The lattice of closed ideals in the Banach algebra of operators on certain Banach spaces

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Abstract

Very few Banach spaces E are known for which the lattice of closed ideals in the Banach algebra $\mathscr{B}(E)$ of all (bounded, linear) operators on E is fully understood. Indeed, up to now the only such Banach spaces are, up to isomorphism, Hilbert spaces and the sequence spaces c_0 and ℓ_p for $1 \leqslant p < \infty$. We add a new member to this family by showing that there are exactly four closed ideals in $\mathscr{B}(E)$ for the Banach space $E := \big(\bigoplus \ell_2^n\big)_{c_0}$, that is, E is the c_0-direct sum of the finite-dimensional Hilbert spaces $\ell_2^1, \ell_2^2, \ldots, \ell_2^n, \ldots$

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1 Introduction

The aim of this paper is to study the lattice of closed ideals in the Banach algebra $\mathscr{B}(E)$ of all (bounded, linear) operators on a Banach space E, and in this way gain new insights into the interrelationship between the geometry of a Banach space E and the structure of its associated Banach algebra $\mathscr{B}(E)$.

The first result of this type is due to Calkin who in [4] classified all the ideals in $\mathscr{B}(\ell_2)$. In particular he proved that the ideal of compact operators is the only non-trivial, closed ideal in $\mathscr{B}(\ell_2)$. For each non-separable Hilbert space H, Gramsch and Luft have independently described all the closed ideals in $\mathscr{B}(H)$ and shown that they are well-ordered by inclusion (see [16] and [25], respectively — or [28, §5.4] for a short account).

Another famous extension of Calkin's result is as follows.

1.1 Theorem. (Gohberg, Markus, and Feldman [12]) *For $E = \ell_p$, where $1 \leqslant p < \infty$, and $E = c_0$, the ideal of compact operators is the only non-trivial, closed ideal in $\mathscr{B}(E)$.* □

A surprising fact that testifies to our limited understanding of Banach algebras of the form $\mathscr{B}(E)$ for a Banach space E is that, to our knowledge, the above-mentioned examples are hitherto the *only* infinite-dimensional Banach spaces E for which the lattice of closed ideals in $\mathscr{B}(E)$ is completely understood. The main purpose of this paper is to add a new member to this family. More precisely, we shall prove that, for the Banach space

$$E := \Big(\bigoplus \ell_2^n\Big)_{c_0} \tag{1.1}$$

(that is, E is the c_0-direct sum of the finite-dimensional Hilbert spaces $\ell_2^1, \ell_2^2, \ldots, \ell_2^n, \ldots$), there are precisely two non-trivial, closed ideals in $\mathscr{B}(E)$, namely the ideal of compact operators and the closure of the ideal of operators that factor through c_0. This theorem is established through 'salami tactics' — we begin with some fairly general results and then gradually specialize until in Section 5 we consider the particular space E given by (1.1).

Even though Banach spaces E for which the lattice of closed ideals in $\mathscr{B}(E)$ is completely understood are rare, quite a few partial results are known. We shall now briefly review some of these.

First, Volkmann has proved that, whenever $p, q \in [1, \infty[$ are distinct, there are exactly two maximal ideals in $\mathscr{B}(\ell_p \oplus \ell_q)$, they are generated by the operators that factor through ℓ_p and ℓ_q, respectively, and their intersection is the ideal of strictly singular operators (see [30] or [28, Theorem 5.3.2]). A similar result holds if either ℓ_p or ℓ_q is replaced with c_0.

Second, building on work of Rosenthal and Schechtman, Pietsch has demonstrated that there are infinitely many closed ideals in $\mathscr{B}\big(L_p[0,1]\big)$ for each $p \in \,]1, \infty[\, \setminus \{2\}$. Moreover, Pietsch has shown that there are uncountably many closed ideals in $\mathscr{B}\big(C[0,1]\big)$ (see [28, Theorems 5.3.9 and 5.3.11]).

Third, Edelstein and Mityagin observed in [11, p. 225] that the ideal of weakly compact operators is a maximal ideal of codimension one in $\mathscr{B}(J)$, where J denotes James's quasi-reflexive Banach space introduced in [18]. Laustsen has proved that this maximal ideal is the only maximal ideal in $\mathscr{B}(J)$, and applied this result to construct Banach spaces E such that $\mathscr{B}(E)$ has any specified finite number of maximal ideals of any specified codimensions (see [23]).

Fourth, while solving the unconditional basic sequence problem, Gowers and Maurey constructed the first example of a hereditarily indecomposable Banach space, and showed that the ideal $\mathscr{S}(E)$ of strictly singular operators is a maximal ideal of codimension one in $\mathscr{B}(E)$ for each such space E (see [14]); once again this maximal ideal is unique (see [23]). Androulakis and Schlumprecht have proved that non-compact, strictly singular operators exist on the particular hereditarily indecomposable Banach space E that Gowers and Maurey constructed (see [1]), and so in this case $\mathscr{S}(E)$ is not the only non-trivial, closed ideal in $\mathscr{B}(E)$. It is a major open problem whether or not there exists a Banach space E such that the ideal of compact operators is a maximal ideal of codimension one in $\mathscr{B}(E)$. The reader is referred to Schlumprecht's paper [29] for the current state of this difficult problem together with an impressive new method of attack.

Fifth, Mankiewicz on the one hand and Dales, Loy, and Willis on the other have found Banach spaces E such that ℓ_∞ is a quotient of $\mathscr{B}(E)$ (see [26] and [8], respectively). It follows that, for each of these spaces E, $\mathscr{B}(E)$ has at least $2^{2^{\aleph_0}}$ maximal ideals of codimension one. Later, when solving Banach's hyperplane problem, Gowers constructed a Banach space G such that ℓ_∞/c_0 is a quotient of $\mathscr{B}(G)$ (see [13] and [15]). Laustsen has classified the maximal ideals in $\mathscr{B}(G)$ by observing that each such ideal is the preimage of a maximal ideal in ℓ_∞/c_0 (see [23]).

We shall next explain how this paper is organized.

Section 2 contains the formal definitions of the direct sums of Banach spaces that we shall be concerned with, together with those of their basic properties that we require.

In Section 3 we modify the techniques known from the proof of Theorem 1.1 to show that, for certain Banach spaces E, the ideals of approximable, compact, strictly singular, and inessential operators in $\mathscr{B}(E)$ coincide, and that there is a unique minimal closed ideal in $\mathscr{B}(E)$ properly containing these ideals. This result applies in particular to each Banach space E that is a c_0- or ℓ_p-direct sum of a sequence of finite-dimensional spaces.

In Section 4 we consider the case where E is the c_0-direct sum of some sequence of Banach spaces, and determine conditions that ensure that the closed ideal $\overline{\mathscr{G}}_{c_0}(E)$ generated by the operators on E that factor through c_0 is a maximal ideal in $\mathscr{B}(E)$.

Section 5 contains our main result: for the Banach space E defined in (1.1), above, the ideal of compact operators $\mathscr{K}(E)$ and the ideal $\overline{\mathscr{G}}_{c_0}(E)$ just defined are the only non-trivial, closed ideals in $\mathscr{B}(E)$.

In our final section, Section 6, we apply this result to give a new proof of the theorem, due to Bourgain, Casazza, Lindenstrauss, and Tzafriri, that each infinite-dimensional, complemented subspace of the Banach space E given by (1.1) is either isomorphic to c_0 or to E.

Before ending this introduction, let us describe some notation and conventions that we rely on throughout the paper.

All Banach spaces are supposed to be over the same scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$

or $\mathbb{K} = \mathbb{C}$. For a Banach space E, we denote by E' the dual Banach space of E, we write $\langle \cdot\,, \cdot \rangle$ for the duality between E and E', and we denote by κ_E the canonical embedding of E into its bidual Banach space E''.

A bounded, linear map between Banach spaces is termed an *operator*. The collection of all operators from a Banach space E to a Banach space F is denoted by $\mathscr{B}(E, F)$, or just $\mathscr{B}(E)$ in the case where $E = F$. We write I_E for the identity operator on E.

An *operator ideal* is an assignment $\mathscr{J}$ which associates to each pair (E, F) of Banach spaces a linear subspace $\mathscr{J}(E, F)$ of $\mathscr{B}(E, F)$ satisfying:

(i) $\mathscr{J}(E, F)$ is non-zero for some Banach spaces E and F;

(ii) for any Banach spaces D, E, F, and G, the composite operator TSR belongs to $\mathscr{J}(D, G)$ whenever R belongs to $\mathscr{B}(D, E)$, S to $\mathscr{J}(E, F)$, and T to $\mathscr{B}(F, G)$.

We usually write $\mathscr{J}(E)$ instead of $\mathscr{J}(E, E)$.

For an operator ideal $\mathscr{J}$ and Banach spaces E and F, we write $\overline{\mathscr{J}}(E, F)$ for the closure (in the operator norm) of $\mathscr{J}(E, F)$ in $\mathscr{B}(E, F)$. The assignment $\overline{\mathscr{J}}$ thus defined is an operator ideal, called the *closure* of $\mathscr{J}$. We say that the operator ideal $\mathscr{J}$ is *closed* if $\mathscr{J} = \overline{\mathscr{J}}$.

We shall consider the following operator ideals (and their closures):

– $\mathscr{F}$, the *finite-rank operators* (the operators in $\overline{\mathscr{F}}$ are termed *approximable*);

– $\mathscr{K}$, the *compact operators;*

– $\mathscr{S}$, the *strictly singular operators;*

– $\mathscr{E}$, the *inessential operators;*

– $\mathscr{I}_\infty$, the ∞-*integral operators;*

– $\mathscr{G}_{\mathscr{C}}$ (where $\mathscr{C}$ is a subset of $\mathscr{B}(E, F)$ for some Banach spaces E and F), the *operator ideal generated by the set* $\mathscr{C}$.

We regard the first two of these operator ideals as so well-known that no definitions are required. We shall define the final four when they first appear in the text.

