.02	-0.02*	0.77	0.76	0.76	-0.01	-0.01	0.77	0.76	0.76	-0.01	-0.01
LPM2	4.06	3.67	3.81	-0.39**	-0.25	4.09	3.70	3.85	-0.39**	-0.24	4.10	3.72	3.87	-0.38*	-0.23
Um	6.94	7.20	7.53	0.26	0.59***	6.76	6.97	7.26	0.21	0.50***	6.69	6.87	7.14	0.18	0.45**
U1	7.87	8.10	8.42	0.23	0.56***	7.69	7.87	8.15	0.18	0.46***	7.62	7.77	8.03	0.16	0.42**
U3	6.71	6.97	7.31	0.26	0.60***	6.53	6.75	7.04	0.21	0.50***	6.45	6.64	6.92	0.19	0.46**
U5	5.55	5.84	6.19	0.30	0.64***	5.37	5.62	5.92	0.25	0.55***	5.29	5.52	5.80	0.22	0.50***
MPPM	3.63	3.90	4.22	0.27	0.59***	3.45	3.68	3.95	0.22	0.50***	3.38	3.58	3.83	0.20	0.46**
CEV3	0.07	0.07	0.07	0.00	0.01***	0.06	0.07	0.07	0.00	0.01***	0.06	0.07	0.07	0.00	0.01**
CEV5	0.05	0.06	0.06	0.00	0.01***	0.05	0.06	0.06	0.00	0.01***	0.05	0.05	0.06	0.00	0.01**
CEV10	0.02	0.02	0.03	0.01*	0.01***	0.01	0.02	0.02	0.01*	0.01**	0.01	0.02	0.02	0.01	0.01**

Table 9: Performance and risk measures of BAB vs SSDBAB and TSDBAB: transaction cost, continued

	I. 1bp					II. 10bps					III. 14bps				
	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB
	Panel B: Proportional trading costs and borrowing costs														
Mean	-0.43	0.54	0.14	0.98***	0.57***	-0.61	0.32	-0.13	0.93***	0.48***	-0.69	0.22	-0.25	0.91***	0.44**
Std	10.80	10.63	10.59	-0.17	-0.22	10.81	10.64	10.59	-0.17	-0.21	10.81	10.64	10.60	-0.17	-0.21
SR	-0.04	0.05	0.01	0.09***	0.05***	-0.06	0.03	-0.01	0.09***	0.04***	-0.06	0.02	-0.02	0.08***	0.04**
Omega	3.35	3.62	3.50	0.27***	0.16***	3.30	3.55	3.43	0.25***	0.13***	3.28	3.52	3.40	0.24***	0.12**
Sortino	-0.05	0.07	0.02	0.12***	0.07***	-0.07	0.04	-0.02	0.12***	0.06***	-0.08	0.03	-0.03	0.11***	0.05**
Upside	1.54	1.69	1.62	0.15***	0.08***	1.52	1.67	1.59	0.15***	0.07**	1.51	1.66	1.58	0.14***	0.07**
VaR	-5.19	-4.73	-5.19	0.46*	0.00	-5.21	-4.76	-5.22	0.44*	-0.01	-5.21	-4.78	-5.23	0.43*	-0.01
ES	-7.96	-7.46	-7.83	0.50**	0.14	-7.98	-7.49	-7.86	0.5**	0.13	-7.99	-7.50	-7.87	0.50**	0.12
MDD	0.71	0.65	0.63	-0.06**	-0.08**	0.75	0.66	0.64	-0.09**	-0.11**	0.77	0.66	0.65	-0.11**	-0.12**
Min	-22.76	-19.80	-18.45	2.96*	4.31	-22.78	-19.83	-18.53	2.96*	4.26	-22.80	-19.84	-18.56	2.95*	4.24
Max	20.37	20.79	14.14	0.42	-6.23***	20.34	20.76	14.11	0.42	-6.23***	20.33	20.75	14.10	0.42	-6.24***
Skewness	-0.63	-0.35	-0.67	0.28**	-0.04	-0.63	-0.35	-0.68	0.28***	-0.04	-0.64	-0.35	-0.68	0.28**	-0.04
Kurtosis	10.49	9.42	7.89	-1.07*	-2.60**	10.49	9.42	7.90	-1.06*	-2.59**	10.49	9.42	7.90	-1.06*	-2.58**
LPM0	0.48	0.47	0.47	0.00	0.00	0.48	0.48	0.48	0.00	-0.01	0.49	0.49	0.48	0.00	0.00
LPM1	1.07	1.03	1.05	-0.04**	-0.02**	1.08	1.04	1.06	-0.04**	-0.02*	1.08	1.05	1.07	-0.04**	-0.02*
LPM2	5.48	4.90	5.18	-0.58**	-0.30*	5.52	4.94	5.24	-0.57***	-0.28	5.53	4.97	5.26	-0.57**	-0.27
Um	-1.95	-0.93	-1.32	1.03***	0.63*	-2.13	-1.15	-1.59	0.98***	0.54***	-2.20	-1.25	-1.71	0.95***	0.50***
U1	-1.02	-0.02	-0.42	1.00***	0.60***	-1.19	-0.25	-0.69	0.95***	0.5***	-1.27	-0.35	-0.81	0.92***	0.46***
U3	-2.19	-1.15	-1.54	1.03***	0.64***	-2.36	-1.38	-1.81	0.98***	0.55***	-2.44	-1.48	-1.93	0.96***	0.51***
U5	-3.35	-2.28	-2.66	1.07***	0.69***	-3.53	-2.51	-2.93	1.02***	0.59***	-3.61	-2.61	-3.05	1.00***	0.55***
MPPM	-5.26	-4.22	-4.62	1.05***	0.64***	-5.44	-4.44	-4.89	1.00***	0.55***	-5.52	-4.54	-5.01	0.97***	0.51***
CEV3	-0.02	-0.01	-0.02	0.01***	0.01***	-0.02	-0.01	-0.02	0.01***	0.01***	-0.03	-0.02	-0.02	0.01***	0.01***
CEV5	-0.04	-0.02	-0.03	0.01***	0.01***	-0.04	-0.03	-0.03	0.01***	0.01***	-0.04	-0.03	-0.03	0.01***	0.01***
CEV10	-0.07	-0.06	-0.06	0.02***	0.01***	-0.08	-0.06	-0.07	0.02***	0.01**	-0.08	-0.06	-0.07	0.01***	0.01**

