The Dade group of a metacyclic p-group

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Abstract

The Dade group $D(P)$ of a finite p-group P, formed by equivalence classes of endo-permutation modules, is a finitely generated abelian group. Its torsion-free rank equals the number of conjugacy classes of non-cyclic subgroups of P and it is conjectured that every non-trivial element of its torsion subgroup $D^t(P)$ has order 2, (or also 4, in case $p = 2$). The group $D^t(P)$ is closely related to the injectivity of the restriction map $\mathrm{Res} : T(P) \to \prod_E T(E)$ where E runs over elementary abelian subgroups of P and $T(P)$ denotes the group of equivalence classes of endo-trivial modules, which is still unknown for (almost) extra-special groups (p odd). As metacyclic p-groups have no (almost) extra-special section, we can verify the above conjecture in this case. Finally, we compute the whole Dade group of a metacyclic p-group.

1 Introduction

In the p-modular representation theory of finite groups, the family of endo-permutation modules seems to occupy a key position. These modules appear indeed as sources of simple modules for p-solvable groups (cf. [Pu2]) and also in the local analysis of derived equivalences between blocks (cf. [Ri]).

Let p be a prime, P a finite p-group and k a field of characteristic p (which we can suppose to be algebraically closed). All the kP-modules we consider are finitely generated. Recall that an endo-permutation kP-module M is a kP-module such that the P-algebra $\mathrm{End}_k M$ is a permutation kP-module (i.e. there exists a P-invariant k-basis). If M is indecomposable, we say it has vertex P if M is not projective relative to any proper subgroup of P. If M is an endo-permutation module with at least one indecomposable summand M_0 with vertex P, then this summand is unique, up to isomorphism, and, following Dade, we call it the *cap* of M. This allows us to define an equivalence relation on these modules:

$$M \sim N \Longleftrightarrow M_0 \cong N_0.$$

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We write $D(P)$ for the set of equivalence classes of such kP-modules. It is an abelian group for the composition given by $[M] + [N] = [M \otimes N]$. The unit element is the class of the trivial kP-module k and the inverse of $[M]$ is $-[M] = [M^*]$, where $M^* = \mathrm{Hom}_k(M, k)$ is the dual of M.

J. Alperin proved recently that, for any P-set X, the relative syzygy $\Omega^1_X(k)$ of the trivial module, that is the kernel of the linear map from kX to k which sends each $x \in X$ to 1, is an endo-permutation kP-module. Moreover, if X is a transitive P-set, then $\Omega^1_X(k)$ is indecomposable with vertex P (cf. [Al2]). To simplify the notations we write Ω_X for the class of $\Omega^1_X(k)$ in $D(P)$.

When dealing with endo-permutation modules, we can't avoid the special family of endo-trivial modules. Recall that these are precisely the kP-modules M such that $\mathrm{End}_k M \cong k \oplus L$, for a free kP-module L. They all have an indecomposable direct summand with vertex P and are obviously endo-permutation modules. So we can consider the same relation as before and look at the corresponding group $T(P)$, which turns out to be a subgroup of $D(P)$ (note that the classes in $T(P)$ contain, in general, less elements than the corresponding ones in $D(P)$). The following theorem is due to Dade. It classifies all endo-permutation modules (up to equivalence) in the abelian case and shows that $T(P)$ is of relevant importance when studying $D(P)$.

Theorem 1.1 *[Dade, [Da]] Let P be an abelian p-group.*

1. *$T(P)$ is generated by Ω_P. Thus $T(P)$ is trivial, if $|P| \le 2$, cyclic of order 2, if P is cyclic of order ≥ 3, and infinite cyclic otherwise.*
2. *$D(P)$ is isomorphic to the direct sum of the groups $T(P/Q)$, where Q runs over the set of subgroups of P.*

