CONNECTED COMPONENTS OF THE CATEGORY OF ELEMENTARY ABELIAN p-SUBGROUPS

NADIA MAZZA

Abstract. We determine the maximal number of conjugacy classes of maximal elementary abelian subgroups of rank 2 in a finite p-group G, for an odd prime p. Namely, it is p if G has rank at least 3 and it is $p+1$ if G has rank 2. More precisely, if G has rank 2, there are exactly 1, 2, $p+1$, or possibly 3 classes for some 3-groups of maximal nilpotency class.

1. Introduction

The elementary abelian p-subgroups of a finite group are ubiquitous in the study of the p-local structure of the entire group. They provide useful information on the modular representations of the group and its cohomology ring. The main motivation of the author for this research is concerned with the classification of endotrivial modules of a finite group over a field of characteristic p. The analysis in [1, 9, 10] and [11], lead to the definition of the category $\mathcal{E}_{\geq 2}(G)$ of elementary abelian p-subgroups of a finite group G of rank at least 2. It turns out that the torsion-free rank of the group of endotrivial modules for G equals the number n_G of connected components of $\mathcal{E}_{\geq 2}(G)$, whence the interest to bound this number. Jon Alperin showed that n_G is bounded and depends on p. Note that if $n_G > 1$, then G has a non abelian Sylow p-subgroup with a cyclic center and some maximal elementary abelian p-subgroup of rank 2. Such p-groups have been thoroughly studied by Blackburn ([5]) and in the works involving endo-permutation and endotrivial modules ([1, 6, 7, 9, 10, 11] and [18]).

The objective of this research is to compute an upper bound for n_G in the case that G is a finite p-group and that p is odd. Equivalently, this problem can be stated as determining the maximal number of conjugacy classes of maximal elementary abelian subgroups of rank 2. This will be explained in the next section, which also contains the proof of the main result:

Theorem 2.7. Let p be an odd prime and G a finite p-group. Then $n_G \leq p+1$. In particular, if G has rank 2, then there are at most $p+1$ conjugacy classes of maximal elementary abelian subgroups of rank 2, and if G has rank at least 3, then there are at most p conjugacy classes of maximal elementary abelian subgroups of rank 2.

In fact, our conclusion is an easy corollary of Héthelyi's work on soft subgroups ([14, 15]), which builds up essentially from [5]. Let us also mention that Jon Carlson recently handled the case for $p = 2$, and obtained that n_G is at most 5 ([8]). He conjectured that, for p odd, $n_G \leq p+1$, which is hence true, as we show in Section 2. We end this note by explictly computing n_G and the conjugacy classes of the elementary abelian subgroups of rank 2 in the case that G has rank 2.

2. Generalities: Results old and new

Throughout, p is an odd prime and G denotes a finite p-group. We write $H \leq G$ (resp. $H < G$) if H is a subgroup of G (resp. proper subgroup). If $g \in G$, we set ${}^gH = gHg^{-1}$ and c_g for the conjugacy map of G; that is, $c_g(x) = {}^gx = gxg^{-1}$, for all $x \in G$. For two groups H and K having an isomorphic central subgroup Z, we write $H *_Z K$ for the central product over Z, or simply $H * K$ if there is no confusion. Finally, G' is the derived subgroup of G, generated by all the commutators $[x, y] = x^{-1}y^{-1}xy$, for all $x, y \in G$. We refer the reader to [16] for further details.

Definition 2.1. Let G be a finite p-group.

(1) An *elementary abelian subgroup* of G is an abelian subgroup E of G of exponent p.

If E has order p^a, then E is a $\mathbb{F}_p$-vector space of dimension a and we call a the *rank* of E. The *rank* of G is the maximum of the ranks of all the elementary abelian subgroups of G.

(2) We set $\mathcal{E}_{\geq 2}(G)$ for the category whose objects are the elementary abelian subgroups of G of rank at least 2 and the morphisms are the compositions of conjugations and inclusions.