2 Preliminaries on direct sums

2.1 Finite direct sums. Let $n \in \mathbb{N}$, and let $E_1, \ldots, E_n$ be Banach spaces. We denote by $E_1 \oplus \cdots \oplus E_n$ the direct sum of $E_1, \ldots, E_n$ equipped with the ℓ_∞^n-norm given by

$$\big\|(x_1, \ldots, x_n)\big\| := \max\big\{\|x_1\|, \ldots, \|x_n\|\big\} \qquad (x_1 \in E_1, \ldots, x_n \in E_n). \tag{2.1}$$

(This particular choice of norm on the direct sum will be important in Section 5.) In the case where $E_1 = \cdots = E_n$, we write $E_1^{\oplus n}$ instead of $E_1 \oplus \cdots \oplus E_n$.

Set $E := E_1 \oplus \cdots \oplus E_n$. For each $m \in \{1, \ldots, n\}$, we write J_m^E for the canonical embedding of E_m into E and Q_m^E for the canonical projection of E onto E_m. When no ambiguity may arise, we omit the superscript E from these operators.

Suppose that $T_1\colon E_1 \to F_1, \dots, T_n\colon E_n \to F_n$ are operators into some Banach spaces $F_1, \dots, F_n$. Then we write $T_1 \oplus \cdots \oplus T_n$ for the *diagonal operator* induced by $T_1, \dots, T_n$, that is,

$$T_1 \oplus \cdots \oplus T_n\colon\ (x_1, \dots, x_n) \mapsto (T_1x_1, \dots, T_nx_n), \quad E_1 \oplus \cdots \oplus E_n \to F_1 \oplus \cdots \oplus F_n.$$

2.2 The *D*-direct sum of an infinite sequence of Banach spaces. Let D be a Banach space with a normalized, 1-unconditional basis (d_n). The $(D,(d_n))$*-direct sum* of a sequence (E_n) of Banach spaces is given by

$$\Big(\bigoplus_{n\in\mathbb{N}} E_n\Big)_{D,\,(d_n)} := \Big\{(x_n) \ \Big|\ x_n \in E_n\ (n \in \mathbb{N}) \text{ and the series } \sum_{n=1}^{\infty} \|x_n\|\, d_n \text{ converges}\Big\}.$$

This is a Banach space for coordinatewise defined addition and scalar multiplication and norm given by

$$\big\|(x_n)\big\| := \Big\|\sum_{n=1}^{\infty} \|x_n\|\, d_n\Big\| \in [0,\infty[\qquad \Big((x_n) \in \Big(\bigoplus_{n\in\mathbb{N}} E_n\Big)_{D,\,(d_n)}\Big).$$

We shall usually suppress the index set $\mathbb{N}$ in this notation. Moreover, in most cases D comes with a 'canonical' basis (d_n), and so we may without ambiguity omit (d_n), thus writing $\big(\bigoplus E_n\big)_D$ instead of $\big(\bigoplus_{n\in\mathbb{N}} E_n\big)_{D,\,(d_n)}$.

Set $E := \big(\bigoplus E_n\big)_D$. As in the finite case (see §2.1), we denote by J_m^E the canonical embedding of E_m into E and by Q_m^E the canonical projection of E onto E_m for each $m \in \mathbb{N}$. Both J_m^E and Q_m^E are operators of norm one; in fact, the former is an isometry, and the latter is a quotient map. Let ν be a non-empty subset of $\mathbb{N}$. Since the basis (d_n) is 1-unconditional, there is an idempotent operator P_ν^E of norm one given by

$$P_\nu^E\colon\ x \mapsto \sum_{m\in\nu} J_m^E Q_m^E x, \quad E \to E.$$

2.3 Duality. Let D be a Banach space with a normalized, 1-unconditional basis (d_n), let (E_n) be a sequence of Banach spaces, and set $E := \big(\bigoplus E_n\big)_D$. Suppose that the basis (d_n) is shrinking, so that the coordinate functionals (d_n') are a normalized, 1-unconditional basis of the dual space D'. Then we can form the D'-direct sum $E^\dagger := \big(\bigoplus E_n'\big)_{D'}$, and it can be shown that the map $\Upsilon_E\colon E^\dagger \to E'$ given by

$$\big\langle (x_n), \Upsilon_E(\varphi_n)\big\rangle := \sum_{n=1}^{\infty} \langle x_n, \varphi_n\rangle \qquad \big((x_n) \in E,\ (\varphi_n) \in E^\dagger\big)$$

is an isometric isomorphism making the diagrams

$$\begin{array}{ccc} E_n' & \xrightarrow{(Q_n^E)'} & E' \\ & {\scriptstyle J_n^{E^\dagger}}\searrow \quad \nearrow{\scriptstyle \Upsilon_E} & \\ & E^\dagger & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} E' & \xrightarrow{(J_n^E)'} & E_n' \\ & {\scriptstyle \Upsilon_E}\nwarrow \quad \nearrow{\scriptstyle Q_n^{E^\dagger}} & \\ & E^\dagger & \end{array}$$

commutative (*e.g.*, see [22, §4]).

2.4 Example. Let $D = c_0$ or $D = \ell_p$ for some $p \in [1, \infty[$. We shall always equip D with its *standard basis* (d_n) given by $d_n = (\delta_{m,n})_{m=1}^{\infty}$ for each $n \in \mathbb{N}$, where $\delta_{m,n}$ is Kronecker's delta symbol. It is well known that (d_n) is a normalized, 1-unconditional basis of D and, moreover, that (d_n) is a shrinking basis for $D = c_0$ and $D = \ell_p$ with $p \in]1, \infty[$, but not for $D = \ell_1$.

Now let (E_n) be a sequence of Banach spaces. Then we have

$$\Big(\bigoplus E_n\Big)_{c_0} = \big\{(x_n) \;\big|\; x_n \in E_n \ (n \in \mathbb{N}) \text{ and } \|x_n\| \to 0 \text{ as } n \to \infty\big\},$$

and $\big\|(x_n)\big\| = \sup\big\{\|x_n\| \;\big|\; n \in \mathbb{N}\big\}$ for each $(x_n) \in \big(\bigoplus E_n\big)_{c_0}$. Similarly, for each $p \in [1, \infty[$, we have

$$\Big(\bigoplus E_n\Big)_{\ell_p} = \Big\{(x_n) \;\Big|\; x_n \in E_n \ (n \in \mathbb{N}) \text{ and } \sum_{n=1}^{\infty} \|x_n\|^p < \infty\Big\},$$

and $\big\|(x_n)\big\| = \big(\sum_{n=1}^{\infty} \|x_n\|^p\big)^{1/p}$ for each $(x_n) \in \big(\bigoplus E_n\big)_{\ell_p}$.

2.5 Diagonal operators. Let D be a Banach space with a normalized, 1-unconditional basis (d_n), and, for each $n \in \mathbb{N}$, let $T_n \colon E_n \to F_n$ be an operator between Banach spaces E_n and F_n. Suppose that $\sup \|T_n\| < \infty$. Then, as in the finite case, we can define the *diagonal operator*

$$\operatorname{diag}(T_n)\colon\ (x_n) \mapsto (T_n x_n), \quad \Big(\bigoplus E_n\Big)_D \to \Big(\bigoplus F_n\Big)_D.$$

Clearly, we have $\big\|\operatorname{diag}(T_n)\big\| = \sup \|T_n\|$.

2.6 Definition. Let D be a Banach space with a normalized, 1-unconditional basis, let (E_n) and (F_n) be sequences of Banach spaces, and let $T\colon \big(\bigoplus E_n\big)_D \to \big(\bigoplus F_n\big)_D$ be an operator. We associate with T the infinite matrix $(T_{m,n})$, where

$$T_{m,n} := Q_m^F T J_n^E\colon\ E_n \to F_m \qquad (m, n \in \mathbb{N}).$$

The *support of the m^{th} row* of T is

$$\operatorname{rowsupp}_m(T) := \big\{n \in \mathbb{N} \;\big|\; T_{m,n} \neq 0\big\} \qquad (m \in \mathbb{N}).$$

We say that T has *finite rows* if each row has finite support, and we say that T has *consecutively supported rows* if $\sup\big(\operatorname{rowsupp}_m(T)\big) < \inf\big(\operatorname{rowsupp}_n(T)\big)$ whenever $m, n \in \mathbb{N}$ with $m < n$ (where we rely on the conventions that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$).

Similarly, the *support of the n^{th} column* of T is

$$\operatorname{colsupp}_n(T) := \big\{m \in \mathbb{N} \;\big|\; T_{m,n} \neq 0\big\} \qquad (n \in \mathbb{N}),$$

T has *finite columns* if each column has finite support, and T has *consecutively supported columns* if $\sup\big(\operatorname{colsupp}_m(T)\big) < \inf\big(\operatorname{colsupp}_n(T)\big)$ whenever $m, n \in \mathbb{N}$ with $m < n$.

If T has both finite rows and finite columns, then we say that T has *locally finite matrix*.

2.7 Lemma. *Let D be a Banach space with a normalized, 1-unconditional basis (d_n), let (E_n) and (F_n) be sequences of Banach spaces, set $E := \left(\bigoplus E_n\right)_D$ and $F := \left(\bigoplus F_n\right)_D$, let $T\colon E \to F$ be an operator, and let $\varepsilon > 0$.*

(i) *Suppose that each of the spaces E_n $(n \in \mathbb{N})$ is finite-dimensional. Then there is an approximable operator $S\colon E \to F$ with $\|S\| \leqslant \varepsilon$ such that $T - S$ has finite columns, and $S_{m,n} = T_{m,n}$ whenever $S_{m,n} \neq 0$ $(m, n \in \mathbb{N})$.*

(ii) *Suppose that the basis (d_n) is shrinking and that each of the spaces F_n $(n \in \mathbb{N})$ is finite-dimensional. Then there is an approximable operator $S\colon E \to F$ with $\|S\| \leqslant \varepsilon$ such that $T - S$ has finite rows, and $S_{m,n} = T_{m,n}$ whenever $S_{m,n} \neq 0$ $(m, n \in \mathbb{N})$.*

(iii) *Suppose that the basis (d_n) is shrinking and that each of the spaces E_n and F_n $(n \in \mathbb{N})$ is finite-dimensional. Then there is an approximable operator $S\colon E \to F$ with $\|S\| \leqslant \varepsilon$ such that $T - S$ has locally finite matrix, and $S_{m,n} = T_{m,n}$ whenever $S_{m,n} \neq 0$ $(m, n \in \mathbb{N})$.*

Proof. For each $M \in \mathbb{N}$, set $\widetilde{P}^E_M := P^E_{\{1,\ldots,M\}}$ and $\widetilde{P}^F_M := P^F_{\{1,\ldots,M\}}$.

