Statistical correlations in a Coulomb gas with a test charge

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Abstract. A recent paper [Jokela *et al.*, arxiv:0806.1491 (2008)] contains a surmise about an expectation value in a Coulomb gas which interacts with an additional charge ξ that sits at a fixed position. Here I demonstrate the validity of the surmised expression and extend it to a certain class of higher cumulants. The calculation is based on the analogy to statistical averages in the circular unitary ensemble of random-matrix theory and exploits properties of orthogonal polynomials on the unit circle.

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1. Purpose and result

In a recent paper Jokela, Järvinen and Keski-Vakkuri studied n-point functions in timelike boundary Liouville theory via the analogy to a Coulomb gas on a unit circle [1]. In this analogy, N unit charges at position t_i interact with additional charges of integer value ξ_a, situated at position τ_a. To illustrate this technique the authors of [1] considered the canonical expectation value

$$\langle \cdot \rangle \equiv \frac{1}{Z} \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \prod_{i<j} \left| \mathrm{e}^{\mathrm{i}t_i} - \mathrm{e}^{\mathrm{i}t_j} \right|^2 \prod_i \left| \mathrm{e}^{\mathrm{i}\tau} - \mathrm{e}^{\mathrm{i}t_i} \right|^{2\xi} (\cdot) \tag{1}$$

(where Z is a normalization factor so that $\langle 1 \rangle = 1$) and surmised that

$$\langle \mathrm{Re}\, a_1 \rangle \equiv \left\langle \sum_i \cos(\tau - t_i) \right\rangle = -\frac{\xi N}{N+\xi}. \tag{2}$$

In this communication I demonstrate the validity of (2), and also compute expectation values of the more general quantities

$$a_n \equiv \sum_{i_1<i_2<\ldots<i_n} \exp\left(\mathrm{i} \sum_{k=1}^{n} (t_{i_k} - \tau) \right). \tag{3}$$

As a result, I find

$$\langle a_n \rangle = (-1)^n \frac{(N-n+1)^{(\xi)} (n+1)^{(\xi-1)}}{(N+1)^{(\xi)} (1)^{(\xi-1)}} \quad \forall\; n = 0, 1, 2, \ldots, N;\; \xi \geq 0, \tag{4}$$

where $(x)^{(y)} = \Gamma(x+y)/\Gamma(x)$ is the generalized rising factorial (Pochhammer symbol). In particular, the validity of (2) follows from (4) by setting $n = 1$.

Expression (4) will be obtained by relating the generating polynomial

$$\varphi_{N,\xi}(\lambda) \equiv \sum_{n=0}^{N} \langle a_n \rangle (-\lambda)^{N-n} \tag{5}$$

to a weighted average of the secular polynomial in the circular unitary ensemble (CUE). This in turn establishes a relation to the Szegő polynomial of a Toeplitz matrix composed of binomial coefficients. This calculation sidesteps Jack polynomials and generalized Selberg integrals, which can be used to tackle general expectation values in multicomponent Coulomb gases [2].

2. Reformulation in terms of random matrices

The CUE is composed of $N \times N$ dimensional unitary matrices U distributed according to the Haar measure. Identify t_i with the eigenphases of such a matrix. The joint probability distribution is then given by [3]

$$P(\{t_i\}_{i=1}^N) = z \prod_{i<j} \left| \mathrm{e}^{\mathrm{i}t_i} - \mathrm{e}^{\mathrm{i}t_j} \right|^2 , \tag{6}$$

where z is again a normalization constant. This expression can also be written as the product of two Vandermonde determinants $\det V^+ \det V^-$ with matrices $V^\sigma_{lm} = \mathrm{e}^{\mathrm{i}\sigma(m-1)t_l}$. Furthermore, we can write

$$\prod_i \left| \mathrm{e}^{\mathrm{i}\tau} - \mathrm{e}^{\mathrm{i}t_i} \right|^{2\xi} = [\det(1 - U\mathrm{e}^{-\mathrm{i}\tau}) \det(1 - U^\dagger \mathrm{e}^{\mathrm{i}\tau})]^\xi . \tag{7}$$

