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Swing-up control of inverted pendulum systems

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SUMMARY

In Part I a technique for the swing-up control of single inverted pendulum system is presented. The requirement is to swing-up a carriage mounted pendulum system from its natural pendent position to its inverted position. It works for *all* carriage balancing single inverted pendulum systems as the swing-up control algorithm does not require knowledge of the system parameters. Comparison with previous swing-up controls shows that the proposed swing-up control is simpler, eaiser, more efficient, and more robust.

In Part II the technique is extended to the case of the swing-up control of double inverted pendulum systems. Use is made of a novel selective partial-state feedback control law. The nonlinear, open-loop unstable, nonminimum-phase, and interactive MIMO pendulum system is actively linearised and decoupled about a neutrally stable equilibrium by the partial-state feedback control. This technique for swing-up control is not at all sensitive to uncertainties such as modelling error and sensor noise, and is both reliable and robust.

KEYWORDS: Inverted pendulum; Nonlinear unstable systems; Swing-up control; Partial-state feedback.

PART I: SINGLE INVERTED PENDULUM SYSTEM

1. Introduction

Inverted pendulum systems[1] are well known in control engineering for verification and practice of control theory and robotics.[2,3] Modern inverted pendulum systems extend the inverted pendulum's working range from around the upright vertical to the full 360° angular range. This introduces an initialisation problem, that is, to swing-up the pendulum from its natural pendent position to its inverted position.

Two approaches to swing-up control have been presented previously. The first uses a feed-forward bang-bang control.[4] This technique is very sensitive to modelling error and noise due to the nature of feed-forward control and is not a reliable technique. The second uses a feedback bang-bang control.[5,6] This was successfully demonstrated on a rotating-arm single inverted pendulum system without rail length restriction. It is not suitable for carriage balancing single inverted pendulum (CBSIP) systems in which the rail length is finite.

In this paper, a new approach to swing-up control is proposed. This approach depends on a sound understanding of the physical behaviour of CBSIP systems. It is based on three observations: (1) when disturbed, the pendulum's natural behaviour is a *nonlinear* vibration around its pendent vertical position (stable equilibrium) with a decreasing amplitude and a *varying* (increasing) circular frequency; (2) by increasing the pendulum's kinetic energy, the amplitude of the pendulum's vibration can be increased; and (3) the kinetic energy can be increased by moving the carriage co-ordinately following the rhythm of the pendulum's vibration.

The major principle of the proposed swing-up control is to follow the pendulum's natural vibration, and gradually turn it from damped vibration into resonance in a controlled manner. This will make the pendulum swing higher and higher. When it is close to the inverted vertical, a full-state feedback controller will take over the control and stabilise the whole system including the inverted pendulum.

The proposed technique has two major advantages: (1) *robustness*, it works for *all* CBSIP systems regardless of the vaues of their parameters, and (2) *simplicity*, it is simple and easy to tune. Furthermore, the principle of the technique can be applied to double and triple pendulum systems, and robotics.[7]

2. Inverted pendulum system

As schematically shown in Figure 1, a CBSIP system consists of a carriage and a pendulum atop the carriage. The carriage has mass M_c and translational friction coefficient k_r along the rail. The pendulum has mass M_p, length L_p, rotational friction coefficient k_o, and gravity centre at C. Control of the system is by means of force u applied horizontally to the carriage, the outputs are the carriage's position x_c and the pendulum's angle θ ($0° \leq \theta < 360°$).

Applying Lagrange's method, the system's nonlinear equations of motion can be readily obtained, and setting $M = M_c + M_p$ can be expressed in the form

$$\begin{cases} \ddot{x}_c = \dfrac{4L_p u - 4L_p k_r \dot{x}_c + 2M_p L_p^2 \sin(\theta)\dot{\theta}^2 + 6\cos(\theta) k_o \dot{\theta} - 1.5 M_p L_p g \sin(2\theta)}{[4M_c + M_p + 3M_p \sin^2(\theta)] L_p} \\ \ddot{\theta} = \dfrac{-6M_p L_p \cos(\theta) u + 6M_p L_p \cos(\theta) k_r \dot{x}_c - 1.5 M_p^2 L_p^2 \sin(2\theta)\dot{\theta}^2 - 12 M k_o \dot{\theta} + 6 M M_p L_p g \sin(\theta)}{[4M_c + M_p + 3M_p \sin^2(\theta)] M_p L_p^2} \end{cases} \tag{1}$$

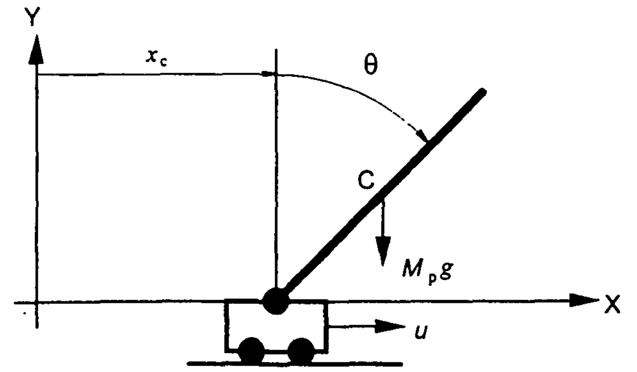

Fig. 1. Carriage balancing single inverted pendulum system.

