Bounded Operators on Banach Spaces: Kernels, Closed Ideals, and Uniqueness of Quotient Algebra Norms

Max Arnott

Submitted for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Lancaster University

June 2023

Abstract

This thesis is comprised of four chapters.

Chapter 1 consists of preliminary definitions and descriptions of the notation we will be using throughout.

In Chapter 2, we ask the following question: 'for a given Banach space X and an arbitrary closed subspace Y of X, is there necessarily an operator $T \in \mathscr{B}(X)$ for which $\ker T = Y$?'

We prove that the answer to this question is yes when $X = c_0(\Gamma)$ or $X = \ell_p(\Gamma)$ for Γ uncountable and $1 < p < \infty$, and that the answer is no for $X = \ell_1(\Gamma)$.

In Chapter 3, we classify the lattice of closed ideals of the space of bounded operators on the direct sums $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{c_0} \oplus c_0(\Gamma)$ and $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{\ell_1} \oplus \ell_1(\Gamma)$ for every uncountable cardinal Γ.

In Chapter 4, we let X be one of the following Banach spaces, for which we know the entire lattice of closed ideals of the Banach algebra $\mathscr{B}(X)$ of bounded operators on X:

- $X = (\ell_2^1 \oplus \ell_2^2 \oplus \cdots \oplus \ell_2^n \oplus \cdots)_{c_0}$ or $X = (\ell_2^1 \oplus \ell_2^2 \oplus \cdots \oplus \ell_2^n \oplus \cdots)_{\ell_1}$,
- $X = (\ell_2^1 \oplus \ell_2^2 \oplus \cdots \oplus \ell_2^n \oplus \cdots)_{c_0} \oplus c_0(\Gamma)$ or $X = (\ell_2^1 \oplus \ell_2^2 \oplus \cdots \oplus \ell_2^n \oplus \cdots)_{\ell_1} \oplus \ell_1(\Gamma)$ for an uncountable index set Γ,
- $X = C_0(K_{\mathcal{A}})$, the Banach space of continuous functions vanishing at infinity on the locally compact Mrówka space $K_{\mathcal{A}}$ associated with an almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$, constructed such that $C_0(K_{\mathcal{A}})$ admits 'few operators'.

We show that in each of these cases, the quotient algebra $\mathscr{B}(X)/\mathscr{I}$ has a unique algebra norm for every closed ideal $\mathscr{I}$ of $\mathscr{B}(X)$.

Acknowledgements

First and foremost, I thank Professor Niels Laustsen, my incredible supervisor. You have been such an important source of knowledge and inspiration. Thank you for giving me this opportunity and for introducing me to the field of bounded operators on Banach spaces. On starting this PhD, there were certain obstacles that I knew that I would have to face. A national lockdown as a response to a global pandemic was admittedly, not something that I expected. That I made it to this point through so much disruption is a testament to how much I enjoyed our collaboration.

Secondly, I thank the late Professor Garth Dales. Not only for your role as my HDC chair until your retirement, but also for your guidance and mentorship throughout my master's dissertation and my PhD application. From you I learned so many valuable lessons on researching and writing mathematics which carried me through this process. It is a privilege to have been able to work with you.

I am also grateful to Doctor Yemon Choi, for stepping in as HDC chair after Garth's retirement, giving me helpful feedback.

To everyone in Lancaster University's Mathematics and Statistics department: the admin staff, the lecturers, my colleagues in office B49, everyone that I taught workshops alongside, and anyone else who I might be forgetting. I feel lucky to have spent these years working in such a positive environment.

Thank you to those in the wider Banach spaces and operator theory communities who I had the pleasure of meeting during my conference visits in the summer of 2022 to both *Banach Algebras and Applications* in Granada and *International Workshop on Operator Theory and Applications* in Krakow. These moments have been real highlights for me.

Last but not least, many thanks to all my friends and family for your continued support, and for providing me with a space away from research where I could always unwind to come back refreshed.

