BRACED TRIANGULATIONS AND RIGIDITY

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Abstract. We consider the problem of finding an inductive construction, based on vertex splitting, of triangulated spheres with a fixed number of additional edges (braces). We show that for any positive integer b there is such an inductive construction of triangulations with b braces, having finitely many base graphs. In particular we establish a bound for the maximum size of a base graph with b braces that is linear in b. In the case that $b = 1$ or 2 we determine the list of base graphs explicitly. Using these results we show that doubly braced triangulations are (generically) minimally rigid in two distinct geometric contexts arising from a hypercylinder in $\mathbb{R}^4$ and a class of mixed norms on $\mathbb{R}^3$.

1. Introduction

A d-dimensional *bar-joint framework* is a pair (G, q), where $G = (V, E)$ is a simple graph and $q \in (\mathbb{R}^d)^V$. We think of a framework as a collection of (fixed length) bars that are connected at their ends by (universal) joints. Loosely speaking, such a framework is called *rigid* if it cannot be deformed continuously into another non-congruent framework while preserving the lengths of all bars. Otherwise, the framework is called *flexible*.

The rigidity and flexibility analysis of bar-joint frameworks and related constraint systems has a rich history which dates back to the work of Euler and Cauchy on the rigidity of polyhedra and to Maxwell's studies of mechanical linkages and trusses in the 19th century. Over the last few decades, the field of geometric rigidity theory has seen significant developments due to a plethora of new applications in pure mathematics and diverse areas of science, engineering and design. We refer the reader to [22, 18], for example, for summaries of key definitions and results.

Triangulations of the 2-sphere play an important role in the rigidity theory of bar-joint frameworks. Gluck has shown that generic realisations of these graphs as bar-joint frameworks in 3-dimensional Euclidean space are minimally rigid ([6]). Whiteley gave an independent proof of Gluck's result by observing that certain vertex splitting moves preserve generic rigidity and are sufficient to construct all sphere triangulations from an easily understood base graph ([21]).

In this paper we consider inductive constructions, based on vertex splitting, for triangulated spheres with a fixed number of additional edges (braces). Our first main result establishes a linear bound for the size of an irreducible (defined below) braced triangulation with b braces (Theorem 7). An easy consequence is that for fixed b there are only

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finitely many irreducible braced triangulations. This is analogous to well known results on irreducible triangulations of surfaces by Barnette, Edelson, Boulch, Nakamoto, Colin de Verdiére and others ([1, 3]).

The case $b = 1$ is quickly dealt with in Section 3 where we show that a triangular bipyramid with a brace connecting the two poles is the unique irreducible. In other words, every unibraced triangulation can be constructed from this single irreducible by a sequence of vertex splitting moves of a specific kind. Triangulated spheres with a single brace have previously been studied in [20] in relation to redundant rigidity and more recently in [7, 11] in relation to global rigidity. The case $b = 2$ is more involved and in Section 4 we show that there are exactly five distinct irreducibles (see Figures 3, 4, 5, 6, 7).

Two major new research strands in geometric rigidity are the rigidity analyses of bar-joint frameworks in Euclidean 3-space whose joints are constrained to move on a surface (such as a cylinder or surface of revolution) [8, 9, 16, 17] and of bar-joint frameworks in non-Euclidean normed spaces [4, 12, 13, 14, 15].

In Section 5 we prove an analogue of Gluck's Theorem for bar-joint frameworks which are constrained to a hypercylinder in $\mathbb{R}^4$ (Theorem 31). In this setting it is clear that doubly braced triangulations have exactly the right number of edges to be minimally rigid and so our inductive construction from Section 4 is a key ingredient in the proof. We introduce the appropriate rigidity matrix for frameworks on the hypercylinder, construct rigid placements for the base graphs and show that vertex splitting preserves rigidity on the hypercylinder.

In Section 6 we prove another analogue of Gluck's Theorem, this time for a class of *mixed norms* on $\mathbb{R}^3$ (Theorem 51). In this setting we first need to establish some key geometric properties of the underlying normed spaces, in particular we characterise the isometries of the spaces. Our inductive construction for doubly braced triangulations is again key to the proof.

2. Braced triangulations

A sphere graph is a simple graph with a fixed embedding in the 2-sphere without edge crossings. A *face* of a sphere graph is the topological closure of a connected component of the complement of the graph in the sphere. In particular a face contains its boundary.

A (sphere) triangulation, P, is a sphere graph with at least 3 vertices that is inclusion-maximal among all sphere graphs with the same vertex set. We say that an edge $e \in E(P)$ is contractible in P if it belongs to precisely two 3-cycles. In other words it does not belong to any non-facial 3-cycle of P. In that case it follows that the simple graph P/e obtained by contracting the edge e is also a triangulation (with the obvious embedding).

The following two lemmas are well known and will be useful for us. See, for example [5].

Lemma 1. *Suppose that P is a triangulation with at least 4 vertices and that F is a face of P. Each vertex of F is incident to a contractible edge of P that is not in F.* □

Lemma 2. *Suppose that P is a triangulation with at least 4 vertices. Each vertex of P is incident to at least two contractible edges of P.* □

A *braced triangulation* is a pair $G = (P, B)$ where P is a triangulation and B is a set of edges on $V(P)$ such that $B \cap E(P) = \emptyset$. We denote by $V(B)$ the set of vertices in $V(G) := V(P)$ which are incident with a brace. Occasionally we shall refer to the *underlying graph* of G, by which we mean the graph $(V(P), E(P) \cup B)$. In other words, the underlying graph is the graph obtained by forgetting the distinction between the braces and the edges of P.

An edge e of P is said to be *contractible* in G if e is contractible in P and e does not belong to any 3-cycle that contains a brace. So in that case $G/e = (P/e, B)$ is also a braced triangulation (we assume that if e is incident to $x \in V(B)$ then we contract e onto x, thus preserving the set $V(B)$ as a subset of $V(P/e)$).

A braced triangulation $G = (P, B)$ is said to be *irreducible* if there is no edge of P that is contractible in G. This is analogous to the notion of irreducible triangulation that is well studied in the literature on triangulations of surfaces. In that context, it is known that for a given surface there are finitely many isomorphism classes of irreducible triangulations - see [1] and [3]. In this section, we will show that for a given number of braces there are finitely many isomorphism classes of irreducible braced triangulations.

For a vertex v of P we write $N_P(v)$ for the set of neighbours of v in P. That is to say $N_P(v) = \{u \in V(P) : uv \in E(P)\}$. For vertices u, v let $X_{uv} = N_P(u) \cap N_P(v)$ and define $r_{uv} = |X_{uv}|$.

Lemma 3. *Suppose that $G = (P, B)$ is an irreducible braced triangulation. Then*

$$V(P) = V(B) \cup \bigcup_{uv \in B} X_{uv} \tag{1}$$

Proof. Suppose that $w \in V(P) \backslash V(B)$. Then by Lemma 1 there is some edge xw in P such that xw is contractible in P. Since G is irreducible, it follows that there is some brace $xy \in B$ such that $yw \in E(P)$. Clearly $w \in X_{xy}$ as required. □

Now suppose that $uv \in B$. Let Q_{uv} be the sphere subgraph of P that is formed by the complete bipartite graph $K(\{u, v\}, X_{uv})$. We will use Q_{uv} frequently in the sequel, so we label its various elements as follows (see Figure 1). Suppose $|X_{uv}| \geq 2$. Let $R_1, \ldots, R_{r_{uv}}$ be the faces of Q_{uv} with the labels chosen so that R_i is adjacent to R_{i+1} for $i = 1, \ldots, r_{uv}$. Here we adopt the convention that $R_{r_{uv}+1} = R_1$. We suppose that the boundary vertices of R_i are y_i, u, y_{i+1}, v for $i = 1, \ldots, r_{uv}$. So $X_{uv} = \{y_1, \ldots, y_{r_{uv}}\}$ and by convention $y_{r_{uv}+1} = y_1$. Note that if y is a point in the sphere and $y \neq u, v$ then y belongs to at most two of $R_1, \ldots, R_{r_{uv}}$. Furthermore if $y \in V(P) - \{u, v\}$ then y belongs to exactly two of $R_1, \ldots, R_{r_{uv}}$ if and only if $y \in X_{uv}$.

Lemma 4. *Suppose that $G = (P, B)$ is irreducible, $uv \in B$, and that some face of Q_{uv} contains no vertices in $V(B)$ other than u, v. Then $r_{uv} \leq 3$.*

Proof. Suppose $r_{uv} \geq 4$ and suppose that R is a face of Q_{uv} satisfying the hypothesis. Let a, u, b, v be the boundary vertices of R. First we claim that there are no vertices of P in the interior of R. If w was such a vertex then by Lemma 1 it is incident with an edge which is contractible in P. Since $w \notin V(B)$, it follows there is some brace $xy \in B$ such that wx

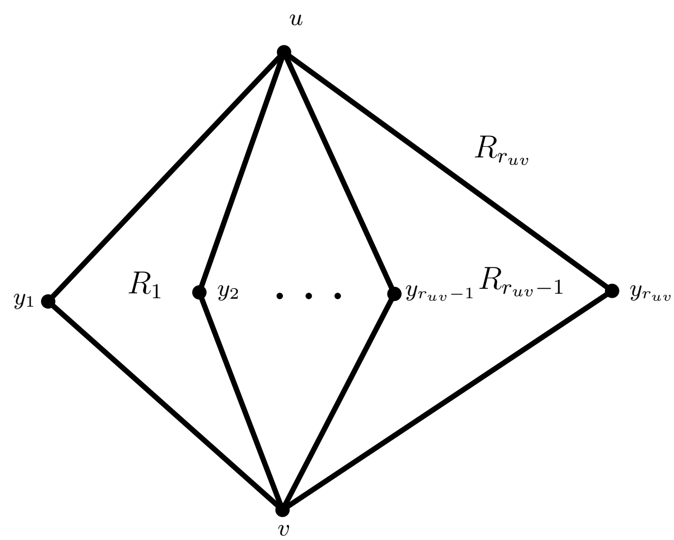

FIGURE 1. The faces of Q_{uv}. Each R_i is a closed quadrilateral region in the sphere.

and wy are edges in P. Now since, by assumption, $w \notin X_{uv}$, at least one of x, y, without loss of generality say x, is not in $\{u, v\}$. Then $x \in R$ and $x \in V(B) - \{u, v\}$ contradicting our hypothesis.

So, in view of this claim, and since P is a triangulation that does not contain the edge uv, it follows that ab is an edge of P that is contained within R. Now since neither a nor b is in $V(B)$ by our hypothesis, it follows that ab is in some non-facial 3-cycle of P. Let c be the third vertex of this 3-cycle. Clearly $c \notin \{u, v\}$. Now let S, T be the distinct faces of Q_{uv} that are adjacent to R. Clearly $c \in S \cap T$. On the other hand, since $r_{uv} \geq 4$ it follows that $S \cap T = \{u, v\}$, a contradiction. □

Theorem 5. *Suppose that $G = (P, B)$ is an irreducible braced triangulation and that $b = |B| \geq 2$. Then $|V(P)| \leq 4b^2 - 2b$.*

Proof. By Lemma 3 we know that $|V(P)| \leq 2b + \sum_{uv \in B} r_{uv}$. Now if $r_{uv} \geq 4$ then it follows from Lemma 4 that every face of Q_{uv} contains an element of $V(B) - \{u, v\}$. Since $|V(B) - \{u, v\}| \leq 2(b-1)$, any element of $V(B) - \{u, v\}$ belongs to at most two faces of Q_{uv}, and r_{uv} is the number of faces of Q_{uv}, we have $r_{uv} \leq 2|V(B) - \{u, v\}| \leq 4b - 4$. Thus $\sum_{uv \in B} r_{uv} \leq b(4b - 4)$ and the result follows. □

We have the following immediate corollary of Theorem 5.

Corollary 6. *For any positive integer b, there are finitely many irreducible braced triangulations with b braces.*

It is natural to wonder if the bound in Theorem 5 can be sharpened. Indeed in the context of triangulations of surfaces of positive genus Boulch et al. in [3] have established that if $f(g, c)$ is the maximum size of an irreducible triangulation of a surface with genus g and c boundary components, then $f(g, c)$ is $\mathcal{O}(g + c)$. Motivated by this we devote the remainder of this section to establishing the following linear bound for the number of vertices of an irreducible braced sphere triangulation in terms of the number of braces.

Theorem 7. *Suppose that $G = (P, B)$ is an irreducible braced triangulation. Then $|V(P)| \leq 11b - 4$.*

Before giving the proof of Theorem 7 we need some lemmas.

Lemma 8. *Suppose that $G = (P, B)$ is a braced triangulation such that $uv \neq xw$ and $r_{uv}, r_{xw} \geq 4$. Then either*
(i) there exists a face R of Q_{uv} that contains Q_{xw}, or,
(ii) $r_{uv} = r_{xw} = 4$ and $Q_{uv} \cup Q_{xw}$ is an octahedral graph.

Proof. Suppose that one of x, w, say x, is contained in the interior of some face R of Q_{uv}. Then $N_P(x) \subset R$ and since at least one of u, v is not in $N_P(x)$ and $r_{xw} \geq 4$ it follows that $w \in R$ also. So $Q_{xw} \subset R$ in this case.

So, using the fact that $xw \notin E(P)$ we can assume that $\{x, w\} \subset X_{uv}$. If $\{x, w\}$ is contained in a face R of Q_{uv} then since $r_{uv} \geq 4$ it follows that $Q_{xw} \subset R$ and we are done. On the other hand suppose that there is no face of Q_{uv} containing both of x, w. Then it follows that $X_{xw} \subset V(Q_{uv})$ and using the assumption that $r_{uv}, r_{xw} \geq 4$ we see that the only possibility is that $Q_{uv} \cup Q_{xw}$ is an octahedral graph. □

Now suppose that $G = (P, B)$ is a braced sphere triangulation. For the remainder of this section it will be convenient to work in the context of plane graphs instead of sphere graphs. So we fix some point in the sphere that is not in P and by removing that point we consider P as a plane graph. In particular any subgraph of P has a unique unbounded face.

We will need the following elementary observations about certain collections of plane graphs.

Suppose that C is a finite set of plane graphs such that

(2) for all $H, K \in C$ with $H \neq K$, there is some face of H that contains K.