The nonlinear equation of motion (1) can be linearised about $\theta = 0°$, and setting $M_1 = 4M_c + M_p$ the result can be expressed in state-space form

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \dfrac{-4k_r}{M_1} & \dfrac{-3M_p g}{M_1} & \dfrac{6k_o}{M_1 L_p} \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{6k_r}{M_1 L_p} & \dfrac{6Mg}{M_1 L_p} & \dfrac{-12Mk_o}{M_1 M_p L_p^2} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{4}{M_1} \\ 0 \\ \dfrac{-6}{M_1 L_p} \end{bmatrix}$$

In equation (2), $x = [x_c\ v_c\ \theta\ \omega]^T$ where v_c is the speed of the carriage and ω is the angular velocity of the pendulum. The nonlinear equations (1) are used for computer simulation of the system, and the linearised equation (2) is used for linear controller design. Because the linearised system shown in equation (2) is unstable but controllable, a full-state feedback control[8] of the form

$$u = k_x x_r - \boldsymbol{k}x \tag{3}$$

with the constant control vector $\boldsymbol{k} = [k_x\ k_v\ k_\theta\ k_\omega]$ exists, which will stabilise the whole system and achieve the carriage's position control. In equation (3), x_r is the commanded carriage position, which is set to zero in this study. The control vector $\boldsymbol{k}$ can be calculated by the LQR (linear-quadratic regulator) design method[8] with the performance criterion

$$J = \int (x^T \mathbf{Q} x + uRu)\, dt \tag{4}$$

where $\mathbf{Q}$ is the state weighting matrix which must be symmetric and positive semi-definite, and R is the control weighting coefficient which must be positive. In order to compare the swing-up control results in this paper with previous results, the system's parameter specifications are taken from Mori's system:[4] $M_c = 0.48$ kg, $k_r = 3.83$ [Ns/m], $M_p = 0.16$ kg, $L_p = 0.5$ m, and $k_o = 0.00218$ [Nms/Rad]. With $\mathbf{Q} = I$ and $R = 0.02$, the optimum control vector is found to be

$$\boldsymbol{k} = [-7.071\ \ -15.73\ \ -59.59\ \ -12.70] \tag{5}$$

Only if the initial pendulum angle $\theta(0)$ is close to the inverted vertical, can the full-state feedback control shown in equation (3) with the control vector (5) balance the pendulum and position the carriage. Evidently, the initialisation of the pendulum is necessary, which is to bring the inverted pendulum up from its natural pendent vertical position.

3. *Swing-up control*

The nonlinear system represented by equation (1) has a stable equilibrium at $\boldsymbol{x}_1 = [x_c\ 0\ 180°\ 0]^T$, and an unstable equilibrium at $\boldsymbol{x}_2 = [x_c\ 0\ 0°\ 0]^T$. Both of these equlibrium points can be obtained with an arbitrary carriage position x_c (therefore the carriage's position control is possible). Without the control ($u = 0$), the system's natural behaviour can be observed by releasing it from the initial state $\boldsymbol{x}(0) = [0\ 0\ \theta(0)\ 0]^T$ where $\theta(0) \neq 0°$ and $\theta(0) \neq 180°$. With $\theta(0) = 10°$ the numerical solution of equation (1) is computed and shown in Figure 2.

The natural behaviour of the system is damped vibration of both the pendulum and the carriage. Two important factors can be observed from the system's natural behaviour: (1) due to the translational and the rotational friction, the amplitudes of both the pendulum and the carriage decrease with the time; and (2) with the decreasing amplitudes, the circular frequencies of the vibrations *increase* with the time, which is a *nonlinear* characteristic.

In order to swing-up the pendulum towards the inverted vertical, its kinetic energy level must be increased. In this system, this can only be achieved by applying force to the carriage, and this force must at least

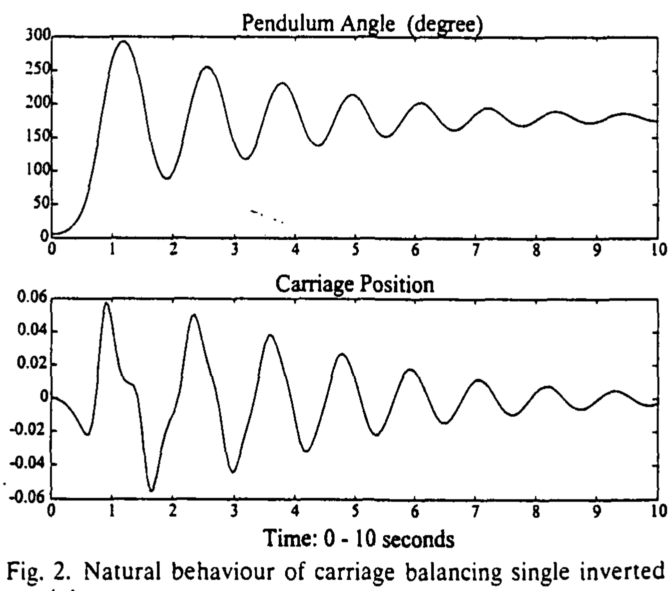

Fig. 2. Natural behaviour of carriage balancing single inverted pendulum system.

be large enough to overcome the friction. The direction and the amplitude of the input force must always be controlled so as to *increase* the pendulum's kinetic energy, despite the difficulties that (1) the force u applies to the pendulum *indirectly* through the carriage, (2) the vibrations of the pendulum and the carriage are nonlinear, and (3) there is a phase-lag between these two vibrations. Failure to do so will at best lead to energy loss and make the initialisation unnecessarily longer and at worst lead to a *forced vibration* of the pendulum at a constant amplitude and make the initialisation impossible. When the total energy of the pendulum equals PE, where

$$PE = M_p L_p g \quad (6)$$

that is, the pendulum's potential energy at the inverted vertical, the pendulum will be swung-up to its upright position. It is shown in this paper that the easiest and most efficient way to achieve this is to apply the force at the frequency of the pendulum's nonlinear vibration, so as to drive the pendulum into a resonance in a controlled manner. This leads directly to the swing-up control law, as linear feedback of ω according to the equation:

$$u_s = k_s \omega \quad (7)$$

Clearly, to start the swing-up control, a nonzero initial $\omega(0)$ is required. This can be obtained either by releasing the pendulum from a small angle away from the pendent vertical, or by applying a force to the carriage for a short time period before switching to the swing-up control. To simplify the discussion, the pendulum is released at $[\theta(0) = 165° \;\; \omega(0) = 0]$.

4. *Tuning of swing-up control*

The tuning of the sole constant k_s in equation (7) is easily achieved by starting from a small value and gradually increasing it, either in a computer simulation or in a real laboratory experiment. There exists a boundary k_b of k_s, which is determined by the friction and the parameters of the system. The relationship between k_s and k_b is as follows: (1) if $k_s < k_b$ then u_s is not strong enough to overcome friction, both the pendulum and the carriage perform damped vibrations: (2) if $k_s = k_b$ then u_s is just strong enough to overcome friction, both the pendulum and the carriage perform undamped vibrations: and (3) if $k_s > k_b$ then u_s is strong enough to swing-up the pendulum towards the inverted vertical.

