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Symmetry and the design of self-stressed structures

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Abstract

The symmetry adapted counting rule for mechanisms and states of self-stress in symmetric frameworks is presented in an accessible and intuitive manner with the aim of empowering engineers who design such structures. By simply counting nodes and bars, it is possible to detect states of self-stress and mechanisms beyond the standard Maxwell-Calladine count. This methodology is first introduced without the need to understand the underlying group theory before being applied to a range of example frameworks. Design problems focusing on gridshells are discussed - it is noted that placing bars on lines of mirror symmetry tend to increase the number of states of self-stress in a framework, which can be desirable. This paper reformulates common symmetric frameworks and introduces simple rules regarding how to obtain a greater number of states of self-stress. By allowing for the design of states of self-stress, the forces in the structure can be designed with greater control.

Keywords

symmetry, self-stress, mechanisms, Maxwell, reciprocal figures, pin-jointed frameworks, gridshell

Introduction

Symmetry and antisymmetry are powerful concepts that can be used in the design of many structures. One application of symmetry is how it can be used to create or avoid states of self-stress and mechanisms (flexes) in pin-jointed frameworks. This paper adapts research from rigidity theory, graph theory and group theory for use by structural designers in detecting and designing mechanisms and states of self-stress that are related to structural symmetry. Although the underlying mathematics relies on group theory and is rather technical, the application to the design of 2D structures is quite simple. Some terminology in this paper refers to these mathematical fields, but jargon that is generally unfamiliar to structural engineers is avoided where possible.

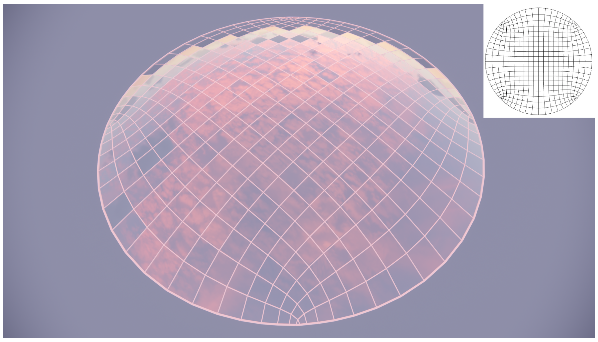

Figure 1. A gridshell structure. Note the 2D projection (form diagram), inset, possesses symmetry. From [15].

Structural "Counts"

In his seminal paper of 1864, Maxwell [11] introduced a counting rule for pin-jointed trusses. For a 2D pin-jointed truss, Maxwell developed a "count" as $2v - b - r$ where v is the number of nodes, b is the number of bars and r is the number of nodal restraints (often $r = 3$ as the minimum number of restraints to prevent rigid body motions). This count is a standard element of engineering education for use as an initial evaluation of static determinacy (or indeterminacy) and kinematic determinacy (mechanisms). This counting rule was refined by Calladine [5] who established the count $2v - b - r = m - s$ where m is the number (count) of mechanisms and s is the number (count) of states of self-stress (a state of self-stress is where truss members contain forces in the absence of external loading). Prior to Calladine, it was known that certain structural geometries could cause a structure, even if it had a nominally statically determinate or indeterminate count ($2v - b - r \leq 0$), to have additional mechanisms and states of self-stress. Calladine's refinement recognises that when "special" geometric positions cause additional mechanisms, m, they also cause additional states of self-stress, s, and that m and s are added in equal measure (see Figure 6 for example). The Maxwell-Calladine count, $2v - b - r = m - s$ applies for any value of $2v - b - r$ and will be further explained on Page 3 of this paper. In this paper, r is taken as 3 and the Maxwell-Calladine count is defined as k where $k = 2v - b - 3$. The term k is called the *freedom number*, but it is also the Maxwell count.

What are these "special" geometries that cause these additional mechanisms and states of self-stress? This is a

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complex question that will be partially addressed in this paper. In particular, this paper will address how to detect the presence of mechanisms and states of self-stress that are related to the symmetry of a structure. This paper reformulates the symmetry adapted counting rule, originally developed in [7], with the aim of making the process simple and avoiding extraneous details. Examples using this counting rule are then given. Discussion of how structures can be *designed* using the counting rule information follows these examples. It is hoped that this paper will make this counting rule more available to practitioners whilst demonstrating how it can be used within structural design. An accompanying paper develops the mathematical background further [22]. This paper is aimed at a more engineering based audience as opposed to the other paper which is written for a more mathematically literate audience. However, readers are referred to the accompanying paper for more rigorous discussion of the underlying mathematics. These papers focus on structural frameworks where no two nodes overlap, no bars have zero length, and bars only cross at nodes - this layout is also referred to as a planar embedding of a graph. The term graph refers to a collection of nodes connected by line segments and is a mathematical representation of the layout of a framework or truss. The methods presented here can be extended to consider non-planar graphs, although this is not explored in this paper.

Utilising States of Self-Stress in the Design of Structures

If a 3D structure is in equilibrium, then any 2D projection of the structure is also in equilibrium. For 3D structures where gravity and other vertical loads are important, it is useful in design to consider the projection of the structure and the forces on to the horizontal xy plane. In this view, the vertical forces are not visible; one sees a 2D structure that appears to be self-stressed against itself or its boundary structure. In this paper, any boundary structure for horizontal reactions is considered to be an integral part of the structure, so the structure is considered to be 'self-tied'. If the structure requires a boundary structure for horizontal equilibrium, then, for this paper, the boundary structure is idealised as a horizontal reaction truss so that all horizontal forces are resolved within the idealised structure [13]. The accompanying paper [22] tackles frameworks with horizontal supports as well.

There are certain types of structures where these 2D states of self-stress are necessary for the 3D structure to function. For example, long span roofs such as tensegrity domes (Geiger domes) depend on the stressing the roof cables against a perimeter compression ring in order to have a stable and stiff structure.

Another class of structures that depend on the ability of the 2D projection of the 3D structure to have states of self-stress are funicular gridshells (Figure 1). A key motivation in this paper is the study of gridshells. Millar *et al* [15] discuss why it is beneficial that the 2D projection possesses many states of self-stress. Each state of self-stress relates to a set of axial loads within the gridshell which is in equilibrium with a companion set of vertical nodal loads. For a 2D projection of a gridshell with s linearly independent states of self-stress and v nodes without vertical supports, the load space which must be taken through bending is of size $v - s$. Transferring load through bending is less efficient than through funicular action (axial forces only). Therefore, it is beneficial to maximise s. For architectural, structural and construction reasons, gridshell layouts often have a great deal of symmetry. Whilst this paper does not maximise s, it does provide methods through which to increase s by understanding the effects of symmetry on states of self-stress.

Increasing the Number of States of Self-Stress

One way to increase s is to add more bars and make k more negative. This would eventually lead to a fully triangulated framework. However, it is often desirable to have a quad-dominant framework in gridshells. This is because nodes with more than 4 connecting bars are seldom torsion free in a gridshell (the members at a node do not share a common axis). Furthermore, triangular glass panels can produce a lot of wastage and therefore increase the cost associated with the design. Whilst some triangular panels and some nodes with more than 4 bars are acceptable, these are typically kept to a minimum for the design of a gridshell.

For a given count, k, it is sometimes possible to increase the number of states of self-stress by having a "special" geometry or layout of the structure. As recognised by the Calladine's refinement of the Maxwell count, these additional states of self-stress will have associated mechanisms. These associated mechanisms are often stabilised by the geometric stiffness of the prestress in cable structures or by flexural stiffness in gridshells.

"Special" Geometries

Maxwell [11] made the significant observation that for a state of self-stress to exist in a 2D structure, the layout must be the projection of a plane-faced polyhedron. If the geometry of the 2D layout can be the projection of one or more different, linearly independent polyhedra, then each polyhedron is related to a different, linearly independent state of self-stress. The layout, thus, represents a "special" layout. It is noted that these plane-faced polyhedra have a meaning in engineering mechanics in that they are discrete Airy stress functions [4], but this aspect is not critical to this paper and is not further discussed.