Observe that C has a partial order defined by $H \preceq K$ if either $H = K$ or H is contained in a bounded face of K. This partial order will be key in the remainder of this section.

Lemma 9. *Suppose that H, K are distinct graphs in C and that there is some z in the plane that does not lie in the unbounded face of H and does not lie in the unbounded face of K. Then either $H \prec K$ or $K \prec H$.*

Proof. Suppose that H, K are incomparable. Then, by (2), H is contained in the unbounded face of K and vice versa. Since z lies in a bounded face of H, it must lie in the unbounded face of K which contradicts our assumption. □

We have the following immediate consequence of Lemma 9.

Corollary 10. *For any point z in the plane let*

$$C^z = \{H \in C : z \text{ is not in the unbounded face of } H\}.$$

Then, with respect to $\preceq$, C^z is a totally ordered subset of C. □

From now on, let $C_G = \{Q_{uv} : uv \in B, r_{uv} \geq 6\}$ and suppose $|C_G| = c$. For convenience we will write $C_G = \{Q_1, \dots, Q_c\}$ where $Q_i = Q_{u_i v_i}$ and $r_i = r_{u_i v_i}$. By Lemma 8, C_G satisfies (2) and so Lemma 9 and Corollary 10 apply to C_G.

Now for each i, let $R_1^i, \dots, R_{r_i}^i$ be the faces of Q_i, labelled so that R_1^i is the unbounded face and so that R_j^i is adjacent to R_{j+1}^i for $j = 1, \dots, r_i - 1$. We choose a set of vertices $Z_i \subset V(B)\backslash\{u_i, v_i\}$ as follows. Start with $Z_i = \emptyset$. Now suppose that t is the smallest integer such that $5 \leq t \leq r_i - 1$ and R_t^i does not contain any vertex in Z_i. Let $Q_i = Q_{s_1} \succ Q_{s_2} \succ \cdots \succ Q_{s_k}$ be a chain in C_G of maximal length such that Q_{s_2} is contained in R_t^i, and for $m \geq 2$, $Q_{s_{m+1}}$ is contained in $R_3^{s_m}$. Moreover we choose so that, for $m \geq 1$, $Q_{s_{m+1}}$ is a maximal element (with respect to $\prec$) in the set $\{Q_l \in C_G : Q_l \prec Q_{s_m}\}$. So for $m \geq 1$ there is no element of C_G strictly between Q_{s_m} and $Q_{s_{m+1}}$.

Since the chain above has maximal length, it follows that

(3) $R_3^{s_k}$, respectively R_t^i, does not contain any $Q_j \in C_G$ if $k \geq 2$, respectively if $k = 1$.

By Lemma 4, $R_3^{s_k}$, or R_t^i if $k = 1$, contains some vertex $z \in V(B) \setminus \{u_{s_k}, v_{s_k}\}$. In particular, since z does not lie in the unbounded face of Q_{s_k} it follows that $z \notin \{u_i, v_i\}$. So $z \in V(B) \setminus \{u_i, v_i\}$ and z lies in R_t^i. We add z to the set Z_i.

We continue choosing elements in this way until each of the faces $R_5^i, \dots, R_{r_i-1}^i$ contains at least one element of Z_i. Since no vertex in $V(P) \setminus \{u_i, v_i\}$ is contained in more than two of $R_5^i, \dots, R_{r_i-1}^1$ it follows that $|Z_i| \geq \frac{r_i-5}{2}$. Also since $Z_i \subset R_5^i \cup \dots \cup R_{r_i-1}^i$ it follows that

(4) for $1 \leq i \leq c$ no element of Z_i lies in R_1^i or in R_3^i.

Lemma 11. $Z_i \cap Z_j = \emptyset$ *for all* $i \neq j$.

Proof. Suppose that $z \in Z_i \cap Z_j$. By Lemma 9 and (4) it follows that Q_i and Q_j are comparable with respect to $\preceq$. Without loss of generality suppose that $Q_j \prec Q_i$. Now consider the sequence $Q_{s_1} \succ \cdots \succ Q_{s_k}$ that is constructed during the selection of z for Z_i. Since z does not lie in the unbounded face of any Q_{s_m}, we see that for $m = 1, \dots, k$, $Q_{s_m} \in C_G^z$ (as defined in Corollary 10). Moreover, using (3), and since there is no element of C_G strictly between Q_{s_m} and $Q_{s_{m+1}}$ it follows that $\{Q_{s_1}, \dots, Q_{s_k}\} = \{Q_l \in C_G^z : Q_l \preceq Q_i\}$. In particular, since $Q_j \in C_G^z$ and $Q_j \prec Q_i$, we have $Q_j = Q_{s_m}$ for some $m \geq 2$. But this implies that $z \in R_3^j$ and, since $z \in Z_j$, this contradicts (4). □

Proof of Theorem 7. Suppose that $r_1, \dots, r_c$ and $Z_1, \dots, Z_c$ are as in the discussion above.

By Lemma 3 we have

$$|V(P)| \leq 2b + \sum_{uv \in B} r_{uv}$$

Now, using $|Z_i| \geq (r_i - 5)/2$, we have

$$\begin{aligned}\sum_{uv\in B} r_{uv} &= \sum_{\{uv: r_{uv}\leq 5\}} r_{uv} + \sum_{i=1}^{c} r_i \\ &\leq 5(b-c) + \sum_{i=1}^{c}(2|Z_i|+5) \\ &= 5b + 2\sum_{i=1}^{c}|Z_i|\end{aligned}$$

By Lemma 11, $\sum_{i=1}^{c}|Z_i| = |\cup_{i=1}^{c} Z_i|$. Now suppose that Q_l is maximal with respect to $\preceq$ in C_G (there is at least one such l). Observe $u_l, v_l \notin \cup_{i=1}^{c} Z_i$ since $u_l, v_l \in R_1^i$ for $i = 1, \ldots, c$. Thus $|\cup_{i=1}^{c} Z_i| \leq |V(B)| - 2 \leq 2b - 2$ and $\sum_{uv\in B} r_{uv} \leq 9b - 4$ as required. □

3. Unibraced triangulations

A unibraced (i.e. $b = 1$) triangulation must have at least five vertices. Up to homeomorphism of the sphere there is a unique triangulation with five vertices. It follows immediately that up to homeomorphisms, there is exactly one unibraced triangulation with five vertices. Observe that in this unibraced triangulation the three vertices that are not in the brace must span a nonfacial triangle of P. The following is implicit in [11]. Also see [7] for related results.

Theorem 12. *Every unibraced triangulation with at least six vertices has a contractible edge. Equivalently, the unibraced triangulation with five vertices is the unique irreducible unibraced triangulation.*

Proof. Suppose that G is an irreducible unibraced triangulation with brace uv. Since G has at least five vertices, it follows from Lemma 3 that $r_{uv} \geq 3$. By Lemma 4, $r_{uv} \leq 3$. Thus, $r_{uv} = 3$. The conclusion now follows from Lemma 3. □

With a little more effort we can strengthen Theorem 12 as follows.

Lemma 13. *Suppose that $G = (P, B)$ is a unibraced triangulation with at least six vertices. Let T be a face of P. There is some edge of P, not in T, that is contractible in G.*

Proof. Let $u_i, i = 1, 2, 3$ be the vertices of T. By Lemma 1 there are edges $e_i, i = 1, 2, 3$ such that e_i is incident with u_i, e_i is not an edge of T and e_i is contractible in P. If any e_i is not adjacent to the brace then we are done. So we may as well assume that each e_i is adjacent to the brace for $i = 1, 2, 3$.

It follows, in particular, that e_1, e_2, e_3 cannot be pairwise non-incident, so there are two cases to consider. Either (a) e_1 and e_2 share a vertex and e_3 is non-adjacent to e_1 and e_2, or, (b) e_1, e_2, e_3 have a common vertex.

In case (b), let p be the common vertex of e_1, e_2, e_3. Clearly p must be one vertex of the brace and it follows from the planarity of P that the other vertex of the brace must lie in

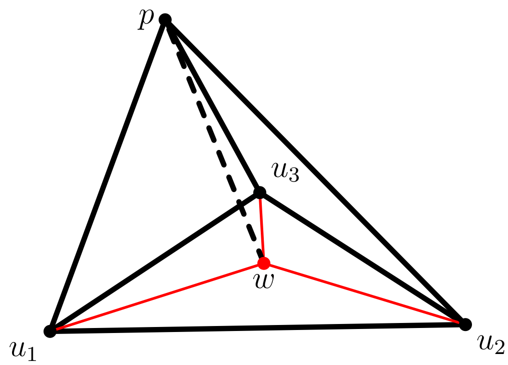

FIGURE 2. If u_1p, u_2p, u_3p are uncontractible edges, then $V(B) = \{p, w\}$ where w lies in the interior of the face T.

the interior of the face T (since it is also adjacent in P to u_1, u_2, u_3). This contradicts the fact that T is a face of P and so has no vertices in its interior, by definition (see Figure 2).

Thus, only case (a) remains. Let p be the common vertex of e_1 and e_2 and let q be the vertex of e_3 that is different from u_3. It is clear that the brace must be either pq or pu_3. If pq is the brace then u_1q, u_2q, pu_3 must all be edges of P. It is not hard to see that this contradicts the planarity of P.

Finally suppose that pu_3 is the brace. Note that $u_1, u_2 \in X_{pu_3}$. Let v be a vertex of G that is not in $\{u_1, u_2, u_3, p\}$. By Lemma 2 there are two edges of P incident to v that are contractible in P. If any such edge is not adjacent to the brace pu_3 then we are done. Thus we may assume that $V(G) = \{p, u_3\} \cup X_{pu_3}$ and that G is in fact a braced n-gonal bipyramid where the brace joins the two poles. Since G has at least six vertices, all of the equatorial edges are contractible in G and so the conclusion of the lemma is true in this case also. □

4. Doubly braced triangulations

In this section we will show that in the case $b = 2$ there are five irreducibles. These are shown in Figures 3, 4, 5, 6 and 7. The braces are indicated with dotted lines.

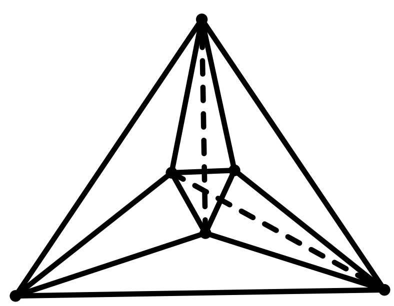

FIGURE 3. The doubly braced octahedron.

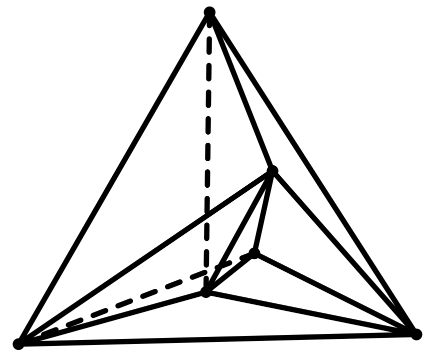

Figure 4. Capped hexahedron with disjoint braces

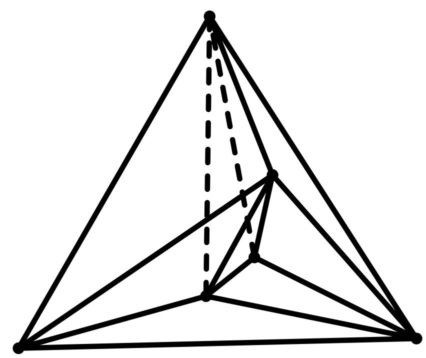

Figure 5. Capped hexahedron with adjacent braces

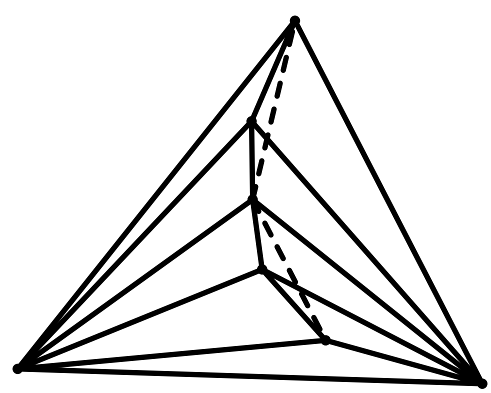

Figure 6. Irreducible with seven vertices and adjacent braces

First we observe that there are precisely two distinct triangulations of the sphere with six vertices. They are the octahedron and the capped hexahedron. It is not hard to deduce that there are three distinct braced triangulations with six vertices and that each of these is irreducible since any braced triangulation with two braces must have at least six vertices. Observe that each of these three irreducibles has underlying graph isomorphic to K_6 with

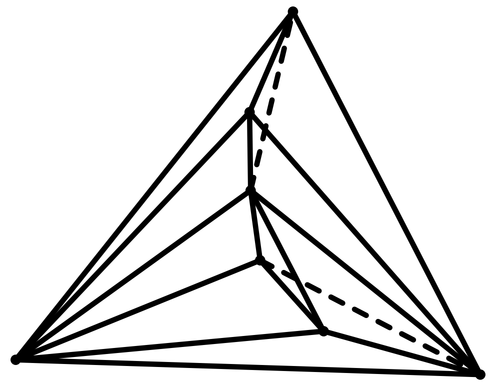

FIGURE 7. Irreducible with seven vertices and non-adjacent braces

one edge removed (which is, of course, the unique six vertex graph with 14 edges). The two seven vertex irreducibles also have isomorphic underlying graphs. In those cases the graph is isomorphic to that obtained by gluing two copies of K_5 together along a K_3.

Theorem 14. *Any irreducible braced triangulation with two braces is isomorphic to one of the examples shown in Figures 3, 4, 5, 6 or 7.*

Clearly it suffices to show that any irreducible with at least seven vertices is isomorphic to one of the examples shown in Figures 6 or 7. The rest of this section is devoted to proving that. We observe that by Theorem 5, any irreducible doubly braced triangulation has at most 12 vertices. Thus, in principle at least, we have a finite list of candidates among which we can search for irreducibles. However since there are a large number of doubly braced triangulations with at most 12 vertices we find it desirable instead to narrow the search space by improving the general bound of Theorem 5 in the case $b = 2$.