(i) Using the compactness of the unit ball of E_n $(n \in \mathbb{N})$, we can construct a strictly increasing sequence (M_n) in $\mathbb{N}$ such that

$$\big\|(I_F - \widetilde{P}^F_{M_n})TJ^E_n\big\| \leqslant \varepsilon/2^n \qquad (n \in \mathbb{N}). \tag{2.2}$$

Set

$$S := \sum_{n=1}^{\infty}(I_F - \widetilde{P}^F_{M_n})TJ^E_nQ^E_n\colon\ E \to F.$$

Then S is an approximable operator with $\|S\| \leqslant \varepsilon$, and we have

$$S_{m,n} = \begin{cases} 0 & \text{for } m \leqslant M_n \\ T_{m,n} & \text{for } m > M_n \end{cases} \qquad (m, n \in \mathbb{N}).$$

This proves (i).

(ii) Dualizing (2.2) (*cf.* §2.3), we obtain a strictly increasing sequence (M_n) in $\mathbb{N}$ such that

$$\big\|Q^F_nT(I_E - \widetilde{P}^E_{M_n})\big\| \leqslant \varepsilon/2^n \qquad (n \in \mathbb{N}).$$

Set

$$S := \sum_{n=1}^{\infty} J^F_nQ^F_nT(I_E - \widetilde{P}^E_{M_n})\colon\ E \to F.$$

As before, it is easy to see that S has the properties listed in (ii).

(iii) This is immediate from (i) and (ii). □

3 The ‘small’ ideals in $\mathscr{B}(E)$

In this section we shall show that, for certain Banach spaces E, the ideals of approximable, compact, strictly singular, and inessential operators on E coincide, and that there is a unique minimal closed ideal in $\mathscr{B}(E)$ properly containing these ideals. We refer to the above-mentioned ideals as ‘small’ because they are proper ideals in $\mathscr{B}(E)$ whenever E is infinite-dimensional.

We proceed by modifying the techniques developed by Herman in his simplified proof of Theorem 1.1 (see [17] or [5, §5.4]). Part of our argument is similar to that outlined in [20, p. 8].

3.1 Definition. (i) A sequence (x_n) in a Banach space is *seminormalized* if $\inf \|x_n\| > 0$ and $\sup \|x_n\| < \infty$.

(ii) A sequence (x_n) in a Banach space E is *complemented* in E if there is an idempotent operator P on E with $\operatorname{im} P = \overline{\operatorname{span}}\{x_n \,|\, n \in \mathbb{N}\}$.

(iii) A basis (d_n) of a Banach space D is *semispreading* if, for each strictly increasing sequence (m_n) in $\mathbb{N}$, there is an operator T on D with $Td_n = d_{m_n}$ for each $n \in \mathbb{N}$.

(iv) Let D and E be Banach spaces with bases (d_n) and (e_n), respectively. We say that *seminormalized blocks of* (e_n) *contain complemented copies of* (d_n) if each seminormalized block basic sequence of (e_n) has a subsequence which is equivalent to (d_n) and complemented in E.

3.2 Theorem. *Let D be a Banach space with a semispreading basis (d_n), and let E be a Banach space with a basis (e_n) such that seminormalized blocks of (e_n) contain complemented copies of (d_n). Then, for each non-compact operator T on E, there are operators $R\colon D \to E$ and $S\colon E \to D$ such that $I_D = STR$.*

The proof of Theorem 3.2 requires some preliminary work. Our first lemma is proved using a standard Cantor-style diagonal argument which we omit.

3.3 Lemma. *Let (T_n) be a sequence of compact operators from a Banach space E to a Banach space F. Then each bounded sequence (x_m) in E has a subsequence (x_{m_k}) such that, for each $n \in \mathbb{N}$, the sequence $(T_n x_{m_k})_{k=1}^{\infty}$ is convergent.* □

Second, we shall improve a classical stability result of Krein, Milman, and Rutman [21] and, independently, Bessaga and Pełczyński [2]; alternatively, see [24, Proposition 1.a.9]. Our proof is inspired by the proof of [27, Proposition 4.3.4].

3.4 Lemma. *Let (x_n) be a basic sequence with basis constant K in a Banach space E, and let (y_n) be a sequence in E such that*

$$\sum_{n=1}^{\infty} \frac{\|x_n - y_n\|}{\|x_n\|} < \frac{1}{2K}.$$

Then (y_n) is a basic sequence equivalent to (x_n).

Suppose that (x_n) is complemented in E. Then (y_n) is also complemented in E.

Proof. For each $m \in \mathbb{N}$, let $\varphi_m \in E'$ with $\|\varphi_m\| \leqslant 2K/\|x_m\|$ be a Hahn–Banach extension of the m^{th} coordinate functional associated with (x_n). Then we can define an operator

$$T\colon\ x \mapsto \sum_{m=1}^{\infty} \langle x, \varphi_m\rangle (x_m - y_m), \quad E \to E,$$

and $\|T\| < 1$, so that the operator $U := I_E - T$ is invertible. It follows that (y_n) is a basic sequence equivalent to (x_n) because $Ux_n = y_n$ for each $n \in \mathbb{N}$.

Now suppose that P is an idempotent operator on E with $\operatorname{im} P = \overline{\operatorname{span}}\{x_n \,|\, n \in \mathbb{N}\}$. Then $Q := UPU^{-1}$ is an idempotent operator on E with $\operatorname{im} Q = \overline{\operatorname{span}}\{y_n \,|\, n \in \mathbb{N}\}$. □

Lemma 3.4 improves its predecessors by asserting that (y_n) is complemented in E whenever (x_n) is, no matter what the norm is of the idempotent operator P with image $\overline{\operatorname{span}}\{x_n \,|\, n \in \mathbb{N}\}$. This enables us to establish the following version of the Bessaga–Pełczyński selection principle, specially tailored to match the set-up in Theorem 3.2.

3.5 Lemma. *Let D and E be Banach spaces with bases (d_n) and (e_n), respectively, such that seminormalized blocks of (e_n) contain complemented copies of (d_n). Let (y_m) be a seminormalized sequence in E such that $\langle y_m, e'_k\rangle \to 0$ as $m \to \infty$ for each (fixed) $k \in \mathbb{N}$, where e'_k denotes the k^{th} coordinate functional associated with the basis (e_n). Then (y_m) has a subsequence which is equivalent to (d_n) and complemented in E.*

Proof. Let K be the basis constant of (e_n). As in the proof of the Bessaga–Pełczyński selection principle (see [2, Theorem 3]), we construct inductively a seminormalized block basic sequence (x_m) of (e_n) and a subsequence $(\bar{y}_m)$ of (y_m) such that

$$\sum_{m=1}^{\infty} \frac{\|x_m - \bar{y}_m\|}{\|x_m\|} < \frac{1}{2K}.$$

By assumption, (x_m) has a subsequence (x_{m_n}) which is equivalent to (d_n) and complemented in E. Now Lemma 3.4 implies that $(\bar{y}_{m_n})$ has the required properties. □

Proof of Theorem 3.2. Let (e'_n) denote the coordinate functionals associated with (e_n). Take a bounded sequence (x_m) in E such that no subsequence of (Tx_m) is convergent. By Lemma 3.3 (applied with $T_n = e'_n$ and the bounded sequence (Tx_m)), (x_m) has a subsequence $(\bar{x}_m)$ such that $\big(\langle T\bar{x}_m, e'_n\rangle\big)_{m=1}^{\infty}$ is convergent for each $n \in \mathbb{N}$. Since $(T\bar{x}_m)$ is divergent, $(\bar{x}_m)$ has a subsequence $(\bar{\bar{x}}_m)$ such that $\inf \|T\bar{\bar{x}}_{m+1} - T\bar{\bar{x}}_m\| > 0$.

Set $z_m := \bar{\bar{x}}_{m+1} - \bar{\bar{x}}_m \in E$. Then (z_m) is bounded, $\inf \|Tz_m\| > 0$, and $\langle Tz_m, e'_n\rangle \to 0$ as $m \to \infty$ for each $n \in \mathbb{N}$. It follows that no subsequence of (Tz_m) can be convergent. Another application of Lemma 3.3 yields a subsequence $(\bar{z}_m)$ of (z_m) such that $\big(\langle \bar{z}_m, e'_n\rangle\big)_{m=1}^{\infty}$ is convergent for each $n \in \mathbb{N}$. Since $(T\bar{z}_m)$ is divergent, we can find a subsequence $(\bar{\bar{z}}_m)$ of $(\bar{z}_m)$ such that $\inf \|T\bar{\bar{z}}_{m+1} - T\bar{\bar{z}}_m\| > 0$.

Set $y_m := \bar{\bar{z}}_{m+1} - \bar{\bar{z}}_m \in E$. Then (y_m) is bounded and $\inf \|Ty_m\| > 0$. This implies that (y_m) and (Ty_m) are seminormalized. Moreover, for each $n \in \mathbb{N}$, we have $\langle y_m, e'_n\rangle \to 0$

and $\langle Ty_m, e_n'\rangle \to 0$ as $m \to \infty$. By Lemma 3.5, (y_m) has a subsequence $(\bar{y}_m)$ which is equivalent to (d_m). Take an operator $U\colon D \to E$ with $Ud_m = \bar{y}_m$ $(m \in \mathbb{N})$. Applying Lemma 3.5 once more shows that $(T\bar{y}_m)$ has a subsequence $(T\bar{y}_{m_n})$ which is equivalent to (d_n) and complemented in E. It follows that there is an operator $S\colon E \to D$ with $S(T\bar{y}_{m_n}) = d_n$ $(n \in \mathbb{N})$. Since (d_n) is semispreading, we can take an operator V on D with $Vd_n = d_{m_n}$ $(n \in \mathbb{N})$. Set $R := UV\colon D \to E$. Then we have

$$STRd_n = STUd_{m_n} = ST\bar{y}_{m_n} = d_n \qquad (n \in \mathbb{N}),$$

and the result follows. □

3.6 Definition. Let D, E, F, and G be Banach spaces. For each subset $\mathscr{C}$ of $\mathscr{B}(E,F)$, set

$$\mathscr{G}_{\mathscr{C}}(D,G) := \operatorname{span}\bigl\{STR \bigm| R \in \mathscr{B}(D,E),\, T \in \mathscr{C},\, S \in \mathscr{B}(F,G)\bigr\} \subseteq \mathscr{B}(D,G). \tag{3.1}$$

Suppose that that $\mathscr{C}$ contains a non-zero operator. Then the assignment $\mathscr{G}_{\mathscr{C}}$ thus defined is an operator ideal, called the *operator ideal generated by* $\mathscr{C}$. It is clearly the smallest operator ideal such that $\mathscr{C} \subseteq \mathscr{G}_{\mathscr{C}}(E,F)$.