Table 10: Abnormal returns of BAB vs SDBAB relative to market conditions

This tables reports the abnormal return of the BAB and SDBAB strategies during non-recession periods (I) and recession periods (II). The alphas are computed relative to the Fama-French 4-factor model (market, size, value, and momentum), Fama-French 4-factor model augmented with PS liquidity, Fama-French 4-factor model augmented with lottery demand FMAX, Fama-French 4-factor model augmented with BAB, BAB, Fama-French 5-factor model (market, size, value, robust operating profitability, and conservative investment), Fama-French 6-factor model (market, size, value, momentum, robust operating profitability, and conservative investment), and q-5 factor model (market, size, investment, return on equity, and expected growth). All values are annualized, and t-statistics are adjusted for heteroscedasticity and autocorrelation. The recession determination follows NBER recession indicators for the United States from the period following the peak through the trough (USREC). *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

	I. Non-recession periods					II. Recession periods				
	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB
FF4	7.95***	8.77***	8.41***	0.82**	0.45**	0.99	2.98	1.37	1.99**	0.38
	[5.33]	[5.93]	[5.62]	[2.07]	[2.23]	[0.26]	[0.74]	[0.36]	[2.43]	[0.97]
FF4+PS	8.45***	9.40***	9.10***	0.95*	0.66***	-3.17	-0.52	-2.31	2.65**	0.86
	[4.37]	[4.94]	[4.73]	[1.79]	[2.85]	[-0.57]	[-0.09]	[-0.41]	[2.40]	[1.48]
FF5	7.18***	6.93***	7.98***	-0.26	0.80***	-11.55**	-7.06	-10.09**	4.49*	1.46**
	[4.17]	[4.03]	[4.69]	[-0.49]	[3.98]	[-2.27]	[-1.16]	[-1.96]	[1.71]	[2.19]
FF6	5.91***	7.04***	6.83***	1.13*	0.92***	-8.63	-6.48	-7.65	2.15*	0.98
	[3.24]	[3.98]	[3.79]	[1.85]	[3.82]	[-1.44]	[-1.02]	[-1.28]	[1.76]	[1.57]
q5	5.70***	7.04***	6.59***	1.33*	0.88***	-11.55**	-6.74	-10.02*	4.81***	1.53**
	[2.80]	[3.51]	[3.32]	[1.65]	[3.19]	[-2.24]	[-1.17]	[-1.90]	[3.23]	[2.53]
FF4+FMAX	3.27**	4.52***	4.32***	1.25**	1.05***	-8.55**	-6.10	-7.80**	2.45**	0.75
	[2.02]	[2.76]	[2.63]	[2.19]	[4.17]	[-2.04]	[-1.37]	[-1.82]	[2.34]	[1.24]
FF4+BAB		0.97**	0.74**	0.97**	0.74**		1.98**	0.38	1.98**	0.38
		[2.00]	[2.45]	[2.00]	[2.45]		[2.44]	[0.96]	[2.44]	[0.96]
BAB		0.33	1.01***	0.33	1.01***		2.18	0.29	2.18	0.29
		[0.72]	[3.14]	[0.72]	[3.14]		[1.62]	[0.47]	[1.62]	[0.47]

Table 11: Performance and risk measures of BAB vs SDBAB relative to market condition

This tables reports the performance and risk measures of the BAB and SDBAB strategies during non-recession periods (I) and recession periods (II). All values are annualized, and one-sided p-values for the differences are obtained using 1,000 bootstrap samples. The recession determination follows NBER recession indicators for the United States from the period following the peak through the trough (USREC). *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

	I. Non-recession periods					II. Recession periods				
	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB
Mean	9.86	9.73	10.45	-0.13	0.59***	0.84	2.97	1.12	2.13	0.27
Std	10.20	10.03	9.93	-0.17	-0.26*	13.31	13.28	13.33	-0.02*	0.02
SR	0.97	0.97	1.05	0.00	0.09***	0.06	0.22	0.08	0.16	0.02
Omega	7.91	7.86	8.31	-0.06	0.40***	3.63	4.10	3.69	0.47	0.06
Sortino	1.57	1.62	1.73	0.05	0.16***	0.08	0.32	0.11	0.23	0.03
Upside	2.79	2.89	2.97	0.10	0.17**	1.83	2.05	1.86	0.22*	0.03
VaR	-3.16	-3.46	-3.35	-0.30	-0.19	-6.42	-5.64	-6.21	0.78	0.21
ES	-6.43	-6.22	-6.35	0.21	0.08	-8.53	-8.04	-8.09	0.49	0.43
MDD	0.47	0.47	0.44	0.00	-0.03	0.41	0.40	0.42	-0.01*	0.01
Min	-22.04	-19.16	-17.57	2.88	4.47	-17.72	-17.00	-16.77	0.72**	0.95
Max	20.77	21.18	14.66	0.41	-6.11***	9.58	12.06	10.20	2.48	0.62
Skewness	-0.51	-0.25	-0.61	0.26	-0.10	-0.66	-0.40	-0.49	0.26***	0.17
Kurtosis	12.62	11.03	9.20	-1.58*	-3.42**	4.87	4.92	4.37	0.05	-0.50*
LPM0	0.32	0.34	0.32	0.02***	0.00	0.42	0.43	0.42	0.01	0.00
LPM1	0.64	0.64	0.62	0.00	-0.02*	1.45	1.34	1.42	-0.10**	-0.02
LPM2	3.29	3.02	3.05	-0.27	-0.25	8.26	7.20	7.95	-1.06***	-0.31
Um	8.51	8.42	9.17	-0.08	0.66***	-1.46	0.68	-1.19	2.13	0.27
U1	9.34	9.23	9.96	-0.11	0.62***	-0.04	2.09	0.23	2.13	0.27
U3	8.30	8.22	8.97	-0.08	0.67***	-1.81	0.32	-1.55	2.14	0.27
U5	7.26	7.22	7.99	-0.04	0.73***	-3.58	-1.44	-3.32	2.14	0.26
MPPM	5.29	5.22	5.95	-0.07	0.66***	-5.29	-3.13	-5.00	2.16	0.29
CEV3	0.08	0.08	0.09	0.00	0.01***	-0.02	0.00	-0.02	0.02	0.00
CEV5	0.07	0.07	0.08	0.00	0.01***	-0.04	-0.02	-0.04	0.02	0.00
CEV10	0.04	0.04	0.05	0.00	0.01***	-0.09	-0.07	-0.09	0.03	0.01