Suppose that $G = (P, B)$ is an irreducible braced triangulation with $B = \{uv, wx\}$. As above, let Q_{uv} be the bipartite sphere graph induced by $K_{\{u,v\},X_{uv}}$ and, when $r_{uv} \geq 2$, let $R_1, \cdots, R_{r_{uv}}$ be the faces of Q_{uv}. Furthermore $X_{uv} = \{y_1, y_2, \cdots, y_{r_{uv}}\}$ where u, y_i, v, y_{i+1} are the boundary vertices of R_i for $i = 1, \cdots, r_{uv}$ (adopting the convention that $y_{r_{uv}+1} = y_1$).

Lemma 15. *Let $G = (P, B)$ be an irreducible braced triangulation with $B = \{uv, wx\}$. Suppose that $\max\{r_{uv}, r_{wx}\} \geq 4$. Then G is the doubly braced octahedron (Figure 3).*

Proof. Without loss of generality, suppose that $r_{uv} \geq 4$. By Lemma 4, we see that each R_i contains some element of $V(B) - \{u, v\} = \{w, x\}$. Note that w (and x) can belong to at most two faces of Q_{uv}. It follows that $w, x \in X_{uv}$ and $r_{uv} = 4$. Now since w, x do not belong to any common face of Q_{uv}, it is clear that $X_{wx} \subset \{u, v\} \cup X_{uv}$. Using Lemma 3 we see that $|V(G)| = 6$ and the conclusion follows easily since the doubly braced octahedron is the only six vertex irreducible that satisfies $\max\{r_{uv}, r_{wx}\} \geq 4$. □

Thus we may assume from now on that $\max\{r_{wx}, r_{uv}\} \leq 3$.

Lemma 16. *Let $G = (P, B)$ be an irreducible braced triangulation with $B = \{uv, wx\}$. Suppose that* $\max\{r_{wx}, r_{uv}\} \leq 3$. *Then at most one of w, x is in $\{u, v\} \cup X_{uv}$.*

Proof. Suppose that $w, x \in \{u, v\} \cup X_{uv}$. Since wx is not an edge of P and $wx \neq uv$, it follows that $\{w, x\} \subset X_{uv}$. Since $r_{uv} \leq 3$, we may assume, without loss of generality, that $\{w, x\} = \{y_1, y_2\}$. Now since P is a triangulation and since neither uv nor wx can be edges of P it follows that there are some vertices in $\mathring{R}_1$ ($\mathring{R}_1$ denotes the interior of R_1). By Lemma 3, all such vertices must be in X_{wx}. However, in this situation $u, v \in X_{wx}$ and since $r_{wx} \leq 3$, it follows that there is exactly one vertex, z, in $\mathring{R}_1$. Since P is a triangulation not containing uv, wx it must be that zx, zw, zu, zv are all edges of P. Thus $z \in X_{uv}$ which contradicts the fact that z is in $\mathring{R}_1$. □

Lemma 17. *If G is an irreducible doubly braced triangulation with braces uv and wx then* $\max\{r_{uv}, r_{wx}\} \geq 3$.

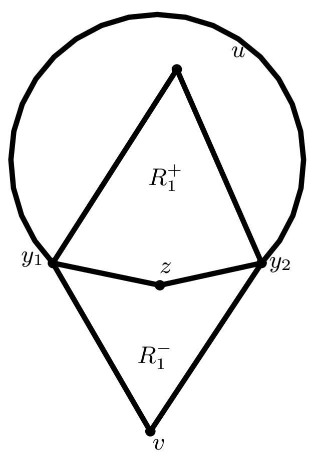

FIGURE 8. A sketch for the proof of Lemma 17.

Proof. For a contradiction assume that $r_{wx} \leq r_{uv} \leq 2$.

Suppose that $r_{wx} + r_{uv} \leq 2$. By Lemma 3 we see that $|V(G)| \leq 6$. However it is easy to see that there are only three irreducible doubly braced triangulations with at most six vertices (these are shown in Figures 3, 4 and 5) and none of these satisfy $r_{wx} + r_{uv} \leq 2$.

Now suppose that $r_{wx} \leq 2$ and $r_{uv} = 2$. By Lemma 16 we may suppose that $w \in \mathring{R}_1$. If $x \notin R_1$ then, by planarity we have $X_{wx} \subseteq \{u, v, y_1, y_2\}$. Thus, by Lemma 3, we have $|V(G)| \leq 6$. As in the previous paragraph, this is not possible since in each of the Figures 3, 4 and 5 we have $\max\{r_{uv}, r_{wx}\} \geq 3$. So we can assume $x \in R_1$. Now since $w \in R_1^\circ$, it follows that $X_{wx} \subseteq R_1$.

By Lemma 3, there are no vertices in $\mathring{R}_2$. It follows that y_1y_2 is an edge of P that is contained in R_2. If this edge is part of a braced triangle then without loss of generality $y_1 \in \{w, x\}$ and $y_2 \in X_{wx}$. Thus, in this case, $|V(G)| = |V(B) \cup X_{uv} \cup X_{wx}| \leq 6$ and we know that no six vertex irreducible satisfies $\max\{r_{uv}, r_{wx}\} \leq 2$.

So y_1y_2 is not part of any braced triangle. It must therefore be part of a non-facial triangle. Thus there is a vertex $z \in \mathring{R}_1$ and edges y_1z and zy_2. See Figure 8 for an illustration. Note that R_1 splits into two closed regions R_1^+ and R_1^- whose intersection is the path y_1, z, y_2 as shown in Figure 8.

Now, we claim that in fact $w = z$. If not, then since $w \in \mathring{R}_1$ it follows that w is contained in the interior of either R_1^+ or R_1^- (see Figure 8), without loss of generality say R_1^+. Suppose x lies in the interior of R_1^-. In this case, by Lemma 3, the interior of R_1^- contains no other vertices. There are now exactly two ways to triangulate the region R_1^-. For each of these triangulations note that xv is an edge of P which is contractible in G, a contradiction. Thus we may assume that x does not lie in the interior of R_1^-. It follows that $zv \in E(P)$. Note that the edge zv cannot lie in a nonfacial triangle of P. Also, since $z \notin X_{uv}$, the edge zv cannot lie in a 3-cycle containing the brace uv. Thus zv must lie in a 3-cycle containing the brace wx. It follows that $x = v$ and that $wz \in E(P)$. Since there can be no other vertices in R_1^+ it follows that wy_1 and wy_2 are edges in P, whence $r_{wx} \geq 3$. Thus our assumption that $w \neq z$ leads to a contradiction.

On the other hand, if $x \in \mathring{R}_1$, then we may assume without loss of generality that x lies in the interior of R_1^-. In this case, there are no vertices in the interior of R_1^+. Moreover, the edge uw lies in P and is contractible in G, a contradiction. Thus we may assume that $x \in \{u, v\}$: without loss of generality $x = u$. Now it is clear that $X_{wx} = X_{uv} = \{y_1, y_2\}$. Thus, by Lemma 3 we have $|V(G)| = 5$, a contradiction. □

Lemma 18. *Suppose that $r_{wx} \leq r_{uv} = 3$ and that there is no face of Q_{uv} that contains both w and x. Then G is a doubly braced capped hexahedron with disjoint braces (see Figure 4).*

Proof. By Lemma 16, we can assume that $w \in \mathring{R}_1$. So $x \notin R_1$ and since, by Lemma 3, $V(G) = \{u, v, w, x\} \cup X_{uv} \cup X_{wx}$ we see that w is the only vertex in $\mathring{R}_1$. Thus $N_P(w) \subset \{y_1, u, y_2, v\}$. Also $|N_P(w)| \geq 3$ since the min degree of any triangulation is at least three. Furthermore $w \notin X_{uv}$, so it follows that w is adjacent to both of y_1, y_2 and exactly one of u, v, say u without loss of generality. In a triangulation an edge incident to a vertex of degree 3 cannot belong to a nonfacial triangle. Thus none of wy_1, wy_2, wu are in nonfacial triangles. Also, since wv is not an edge of G, the edge wu is not in a triangle that contains the brace uv. It follows that the edges wy_1, wy_2, wu must all belong to triangles containing the brace wx. Thus x is a common neighbour in P of y_1, y_2 and u that is not in R_1. Since $r_{uv} = 3$ we must have by planarity that $X_{uv} = \{x, y_1, y_2\}$. It now follows easily that G is the doubly braced capped hexahedron with disjoint braces as claimed. □

Lemma 19. *Suppose that $r_{wx} \leq r_{uv} = 3$. Then X_{uv} spans a 3-cycle in P.*

Proof. If no face of Q_{uv} contains both w, x then by Lemma 18, G is isomorphic to the capped hexahedron shown in Figure 4 and the conclusion is true. Using this and Lemma 16 we may assume that $w \in \mathring{R}_1$ and $x \in R_1$. Now it is clear that there are no vertices in $\mathring{R}_2$ or in $\mathring{R}_3$ since such vertices would have to be in X_{wx}, by Lemma 3 and this contradicts $w \in \mathring{R}_1$. Now, since uv is not an edge of P, it follows that y_2y_3 and y_3y_1 are both edges of P. Since $y_3 \notin X_{wx}$ we also conclude that both of these edges must lie in nonfacial triangles of P. Furthermore, it follows that $N_P(y_3) = \{u, v, y_1, y_2\}$. Therefore y_1y_2 must also be an edge of P as required. □

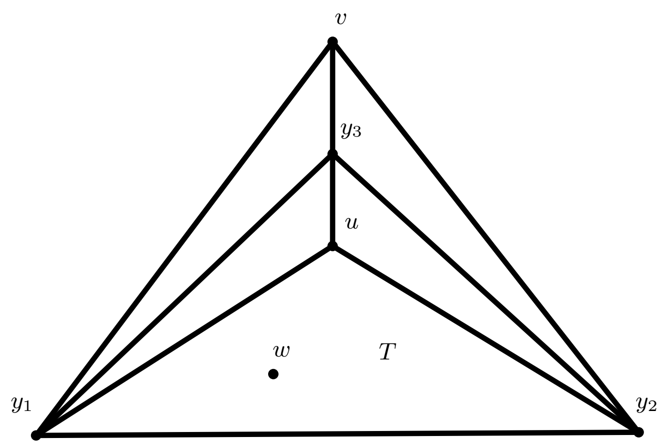

FIGURE 9. The case $r_{wx} \leq r_{uv} = 3$. Here R_1 is split into two triangular regions by the edge y_1y_2. One is the unbounded region (in this plane embedding) and one is the region containing w.

Proof of Theorem 14. If $\max\{r_{uv}, r_{wx}\} \geq 4$ then, by Lemma 15, G is isomorphic to the example of Figure 3. So we assume that $r_{wx} \leq r_{uv} \leq 3$. By Lemma 17 we have $r_{uv} = 3$.

Now using Lemmas 16 and 19 we have the situation illustrated in Figure 9. Note that any vertices that are not in $X_{uv} \cup \{u, v, w, x\}$ must lie in the interior of the triangular region labelled T in Figure 9, since any such vertex must be in X_{wx}.

Now suppose that x does not lie in the closed region T. If $x \notin R_1$ then, by Lemma 18, G is isomorphic to the doubly braced capped hexahedron in Figure 4. If $x \in R_1$ then, using the observation in the paragraph above, we see that there are no other vertices in the interior of the region $R_1 \backslash T$. If $x \neq v$ then the triangulation P contains the edges xv, xy_1 and xy_2. Now note that the edge xv is contractible in G, a contradiction. Thus, we conclude that $x = v$ and so $V(G) = \{u, v, w\} \cup X_{uv}$. In other words G has six vertices and adjacent braces and so it is isomorphic to the example shown in Figure 5.

Finally suppose that x also lies in the closed triangular region T. We can construct a unibraced triangulation H by deleting the vertices v, y_3 and all their incident edges. Now H is a unibraced triangulation with a triangular face bounded by edges uy_1, y_1y_2, y_2u. If H has six or more vertices, then by Lemma 13 there is some edge of H that is contractible that is not one of uy_1, y_1y_2, y_2u. Such an edge would also be contractible in G, contradicting our assumption that G is irreducible. Therefore H has only five vertices and it follows easily that G is isomorphic to one of the examples shown in Figures 6 or 7. This completes the proof of Theorem 14. □

If $G' = (P, B)$ is a braced triangulation and e is an edge of P that is contractible in G', then $G = (P/e, B)$ is a braced triangulation and we say that G' is obtained from G by a *topological vertex splitting move.*

Combining the above results we obtain the following theorem:

Theorem 20. *Let G be a doubly braced triangulation. Then G can be constructed from one of the examples in Figures 3, 4, 5, 6 or 7 by a sequence of topological vertex splitting moves.*

In general, a *d-dimensional vertex splitting move* on a graph is defined as follows (see [21], for example). Let $G = (V, E)$ be a graph, and let $v_1 \in V$ and $v_1v_i \in E$ for $i = 2, \dots, d$. If a graph G' is obtained from G by

- adding a new vertex v_0 and edges $v_0v_1, v_0v_2, \dots, v_0v_d$ to G, and
- for every edge $v_1x \in E$ with $x \notin \{v_2, \dots, v_d\}$, either leaving the edge unchanged or replacing it with the edge v_0x,

then G' is said to be obtained from G by a *d-dimensional vertex split* at v_1 (on the edges $v_1v_2, \dots, v_1v_d$).

Of course, topological vertex splitting is a special case of (3-dimensional) vertex splitting for graphs, and it is known that vertex splitting for graphs preserves rigidity for generic frameworks in a wide variety of settings [21, 4, 7, 8]. Therefore it is natural to look for geometric rigidity applications of Theorem 20. We provide two such applications in Sections 5 and 6.

5. Application: rigidity in the hypercylinder

There is a sizable literature on the rigidity of bar-joint frameworks in 3-dimensional Euclidean space whose points are constrained to lie on a surface (see, for example, [8, 9, 16, 17]. In this section, we will consider the rigidity of bar-joint frameworks in 4-dimensional Euclidean space where the points are constrained to lie on a hypercylinder.

5.1. The hypercylinder in $\mathbb{R}^4$.

Let $\Sigma = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 = 1\}$ be the hypercylinder in $\mathbb{R}^4$. Observe that Σ is a smooth three-dimensional manifold that inherits a natural metric as a subspace of the Euclidean space $\mathbb{R}^4$. The group of isometries of Σ with respect to this metric is a Lie group of real dimension 4. Indeed this group is canonically isomorphic to $O(3) \times E(1)$ where $O(3)$ is the group of 3×3 orthogonal matrices and $E(1)$ is the group of Euclidean isometries of $\mathbb{R}$.