For short, we write $E \in \mathcal{E}_{\geq 2}(G)$ if E is an object of $\mathcal{E}_{\geq 2}(G)$. The definition says that for $E, F \in \mathcal{E}_{\geq 2}(G)$, then $\mathrm{Hom}_G(E, F) = \{c_g \mid g \in G \ : \ {}^gE \leq F\}$. In particular, $\mathrm{Hom}_G(E, F)$ is not empty if and only if E is conjugate to a subgroup of F. We define a *connected component* of $\mathcal{E}_{\geq 2}(G)$ as follows: $E, F \in \mathcal{E}_{\geq 2}(G)$ are connected if there are subgroups $E_0, \ldots, E_n \in \mathcal{E}_{\geq 2}(G)$ with the properties that $E_0 = E$, $E_n = F$ and, for each $0 \leq i < n$, one of $\mathrm{Hom}_G(E_i, E_{i+1})$ or $\mathrm{Hom}_G(E_{i+1}, E_i)$ is not empty. We call $E \in \mathcal{E}_{\geq 2}(G)$ *maximal* if the condition $\mathrm{Hom}_G(E, F) \neq \emptyset$ implies $E =_G F$; that is, E is not properly contained in an elementary abelian subgroup of G. We let n_G be the number of connected components of $\mathcal{E}_{\geq 2}(G)$. Recall that we want to find an upper bound for n_G. Since $n_G \leq 1$ whenever G is abelian or has a non cyclic center, let us exclude these cases from now on. Hence, we set Z for the unique central subgroup of G of order p and E_0 for a normal elementary abelian subgroup of G of rank 2, which exists by [16, Hilfsatz III 7.5]. Define $G_0 = C_G(E_0)$. Note that G_0 is a maximal subgroup of G.

If G has rank 2, then a component of $\mathcal{E}_{\geq 2}(G)$ is a G-conjugacy class of elementary abelian subgroups of rank 2. By [1, § 5], if G has rank 3 or more, then all the objects of $\mathcal{E}_{\geq 2}(G)$ of rank at least 3 are connected and lie in the *big component*, denoted $\mathcal{B}$. Moreover, $E_0 \in \mathcal{B}$. The remaining components of $\mathcal{E}_{\geq 2}(G)$ are conjugacy classes of maximal elementary abelian subgroups of rank 2, that we call *isolated*. Hence, $E \in \mathcal{E}_{\geq 2}(G)$ is isolated if E is a maximal elementary abelian subgroup of G that does not lie in $\mathcal{B}$. In particular, $Z < E$ and E has rank 2. Note that $n_G > 1$ if and only if G has an isolated subgroup.

Lemma 2.2. *Let $E \in \mathcal{E}_{\geq 2}(G)$ be isolated and set $L = C_{G_0}(E)$.*

(1) *L is cyclic and $E \cap G_0 = Z$.*
(2) *$N_G(E) = L * EE_0$, with $L = Z(N_G(E))$ and EE_0 extraspecial of order p^3 and exponent p.*

Proof. Since E is isolated, the intersection $L = G_0 \cap C_G(E)$ is necessarily cyclic. It follows that $C_G(E) = L * E$ is metacyclic abelian. Moreover, $E_0 \lhd G$ implies that $Z < E_0$, whence $E \cap E_0 = Z$ and EE_0 is non abelian of order p^3 and exponent p, since E normalises E_0. Thus, EE_0 is extraspecial of order p^3 and exponent p. In particular, E is normal in $L * EE_0$, which shows that $N_G(E) = L * EE_0$. Clearly, $L = Z(N_G(E))$. □

Lemma 2.2 has the following immediate consequence.

Corollary 2.3. *Let $E \in \mathcal{E}_{\geq 2}(G)$ be isolated and assume that E is normal in G. Then G is a central product $X *_Z Z(G)$, with X extraspecial of order p^3 and exponent p, and where $Z(G)$ is cyclic. In particular, G has rank 2 and $n_G = p + 1$.*

Hence, from now on, we may assume that if $E \in \mathcal{E}_{\geq 2}(G)$ is isolated, then $N_G(E) < G$, and thus $C_G(E)$ has index at least p^2 in G. Indeed, by Lemma 2.2, either all isolated subgroups are normal, or none is normal. Then, we appeal to Héthelyi's work on soft subgroups ([14, 15]), since so are the centralisers of the isolated subgroups.