More than 20 years after Dade's result, the question concerning an arbitrary finite p-group P is still open. Only partial results have been obtained in the general case. First, L. Puig proved that $D(P)$ is always finitely generated (cf. [Pu1]). More recently, J. Alperin determined the torsion-free rank of $T(P)$ (cf. [Al2]) and S. Bouc and J. Thévenaz calculated the torsion-free rank of $D(P)$ (cf. [BoTh]). Following [CaTh], the problem of determining the torsion subgroup $D^t(P)$ comes down to finding a detecting family $\mathcal{F}$ of groups for which the restriction map $\mathrm{Res} : T(P) \to \prod_{E\in\mathcal{F}} T(E)$ is injective. That would then allow us to recover all torsion elements of $D(P)$ through restriction-deflation maps to all sections of P belonging to $\mathcal{F}$ (see Theorem 10.1 in [CaTh]). By now, for an odd p, we have to take for $\mathcal{F}$ all elementary abelian groups of rank 1 or 2 and also all extraspecial groups of exponent p. However, as it is hoped that we can take for $\mathcal{F}$ just the elementary abelian groups of rank 1 or 2, we will discuss here the case of p-metacyclic groups for odd p, where no extraspecial group of exponent p appears in the set of all sections of P. The final aim is to prove the following result.

Theorem 1.2 *Let p be an odd prime and P be a metacyclic p-group. Then we have an isomorphism of abelian groups:*

$$\alpha_P : D(P) \longrightarrow \prod_{H\in[\mathcal{X}/P]} T(H/\Phi(H)) \cong \mathbb{Z}^r \times (\mathbb{Z}/2\mathbb{Z})^s,$$

where $\Phi(H)$ is the Frattini subgroup of H, for all subgroup H of P, and where $[\mathcal{X}/P] = [\mathcal{S}/P] \cup [\mathcal{C}/P]$ is a set of representatives of conjugacy classes of non trivial subgroups of P, divided into $\mathcal{S}/P$, *the set of conjugacy classes of non-cyclic subgroups of P, of cardinality r, and $\mathcal{C}/P$, the set of conjugacy classes of non-trivial cyclic subgroups of P, of cardinality s.*

Moreover, the set $\{\Omega_{P/\Phi(H)},\ H \in [\mathcal{S}/P]\}$ form a basis of the torsion-free part and the set $\{\mathrm{Teninf}^P_{H/\Phi(H)}\,\Omega_{H/\Phi(H)},\ H \in [\mathcal{C}/P]\}$ a basis (over $\mathbb{Z}/2\mathbb{Z}$) of the torsion subgroup $D^t(P)$ of $D(P)$.

Note that it will also prove that the injection of Theorem 4.1 of [BoTh] is, in fact, a bijection.

Let us finally recall some of the notions we are going to use. A group is called metacyclic if it is an extension of a cyclic group by a cyclic group. Let P be a finite p-group. We write $Q \leq P$, if Q is a subgroup of P, and we write $Q \leq_P R$, if Q and R are subgroups of P such that $Q \leq {}^gR$, for a $g \in P$, and where ${}^gR = gRg^{-1}$. If $Q \leq P$, we have an obvious restriction map Res^P_Q, from $D(P)$ to $D(Q)$, and we can also consider the tensor induction Ten^P_Q from $D(Q)$ to $D(P)$. If, moreover, Q is normal in P, we have an inflation map $\mathrm{Inf}^P_{P/Q}$ from $D(P/Q)$ to $D(P)$ and a deflation map $\mathrm{Def}^P_{P/Q}$ from $D(P)$ to $D(P/Q)$. This map is obtained as follows. If M is an endo-permutation kP-module, then we consider the Brauer construction $A[Q] = A^Q/(\sum_{R<Q} A^Q_R)$ of the P-algebra $A = \mathrm{End}_k M$, where $A^Q = \mathrm{End}_{kQ} M$ is the set of fixed points of A under the action of Q, and $\forall R < Q$, A^Q_R denotes the image of the relative trace from A^R to A^Q. As P is a p-group, we know that $A[Q] \cong \mathrm{End}_k N$ for a unique (up to isomorphism) endo-permutation $k[P/Q]$-module N. So we define $\mathrm{Def}^P_{P/Q}([M]) = [N] \in D(P/Q)$. If $R \lhd Q < P$, we write $\mathrm{Defres}^P_{Q/R}$ instead of $\mathrm{Def}^Q_{Q/R} \circ \mathrm{Res}^P_Q$, and $\mathrm{Teninf}^P_{Q/R}$ instead of $\mathrm{Ten}^P_Q \circ \mathrm{Inf}^Q_{Q/R}$. We assume the reader to be familiar with all the properties of these notions and with basic results of representation theory of finite groups.