Let $\mathcal{T}(\mathbb{R}^4)$ denote the real linear space of infinitesimal rigid motions of the Euclidean space $\mathbb{R}^4$. Recall that each infinitesimal rigid motion $\eta \in \mathcal{T}(\mathbb{R}^4)$ is an affine map $\eta : \mathbb{R}^4 \to \mathbb{R}^4$ of the form $\eta(x) = B(x) + c$ where the linear part B is a 4×4 skew-symmetric matrix and the translational part c is a vector in $\mathbb{R}^4$.

Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be the projection $(x, y, z, w) \mapsto (x, y, z)$ and let $\mathcal{T}(\Sigma)$ denote the following subspace of $\mathcal{T}(\mathbb{R}^4)$,

$$\mathcal{T}(\Sigma) = \{\eta \in \mathcal{T}(\mathbb{R}^4) : \pi(\eta(x)) \cdot \pi(x) = 0, \quad \forall\, x \in \Sigma\}.$$

We refer to the elements of $\mathcal{T}(\Sigma)$ as infinitesimal rigid motions of Σ.

Lemma 21. *Let $\eta \in \mathcal{T}(\mathbb{R}^4)$ take the form $\eta(x) = B(x) + c$ where B is a 4×4 skew-symmetric matrix and $c \in \mathbb{R}^4$. Then $\eta \in \mathcal{T}(\Sigma)$ if and only if*

$$B = \begin{bmatrix} \tilde{B} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\tilde{B}$ is a 3×3 skew-symmetric matrix, and $\pi(c) = 0$.

Proof. Let e_1, e_2, e_3, e_4 denote the standard basis vectors in $\mathbb{R}^4$.

If $\eta \in \mathcal{T}(\Sigma)$ then

$$\pi(c) = \sum_{i=1}^{3} \left(\pi(\eta(e_i)) \cdot \pi(e_i)\right) \pi(e_i) = 0.$$

Also, note that $e_i + e_4 \in \Sigma$ for $i = 1, 2, 3$ and so,

$$B(e_4) \cdot e_i = \pi(\eta(e_i + e_4)) \cdot \pi(e_i + e_4) = 0.$$

Thus B has the form,

$$B = \begin{bmatrix} \tilde{B} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\tilde{B}$ is a 3×3 skew-symmetric matrix.

For the converse, suppose the linear part of η has the form

$$B = \begin{bmatrix} \tilde{B} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\tilde{B}$ is a 3×3 skew-symmetric matrix, and the translational part c satisfies $\pi(c) = 0$. Since $\tilde{B}$ is skew-symmetric, if $x \in \Sigma$ then

$$\pi(\eta(x)) \cdot \pi(x) = \tilde{B}(\pi(x)) \cdot \pi(x) = 0.$$

Thus, $\eta \in \mathcal{T}(\Sigma)$. □

5.2. **Frameworks in the hypercylinder.** For a graph $G = (V, E)$, a *placement* of G in Σ is a vector $q = (q_v)_{v \in V} \in \Sigma^V$. A pair (G, q) consisting of a graph G and a placement q is called a *(bar-joint) framework* in Σ. A *subframework* of (G, q) is a framework (H, q^H) where H is a subgraph of G and $q_v^H = q_v$ for all $v \in V(H)$.

We say that (G, q) is *full* in Σ if the restriction map,

$$\rho : \mathcal{T}(\Sigma) \to (\mathbb{R}^4)^V, \quad \eta \mapsto (\eta(q_v))_{v \in V},$$

is injective. In this case we refer to q as a *full placement* of G in Σ. We say that (G, q) is *completely full* in Σ if every subframework of (G, q) with at least 6 vertices is full in Σ.

Lemma 22. *Let (G, q) be a framework in Σ and let $S = \{q_v : v \in V\}$. Then (G, q) is full in Σ if and only if $\pi(S)$ contains at least two linearly independent vectors.*

Proof. Suppose $\pi(S)$ does not contain two linearly independent vectors. Let $A(\theta) \in O(3)$ be the rotation matrix with rotation axis spanned by $\pi(S)$, where θ denotes the angle of rotation. Let B denote the skew-symmetric matrix

$$B = \begin{bmatrix} A'(0) & 0 \\ 0 & 0 \end{bmatrix}$$

Let $\eta \in \mathcal{T}(\mathbb{R}^4)$ be the infinitesimal rigid motion with $\eta(x) = B(x)$. Then, by Lemma 21, $\eta \in \mathcal{T}(\Sigma)$. Also, the rotation axis for $A(\theta)$ lies in the kernel of $A'(0)$ and so $\eta(S) = (A'(0)(\pi(S)), 0) = 0$. Thus (G, q) is not full.

For the converse, suppose there exists a non-zero $\eta \in \mathcal{T}(\Sigma)$ such that $\eta(S) = 0$. Note that, by Lemma 21, $\eta(x) = \tilde{B}(\pi(x)) + c$ for some non-zero 3×3 skew-symmetric matrix $\tilde{B}$ and some $c = (0, 0, 0, w) \in \mathbb{R}^4$. Now $\tilde{B}(\pi(S)) = \pi(\eta(S)) = 0$. The rank of a skew-symmetric matrix is always even and so the kernel of $\tilde{B}$ must have dimension 1. We conclude that $\pi(S)$ does not contain two linearly independent vectors. □

We denote by $\text{Full}(G; \Sigma)$ the set of all full placements of G in Σ. Note that by the above lemma, $\text{Full}(G; \Sigma)$ is an open and dense subset of Σ^V.

5.3. **Rigidity in the hypercylinder.** Let (G, q) be a framework in Σ. An *infinitesimal flex* of (G, q) is a vector $m = (m_v)_{v \in V} \in (\mathbb{R}^4)^V$ that satisfies,

$$(m_u - m_v) \cdot (q_u - q_v) = 0 \qquad \text{for every } uv \in E, \tag{5}$$

and,

$$\pi(m_v) \cdot \pi(q_v) = 0 \qquad \text{for every } v \in V. \tag{6}$$

The constraints in (5) are the standard Euclidean first-order length constraints for the edges of G, and the constraints in (6) ensure that the velocity vectors of an infinitesimal flex lie in the tangent hyperplanes of Σ at the corresponding points.

Lemma 23. *Let (G, q) be a framework in Σ and let $\eta \in \mathcal{T}(\Sigma)$. Then the vector $m \in (\mathbb{R}^4)^V$ where,*

$$m_v = \eta(q_v), \quad \forall\, v \in V,$$

is an infinitesimal flex of (G, q).

Proof. The conditions (5) and (6) are readily verified using Lemma 21. □

We refer to the infinitesimal flexes described in Lemma 23 as *trivial* infinitesimal flexes of (G, q). The set of all trivial infinitesimal flexes of (G, q) is a linear subspace of $(\mathbb{R}^4)^V$, which we denote by $\mathcal{T}(q)$. The orthogonal projection of $(\mathbb{R}^4)^V$ onto $\mathcal{T}(q)$ will be denoted P_q. We say that (G, q) is *infinitesimally rigid* if every infinitesimal flex of (G, q) is trivial. Otherwise (G, q) is *infinitesimally flexible.*

Lemma 24. *Let $G = (V, E)$ be a graph and let $x \in (\mathbb{R}^4)^V$. Then the map*

$$\phi_x : \text{Full}(G; \Sigma) \to \mathcal{T}(q), \quad q \mapsto P_q(x),$$

is continuous.

Proof. If (G,q) is full in Σ then a basis for $\mathcal{T}(q)$ is given by the vectors $m_1(q),\dots,m_4(q)$ where for each $v\in V$, we have $m_1(q)_v=(q_v^3,0,-q_v^1,0)$, $m_2(q)_v=(q_v^2,-q_v^1,0,0)$, $m_3(q)_v=(0,q_v^3,-q_v^2,0)$, $m_4(q)_v=(0,0,0,1)$. The result follows since $P_q(x)$ depends continuously on the basis vectors $m_1(q),\dots,m_4(q)$. □

5.4. **The rigidity matrix.** Let (G,q) be a framework in Σ. The *rigidity matrix* $R(G,q)$ for (G,q) in Σ is the matrix corresponding to the linear system in (5) and (6). This matrix is a $(|E|+|V|)\times 4|V|$ matrix of the following form. The rows are indexed by the set $E\cup V$ and the columns are indexed in collections of four by the set V. For an edge $uv\in E$ the corresponding row has entries q_u-q_v in the collection of columns corresponding to u and q_v-q_u in the collection of columns corresponding to v and zeroes in all other columns. For a vertex $v\in V$ the corresponding row has entries $(\pi(q_v),0)$ in the collection of columns indexed by v and zeroes in all other columns.

Lemma 25. *Let (G,q) be a full framework in Σ. Then (G,q) is infinitesimally rigid if and only if* $\operatorname{rank} R(G,q)=4|V|-4$.

Proof. Note that the kernel of $R(G,q)$ is the linear space of infinitesimal flexes of (G,q). Thus, (G,q) is infinitesimally rigid if and only if $\ker R(G,q)=\mathcal{T}(q)$. Also, note that $\operatorname{rank} R(G,q)=4|V|-\dim\ker R(G,q)$. Since (G,q) is full in Σ, $\dim\mathcal{T}(q)=\dim\mathcal{T}(\Sigma)=4$. The result now follows. □

All of the above discussion is by way of context for the following result, which provides necessary conditions for a full framework in Σ to be *minimally* infinitesimally rigid. We say that a graph $G=(V,E)$ is $(3,4)$-*tight* if $|E|=3|V|-4$ and $|E'|\le 3|V'|-4$ for every subgraph $G'=(V',E')$ containing at least one edge.

Theorem 26. *Suppose that $G=(V,E)$ has at least six vertices and that (G,q) is completely full and infinitesimally rigid in Σ. Furthermore suppose that for any $e\in E$, $(G-e,q)$ is not infinitesimally rigid. Then G is $(3,4)$-tight.*

Proof. Since (G,q) is full in Σ we have $\dim\mathcal{T}(q)=4$. If $|E|<3|V|-4$, then the rigidity matrix $R(G,q)$ has rank less than $4|V|-4$. It follows that $\mathcal{T}(q)$ is a proper subspace of $\ker R(G,q)$ and so (G,q) is infinitesimally flexible, a contradiction. If $|E|>3|V|-4$, then $R(G,q)$ has a non-trivial row dependence $\omega\in\mathbb{R}^{E\cup V}$. By the structure of $R(G,q)$, the rows of $R(G,q)$ indexed by V are linearly independent, and hence $\omega_e\neq 0$ for some edge $e\in E$. It follows that the removal of the edge e does not decrease the rank of the rigidity matrix and so $(G-e,q)$ is still infinitesimally rigid, a contradiction.

Similarly, if there is a non-trivial subgraph $G'=(V',E')$ with $|E'|>3|V'|-4$, then, by the simplicity of G, $|V'|\ge 6$. Since (G,q) is completely full in Σ, the subframework $(G',q_{G'})$ is full in Σ. The $(|E'|+|V'|)\times 4|V'|$ submatrix of $R(G,q)$ corresponding to G' has a non-trivial row dependence with a non-zero support on one of the edges of G'. Thus, as above, it follows that the removal of this edge from G leaves the framework infinitesimally rigid, a contradiction. This gives the result. □

On the other hand we may ask if, given a $(3,4)$-tight graph G, there is a placement q of G in Σ such that (G,q) is minimally infinitesimally rigid. In general this is open. However, in the following section we show this to be true whenever G is the underlying graph of a doubly braced triangulation.

5.5. **Minimal rigidity of doubly braced triangulations.** We say that a placement $q \in \Sigma^V$ of a graph $G = (V, E)$ in Σ is *regular* if the function,

$$r_G : \Sigma^V \to \mathbb{N}, \quad x \mapsto \operatorname{rank} R(G, x)$$

achieves its maximum value at q. Note that the set of regular placements of G in Σ is an open and dense subset of Σ^V. Moreover, if (G, q) is infinitesimally rigid (respectively, flexible) in Σ for some regular placement q then every regular placement of G in Σ is infinitesimally rigid (respectively, flexible). In this case, we say that the graph G is rigid (respectively, flexible) in Σ.

Lemma 27. *The graph $K_5 \cup_{K_3} K_5$ is (minimally) rigid in the hypercylinder Σ.*

Proof. By Lemma 25, it suffices to find a particular placement of the graph whose associated rigidity matrix has rank 24. Since a randomly chosen matrix will, with probability 1, yield a rigidity matrix with maximum rank it is easy to find such a placement. For example we have verified that the following placement yields the required rank:

$$q_1 = \frac{1}{\sqrt{3}}(1,1,1,1) \qquad q_2 = \frac{1}{\sqrt{17}}(3,2,2,1) \qquad q_3 = \frac{1}{\sqrt{17}}(2,3,2,3)$$

$$q_4 = \frac{1}{\sqrt{11}}(1,3,1,1) \qquad q_5 = \frac{1}{\sqrt{17}}(3,2,2,2)$$

$$q_6 = \frac{1}{\sqrt{14}}(2,3,1,1) \qquad q_7 = \frac{1}{\sqrt{14}}(2,1,3,2)$$

where one K_5 is induced by $q_1, \ldots, q_5$ and the other is induced by $q_3, \ldots, q_7$. □

Lemma 28. *The graph $K_6 - e$ is (minimally) rigid in the hypercylinder Σ.*

Proof. As for Lemma 27, it suffices to find one placement that yields a rigidity matrix of rank 20. In this case the placement

$$q_1 = \frac{1}{\sqrt{11}}(1,3,1,1) \qquad q_2 = \frac{1}{3}(2,2,1,1) \qquad q_3 = \frac{1}{\sqrt{22}}(3,2,3,3)$$

$$q_4 = \frac{1}{\sqrt{22}}(3,3,2,2) \qquad q_5 = \frac{1}{\sqrt{27}}(3,3,3,1) \qquad q_6 = \frac{1}{\sqrt{14}}(1,2,3,2)$$

where the missing edge is between q_5 and q_6, yields the required rank. □

For each $k \in \mathbb{N}$ define,

$$H_k = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \, x_1^2 + x_2^2 + x_3^2 < \frac{1}{k^4} \text{ and } \frac{1}{2k} < x_4 < \frac{1}{k} \right\}.$$

Note that H_k is the interior of a truncated hypercylinder with radius $\frac{1}{k^2}$ and height $\frac{1}{2k}$.