Statement of Originality

The content of this thesis has not been submitted for a degree elsewhere. All information within is either clearly referenced or is to my knowledge original.

Chapter 2 contains unpublished, joint work with N. J. Laustsen.

Chapter 3 is a version of a co-authored paper with N. J. Laustsen, published as [4], augmented with some content due to appear in [5].

Chapter 4 contains joint work with N. J. Laustsen, and a version of it has been submitted for publication as [5].

Contents

Chapter 1

Preliminaries

The purpose of this chapter is to detail the notation and constructions that will appear frequently, and to give some classical results of Banach space theory used throughout this thesis. In each of Section 1.4, Section 1.5, and Section 1.6, we will give technical details pertaining to specific Banach spaces that will be key points of focus to us in later chapters.

1.1 Basic and miscellaneous terminology

We begin by detailing the terminology and notation that will repeatedly appear over the course of this thesis.

The blackboard letters $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ will denote the set of natural, rational, real, and complex numbers respectively. We abide by the convention that $0 \notin \mathbb{N}$ and write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By a *Banach space* we mean a complete, normed vector space. The choice of scalar field for all vector spaces in this thesis will be denoted $\mathbb{K}$, and can always be taken as equal to either of $\mathbb{R}$ or $\mathbb{C}$ unless otherwise stated. A *Banach algebra* is a Banach space equipped with a multiplication between its elements which is submultiplicative in the given norm. We define a *bounded operator* to be a bounded, linear function between two Banach spaces, recalling that in this setting, boundedness is equivalent to continuity. We sometimes simply refer to bounded operators as *operators* since we will never be involving their unbounded counterparts. The following Banach spaces appear repeatedly over the course of this thesis:

(a) Let $\mathscr{B}(X;Y)$ denote the set of all bounded operators from a Banach space X to a Banach space Y. Then $\mathscr{B}(X;Y)$ is a Banach space with respect to the *operator norm*, defined as

$$\|T\|_{\mathrm{op}} = \sup\{\|Tx\| : (x \in X) \wedge (\|x\| \leqslant 1)\} .$$

We use the abbreviation $\mathscr{B}(X) := \mathscr{B}(X;X)$ and note that $\mathscr{B}(X)$ is a Banach algebra with respect to the operator norm, and with multiplication given by operator composition.

(b) Let K be a compact Hausdorff space. We define $C(K)$ to be the Banach space of continuous functions from K to the scalar field $\mathbb{K}$, equipped with the *supremum norm*:

$$\|f\|_\infty = \sup\{|f(x)| : x \in K\} .$$

The specific topological spaces K that we will be using to define $C(K)$ spaces will be introduced in their relevant sections.

(c) For a locally compact Hausdorff space K (i.e. every element in K has a compact neighbourhood), $C_0(K)$ will denote the space of all continuous functions $K \to \mathbb{K}$ *vanishing at infinity*, in the sense that the set $\{x \in K : |f(x)| \geqslant \epsilon\}$ is compact for every $\epsilon > 0$. These spaces are also equipped with the supremum norm.

(d) Let X be a Banach space, and let Y be a closed subspace of X. The *quotient space* X/Y is defined as the set $\{x + Y : x \in X\}$ of all cosets of Y in X, equipped with the norm

$$\|x + Y\|_{X/Y} = \inf\{\|x + y\| : y \in Y\} .$$

We say that a Banach space Z is a *quotient* of X if Z is isometrically isomorphic to X/Y for some closed subspace Y of X.

Seminorms on X which have kernel Y naturally induce norms on the quotient space X/Y, and vice versa. As such, we will use the same notation $\|\cdot\|_{X/Y}$ for the quotient norm on X/Y and the corresponding seminorm on X.