By substituting equation (7) into the nonlinear equation (1), k_b could be obtained analytically, however this is both difficult and unnecessary. The value of k_b can be found by numerical solution of equations (1) and (7) during the tuning of the swing-up control. For example, it is found that $k_b = 0.31$ for the parameter values used in this paper. The system's time responses with $k_s = 0.1$, 0.31, and 0.4 are shown in Figure 3. Once $k_s > k_b$, the pendulum will be swung-up towards the upward vertical by the control u_s given by equation (7).

The tuning of k_s is judged by two factors, the required rail space x_{rs} and the required swing-up time t_s at which the pendulum is close to the inverted vertical (normally within $\pm 10°$). Generally, increasing k_s leads to decreasing t_s and increasing x_{rs}. For example, a swing-up time $t_s < 2$ s can be achieved by choosing $k_s = 2$. The results of the swing-up control shown in Figure 4, in which the pendulum is successfully swung-up towards the inverted vertical, are obtained with $k_s = 2$. In Figure 4, the control u_s is limited by a maximum force $u_{max} = 10.8$ N, which will be discussed shortly.

5. *Switching conditions*

There are three important factors which must be considered when switching between the swing-up control and the full-state feedback control. First, the closed-loop system with the full-state feedback control expressed in equation (3) has a restricted stable state region around the origin, because the controller design is based on the linearised system represented in equation (2). Second, the final states of the swing-up control must not force the

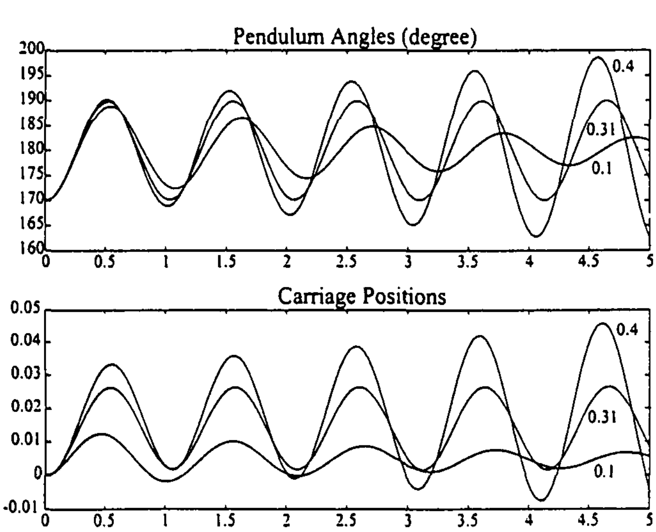

Fig. 3. The vibrations of the carriage balancing single inverted pendulum system.

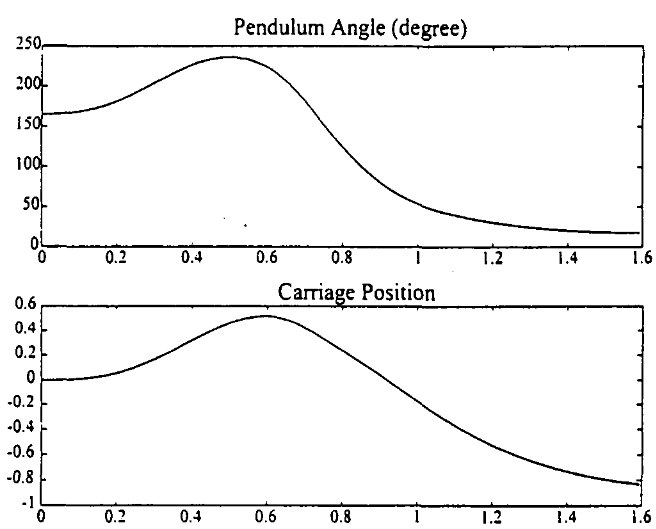

Fig. 4. Swing-up control of the carriage balancing single inverted pendulum system.

carriage beyond the available rail space. Third, during the swing-up control the pendulum should not be allowed to go over the top. Once it crosses the upward vertical, the pendulum will perform a nonlinear *rotation* around its pivot with a nonzero angular velocity. In this situation, even if the system is switched when the pendulum is vertical, it is most likely that either the system is outside the stable region, or the full-state feedback control runs out of rail space.

To prevent the pendulum from crossing the upward vertical, its total energy must be less than PE as given by equation (6) (Clearly a larger M_p or L_p leads to a larger PE, and hence a larger force or a longer switching time.). This can be achieved by saturating the control to a maximum u_{max}. If the pendulum starts rotating, the rotation can be controlled back to a vibration by resetting u_{max}.

During swing-up control, the *best* switching moment is when the pendulum is close to the upward vertical and at its maximum angle of vibration, then, the system's states are nearest to the equilibrium condition. Although x_c is at its maximum position, it does not affect the system's stability. v_c is close to zero (there is a phase lag between the vibrations of the carriage and the pendulum). $|\theta|$ is at the minimum ($\theta = 360°$ also stands for the inverted vertical), and $\omega = 0$. This case can be observed in Figure 5 which shows the states of the swing-up control with $k_s = 2$ and $u_{max} = 10.8$ N.

Moreover, when close to the upward vertical, the full-state feedback control u_f can quickly stabilise the pendulum (small θ and $\omega = 0$) and then drive the carriage towards the middle of the rail ($x_s = 0$), so that the recovery does not require much extra rail length.

Thus, the switching conditions ate $|\theta| \geq \theta_s$ and $\omega = 0$, where the boundary θ_s is determined mainly by the constants of the full-state feedback control. Generally, θ_s can be set to 10° to ensure smooth switching. However, the results shown in Figure 5 indicate that the system's best switching time is $t_s = 1.58$ s, at which time $\theta = 17.5°$. Even with this larger angle ($\theta > 10°$), the system can be stabilised easily and smoothly, by switching on the full-state feedback control. The complete swing-up control of the system is computed and shown in Figure 6.

The flow chart of a computer algorithm that automatically tunes and switches the swing-up control is shown in Figure 7. In the algorithm, H is the step size with which to increase k_s from zero, and h is the step size with which to decrease u_{max} from the real maximum force of the system.