Lemma 29. *Let $k \in \mathbb{N}$, let $x \in H_k$ and let $e_4 = (0,0,0,1) \in \mathbb{R}^4$. Then*

$$\left\| \frac{x}{\|x\|} - e_4 \right\| < \frac{2\sqrt{2}}{k}.$$

Proof. Note that $0 < x_4 \leq \|x\|$ and $\frac{1}{\|x\|} < 2k$. We have,

$$\begin{aligned}
\left\| \frac{x}{\|x\|} - e_4 \right\|^2 &= \left\| \frac{(x_1, x_2, x_3, x_4 - \|x\|)}{\|x\|} \right\|^2 \\
&< \left(\frac{1}{k^4} + (x_4 - \|x\|)^2 \right) 4k^2 \\
&= \left(\frac{1}{k^4} + \|x\|^2 + x_4^2 - 2x_4\|x\| \right) 4k^2 \\
&\leq \left(\frac{1}{k^4} + \|x\|^2 - x_4^2 \right) 4k^2 \\
&= \left(\frac{1}{k^4} + x_1^2 + x_2^2 + x_3^2 \right) 4k^2 \\
&< \frac{8}{k^2}.
\end{aligned}$$

□

Proposition 30. Suppose that G is (minimally) rigid in the hypercylinder Σ and that G' is obtained from G by a 3-dimensional vertex splitting move. Then G' is also (minimally) rigid in Σ

Proof. We adapt the proof of Lemma 5.1 in [16].

Suppose $G = (V, E)$ has n vertices $v_1, v_2, \ldots, v_n$. Let $G' = (V', E')$ be obtained from G by a 3-dimensional vertex splitting move at the vertex v_1 on the edges v_1v_2 and v_1v_3. Let $V' = V \cup \{v_0\}$. We will show that if G' is flexible in Σ then G is also flexible in Σ.

Suppose G' is flexible in Σ and let $q \in \Sigma^V$ be a regular placement of G in Σ. For convenience we will write,

$$q = (q_{v_1}, q_{v_2}, \ldots, q_{v_n}) = (q_1, q_2, \ldots, q_n).$$

Let $n_1 \in \mathbb{R}^4$ be a normal vector to the tangent plane of Σ at q_1 and let $e_4 = (0,0,0,1) \in \mathbb{R}^4$. Since the set of regular placements of G in Σ is open in Σ^V we may assume that the vectors $q_1 - q_2$, $q_1 - q_3$, n_1 and e_4 are linearly independent in $\mathbb{R}^4$.

Define $q'_v = q_v$ for all $v \in V$ and $q'_{v_0} = q_{v_1}$. Then $q' = (q'_v)_{v \in V'}$ is a non-regular placement of G' in Σ. Again for convenience we write,

$$q' = (q'_{v_0}, q'_{v_1}, q'_{v_2}, \ldots, q'_{v_n}) = (q_1, q_1, q_2, \ldots, q_n).$$

For each $k \in \mathbb{N}$, let B_k denote the open ball in $\mathbb{R}^4$ with centre 0 and radius $\frac{1}{k}$ and consider the following subset of $\mathbb{R}^{4(n+1)}$,

$$U_k = H_k \times \overbrace{B_k \times \cdots \times B_k}^{n}.$$

Let $N_k = (q' + U_k) \cap \Sigma^{V'}$ and note that N_k is a non-empty open subset of $\Sigma^{V'}$. Since the set of regular placements of G' in Σ is dense in $\Sigma^{V'}$, for each $k \in \mathbb{N}$ there exists a regular placement q^k of G' in Σ such that $q^k \in N_k$. Moreover, by applying an isometry of Σ to the components of q^k we may assume that $q^k_{v_1} = q_{v_1}$ for each $k \in \mathbb{N}$. For convenience we write,

$$q^k = (q^k_{v_0}, q^k_{v_1}, \ldots, q^k_{v_n}) = (q^k_0, q_1, q^k_2, \ldots, q^k_n).$$

Note that the sequence (q^k) of regular placements of G' in Σ converges to the non-regular placement q'. Also note that for each $k \in \mathbb{N}$, we have $q^k - q' \in U_k$. In particular, $q^k_0 - q_1 \in H_k$ and so, by Lemma 29,

$$\left\| \frac{q^k_0 - q_1}{\|q^k_0 - q_1\|} - e_4 \right\| < \frac{2\sqrt{2}}{k}.$$

It follows that the sequence of unit vectors $\frac{q^k_0 - q_1}{\|q^k_0 - q_1\|}$ converges to $e_4 = (0, 0, 0, 1) \in \mathbb{R}^4$.

For each $k \in \mathbb{N}$, the framework (G', q^k) is infinitesimally flexible in Σ and so there exists a unit vector $m^k = (m^k_0, m^k_1, \ldots, m^k_n) \in (\mathbb{R}^4)^{V'}$ which is a non-trivial infinitesimal flex of (G', q^k). We may assume, without loss of generality, that m^k has no trivial flex component, in the sense that $P_{q^k}(m^k) = 0$. By passing to a subsequence (using the Bolzano-Weierstrass Theorem), we may assume that the sequence (m^k) converges to a unit norm vector $m' = (m_0, m_1, \ldots, m_n) \in (\mathbb{R}^4)^{V'}$. Note that for each edge v_iv_j in G', we have,

$$m_i \cdot (q'_{v_i} - q'_{v_j}) = \lim_{k\to\infty} m^k_i \cdot (q^k_{v_i} - q^k_{v_j}) = \lim_{k\to\infty} m^k_j \cdot (q^k_{v_i} - q^k_{v_j}) = m_j \cdot (q'_{v_i} - q'_{v_j}),$$

and for each vertex v_i in G' we have,

$$\pi(m_i) \cdot \pi(q'_{v_i}) = \lim_{k\to\infty} \pi(m^k_i) \cdot \pi(q^k_{v_i}) = 0.$$

Moreover, by Lemma 24,

$$P_{q'}(m') = \lim_{k\to\infty} P_{q^k}(m^k) = 0.$$

Thus, m' is a non-trivial infinitesimal flex of (G', q').

We claim that $m_0 = m_1$. To see this, note that since m' is an infinitesimal flex of (G', q') we have, for $i = 2, 3$,

$$m_1 \cdot (q_1 - q_i) = m_1 \cdot (q'_{v_1} - q'_{v_i}) = m_i \cdot (q'_{v_1} - q'_{v_i}) = m_i \cdot (q_1 - q_i),$$

$$m_0 \cdot (q_1 - q_i) = m_0 \cdot (q'_{v_0} - q'_{v_i}) = m_i \cdot (q'_{v_0} - q'_{v_i}) = m_i \cdot (q_1 - q_i).$$

Thus, $(m_0 - m_1) \cdot (q_1 - q_i) = 0$ for $i = 2, 3$. We also have,

$$(m_0 - m_1) \cdot n_1 = \pi(m_0) \cdot \pi(q'_{v_0}) - \pi(m_1) \cdot \pi(q'_{v_1}) = 0,$$

and since m^k is an infinitesimal flex of (G', q^k),

$$(m_0 - m_1) \cdot e_4 = \lim_{k\to\infty} (m^k_0 - m^k_1) \cdot \frac{q^k_0 - q_1}{\|q^k_0 - q_1\|} = 0.$$

Thus, $m_0 - m_1$ is orthogonal to the four linearly independent vectors $q_1 - q_2$, $q_1 - q_3$, n_1 and e_4 and hence $m_0 = m_1$.

It now follows that the vector $m = (m_1, m_2, \dots, m_n)$ is a non-trivial infinitesimal flex of (G, q). We conclude that G is flexible in Σ. □

Theorem 31. *Let G be the graph of a doubly braced triangulation. Then G is (minimally) rigid in the hypercylinder Σ.*

Proof. This follows immediately from Theorem 14, Lemmas 27, 28 and Proposition 30. □

6. Application: rigidity for mixed norms on $\mathbb{R}^3$

The rigidity theory of bar-joint frameworks in non-Euclidean finite dimensional real normed linear spaces was first considered in [14]. This and subsequent work has explored special classes of norms, particularly the classical ℓ_p-norms, polyhedral norms, unitarily invariant matrix norms and product norms (eg. [4, 12, 13, 14, 15]). In this section, we consider a new context provided by a class of *mixed norms* on $\mathbb{R}^3$.

6.1. **The normed space $\ell^3_{2,p}$.** For $p \in (1, \infty)$, define the *mixed $(2, p)$-norm* on $\mathbb{R}^3$ by,

$$\|(x, y, z)\|_{2,p} = ((x^2 + y^2)^{\frac{p}{2}} + |z|^p)^{\frac{1}{p}}.$$

We denote the normed spaces $(\mathbb{R}^3, \|\cdot\|_{2,p})$ by $\ell^3_{2,p}$. Note that the $(2, 2)$-norm is the standard Euclidean norm on $\mathbb{R}^3$. Our main interest will be the non-Euclidean $(2, p)$-norms (i.e. when $p \neq 2$). The main result in this section states that the graph of a doubly braced triangulation is minimally rigid in $\ell^3_{2,p}$ for all $p \in (1, \infty)$, $p \neq 2$ (Theorem 51).

Remark 32. We have excluded the extreme case where $p = 1$ as our geometric techniques are not applicable in that setting. In particular, the $(2, 1)$-norm is neither smooth nor strictly convex (note that the unit sphere is a double cone). For similar reasons we will not consider the $(2, \infty)$-norm,

$$\|(x, y, z)\|_{2,\infty} = \max\{(x^2 + y^2)^{\frac{1}{2}}, |z|\}.$$

(Note that in this case the unit sphere is cylindrical.) Rigidity theory for the $(2, \infty)$-norm is developed in [13, §5.1] using different techniques.

Our first goal is to collect some preliminary geometric results which will be required later on.

Lemma 33. *Let $p \in (1, \infty)$. Then the dual space of $\ell^3_{2,p}$ is $\ell^3_{2,q}$, where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Given $y \in \ell^3_{2,q}$, define $f_y : \ell^3_{2,p} \to \mathbb{R}$, $x \mapsto x \cdot y$. Note that for every $x \in \ell^3_{2,p}$, the Cauchy-Schwarz and Hölder inequalities imply that

$$|f_y(x)| \leq \sum_{i=1}^{3} |x_i y_i| \leq \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\|_2 + |x_3 y_3| \leq \left\| \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\|_{2,p} \left\| \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\|_{2,q}.$$

Hence it suffices to show that the contraction,

$$T : \ell^3_{2,q} \to (\ell^3_{2,p})^*, \quad y \mapsto f_y,$$

is an isometry. Let $y \in \ell^3_{2,q}$ be non-zero. There exists $\theta \in [0, 2\pi)$ such that,

$$y^\theta = R_\theta y = \begin{bmatrix} y_1^\theta \\ 0 \\ y_3^\theta \end{bmatrix},$$

where R_θ is the isometry given by clockwise rotation by θ about the z-axis. Choose x_θ, such that $x_k^\theta = |y_k^\theta|^{q-1}\operatorname{sgn}(y_k^\theta)$. Note that $f_y(R_{-\theta}x^\theta) = y \cdot R_{-\theta}x^\theta = R_\theta y \cdot x^\theta = f_{y^\theta}(x^\theta)$. Also,

$$\|x^\theta\|_{2,p} = \left(\sum_{k=1}^3 |y_k^\theta|^q\right)^{1/p}.$$

Hence we have,

$$\|f_y\|_{2,p}^* \geq \frac{f_y(R_{-\theta}x^\theta)}{\|R_{-\theta}x^\theta\|_{2,p}} = \frac{f_{y^\theta}(x^\theta)}{\|x^\theta\|_{2,p}} = \frac{\sum_{k=1}^3 |y_k^\theta|^q}{\left(\sum_{i=1}^3 |y_k^\theta|^q\right)^{1/p}} = \left(\sum_{k=1}^3 |y_k^\theta|^q\right)^{1/q} = \|y_\theta\|_{2,q} = \|y\|_{2,q},$$

and so $\|f_y\|_{2,p}^* = \|y\|_{2,q}$. □

Lemma 34. *The space $\ell^3_{2,p}$ is smooth and strictly convex for every $p \in (1, \infty)$.*

Proof. By [15, Lemma 1], it suffices to show that for all non-zero (x, y, z) and (a, b, c) in $\ell^3_{2,p}$, the function

$$\zeta : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \|(x, y, z) + t(a, b, c)\|_{2,p}$$

is differentiable at zero. Note that,

$$\zeta = h \circ (f + g),$$

where $f(t) = ((x + ta)^2 + (y + tb)^2)^{\frac{p}{2}}$, $g(t) = |z + tc|^p$ and $h(t) = t^{\frac{1}{p}}$ $(t > 0)$. Applying the chain rule, it suffices to show that f, g are differentiable at 0. We shall prove it only for f, since the same arguments will work for g. When $(x, y) \neq (0, 0)$, then $(x+ta)^2+(y+tb)^2 > 0$ for t sufficiently close to zero, and hence f is differentiable. So it remains to check the case for $(x, y) = 0$. Then

$$\left|\frac{((ta)^2 + (tb)^2)^{p/2}}{t}\right| = |t|^{p-1}(a^2 + b^2)^{p/2}.$$

Since

$$\lim_{t \to 0} |t|^{p-1}(a^2 + b^2)^{p/2} = 0,$$

it follows that f is differentiable with $f'(0) = 0$.