In the case where $E = F$ and $\mathscr{C} = \{I_E\}$, we write $\mathscr{G}_E$ instead of $\mathscr{G}_{\mathscr{C}}$.

Suppose that the set $\mathscr{C}$ satisfies: for each $T_1, T_2 \in \mathscr{C}$, there are operators $U\colon E \oplus E \to E$, $V \in \mathscr{C}$, and $W\colon F \to F \oplus F$ such that $T_1 \oplus T_2 = WVU$. Then the set

$$\bigl\{STR \bigm| R \in \mathscr{B}(D,E),\, T \in \mathscr{C},\, S \in \mathscr{B}(F,G)\bigr\}$$

is already a linear subspace of $\mathscr{B}(D,G)$, and so the 'span' appearing in (3.1) is superfluous.

In particular, in the case where E is a Banach space containing a complemented subspace isomorphic to $E \oplus E$, then

$$\mathscr{G}_E(D,G) = \bigl\{SR \bigm| R \in \mathscr{B}(D,E),\, S \in \mathscr{B}(E,G)\bigr\}$$

for each pair (D,G) of Banach spaces.

3.7 Definition. Let E and F be Banach spaces, and let $T\colon E \to F$ be an operator. We say that T is *strictly singular* if T is not bounded below on any infinite-dimensional subspace of E, and we say that T is *inessential* if $I_E - ST$ is a Fredholm operator for each operator $S\colon F \to E$. We write $\mathscr{S}(E,F)$ and $\mathscr{E}(E,F)$ for the sets of strictly singular and inessential operators from E to F, respectively. The assignments $\mathscr{S}$ and $\mathscr{E}$ thus defined are closed operator ideals (*e.g.*, see [28, §1.9 and §4.3]).

In general, the inclusions

$$\overline{\mathscr{F}}(E,F) \subseteq \mathscr{K}(E,F) \subseteq \mathscr{S}(E,F) \subseteq \mathscr{E}(E,F) \subseteq \mathscr{B}(E,F) \tag{3.2}$$

hold; the first inclusion can be replaced with equality if F has the approximation property. However, in the case where $E = F$ and this is a Banach space of the form considered in Theorem 3.2, much more is true.

3.8 Corollary. *Let D be a Banach space with a semispreading basis (d_n), and let E be a Banach space with a basis (e_n) such that seminormalized blocks of (e_n) contain complemented copies of (d_n). Suppose that $\mathscr{J}$ is an ideal in $\mathscr{B}(E)$ not contained in $\overline{\mathscr{F}}(E)$. Then $\mathscr{J}$ contains the ideal $\mathscr{G}_D(E)$.*

It follows that

$$\overline{\mathscr{F}}(E) = \mathscr{K}(E) = \mathscr{S}(E) = \mathscr{E}(E) \subsetneq \overline{\mathscr{G}}_D(E),$$

and there are no closed ideals $\mathscr{J}$ in $\mathscr{B}(E)$ such that $\overline{\mathscr{F}}(E) \subsetneq \mathscr{J} \subsetneq \overline{\mathscr{G}}_D(E)$.

Proof. This is immediate from Theorem 3.2 and (3.2). □

3.9 Example. Let $D = c_0$ or $D = \ell_p$, where $1 \leqslant p < \infty$, and, for each $n \in \mathbb{N}$, let E_n be a non-zero, finite-dimensional Banach space with a normalized, monotone basis $(e_1^{(n)}, \ldots, e_{N_n}^{(n)})$. Then

$$(e_n)_{n=1}^{\infty} := \big(J_1^E(e_1^{(1)}), J_1^E(e_2^{(1)}), \ldots, J_1^E(e_{N_1}^{(1)}), J_2^E(e_1^{(2)}), J_2^E(e_2^{(2)}), \ldots, J_2^E(e_{N_2}^{(2)}), \ldots$$
$$\ldots, J_n^E(e_1^{(n)}), J_n^E(e_2^{(n)}), \ldots, J_n^E(e_{N_n}^{(n)}), \ldots\big)$$

is a normalized, monotone basis of $E := \big(\bigoplus E_n\big)_D$. We *claim* that seminormalized blocks of (e_n) contain complemented copies of the standard basis (d_n) of D. (We note in passing that, in the case where $E_n = \ell_q^n$ for each $n \in \mathbb{N}$ and some $q \in [1, \infty]$, this is an easy consequence of a theorem of Casazza and Lin (see [7, Theorem 38] or [24, Proposition 2.a.12]).)

To prove the claim, let (x_n) be a seminormalized block basic sequence of (e_n). For each $x \in E$, set

$$\operatorname{supp} x := \big\{m \in \mathbb{N} \bigm| Q_m^E(x) \neq 0\big\}.$$

Inductively we choose a subsequence (x_{n_k}) of (x_n) such that

$$\max(\operatorname{supp} x_{n_k}) < \min(\operatorname{supp} x_{n_{k+1}}) \qquad (k \in \mathbb{N}).$$

For each $k \in \mathbb{N}$, take $\varphi_k \in E'$ such that $\|\varphi_k\| = 1/\|x_{n_k}\|$, $\langle x_{n_k}, \varphi_k\rangle = 1$, and $\langle x, \varphi_k\rangle = 0$ whenever $x \in E$ with $\operatorname{supp} x \cap \operatorname{supp} x_{n_k} = \emptyset$. Since the sequence (x_{n_k}) is seminormalized, we can define operators

$$S\colon\ x \mapsto \big(\langle x, \varphi_k\rangle\big)_{k=1}^{\infty}, \quad E \to D, \qquad \text{and} \qquad T\colon\ (\alpha_k) \mapsto \sum_{k=1}^{\infty} \alpha_k x_{n_k}, \quad D \to E.$$

Clearly, we have $Sx_{n_k} = d_k$ and $Td_k = x_{n_k}$ for each $k \in \mathbb{N}$. This implies that (x_{n_k}) is a complemented basic sequence equivalent to (d_k), and the claim follows.

The basis (d_n) is obviously semispreading, and so we conclude from Corollary 3.8 that

$$\overline{\mathscr{F}}(E) = \mathscr{K}(E) = \mathscr{S}(E) = \mathscr{E}(E) \subsetneq \overline{\mathscr{G}}_D(E),$$

and for each non-zero, closed ideal $\mathscr{J}$ in $\mathscr{B}(E)$, either $\mathscr{J} = \overline{\mathscr{F}}(E)$ or $\overline{\mathscr{G}}_D(E) \subseteq \mathscr{J}$. □

4 Operators on c_0-direct sums

In this section we shall concentrate on the special situation where $E = \big(\bigoplus E_n\big)_{c_0}$ for certain sequences (E_n) of Banach spaces. In particular we shall determine conditions which are sufficient for $\overline{\mathscr{G}}_{c_0}(E)$ to be a maximal ideal in $\mathscr{B}(E)$.

Our first lemma characterizes those matrices with finite columns that induce operators between c_0-direct sums. The proof is straightforward and thus omitted.

4.1 Lemma. *Let (E_n) and (F_n) be sequences of Banach spaces, and let $(V_{m,n})$ be a matrix with $V_{m,n} \in \mathscr{B}(E_n, F_m)$ for each $m, n \in \mathbb{N}$ and at most finitely many non-zero entries in each column. Then there is an operator $V\colon \big(\bigoplus E_n\big)_{c_0} \to \big(\bigoplus F_n\big)_{c_0}$ with matrix $(V_{m,n})$ (that is, $V_{m,n} = Q_m^F V J_n^E$ for each $m, n \in \mathbb{N}$) if and only if there is a constant $c \geqslant 0$ such that*

$$\Big\| \sum_{n=1}^{N} V_{m,n} x_n \Big\| \leqslant c \max_{1 \leqslant n \leqslant N} \|x_n\| \qquad (m, N \in \mathbb{N},\ x_1 \in E_1, \ldots, x_N \in E_N). \tag{4.1}$$

In this case, $\|V\| = \inf c$, where the infimum is taken over the set of all $c \geqslant 0$ such that (4.1) holds. □

The following construction will be important in the proof of the main result (Theorem 4.4) of this section.

4.2 Construction. Let (E_n) and (F_n) be sequences of Banach spaces, and set

$$E := \Big(\bigoplus E_n\Big)_{c_0} \qquad \text{and} \qquad F := \Big(\bigoplus F_n\Big)_{c_0}.$$

Let $T\colon E \to F$ be an operator with finite columns. Define

$$\nu_m := \operatorname{rowsupp}_m(T), \qquad B_m := \begin{cases} \{0\} & \text{if } \nu_m = \emptyset \\ \big(\bigoplus_{n \in \nu_m} E_n\big)_{c_0} & \text{otherwise} \end{cases} \qquad (m \in \mathbb{N}),$$

$$B := \Big(\bigoplus_{m \in \mathbb{N}} B_m\Big)_{c_0}, \qquad V_{m,n} := \begin{cases} J_n^{B_m} & \text{if } n \in \nu_m \\ 0 & \text{otherwise} \end{cases} \in \mathscr{B}(E_n, B_m) \qquad (m, n \in \mathbb{N}),$$

where $\big(\bigoplus_{n \in \nu_m} E_n\big)_{c_0}$ is the obvious generalization of $\big(\bigoplus_{n \in \mathbb{N}} E_n\big)_{c_0}$ to index sets $\nu_m \subsetneq \mathbb{N}$, and $J_n^{B_m}$ denotes the natural embedding of E_n into B_m for each $n \in \nu_m$.