2 Structure of metacyclic p-groups

Let P be a metacyclic p-group (p odd) such that the sequence $1 \to C_{p^n} \to P \to C_{p^m} \to 1$ is exact. So we can choose $u, v \in P$ such that $u^{p^n} = 1$, $v^{p^m} \in \, <u>$ (including $m = 0$, in case P is cyclic) and ${}^vu = u^{p^l+1}$, for an integer l such that $0 < l \leq n$ (we have $l = n$ if P is abelian) and where ${}^vu = vuv^{-1}$. So $|P| = p^{m+n}$, $<u> \lhd P$ and any element of P can be expressed as $u^a v^b$, with $0 \leq a < p^n$ and $0 \leq b < p^m$. It is easy to verify that $<u^{p^{n-1}}> \leq Z(P)$. Indeed, this is the only subgroup of $<u>$ of order p. Moreover, since $<u> \lhd P$, one has that $<u> \cap Z(P) \neq \{1\}$. If P is not cyclic, we have the following result.

Lemma 2.1 *Let P be a non-cyclic metacyclic p-group for an odd prime p. Then P has a unique elementary abelian subgroup of rank 2.*

Proof. Let E be an elementary abelian subgroup of P of rank 2. Then $E \cap {<u>} \neq \{1\}$, because $E/E \cap {<u>}$ is cyclic (it is isomorphic to the subgroup $E{<u>}/{<u>}$ of the cyclic group $P/{<u>}$). Thus, E contains $u^{p^{n-1}}$. The image of E in $P/{<u>}$ is non-trivial, thus E contains an element of the form $u^a v^{p^{m-1}}$, and in fact $E = \, <u^{p^{n-1}}, u^a v^{p^{m-1}}>$ (since $<u^{p^{n-1}}> \, < \, <u^{p^{n-1}}, u^a v^{p^{m-1}}> \, \leq E$). Set $w = v^{p^{m-1}}$. The action of w on $<u>$ is given by ${}^wu = u^r$, where $r = sp^{n-1} + 1$, for some integer $s > 0$: if w and u commute, take $s = p$. This will be the case in particular if $n = 1$, since this implies that P is abelian.

Moreover, there is an integer d such that $w^p = u^d$, and d is a multiple of p if P is non-cyclic (otherwise d is prime to p and there exists an integer e, prime to p, such that $de \equiv 1 \pmod{p^n}$, but then $u = u^{de} = w^{pe} \in <v>$). Then

$$(u^a w)^p = u^a({}^{w}u)^a({}^{w^2}u)^a \cdots ({}^{w^{p-1}}u)^a w^p = u^{a(r+r^2+\cdots+r^{p-1})+d} = u^{a\frac{r^p-1}{r-1}+d}.$$

Finally,

$$\frac{r^p-1}{r-1} = (sp^{n-1})^{p-1} + \binom{p}{1}(sp^{n-1})^{p-2} + \cdots + \binom{p}{p-2} sp^{n-1} + \binom{p}{p-1}.$$

Since $(sp^{n-1})^{p-1} \equiv 0 \pmod{p^n}$ (even if $n = 1$, with the above convention on s), and since, for p odd, the binomial coefficients are all multiple of p, it follows that $\frac{r^p-1}{r-1} \equiv p \pmod{p^n}$. Hence, $(u^a w)^p = u^{ap+d}$ and this is equal to 1 if and only if $ap + d$ is a multiple of p^n. Equivalently, a is congruent to $-\frac{d}{p}$ modulo p^{n-1}, and $E = <u^{p^{n-1}}, u^{-\frac{d}{p}}v^{p^{m-1}}>$. Conversely, the previous argument shows that the group E defined by this formula is elementary abelian of rank 2. Hence P has an elementary abelian subgroup of rank 2, and it is unique.

□

We can deduce immediately from Lemma 2.1 the following consequence, which is fundamental for the motivation of this paper.