By Lemma 33, for each $p \in (1, \infty)$ the space $\ell^3_{2,p}$ is reflexive and the dual of a smooth space. Thus $\ell^3_{2,p}$ is also strictly convex (see [2, p. 184] eg.). □

6.2. **Isometries of $\ell^3_{2,p}$.** Next, we determine the identity component $\mathrm{Isom}_0(\ell^3_{2,p})$ of the isometry group $\mathrm{Isom}(\ell^3_{2,p})$ for $p \neq 2$. It is known, by the Mazur-Ulam Theorem ([19, Theorem 3.1.2]), that for every isometry ϕ on a real finite dimensional normed space X, there exists a linear isometry T_ϕ on X and $t_\phi \in X$, such that

$$\phi(x) = T_\phi x + t_\phi, \text{ for all } x \in X.$$

Moreover, the map $\phi \mapsto T_\phi$ is a group homomorphism with kernel equal to the group of translations on X. Hence we can focus on linear isometries. To do this, we recall John's theorem regarding the Löwner-John ellipsoid, that is the ellipsoid of maximal volume, inside the unit ball of X (see [10] and [19, Theorem 3.3.1]).

Theorem 35 (John)**.** *Each convex body K in $\mathbb{R}^n$ contains a unique ellipsoid of maximal volume, called the inner Löwner-John ellipsoid. This ellipsoid is equal to the Euclidean unit ball B_2^n if and only if B_2^n is contained in K and there exist unit vectors $u_i \in \partial K$ and positive numbers c_i, $i = 1, \ldots, m$, such that:*

(i) $\sum_{i=1}^{m} c_i u_i = 0$;

(ii) $\sum_{i=1}^{m} c_i \langle x, u_i \rangle^2 = \|x\|_2^2$ *for all* $x \in \mathbb{R}^n$.

Corollary 36. *Let X be a normed space that is associated with an inner Löwner-John ellipsoid E. Then the group* $\mathrm{Isom}(X)$ *of isometries of X is a subgroup of the isometry group of the Euclidean space with unit ball E.*

Proof. Given a linear isometry T on X, note that T maps B_X to itself. Since T is volume preserving, it follows by the uniqueness of the inner Löwner-John ellipsoid that $T(E) = E$, so T is also an isometry of the Euclidean space associated with E. □

Let $B^3_{2,p}$ denote the closed unit ball in $\ell^3_{2,p}$ and let B^3_2 denote the closed unit ball in Euclidean space $\mathbb{R}^3$.

Lemma 37. *Let $p \in (2, \infty)$. Then the inner Löwner-John ellipsoid for $B^3_{2,p}$ is B^3_2.*

Proof. We apply Theorem 35 with $K = B^3_{2,p}$. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis in $\mathbb{R}^3$. Note that for $p > 2$, B^3_2 is contained in $B^3_{2,p}$ and each vector e_i lies on $\partial B^3_{2,p}$, $i = 1, 2, 3$. Define for $i = 1, 2, 3$ the vectors $u_i = e_i$, $u_{i+3} = -e_i$ and the scalars $c_i = \frac{1}{2}$, $c_{i+3} = c_i$. Property (i) of Theorem 35 is evident, while property (ii) is satisfied by Parseval's identity. The result follows. □

Recall that the orientation preserving isometries on the Euclidean space $\mathbb{R}^n$ are of the form $\phi(x) = T_\phi x + t_\phi$ with $T_\phi \in \mathrm{SO}(n)$, meaning that $\det T_\phi = 1$. Hence the identity component $\mathrm{Isom}_0(\mathbb{R}^n)$ is generated by translations and rotations.

Proposition 38. Let $p \in (1, \infty)$, $p \neq 2$. Then $\mathrm{Isom}_0(\ell^3_{2,p})$ is the group generated by rotations R about the z-axis and translations T_t, $t \in \ell^3_{2,p}$.

Proof. Let T be a linear isometry that lies in $\text{Isom}_0(\ell^3_{2,p})$. We consider first the case $p > 2$. It follows by Corollary 36 and Lemma 37 that the linear isometries of $\ell^3_{2,p}$ are a subgroup of the group of linear isometries of Euclidean space ℓ^3_2. Hence T leaves invariant the set

$$\partial B^3_2 \cap \partial B^3_{2,p} = \{(x, y, 0) : x^2 + y^2 = 1\} \cup \{(0, 0, \pm 1)\}.$$

Since T fixes both poles $(0, 0, \pm 1)$ it also fixes the z-axis. Thus, T is a rotation operator about the z-axis.

Now suppose $p \in (1, 2)$. Note that in this case the dual operator T^* is a linear isometry on $\ell^3_{2,q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, T^* lies in the identity component $\text{Isom}_0(\ell^3_{2,q})$. Since $q = 1 + \frac{1}{p-1} > 2$, the above argument shows that T^* is a rotation operator about the z-axis. It follows that T is also a rotation operator about the z-axis. □

6.3. **Rigid motions of $\ell^3_{2,p}$.** Let X be a finite dimensional real normed linear space. A *rigid motion* of X is a collection $\alpha = \{\alpha_x : [-1, 1] \to X\}_{x \in X}$ with the properties that:

(i) α_x is a continuous path, for all $x \in X$;
(ii) $\alpha_x(0) = x$ for any $x \in X$;
(iii) $\|\alpha_x(t) - \alpha_y(t)\| = \|x - y\|$ for all $x, y \in X$ and all $t \in [-1, 1]$.

Proposition 39. Let X be a finite dimensional real normed linear space and let $\alpha = \{\alpha_x\}_{x \in X}$ be a rigid motion of X. For each $t \in [-1, 1]$, define

$$\beta_t : X \to X, \quad \beta_t(x) = \alpha_x(t) - \alpha_0(t).$$

Then,

(i) $\beta_t \in \text{Isom}_0(X)$ for each $t \in [-1, 1]$.
(ii) The map $\beta : [-1, 1] \to \text{Isom}_0(X)$, $t \mapsto \beta_t$, is continuous.

Proof. Note that, for each $t \in [-1, 1]$, β_t is isometric and $\beta_t(0) = 0$. It follows, by the Mazur-Ulam Theorem, that β_t is a linear isometry. Let $t_0 \in [-1, 1]$ and let $\epsilon > 0$. Since the unit ball B_X is compact, we can choose $x_1, x_2, \ldots, x_n \in B_X$, such that

$$B_X \subseteq \bigcup_{i=1}^{n} B\left(x_i, \frac{\epsilon}{4}\right) \quad \text{and} \quad 0 \in \{x_i\}_{i=1}^n.$$

Since the paths $\alpha_{x_1}, \ldots, \alpha_{x_n}$ are continuous we can choose $\delta > 0$ such that for all $t \in [-1, 1]$,

$$|t - t_0| < \delta \quad \implies \quad \max_{1 \le i \le n} \|\alpha_{x_i}(t) - \alpha_{x_i}(t_0)\| < \frac{\epsilon}{4}$$

Let $x \in B_X$. Then there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\|x - x_{i_0}\|_X \le \frac{\epsilon}{4}$. For each $t \in [-1, 1]$ we have,

$$\begin{aligned} \|\alpha_x(t) - \alpha_x(t_0)\| &\le \|\alpha_x(t) - \alpha_{x_{i_0}}(t)\| + \|\alpha_{x_{i_0}}(t) - \alpha_{x_{i_0}}(t_0)\| + \|\alpha_{x_{i_0}}(t_0) - \alpha_x(t_0)\| \\ &\le 2\|x - x_{i_0}\| + \frac{\epsilon}{4} \le \frac{3\epsilon}{4} \end{aligned}$$

Hence for all $t \in (t_0 - \delta, t_0 + \delta)$ we have

$$\begin{aligned} \|\beta_t(x) - \beta_{t_0}(x)\| &= \|(\alpha_x(t) - \alpha_0(t)) - (\alpha_x(t_0) - \alpha_0(t_0))\| \\ &\leq \|\alpha_x(t) - \alpha_x(t_0)\| + \|\alpha_0(t) - \alpha_0(t_0)\| \\ &\leq \frac{3\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

Since δ is independent of $x \in B_X$, it follows that for all $t \in [-1,1]$,

$$|t - t_0| < \delta \quad \Longrightarrow \quad \|\beta_t - \beta_{t_0}\|_{op} < \epsilon.$$

Thus the map $\beta : [-1,1] \to \mathrm{Isom}(X)$, $t \mapsto \beta_t$, is continuous.

Finally, note that $\beta([-1,1])$ is a connected subset of $\mathrm{Isom}(X)$ which contains the identity on X. Hence β_t lies in $\mathrm{Isom}_0(X)$ for all $t \in [-1,1]$.

□

Corollary 40. *Let $p \in (1,\infty)$, $p \neq 2$. A collection of maps $\alpha = \{\alpha_x : [-1,1] \to \ell^3_{2,p}\}_{x \in \ell^3_{2,p}}$ is a rigid motion of $\ell^3_{2,p}$ if and only if there exists a continuous map $\theta : [-1,1] \to \mathbb{R}$ which satisfies $\theta(0) = 0$ such that for each $x = (x_1, x_2, x_3) \in \ell^3_{2,p}$ and $t \in [-1,1]$,*

$$\alpha_x(t) = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \alpha_0(t). \tag{7}$$

Proof. Suppose α is a rigid motion of $\ell^3_{2,p}$. Then equation (7) follows directly from Propositions 38 and 39. Moreover, since $\alpha_x(t)$ is continuous on $[-1,1]$, the same also holds for the map $t \mapsto \theta(t)$. Note also that we can take $\theta(0) = 0$. The converse direction is clear. □

Let $\alpha = \{\alpha_x\}_{x \in X}$ be a rigid motion of a normed space X. If each α_x is differentiable at $t = 0$ then the map $\eta : X \to X$, $\eta(x) = \alpha'_x(0)$, is called an *infinitesimal rigid motion* of X. The collection of all infinitesimal rigid motions of X is a real vector space, denoted $\mathcal{T}(X)$.

Theorem 41. *Let $p \in (1,\infty)$, $p \neq 2$, and let $\eta : \ell^3_{2,p} \to \ell^3_{2,p}$ be an affine map. Then $\eta \in \mathcal{T}(\ell^3_{2,p})$ if and only if there exists a scalar $\lambda \in \mathbb{R}$ and a vector $c \in \mathbb{R}^3$ such that,*

$$\eta(x_1, x_2, x_3) = \lambda(-x_2, x_1, 0) + c,$$

for all $(x_1, x_2, x_3) \in \mathbb{R}^3$.

In particular, $\dim \mathcal{T}(\ell^3_{2,p}) = 4$.

Proof. By Corollary 40, if η is an infinitesimal rigid motion of $\ell^3_{2,p}$ then there exists θ such that, for each $x \in \mathbb{R}^3$, $\eta(x)$ is given by,

$$\eta(x) = \frac{d}{dt}\left(\begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \alpha_0(t) \right)\Bigg|_{t=0} = \theta'(0) \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} + \alpha'_0(0).$$

For the converse, suppose η is an affine map of the form

$$\eta(x_1, x_2, x_3) = \lambda(-x_2, x_1, 0) + c,$$

for some scalar $\lambda \in \mathbb{R}$ and some vector $c \in \mathbb{R}^3$. Consider the collection of continuous paths,

$$\alpha_x(t) = \begin{bmatrix} \cos(\lambda t) & -\sin(\lambda t) & 0 \\ \sin(\lambda t) & \cos(\lambda t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + tc.$$

Then α is a rigid motion of $\ell^3_{2,p}$ that satisfies $\eta(x) = \alpha_x'(0)$ for each $x \in \ell^3_{2,p}$. □

6.4. **Full sets in $\ell^3_{2,p}$.** Let X be a normed linear space and let $S \subseteq X$ be a non-empty set. We say that S is *isometrically full* in X if the only isometry in $\mathrm{Isom}_0(X)$ which fixes every point in S is the identity map.

Lemma 42. *If S has full affine span in X then S is isometrically full in X.*

Proof. Suppose that there exists $\phi \in \mathrm{Isom}_0(X)$ such that $\phi(s) = s$ for every $s \in S$. Note that ϕ is of the form $\phi(x) = Ax + b$, for some linear operator A and $b \in X$. Fix some element $s_0 \in S$. Then the operator A also lies in $\mathrm{Isom}_0(X)$ and it is the identity on the linear span of the set $\{s - s_0 : s \in S\}$. Since S has full affine span, it follows that A is the identity. Since $b = \phi(s) - s = 0$ we see that ϕ is the identity map. □

Define the restriction map,

$$\rho_S : \mathcal{T}(X) \to X^S : \eta \mapsto (\eta(s))_{s\in S}.$$

We say that S is *full* in X if ρ_S is injective (see [13]).

Proposition 43. Let $p \in (1, \infty)$, $p \neq 2$, and let S be a non-empty subset of $\ell^3_{2,p}$. The following statements are equivalent.

(i) S is full in $\ell^3_{2,p}$.
(ii) S is isometrically full in $\ell^3_{2,p}$.
(iii) The orthogonal projection of S onto the xy-plane contains at least two points.

Proof. Let P_{xy} denote the projection of $\ell^3_{2,p}$ onto the xy-plane along the z-axis.

$(i) \Leftrightarrow (iii)$ Suppose that $P_{xy}(S) = \{s\}$. Say $s = (s_1, s_2, 0)$ and define

$$\eta : \ell^3_{2,p} \to \ell^3_{2,p}, \quad x \mapsto \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} s_2 \\ -s_1 \\ 0 \end{bmatrix}.$$

By Theorem 41, η is an infinitesimal rigid motion of $\ell^3_{2,p}$. Note that $\rho_S(\eta) = 0$ and so S is not full in $\ell^3_{2,p}$.

Let us now assume that there exist $s, r \in S$ such that $P_{xy}(s) \neq P_{xy}(r)$. If S is not full, then there exists a non-zero $\eta \in \mathcal{T}(\ell^3_{2,p})$ that satisfies $\eta(s) = \eta(r) = 0$. Write $s = (s_1, s_2, s_3)$ and $r = (r_1, r_2, r_3)$. Then, by Theorem 41, it follows that $(-s_2, s_1, 0) = (-r_2, r_1, 0)$. Hence $P_{xy}(s) = P_{xy}(r)$, a contradiction.

$(ii) \Leftrightarrow (iii)$ Suppose first that S is isometrically full and that the set $P_{xy}(S) = \{P_{xy}(s) : s \in S\}$ is a singleton $\{a\}$. Then for every rotation R about the z-axis we have

$$T_a R T_{-a} s = T_a R(s - a) = T_a(s - a) = s, \quad \forall\, s \in S,$$

a contradiction.