Definition 2.4. A subgroup A of G is *soft* if $A = C_G(A)$ and $|N_G(A) : C_G(A)| = p$.

Let us collect some useful facts on soft subgroups.

Theorem 2.5. [15, § 1, 2 and 3] *Let G be a finite p-group, and let A be a soft subgroup of G.*

(1) *The subgroups containing A form a chain $A = N_0, N_1, \ldots, N_k = M$, where $N_i = N_G(N_{i-1})$, for $1 \leq i \leq r$, and $|G : M| = p$.*
(2) *M is the unique maximal subgroup of G containing A, and M has nilpotency class $k + 1$.*
(3) *Any two soft subgroups of G contained in M are G-conjugate.*
(4) *M has exactly p maximal subgroups containing a soft subgroup of G.*
(5) *If $|G : A| \geq p^2$, then $H = G'Z(N_G(A))$ is independent of A and the factor group G/H is elementary abelian of order p^2. Moreover, for $x \in H$, then the subgroup $C_G(x)$ of G is not soft, or $|G : C_G(x)| < p^2$. The conjugates of A are the subgroups $C_G(s)$, for $s \in M - H$.*

As an easy corollary of Theorem 2.5, we get the following.

Proposition 2.6. *Assume that G has some isolated subgroup which is not normal in G. Then, $2 \leq n_G \leq p + 1$ and G has at most p maximal elementary abelian subgroups of rank 2.*

Proof. Clearly, $n_G > 1$. Let $E \in \mathcal{E}_{\geq 2}(G)$ be isolated. Set $A = C_G(E)$ and $H = G'Z(N_G(A))$. Then, A satisfies the hypothesis of Theorem 2.5. Thus, H is a normal subgroup of G, independent of the choice of E, and G/H is elementary abelian of rank 2. In particular, G has $p + 1$ maximal subgroups that contain H and hence might contain some isolated subgroup of G. Now, $H < G_0$, since $Z(N_G(A))$ centralizes E_0 and since G' is contained in any maximal subgroup of G. Since G_0 does not contain any isolated subgroup of G, there are at most p maximal subgroups containing an isolated subgroup, whence the claim by Theorem 2.5 (3). □

Observe that Theorem 2.5 provides more information on the structure of G than the fact deducted in Proposition 2.6. In particular, under the same assumptions, the nilpotency class of G is $n - r$ and that of H is $n - r - 1$, where r is defined by $|Z(N_G(E))| = p^r$. However, for the purpose of this research, let us focus on the consequences of Corollary 2.2 and Proposition 2.6.

Theorem 2.7. *Let p be an odd prime and G a finite p-group. Then $n_G \leq p + 1$. In particular, if G has rank 2, then there are at most $p + 1$ conjugacy classes of maximal elementary abelian subgroups of rank 2, and if G has rank at least 3, then there are at most p conjugacy classes of maximal elementary abelian subgroups of rank 2.*

Corollary 2.8. *The torsion-free rank of the group of endotrivial modules of a finite p-group, for an odd prime p, is at most $p + 1$.*

We refer the reader to [8, 9] for further details on Corollary 2.8. We end this section with a criteria stating when $n_G = 1$.