Finally, the expressions a_n in (3) arise as the expansion coefficients of the secular polynomial

$$\det(U\mathrm{e}^{-\mathrm{i}\tau} - \lambda) = \sum_{n=0}^{N} a_n (-\lambda)^{N-n} . \tag{8}$$

Note that in all these expressions τ can be shifted to any fixed value by a uniform shift of all t_i's, which leaves the unitary ensemble invariant. Therefore the expectation values are independent of τ. Collecting all results, we have the identity

$$\varphi_{N,\xi}(\lambda) = \frac{\left\langle [\det(1-U)\det(1-U^\dagger)]^\xi \det(U-\lambda) \right\rangle_{\mathrm{CUE}}}{\langle [\det(1-U)\det(1-U^\dagger)]^\xi \rangle_{\mathrm{CUE}}} . \tag{9}$$

This can be interpreted as a weighted average of the secular polynomial in the CUE.

3. Random-matrix average

Statistical properties of the secular polynomial without the weight factor ($\xi = 0$) have been considered in [4]. Clearly, $\varphi_{N,0} = (-\lambda)^N$, so that in this case the attention quickly moves on to higher moments of the a_n. The main technical observation in [4] which allows to address the case of finite ξ concerns averages of expressions $g(\{t_i\}_{i=1}^N)$ that are completely symmetric in all eigenphases. In this situation the average can be found via

$$\langle g(\{t_i\}_{i=1}^N) \rangle_{\mathrm{CUE}} = \int \prod_i \frac{\mathrm{d}t_i}{2\pi} g(\{t_l\}_{l=1}^N) \det W , \tag{10}$$

where $W_{lm} = \mathrm{e}^{\mathrm{i}t_m(l-m)}$. Equation (10) is simpler than the general expression involving the product of two Vandermonde matrices, since each eigenphase only appears in a single column of W.

In the present problem, the numerator in (9) is represented by the completely symmetric function

$$g_1(\{t_i\}_{i=1}^N) = \prod_{i=1}^N [(\mathrm{e}^{\mathrm{i}t_i} - \lambda)(1 - \mathrm{e}^{\mathrm{i}t_i})^\xi (1 - \mathrm{e}^{-\mathrm{i}t_i})^\xi], \tag{11}$$

while for the denominator we need to consider the similar expression

$$g_2(\{t_i\}_{i=1}^N) = \prod_{i=1}^N [(1 - \mathrm{e}^{\mathrm{i}t_i})^\xi (1 - \mathrm{e}^{-\mathrm{i}t_i})^\xi]. \tag{12}$$

Using the multilinearity of the determinant we can now pull each factor into the ith column and perform the integrals. This delivers the representation

$$\varphi_{N,\xi}(\lambda) = \frac{\det(B - \lambda A)}{\det A}, \tag{13}$$

where the matrices $A_{lm} = (-1)^{l-m}\binom{2\xi}{\xi+l-m}$, $B_{lm} = (-1)^{l-m+1}\binom{2\xi}{\xi+l-m+1}$ have entries given by binomial coefficients. We now exploit the regular structure of these matrices in two steps.

1) Matrix B contains the same entries as matrix A, but shifted to the left by one column index. In order to exploit this, let us expand the determinant in the numerator into a sum of determinants of matrices labeled by $X = (x_m)_{m=1}^N$, where we select each column either from A ($x_m = \mathrm{A}$) or from B ($x_m = \mathrm{B}$). [Note that we set these symbols in roman letters.] The related structure of A and B then entails that $\det X$ vanishes if X contains a subsequence $(x_m, x_{m+1}) = (\mathrm{A}, \mathrm{B})$. Consequently we only need to consider determinants of matrices $X_n \equiv (\mathrm{B})_{m=1}^n \oplus (\mathrm{A})_{m=n+1}^N$, associated to sequences that contain n leading B's and $N - n$ trailing A's. As A is multiplied by $-\lambda$, $\det X_n$ contributes to order $(-\lambda)^{N-n}$. (Note that $X_0 = A$ and $X_N = B$.)