For the converse, suppose there exists $s_1, s_2 \in S$ such that $P_{xy}(s_1) \neq P_{xy}(s_2)$. Note that, by Proposition 38, it follows that every isometry in $\mathrm{Isom}_0(\ell^3_{2,p})$ can be written in the form $T_t R$ for some rotation R about the z-axis and some translation T_t. Suppose $T_t R(s) = s$ for each $s \in S$ and let $s_1, s_2 \in S$ be such that $P_{xy}(s_1) \neq P_{xy}(s_2)$. Then $s_1 - s_2 = T_t R(s_1) - T_t R(s_2) = R(s_1 - s_2)$ and so,

$$P_{xy}(s_1 - s_2) = P_{xy} R(s_1 - s_2) = R P_{xy}(s_1 - s_2).$$

If R is not the identity map then $P_{xy}(s_1 - s_2) = 0$, a contradiction. It follows that $T_t R$ is the identity map and so S is full. □

Remark 44. We expect that in any finite dimensional real normed linear space a subset S is full if and only if it is isometrically full, but we are currently unaware of such a proof.

6.5. **Frameworks in $\ell^3_{2,p}$.** Let $G = (V, E)$ be a finite simple graph. A *(bar-joint) framework* in X is a pair (G, q) where $q = (q_v)_{v \in V} \in X^V$ and $q_v \neq q_w$ whenever $vw \in E$. A *subframework* of (G, q) is a framework (H, q_H) where $H = (V(H), E(H))$ is a subgraph of G and $q_H(v) = q(v)$ for all $v \in V(H)$.

A framework (G, q) is said to be *full* in X if the set $S = \{q_v : v \in V\}$ is full in X. A framework (G, q) is *completely full* in X if it is full in X and every subframework of (G, q) containing at least $2\dim(X)$ vertices is also full in X.

The *rigidity map* for G and X is defined by $f_G : X^V \to \mathbb{R}^E$, $x \mapsto (\|x_v - x_w\|)_{vw \in E}$. An *infinitesimal flex* of a bar-joint framework (G, q) in X is a vector $m \in X^V$ such that, for each edge $vw \in E$, the directional derivative of the rigidity map f_G in the direction of m vanishes,

$$\lim_{t \to 0} \frac{1}{t}(f_G(q + tm) - f_G(q)) = 0.$$

An infinitesimal flex $m \in X^V$ is said to be *trivial* if there exists an infinitesimal rigid motion $\eta \in \mathcal{T}(X)$ such that $m_v = \eta(q_v)$ for all $v \in V$. A bar-joint framework (G, q) in X is *infinitesimally rigid* if and only if every infinitesimal flex of (G, q) is trivial.

Lemma 45. *Let (G, q) be a bar-joint framework in $\ell^3_{2,p}$, where $p \in (1, \infty)$. Then a vector $m \in X^V$ is an infinitesimal flex of (G, q) if and only if for each edge $vw \in E$ we have,*

$$\begin{cases} (x, y, \frac{\mathrm{sgn}(z)|z|^{p-1}}{d^{p-2}}) \cdot (a, b, c) = 0, & \text{if } d \neq 0, \\ c = 0, & \text{otherwise.} \end{cases}$$

where $q_v - q_w = (x, y, z)$, $m_v - m_w = (a, b, c)$ and $d = (x^2 + y^2)^{\frac{1}{2}}$.

Proof. For each edge $vw \in E$, consider the function

$$\zeta_{vw} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \|(q_v + tm_v) - (q_w + tm_w)\|_{2,p}.$$

Note that m is an infinitesimal flex for (G, q) if and only if $\zeta'_{vw}(0) = 0$ for each edge $vw \in E$. As in the proof of Lemma 34, by expressing ζ_{vw} in the form $\zeta_{vw} = h \circ (f + g)$ we can show that ζ_{vw} is differentiable at 0. If $d \neq 0$ then using the chain rule we compute,

$$\zeta'_{vw}(0) = (d^p + |z|^p)^{\frac{1}{p} - 1}(d^{p-2}(xa + yb) + \mathrm{sgn}(z)|z|^{p-1}c),$$

Rearranging the above we obtain the desired equation. If $d = 0$, then $z \neq 0$, so we have,

$$\zeta'_{vw}(0) = h'(f(0) + g(0))(f'(0) + g'(0)) = (|z|^p)^{\frac{1-p}{p}} \operatorname{sgn}(z)|z|^{p-1}c = \operatorname{sgn}(z)c.$$

The result follows. ☐

6.6. **The rigidity matrix.** We define the *rigidity matrix* $R(G, q)$ for a graph G and a vector $q \in \left(\ell^3_{2,p}\right)^V$ to be the $|E| \times 3|V|$ matrix with rows indexed by E, columns indexed by $V \times \{1, 2, 3\}$ and entries defined as follows: Let $vw \in E$. Write $q_v - q_w = (x, y, z)$ and $d = (x^2 + y^2)^{\frac{1}{2}}$. When $d \neq 0$ then the entries of the row indexed by vw are given by,

$$\begin{array}{c} \\ vw \end{array} \begin{bmatrix} & & & (v,1) & (v,2) & (v,3) & & & & (w,1) & (w,2) & (w,3) & & & \\ 0 & \cdots & 0 & x & y & \frac{\operatorname{sgn}(z)|z|^{p-1}}{d^{p-2}} & 0 & \cdots & 0 & -x & -y & -\frac{\operatorname{sgn}(z)|z|^{p-1}}{d^{p-2}} & 0 & \cdots & 0 \end{bmatrix}.$$

When $d = 0$, then the entries of the row indexed by vw are given by,

$$\begin{array}{c} \\ vw \end{array} \begin{bmatrix} & & & (v,1) & (v,2) & (v,3) & & & & (w,1) & (w,2) & (w,3) & & & \\ 0 & \cdots & 0 & 0 & 0 & z & 0 & \cdots & 0 & 0 & 0 & -z & 0 & \cdots & 0 \end{bmatrix}.$$

Lemma 46. *Let $p \in (1, \infty)$, $p \neq 2$. A full bar-joint framework (G, q) in $\ell^3_{2,p}$ is infinitesimally rigid if and only if* $\operatorname{rank} R(G, q) = 3|V| - 4$.

Proof. By Lemma 45, the kernel of $R(G, q)$ is the linear space of infinitesimal flexes of (G, q). Also, $\operatorname{rank} R(G, q) = 3|V| - \dim\ker R(G, q)$. Since (G, q) is full, by Theorem 41 the infinitesimal rigid motions of $\ell^3_{2,p}$ induce a 4-dimensional space of trivial infinitesimal flexes on (G, q). The result now follows. ☐

This gives the following analogue of Theorem 26.

Theorem 47. *Let $p \in (1, \infty)$, $p \neq 2$. Suppose that $G = (V, E)$ has at least six vertices and that (G, q) is an infinitesimally rigid and completely full framework in $\ell^3_{2,p}$. Furthermore, suppose that for any $e \in E$, $(G - e, q)$ is not infinitesimally rigid. Then G is $(3, 4)$-tight.*

Proof. If $|E| < 3|V|-4$, then the rigidity matrix $R(G, q)$ has rank less than $3|V|-4$. Hence, by Lemma 46, (G, q) is infinitesimally flexible, a contradiction. Now suppose $|E| > 3|V|-4$. By Lemma 46, $R(G, q)$ has a non-trivial row dependence and hence there is an edge whose removal does not decrease the rank of the rigidity matrix. Thus $G-e$ is still infinitesimally rigid, a contradiction.

Similarly, if there is a non-trivial subgraph $G' = (V', E')$ with $|E'| > 3|V'| - 4$, then, by the simplicity of G, $|V'| \geq 6$. Since (G, q) is completely full, the subframework $(G', q_{G'})$ is full in $\ell^3_{2,p}$. Thus, by Lemma 46, the $|E'| \times 3|V'|$ submatrix of $R(G, q)$ corresponding to G' has a non-trivial row dependence. Thus, there is an edge of G' whose removal from G leaves the framework infinitesimally rigid, a contradiction. ☐

It is open as to whether every $(3, 4)$-tight graph can be realised as a minimally infinitesimally rigid framework in $\ell^3_{2,p}$, when $p \in (1, \infty)$ and $p \neq 2$. However, we will now show that if G is the graph of a doubly braced triangulation, then such a realisation of G always exists.

6.7. **Minimal rigidity of doubly braced triangulations in $\ell^3_{2,p}$.** We first show that the irreducible base graphs given in Theorem 14 can be realised as minimally infinitesimally rigid bar-joint frameworks in $\ell^3_{2,p}$ whenever $p \in (1,\infty)$ and $p \neq 2$.

Example 48. Consider the base graph $K_6 - e$. Let $V(K_4 - e) = \{s_1, s_2, s_3, s_4\}$ where $e = s_2s_4$ is the deleted edge. To obtain $K_6 - e$ we cone $K_4 - e$ with a vertex v_0 and the resulting graph with another vertex v_1. Note that $K_6 - e$ is the underlying graph of the irreducible doubly braced triangulations given in Figures 3, 4 and 5 (see also Figure 10).

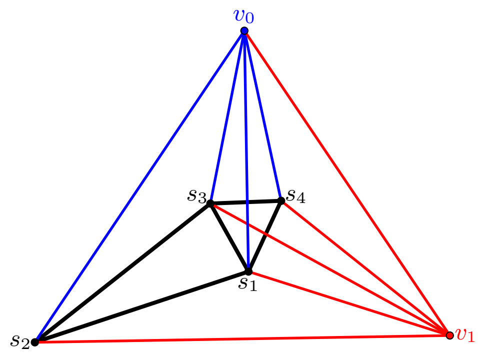

FIGURE 10. The base graph $K_6 - e$.

Let q be the following placement:

$$s_1 = (1,0,0), \quad s_2 = (0,1,0), \quad s_3 = (-1,0,0), \quad s_4 = (0,-1,0),$$

$$v_0 = (1,1,1), \quad v_1 = (0,0,-1).$$

Then the rigidity matrix is of the form

$$R(K_6 - e, q) = \begin{bmatrix} A & 0 \\ * & D \end{bmatrix},$$

where the submatrix A contains the entries arising from the x and y coordinates of the edges of $K_4 - e$ and is of the form

$$A = \begin{array}{c} \\ s_1s_2 \\ s_1s_3 \\ s_1s_4 \\ s_2s_3 \\ s_3s_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} (s_1{,}1) & (s_1{,}2) & (s_2{,}1) & (s_2{,}2) & (s_3{,}1) & (s_3{,}2) & (s_4{,}1) & (s_4{,}2) \end{array} \\ \left[\begin{array}{cccccccc} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \end{array}\right] \end{array}$$

and D, which lies in $M_{9\times 10}(\mathbb{R})$, is given below;

$$D = \begin{array}{c} \\ s_1v_0 \\ s_2v_0 \\ s_3v_0 \\ s_4v_0 \\ s_1v_1 \\ s_2v_1 \\ s_3v_1 \\ s_4v_1 \\ v_0v_1 \end{array} \begin{array}{cccccccccc} (s_1,3) & (s_2,3) & (s_3,3) & (s_4,3) & (v_0,1) & (v_0,2) & (v_0,3) & (v_1,1) & (v_1,2) & (v_1,3) \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{5}^{2-p} & 0 & 2 & 1 & \sqrt{5}^{2-p} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{5}^{2-p} & 1 & 2 & \sqrt{5}^{2-p} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & \sqrt{2}^p & -1 & -1 & -\sqrt{2}^p \end{array}$$

Since the rows of the matrix A are evidently linearly independent, it suffices to show that the rows of the matrix D are also linearly independent. In the remaining argument, the row operations will be indicated with the standard notation. For example, the fifth row of the matrix D_1 below is the sum of the first and the fifth row of the matrix D, so we write $R_5 = r_5 + r_1$.

$$D_1 = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ R_5=r_5+r_1 \\ R_6=r_6+r_2 \\ R_7=\sqrt{5}^{2-p}r_7+r_3 \\ R_8=\sqrt{5}^{2-p}r_8+r_4 \\ r_9 \end{array} \begin{array}{cccccccccc} (s_1,3) & (s_2,3) & (s_3,3) & (s_4,3) & (v_0,1) & (v_0,2) & (v_0,3) & (v_1,1) & (v_1,2) & (v_1,3) \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{5}^{2-p} & 0 & 2 & 1 & \sqrt{5}^{2-p} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{5}^{2-p} & 1 & 2 & \sqrt{5}^{2-p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 1 & \sqrt{5}^{2-p} & \sqrt{5}^{2-p} & 0 & -\sqrt{5}^{2-p} \\ 0 & 0 & 0 & 0 & 1 & 2 & \sqrt{5}^{2-p} & 0 & \sqrt{5}^{2-p} & -\sqrt{5}^{2-p} \\ 0 & 0 & 0 & 0 & 1 & 1 & \sqrt{2}^p & -1 & -1 & -\sqrt{2}^p \end{array}$$

It is evident now that the first four rows of this matrix are linearly independent, so we may focus on the submatrix:

$$D_2 = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{array} \begin{array}{cccccc} (v_0,1) & (v_0,2) & (v_0,3) & (v_1,1) & (v_1,2) & (v_1,3) \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & -1 \\ 2 & 1 & \sqrt{5}^{2-p} & \sqrt{5}^{2-p} & 0 & -\sqrt{5}^{2-p} \\ 1 & 2 & \sqrt{5}^{2-p} & 0 & \sqrt{5}^{2-p} & -\sqrt{5}^{2-p} \\ 1 & 1 & \sqrt{2}^p & -1 & -1 & -\sqrt{2}^p \end{array}$$

Next, we eliminate the matrix elements below the first entry in the main diagonal of the above matrix, and get the equivalent matrix

$$
D_3 = \begin{array}{c} \\ R_1=r_2 \\ R_2=r_1 \\ R_3=r_3-2r_2 \\ R_4=r_4-r_2 \\ R_5=r_5-r_2 \end{array}
\begin{array}{c} \begin{array}{cccccc} (v_0,1) & (v_0,2) & (v_0,3) & (v_1,1) & (v_1,2) & (v_1,3) \end{array} \\
\begin{bmatrix}
1 & 0 & 1 & 0 & -1 & -1 \\
0 & 1 & 1 & -1 & 0 & -1 \\
0 & 1 & -2+\sqrt{5}^{2-p} & \sqrt{5}^{2-p} & 2 & 2-\sqrt{5}^{2-p} \\
0 & 2 & -1+\sqrt{5}^{2-p} & 0 & 1+\sqrt{5}^{2-p} & 1-\sqrt{5}^{2-p} \\
0 & 1 & -1+\sqrt{2}^{p} & -1 & 0 & 1-\sqrt{2}^{p}
\end{bmatrix} \end{array}
$$