Refer to Figure 7, when the switch conditions are met, the full-state feedback control is switched on to complete the swing-up control. If the pendulum starts rotating ($\theta = 0°$), u_{max} will be decreased by h and the experiment or simulation will start again. On the other hand, if the switching conditions are not met by $t = t_s$, k_s will be increased by H and the experiment or simulation will start again.

6. *Discussion of swing-up control*

The results of the swing-up control shown in Figure 4 are almost identical to those in Mori's nonlinear optimal swing-up control.[4] Mori's approach uses a bang-bang type feed-forward control, this is very sensitive to changes of the parameter values, the experiment conditions, and noise. Also the recovery can be difficult, as the system might not be at the desired state at the end of the *preset* control sequence. On the other hand, the approach in this paper uses a linear feedback control equation (7), which does not require detailed knowledge of the system. As the system is switched at a state near the equilibrium, the whole control is smooth and reliable. Therefore, the proposed swing-up control approach is simple and robust.

Based on a rotating-arm balancing inverted pendulum system, Furuta's approach used a feedback control[5,6] and was therefore more robust than Mori's approach. However, as it was still of the bang-bang type, the result was not as smooth as the result of the approach presented in this paper. Furthermore, Furuta's approach was not as simple as that presented in this paper. Because the switching of the bang-bang control was

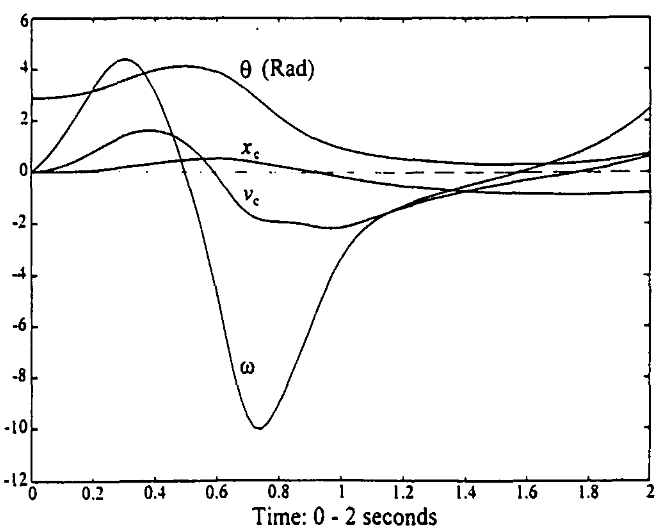

Fig. 5. Complete responses of the carriage balancing single inverted pendulum system.

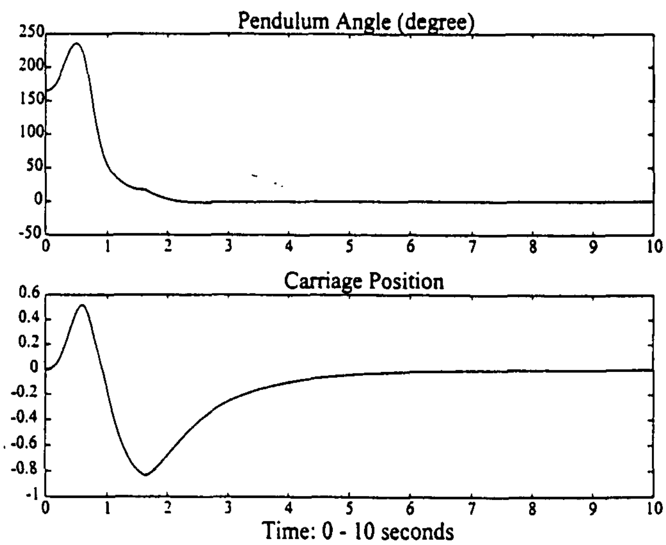

Fig. 6. Complete initialisation process.

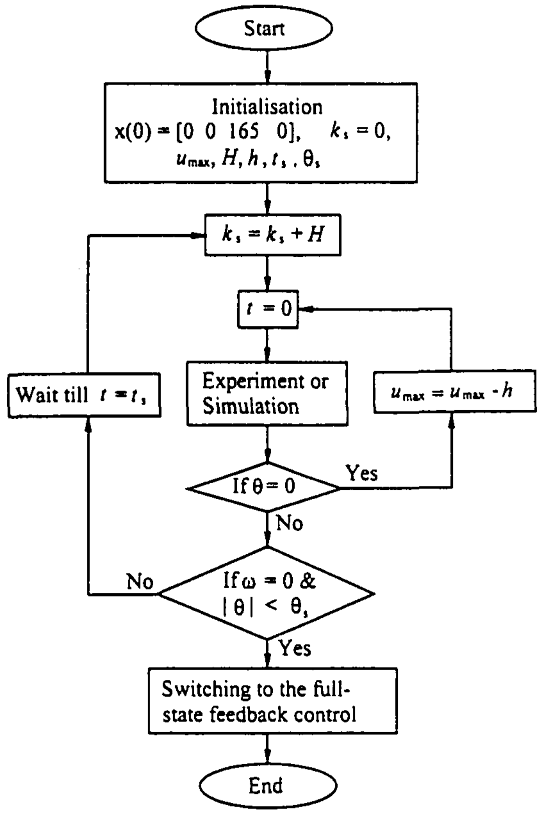

Fig. 7. The Algorithm of the swing-up control.

based on the phase plane of ($\theta\omega$), knowledge of the system parameter values was requried in Furuta's approach. Finally unlike CBSP systems, rail length is not an issue in the rotating-arm type of inverted pendulum system.

PART II: DOUBLE INVERTED PENDULUM SYSTEM

1. Introduction

Except for the case of swing-up control,[4–6] nearly all previous theoretical and experimental studies of inverted pendulum systems have been based on linear analyses. All reported double pendulum systems, such as,[9,10] have been single-input systems. The only two MIMO (multi-input multi-output) pendulum systems were both triple pendulum systems.[11,12] Although these two systems were able to achieve nonlinear pendulum position control, they were both limited to small perturbations from the vertical equilibrium position, and so were essentially examples of linear control systems.