There are other geometric aspects of the 2D layout that should be considered. Gridshells are often subjected to uniform symmetric loads (for example, self-weight or a snow load). Assuming a symmetric layout, a symmetric load requires a symmetric state of self-stress if it is to be funicular. Similarly, if an antisymmetric load is applied then an antisymmetric state of self-stress is required. Half-loads, such as snow drifts, can be decomposed into a symmetric and antisymmetric load, as is discussed by McRobie *et al* [12]. The authors note that even if the form diagram possesses a symmetric state of self-stress, the gridshell may not be funicular for the desired load case.

The mechanisms and states of self-stress that are created by "special" geometries are interrelated. Often, but not always, the mechanisms and associated states of self-stress that are related to symmetry will have the property that if the state of self-stress is symmetric then the mechanism will be antisymmetric, and vice-versa. These will be detected by the process described below. In the situation where the mechanism and associated state of self-stress are both symmetric or both antisymmetric, these are not detected. One way to determine the exact values of s and m is to investigate the rank of the equilibrium matrix, $\mathbf{A}$. However, this often does not aid the designer in obtaining geometries with additional states of self-stress. Similarly, it does not help to design symmetric and antisymmetric states of self-stress which are desirable in gridshells, as described on Page 8.

States of self-stress are a *projective* property. If the 2D layout of a structure is *projected* to a different 2D geometry through an affine (shearing) or projective (perspective) transformation, the 2D infinitesimal rigidity properties of equilibrium (states of self-stress and mechanisms) are all retained [9] [4] [16]. This can be useful in design when a structural layout can be developed in a highly symmetric layout and then projected to match the project requirements (often projective transformations destroy symmetries). See Figure 2 for an example.

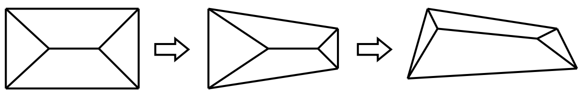

Figure 2. When a Desargues configuration is projected, it remains a Desargues configuration so that $m = s = 1$.

Prior Work on Mechanisms and States of Self-Stress in Symmetric and Periodic Structures

Fowler and Guest [7] introduced a symmetry adapted counting rule based on the Maxwell-Calladine count [5]. This paper uses this counting rule and applies it to examples from the field of gridshell design. Additional work in this field was done by Connelly *et al* [6] and Schulze [20]. Other contributions are well referenced in the twpo referenced papers. One of the foci of this earlier work was on establishing the conditions where a symmetric or periodic structure could be isostatic (where there are no mechanisms or states or self-stress). This paper takes a different approach - its focus is on detecting and designing states of self-stress with the goal of increasing their number.

An engineer may want to know how many states of self-stress and how many mechanisms a given structural layout possesses, what they look like, and how to manipulate them. Singular value decomposition of the equilibrium or compatibility matrix [5] is not easy to do by hand and may not give the insight which the designer is seeking. Group theory can provide some insight into the symmetrical and antisymmetrical states of self-stress and mechanisms. Using information from group theory, one can perform a block diagonlisation on the equilibrium matrix and observe the impact of structural symmetry on the states of self-stress and mechanisms present within the framework.

States of Self-Stress Beyond Symmetry

As it is a projective property, the symmetry adapted count discussed in this paper detects only the states of self-stress that are associated with the symmetry. There are other special conditions which might lead to a greater number of states of self-stress [24]. Therefore, this paper presents a method which finds *symmetry detectable states of self-stress* rather than *all* states of self-stress. This paper restricts focus to 2D pin-jointed frameworks, but the methods presented can be extended to 3D trusses, such as space frames.

Theory

In this section, the underlying theory of the symmetry adapted count is developed. It is based on an area of mathematics called group theory. The resulting counts, as shown in Table 1, and its implications are discussed on Page 5. It is possible to skip the next section, which provides background information only, and just use the results.

Group Theory

The symmetry adapted counting rules are derived from group theory. McWeeny [14] provides a good introduction to group theory and presents applications of it. This section tries to derive the counts with minimal use of group theory terminology. A group is a fundamental algebraic structure which can be used to formalise the notion of symmetry mathematically (see Page 11 for a definition). Since symmetry is a ubiquitous concept in mathematics and in the applied sciences, group theory is a very large and well-studied mathematical field.

A symmetry group is a collection of symmetry operations (see [14] or Page 11 for the detailed mathematical definition). A symmetry operation is a mirror reflection or rotation of the framework which yields a framework with an identical geometry. For example, the framework in Figure 7 has reflection symmetry in a vertical (σ_v) and horizontal mirror (σ_h), and a 180° (C_2) rotation symmetry. Together with the trivial identity operation E, this forms a common symmetry group with four symmetry operations labelled $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$. A group is an unordered collection of operations and is similar to a set, hence the use of curly brackets $\{\}$. Examples of symmetry operations are given in Figure 3. In this paper, all groups are labelled G.

All symmetry operations can be thought of through transformation matrices, as well as through their physical meaning. The identity operation, E, is effectively a zero degree rotation as the transformation matrix is just the identity matrix, $\mathbf{I}$. The framework maps to itself under this operation. This is the most fundamental of the transformations and must be included in every group of symmetries.

Each of these symmetry operations has a symbol (σ for reflection or C_2 for 180° rotation) and a 'count' that is given in Table 1. The framework also has the Maxwell count, k, regardless of the symmetry properties of the framework. This basic property can be viewed as a count associated with the zero degree rotation, or the identity operation, and is labelled E. This combines with the symmetry operations of a framework to form a symmetry group, G.

A count (number) is associated with each symmetry operation. These are combined into an array of counts, as in Table 1. If there are no symmetries (other than the identity operation, E), then the only count is the Maxwell-Calladine count. For each additional symmetry operation, there is an additional count, as given in Table 1.

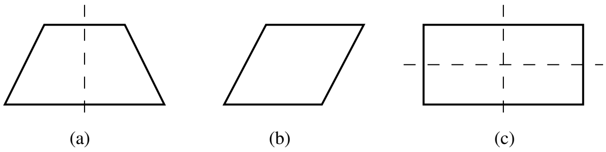

Figure 3. Some symmetry operations. (a) Isosceles trapezoid has reflectional symmetry $G = \{E, \sigma_v\}$. (b) Parallelogram has C_2 symmetry (180° rotation) $G = \{E, C_2\}$.
(c) Rectangles have $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$ symmetry. Higher order symmetries, such as that in a regular hexagon, can be found but this paper focuses on smaller sets of symmetry operations as the ideas easily extend to cover these larger symmetry groups.

Before developing the idea of the symmetry adapted count, it is worthwhile to examine the Maxwell-Calladine count, $2v - b - r = m - s$. The Maxwell-Calladine count is derived from considerations of the equilibrium matrix, $\mathbf{A}$ [5]. The equilibrium matrix relates to bar forces via $\mathbf{A}\vec{T} = \vec{P}$ where $\vec{T}$ is a vector of bar forces and $\vec{P}$ is a vector of applied nodal loads. The equilibrium matrix, $\mathbf{A}$, is of size $(2v - r) \times b$ and has a rank of R. Therefore, the size of the space of states of self-stress is $s = b - R$ as they lie in the null-space of $\mathbf{A}$. Similarly, the size of the space of mechanisms is $m = 2v - r - R$ as they lie in the left-null-space of $\mathbf{A}$. Substituting to eliminate R gives the count $2v - b - r = m - s$. The Maxwell-Calladine count, $m - s = 2v - b - r$ is adapted with $r = 3$ to give $k = 2v - b - 3$, that is the count for the identity operation, E.