Corollary 2.2 *Let P and p be as in the previous lemma. Then P has no extraspecial section of exponent p.*

Proof. Any non-trivial extraspecial p-group of exponent p has many elementary abelian subgroups of rank 2. As any subgroup and any quotient of a metacyclic group is still metacyclic, it follows that any section of P is a metacyclic group and thus it cannot be an extraspecial group of exponent p.

□

Remark 2.3 *Note that this lemma doesn't apply to metacyclic 2-groups, as, for instance, the dihedral group of order 8 is metacyclic but has 2 elementary abelian subgroups of rank 2.*

Here is another property of non-cyclic metacyclic p-groups which will be of relevant importance for the proof of Theorem 1.2.

Lemma 2.4 *Assume P to be non-cyclic and let H be a non-cyclic subgroup of P. Then H is uniquely determined by $\Phi(H)$. In other words, if H and K are two non-cyclic subgroups of P, with $\Phi(K) = \Phi(H)$, then $K = H$.*

Proof. Consider $N = N_P(\Phi(H))$. The quotient group $N/\Phi(H)$ is also metacyclic and so has a unique elementary abelian subgroup of rank 2, by the previous lemma and the fact that it is non-cyclic (since it contains $H/\Phi(H) \cong C_p \times C_p$ because H is metacyclic, hence generated by two elements, and non-cyclic). So N has a unique subgroup containing $\Phi(H)$ and such that the quotient is elementary abelian of rank 2. Thus H is unique.

□

We conclude this section with an immediate consequence of this lemma.

Corollary 2.5 *Under the same hypothesis of Lemma 2.4, we have* $N_P(H) = N_P(\Phi(H))$.

Proof. $\Phi(H)$ is a characteristic subgroup of $N_P(H)$. Thus we have $N_P(H) \leq N_P(\Phi(H))$. Conversely, if $u \in N_P(\Phi(H))$, then $\Phi({}^uH) = {}^u(\Phi(H)) = \Phi(H)$ implies $H = {}^uH$ by Lemma 2.4.

□

3 The Dade group of a metacyclic p-group

Let P be a metacyclic p-group (p odd) and consider the homomorphism of abelian groups

$$\Psi_P = \prod_{Q\in[\mathcal{X}/P]} \mathrm{Defres}^P_{Q/\Phi(Q)} : D(P) \longrightarrow \prod_{Q\in[\mathcal{X}/P]} D(Q/\Phi(Q)),$$

where $[\mathcal{X}/P]$ is a set of representatives of the conjugacy classes of the non-trivial subgroups of P. We can write $\mathcal{X}/P$ as the disjoint union of $\mathcal{S}/P$ and $\mathcal{C}/P$, where $\mathcal{S}/P$ is the set of conjugacy classes of non-cyclic subgroups of P, and $\mathcal{C}/P$ is the set of conjugacy classes of non-trivial cyclic subgroups of P.

Proposition 3.1 *The map* Ψ_P *is injective.*

Proof. For all $Q \in [\mathcal{X}/P]$, the quotient $Q/\Phi(Q)$ is cyclic of order p, if Q is cyclic, and elementary abelian of rank 2 otherwise. So we have respectively $D(Q/\Phi(Q)) = T(Q/\Phi(Q)) \cong \mathbb{Z}/2\mathbb{Z}$ or $D(Q/\Phi(Q)) \cong \bigoplus_{R<Q} T(Q/R) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{p+1}$. As P doesn't have any extraspecial section of exponent p (cf. Lemma 2.2), Theorem 10.1 of [CaTh] implies that the map

$$\prod_{H/K\in[\mathcal{Y}/P]} \mathrm{Defres}^P_{H/K} : D(P) \longrightarrow \prod_{H/K\in[\mathcal{Y}/P]} D(H/K) \text{ is injective,}$$