Proposition 2.9. *Let G be a finite p-group, p odd, and let V be a normal elementary abelian subgroup of rank r, with $r > p$. Then $n_G = 1$.*

Proof. We need to show that any $E \in \mathcal{E}_{\geq 2}(G)$ is contained in an elementary abelian subgroup F of rank at least 3. Let $E \in \mathcal{E}_{\geq 2}(G)$ have rank 2. If E does not contain a central subgroup Z of order p, then we set $F = EZ$. Otherwise, consider V as $\mathbb{F}_p$-vector space. The conjugation action of G induces a map $E \to \operatorname{Aut} V$. Via the choice of a basis for V, we have that $\operatorname{Aut} V \cong \mathrm{SL}_r(p)$ and any $x \in E$ identifies with a matrix m_x in $\mathrm{SL}_r(p)$ with order 1 or p. If $m_x = 1$, for all $x \in E$, then $E \leq C_G(V)$, whence $E < VE = F$, which is elementary abelian of rank at least 3. Otherwise, there is $x \in E$ with $x \notin V$, and $E = Z \times \langle x \rangle$, with $Z = V \cap E$. Then, m_x has two non collinear eigenvectors $v, w \in V$, since m_x is an element of $\mathrm{SL}_r(p)$ with order p. That is, $x \in C_G(\langle v, w\rangle)$, and hence $F = E\langle v, w\rangle \in \mathcal{E}_{\geq 2}(G)$ has rank three (note that $Z < \langle v, w\rangle$). □

Let us point out that Proposition 2.9 is in fact a trivial consequence of a more general result of Y. Berkovich ([3, Proposition 20]). This discussion naively leads us to the question: "Let G be a p-group of rank at least $p+1$, for an odd prime p. Is $n_G = 1$?"

The answer is not yet known to the author, except in case $p = 3$.

Proposition 2.10. *If $p = 3$ and G is a 3-group of rank at least 4, then $n_G = 1$.*

Proof. By [17, Theorem], if G contains an elementary abelian subgroup of rank 4, then G contains a normal elementary abelian subgroup of the same rank. The result follows by Proposition 2.9. □

If $p \geq 5$, then a p-group of rank $r > p$ does not contain any normal elementary abelian subgroup of rank r in general, as shown by Professor Glauberman ([13]). Note that the same conclusion is suggested by [2, Theorem A]. However, in the examples considered, it is always true that if G has rank at least $p+1$, then $n_G = 1$.

3. p-groups of rank 2

Throughout this section, we let G be a finite p-group of rank 2, for an odd prime p. We compute n_G and describe the conjugacy classes of elementary abelian subgroups of G of rank 2, that is, the connected components of $\mathcal{E}_{\geq 2}(G)$. Let us recall the classification of the p-groups established by Blackburn ([4]). For convenience, we use the notation of [12].

There are four types of p-groups of rank 2 ([12, Theorem A.1]):

- The non-cyclic metacyclic groups $M(p, r)$, of order p^r, $r \geq 2$.
- The groups $C(p, r)$, of order p^r, $r \geq 3$, defined by
$$C(p,r) = \langle a, b, c \mid a^p = b^p = c^{p^{r-2}} = [a,c] = [b,c] = 1,\ [a,b] = c^{p^{r-3}} \ \rangle \ .$$
- The groups $G(p, r, \varepsilon)$, with $\varepsilon = 1$ or ε is not congruent to a square modulo p, of order p^r, $r \geq 4$, defined by
$$G(p,r,\varepsilon) = \langle a, b, c \mid a^p = b^p = c^{p^{r-2}} = [b,c] = 1,\ [a, b^{-1}] = c^{\varepsilon p^{r-3}} \ ,\ [a,c] = b \ \rangle \ .$$
- If $p = 3$, the following 3-groups of maximal nilpotency class of the form $\mathrm{B}(3, r; \beta, \gamma, \delta)$, of order 3^r, $r \geq 4$. Fix a set $\{s, s_1, \ldots, s_{r-1}\}$ of generators of G, subject to the relations:
  1. $s_i = [s_{i-1}, s]$, $2 \leq i \leq r-1$;
  2. $[s_1, s_2] = s_{r-1}^{\beta}$;
  3. $[s_1, s_i] = 1$, $3 \leq i \leq r-1$;

(4) $s^3 = s_{r-1}^{\delta}$;
(5) $s_1^3 s_2^3 s_3 = s_{r-1}^{\gamma}$
(6) $s_i^3 s_{i+1}^3 s_{i+2} = 1,\ 2 \leq i \leq r-1$ and with $s_r = s_{r+1} = 1$.