Table 12: Abnormal return of BAB vs SDBAB relative to liquidity condition

This tables reports the abnormal return of the BAB and SDBAB strategies during high liquidity periods (I) and low liquidity periods (II). The alphas are computed relative to the Fama-French 4-factor model (market, size, value, and momentum), Fama-French 4-factor model augmented with Pastor and Stambaugh (2003) liquidity, Fama-French 4-factor model augmented with lottery demand FMAX, Fama-French 4-factor model augmented with BAB, BAB, Fama-French 5-factor model (market, size, value, robust operating profitability, and conservative investment), Fama-French 6-factor model (market, size, value, momentum, robust operating profitability, and conservative investment), and q-5 factor model (market, size, investment, return on equity, and expected growth). All values are annualized, and t-statistics are adjusted for heteroscedasticity and autocorrelation. The liquidity determination follows the Pastor and Stambaugh (2003) traded liquidity factor, for which high liquidity periods are with factor returns above the median and low liquidity periods are with factor returns below the median. *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

	I. High liquidity periods					II. Low liquidity periods				
	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB
FF4	9.75***	10.13***	11.04***	0.38	1.29***	12.17***	12.25***	13.08***	0.08	0.91**
	[4.13]	[4.26]	[4.73]	[0.58]	[3.04]	[4.10]	[4.32]	[4.40]	[0.12]	[2.25]
FF4+PS	12.01***	11.03***	12.40***	-0.98	0.39	12.25***	11.94***	12.48***	-0.30	0.23
	[2.99]	[2.78]	[3.18]	[-0.88]	[0.51]	[3.48]	[3.41]	[3.50]	[-0.38]	[0.50]
FF5	9.85***	10.25***	11.29***	0.40	1.45***	11.68***	11.62***	12.56***	-0.06	0.88**
	[4.20]	[4.14]	[4.83]	[0.63]	[3.53]	[3.93]	[3.97]	[4.18]	[-0.09]	[2.41]
FF6	10.10***	10.54***	11.49***	0.43	1.38***	12.12***	12.15***	12.98***	0.02	0.86**
	[4.30]	[4.31]	[4.92]	[0.67]	[3.36]	[4.11]	[4.26]	[4.39]	[0.04]	[2.23]
q5	9.92***	10.12***	11.24***	0.20	1.32***	12.92***	13.00***	13.83***	0.08	0.91**
	[3.66]	[3.67]	[4.22]	[0.31]	[3.01]	[3.72]	[3.87]	[4.06]	[0.10]	[2.18]
FF4+FMAX***	9.54***	9.92***	10.87***	0.38	1.34***	11.19***	11.35***	12.07***	0.16	0.89**
	[4.01]	[4.01]	[4.58]	[0.55]	[3.14]	[3.79]	[4.02]	[4.06]	[0.23]	[2.14]
FF4+BAB		1.45	1.73***	1.45	1.73***		0.07	0.73	0.07	0.73
		[1.36]	[3.48]	[1.36]	[3.48]		[0.08]	[1.62]	[0.08]	[1.62]
BAB		1.48	1.72***	1.48	1.72***		0.17	0.85**	0.17	0.85**
		[1.37]	[3.51]	[1.37]	[3.51]		[0.22]	[2.04]	[0.22]	[2.04]

Table 13: Sub-sample analysis of BAB vs SDBAB relative to liquidity condition

This tables reports the performance and risk measures of the BAB and SDBAB strategies during high liquidity periods (I) and low liquidity periods (II). All values are annualized, and one-sided p-values for the differences are obtained using 1,000 bootstrap samples. The liquidity determination follows the Pastor and Stambaugh (2003) traded liquidity factor, for which high liquidity periods are with factor returns above the median and low liquidity periods are with factor returns below the median. *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