The accompanying paper [22] describes the derivation of the symmetry adapted count using a block diagonalisation of the equilibrium matrix. This paper tries to explain this in a simple way. Any matrix is written with respect to a fixed choice of basis vectors, or coordinate system. The equilibrium matrix is normally written in terms of the standard xy coordinate system and the force in each bar taken one at a time, but can be rewritten in a different coordinate system which is based on symmetry and antisymmetry. This then gives the block diagonalisation. For example, consider two bars that are images of each other under a reflection symmetry, as shown in Figure 4. The basis vectors for the axial forces would normally be $[1, 0]^T$ and $[0, 1]^T$, but they are rewritten as $[1, 1]^T$ and $[1, -1]^T$ to leverage symmetry. Note these vectors remain linearly independent. Such an example is given in Figure 4 where each value could be considered the force in the bar. An example based on a Desargues framework is given on Page 12. A similar basis change can be applied to the mechanism space (this is explored more in the accompanying paper [22]). This block diagonalisation of the equilibrium matrix was first described by Kangwai and Guest [10] (see also [19] and [17]). This block diagonalised matrix can be found through one of two methods; it can be found computationally, as in Kangwai and Guest [10], or one can manually write down the symmetry based basis sets and construct the equilibrium matrix directly from this. Further discussion of the block diagonalised matrix using a Desargues framework as an example is given on Page 12.

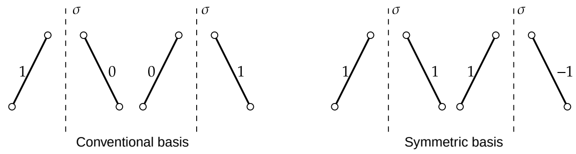

Figure 4. A set of conventional basis vectors and the symmetry based basis vectors.

Once the equilibrium matrix is rewritten in this form, it is block diagonalised and one can consider the rank of each block of the equilibrium matrix [19]. Each block is related to a particular type of symmetry. A count related to the size of each block is then combined to form an array of counts, γ, which is called the symmetry adapted count (this is similar to the rank argument used by Calladine [5]).

Symmetry Adapted Counts

For each symmetry operation, it is possible to write down a number (from a counting system). A 2D structure can have symmetries based on reflection operations or rotational operations. Reflectional symmetries are labelled σ (σ_v is a vertical mirror and σ_h is a horizontal mirror in this paper), a rotational symmetry of 180° is labelled C_2, and all other rotational symmetries by repeated rotation ϕ are labelled C_n (rotation by $\phi = \frac{2\pi}{n}$ - for example, a 90° rotation symmetry is labelled C_4 as four of these operations returns it to its original state). The identity operation, E, leaves the framework unchanged and its count is the Maxwell count, $k = 2v - b - 3$. This is only related to topology (the number of interconnecting nodes and bars) and is independent of any other symmetries. The array of symmetry adapted counts is labelled γ. The length of this array is equal to the number of symmetry operations (including E).

The terms in the counts are defined by:

- v is the total number of nodes.
- v_c is the number of nodes lying on the centre of rotation.
- v_σ is the number of vertices lying on a given mirror.
- b is the total number of bars.
- b_2 is the number of bars left unshifted by a C_2 symmetry operation. Such a bar must have its midpoint lying at the centre of rotation.
- b_σ is the number of bars left unshifted by a reflection. These bars must lie within the mirror plane or be a perpendicular bisector of the mirror plane.

Some notes on the rotational symmetry:

- The centre of rotation is the same for every rotational symmetry.
- v_c can only be 0 or 1 as nodes cannot be coincident.
- b_2 can only be 0 or 1 as this paper only considers planar graphs.
- v_c and b_2 cannot both be equal to 1 as this paper only considers planar graphs.

Concept	Identity operation	n-fold rotational symmetry	180° rotational symmetry	Mirror symmetry
Symbol	E	C_n	C_2	σ
γ	$2v - b - 3$	$2(v_c - 1)\cos\phi - 1$	$-2v_c - b_2 + 1$	$-b_\sigma + 1$

Table 1. Symmetry adapted counts, γ, for all possible symmetry operations in the plane.

- It is not possible for a bar to be unshifted by a rotational symmetry that is not 180°. Therefore, for a C_n symmetry operation $b_n = 0$.
- If $b_2 = 1$, then it is not possible to have any rotational symmetry other than C_2.

Some rotational operations introduce complex numbers in γ which are avoided in this paper for the sake of simplicity. This is a consequence of the $\cos\phi$ term which can give a non-integer value. Note that the coefficients of the symmetry adapted count basis vectors, β_i, described below, are necessarily integer values even if the count terms are not.

Using the Symmetry Adapted Count

The symmetry adapted count, γ, gives an array of numbers or counts. There is one entry for each symmetry operation, including E. For example, Figure 6(b) has $G = \{E, \sigma_h\}$ and $\gamma = (2v - b - 3, -b_\sigma + 1) = (2 \times 6 - 9 - 3, -3 + 1) = (0, -2)$. It can be shown that this array is a linear combination of *symmetry adapted count basis vectors* where the coefficient of each symmetry adapted count basis vector has an integer value. These symmetry adapted count basis vectors are *irreducible characters* in group theory.

The counting rule gives an array of counts, γ; group theory literature labels this array $\gamma = \Gamma(m) - \Gamma(s)$, but this paper has labelled the count γ for the sake of brevity. A lot of information can be gleaned by rewriting γ as a linear combination of symmetry adapted count basis vectors. This is essentially a change in basis vectors. Tables of symmetry adapted count basis vectors, also called irreducible characters, can be found in [1] [2] and the coefficients appearing in the linear combination of these vectors can be found through a simple formula. The examples given on Page 6 show how this can be done. From here on, irreducible characters are called symmetry adapted count basis vectors as this more accurately describes their role in this paper. It is worth noting the symmetry adapted count basis vectors of the group which contains E and only one other symmetry operation - these vectors are shown in Table 2 where A_1 is symmetric and A_2 is antisymmetric (see Figure 5). Finding the coefficients, β_i, can be done through multiple means, including by considering simultaneous equations. In this paper, all symmetry adapted count basis vectors are, for simplicity, labelled $A_1, A_2, ..., A_i$ in contrast to other notations. The reason for introducing these symmetry adapted count basis vectors is because the count, γ, can be rewritten in terms of them with the coefficients, β_i, providing information on the number of mechanisms and states of self-stress, as discussed later in this paper.

$$\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_N) = \beta_1 A_1 + \beta_2 A_2 + \ldots + \beta_j A_j \quad (1)$$

Each symmetry adapted count basis vector can be thought of as a $1 \times N$ array, matrix, or row vector, where N is the number of symmetry operations, including E. These row vectors form a basis of a certain vector space (so called class-functions). Each of these basis vectors describes a fundamental pattern that axial forces or displacement vectors at the nodes of the framework may exhibit with respect to the symmetry operations. Note that these symmetry adapted count basis vectors are independent and necessarily orthogonal.

Each symmetry adapted count, γ, lies in this space and so can be written uniquely as a linear combination of these symmetry adapted count basis vectors. The coefficient of A_i, here labelled β_i, is $\frac{1}{N}\gamma \cdot A_i$ where $\frac{1}{N}$ normalises A_i and $\cdot$ is the dot product of the two arrays γ and A_i. It is important to note that γ can have non-integer entries but when written as a linear combination of symmetry adapted count basis vectors, the coefficients, β_i, are always integers. It is these coefficients that provide information on the number of states of self-stress and mechanisms. This is why the count, γ, is typically written as $\Gamma(m) - \Gamma(s)$ in the rigidity theory literature.

The coefficients, β_i, can be rewritten as $m_i - s_i$. This is because the coefficient, β_i, gives the *difference* between the number of mechanisms and number of states of self-stress. Whilst this is rarely explicitly written, it is an important feature of this symmetry adapted count. m_i is the number of mechanisms of symmetry type A_i and s_i is the number of states of self-stress of symmetry type A_i. The reason for writing $\beta_i = m_i - s_i$ instead of $\gamma = \Gamma(m) - \Gamma(s)$, which is common in previous literature, is because the number of mechanisms and states of self-stress is described by the coefficients and not the array.