where $\mathcal{Y}$ denotes the set of all elementary abelian sections of P of rank 1 or 2. Let us prove the result by induction on $|P|$. If $|P| = p$, then $\mathcal{S} = \emptyset$, $\mathcal{C} = \{P\}$ and so $\Psi_P = \mathrm{Id}_{D(P)}$ is injective. Assume $|P| > p$ and let $a \in \mathrm{Ker}(\Psi_P)$. We must prove that $\mathrm{Defres}^P_{H/K}(a) = 0$, $\forall\, H/K \in \mathcal{Y}$. Let $H < P$. Then it is clear that $\mathrm{Res}^P_H(a) \in \mathrm{Ker}(\Psi_H)$. Thus, by induction hypothesis, we have $\mathrm{Res}^P_H(a) = 0$ and so $\mathrm{Defres}^P_{H/K}(a) = 0$. If $H = P$ and $K \lhd P$ is such that P/K is elementary abelian of rank 1 or 2, then we must show that $\mathrm{Def}^P_{P/K}(a) = 0$. If P/K is elementary abelian of rank 2, then $K = \Phi(P)$, and $\mathrm{Def}^P_{P/K}(a) = 0$ by assumption. Suppose $|P : K| = p$. Then $\Phi(P) < K < P$ and we have $\mathrm{Def}^P_{P/K}(a) = \mathrm{Def}^{P/\Phi(P)}_{P/K}\left(\mathrm{Def}^P_{P/\Phi(P)}(a)\right) = \mathrm{Def}^{P/\Phi(P)}_{P/K}(0) = 0$, by hypothesis on a.

□

In order to prove the injectivity of the map given in Theorem 1.2, we first need two more lemmas.

Lemma 3.2 *Let P be a non-cyclic metacyclic p-group (p odd). Then, the map*

$$\prod_{1<Q<P} \mathrm{Res}^P_Q : D^t(P) \longrightarrow \prod_{1<Q<P} D^t(Q) \quad \textit{is injective.}$$

Proof. Proposition 3.1 implies that $\prod_{1<Q\leq P} \mathrm{Defres}^P_{Q/\Phi(Q)} : D^t(P) \longrightarrow \prod_{1<Q\leq P} D^t(Q/\Phi(Q))$ is injective. Moreover, $\prod_{\Phi(Q)<R<Q} \mathrm{Res}^{Q/\Phi(Q)}_{R/\Phi(Q)} : D^t(Q/\Phi(Q)) \longrightarrow \prod_{\Phi(Q)<R<Q} D^t(R/\Phi(Q))$ is injective for any non-cyclic subgroup Q of P, by Lemma 6.1 of [BoTh]. So, there is a commutative square

$$\begin{array}{ccc} D^t(P) & \longrightarrow & \prod_{1<S<P} D^t(S) \\ \downarrow & & \downarrow \\ \prod_{1<Q\leq P} D^t(Q/\Phi(Q)) & \longrightarrow & \prod_{\Phi(Q)<R<Q} D^t(R/\Phi(Q)), \end{array}$$

where the upper horizontal map is $\prod_{1<S<P} \mathrm{Res}^P_S$, the bottom map is $\prod_{\Phi(Q)<R<Q} \mathrm{Res}^{Q/\Phi(Q)}_{R/\Phi(Q)}$, which is injective by Lemma 6.1 of [BoTh], and the left vertical map is $\prod_{1<Q\leq P} \mathrm{Defres}^P_{Q/\Phi(Q)}$, which is injective by the previous proposition. Hence the top map is injective.

□

Lemma 3.3 *Let P be a metacyclic p-group (p odd) and $\mathcal{C}$ the set of all non-trivial cyclic subgroups of P. Then, the map $\prod_{[C\in\mathcal{C}/P]} \mathrm{Defres}^P_{C/\Phi(C)} : D^t(P) \longrightarrow \prod_{C\in[\mathcal{C}/P]} T(C/\Phi(C))$ is injective.*

Proof. Note that this map is well defined, as $C/\Phi(C)$ is cyclic of order p and so $D(C/\Phi(C)) = T(C/\Phi(C))$ is cyclic of order 2, for all $C \in \mathcal{C}$. Let us proceed by induction on $|P|$ and start with $|P| = p$. Then $\mathcal{C} = \{P\}$ and the above map is the identity map. Assume now $|P| > p$. If P is cyclic, then, the lemma coincides with Proposition 3.1, and so there is nothing to prove. If P is not cyclic, then, by induction and the above lemma, it follows that $\prod_{C\in\mathcal{C}} \mathrm{Defres}^P_{C/\Phi(C)}$ maps $D^t(P)$ into $\prod_{C\in\mathcal{C}} T(C/\Phi(C))$. As $\mathrm{Ker}\left(\mathrm{Defres}^P_{Q/\Phi(Q)}\right) = \mathrm{Ker}\left(\mathrm{Defres}^P_{{}^gQ/\Phi({}^gQ)}\right)$, $\forall Q \leq P$, we can restrict this map to the conjugacy classes of non-trivial cyclic subgroups without loosing the injectivity and so the lemma is proved.