Hence, the groups of maximal nilpotency class are given by the sets of parameters (β, γ, δ) taking the values:

– For $r \geq 5$, $(\beta, \gamma, \delta) = (1, 0, \delta)$ and $\delta \in \{0, 1, 2\}$;
– For even $r \geq 4$, $(\beta, \gamma, \delta) \in \{(0, 1, 0), (0, 2, 0), (0, 0, 0), (0, 0, 1)\}$, except $\mathrm{B}(3, 4; 0, 1, 0)$;
– For odd $r \geq 5$, $(\beta, \gamma, \delta) \in \{(0, 1, 0), (0, 0, 0), (0, 0, 1)\}$.

In particular, we have isomorphisms $G(3, 4, 1) \cong \mathrm{B}(3, 4; 0, 0, 0)$ and $G(3, 4, -1) \cong \mathrm{B}(3, 4; 0, 2, 0)$. The group $\mathrm{B}(3, 4; 0, 1, 0)$ is the wreath product $C_3 \wr C_3$, which has rank 3.

3.1. **The non-cyclic metacyclic groups $M(p, r)$, $r \geq 3$.**
If $G = M(p, r)$, then G has a unique elementary abelian subgroup of rank 2, by [19, Lemma 2.1].

Proposition 3.1. *If $G = C(p, r)$, then $n_G = 1$ and $\mathcal{E}_{\geq 2}(G) = \mathcal{B} = \{E_0\}$.*

3.2. **The groups $C(p, r)$, $r \geq 3$.**
By [12, Lemma A.5], $C(p, r)$ is a non-split central extension

$$1 \longrightarrow Z(C(p, r)) \longrightarrow C(p, r) \longrightarrow C(p, 3) \longrightarrow 1 ,$$

where $Z(C(p, r)) = \langle c \rangle$ is cyclic of order p^{r-2}. In addition, $C(p, 3)$ is an extraspecial group of order p^3 and exponent p and contains all the elements of order p of $C(p, r)$. Thus, we fall into the situation of Corollary 2.3, and so any $E \in \mathcal{E}_{\geq 2}(C(p, r))$ is normal in $C(p, r)$.

Proposition 3.2. *If $G = C(p, r)$, then $n_G = p + 1$, and each component of $\mathcal{E}_{\geq 2}(G)$ is a single subgroup.*

3.3. **The groups $G(p, r, \varepsilon)$, $r \geq 4$ and $\varepsilon = 1$ or ε is not congruent to a square modulo p.**

Write $G = G(p, r, \varepsilon) = \langle a, b, c \mid a^p = b^p = c^{p^{r-2}} = [b, c] = 1,\ [a, b^{-1}] = c^{\varepsilon p^{r-3}}\ ,\ [a, c] = b\ \rangle$.
We have

$$1 \longrightarrow \langle b, c \rangle \longrightarrow G \longrightarrow \langle a \rangle \longrightarrow 1 ,$$

and $z = c^{p^{r-3}}$ generates the unique central subgroup of order p. Hence, we can choose $E_0 = \langle b, z \rangle$, and the other objects of $\mathcal{E}_{\geq 2}(G)$ are $E_i = \langle ab^i, z \rangle$, $1 \leq i \leq p$. Indeed, it follows from [12, Lemma A.6], or from a routine computation, that the elements of order p of $G - E_0$ are of the form $a^i b^j z^k$, with $0 \leq i, j, k \leq p - 1$. Moreover, ${}^{c}E_i = E_{i-1}$, where the index is taken modulo p, in the range $1, \ldots, p$. Thus $\{E_i \mid 1 \leq i \leq p\}$ is a single conjugacy class, i.e. a component of $\mathcal{E}_{\geq 2}(G)$.