If a coefficient, β_i is negative then it indicates a minimum number of states of self-stress. Similarly, if it is positive then it indicates a minimum number of mechanisms. Say $\beta_1 = -3$; this would indicate the presence of at least 3 states of self-stress which are A_1 symmetric. Say $\beta_2 = +2$; this would indicate the presence of at least 2 mechanisms which are A_2 symmetric. For $G = \{E, \sigma\}$ and using Table 2, A_1 is symmetric and A_2 is antisymmetric (see Figure 5). For symmetric states of self-stress, axial force terms are preserved by symmetry. For symmetric mechanisms, the

	E	σ or C_2
A_1	1	1
A_2	1	-1

Table 2. Symmetry adapted count basis vectors for a group with E and one other symmetry operation.

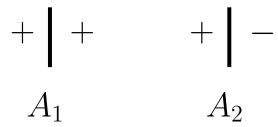

Figure 5. The symmetry conditions of the symmetry adapted count basis vectors in Table 2

magnitude of the velocity vectors are preserved by reflection and the direction is mirrored.

It is possible to obtain structures which contain a symmetric state of self-stress and a symmetric mechanism (this is related to the Maxwell-Calladine count of $2v - b - 3 = m - s$; note the minus sign on s and the plus sign on m). The coefficient, β_i, actually counts the number of mechanisms minus the number of states of self-stress for symmetry type A_i (note $\beta_i = m_i - s_i$). This can lead to frameworks where $\gamma = +A_1 - A_1 = 0$. Therefore, the mechanism and state of self-stress cannot be immediately detected using this count. One way to determine the exact values of s and m is to investigate the rank of the equilibrium matrix, $\mathbf{A}$. However, this does not aid the designer in obtaining geometries with additional states of self-stress. Similarly, it does not help to design symmetric and antisymmetric states of self-stress which are desirable in gridshells, as described on Page 8.

Note that the count for the identity operation, E, returns the count $2v - b - 3 = k = m - s$. Therefore, the sum of all coefficients is equal to k (succinctly written as $\sum_i \beta_i = k$). The sum of all negative coefficients gives the total number of symmetry detectable states of self-stress. Similarly, the sum of all positive coefficients gives the total number of symmetry detectable mechanisms. Therefore, the Maxwell-Calladine count is contained within the symmetry adapted count, but this count may find additional states of self-stress and mechanisms and provides information on the type of self-stress or mechanism present.

Examples

This section introduces a number of examples of how the count can be used to detect and design symmetric and antisymmetric states of self-stress and mechanisms.

Desargues Configuration

The Desargues configuration involves two triangles connected by three straight bars in a 'special' geometry such that it contains $m = s = 1$. As noted on Page 2, this special geometry is achieved if the 2D configuration is a projection of a 3D plane-faced polyhedron. By enforcing a horizontal mirror symmetry on the topology, a Desargues configuration is necessarily obtained.

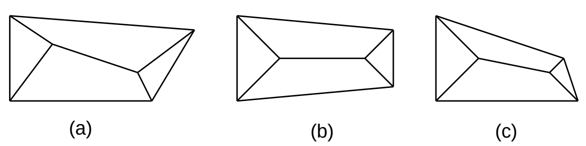

Figure 6. The Desargues configuration. (a) Geometry with $m = s = 0$ - this is not a Desargues configuration despite having the same topology. (b) Symmetric configuration (horizontal mirror, σ_h), necessarily with $s = 1$. (c) Configuration with $s = 1$ which is not symmetric.

The graph of the Desargues configuration satisfies $2v - b - 3 = 0$ and since for the framework in Figure 6(b) there are exactly three bars that are unshifted by the horizontal reflection, we have $b_{\sigma_h} = 3$. Figure 6(b) has the symmetry group $G = \{E, \sigma_h\}$ and the symmetry adapted count is given in Equation (2). There is only one line of symmetry so the symmetry adapted count basis vectors of Table 2 are used (see also Figure 5). There is at least one symmetric state of self-stress (since $\beta_1 = -1$) and at least one antisymmetric mechanism (since $\beta_2 = +1$). Thus, the simple calculation in Equation (2) shows that there is at least one symmetric state of self-stress.

$$\begin{aligned} G &= \{E, \sigma_h\} \\ \gamma &= (2v - b - 3, -b_{\sigma_h} + 1) \\ &= (0, -2) \\ &= \beta_1 A_1 + \beta_2 A_2 \\ &= \beta_1(1,1) + \beta_2(1,-1) \\ &= -A_1 + A_2 \end{aligned} \tag{2}$$

A further discussion of the Desargues configuration is given on Page 12.

Rectangular Boundary

Consider the grid for a rectangular boundary shown in Figure 7. The triangulation around the perimeter acts like a truss and allows forces from a state of self-stress to be equilibriated. This has two perpendicular mirror lines and a C_2 symmetry (this 180° symmetry is the result of the two mirror symmetries) - this set of symmetries, which is very common and important, is referred to as a $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$ (Schoenflies notation). There are four symmetry adapted count basis vectors, as given in Table 3. The meaning of each of the symmetry adapted count basis vectors is shown in Figure 8. The symmetry adapted count is given in Equation (3). In this example, engineering intuition shows that $m = 0$ so it is known that all states of self-stress have been detected. Furthermore, because of the symmetry adapted count the engineer knows more information on the states of self-stress in the framework - 5 are fully symmetric (A_1 type - doubly symmetric), 4 are antisymmetric about both mirrors (A_2 type), 4 are antisymmetric about the vertical mirror only (A_3 type), and 4 are antisymmetric about the horizontal mirror only (A_4 type).

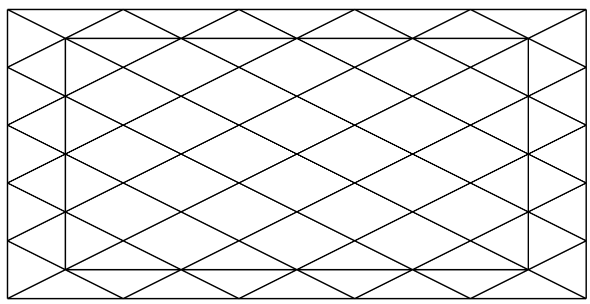

Figure 7. Grid for a rectangular boundary. The count follows from $v_c = 1$, $b_2 = 0$ and $b_{\sigma_h} = b_{\sigma_v} = 2$.

$$\begin{aligned} G &= \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\} \\ \gamma &= (2v - b - 3, -2v_c - b_2 + 1, -b_{\sigma_h} + 1, -b_{\sigma_v} + 1) \\ &= (-17, -1, -1, -1) \\ &= -5A_1 - 4A_2 - 4A_3 - 4A_4 \end{aligned} \tag{3}$$

	E	C_2	σ_h	σ_v
A_1	1	1	1	1
A_2	1	1	-1	-1
A_3	1	-1	1	-1
A_4	1	-1	-1	1

Table 3. Symmetry adapted count basis vectors of the group $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$.

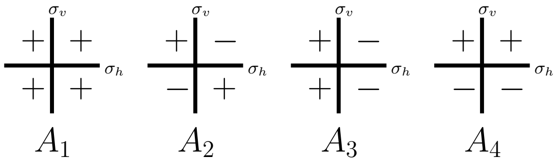

Figure 8. The symmetry conditions of the symmetry adapted count basis vectors in Table 3.

The authors note that it is possible to consider many symmetry groups and the symmetry adapted count basis vectors (irreducible characters) for each symmetry group can be found in [1] [2]. This paper does not go beyond three symmetry operations, in addition to the identity operation E, so as to demonstrate the simplicity of the method. Examples containing a greater number of symmetry operations is given in other papers [8] [22].

A quad-dominant gridshell

Consider the 2D projection of a 3D quad-dominant gridshell shown in Figure 9. It has a horizontal mirror symmetry, σ_h, a vertical mirror symmetry, σ_v, and a C_2 symmetry (this is again the $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$ symmetry group). The count is given in Equation (4).