Observe that, for each $m, n \in \mathbb{N}$, $T_{m,n} = 0$ if and only if $V_{m,n} = 0$, and so $(V_{m,n})$ has at most finitely many non-zero entries in each column. Since $(V_{m,n})$ clearly satisfies condition (4.1) in Lemma 4.1 with $c = 1$, we conclude that there is an operator $V\colon E \to B$ with matrix $(V_{m,n})$, and $\|V\| \leqslant 1$.

For each $m \in \mathbb{N}$, let L_m be the canonical embedding of B_m into E. This is an isometry, and so $\widetilde{T}_m := Q_m^F T L_m\colon B_m \to F_m$ is an operator of norm at most $\|T\|$. It follows that there is a diagonal operator $\operatorname{diag}(\widetilde{T}_m)\colon B \to F$, as defined in §2.5. We *claim* that

$$T = \operatorname{diag}(\widetilde{T}_m) V. \tag{4.2}$$

Indeed, for each $m, n \in \mathbb{N}$ and $x \in E_n$, we have

$$Q_m^F \operatorname{diag}(\widetilde{T}_k) V J_n^E x = \widetilde{T}_m Q_m^B (V_{k,n} x)_{k=1}^{\infty} = Q_m^F T L_m V_{m,n} x$$
$$= \left\{ \begin{array}{ll} Q_m^F T L_m J_n^{B_m} x = Q_m^F T J_n^E x & \text{if } n \in \nu_m \\ 0 & \text{otherwise} \end{array} \right\} = T_{m,n} x,$$

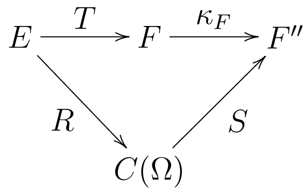
and (4.2) follows. □

4.3 Definition. Let E and F be Banach spaces, and let $T \colon E \to F$ be an operator.

(i) Let $\varepsilon \geqslant 0$. To measure the ε-approximate factorization of the operator T through the finite-dimensional spaces ℓ_∞^M ($M \in \mathbb{N}$), we define

$$\operatorname{fac}_\infty^\varepsilon(T) := \inf\bigl\{ \|S\|\,\|R\| \bigm| M \in \mathbb{N},\ R \in \mathscr{B}(E, \ell_\infty^M),\ S \in \mathscr{B}(\ell_\infty^M, F),\ \|T - SR\| \leqslant \varepsilon \bigr\} \in [0, \infty].$$

(ii) The operator T is *∞-integral* if there is a compact Hausdorff space Ω, and two operators $R \colon E \to C(\Omega)$ and $S \colon C(\Omega) \to F''$ such that the diagram



is commutative. We write $\mathscr{I}_\infty(E, F)$ for the set of all ∞-integral operators from E to F. The assignment $\mathscr{I}_\infty$ thus defined is an operator ideal.

Before stating our next theorem, we recall that, for an operator ideal $\mathscr{J}$, $\overline{\mathscr{J}}$ denotes the closure of $\mathscr{J}$.

4.4 Theorem. *Let (E_n) be a sequence of finite-dimensional Banach spaces, (F_n) a sequence of dual Banach spaces, and set $E := \bigl(\bigoplus E_n\bigr)_{c_0}$ and $F := \bigl(\bigoplus F_n\bigr)_{c_0}$. For each operator $T \colon E \to F$ with locally finite matrix, the following three assertions are equivalent:*

(a) $T \in \overline{\mathscr{G}}_{c_0}(E, F)$;

(b) $T \in \overline{\mathscr{I}}_\infty(E, F)$;

(c) $\sup\bigl\{\operatorname{fac}_\infty^\varepsilon(Q_m^F T) \bigm| m \in \mathbb{N}\bigr\} < \infty$ *for each* $\varepsilon > 0$.

Proof. (a)⇒(b). This is clear because the Banach spaces c_0, c, and $C(\mathbb{N}_\infty)$ are isomorphic, where $\mathbb{N}_\infty$ denotes the one-point compactification of $\mathbb{N}$.

(b)⇒(c). Suppose that $T \in \overline{\mathscr{I}}_\infty(E, F)$. Given $\varepsilon > 0$, take a compact Hausdorff space Ω and operators $R \colon E \to C(\Omega)$ and $S \colon C(\Omega) \to F''$ such that $\|\kappa_F T - SR\| \leqslant \varepsilon$. We *claim* that $\operatorname{fac}_\infty^\varepsilon(Q_m^F T) \leqslant 2\,\|S\|\,\|R\|$ for each $m \in \mathbb{N}$.

If $\operatorname{rowsupp}_m(T) = \emptyset$, then $Q_m^F T = 0$, and so the claim trivially holds in this case.

Now suppose that $\nu := \operatorname{rowsupp}_m(T)$ is non-empty. By assumption, ν is a finite set, and so P_ν^E is a finite-rank operator which clearly satisfies $Q_m^F T = Q_m^F T P_\nu^E$. In particular, $\operatorname{im}(RP_\nu^E)$ is a finite-dimensional subspace of $C(\Omega)$. Since $C(\Omega)$ is an $\mathscr{L}_{\infty,2}$-space (*e.g.*, see [10, Theorem 3.2(II)]), we can take $M \in \mathbb{N}$ and an M-dimensional subspace C of $C(\Omega)$ such that there is an isomorphism $U\colon C \to \ell_\infty^M$ with $\|U\|\,\|U^{-1}\| \leqslant 2$ and $\operatorname{im}(RP_\nu^E) \subseteq C$.

Let G_m be a predual Banach space of F_m, so that $G_m' = F_m$, and define operators $\widetilde{R}$ and $\widetilde{S}$ by

$$\begin{array}{ccc} E & \overset{\widetilde{R}}{\dashrightarrow} & \ell_\infty^M \\ & {\scriptstyle RP_\nu^E}\searrow \quad \nearrow{\scriptstyle U,\ \cong} & \\ & C & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} \ell_\infty^M & & \overset{\widetilde{S}}{\dashrightarrow} & & F_m \\ {\scriptstyle U^{-1}}\downarrow{\scriptstyle \cong} & & & & \uparrow{\scriptstyle \kappa_{G_m}'} \\ C \subseteq C(\Omega) & \xrightarrow{S} & F'' & \xrightarrow{(Q_m^F)''} & F_m''. \end{array}$$

Then, using the facts that $\kappa_{G_m}'\kappa_{F_m} = I_{F_m}$ and $\kappa_{F_m} Q_m^F = (Q_m^F)''\kappa_F$, we obtain

$$\begin{aligned} \|Q_m^F T - \widetilde{S}\widetilde{R}\| &= \big\|\kappa_{G_m}'\kappa_{F_m} Q_m^F T P_\nu^E - \widetilde{S}\widetilde{R}\big\| \\ &= \big\|\kappa_{G_m}'(Q_m^F)''\kappa_F T P_\nu^E - \kappa_{G_m}'(Q_m^F)'' S R P_\nu^E\big\| \\ &\leqslant \|\kappa_{G_m}'\|\,\|(Q_m^F)''\|\,\|\kappa_F T - SR\|\,\|P_\nu^E\| \leqslant \varepsilon. \end{aligned}$$

This implies that

$$\operatorname{fac}_\infty^\varepsilon(Q_m^F T) \leqslant \|\widetilde{S}\|\,\|\widetilde{R}\| \leqslant \|\kappa_{G_m}'\|\,\|(Q_m^F)''\|\,\|S\|\,\|U^{-1}\|\,\|U\|\,\|R\|\,\|P_\nu^E\| \leqslant 2\,\|S\|\,\|R\|,$$

as claimed, and consequently (c) is satisfied.

(c)$\Rightarrow$(a). Let $\varepsilon > 0$ be given, and suppose that $\sup\big\{\operatorname{fac}_\infty^\varepsilon(Q_m^F T) \,\big|\, m \in \mathbb{N}\big\} < \infty$. Then, for each $m \in \mathbb{N}$, we can take $M_m \in \mathbb{N}$ and operators $R_m\colon E \to \ell_\infty^{M_m}$ and $S_m\colon \ell_\infty^{M_m} \to F_m$ such that $\sup\|R_m\| < \infty$, $\sup\|S_m\| < \infty$, and $\|Q_m^F T - S_m R_m\| \leqslant \varepsilon$. Set $D := \big(\bigoplus \ell_\infty^{M_m}\big)_{c_0}$. We shall use the notation and results of Construction 4.2. Since $\sup\|R_m\| < \infty$ and $\sup\|S_m\| < \infty$, there are diagonal operators $\operatorname{diag}(R_m L_m)\colon B \to D$ and $\operatorname{diag}(S_m)\colon D \to F$, and we have

$$\begin{aligned} \big\|\operatorname{diag}(\widetilde{T}_m) - \operatorname{diag}(S_m)\operatorname{diag}(R_m L_m)\big\| &= \sup\|\widetilde{T}_m - S_m R_m L_m\| \\ &\leqslant \sup\|Q_m^F T - S_m R_m\|\,\|L_m\| \leqslant \varepsilon. \end{aligned}$$

It follows that $\operatorname{diag}(\widetilde{T}_m) \in \overline{\mathscr{G}}_{c_0}(B, F)$ because D is isomorphic to c_0 and ε is arbitrary, and so we conclude that $T = \operatorname{diag}(\widetilde{T}_m)V \in \overline{\mathscr{G}}_{c_0}(E, F)$. $\square$

Combining Lemma 2.7(iii) and Theorem 4.4 yields the following result.