The failure to extend inverted pendulum system studies into large amplitude, nonlinear regions is mainly due to the fact that a nonlinear unstable pendulum system is extremely difficult to balance and control. Furthermore, at interactive MIMO pendulum system also requires decoupling. With the uncertainties of modelling error and noise, decoupling is difficult, even in linear unstable systems.

In nonlinear MIMO double pendulum systems, a challenging control object is to swing-up the double pendulum from its natural pendent position to the inverted position. As these systems are nonlinear, open-loop unstable, nonminimum-phase, and interactive, it is impossible to achieve swing-up control by using a linear full-state feedback control law. As a result, swing-up control of double pendulum system has not been studied previously in the literature.

For nonlinear, open-loop unstable, and interacting MIMO systems, full-state feedback is not always the best solution. In fact, by a suitable partitioning of the state-space, it is possible to achieve several control objectives. Indeed, a new controller design strategy based on a selective partial-state feedback control approach is presented in this paper. A special feature of the approach is that it can actively decouple and linearise interacting nonlinear unstable MIMO systems. Furthermore, as the partial-state feedback control itself is linear, linear control theory can be used for system analysis and controller design.

Based on both the partial-state feedback control and the single pendulum swing-up control presented in Part I, the swing-up control of a double pendulum system is achieved successfully in this study. The theoretical studies and simulations in this paper show that the once complicated control problem, which is associated with nonlinearity and interaction, is solved in a simple and easy manner. Unlike conventional linearisation and decoupling which are associated with fixed operating points, the double pendulum is actively linearised and decoupled about its neutrally stable equilibrium[8] by the partial-state feedback control. This makes the double pendulum swing-up control very reliable and robust.

2. Selective partial-state feedback control

To date, the most common procedure for stabilising and controlling a nonlinear unstable MIMO system is: (1) linearise the system at its operating point which must be an equilibrium state of the system, (2) decouple, stabilise, and control the linearised system by using full-state feedback, and (3) use the resulting linear controller to control the original nonlinear system around the operating point. As linear control theory has been well developed, the above procedure often works successfully.

There are three major disadvantages of the above procedure which are now discussed.

(i) The resulting linear controller can only stabilise the nonlinear system around the operating point. To cover a wide range of the state-space, the system has to be linearised at several operating points. Also the linear controller design has to be carried out at each operating point. When the system works over large regions of the state-space, the number of required operating points increase sharply.

(ii) The decoupling control is based on the linearised model of the nonlinear system, and is therefore very sensitive to the system's modelling error and noise. Moreover, unwanted steady-state errors are introduced when the system is stabilised between the operating points.

(iii) The idea of "stability" and "full-state feedback" have greatly restricted the search for better control, as control engineers are used to think in these terms. For example, the first double pendulum system was built more than 30 years ago in 1963,[9] its control system was designed using a linearised model and full-state feedback. Moreover, there has been no significant progress in the control of double pendulum systems since.

A system is defined as unstable if some of its output increase without a boundary or outside a preset boundary when time tends to infinity. However, many real controls are only required temporarily, such as the initialisation of an inverted pendulum. By using the concept of BIBS (bounded-input bounded-state) stability,[8] the behaviour of those unstable outputs is often well understood and predictable. For a nonlinear unstable MIMO system with some equilibrium positions, instead of linearising and decoupling the nonlinear system around some operating points, partial-state feedback could be used directly to decouple, linearise, and stabilise the system at the equilibrium. A nonlinear, unstable, and interactive MIMO system with linear sub-state x_l and nonlinear sub-state x_n can be expressed in the partitioned state-space form

$$\begin{bmatrix} \dot{x}_l \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_l(x_l, x_n, u_l, u_n) \\ f_n(x_l, x_n, u_l, u_n) \end{bmatrix} \qquad (8)$$

where u_l is the control of x_l and u_n is the control of x_n. If equation (8) has a neutrally stable equilibrium $[x_{le}\ x_{ne}]^T$, then it is possible to find a partial-state feedback

$$u_n = r_n - K_n x_n \qquad (9)$$

to stabilise the subsystem of x_n at x_{ne}. In equation (9), $r_n = f(x_{ne})$ is the reference input, and K_n is the controller constant matrix. Once the subsystem is stabilised at x_{ne} by u_n, there is no movement within the subsystem. Therefore the nonlinear subsystem is effectively linearised at x_{ne}, and it can be described by a reduced order state x_d. To stabilise the whole system, another partial-state feedback

$$u_l = r_l - K_l x_l - K_{ld} x_d + r_{ln} \qquad (10)$$

is required, where $r_l = f(x_{le})$ is the reference input, K_l is the controller constant matrix of x_l, K_{ld} is the controller constant matrix of x_d, and $r_{ln} = f(x_{ne})$ is to compensate for the effect of x_d on x_n. As the position control of x_n is achieved by u_n, and the position control of x_l is achieved by u_l, the nonlinear system expressed in equation (8) is effectively decoupled, linearised, and stabilised. The overall control u of equation (8) is

$$u = \begin{bmatrix} u_l \\ u_n \end{bmatrix} = \begin{bmatrix} r_l - K_l x_l - K_{ld} x_d + r_{ln} \\ r_n - K_n x_n \end{bmatrix} \qquad (11)$$

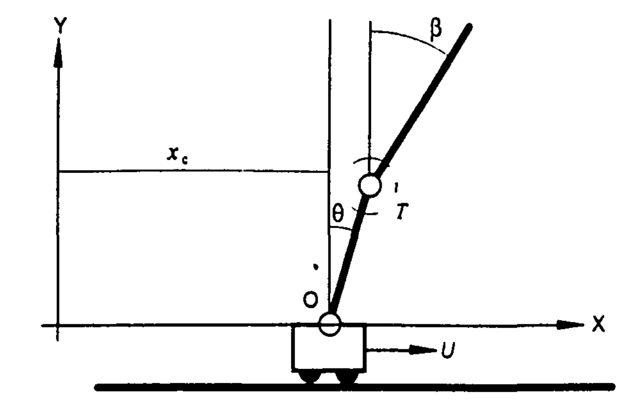

Fig. 8. Nonlinear double pendulum system.