	I. High liquidity periods					II. Low liquidity periods				
	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB
Mean	9.01	9.49	10.33	0.48	1.32***	11.54	11.71	12.55	0.17	1.01***
Std	11.20	10.56	10.85	-0.64**	-0.34**	11.04	11.48	11.34	0.43**	0.30**
SR	0.80	0.90	0.95	0.09	0.15***	1.04	1.02	1.11	-0.02	0.06**
Omega	6.67	7.10	7.38	0.44	0.71***	7.81	7.73	8.22	-0.08	0.41**
Sortino	1.29	1.56	1.60	0.27*	0.31***	1.67	1.65	1.80	-0.02	0.13**
Upside	2.68	3.05	3.02	0.37*	0.34***	3.00	2.99	3.11	-0.01	0.11
VaR	-4.68	-4.27	-4.82	0.40	-0.15	-4.70	-4.84	-4.88	-0.14	-0.18
ES	-7.21	-6.24	-6.63	0.97**	0.58**	-7.21	-7.24	-7.14	-0.04	0.07
MDD	0.35	0.36	0.31	0.01	-0.03*	0.36	0.34	0.34	-0.02	-0.01
Min	-13.81	-11.86	-12.37	1.96	1.44***	-15.69	-16.32	-17.57	-0.63***	-1.88
Max	11.98	12.06	11.30	0.08	-0.68	14.95	13.71	13.68	-1.24	-1.27
Skewness	-0.29	0.09	-0.16	0.38*	0.13	-0.64	-0.61	-0.73	0.03	-0.09
Kurtosis	6.01	5.10	5.05	-0.91	-0.96***	6.61	6.24	6.81	-0.37	0.20
LPM0	0.33	0.37	0.33	0.04***	0.00	0.31	0.33	0.31	0.02**	0.01
LPM1	0.81	0.75	0.76	-0.06	-0.05***	0.77	0.79	0.76	0.03	0.00
LPM2	4.07	3.08	3.46	-0.99**	-0.61***	3.98	4.20	4.07	0.22	0.09
Um	7.38	8.04	8.80	0.66	1.42***	9.95	10.00	10.88	0.05	0.93***
U1	8.38	8.94	9.74	0.55	1.36***	10.93	11.05	11.91	0.12	0.98***
U3	7.13	7.82	8.56	0.69	1.43***	9.71	9.74	10.62	0.03	0.91***
U5	5.88	6.71	7.38	0.83	1.51***	8.49	8.42	9.34	-0.07	0.85***
MPPM	2.55	3.23	3.97	0.68	1.42***	5.12	5.17	6.03	0.04	0.91***
CEV3	0.07	0.08	0.09	0.01	0.01***	0.10	0.10	0.11	0.00	0.01***
CEV5	0.06	0.07	0.07	0.01	0.02***	0.08	0.08	0.09	0.00	0.01***
CEV10	0.02	0.04	0.04	0.02*	0.02***	0.05	0.04	0.05	0.00	0.00

Table 14: Abnormal returns of BAB vs SDBAB relative to lottery demand condition

This tables reports the abnormal return of the BAB and SDBAB strategies during high lottery demand periods (I) and low lottery demand periods (II). The alphas are computed relative to the Fama-French 4-factor model (market, size, value, and momentum), Fama-French 4-factor model augmented with PS liquidity, Fama-French 4-factor model augmented with lottery demand FMAX, Fama-French 4-factor model augmented with BAB, BAB, Fama-French 5-factor model (market, size, value, robust operating profitability, and conservative investment), Fama-French 6-factor model (market, size, value, momentum, robust operating profitability, and conservative investment), and q-5 factor model (market, size, investment, return on equity, and expected growth). All values are annualized, and t-statistics are adjusted for heteroscedasticity and autocorrelation. The lottery demand determination follows FMAX factor, for which high lottery demand periods are with factor returns above the median and low lottery demand periods are with factor returns below the median. *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

	I. High lottery demand periods					II. Low lottery demand periods				
	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB
FF4	10.76***	9.94***	11.00***	-0.81	0.25	10.39***	10.39***	11.38***	0.00	0.99**
	[3.43]	[3.00]	[3.61]	[-0.85]	[0.48]	[3.64]	[3.65]	[4.01]	[0.01]	[2.24]
FF4+PS	10.45***	9.57***	10.70***	-0.88	0.24	10.98***	11.00***	12.00***	0.02	1.02**
	[3.04]	[2.64]	[3.17]	[-0.85]	[0.44]	[3.50]	[3.54]	[3.86]	[0.03]	[2.14]
FF5	9.51***	9.01***	10.02***	-0.50	0.50	11.27***	10.92***	12.22***	-0.36	0.95**
	[3.05]	[2.70]	[3.29]	[-0.56]	[0.96]	[4.15]	[3.86]	[4.43]	[-0.60]	[2.24]
FF6	10.91***	10.11***	11.17***	-0.80	0.26	10.93***	10.88***	11.93***	-0.05	1.00**
	[3.51]	[3.07]	[3.70]	[-0.82]	[0.50]	[3.92]	[3.80]	[4.26]	[-0.08]	[2.32]
q5	8.98**	7.82**	9.12***	-1.16	0.15	12.66***	12.96***	13.91***	0.30	1.25***
	[2.44]	[2.04]	[2.59]	[-1.08]	[0.25]	[3.89]	[3.91]	[4.35]	[0.41]	[2.63]
FF4+FMAX	10.89***	9.95***	11.08***	-0.94	0.19	8.67***	8.25**	9.47***	-0.42	0.80*
	[3.32]	[2.96]	[3.50]	[-0.88]	[0.35]	[2.76]	[2.50]	[2.92]	[-0.64]	[1.80]
FF4+BAB		-0.36	0.39	-0.36	0.39		0.67	1.15**	0.67	1.15**
		[-0.37]	[0.68]	[-0.37]	[0.68]		[0.72]	[2.43]	[0.72]	[2.43]
BAB		0.98	1.16***	0.98	1.16***		0.76	1.38***	0.76	1.38***
		[1.11]	[2.72]	[1.11]	[2.72]		[0.79]	[3.27]	[0.79]	[3.27]