□

Notice that this lemma gives us an upper bound for the rank of the torsion subgroup of $D(P)$. We will show later that this rank is, in fact, equal to this upper bound. Let us turn back now to the injectivity question. The result of Proposition 3.1 can be improved in the following sense. If P is an abelian group, we can consider the "projection" $\rho_P : D(P) \twoheadrightarrow T(P)$, defined thanks to the isomorphism given in Theorem 1.1, where the reverse isomorphism is equal to

$$\bigoplus_{Q<P} \mathrm{Inf}^P_{P/Q} : \bigoplus_{Q<P} T(P/Q) \longrightarrow D(P).$$

Let P be a metacyclic p-group (p odd) and let α_P be the homomorphism of abelian groups defined as the composition $\left(\prod_{Q\in[\mathcal{X}/P]} \rho_{Q/\Phi(Q)}\right) \circ \Psi_P$.

Theorem 3.4 *The map $\alpha_P : D(P) \longrightarrow \prod_{Q\in[\mathcal{X}/P]} T(Q/\Phi(Q))$ is injective.*

Proof. Ψ_P composed with the isomorphism in Theorem 1.1 gives an injective map

$$\Psi'_P : D(P) \longrightarrow \prod_{Q\in[\mathcal{S}/P]} \big(\oplus_{\Phi(Q)\leq R<Q} T(R/\Phi(Q))\big) \times \prod_{C\in[\mathcal{C}/P]} T\big(C/\Phi(C)\big).$$

Let $x \in \mathrm{Ker}(\alpha_P)$. Then we have $\Psi'_P(x) = (x_Q)_{Q\in[\mathcal{X}/P]}$, with $x_Q = 0$, if Q is cyclic, or $x_Q \in \oplus_{\Phi(Q)<R<Q} T(R/\Phi(Q))$, otherwise. Thus we have $2x \in \mathrm{Ker}(\Psi'_P) = \mathrm{Ker}(\Psi_P)$ and so $x \in D^t(P)$, since Ψ_P is injective.

The previous Lemma implies that the product of deflation-restriction maps from $D^t(P)$ to $\prod_{C\in[\mathcal{C}/P]} T\big(C/\Phi(C)\big)$ is injective. But this map coincide with the restriction of α_P to $D^t(P)$, and, by hypothesis, $x \in \mathrm{Ker}(\alpha_P)$. Hence, $x = 0$. Thus, the map α_P is injective.

□

To prove the surjectivity of α_P, we are going to choose a subset of $D(P)$ which is mapped onto a set of generators of the target. In order to do this, we must first define an order on the set $[\mathcal{X}/P]$ of representatives of the conjugacy classes of the non-trivial subgroups of P. Consider separately the representatives $[\mathcal{C}/P]$ of the conjugacy classes of non-trivial cyclic subgroups and $[\mathcal{S}/P]$ of the non-cyclic subgroups. Note that $\mathcal{S} = \emptyset$ iff P is cyclic, and $\mathcal{C} \neq \emptyset$, $\forall\, P \neq \{1\}$. On both sets, there is a "natural" order $\preceq$, induced by the inclusion (that is $H \preceq K$, if $H \leq_P K$, and H and K are both in $\mathcal{C}$, or both in $\mathcal{S}$) and we can extend them to total orders, still denoted by $\preceq$, on $[\mathcal{C}/P]$ and on $[\mathcal{S}/P]$. Let now H and K be in $[\mathcal{X}/P]$ and write $H \preceq K$ if exactly one of the following condition is satisfied:

1. $H \in [\mathcal{C}/P]$ and $K \in [\mathcal{S}/P]$.
2. $H, K \in [\mathcal{C}/P]$ and $H \preceq K$.
3. $H, K \in [\mathcal{S}/P]$ and $H \preceq K$.