2) Next, consider the matrix A_{N+1}, where the subscript denotes the dimension, and strike out the first row and the $n+1$st column ($n = 0, 1, 2, \cdots, N$). This takes exactly the form of the matrix X_n of dimension N. Therefore, the expressions $(-1)^n \det X_n$ are the cofactors of the first row of A_{N+1}. These, in turn, are proportional to the first column of A_{N+1}^{-1}, where the proportionality factor is given by $\det A_{N+1}$. Consequently, taking care of all alternating signs,

$$\varphi_{N,\xi}(\lambda) = (-1)^N \frac{\det A_{N+1}}{\det A_N} \sum_{n=0}^N (A_{N+1}^{-1})_{1,1+n} \lambda^{N-n}. \tag{14}$$

Via steps 1) and 2) we have eliminated any reference to the matrix B.

4. Orthogonal polynomials

Matrix A is a Toeplitz matrix, $A_{lm} = c_{l-m}$. In order to find the explicit expression (4) we now make contact to the theory of orthogonal polynomials on the unit circle [5]. Among its many applications, this theory provides a general expression for the inverse of any Toeplitz matrix in terms of Szegő polynomials $\psi_N(\lambda)$. For the case of real symmetric coefficients, the inverse is generated via

$$\frac{\lambda\mu^N\psi_N(\lambda)\psi_N(\mu^{-1})-\lambda^N\mu\psi_N(\lambda^{-1})\psi_N(\mu)}{\lambda-\mu}=\frac{\det A_{N+1}}{\det A_N}\sum_{n,m=0}^{N}(A_{N+1}^{-1})_{m+1,n+1}\lambda^{N-n}\mu^m. \quad (15)$$

Comparison of this equation with $m=0$ to (14) immediately leads to the identification of $(-1)^N\varphi_{N,\xi}(\lambda)$ with the Szegő polynomial $\psi_N(\lambda)$ of degree N. These polynomials satisfy recursion relations which for real symmetric coefficients take the form

$$\gamma_N=-\frac{1}{\delta_{N-1}}\oint\frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\psi_{N-1}(\lambda)\sum_{n=-\infty}^{\infty}c_n\lambda^n, \quad (16a)$$

$$\psi_N(\lambda)=\lambda\psi_{N-1}(\lambda)+\gamma_N\lambda^{N-1}\psi_{N-1}(\lambda^{-1}), \quad (16b)$$

$$\delta_N=\delta_{N-1}(1-\gamma_N^2). \quad (16c)$$

The initial conditions are $\delta_0=c_0$, $\psi_0(\lambda)=1$. The numbers γ_N are known as the Schur or Verblunsky coefficients.

It can now be seen in an explicit if tedious calculation that the polynomials

$$\psi_N(\lambda)=(-1)^N\varphi_{N,\xi}(\lambda)=\sum_{n=0}^{N}\frac{(N-n+1)^{(\xi)}(n+1)^{(\xi-1)}}{(N+1)^{(\xi)}(1)^{(\xi-1)}}\lambda^{N-n} \quad (17a)$$

$$=\lambda^N{}_2F_1(-N,\xi;-N-\xi;\lambda^{-1}) \quad (17b)$$

[with coefficients and expansion given in (4), (5)] indeed fulfill the Szegő recursion generated by the binomial coefficients $c_n=(-1)^n\binom{2\xi}{\xi-n}$. The recursion coefficients take the simple form

$$\gamma_N=\frac{\xi}{\xi+N}, \qquad \delta_N=\frac{N!(2\xi+1)^{(N)}}{[(\xi+1)^{(N)}]^2}. \quad (17c)$$

This completes the proof of (4), and also entails the validity of (2).

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