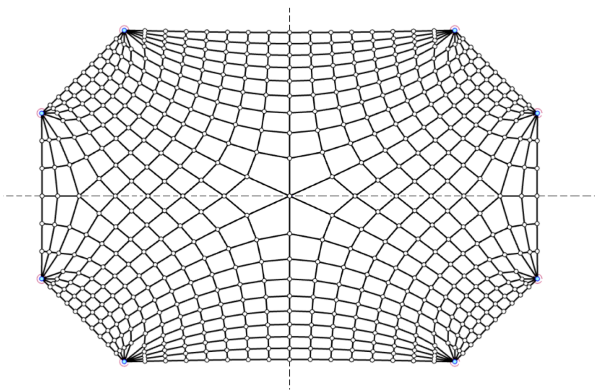

Figure 9. Gridshell roof layout with a horizontal mirror, σ_h, a vertical mirror, σ_v and C_2 symmetry. Symmetry group $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$.
(The structure has $v = 561$, $b = 1102$, $v_c = 1$, $b_2 = 0$, $b_{\sigma_h} = 4$, $b_{\sigma_v} = 18$.)

$$\begin{aligned} G = \mathcal{C}_{2v} &= \{E, C_2, \sigma_h, \sigma_v\} \\ \gamma &= (2v - b - 3, \quad -2v_c - b_2 + 1, -b_{\sigma_h} + 1, -b_{\sigma_v} + 1) \\ &= (2 \times 561 - 1102 - 3, -2 - 0 + 1, -4 + 1, -18 + 1) \\ &= (17, -1, -3, -17) \\ &= -A_1 + 9A_2 + 8A_3 + A_4 \end{aligned} \tag{4}$$

The symmetry adapted count detects at least one state of self-stress which is fully symmetric (A_1 type). This framework was form-found using the force density method [18] so it is already known that it possessed at least one fully symmetric state of self-stress. This symmetry adapted count verifies this observation. Note that there are some 'T' connections along the structural perimeter. These necessarily are zero force members and could be removed from the framework during analysis.

If a designer wishes to increase the number of states of self-stress (in order to increase the nodal load cases which can be taken with only axial forces), one can add bars along the line of the horizontal mirror symmetry, as shown in Figure 10. The new counts are shown in Equation (5). The revised structure has at least 6 fully symmetric (A_1 type) states of self-stress, as opposed to 1 previously. It is noted that although the new bars create additional triangular panels, the nodes are still not twisted because they occur on a line of symmetry.

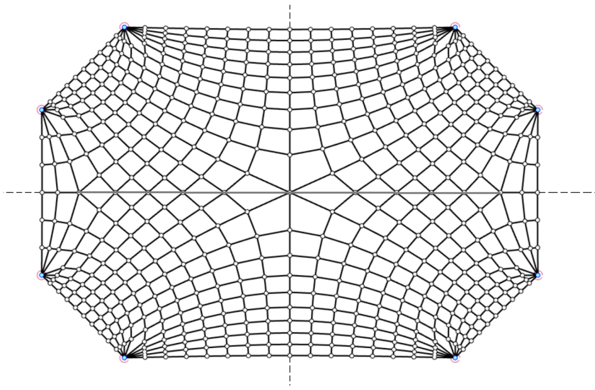

Figure 10. Modified gridshell layout based on Figure 9. The new bars are shown in grey.
(The structure has $v = 561$, $b = 1112$, $v_c = 1$, $b_2 = 0$, $b_{\sigma_h} = 14$, $b_{\sigma_v} = 18$.)

$$\begin{aligned} G = \mathcal{C}_{2v} &= \{E, C_2, \sigma_h, \sigma_v\} \\ \gamma &= (2v - b - 3, \quad -2v_c - b_2 + 1, -b_{\sigma_h} + 1, -b_{\sigma_v} + 1) \\ &= (2 \times 561 - 1112 - 3, -2 - 0 + 1, -14 + 1, -18 + 1) \\ &= (7, -1, -13, -17) \\ &= -6A_1 + 9A_2 + 3A_3 + A_4 \end{aligned} \tag{5}$$

Increasing the number of states of self-stress

One aim of this paper is to present a simple method based on the symmetry adapted count through which the number of states of self-stress can be increased. Each type of symmetry will be considered separately.

The authors note that equilibrium as well as infinitesimal and static rigidity are projectively invariant. That is, for a 2D pin-jointed truss, an affine or projective transformation preserves m and s, as discussed on Page 2. Therefore, one can design a highly symmetric geometry which possesses many states of self-stress and then project it to obtain a different geometry with the same number of states of self-stress. The symmetries might be destroyed by the transformation, but this might not be important to the designer (in fact, the designer may want to destroy certain symmetries). Larger groups of symmetry operations can help to detect more states of self-stress and more mechanisms which are not otherwise detected - this is discussed in greater depth in Schulze *et al* [22]. In that paper, an example with higher order symmetry; 4 mirror symmetries and 90° rotational symmetry - $G = \mathcal{C}_{4v} = \{E, 2C_4, C_2, 2\sigma_v, 2\sigma_d\}$ in Schoenflies notation is given which is then transformed into a framework with only $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$ symmetry. Note the $2C_4$ because there is a $90°$ and $270°$ rotational symmetry with the same associated count. Similarly, there are two diagonal mirrors, σ_d, with the same count and a vertical and horizontal mirror with the same count, hence $2\sigma_v$ and $2\sigma_d$. The integers m and s are preserved, but the count of the new framework may not be able to detect all the states of self-stress previously detected. Therefore, working with frameworks with greater levels of symmetry can sometimes detect more states of self-stress and more mechanisms.

Reflection Symmetry: σ

For a reflection symmetry operation, σ, the count is shown in Equation (6). Here, A_1 is the symmetric and A_2 is the antisymmetric symmetry adapted count basis vector.

$$\gamma = (k, -b_\sigma + 1) = \frac{k - b_\sigma + 1}{2} A_1 + \frac{k + b_\sigma - 1}{2} A_2 \quad (6)$$

k	b	b_σ
Even	Odd	Odd
Odd	Even	Even

Table 4. Table of bar counts for mirror symmetry.

Table 4 gives information on how the number of bars must be arranged for a framework with reflectional symmetry.

Assume that k is fixed by a chosen topology. To increase the number of symmetric states of self-stress, then one must revise the geometry to increase b_σ to make the coefficient of A_1 more negative. In turn, this creates more antisymmetric mechanisms. For each symmetric state of self-stress gained, an antisymmetric mechanism accompanies it. This maintains the overall Maxwell-Calladine count. If one wants more antisymmetric states of self-stress, then b_σ should be kept small to make the coefficient of A_2 negative. It is not possible to detect an antisymmetric state of self-stress unless $k \leq -1$. When one performs a symmetry extended count on a framework, not only is it possible to detect additional states of self-stress and mechanisms beyond the Maxwell-Calladine count, but information on the symmetry properties of the states of self-stress and mechanisms is also readily obtained.

As an example, consider the frameworks in Figure 11. Both have mirror symmetry and $G = \{E, \sigma_v\}$. (a) has been designed so that $b_{\sigma_v} = 0$ and $k = -1$. By Equation (6), one obtains $\gamma = 0A_1 - A_2$ and so it contains one antisymmetric state of self-stress and no mechanisms. In contrast, (b) has $b_{\sigma_v} = 4$ and $k = -1$ (same underlying topology). Therefore, $\gamma = -2A_1 + A_2$ so it contains at least two symmetric states of self-stress and at least one antisymmetric mechanism. This shows that increasing the number of unshifted bars in a framework increases the number of symmetric states of self-stress and the number of anti-symmetric mechanisms, for a fixed value of k.