Proposition 3.3. *If $G = G(p, r, \varepsilon)$, then $n_G = 2$, with components $\mathcal{B} = \{E_0\}$ and $\{E_i \mid 1 \leq i \leq p\}$.*

3.4. **The 3-groups of maximal nilpotency class $\mathrm{B}(3, r; \beta, \gamma, \delta)$, $r \geq 4$.**
In addition to the above notation, let us set $G_i = \langle s_i, \ldots, s_{r-1} \rangle$, for all $1 \leq i \leq r - 1$. Then, $G \triangleright G_2 \triangleright G_3 \triangleright \cdots \triangleright G_{r-1}$ is the lower central series of G and $G_{r-1} = Z(G)$ has order p. Moreover, $G_1 = \langle s_1\ ,\ s_2 \rangle$ is metacyclic non cyclic, of index p in G, and G_1 is abelian if and only if $\beta = 0$. In this case, if $r = 2k$, then $G_1 = \langle s_1 \rangle \times \langle s_2 \rangle \cong C_k \times C_{k-1}$ and if $r = 2k + 1$, then $G_1 = \langle s_1 \rangle \times \langle s_2 \rangle \cong C_k \times C_k$. Also, if $\langle \bar{s} \rangle = G/G_1 \cong C_3$, then the extension

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow \langle \bar{s} \rangle \longrightarrow 1$$

splits if and only if $\delta = 0$.

In any case, G_1 has a unique elementary abelian subgroup of rank 2, which is hence characteristic in G. Clearly, this subgroup is $G_{r-2} = \Omega_1(G_1)$, and we have $C_G(G_{r-2}) = G_1$.

Now, we want to determine the conjugacy classes of the elements $E \in \mathcal{E}_{\geq 2}(G)$, with $E \neq G_{r-2}$. Therefore, we first find the elements of order 3. Observe that any element $x \in G$ can be uniquely written as $x = s^{\zeta} s_1^{\zeta_1} \cdots s_{r-1}^{\zeta_{r-1}}$, for integers $\zeta, \zeta_i \in \{-1, 0, 1\}$, for all $1 \leq i \leq r-1$. From [12, Proposition A.9], we obtain that the elements of order 3 in G are either in G_{r-2} or belong to a coset of G/G_2 satisfying $\zeta(\delta + \beta\zeta_1^2) + \gamma\zeta_1 \equiv 0 \mod 3$, with $\zeta = \pm 1$. The solutions for ζ_1 are:

r	(β, γ, δ)	ζ_1
$r \geq 4$	$(0, 1, 0)$	0
$r \geq 4$	$(0, 0, 0)$	$-1, 0, 1$
$r \geq 4$	$(0, 0, 1)$	no solution
r even	$(0, 2, 0)$	0
$r \geq 5$	$(1, 0, 0)$	0
$r \geq 5$	$(1, 0, 1)$	no solution
$r \geq 5$	$(1, 0, 2)$	$-1, 1$

From this table, we immediately deduce the following.

Proposition 3.4. *If $\delta = 1$, then $n_G = 1$ and $\mathcal{E}_{\geq 2}(G) = \mathcal{B} = \{G_{r-2}\}$.*

Assume now that $\delta = 0$, and that $G \ncong \mathrm{B}(3, 4; 0, 1, 0)$. Then, the extension

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow \langle \bar{s} \rangle \longrightarrow 1$$

splits, and we have that $\Omega_1(G) = G_{r-2} \rtimes \langle s \rangle$ is extraspecial of order 27 and exponent 3. Thus, except G_{r-2}, the group G has 3 elementary abelian subgroups E_i of rank 2, generated by a central element z of order 3 and $x_i = s s_{r-2}^i$, for $i \in \{1, 2, 3\}$. Moreover, E_1, E_2 and E_3 are conjugate.