3. Double pendulum system

A nonlinear double pendulum system is shown in Figure 8. The carriage has mass M_c and translational friction coefficient k_r. The lower pendulum has mass M_{p1}, length L_{p1}, and rotational friction coefficient k_{o1} at the pivot O. The higher pendulum has mass M_{p2}, length L_{p2}, and rotational friction coefficient k_{o2} at the joint between the pendulums. There are two control inputs: a horizontal force U on the carriage, and a torque T between the pendulums.

The coordinates of the system are the carriage position x_c, the lower pendulum angle θ, and the higher pendulum angle β. By using Lagrange's equations, the system's nonlinear equations of motion can be expressed in the form

$$\begin{cases} M\ddot{x}_c + P_1\cos(\theta)\ddot{\theta} + P_2\cos(\beta)\ddot{\beta} \\ \quad = U - k_r\dot{x}_c + P_1\sin(\theta)\dot{\theta}^2 + P_2\sin(\beta)\dot{\beta}^2 \\ P_1\cos(\theta)\ddot{x}_c + 2P_3\ddot{\theta} + P_4\cos(\beta-\theta)\ddot{\beta} \\ \quad = -T - (k_{o1} + k_{o2})\dot{\theta} + k_{o2}\dot{\beta} \\ \qquad + P_4\sin(\beta-\theta)\dot{\beta}^2 + P_1 g\sin(\theta) \\ P_2\cos(\beta)\ddot{x}_c + P_4\cos(\beta-\theta)\ddot{\theta} + 2P_5\ddot{\beta} \\ \quad = T + k_{o2}\dot{\theta} - k_{o2}\dot{\beta} - P_4\sin(\beta-\theta)\dot{\theta}^2 + P_2 g\sin(\beta) \end{cases} \qquad (12)$$

where

$$M = M_c + M_{p1} + M_{p2}$$
$$P_1 = (0.5M_{p1} + M_{p2})L_{p1}$$
$$P_2 = 0.5M_{p2}L_{p2}$$
$$P_3 = (M_{p1}/6 + 0.5M_{p2})L_{p1}^2$$
$$P_4 = 0.5M_{p2}L_{p1}L_{p2}$$
$$P_5 = M_{p2}L_{p2}^2/6$$

Setting $\dot{x}_c = \ddot{x}_c = \dot{\theta} = \ddot{\theta} = \dot{\beta} = \ddot{\beta} = 0$, the equilibrium of equation (12) is determined by

$$\begin{cases} U_e = 0 \\ T_e = P_1 g\sin(\theta_e) \\ P_1\sin(\theta_e) + P_2\sin(\beta_e) = 0 \end{cases} \qquad (13)$$

where U_e and T_e are the required controls to achieve the equilibrium, θ_e and β_e are the equilibrium pendulum angles. Equation (13) is very important in any double pendulum system study, as it shows that: (1) the equilibrium can occur at an arbitrary x_c, therefore the carriage position control can be achieved: (2) as there is only one equation governing θ_e and β_e, one of the pendulums can be positioned at an arbitrary angle, however the other pendulum must meet equation (13) to keep both of the pendulums balanced; and (3) a nonzero steady-state torque T_e is required to keep the pendulums at the required angles. Equation (13) defines a neutrally stable equilibrium, as the carriage position and one of the pendulum angles can be arbitrary. It can be seen from Figure 8 that the maximum moment of the higher pendulum about O occurs at $\beta = -90°$. Further, it can be seen from equation (13) that, in order to balance the pendulums, the maximum lower pendulum angle is

$$\theta_{max} = \sin^{-1}\left(\frac{P_2}{P_1}\right) \tag{14}$$

which is defined by the pendulums' lengths and masses. It follows from equations (12) and (13), in order to achieve the full 360° angle pendulum control, that

$$\frac{M_{p2}L_{p2}}{(0.5M_{p1} + M_{p2})L_{p1}} \geq 1 \tag{15}$$

For demonstration purposes, the following values are assigned to the system parameters: $M_c = 0.5$ kg, $k_r = 3.8$ [Ns/m], $M_{p1} = 0.1$ kg, $L_{p1} = 0.25$ m, $k_{a1} = 0.002$ [Nms/Rad], $M_{p2} = 0.1$ kg, $L_{p2} = 0.25$ m, $k_{a2} = 0.002$ [Nms/Rad], and $g = 9.8$ [m/s^2].

4. Swing-up control

The natural behaviour of the nonlinear double pendulum system can be computed and observed from equation (12) by setting $U = T = 0$ and $x(0) = [0\ 0\ 10°\ 0\ 10°\ 0]^T$, as shown in Figure 9. The carriage and the pendulums all perform damped nonlinear vibrations. At the beginning the two pendulums vibrate differently, therefore the interaction between the pendulums makes the system's motion complicated. However, the pendulums tend to vibrate at the same circular frequency, and the vibrations are towards the stable equilibrium ($\theta = 180°$ $\beta = 180°$). The system's responses are much smoother when the pendulums vibrate at the same frequency.

Clearly, in order to swing-up the two pendulums, the higher pendulum has to follow the lower pendulum as it swings up towards the inverted vertical. Therefore, the angle $(\beta - \theta)$ between the pendulums has to be controlled. Because (1) the pendulum system's state-variables can be partitioned as

$$x_l = [x_c\ \dot{x}_c]^T \quad x_n = [\theta\ \dot{\theta}\ \beta\ \dot{\beta}]^T \tag{16}$$

and (2) there is a neutrally stable equilibrium described by equation (13), the partial-state feedback expressed in equation (9) exists in the double pendulum system with $r_n = T_{rg}K_n = [-K_\beta\ -K_\gamma\ K_\beta\ K_\gamma]$, and $u_n = T_p$, where

$$T_p = T_r - K_\beta(\beta - \theta) - K_\gamma(\dot{\beta} - \dot{\theta}) \tag{17}$$

By substituting equation (17) into equation (12), the nonlinear system's steady-state responses under the partial-state feedback are given by

$$\begin{cases} P_1 \sin(\theta) + P_2 \sin(\beta) = 0 \\ T_r = K_\beta(\beta - \theta) + P_1 g \sin(\theta) \end{cases} \tag{18}$$

Comparison of equations (13) and (18) shows that T_p can balance the pendulums at the equilibrium angles θ_e and β_e. Therefore, the angle between the pendulums is controlled, and the whole nonlinear system is both linearised and decoupled actively. For example, by setting $K_\beta = 10$, $K_\gamma = 5$, and $\theta_e = 10°$, it follows from equation (18) that $\beta_e = -31.40°$ and $T_r = -7.162$. If the nonlinear system described by equation (12) is released from $x(0) = [0\ 0\ 0\ 0\ 10°\ 0]^T$, with $U = 0$ and $T = T_p$ the system's time responses can be computed and are shown in Figure 10. Although the pendulums vibrate towards the stable equilibrium ($\theta_e + 180°$ $\beta_e + 180°$), the angle $(\beta - \theta)$ between the pendulums is balanced at $(\beta_e - \theta_e) = -41.40°$. With the control T_p, the double pendulum can be safely regarded as an *equivalent single pendulum*, which can be described by the lower pendulum's state $x_d = [\theta\ \dot{\theta}]^T$.