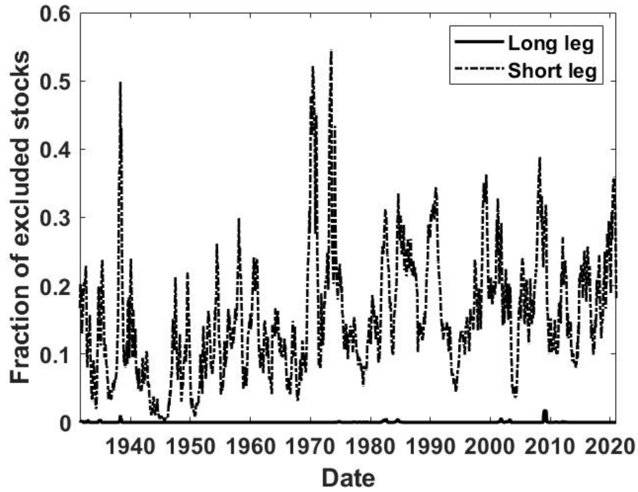
Table 15: Performance and risk measures of BAB vs SDBAB relative to lottery demand condition

This tables reports the performance and risk measures of the BAB and SDBAB strategies during high lottery demand periods (I) and low lottery demand periods (II). All values are annualized, and one-sided p-values for the differences are obtained using 1,000 bootstrap samples. The lottery demand determination follows FMAX factor, for which high lottery demand periods are with factor returns above the median and low lottery demand periods are with factor returns below the median. *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

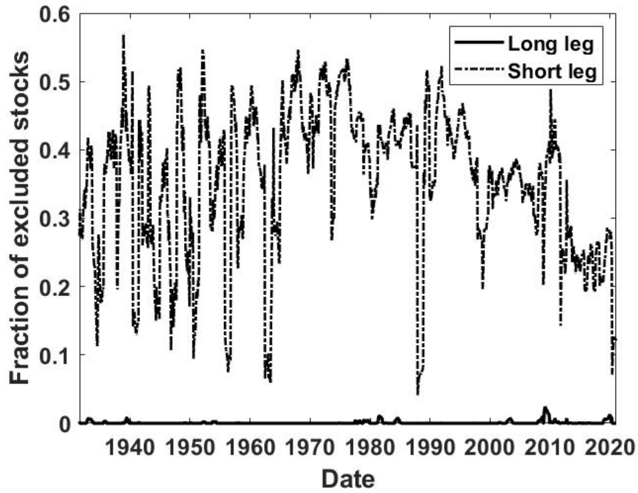
	I. High lottery demand periods					II. Low lottery demand periods				
	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB	BAB	SSDBAB	TSDBAB	SSDBAB-BAB	TSDBAB-BAB
Mean	8.82	9.40	9.84	0.58	1.02***	11.97	11.97	13.17	0.00	1.20***
Std	11.39	11.47	11.36	0.08	-0.02	10.22	9.95	10.21	-0.27	-0.01
SR	0.77	0.82	0.87	0.04	0.09***	1.17	1.20	1.29	0.03	0.12***
Omega	6.41	6.70	6.86	0.29	0.44***	8.77	8.79	9.52	0.02	0.76***
Sortino	1.17	1.28	1.34	0.11	0.17***	2.06	2.26	2.37	0.19	0.31***
Upside	2.54	2.65	2.71	0.11	0.17**	3.41	3.73	3.73	0.31	0.31***
VaR	-5.37	-5.01	-5.23	0.36	0.14	-4.04	-3.81	-4.22	0.23	-0.17
ES	-7.86	-7.61	-7.52	0.25	0.34	-6.08	-5.47	-5.88	0.62	0.20
MDD	0.41	0.42	0.37	0.01	-0.04	0.30	0.32	0.31	0.03	0.01
Min	-15.69	-16.32	-17.57	-0.63	-1.88	-13.81	-8.93	-12.37	4.89	1.44
Max	11.98	12.06	11.06	0.08	-0.92*	14.95	13.71	13.68	-1.24	-1.27
Skewness	-0.67	-0.57	-0.72	0.10	-0.05	-0.17	0.14	-0.08	0.31	0.09
Kurtosis	6.23	6.35	6.35	0.12	0.12	6.53	4.69	5.52	-1.83***	-1.00**
LPM0	0.31	0.36	0.32	0.05***	0.01	0.32	0.34	0.32	0.01*	0.00
LPM1	0.86	0.84	0.84	-0.03	-0.03*	0.65	0.65	0.63	0.00	-0.02*
LPM2	4.75	4.50	4.49	-0.25	-0.25	2.80	2.34	2.57	-0.46	-0.23*
Um	7.14	7.69	8.16	0.55	1.02***	10.61	10.68	11.82	0.07	1.21***
U1	8.18	8.75	9.19	0.57	1.02***	11.45	11.47	12.65	0.02	1.20***
U3	6.88	7.43	7.90	0.55	1.02***	10.40	10.48	11.61	0.08	1.21***
U5	5.58	6.11	6.61	0.53	1.03***	9.36	9.49	10.56	0.13	1.21***
MPPM	2.80	3.35	3.81	0.56	1.01***	5.65	5.73	6.85	0.08	1.20***
CEV3	0.07	0.07	0.08	0.01	0.01***	0.10	0.10	0.12	0.00	0.01***
CEV5	0.05	0.06	0.06	0.01	0.01***	0.09	0.10	0.11	0.00	0.01***
CEV10	0.02	0.02	0.02	0.01	0.01*	0.06	0.07	0.08	0.01	0.01***

Figure 1: Fraction of stocks to be excluded from the long and the short legs of the BAB strategy

This figure plots the time series of the fractions of stocks excluded from the long leg (solid line) due to them being dominated by the market index and the short leg (dashed line) for the stocks that dominate the market index. The exclusion determination is based on Davidson (2009)'s non-dominance test if non-dominance is rejected at the 1% significance level. Subplot (a) uses the SSD criterion while subplot (b) uses the TSD criterion.