Consider the following set of elements of $D(P)$ and order it according to the increasing order of the subgroups which appear in the subscripts.

$$\mathcal{B} = \Big\{ \mathrm{Teninf}_{C/\Phi(C)}^{P}\, \Omega_{C/\Phi(C)},\, C \in [\mathcal{C}/P] \Big\} \bigcup \Big\{ \Omega_{P/\Phi(Q)},\, Q \in [\mathcal{S}/P] \Big\}.$$

Let us prove that $\mathcal{B}$ form a basis of the abelian group $D(P)$, viewed as the direct sum of a free $\mathbb{Z}$-module of rank $|[\mathcal{S}/P]|$ and a $\mathbb{F}_2$-vector space of dimension $|[\mathcal{C}/P]|$. Let $C \in [\mathcal{C}/P]$ and consider $\mathrm{Teninf}_{C/\Phi(C)}^{P}\, \Omega_{C/\Phi(C)}$. It is a torsion element so the component of its image by α_P in any $T(Q/\Phi(Q))$ for $Q \in [\mathcal{S}/P]$ must be zero as $T(Q/\Phi(Q)) \cong \mathbb{Z}$. Moreover, in Theorem 6.2 of [BoTh], it is proved that $\mathrm{Defres}_{C'/\Phi(C')}^{P}(\mathrm{Teninf}_{C/\Phi(C)}^{P}\, \Omega_{C/\Phi(C)}) = \delta_{C,C'}\Omega_{C/\Phi(C)}$, $\forall\, C, C' \in [\mathcal{C}/P]$. Let $H \in [\mathcal{S}/P]$ (and hence assume that P is not cyclic, i.e. $\mathcal{S}$ is not empty) and prove that for all $K \in [\mathcal{X}/P]$ such that $H \preceq K$ we have $\mathrm{Defres}_{K/\Phi(K)}^{P}\, \Omega_{P/\Phi(H)} = \delta_{H,K}\Omega_{H/\Phi(H)}$. By definition of the order, the condition $H \preceq K$ implies $K \in [\mathcal{S}/P]$. We have $\mathrm{Defres}_{K/\Phi(K)}^{P}\big(\Omega_{P/\Phi(H)}\big) = \mathrm{Def}_{K/\Phi(K)}^{K}\bigg(\mathrm{Res}_{K}^{P}\big(\Omega_{P/\Phi(H)}\big)\bigg)$. Let us recall some useful results of [Bo]: if X is a finite P-set, K and N two subgroups of P with $N \lhd P$, then $\mathrm{Def}_{P/N}^{P}\, \Omega_X = \Omega_{X^N}$, where X^N denotes the set of fixed point of X under the action of N. We also have $\mathrm{Res}^P_K \Omega_X = \Omega_{\mathrm{Res}^P_K X}$. Then, using Mackey's formula and the isomorphism of P-sets $\mathrm{Ind}^P_K\{*\} \cong P/K$, it follows that

$$\mathrm{Res}^P_K(P/\Phi(H)) = \Big(\coprod_{g\in[K\backslash P/\Phi(H)],\, g\in N} \underbrace{K/(K\cap\Phi(H))}_{(I)}\Big) \coprod \Big(\coprod_{g\in[K\backslash P/\Phi(H)],\, g\notin N} \underbrace{K/(K\cap {}^g\Phi(H))}_{(II)}\Big),$$

where $N = N_P(\Phi(H)) = N_P(H)$. If $H = K$, then all the elements of (I) are fixed by $\Phi(H)$. But there is no fixed point in (II). Indeed, if

$$x(H\cap {}^g\Phi(H)) \in \big(H/H\cap {}^g\Phi(H)\big)^{\Phi(H)} = \{x(H\cap {}^g\Phi(H)) \mid y^x \in H\cap {}^g\Phi(H),\ \forall\, y\in\Phi(H)\},$$