Proposition 3.5. *If $\delta = 0$ and $G \ncong \mathrm{B}(3, 4; 0, 1, 0)$, then $n_G = 2$, with components $\mathcal{B} = \{G_{r-2}\}$ and $\{E_1, E_2, E_3\}$.*

Proposition 3.3 handles the cases $\mathrm{B}(3, 4; 0, 0, 0) \cong G(3, 4, 1)$ and $\mathrm{B}(3, 4; 0, 2, 0) \cong G(3, 4, -1)$, whereas $\mathrm{B}(3, 4; 0, 1, 0)$ has rank 3. Hence, we are left with the case that $r \geq 5$ and $(\beta, \gamma, \delta) = (1, 0, 2)$. By the above discussion, the elements of order 3 that do not belong to G_1 are all the elements of the cosets of G/G_2 of the form $s^{\pm 1} s_1^{\pm 1} G_2$. Set $x = ss_1$, $z = s_{r-1}$ and let $E = \langle x, z \rangle = \{1,\ x = ss_1\ ,\ x^2 = s^2 s_1^2 s_2 z^{-1}\ ,\ z\ ,\ xz = ss_1 z\ ,\ x^2 z = s^2 s_1^2 s_2\ ,\ z^2\ ,\ xz^2 = ss_1 z^2\ ,\ x^2 z^2 = s^2 s_1^2 s_2 z\}$. Since $x \notin G_1$, [16, Hilfsatz III.14.3] shows that $E = C_G(E)$, and so, the conjugacy class of x is the coset $xG_2 \subset G/G_2$. Notice also that $N_G(E) = EG_{r-2}$ is extraspecial of order 27 and exponent 3. It follows that there are exactly two conjugacy classes of elementary abelian subgroups that are not normal in G.

Proposition 3.6. *If $\delta = 2$, then $n_G = 3$, with components $\mathcal{B} = \{G_{r-2}\}$ and the conjugacy classes of $\langle ss_1, s_{r-1} \rangle$ and of $\langle ss_1^{-1}, s_{r-1} \rangle$.*

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References

[1] J. Alperin, *A construction of endo-permutation modules*, J. Group Theory **4** (2001), no. 1, 3–10.
[2] J. Alperin, G. Glauberman, *Limits of Abelian Subgroups of Finite p-Groups*, J. Algebra, **203** (1998), 533–566.
[3] Y. Berkovich, *On abelian subgroups of p-groups*, J. Algebra, **199** (1998), 262–280.
[4] N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45–92.
[5] N. Blackburn, *Groups of prime-power order having an abelian centralizer of type* $(r, 1)$, Mh. Math. **99** (1985), 1–18.
[6] S. Bouc, *The Dade group of a p-group*, Invent. Math., **164** (2006), 189–231.
[7] S. Bouc and J. Thévenaz, *Gluing torsion endo-permutation modules*, J. London Math. Soc., to appear.
[8] J. Carlson, *Maximal elementary abelian subgroups of rank 2*, J. Group Theory **10** (2007), 5–13.
[9] J. Carlson, *Constructing endotrivial modules*, J. of pure and applied Algebra **206** (2006), 83–110.
[10] J. Carlson, N. Mazza and D. Nakano, *Endotrivial modules for finite groups of Lie type*, J. Reine Angew. Math., **595** (2006) 93–120.
[11] J. Carlson and J. Thévenaz, *The classification of endo-trivial modules*, Invent. Math. **158** (2004), no. 2, 389–411.
[12] A. Diaz, A. Ruiz, A. Viruel, *All p-local finite groups of rank two for odd prime p*, Trans. Amer. Math. Soc. **359** (2007), no. 4, 1725–1764.
[13] G. Glauberman, *Private communication*, Chicago, 2008.
[14] L. Héthelyi, *Soft subgroups of p-groups*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **27** (1984), 81–85.
[15] L. Héthelyi, *On subgroups of p-groups having soft subgroups*, J. London Math. Soc. (2) **41** (1990), no.3, 425–437.
[16] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin, 1967.
[17] D. Jonah, M. Konvisser, *Counting abelian subgroups of p-groups. A projective approach*, J. Algebra, **34** (1975), 309–330.
[18] M. Linckelmann, N. Mazza, *The Dade group of a fusion system*, J. Group Theory, to appear.
[19] N. Mazza, *The Dade group of a metacyclic p-group*, J. Algebra, **266** (2003), no. 1, 102–111.