4.5 Corollary. *Let (E_n) and (F_n) be sequences of finite-dimensional Banach spaces. Then*

$$\overline{\mathscr{G}}_{c_0}\Big(\Big(\bigoplus E_n\Big)_{c_0}, \Big(\bigoplus F_n\Big)_{c_0}\Big) = \overline{\mathscr{I}}_\infty\Big(\Big(\bigoplus E_n\Big)_{c_0}, \Big(\bigoplus F_n\Big)_{c_0}\Big). \qquad \square$$

4.6 Corollary. *Let (E_n) be a sequence of finite-dimensional Banach spaces, (F_n) a sequence of dual Banach spaces, set $E := \big(\bigoplus E_n\big)_{c_0}$ and $F := \big(\bigoplus F_n\big)_{c_0}$, and let $T\colon E \to F$ be an operator with locally finite matrix. Then $T \notin \overline{\mathscr{G}}_{c_0}(E,F)$ if and only if there is a non-empty subset ν of $\mathbb{N}$ such that the operator $P_\nu^F T$ has consecutively supported rows and $P_\nu^F T \notin \overline{\mathscr{G}}_{c_0}(E,F)$.*

Proof. Suppose that $T \notin \overline{\mathscr{G}}_{c_0}(E,F)$. Then Theorem 4.4 implies that

$$\sup\big\{\mathrm{fac}_\infty^\varepsilon(Q_m^F T) \;\big|\; m \in \mathbb{N}\big\} = \infty$$

for some $\varepsilon > 0$. Inductively we choose a strictly increasing sequence (M_m) in $\mathbb{N}$ such that

$$\mathrm{fac}_\infty^\varepsilon(Q_{M_m}^F T) \geqslant m \quad \text{and} \quad \sup\big(\mathrm{rowsupp}_{M_m}(T)\big) < \inf\big(\mathrm{rowsupp}_{M_{m+1}}(T)\big) \qquad (m \in \mathbb{N}).$$

Set $\nu := \{M_m \,|\, m \in \mathbb{N}\}$. We observe that the k^{th} row of the matrix of $P_\nu^F T$ is equal to the k^{th} row of the matrix of T if $k \in \nu$ and zero otherwise. It follows that the operator $P_\nu^F T$ has consecutively supported rows, and Theorem 4.4 implies that $P_\nu^F T \notin \overline{\mathscr{G}}_{c_0}(E,F)$ because $\mathrm{fac}_\infty^\varepsilon(Q_{M_m}^F P_\nu^F T) \geqslant m$ for each $m \in \mathbb{N}$.

The converse implication is immediate from the fact that $\overline{\mathscr{G}}_{c_0}$ is an operator ideal. □

4.7 Lemma. *Let E and F be Banach spaces, and let P be an idempotent operator on E. Then $P \in \overline{\mathscr{G}}_F(E)$ if and only if, for some $n \in \mathbb{N}$, there is an idempotent operator Q on $F^{\oplus n}$ with $\operatorname{im} Q \cong \operatorname{im} P$.*

Proof. '⇒'. Suppose that $P \in \overline{\mathscr{G}}_F(E)$. Then in fact $P \in \mathscr{G}_F(E)$ by [23, Proposition 3.4], and so $P = \sum_{j=1}^n S_j R_j$ for some $n \in \mathbb{N}$, $R_1, \ldots, R_n \in \mathscr{B}(E,F)$, and $S_1, \ldots, S_n \in \mathscr{B}(F,E)$. Clearly, the operators

$$R\colon x \mapsto (R_1x, \ldots, R_nx),\; E \to F^{\oplus n}, \quad \text{and} \quad S\colon (x_1, \ldots, x_n) \mapsto \sum_{j=1}^n S_j x_j,\; F^{\oplus n} \to E,$$

satisfy $P = SR$. This implies by [23, Lemma 3.6(ii)] that $Q := RSRS \in \mathscr{B}(F^{\oplus n})$ is idempotent with $\operatorname{im} Q \cong \operatorname{im} P$.

'⇐'. Suppose that Q is an idempotent operator on $F^{\oplus n}$ with $\operatorname{im} Q \cong \operatorname{im} P$. By [23, Lemma 3.6(i)], there are operators $R\colon E \to F^{\oplus n}$ and $S\colon F^{\oplus n} \to E$ such that $P = SR$ and $Q = RS$. For each $j \in \{1, \ldots, n\}$, set $R_j := Q_j R \in \mathscr{B}(E,F)$ and $S_j := SJ_j \in \mathscr{B}(F,E)$, where $J_j\colon F \to F^{\oplus n}$ and $Q_j\colon F^{\oplus n} \to F$ are the j^{th} coordinate embedding and projection, respectively. Then we have

$$\sum_{j=1}^n S_j R_j = S\Bigg(\sum_{j=1}^n J_j Q_j\Bigg) R = P,$$

and so $P \in \mathscr{G}_F(E)$. □

4.8 Corollary. *Let P be an idempotent operator on a Banach space E. Then $P \in \overline{\mathscr{G}}_{c_0}(E)$ if and only if $\operatorname{im} P$ is either finite-dimensional or isomorphic to c_0.*

Proof. Suppose that $P \in \overline{\mathscr{G}}_{c_0}(E)$. Then Lemma 4.7 implies that, for some $n \in \mathbb{N}$, there is an idempotent operator Q on $c_0^{\oplus n}$ with $\operatorname{im} Q \cong \operatorname{im} P$. Since $c_0^{\oplus n} \cong c_0$, Pełczyński's theorem [24, Theorem 2.a.3] shows that either $\operatorname{im} Q$ is finite-dimensional or $\operatorname{im} Q \cong c_0$, and so the same is true for $\operatorname{im} P$.

The converse implication is clear. □

Applying this result with P being the identity operator yields the following conclusion.

4.9 Corollary. *Let E be a Banach space. Then $\overline{\mathscr{G}}_{c_0}(E) = \mathscr{B}(E)$ if and only if E is either finite-dimensional or isomorphic to c_0.* □

4.10 Theorem. *Let (E_n) be a sequence of non-zero, finite-dimensional Banach spaces, each having a normalized, monotone basis, and set $E := \left(\bigoplus E_n\right)_{c_0}$. Then the lattice of closed ideals in $\mathscr{B}(E)$ is given by*

$$\{0\} \subsetneq \overline{\mathscr{F}}(E) \subsetneq \overline{\mathscr{G}}_{c_0}(E) \subsetneq \mathscr{B}(E) \tag{4.3}$$

if and only if the following two conditions are satisfied:

(i) $E \not\cong c_0$;

(ii) *for each operator T on E with locally finite matrix and consecutively supported rows, either $T \in \overline{\mathscr{G}}_{c_0}(E)$ or $\overline{\mathscr{G}}_{\{T\}}(E) = \mathscr{B}(E)$.*

Proof. Suppose that the lattice of closed ideals in $\mathscr{B}(E)$ is given by (4.3). Then $E \not\cong c_0$ because otherwise we would have $\overline{\mathscr{G}}_{c_0}(E) = \mathscr{B}(E)$, contradicting (4.3). Moreover, if T is any operator on E such that $T \notin \overline{\mathscr{G}}_{c_0}(E)$, then necessarily $\overline{\mathscr{G}}_{\{T\}}(E) = \mathscr{B}(E)$ by (4.3).

Conversely, suppose that $E \not\cong c_0$ and that the lattice of closed ideals in $\mathscr{B}(E)$ is not given by (4.3). Corollary 4.9 shows that $\overline{\mathscr{G}}_{c_0}(E)$ is a proper ideal in $\mathscr{B}(E)$, and so Example 3.9 implies that there is a proper closed ideal $\mathscr{J}$ in $\mathscr{B}(E)$ such that $\overline{\mathscr{G}}_{c_0}(E) \subsetneq \mathscr{J}$. Pick $R \in \mathscr{J} \setminus \overline{\mathscr{G}}_{c_0}(E)$. By Lemma 2.7(iii), we can find an approximable operator S on E such that $R - S$ has locally finite matrix. Since $R - S \notin \overline{\mathscr{G}}_{c_0}(E)$, Corollary 4.6 implies that there is a subset ν of $\mathbb{N}$ such that the operator $T := P_\nu^E(R - S)$ has consecutively supported rows and $T \notin \overline{\mathscr{G}}_{c_0}(E)$. The ideal $\overline{\mathscr{G}}_{\{T\}}(E)$ is proper because $T \in \mathscr{J}$ and $\mathscr{J}$ is proper. □

Finally in this section we shall characterize the approximable operators between certain c_0-direct sums. The proof is an easy combination of standard methods, but for the convenience of the reader we have included it.

4.11 Proposition. *Let (E_n) be a sequence of Banach spaces, (F_n) a sequence of finite-dimensional Banach spaces, and set $E := \left(\bigoplus E_n\right)_{c_0}$ and $F := \left(\bigoplus F_n\right)_{c_0}$. Then, for each operator $T\colon E \to F$, the following three assertions are equivalent:*

(a) $T \in \overline{\mathscr{F}}(E, F)$;

(b) $\|T - P_{\{1,\ldots,n\}}^F T\| \to 0$ *as* $n \to \infty$;

(c) $\|Q_n^F T\| \to 0$ *as* $n \to \infty$.

Proof. (a)⇒(c). Clearly, it suffices to verify that (c) holds for each non-zero finite-rank operator $T\colon E\to F$. Take a basis $(y_1,\dots,y_m)$ for $\operatorname{im}T$, and let $(y_1',\dots,y_m')$ be the associated coordinate functionals, so that $Tx=\sum_{k=1}^m\langle Tx,y_k'\rangle\,y_k$ for each $x\in E$. Then we have

$$\|Q_n^F T\|\leqslant\sum_{k=1}^m\|Q_n^F y_k\|\,\|T'y_k'\|\to 0 \text{ as } n\to\infty,$$

as required.

(c)⇔(b). For each $n\in\mathbb{N}$, we have

$$\begin{aligned}\|T-P^F_{\{1,\dots,n\}}T\| &= \sup\bigl\{\|(T-P^F_{\{1,\dots,n\}}T)x\|\bigm| x\in E,\ \|x\|\leqslant 1\bigr\}\\ &= \sup\bigl\{\|Q_m^F(T-P^F_{\{1,\dots,n\}}T)x\|\bigm| m\in\mathbb{N},\ x\in E,\ \|x\|\leqslant 1\bigr\}\\ &= \sup\bigl\{\|Q_m^F Tx\|\bigm| m>n,\ x\in E,\ \|x\|\leqslant 1\bigr\}=\sup\bigl\{\|Q_m^F T\|\bigm| m>n\bigr\},\end{aligned}$$

and so (c) and (b) are equivalent.

(b)⇒(a). This is clear because $P^F_{\{1,\dots,n\}}\in\mathscr{F}(F)$ for each $n\in\mathbb{N}$. □

5 Classification of the closed ideals in $\mathscr{B}\bigl((\bigoplus\ell_2^n)_{c_0}\bigr)$

In this section we shall join the ends together to prove our main result: the Banach space $E:=\bigl(\bigoplus\ell_2^n\bigr)_{c_0}$ satisfies the two conditions in Theorem 4.10, and so the lattice of closed ideals in $\mathscr{B}(E)$ is given by (4.3).