then $\Phi(H) = \Phi(H)^x \le H\cap {}^g\Phi(H) \le {}^g\Phi(H)$. Thus $\Phi(H) = {}^g\Phi(H)$, i.e. $g\in N$. It follows that $\mathrm{Defres}^P_{H/\Phi(H)}\Omega_{P/\Phi(H)} = \Omega_{H/\Phi(H)} \in D(H/\Phi(H))$, as $\Omega^1_{H/\Phi(H)}(k)$ is a direct summand of the kernel of the map $\left(\bigoplus_{g\in[N/H]} k[H/\Phi(H)] \to k\right)$ (and so is "the" cap of this kernel). If $|H| < |K|$, then there is no fixed point at all and so $\mathrm{Defres}^P_{K/\Phi(K)}\Omega_{P/\Phi(K)} = 0$. If $|H| = |K|$, then H and K are two non conjugate non-cyclic subgroups of P having the same order. That implies $\Phi(H)$ and $\Phi(K)$ have same order, but are not conjugate, by Lemma 2.4. It follows that $X^{\Phi(K)} = \emptyset$ and so $\mathrm{Defres}^P_{K/\Phi(K)}\Omega_{P/\Phi(H)} = 0$. Let us sum up all these calculations: we proved that α_P maps $\mathcal{B}$ onto a set of generators of the target group and in terms of matrix, taking the set $\{\Omega_{H/\Phi(H)},\ H\in[\mathcal{X}/P]\}$ as basis for the target and $\mathcal{B}$ for the source, we get an upper triangular matrix with ones on the diagonal. So we have proved the following result.

Theorem 3.5 *Let P be a metacyclic p-group (p odd). Then,*

$$\alpha_P = \prod_{H\in[\mathcal{X}/P]} \big(\rho_{H/\Phi(H)} \circ \mathrm{Defres}^P_{H/\Phi(H)}\big) : D(P) \longrightarrow \prod_{H\in[\mathcal{X}/P]} T(H/\Phi(H))$$

is an isomorphism of abelian groups.

Moreover, the set $\{\Omega_{P/\Phi(H)},\ H\in[\mathcal{S}/P]\}$ forms a basis of the torsion-free part and the set $\{\mathrm{Teninf}^P_{H/\Phi(H)}\Omega_{H/\Phi(H)},\ H\in[\mathcal{C}/P]\}$ a basis (over $\mathbb{Z}/2\mathbb{Z}$) of $D^t(P)$.

Remark 3.6 *Finally some remarks on endo-permutation modules.*

1. *If P is a metacyclic p-group (p odd), it is immediate, from Lemma 2.1, that $T(P) = <\Omega_P>$ and so it is cyclic of order 2, if P is cyclic, and infinite cyclic, otherwise.*
2. *Take the same notations as in Theorem 3.5. The matrix of the map α_P, in the given ordered basis, is upper triangular with 1 on the diagonal. But, in general, it is not the identity matrix. Indeed, consider for instance a non-cyclic metacyclic p-group P of order $\ge p^3$ and $E < P$ the unique elementary abelian subgroup of P. Then $\Phi(E) = \{1\}$ and we have $\alpha_P(\Omega_{P/\Phi(E)}) = \Omega_E + \sum_{C\in[\mathcal{C}/P]}(\Omega_C)$.*
3. *Let us finish with a remark about splendid equivalences between blocks. Recall that a kP-module M has an endo-split permutation resolution X_M if there exists a bounded complex X_M of permutation kP-modules and an isomorphism $M \cong H_0(X_M)$, such that $H_n(X_M) =$*

$0, \forall\, n \neq 0$ and the complex $\mathrm{End}_k(X_M)$ is split. It follows from Theorem 3.5, using Lemma 2.3.7 of [Bo] and Lemma 7.3 of [Ri], that if P is a metacyclic p-group (p odd), then every endo-permutation kP-module M with a direct summand of vertex P has an endo-split permutation resolution X_M. Thus, Theorem 7.8 of [Ri] can be applied to p-nilpotent groups G with a metacyclic p-Sylow subgroup P. In other words (assume k to be a splitting field for G), if kB is the principal block of kG, M an endo-permutation module belonging to kB and X_M an endo-split permutation resolution of M, then X_M is a splendid tilting complex inducing a splendid derived equivalence between the derived categories $D^b(kP)$ and $D^b(kB)$.

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