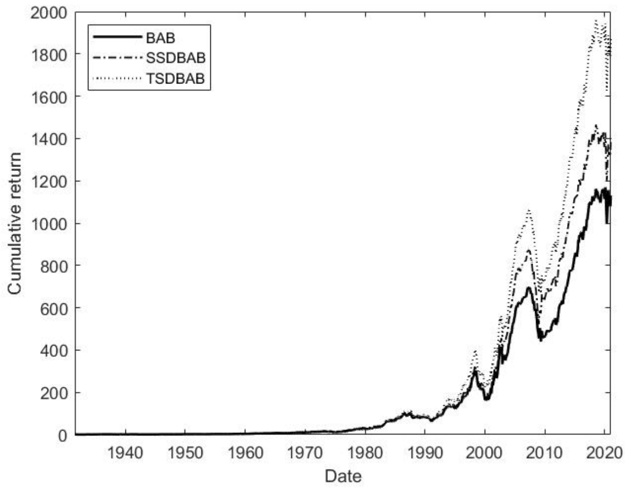
(a) SSD



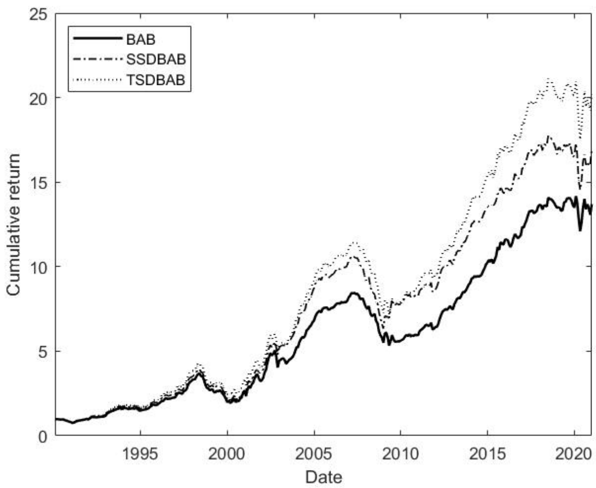
(b) TSD

Figure 2: Cumulative performance of BAB, SSDBAB, and TSDBAB portfolios

This figure plots the cumulative return of the BAB and SD-enhanced BAB portfolios from July 1931 to December 2020 (sub-figure (a)) and from January 1990 to December 2020 (sub-figure (b)), given an initial investment of \$1. In SSDBAB (TSDBAB), stocks that have dominated the market during the preceding year by the second (third) order are excluded from the short leg and stocks that have been previously dominated by the market are excluded from the long leg. The exclusion determination is based on Davidson (2009)'s non-dominance test if non-dominance is rejected at the 1% significance level.



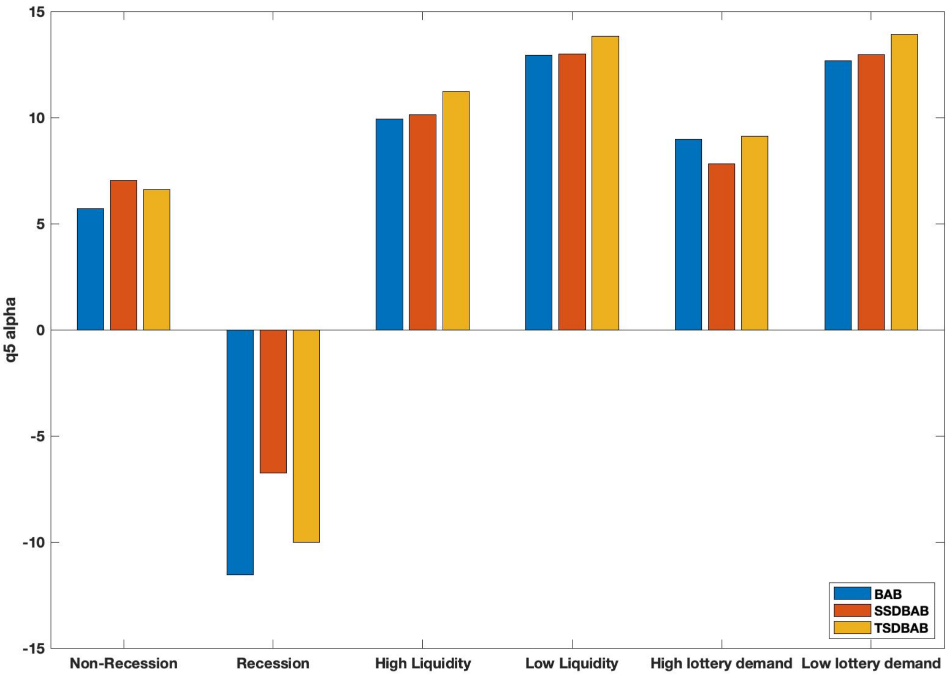
(a) 1931-2020



(b) 1990-2020

Figure 3: Alpha relative to the q5 model: subsample analysis

This figure plots the estimated alpha relative to the q5 model for the BAB, SSDBAB, and TSDBAB portfolios computed using subsamples with different conditions. The recession determination follows NBER recession indicators for the United States from the period following the peak through the trough (USREC). The liquidity determination follows the Pastor and Stambaugh (2003) traded liquidity factor, for which high liquidity periods are with factor returns above the median and low liquidity periods are with factor returns below the median. The lottery demand determination follows FMAX factor, for which high lottery demand periods are with factor returns above the median and low lottery demand periods are with factor returns below the median. In SSDBAB (TSDBAB), stocks that have dominated the market during the preceding year by the second (third) order are excluded from the short leg and stocks that have been previously dominated by the market are excluded from the long leg. The exclusion determination is based on Davidson (2009)'s non-dominance test if non-dominance is rejected at the 1% significance level.