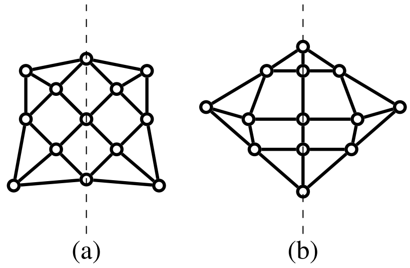

Figure 11. Reflection-symmetric frameworks with an anti-symmetric self-stress (a) and fully-symmetric self-stresses (b). Note that (b) has four bars that are unshifted by the reflection, whereas (a) has none. Note that the two frameworks have the same topology (rotate (b) anticlockwise $45°$ and compare).

Rotational Symmetry: C_2 & C_n

Table 5 gives information on how the number of bars must be arranged for a framework with rotational symmetry. For a C_2 symmetry operation ($180°$ rotation), Table 6 is obtained. Again, A_1 is symmetric and A_2 is antisymmetric.

k	b	b_2
Even	Odd	1
Odd	Even	0

Table 5. Table of bar counts for rotational symmetry.

	$v_c = 0$	$v_c = 1$
$b_2 = 0$	$\frac{k+1}{2} A_1 + \frac{k-1}{2} A_2$	$\frac{k-1}{2} A_1 + \frac{k+1}{2} A_2$
$b_2 = 1$	$\frac{k}{2} A_1 + \frac{k}{2} A_2$	

Table 6. C_2 symmetry count

If one wants to increase s, then one must make k more negative. It turns out that for C_n, with $n \geq 3$, the symmetry adapted count does not reveal any self-stresses in addition to the ones that are detected with the standard Maxwell-Calladine count, but additional information on the symmetry type is obtained (see [22] for details).

Designing Symmetric and Antisymmetric states of self-stress

Layouts for gridshells possessing both symmetric and antisymmetric states of self-stress can be desirable, as discussed on Page 1. It is often a design preference to have a symmetric 'spider-net' state of self-stress where all interior members have forces of the same sign. This corresponds to a compression-only gridshell. Antisymmetric states of self-stress can be hard to design as any bar which is bisected by a mirror line must have zero force.

An example of how to design this is given below for a framework with a horizontal mirror, σ_h, vertical mirror, σ_v, and C_2 symmetry (symmetry group $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$). The count is given in Equation (7). The symmetry adapted count basis vectors are given in Table 3. The count expressed as a linear combination of symmetry adapted count basis vectors is given in Table 7.

$$\gamma = (2v - b - 3, -2v_c - b_2 + 1, -b_{\sigma_h} + 1, -b_{\sigma_v} + 1) \tag{7}$$

An analysis of Table 7 shows that to increase the number of symmetric states of self-stress for any mirror, one should increase the number of unshifted bars associated with that mirror (this complements the discussion on Page 8). This has the impact of introducing more mechanisms of type A_2 (antisymmetric about each mirror but rotationally symmetric). The meaning of each symmetry adapted count basis vector is shown in Figure 8. Most pattern loads on gridshells relate to A_3 and A_4 and not A_2 so focus is given to the coefficients of these. Table 8 gives information on how the number of bars must be arranged for a framework with $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$ symmetry.

Assume that $b_{\sigma_h} \geq b_{\sigma_v}$ for this section. If one were to increase the value of $b_{\sigma_h} - b_{\sigma_v}$, then one obtains more antisymmetric states of self-stress about the vertical mirror (A_3 type) and more antisymmetric mechanisms about the horizontal mirror (A_4 type). It is often preferable to maintain a given topology and, therefore, k is fixed. One can therefore design frameworks by placing bars and nodes on lines of symmetry as desired. For example, consider the problem where one wants to obtain at least one fully symmetric state of self-stress and at least one antisymmetric state of self-stress for each mirror whilst maintaining k as positive as possible. Assuming $v_c = 0$ and $b_2 = 0$, then $b_{\sigma_h} = b_{\sigma_v}$ and $k = -3$ to give $\beta_3 = \beta_4 = -1$ (see the expressions for the coefficients, β_i, shown in the Table 8). Increasing b_{σ_h} (and b_{σ_v}) will make β_1 more negative and thus produce more fully symmetric states of self-stress. This is discussed in more detail in the accompanying paper [22].

Limitations in the Design of Gridshells

Not all states of self-stress are detectable using these counts; only *symmetry detectable* ones are. There are special conditions which can lead to a greater number of states of self-stress [24]. This is because states of self-stress relate to the projection of plane-faced polyhedra [4] and not symmetry.

Methods beyond symmetry

As has previously been discussed, this method does not detect all states of self-stress, nor all symmetric states of self-stress. The symmetry adapted count only detects states of self-stress which exist *because* of a relationship to symmetry. For example, consider the example shown in Figure 12 which consists of two frameworks 'glued' together. By inspection, this framework has two states of self-stress, one symmetric and the other antisymmetric. The count, given in Equation (8), does not detect either state of self-stress. This is because the states of self-stress are not related to the symmetry of the framework. In practice, $\beta_1 = -1 + 2$ (so $s_1 = 1$ and $m_1 = 2$) and $\beta_2 = -1 + 2$ (so $s_2 = 1$ and $m_2 = 2$). Similarly, states of self-stress which lie within a portion of the framework (and are then replicated by symmetry) will not be detected.

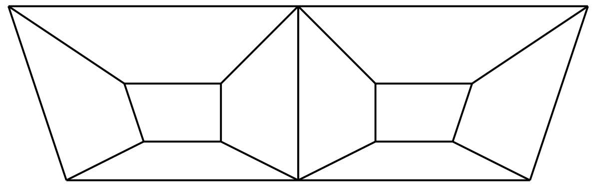

Figure 12. Two frameworks 'glued' together. $G = \{E, \sigma_v\}$. Note that no states of self-stress are detected by the symmetry adapted count even though $s = 2$. The count gives $b_{\sigma_v} = 1$.

$$\begin{aligned} G &= \{E, \sigma_v\} \\ \gamma &= (2v - b - 3, -b_{\sigma_v} + 1) \\ &= (2, 0) \\ &= A_1 + A_2 \end{aligned} \tag{8}$$

One can create a framework with many states of self-stress by 'gluing' primitive frameworks together, as in Figure 12. This can create symmetric and antisymmetric states of self-stress as needed. However, the symmetric state of self-stress may not be a 'spider web' in that the interior bars may have forces of varying signs and, therefore, it might not be useful in the design of a gridshell which tend to be compression only, where possible.

Future Work

This methodology provides tools for a designer to increase the number of states of self-stress and to design states of self-stress which are symmetric or antisymmetric. This has direct applications in gridshell design. However, there are still avenues for future work.

- How to maximise the number of states of self-stress, s, for a given topology (without changing the graph connectivity) under the restriction of non-degeneracy of the framework. There are multiple avenues to explore related to this: maximising the number of states of self-stress or planar liftings, maximising the number of mechanisms or parallel redrawings, or maximising the decomposability of the discrete Airy stress function polyhedron [21]. The authors note the results of Smilansky [23] whose plot of decomposability directly aligns with the Maxwell-Calladine count for 2D frameworks.

- The special projective conditions which provide additional states of self-stress have been studied [24]. An area for future research is to expand the knowledge and understanding of these special conditions so that one can design them into frameworks, if desired.

- For structures containing multiple states of self-stress, it is desirable to be able to perform subdivisions without losing the states of self-stress. Therefore, during the subdivision linearly independent states of self-stress should not be directly connected. Further development and research into this is left to future work.