Obviously, for the swing-up control of the double pendulum system, in equation (17) $\theta_e = 0$, and therefore it follows from equation (18) that $\beta_e = T_r = 0$. Moreover, from the single pendulum swing-up control presented in

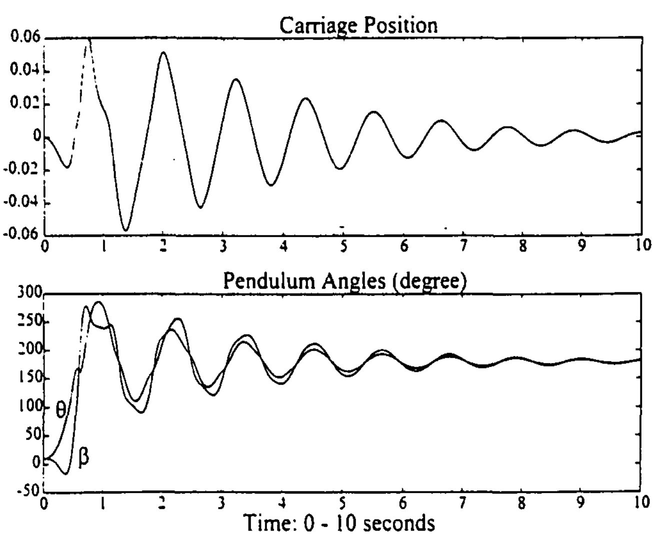

Fig. 9. Natural behaviour of the pendulum system.

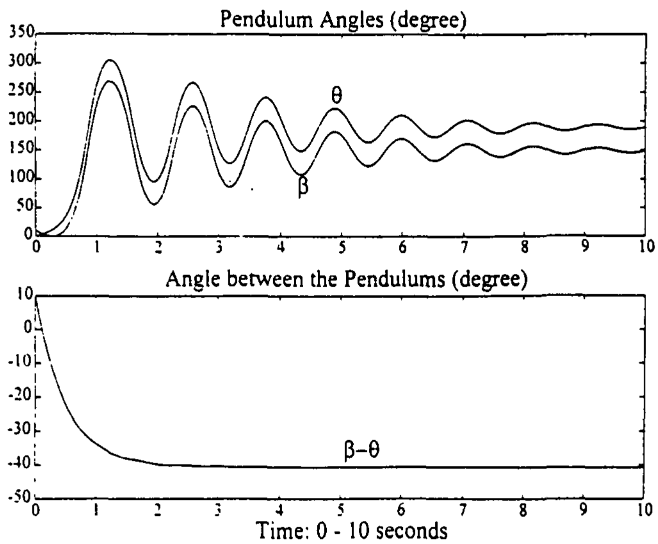

Fig. 10. Angle control between the pendulums.

Part I (Section 3), it can be seen that the control U_s to swing-up the double pendulum is

$$\begin{cases} U_s = K_s\dot{\theta} & K_s\dot{\theta} < U_{max} \\ U_s = U_{max} & K_s\dot{\theta} \geq U_{max} \end{cases} \quad (19)$$

where K_s is the controller constant, and U_{max} is the maximum force. Obviously, U_s is a positive partial-state feedback. The swing-up control tuning process described in Part I (Section 4) gives the result that $K_s = 4.1$ and $U_{max} = 10.6$ N. Substitution of equations (17) and (19) into equation (12) gives the swing-up control of the double pendulum system which has been computed and is shown in Figure 11.

As shown in Figure 11, both of the pendulums are swung-up from (around) the pendent vertical to near the inverted vertical. The best switching time is at $t_s = 2.07$ s with $\theta(t_s) = 11.03°$ and $\beta(t_s) = 10.78°$. Therefore, the originally nonlinear swing-up control of the double pendulum system has been successfully achieved by using two simple linear partial-state feedback control laws simultaneously.

5. Stabilisation of the system

Following the discussion in Section 2, after the swing-up control has been accomplished, the whole system can be stabilised by using equation (17) and another partial-state feedback U_b

$$U_b = U_r - K_x x_c - K_v \dot{x}_c - K_\theta \theta - K_\omega \dot{\theta} \quad (20)$$

where K_x, K_v, K_θ, and K_ω are the controller constants, U_r is the reference input. The purpose of U_r is to effect the desired carriage and pendulum positions in the steady state. It can be deduced by consideration of the steady state behaviour of the system obtained when $U = U_b$ in equation (12). Thus

$$U_r = K_x x_r + K_\theta \theta_r \quad (21)$$

where x_r is the commanded carriage position and θ_r is the commanded lower pendulum angle. In equation (21) $K_\theta \theta_r$ is for removing the interaction from the double pendulum to the carriage. Comparing equations (20) and (21) with equation (10), it can be seen that $\boldsymbol{K}_l = [K_x \;\; K_v]$, $\boldsymbol{K}_{ld} = [K_\theta \;\; K_\omega]$, $r_l = K_x x_r$, and $r_{ld} = K_\theta \theta_r$. As the nonlinear double pendulum system is dynamically converted into an equivalent linear single pendulum system by T_p, the controller constants of equation (20) can be obtained by following the design procedure of the single pendulum system in Section 2 (Part I). The equivalent single pendulum has the parameters $M_p = 0.2$ kg, $L_p = 0.5$ m, and $k_o = 0.002$ [Nms/Rad]. Setting $\mathbf{Q} = \mathbf{I}$ and $R = 0.01$ in the LQR design,[8] the controller constants are obtained as

$$\boldsymbol{K}_l = [-10 \quad -19.96] \quad \boldsymbol{K}_{ld} = [-78.74 \quad -17.20] \quad (22)$$

Using the partial-state feedback equations (17) and (20) together, asymptotically stable nonlinear position control of the double pendulum system can be achieved. By switching the control from equations (17) and (19) to equations (17) and (20) at $t_s = 2.07$ s, the double pendulum system is initialised and then stabilised, as shown in Figure 12.