We begin with the fundamental observation that $\bigl(\bigoplus\ell_2^n\bigr)_{c_0}$ is not isomorphic to c_0. This result is well known, but by no means easy, its proof relying either on Grothendieck's theorem (see [24, p. 73] for details) or on the fact that the second dual of c_0 has the Dunford–Pettis property, whereas the second dual of $\bigl(\bigoplus\ell_2^n\bigr)_{c_0}$ does not (see [9, p. 22]).

5.1 Theorem. *The Banach space $\bigl(\bigoplus\ell_2^n\bigr)_{c_0}$ is not isomorphic to c_0.* □

At this point, we should like to recall our convention from §2.1 that finite direct sums are always equipped with the ℓ_∞^n-norm, so that even in the case where $H_1,\dots,H_n$ are Hilbert spaces, the norm of an element $(x_1,\dots,x_n)\in H_1\oplus\cdots\oplus H_n$ is given by (2.1).

5.2 Definition. (i) Suppose that G is a closed subspace of a Hilbert space H. We denote by $G^\perp$ the orthogonal complement of G, and write proj_G^H for the orthogonal projection of H onto G (so that proj_G^H is the idempotent operator on H with $\operatorname{im}\operatorname{proj}_G^H=G$ and $\ker\operatorname{proj}_G^H=G^\perp$).

(ii) Let $n\in\mathbb{N}$, let $H_1,\dots,H_n$ be Hilbert spaces, and let E be a Banach space. For each $\varepsilon\geqslant 0$ and each operator $T\colon H_1\oplus\cdots\oplus H_n\to E$, we define

$$m_\varepsilon(T):=\sup\Bigl\{m\in\mathbb{N}_0\Bigm| \bigl\|T\bigl((I_{H_1}-\operatorname{proj}_{G_1}^{H_1})\oplus\cdots\oplus(I_{H_n}-\operatorname{proj}_{G_n}^{H_n})\bigr)\bigr\|>\varepsilon \text{ whenever } G_j \text{ is a subspace of } H_j \text{ with } \dim G_j\leqslant m \text{ for each } j=1,\dots,n\Bigr\}\in\mathbb{N}_0\cup\{\pm\infty\}.$$

(By convention, we have $\sup\emptyset=-\infty$.)

Hence, $m_\varepsilon(T)$ is the largest number m such that, no matter what subspace G_j of H_j of dimension at most m that we remove for $j = 1, \dots, n$, the restriction of the operator T to the complement has norm greater than ε. We shall now show that this number $m_\varepsilon(T)$ is closely related to the ε-approximate factorization number $\mathrm{fac}_\infty^\varepsilon(T)$ that we introduced in Section 4.

5.3 Lemma. *Let $n \in \mathbb{N}$, let $H_1, \dots, H_n$, and K be Hilbert spaces, let $T\colon H_1 \oplus \cdots \oplus H_n \to K$ be an operator, and let $0 < \varepsilon < \|T\|$. Then:*

(i) $\mathrm{fac}_\infty^\varepsilon(T) \leqslant \|T\| \sqrt{m_\varepsilon(T) + 1}$;

(ii) *for each $m \in \mathbb{N}$ with $m \leqslant m_\varepsilon(T)/2 + 1$, there are operators $R\colon \ell_2^m \to H_1 \oplus \cdots \oplus H_n$ and $S\colon K \to \ell_2^m$ such that $\|R\| \leqslant 1$, $\|S\| \leqslant 1/\varepsilon$, and $I_{\ell_2^m} = STR$.*

Proof. (i) The fact that $\varepsilon < \|T\|$ ensures that $m_\varepsilon(T) \geqslant 0$. If $m_\varepsilon(T) = \infty$, then the inequality is trivial. Otherwise set $m := m_\varepsilon(T) + 1 \in \mathbb{N}$. By the definition of $m_\varepsilon(T)$, there are subspaces $G_1, \dots, G_n$ of $H_1, \dots, H_n$, respectively, each of dimension at most m, such that

$$\big\| T\big((I_{H_1} - \mathrm{proj}_{G_1}^{H_1}) \oplus \cdots \oplus (I_{H_n} - \mathrm{proj}_{G_n}^{H_n})\big)\big\| \leqslant \varepsilon. \tag{5.1}$$

Let $j = 1, \dots, n$. Since the formal identity operators $\ell_2^m \to \ell_\infty^m$ and $\ell_\infty^m \to \ell_2^m$ have norms 1 and $\sqrt{m}$, respectively, we can find operators $R_j\colon H_j \to \ell_\infty^m$ and $S_j\colon \ell_\infty^m \to H_j$ such that $\|R_j\| = 1$, $\|S_j\| \leqslant \sqrt{m}$, and $\mathrm{proj}_{G_j}^{H_j} = S_j R_j$. Set

$$R := R_1 \oplus \cdots \oplus R_n\colon\ H_1 \oplus \cdots \oplus H_n \to (\ell_\infty^m)^{\oplus n} = \ell_\infty^{mn}$$

and

$$S := S_1 \oplus \cdots \oplus S_n\colon\ \ell_\infty^{mn} = (\ell_\infty^m)^{\oplus n} \to H_1 \oplus \cdots \oplus H_n.$$

Then $\|R\| = 1$, $\|S\| \leqslant \sqrt{m}$, and $\|T - TSR\| \leqslant \varepsilon$ by (5.1). It follows that

$$\mathrm{fac}_\infty^\varepsilon(T) \leqslant \|TS\|\,\|R\| \leqslant \|T\|\sqrt{m},$$

as required.

(ii) By finite induction, we choose vectors $x_1 = (x_1^{(1)}, \dots, x_1^{(n)}), \dots, x_m = (x_m^{(1)}, \dots, x_m^{(n)})$ in $H_1 \oplus \cdots \oplus H_n$ such that

(1) $\|x_i\| \leqslant 1$ and $\|Tx_i\| \geqslant \varepsilon$ for each $i = 1, \dots, m$;

(2) $x_1^{(j)}, \dots, x_m^{(j)}$ are orthogonal in H_j for each $j = 1, \dots, n$;

(3) $Tx_1 \dots, Tx_m$ are orthogonal in K.

To start the induction, take a unit vector $x_1 \in H_1 \oplus \cdots \oplus H_n$ such that $Tx_1 \in K$ has norm at least ε; this is possible because $\|T\| > \varepsilon$.

Now suppose that $k \in \{1, \dots, m-1\}$ and that $x_1, \dots, x_k \in H_1 \oplus \cdots \oplus H_n$ have been chosen in accordance with (1)–(3). For each $j = 1, \dots, n$, set

$$G_j := \mathrm{span}\big\{x_1^{(j)}, \dots, x_k^{(j)}, (TJ_j)^*Tx_1, \dots, (TJ_j)^*Tx_k\big\} \subseteq H_j,$$

where $J_j\colon H_j \to H_1 \oplus \cdots \oplus H_n$ is the j^{th} coordinate embedding, and $(TJ_j)^*\colon K \to H_j$ is the (Hilbert space) adjoint operator of $TJ_j\colon H_j \to K$. Then we have

$$\dim G_j \leqslant 2k \leqslant 2m-2 \leqslant m_\varepsilon(T),$$

and so there is a unit vector $w = (w_1, \ldots, w_n) \in H_1 \oplus \cdots \oplus H_n$ such that

$$\big\|T\big((I_{H_1} - \operatorname{proj}_{G_1}^{H_1}) \oplus \cdots \oplus (I_{H_n} - \operatorname{proj}_{G_n}^{H_n})\big)w\big\| > \varepsilon$$

by the definition of $m_\varepsilon(T)$. Set

$$x_{k+1} := \big((I_{H_1} - \operatorname{proj}_{G_1}^{H_1}) \oplus \cdots \oplus (I_{H_n} - \operatorname{proj}_{G_n}^{H_n})\big)w \in H_1 \oplus \cdots \oplus H_n.$$

Then clearly (1) is satisfied. If we write $x_{k+1} = (x_{k+1}^{(1)}, \ldots, x_{k+1}^{(n)})$, then we see that

$$x_{k+1}^{(j)} = (I_{H_j} - \operatorname{proj}_{G_j}^{H_j})w_j \in G_j^\perp \qquad (j = 1, \ldots, n),$$

and so (2) is satisfied. Finally, (3) holds because

$$(Tx_{k+1} \,|\, Tx_i) = \sum_{j=1}^n (TJ_j x_{k+1}^{(j)} \,|\, Tx_i) = \sum_{j=1}^n \big(x_{k+1}^{(j)} \,\big|\, (TJ_j)^* Tx_i\big) = 0 \qquad (i = 1, \ldots, k),$$

where $(\cdot\,|\,\cdot)$ denotes the the inner product in the appropriate Hilbert spaces. Hence the induction continues.

Define $R\colon \ell_2^m \to H_1 \oplus \cdots \oplus H_n$ by $Re_k = x_k$ for each $k = 1, \ldots, m$. Using (1), (2), and Pythagoras's formula, we deduce that $\|R\| \leqslant 1$. Next, define

$$S_1\colon\ y \mapsto \sum_{k=1}^m \frac{(y \,|\, Tx_k)}{\|Tx_k\|}\, e_k, \quad K \to \ell_2^m.$$

By (3) and Bessel's inequality, we obtain $\|S_1\| = 1$. Finally, we define $S_2\colon \ell_2^m \to \ell_2^m$ by

$$S_2 e_k := \frac{1}{\|Tx_k\|} e_k \qquad (k = 1, \ldots, m).$$

Then (1) implies that $\|S_2\| \leqslant 1/\varepsilon$, and so $S := S_2 S_1\colon K \to \ell_2^m$ satisfies $\|S\| \leqslant 1/\varepsilon$. Clearly we have $STRe_k = e_k$ for each $k = 1, \ldots, m$, and the result follows. □

5.4 Remark. Let (H_n) be a sequence of Hilbert spaces, set $E := \big(\bigoplus H_n\big)_{c_0}$, and let T be an operator on E with finite rows. Then, for each $\varepsilon \geqslant 0$ and each $n \in \mathbb{N}$, there is a natural way to define $m_\varepsilon(Q_n^E T)$, namely by 'forgetting' the cofinite number of Hilbert spaces on which $Q_n^E T$ acts trivially. To be specific, if $Q_n^E T = 0$, then we set $m_\varepsilon(Q_n^E T) := -\infty$. Otherwise $\nu := \operatorname{rowsupp}_n(T)$ is a finite, non-empty set, and so $F := \bigoplus_{j\in\nu} H_j$ is a finite direct sum of Hilbert spaces. Let $L\colon F \to E$ be the natural inclusion operator, and define

$$m_\varepsilon(Q_n^E T) := m_\varepsilon(Q_n^E TL),$$

where the quantity on the right-hand side is defined as in Definition 5.2(ii). We note in passing that $Q_n^E T = Q_n^E TLP$, where $P\colon E \to F$ is the natural projection.