Conditions	γ
$v_c = 0$ & $b_2 = 0$	$\frac{k-b_{\sigma_h}-b_{\sigma_v}+3}{4}A_1 + \frac{k+b_{\sigma_h}+b_{\sigma_v}-1}{4}A_2 + \frac{k-b_{\sigma_h}+b_{\sigma_v}-1}{4}A_3 + \frac{k+b_{\sigma_h}-b_{\sigma_v}-1}{4}A_4$
$v_c = 0$ & $b_2 = 1$	$\frac{k-b_{\sigma_h}-b_{\sigma_v}+2}{4}A_1 + \frac{k+b_{\sigma_h}+b_{\sigma_v}-2}{4}A_2 + \frac{k-b_{\sigma_h}+b_{\sigma_v}}{4}A_3 + \frac{k+b_{\sigma_h}-b_{\sigma_v}}{4}A_4$
$v_c = 1$ & $b_2 = 0$	$\frac{k-b_{\sigma_h}-b_{\sigma_v}+1}{4}A_1 + \frac{k+b_{\sigma_h}+b_{\sigma_v}-3}{4}A_2 + \frac{k-b_{\sigma_h}+b_{\sigma_v}+1}{4}A_3 + \frac{k+b_{\sigma_h}-b_{\sigma_v}+1}{4}A_4$

Table 7. Symmetry count, γ, for a framework with $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$ symmetry.

k	b_2	b_{σ_h}	b_{σ_v}
Even	1	Odd	Odd
Odd	0	Even	Even

Table 8. Table of bar counts for $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$ symmetry.

- McRobie *et al* [13] describes the relationship between mechanisms and states of self-stress in the dual form and force diagrams. An investigation into this relationship with an emphasis on symmetry could yield interesting results.

Discussion of states of self-stress and mechanisms is common within the field of graphic statics [13]. Graphic statics relies upon the reciprocal relationship between the form diagram, χ, and the force diagram, χ^*, which describe the structural form and forces within the structure respectively. The number of mechanisms, m^*, and states of self-stress, s^*, in the reciprocal diagram are related to those in the original diagram via $s = m^* + 1$ and $s^* = m + 1$ [13]. It is sometimes easier to design the reciprocal force diagram than the form diagram of the structure.

What is the topological maximum number of states of self-stress?

Given a self-stressable 2D (planar) framework, one can lift it to form a 3D plane-faced polyhedron. Defining the first three points on a single face will position the plane in 3D space. The z-coordinates of all nodes of the face will be then known. Defining an additional node belonging to a different face with known elevations of 2 nodes will position the plane for that face in space. Defining elevation of a single node may define more than one plane. If not all planes for the faces are defined, an elevation of an additional node is required. Each of these additional points can correspond to an additional state of self-stress. However, completing the definitions of all planes one may encounter planes defined by 4 or more known nodes. These situations, that we call “conflicts”, possibly reduce the number of states of self-stress. Each conflict gives an additional condition which may be satisfied by making a lift node dependant on others (thereby reducing the number of states of self-stress), or by moving nodes to special locations (generating an additional mechanism in the process). The number of possible states of self-stress and conflicts depends on the selection of the nodes for which we define z-coordinates. It is not possible to define more independent states of self-stress than the minimum number of nodes, reduced by 3, needed to be defined in order to obtain all faces of the polyhedron. Therefore, this method gives us an upper bound on the number of states of self-stress.

Let the number of points which need to be defined be $d + 3$ and the number of conflicts be c. The upper-bound maximum number of states of self-stress is $s_{max} = d$, although it may not be possible to achieve this. The lowest possible number of states of self-stress is $s_{min} = d - c$ (note that $s \geq 0$). An example is shown in Figure 13 [3]. The authors note that it is always possible to get one state of self-stress for any topology with a restraining frame using the force density method [18].

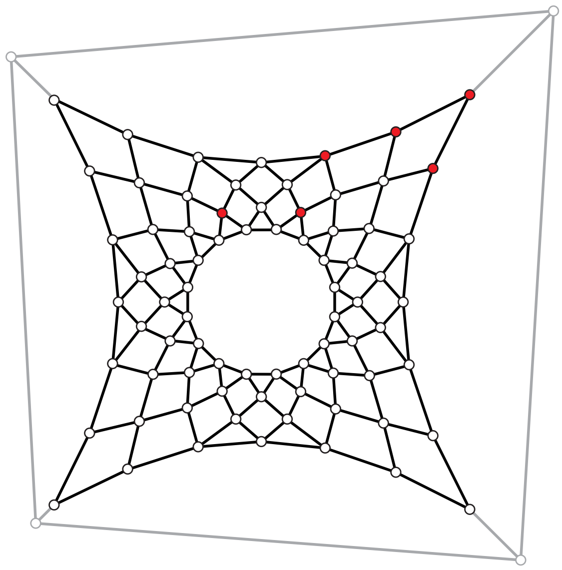

Figure 13. The spider web geometry shown requires a total of six points (shown in red with $d = 3$) to be defined in order to know the full polyhedron, but there are also seven conflicts. Therefore, the net has an upper-bound maximum of three states of self-stress. This spider web geometry was first discussed by Baker *et al* [3].

Conclusions

This paper introduced the symmetry adapted counting rule with the aim of making it simple and avoiding extraneous details. Examples using this counting rule were then given, including a focus on the gridshell design. Discussion of how structures can be *designed* using the information contained within the counting rule followed these examples. The methods presented are easy to use, as they rely only on counting nodes and bars, but also provide more information than the standard Maxwell-Calladine count. It is noted that by placing more bars on a line of symmetry, the number of states of self-stress in a framework can be increased. By increasing the accessibility of these counting rules, it is hoped that more engineers will understand how states of self-stress and mechanisms manifest themselves in symmetric frameworks and how they can be used in the design of structural layouts.

Acknowledgements

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Glossary

- **Graph** - A mathematical structure of a set of nodes connected by edges. Such graphs are often used to represent pin-jointed frameworks with bars represented by straight edges.
- **Planar graph** - A graph which can be embedded in the plane (it can be drawn in a way so that no two edges cross each other).
- **Group** - A group is a set of objects, G, together with an operation $\circ$ so that for any two elements $a, b \in G$, then $a \circ b$ is again in G. The operation $\circ$ satisfies the associativity law (brackets can be placed arbitrarily). Moreover, each group has a special element e, called the identity element, which has no effect on any of the elements of G under $\circ$, that is $a \circ e = e \circ a = a$ (the identity operation for symmetry is defined below). Each element a of G also has an inverse a' in G such that $a \circ a' = a' \circ a = e$ (note that a' may be written as a^{-1} in other papers).
- **Symmetry operation** - A symmetry operation of a 2D structure is an isometry (distance-preserving transformation) of the plane that leaves the structure invariant. A symmetry operation of a 2D structure is either a reflection or a rotation.
- **Symmetry group** - A symmetry group (sometimes also called a point group) of a 2D structure is a group G, where each element of G is a symmetry operation of the structure, and $\circ$ is the composition of symmetry operations.
- **Identity operation**, E - The identity element of a symmetry group is also called the identity operation. This is simply the identity map which maps every

point of the plane to itself. The identity operation is often denoted by E. It can be considered a zero degree rotation. Similarly, if symmetry transformations are considered through a transformation matrix, then the identity operation is the identity matrix, $\mathbf{I}$.

- **Group representation** - A group representation of a group G describes the group G by assigning an invertible matrix to each element of G.
- **Character of a group representation** - Given a group representation of G, the row vector of length $|G|$ that has the trace of the matrix corresponding to the ith element of G in the ith component is called the character of the representation. (Here $|G|$ denotes the number of elements in G.)
- **Irreducible characters** - Each symmetry group G has a set of irreducible characters (corresponding to the most basic group representations of G) which can be found in standard character tables [1] [2]. The character of any group representation of G can always be written uniquely as a linear combination (with integer coefficients) of the irreducible characters of G. These irreducible characters for a symmetry group are a set of linearly independent orthogonal vectors - in this paper they are referred to as the symmetry adapted count basis vectors.
- **Block diagonalised matrix** - This is a matrix that has been broken up into submatrices, where all submatrices are zero matrices, except possibly the ones along the diagonal.
- **Freedom number**, k - A count associated with the identity operation, E. It is essentially the Maxwell count with $r = 3$ so that $k = 2v - b - 3$. This is a fundamental part of the symmetry adapted count. This paper has focused on unpinned frameworks with $r = 3$, but the methods can easily be extended to consider pinned frameworks with $r > 3$, as discussed in the accompanying paper [22].
- The count, γ, is often written as $\gamma = \Gamma(m) - \Gamma(s)$ to indicate how it only detects the difference between the number of mechanisms and number of states of self-stress. In this paper, the difference is denoted by the coefficient of the symmetry adapted count basis vectors, β_i, which is then broken down to $\beta_i = m_i - s_i$.
- Schoenflies notation is often used to describe common symmetry groups such as $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$ and $G = \mathcal{C}_{4v} = \{E, C_4, C_2, C_4^2, \sigma_h, \sigma_{45}, \sigma_v, \sigma_{-45}\}$.