Online Appendix

Enhancing Betting Against Beta with Stochastic Dominance

SUPPLEMENTARY RESULTS

A1 Alternative SD tests

There exist alternative tests to Davidson (2009) which could be potentially used to determine SD dominating and dominated stocks. These tests often exhibit better statistical properties than the test used in this paper, however, their restricted computational feasibility in a large cross-section of assets presents an implementational challenge.

Prior studies, such as Post (2003), Kuosmanen (2004), and Fang and Post (2017) propose tests to evaluate if a benchmark portfolio (e.g., the market portfolio) is SD efficient, that is, if the portfolio can be dominated in SD sense by a linear combination of its underlying assets. Their empirical applications usually involve samples with a large number of return observations (T) and a small number of assets (N). For example, Post (2003) analyses if the market portfolio is SSD efficient relative to the 25 Fama and French portfolios sorted on size and book-to-market based on a sample of 460 months. Fang and Post (2017) expand the examination to different sets of portfolios and the size of risky assets is up to 30 over a sample period from the late

1920s to 2015. Kuosmanen (2004, page 1397) highlights the computation aspects related to the specification of T and N. He notes that a relatively large T and a small N achieve a good balance between statistical inference and computation burden. For example, he recommends specifying T from 100 to 500 when N = 26, which gives reasonable statistical size and power. For larger N, one needs much greater T to guarantee reliable statistical inference on cross-sectional SD efficiency. Post (2003, Figure 3) illustrates that the statistical size deteriorates when T is small.

By contrast, our paper performs the SD analysis over the entire CRSP stock universe rather than a few aggregated portfolios, and N can go beyond 7,000. Suppose there are 250 trading days in a year, such a combination of T and N renders 2 million observations that remain to be solved in Kuosmanen (2004) SSD efficiency test. This falls into the category of huge linear programming complexity as Kuosmanen (2004, Section 5) documents. Additionally, such SD efficiency tests should be repeated more than one thousand times due to monthly portfolio rebalancing from the late 1920s to 2020 in our CRSP sample. The heavy computation burden precludes us from implementation of the SD efficiency tests, let alone the aforementioned statistical size concerns associated with a large N relative to T.

There is another set of tests that consider the pairwise relations between distributions of stock returns, similar to the test adopted in our paper. The null hypothesis in these tests is dominance of one distribution over the other, as opposed to non-dominance as in the Davidson (2009) test. The bootstrap SD tests introduced by Linton et al. (2005) and Linton et al. (2010) also require a large T to guarantee the efficiency of subsampling. These tests are suitable to test SD relationship among a selection of portfolios over a relatively long period, for example as in Kolokolova et al. (2022) in the context of industry portfolios. Linton et al. (2005, page 753) point out that T has to be greater than 500 for subsampling, and they compare the DJIA and S&P 500 index daily returns from 1988 to 2000. Nonetheless, stock beta is predominantly estimated with one-year daily returns (Welch 2022), which poses a challenge for the application of the bootstrap SD tests using shorter time series. Further, Linton et al. (2010) suggest the number of bootstrap replications to be 400, which increases the computation time for a large number of SD comparisons.

To illustrate the differences in the computational efficiency between different tests, we generate 10, 100, 500, 1000, 2000, and 5000 pairs of random vectors from a standard normal distribution with the length of 500 observations each. We implement the Linton et al. (2010) test following their parameter specifications and the Davidson (2009) test. The corresponding computation time is summarized in Table A1. The Linton et al. (2010) test specification is feasible for a small number of comparisons. However, if investors have to conduct 5000 comparisons, the test requires about 6000 seconds, which is about one hour and half. In contrast, the Davidson (2009) test delivers the results within 1 minute. Since the number of CRSP stocks is often greater than 5000, and monthly rebalancing over around 95 years requires a repetition of the test implementation for 1140 times, running the Linton et al. (2010) type tests in our setting turns computationally unpractical. Hence, we resort to the Davidson (2009) test as the key tool for our analysis.

Table A1: Computational time of SD tests

This table summarizes the corresponding computation time in second for conducting 10, 100, 500, 1000, 2000, and 5000 pairwise tests following Linton et al. (2010) and Davidson (2009). The pairs of random vectors have the length of 500 and are generated from a standard normal distribution.

	10	100	500	1000	2000	5000
Linton et al. (2010)	9.86	112.87	595.94	1192.84	2392.89	5587.57
Davidson (2009)	0.19	0.92	5.03	10.56	20.28	52.25

A2 Performace of the long and shot legs of the portfolio

In this appendix, we decompose the strategy performance into that of the long and the short leg separately. Consistent with the observation that SD-prefiltering impacts the short leg the most, the results in Table A2 show that the improvement of the overall portfolio alphas is due to the more negative performance of the short leg. A higher Sharpe ratio of the TSDBAB portfolio is also due to significant variance reduction of the short leg (Table A3). These results further highlight the asymmetry of the low beta anomaly. Low beta stocks tend to be rarely dominated by the market, whereas a substantial fraction of high-beta stocks may turn out to dominate the market returns in SD sense.