We are now ready to prove the following trichotomy theorem for operators on $\left(\bigoplus \ell_2^n\right)_{c_0}$ with locally finite matrix.

5.5 Theorem. *Set* $E := \left(\bigoplus \ell_2^n\right)_{c_0}$*, and let* T *be an operator on* E *with locally finite matrix. Then:*

(i) $T \in \overline{\mathscr{F}}(E)$ *if and only if* $\|Q_n^E T\| \to 0$ *as* $n \to \infty$;

(ii) $T \in \overline{\mathscr{G}}_{c_0}(E)$ *if and only if* $\sup\{m_\varepsilon(Q_n^E T) \mid n \in \mathbb{N}\} < \infty$ *for each* $\varepsilon > 0$;

(iii) *there are operators* R *and* S *on* E *such that* $STR = I_E$ *if and only if*

$$\sup\{m_\varepsilon(Q_n^E T) \mid n \in \mathbb{N}\} = \infty$$

for some $\varepsilon > 0$.

Proof. (i). This is a special case of Proposition 4.11.

(ii), $\Leftarrow$. Let $\varepsilon > 0$ be given, and suppose that $\sup\{m_\varepsilon(Q_n^E T) \mid n \in \mathbb{N}\} < \infty$. Then it follows from Lemma 5.3(i) that $\sup\{\mathrm{fac}_\infty^\varepsilon(Q_n^E T) \mid n \in \mathbb{N}\} < \infty$ as well, and so $T \in \overline{\mathscr{G}}_{c_0}(E)$ by Theorem 4.4.

(iii), $\Leftarrow$. Suppose that $\sup\{m_\varepsilon(Q_n^E T) \mid n \in \mathbb{N}\} = \infty$ for some $\varepsilon > 0$. Inductively we construct a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $m_\varepsilon(Q_{n_k}^E T) \geqslant 2k - 2$ and $\sup(\mathrm{rowsupp}_{n_k} T) < \inf(\mathrm{rowsupp}_{n_{k+1}} T)$ for each $k \in \mathbb{N}$. Set

$$M_0 := 0, \quad M_k := \sup(\mathrm{rowsupp}_{n_k} T) \in \mathbb{N}, \quad \text{and} \quad F_k := \bigoplus_{n=M_{k-1}+1}^{M_k} \ell_2^n,$$

and let $L_k \colon F_k \to E$ be the natural inclusion map for each $k \in \mathbb{N}$. Then

$$m_\varepsilon(Q_{n_k}^E T L_k) = m_\varepsilon(Q_{n_k}^E T) \geqslant 2k - 2,$$

and so Lemma 5.3(ii) implies that there are operators $R_k \colon \ell_2^k \to F_k$ and $S_k \colon \ell_2^{n_k} \to \ell_2^k$ such that $\|R_k\| \leqslant 1$, $\|S_k\| \leqslant 1/\varepsilon$, and $I_{\ell_2^k} = S_k Q_{n_k}^E T L_k R_k$.

Set $R := \mathrm{diag}(R_k) \colon E \to \left(\bigoplus F_k\right)_{c_0}$. By ignoring parentheses, we identify $\left(\bigoplus F_k\right)_{c_0}$ with E, and thus we regard R as an operator mapping into E. Define $S \colon (x_n) \mapsto (S_k x_{n_k})$, $E \to E$. Then S is an operator of norm at most $1/\varepsilon$, and for each $j, k \in \mathbb{N}$, we have

$$Q_j^E STR J_k^E = S_j Q_{n_j}^E T L_k R_k = \begin{cases} I_{\ell_2^k} & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $STR = I_E$, as desired.

Finally, the implications $\Rightarrow$ in (ii) and (iii) follow from what we have already shown together with the fact that $\overline{\mathscr{G}}_{c_0}(E) \neq \mathscr{B}(E)$ (*cf.* Corollary 4.9 and Theorem 5.1). □

In particular, we see that condition (ii) in Theorem 4.10 is satisfied, and so we obtain the following result.

5.6 Corollary. *For the Banach space* $E := \left(\bigoplus \ell_2^n\right)_{c_0}$*, there are exactly four distinct closed ideals in* $\mathscr{B}(E)$*, and they are totally ordered by inclusion. More specifically, the lattice of closed ideals in* $\mathscr{B}(E)$ *is given by* (4.3). □

6 A new proof of a theorem of Bourgain, Casazza, Lindenstrauss, and Tzafriri

Our first clue that the classification of the closed ideals in $\mathscr{B}(E)$ for $E := \left(\bigoplus \ell_2^n\right)_{c_0}$ obtained in Section 5 might be true came from the following theorem which, roughly speaking, asserts that E has no 'exotic' complemented subspaces.

6.1 Theorem. (Bourgain, Casazza, Lindenstrauss, and Tzafriri [3]) *Let F be an infinite-dimensional, complemented subspace of the Banach space $E := \left(\bigoplus \ell_2^n\right)_{c_0}$. Then F is either isomorphic to c_0 or to E.* □

In this section we shall show how one can apply Corollary 5.6 to give a new and, we feel, more elementary proof of this theorem. To do so, we require a few preparations.

6.2 Definition. A Banach space E is *primary* if, for each idempotent operator P on E, either $\operatorname{im} P \cong E$ or $\ker P \cong E$ (or both).

6.3 Lemma. *Let E and F be Banach spaces. Suppose that E is primary and that E is isomorphic to $F^{\oplus n}$ for some $n \in \mathbb{N}$. Then E and F are isomorphic.*

Proof. We may suppose that $n \in \mathbb{N}$ is chosen to be the smallest integer such that $E \cong F^{\oplus n}$. Since $F^{\oplus n} = F \oplus F^{\oplus(n-1)}$ and E is primary, this implies that either $E \cong F$ or $E \cong F^{\oplus(n-1)}$. The latter case contradicts the minimality of n, and so we conclude that $E \cong F$. □

6.4 Proposition. (Casazza, Kottman, and Lin [6]) *Set $E := \left(\bigoplus \ell_2^n\right)_{c_0}$. Then:*

(i) *E is isomorphic to $E \oplus E$;*

(ii) *E is primary.*

Proof. This follows immediately from [6, Corollary 7 and Theorem 10]. □

Proof of Theorem 6.1. Let P be an idempotent operator on $E := \left(\bigoplus \ell_2^n\right)_{c_0}$ with infinite-dimensional image. Proposition 6.4(ii) implies that either $\operatorname{im} P \cong E$ or $\ker P \cong E$. If $\operatorname{im} P \cong E$, then there is nothing to prove, and so we may suppose that $\ker P \cong E$. Since P is idempotent and has infinite-dimensional image, P is non-compact. By Corollary 5.6, there are two cases to consider:

(i) $\overline{\mathscr{G}}_{\{P\}}(E) = \overline{\mathscr{G}}_{c_0}(E)$;

(ii) $\overline{\mathscr{G}}_{\{P\}}(E) = \mathscr{B}(E)$.

In case (i), Corollary 4.8 shows that $\operatorname{im} P \cong c_0$.

In case (ii), it follows from Lemma 4.7 (applied with the Banach space $F := \operatorname{im} P$ and the idempotent operator $I_E \in \overline{\mathscr{G}}_F(E)$) that we can take $n \in \mathbb{N}$ and an idempotent operator Q on $(\operatorname{im} P)^{\oplus n}$ such that $\operatorname{im} Q \cong E$. Then we have

$$E \cong E^{\oplus n} \cong (\operatorname{im} P)^{\oplus n} \oplus (\ker P)^{\oplus n} \cong \operatorname{im} Q \oplus \ker Q \oplus E^{\oplus n} \cong E \oplus \ker Q \cong (\operatorname{im} P)^{\oplus n}$$

by repeated use of Proposition 6.4(i). Now Lemma 6.3 and Proposition 6.4(ii) imply that $\operatorname{im} P \cong E$. □

6.5 Remark. In fact, Bourgain, Casazza, Lindenstrauss, and Tzafriri prove analogues of Theorem 6.1 for other Banach spaces than $\left(\bigoplus \ell_2^n\right)_{c_0}$. To state their results in a unified way, set $E := \left(\bigoplus E_n\right)_D$, where D and E_n are given in one of the following four ways:

(i) $D = c_0$ and $E_n = \ell_2^n$ for each $n \in \mathbb{N}$;

(ii) $D = c_0$ and $E_n = \ell_1^n$ for each $n \in \mathbb{N}$;

(iii) $D = \ell_1$ and $E_n = \ell_2^n$ for each $n \in \mathbb{N}$;

(iv) $D = \ell_1$ and $E_n = \ell_\infty^n$ for each $n \in \mathbb{N}$.

Then it is shown in [3, §8] that, for each infinite-dimensional, complemented subspace F of E, either F is isomorphic to D or F is isomorphic to E.

In the light of these results and Corollary 5.6, it is natural to ask what the closed ideals in $\mathscr{B}(E)$ are in the cases (ii)–(iv).

Another Banach space for which this question attracts attention is $E := \left(\bigoplus \ell_p^n\right)_{c_0}$ for a fixed $p > 1$. It follows from [24, p. 72f] that E contains a complemented subspace isomorphic to $\left(\bigoplus \ell_2^n\right)_{c_0}$, as well as the 'trivial' complemented subspaces isomorphic to c_0 and of finite dimension. Consequently, for $p \neq 2$, $\mathscr{B}(E)$ contains at least five distinct closed ideals, but we do not know if there are any others.

We intend to address these questions in future work.

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