Thus, we can remove the first row and the first column. Working in a similar manner we obtain

$$
D_4 = \begin{array}{c} \\ r_1 \\ R_2=r_2+r_1 \\ R_3=r_3+2r_1 \\ R_4=r_4-r_1 \end{array}
\begin{array}{c} \begin{array}{ccccc} (v_0,2) & (v_0,3) & (v_1,1) & (v_1,2) & (v_1,3) \end{array} \\
\begin{bmatrix}
1 & 1 & -1 & 0 & -1 \\
0 & 3-\sqrt{5}^{2-p} & -1-\sqrt{5}^{2-p} & -2 & -3+\sqrt{5}^{2-p} \\
0 & 3-\sqrt{5}^{2-p} & -2 & -1-\sqrt{5}^{2-p} & -3+\sqrt{5}^{2-p} \\
0 & -2+\sqrt{2}^{p} & 0 & 0 & 2-\sqrt{2}^{p}
\end{bmatrix} \end{array}.
$$

Note that the last row of the matrix D_4 becomes zero for $p = 2$. For $p \neq 2$, we remove again the first row and the first column and rearrange

$$
D_5 = \begin{array}{c} \\ R_1=r_3 \\ R_2=r_1-xr_3 \\ R_3=r_2-xr_3 \end{array}
\begin{array}{c} \begin{array}{cccc} (v_0,3) & (v_1,1) & (v_1,2) & (v_1,3) \end{array} \\
\begin{bmatrix}
-2+\sqrt{2}^{p} & 0 & 0 & 2-\sqrt{2}^{p} \\
0 & -1-\sqrt{5}^{2-p} & -2 & 0 \\
0 & -2 & -1-\sqrt{5}^{2-p} & 0
\end{bmatrix} \end{array}
$$

where $x = \frac{3-\sqrt{5}^{2-p}}{-2+\sqrt{2}^{p}}$. Thus, it suffices to show that the matrix

$$
D_6 = \begin{array}{c} \\ R_1=r_2 \\ R_2=r_3 \end{array}
\begin{array}{c} \begin{array}{cc} (v_1,1) & (v_1,2) \end{array} \\
\begin{bmatrix}
-1-\sqrt{5}^{2-p} & -2 \\
-2 & -1-\sqrt{5}^{2-p}
\end{bmatrix} \end{array}
$$

has linearly independent rows, which is true for every $p \neq 2$.

Example 49. Consider the base graph $K_5 \cup_{K_3} K_5$. This graph can be obtained from $K_4 - e$ by repeatedly adding three degree 4 vertices (see Figure 11). We denote $V(K_4 - e) = \{s_1, s_2, s_3, s_4\}$ and the extra vertices by v_1, v_2, v_3. Note that $K_5 \cup_{K_3} K_5$ is the underlying graph of the irreducible doubly braced triangulations given in Figures 6 and 7. The two K_5 subgraphs will be described by the respective vertex sets $\{s_1, s_2, s_3, v_1, v_2\}$ and $\{s_1, s_3, s_4, v_1, v_3\}$. The intersection of those subgraphs is the graph K_3 indicated by the dashed edges in Figure 11.

The placement q of $V(K_4 - e)$ is the same as in Example 48, while the vertices v_1, v_2, v_3 are placed at the respective points $(0, 0, 1)$, $(-1, 1, -1)$, $(-1, -1, -1)$. Following the same

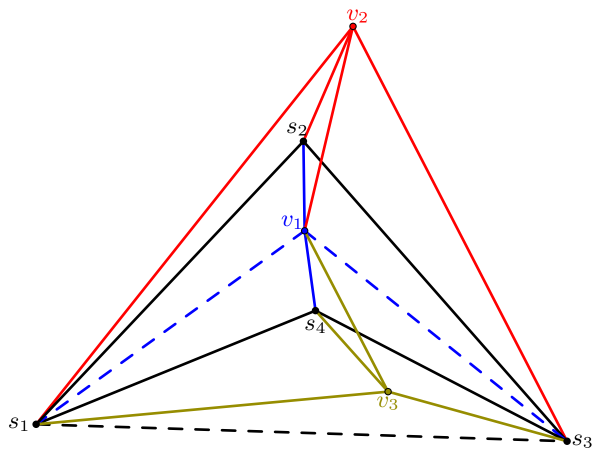

FIGURE 11. The base graph $K_5 \cup_{K_3} K_5$.

procedure as in the previous example, the rigidity matrix of the framework $(K_5 \cup_{K_3} K_5, q)$ is a lower triangular block matrix

$$R(K_5 \cup_{K_3} K_5, q) = \begin{bmatrix} A & 0 \\ * & D \end{bmatrix},$$

where the submatrix A contains the entries arising from the x and y coordinates of the edges of $K_4 - e$, and D is the following matrix

$$D = \begin{array}{c} \\ s_1v_1 \\ s_2v_1 \\ s_3v_1 \\ s_4v_1 \\ s_1v_2 \\ s_2v_2 \\ s_3v_2 \\ v_1v_2 \\ s_1v_3 \\ s_3v_3 \\ s_4v_3 \\ v_1v_3 \end{array} \begin{array}{ccccccccccccc} (s_1,3) & (s_2,3) & (s_3,3) & (s_4,3) & (v_1,1) & (v_1,2) & (v_1,3) & (v_2,1) & (v_2,2) & (v_2,3) & (v_3,1) & (v_3,2) & (v_3,3) \\ -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{5}^{2-p} & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & -\sqrt{5}^{2-p} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & \sqrt{2}^{p} & -1 & 1 & -\sqrt{2}^{p} & 0 & 0 & 0 \\ \sqrt{5}^{2-p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & -\sqrt{5}^{2-p} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & \sqrt{2}^{p} & 0 & 0 & 0 & -1 & -1 & -\sqrt{2}^{p} \end{array}$$

It suffices to show again that the rows of the matrix D are linearly independent. Performing row operations in order to eliminate the subdiagonal elements of the first four columns, we obtain the equivalent matrix

$$\begin{bmatrix} D_1 & D_2 \\ 0 & E \end{bmatrix}$$

where the blocks D_1, D_2 form the first 4 rows of the matrix D and E is given by

$$E = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{array} \begin{array}{ccccccccc} (v_1,1) & (v_1,2) & (v_1,3) & (v_2,1) & (v_2,2) & (v_2,3) & (v_3,1) & (v_3,2) & (v_3,3) \\ -\sqrt{5}^{2-p} & 0 & \sqrt{5}^{2-p} & -2 & 1 & -\sqrt{5}^{2-p} & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & \sqrt{2}^{p} & -1 & 1 & -\sqrt{2}^{p} & 0 & 0 & 0 \\ -\sqrt{5}^{2-p} & 0 & \sqrt{5}^{2-p} & 0 & 0 & 0 & -2 & -1 & -\sqrt{5}^{2-p} \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 1 & \sqrt{2}^{p} & 0 & 0 & 0 & -1 & -1 & -\sqrt{2}^{p} \end{array}$$

We work now simultaneously on 2 different blocks of E, the first one is formed by the first 4 rows of E and the second one is given from the remaining rows.

$$E_1 = \begin{array}{c} \\ R_1=-r_2 \\ R_2=r_1-2r_2 \\ r_3 \\ R_4=r_4-r_2 \\ R_5=-r_7 \\ R_6=r_6 \\ R_7=r_5-2r_7 \\ R_8=r_8-r_7 \end{array} \begin{array}{ccccccccc} (v_1,1) & (v_1,2) & (v_1,3) & (v_2,1) & (v_2,2) & (v_2,3) & (v_3,1) & (v_3,2) & (v_3,3) \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -\sqrt{5}^{2-p} & 2 & -2+\sqrt{5}^{2-p} & 0 & 1 & 2-\sqrt{5}^{2-p} & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1+\sqrt{2}^{p} & 0 & 1 & 1-\sqrt{2}^{p} & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ -\sqrt{5}^{2-p} & -2 & -2+\sqrt{5}^{2-p} & 0 & 0 & 0 & 0 & -1 & 2-\sqrt{5}^{2-p} \\ 1 & 0 & -1+\sqrt{2}^{p} & 0 & 0 & 0 & 0 & -1 & 1-\sqrt{2}^{p} \end{array}$$

Hence we may remove the first and the fifth row and the columns $(v_2,1)$ and $(v_3,1)$ to obtain the matrix E_2

$$E_2 = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} \begin{array}{ccccccc} (v_1,1) & (v_1,2) & (v_1,3) & (v_2,2) & (v_2,3) & (v_3,2) & (v_3,3) \\ -\sqrt{5}^{2-p} & 2 & -2+\sqrt{5}^{2-p} & 1 & 2-\sqrt{5}^{2-p} & 0 & 0 \\ 1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1+\sqrt{2}^{p} & 1 & 1-\sqrt{2}^{p} & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ -\sqrt{5}^{2-p} & -2 & -2+\sqrt{5}^{2-p} & 0 & 0 & -1 & 2-\sqrt{5}^{2-p} \\ 1 & 0 & -1+\sqrt{2}^{p} & 0 & 0 & -1 & 1-\sqrt{2}^{p} \end{array}$$

We continue with the following row operations:

$$E_3 = \begin{array}{c} \\ R_1{=}r_2 \\ R_2{=}r_1{-}r_2 \\ R_3{=}r_3{-}r_2 \\ R_4{=}r_4 \\ R_5{=}r_5{-}r_4 \\ R_6{=}r_6{-}r_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} (v_1,1) & (v_1,2) & (v_1,3) & (v_2,2) & (v_2,3) & (v_3,2) & (v_3,3) \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 1 & 1 & -1 & 0 & 0 \\ -1-\sqrt{5}^{2-p} & 2 & -3+\sqrt{5}^{2-p} & 0 & 3-\sqrt{5}^{2-p} & 0 & 0 \\ 0 & 0 & -2+\sqrt{2}^p & 0 & 2-\sqrt{2}^p & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & -1 \\ -1-\sqrt{5}^{2-p} & -2 & -3+\sqrt{5}^{2-p} & 0 & 0 & 0 & 3-\sqrt{5}^{2-p} \\ 0 & 0 & -2+\sqrt{2}^p & 0 & 0 & 0 & 2-\sqrt{2}^p \end{array}\right] \end{array}$$

Again note that for $p = 2$ all the entries of the rows R_3 and R_6 of E_3 are equal to zero, so the matrix fails to have independent rows. For $p \neq 2$, it suffices to show that the rows of the matrix E_4, given below, are linearly independent.

$$E_4 = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \begin{array}{c} \begin{array}{ccccc} (v_1,1) & (v_1,2) & (v_1,3) & (v_2,3) & (v_3,3) \end{array} \\ \left[\begin{array}{ccccc} -1-\sqrt{5}^{2-p} & 2 & -3+\sqrt{5}^{2-p} & 3-\sqrt{5}^{2-p} & 0 \\ 0 & 0 & -2+\sqrt{2}^p & 2-\sqrt{2}^p & 0 \\ -1-\sqrt{5}^{2-p} & -2 & -3+\sqrt{5}^{2-p} & 0 & 3-\sqrt{5}^{2-p} \\ 0 & 0 & -2+\sqrt{2}^p & 0 & 2-\sqrt{2}^p \end{array}\right] \end{array}.$$

Define again $x = \frac{3-\sqrt{5}^{2-p}}{-2+\sqrt{2}^p}$. Since the equivalent matrix

$$E_5 = \begin{array}{c} \\ R_1{=}r_1{+}xr_2 \\ r_2 \\ R_3{=}r_3{+}xr_4 \\ r_4 \end{array} \begin{array}{c} \begin{array}{ccccc} (v_1,1) & (v_1,2) & (v_1,3) & (v_2,3) & (v_3,3) \end{array} \\ \left[\begin{array}{ccccc} -1-\sqrt{5}^{2-p} & 2 & 0 & 0 & 0 \\ 0 & 0 & -2+\sqrt{2}^p & 2-\sqrt{2}^p & 0 \\ -1-\sqrt{5}^{2-p} & -2 & 0 & 0 & 0 \\ 0 & 0 & -2+\sqrt{2}^p & 0 & 2-\sqrt{2}^p \end{array}\right] \end{array}.$$

has evidently linearly independent rows, it follows that the framework $(K_5 \cup_{K_3} K_5, q)$ is infinitesimally rigid.

We now recall the following result.

Proposition 50. [4, Proposition 4.7] Let X be a strictly convex and smooth finite dimensional real normed linear space with dimension d. Suppose G' is a graph which is obtained from G by a d-dimensional vertex splitting move. If there exists q such that (G, q) is (minimally) infinitesimally rigid in X then there exists q' such that (G', q') is (minimally) infinitesimally rigid in X.

A framework (G, q) in $\ell^3_{2,p}$ is said to be *regular* if the function

$$r_G : \left(\ell^3_{2,p}\right)^V \to \mathbb{N}, \quad x \mapsto \operatorname{rank} R(G, x),$$

achieves its maximum value at q. Note that if (G, q) is infinitesimally rigid in $\ell^3_{2,p}$ for some regular placement q then every regular placement of G in $\ell^3_{2,p}$ is infinitesimally rigid. In this case, we say that the graph G is rigid in $\ell^3_{2,p}$.

Theorem 51. *Let G be the graph of a doubly braced triangulation and let $p \in (1, \infty)$, $p \neq 2$. Then G is (minimally) rigid in $\ell^3_{2,p}$.*

Proof. Let G be the graph of a doubly braced triangulation. By Theorem 14, G can be constructed from the graph of one of the irreducible doubly braced triangulations by a sequence of 3-dimensional vertex splitting moves. Examples 48 and 49 show that the graphs of these irreducible doubly braced triangulations have an infinitesimally rigid placement in $\ell^3_{2,p}$. (In fact these placements are minimally infinitesimally rigid since they have exactly $3|V| - 4$ edges.) By Lemma 34, $\ell^3_{2,p}$ is strictly convex and smooth for all $p \in (1, \infty)$. Thus the result follows from Proposition 50. □

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