Table A2: Abnormal returns of BAB vs SSDBAB and TSDBAB: Long and short legs

This table reports the abnormal return of the long and short legs of BAB and SDBAB strategies based on the Fama-French 4-factor model (market, size, value, and momentum), Fama-French 4-factor model augmented with PS liquidity, Fama-French 4-factor model augmented with lottery demand FMAX, Fama-French 4-factor model augmented with BAB, BAB, Fama-French 5-factor model (market, size, value, robust operating profitability, and conservative investment), Fama-French 6-factor model (market, size, value, momentum, robust operating profitability, and conservative investment), and q-5 factor model (market, size, investment, return on equity, and expected growth). Values are annualized and t-statistics are robust to heteroscedasticity and autocorrelation. *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

	I. Long					II. Short				
	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB
FF4	6.12***	6.12***	6.13***	0.00	0.01	-0.36	-1.47**	-0.81	-1.10***	-0.45**
	[4.89]	[4.88]	[4.88]	[-0.10]	[0.50]	[-0.44]	[-2.18]	[-1.05]	[-3.27]	[-2.52]
FF4+PS	6.03***	6.02***	6.05***	-0.01	0.01	-0.20	-1.53*	-0.95	-1.33***	-0.76***
	[3.27]	[3.26]	[3.27]	[-0.53]	[0.50]	[-0.19]	[-1.94]	[-0.99]	[-3.04]	[-3.63]
FF4+FMAX	3.53**	3.53**	3.56**	0.00	0.03	2.00**	0.46	0.96	-1.54***	-1.04***
	[2.07]	[2.06]	[2.07]	[0.13]	[0.81]	[2.00]	[0.65]	[1.06]	[-3.41]	[-4.80]
FF4+BAB		0.00	0.00	0.00	0.00		-1.18***	-0.48***	-1.18***	-0.48***
		[-0.32]	[0.00]	[-0.32]	[0.00]		[-4.33]	[-2.94]	[-4.33]	[-2.94]
BAB		-0.01	-0.01	-0.01	-0.01		0.26	-0.21	0.26	-0.21
		[-1.60]	[-1.13]	[-1.60]	[-1.13]		[1.15]	[-1.28]	[1.15]	[-1.28]
FF5	4.54***	4.53***	4.54***	-0.01	-0.01	-0.60	-0.76	-1.45	-0.17	-0.86***
	[2.64]	[2.63]	[2.63]	[-1.26]	[-0.34]	[-0.59]	[-1.15]	[-1.55]	[-0.30]	[-4.36]
FF6	5.17***	5.17***	5.18***	0.00	0.01	1.41	-0.09	0.40	-1.50***	-1.01***
	[2.95]	[2.94]	[2.95]	[-0.01]	[0.53]	[1.30]	[-0.13]	[0.41]	[-2.95]	[-4.70]
q5	5.87***	5.85***	5.88***	-0.02	0.01	2.23*	0.40	1.26	-1.83***	-0.97***
	[3.07]	[3.06]	[3.07]	[-1.11]	[0.65]	[1.71]	[0.50]	[1.09]	[-2.68]	[-3.89]

Table A3: Performance and risk measures of BAB vs SSDBAB and TSDBAB: Long and short legs

This table reports the performance and risk measures of the long and the short legs of the BAB and SDBAB strategies. Values are annualized and one-sided p-values for the differences are obtained using 1,000 bootstrap samples. *, **, and *** denote significance at the 10%, 5%, and 1% levels respectively.

	I. Long					II. Short				
	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB	BAB	SSDBAB	TSDBAB	SSDBAB -BAB	TSDBAB -BAB
Mean	16.28	16.28	16.29	0.00	0.01	7.81	7.59	7.28	-0.22	-0.53***
Std	23.47	23.48	23.50	0.01***	0.04***	22.55	21.30	21.68	-1.25***	-0.87***
SR	0.69	0.69	0.69	-0.01**	-0.01*	0.35	0.36	0.34	0.01	-0.01**
Omega	6.35	6.34	6.35	-0.01*	0.00	4.70	4.72	4.66	0.02	-0.04
Sortino	1.16	1.16	1.16	-0.01*	0.00	0.59	0.60	0.58	0.01	-0.01
Upside	2.55	2.55	2.55	-0.01*	0.00	2.25	2.24	2.24	0.00	0.00
VaR	-8.56	-8.56	-8.56	0.00	0.00	-8.20	-8.01	-7.98	0.19	0.22
ES	-14.29	-14.31	-14.31	-0.01***	-0.02***	-12.74	-12.22	-12.13	0.52***	0.61***
MDD	0.70	0.70	0.70	0.00	0.00	0.66	0.68	0.66	0.02*	0.00
Min	-41.03	-41.03	-41.02	-0.01***	0.01	-25.34	-24.71	-23.45	0.63***	1.89***
Max	65.90	65.90	66.30	0.00	0.40***	59.11	55.07	56.71	-4.04***	-2.40
Skewness	0.97	0.97	0.98	0.00	0.01	1.67	1.58	1.83	-0.09*	0.15
Kurtosis	18.65	18.63	18.75	-0.03**	0.10	16.29	16.53	17.78	0.24	1.49
LPM0	0.37	0.37	0.37	0.01***	0.01***	0.44	0.44	0.45	0.00	0.00
LPM1	1.63	1.63	1.63	0.01***	0.01***	1.82	1.74	1.75	-0.08***	-0.07***
LPM2	16.50	16.53	16.54	0.02***	0.04***	14.55	13.45	13.28	-1.10***	-1.27***
Um	9.12	9.11	9.11	-0.01**	-0.01	1.20	1.69	1.17	0.49**	-0.03
U1	13.53	13.52	13.53	-0.01	0.00	5.27	5.32	4.93	0.05	-0.34**
U3	8.02	8.01	8.01	-0.01**	-0.01	0.19	0.78	0.23	0.6***	0.05
U5	2.51	2.49	2.48	-0.02***	-0.03***	-4.90	-3.76	-4.47	1.14***	0.43**
MPPM	5.85	5.84	5.85	-0.01*	-0.01	-1.58	-1.20	-1.61	0.38*	-0.03
CEV3	0.08	0.08	0.08	-0.01**	0.00	0.01	0.01	0.01	0.01**	0.00
CEV5	0.02	0.02	0.02	-0.01***	-0.01**	-0.04	-0.03	-0.03	0.01***	0.01***
CEV10	-0.25	-0.25	-0.25	-0.01***	-0.01*	-0.16	-0.14	-0.14	0.02***	0.02***