Block Diagonalised Kinematic Matrix for Desargues Framework

Here, the doubly symmetric ($G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$ symmetry) Desargues layout framework example shown in Figure 15 is considered. The kinematic matrix, $\mathbf{A}^T$, is formed based on a set of symmetry adapted basis. The left-hand column shows the member extensions basis and the right hand column shows the nodal displacement basis which are all of unit magnitude. Note that the same could be done for the equilibrium matrix, $\mathbf{A}$, using the same bases for external loads and internal forces. The equilibrium matrix is the transpose of the kinematic matrix [13] so it follows easily.

The Desargues example has $G = \mathcal{C}_{2v} = \{E, C_2, \sigma_h, \sigma_v\}$ symmetry, as discussed in on Page 6. As the framework is symmetric, a symmetric loading/displacement creates a similarly symmetric set of internal forces/extensions.

Once the basis sets have been written out in terms of symmetry, as in Figure 15, one can construct the corresponding kinematic matrix, as shown in Figure 14. This consists of ‘blocks’ for each symmetry type of the symmetry adapted count basis vectors (see Page 6 for the symmetry adapted count basis vectors and how they relate to the different symmetry operations). The top left block is rectangular and contains the single state of self-stress (extra row) found in this framework - this block relates to fully symmetric (A_1 type) axial forces and displacement vectors so shows that the state of self-stress is fully symmetric. The second block is also rectangular and contains the single mechanism (extra column) in the framework. The mechanism is of A_2 symmetry type (rotational), as expected. The third block (A_3 type) is square and has no obvious rank deficiencies meaning that it contains no mechanisms or states of self-stress. The fourth block (A_4 type) is rectangular (2×3 in size - column λ has no on-zero entries) and contains at least one mechanism - this mechanism is clearly λ as is has no non-zero entries in the matrix. This is shown in Table 9.

The symmetry adapted count gives Equation (9). This shows the state of self-stress of symmetry type A_1 and the mechanism present of type A_2. There is an additional mechanism of type A_4 (motion of the central node up and down). Note that $k = 1$ and this is why the rigidity matrix is of size 10×11. A more traditional analysis of a similar framework is discussed on Page 6.

$$\begin{aligned} G = \mathcal{C}_{2v} &= \{E, C_2, \sigma_h, \sigma_v\} \\ \gamma &= (2v - b - 3, -2v_c - b_2 + 1, -b_{\sigma_h} + 1, -b_{\sigma_v} + 1) \\ &= (1, -1, -3, -1) \\ &= -A_1 + A_2 + 0A_3 + A_4 \end{aligned} \tag{9}$$

The symmetry adapted count discussed in this paper finds the size of each of these submatrices. The size is related to how many basis vectors can be obtained for each symmetry type. As in this example, it is possible to create more fully symmetric sets of extensions than fully symmetric nodal motions so the submatrix associated with A_1 symmetry is rectangular. If one of the submatrices is rank deficient, then there is an additional state of self-stress and an additional mechanism but this is not detected by the symmetry adapted count - a full analysis of the equilibrium matrix is required.

Symmetry type	A_1	A_2	A_3	A_4
Bar elongations	a, b, c, d	e	f, g, h	i, j
Nodal motions	α, β, γ	δ, ϵ	ζ, η, θ	ι, κ, λ

Table 9. Symmetry adapted counts, (γ), for all possible symmetry operations in the plane.

The kinematic matrix, $\mathbf{A}^T$, in Figure 14 relates the bar extensions, $\vec{\delta l}$ and the nodal displacements $\vec{u}$ via $\mathbf{A}^T\vec{u} = \vec{\delta l}$. Also, the equilibrium matrix, $\mathbf{A}$, relates the bar forces, $\vec{T}$, and the nodal forces, $\vec{P}$ via $\mathbf{A}\vec{T} = \vec{P}$.

$$
\begin{array}{c|ccccccccccc}
 & \alpha & \beta & \gamma & \delta & \epsilon & \zeta & \eta & \theta & \iota & \kappa & \lambda \\
\hline
a & 1 & & & & & & & & & & \\
b & & 1 & & & & & & & & & \\
c & & & 1 & & & & & & & & \\
d & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & & & & & & & & \\
e & & & & 1/\sqrt{2} & 1/\sqrt{2} & & & & & & \\
f & & & & & & & 1 & & & & \\
g & & & & & & & & -1 & & & \\
h & & & & & & 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} & & & \\
i & & & & & & & & & 1 & & \\
j & & & & & & & & & 1/\sqrt{2} & 1/\sqrt{2} &
\end{array}
$$

Figure 14. Construction of a block diagonalised kinematic matrix, $\mathbf{A}^T$, from a set of symmetric basis sets for a Desargues framework.

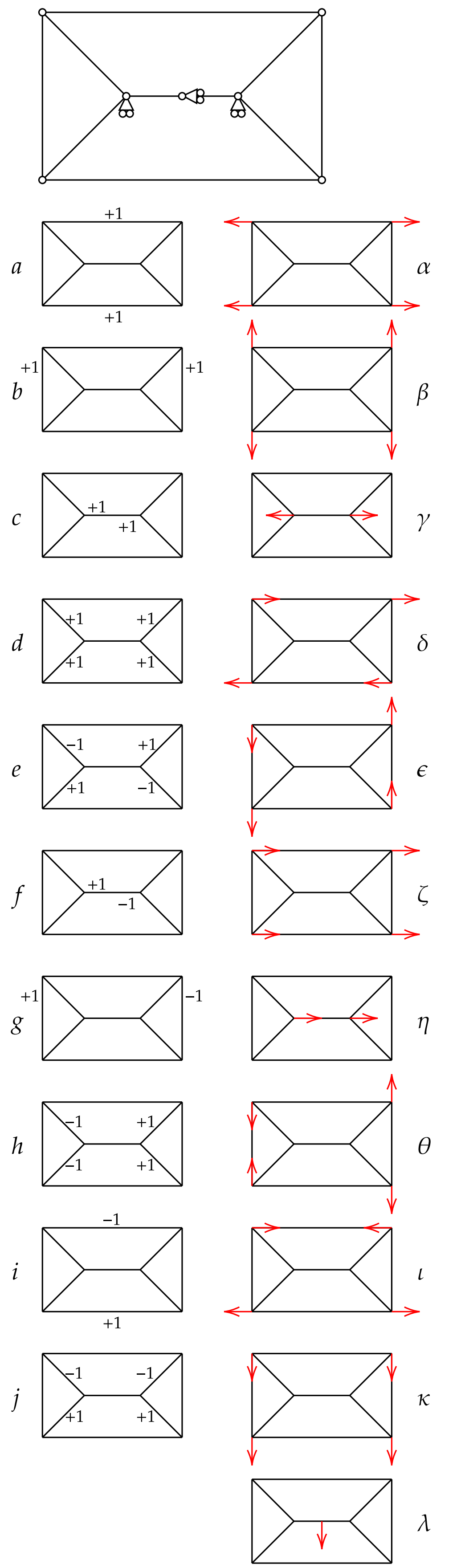

Figure 15. Construction of a block diagonalised kinematic matrix from a set of symmetric basis sets for a Desargues framework.