Supported by the robust partial-state feedback control approach, the swing-up control can also be achieved with an angle between the pendulums. With $\theta_r = 10°$, $\beta_r = -31.40°$, $T_r = -7.162$, $K_s = 4.1$, and $U_{max} = 10.3$ N, the partial-state feedback equations (17) and (19) can swing-up the pendulums towards θ_r and β_r from $\theta(0) = \beta(0) = 180°$. The best switching time is found at $t_s = 1.606$ s, with $x_c(t_s) = -0.8$ m, $\theta(t_s) = 24.69°$, and $\beta(t_s) = -14.87°$. Afterwards, with $x_r = 0$ and $U_r = -13.74$, equations (17) and (20) can stabilise the whole system at the required positions: $x_r = 0$, $\theta_r = 10°$, and $\beta_r = -31.40°$. The complete swing-up control is shown in Figure 13. The large angle $\beta_r - \theta_r = -41.40°$ is achieved and kept throughout the whole initialisation process.

PART III: CONCLUSION

A practical and robust swing-up control approach is presented in Part I. The robust approach gives a smooth performance, and is able to bring a pendulum up from its

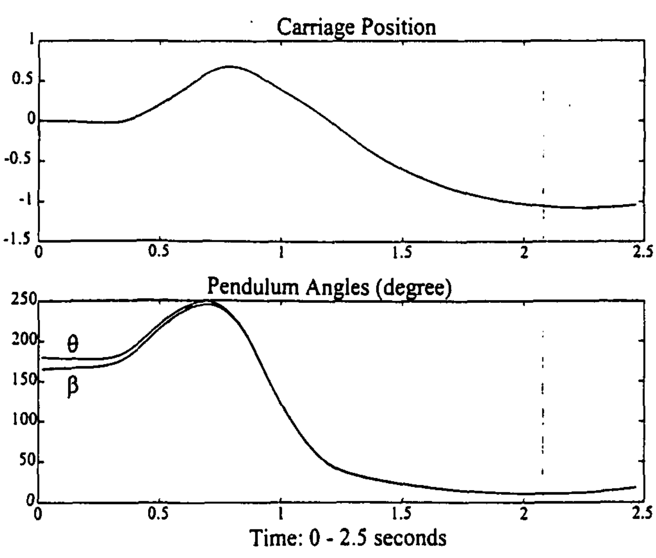

Fig. 11. Swinging-up the pendulums.

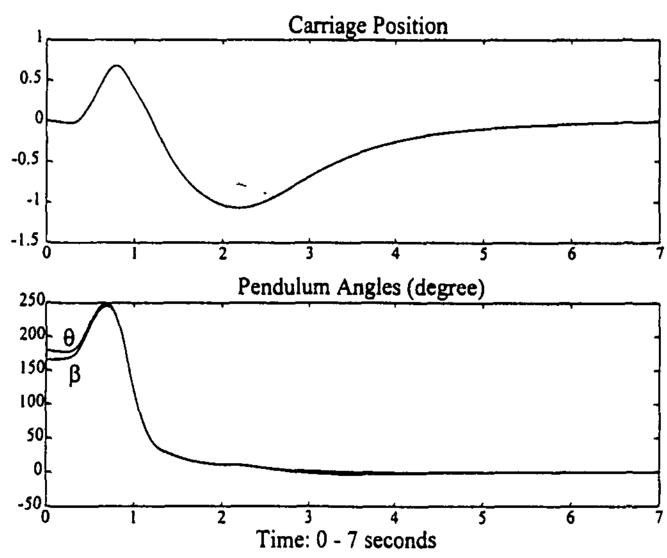

Fig. 12. Repositioning the double pendulum system.

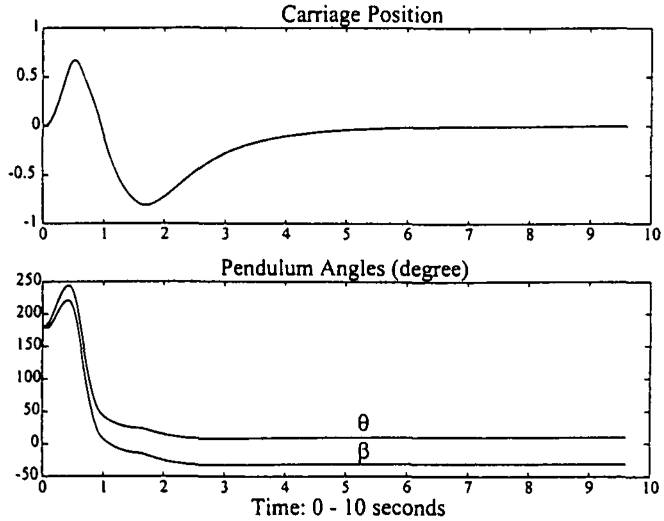

Fig. 13. Pendulum control with a large angle.

natural pendent position and balance it in the inverted position. It works for all CBSIP systems regardless of the parameter values. The principle employed in the approach can be used to swing-up a double or triple pendulum. Furthermore, the principle can also be applied to new kinds of artificial intelligent machines.[7]

As presented in Part II, the selective partial-state feedback control approach has been proved extremely successful in the nonlinear, unstable, and interactive double pendulum system. It can also be applied to triple pendulum systems for both swing-up control and nonlinear pendulum position control. Indeed, partial-state feedback control can be applied with advantage to a wide range of nonlinear, unstable systems. All that is required is that the system equations can be partitioned as in equation (8). Many machines and processes are governed by such equations. In particular, many dynamical models used in the study of legged robots[2,3] are governed by partitionable, nonlinear systems of equations.

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