Let $\mathcal{A}$ be a Banach algebra. An *ideal* of $\mathcal{A}$ is a non-empty subset $\mathcal{I}$ of $\mathcal{A}$ for which $a, b \in \mathcal{I} \implies a + b \in \mathcal{I}$, and $a \in \mathcal{I}$, $b, c \in \mathcal{A} \implies ca, ab, cab \in \mathcal{I}$. We say that $\mathcal{I}$ is a *closed ideal* if it is closed in $\mathcal{A}$.

Suppose that $\mathcal{A}$ is a Banach algebra, and that $\mathcal{I}$ is a closed ideal of $\mathcal{A}$. Then $\mathcal{A}/\mathcal{I}$ is a Banach algebra with respect to the quotient norm $\|\cdot\|_{\mathcal{A}/\mathcal{I}}$, with multiplication defined by

$$(a + \mathcal{I}) \cdot (b + \mathcal{I}) = ab + \mathcal{I}$$

for every $a, b \in \mathcal{A}$.

(e) We require the Banach sequence spaces c_0 and ℓ_p for $1 \leqslant p \leqslant \infty$. These spaces appear in the subsequent two sections, but we defer their precise definition until Section 1.4 where we explain their generalised, transfinite counterparts. They are defined in the usual way.

We omit subscripts on norms in this thesis, since this practice is unlikely to cause confusion for us. Let $\overline{Y}$ denote the closure of a subset Y of a topological space X. The *density character* of a topological space X is the least cardinality of any subset Y which is *dense* in X, meaning that $\overline{Y} = X$. A topological space is *separable* if it has density character at most $\aleph_0$, and is *non-separable* otherwise.

For a Banach space X, we write $I_X \in \mathscr{B}(X)$ to denote the *identity operator* on X, with action $x \mapsto x$, and as usual we allow ourselves to abandon the subscript X when it is not necessary to distinguish between different identity operators. Our notation for the *closed unit ball* of a Banach space X will be $B_X := \{x \in X : \|x\| \leqslant 1\}$.

For a Banach space X, we write X^* to mean the *(topological) dual space* of X, defined as the set of all functionals on X (i.e. the continuous linear functions from X to its scalar field). We will use the duality bracket notation $\langle x, f\rangle = f(x)$ whenever $x \in X$ and $f \in X^*$. For each pair of Banach spaces X and Y, and each $T \in \mathscr{B}(X;Y)$, we define the *adjoint* (or *dual*) T^* of T as the unique operator $Y^* \to X^*$ for which $\langle Tx, f\rangle = \langle x, T^*f\rangle$ for every $x \in X$ and every $f \in Y^*$.

Let $T \in \mathscr{B}(X;Y)$ for Banach spaces X and Y, and let $c > 0$. We say that T is *bounded below by* c if $\|Tx\| \geqslant c\|x\|$ for every $x \in X$. If there exists $c > 0$ such that T is bounded below by c, we say that T is *bounded below*. We will use repeatedly the fact that an operator between Banach spaces is bounded below if and only if it is injective and has closed range. We write $\operatorname{im} T$ for the image of an operator T, and $\ker T$ for the kernel.

For Banach spaces X and Y, an invertible operator $X \to Y$ is an *isomorphism.* If such an operator exists, we say that X and Y are *isomorphic*, and we write $X \cong Y$. If an operator preserves norms, it is an *isometry.* If there is an isometric isomorphism between X and Y, we say that the two Banach spaces are *isometric*, and we write $X \equiv Y$.

Let X be a Banach space, and let Y be a closed subspace of X. An operator $P \in \mathscr{B}(X)$ is a *projection* onto Y if $P^2 = P$ and $\operatorname{im}(P) = Y$. If such an operator exists, we say that Y is *complemented* in X. On occasion, we will treat a projection P onto Y as an element of $\mathscr{B}(X;Y)$ by identifying it with its corestriction to Y. This is unlikely to cause confusion because each time a projection is introduced, its domain and codomain will be clear.

1.2 Operators on direct sums of Banach spaces

Let N be an infinite subset of $\mathbb{N}$ and let $p \in [1,\infty)$. When $D = c_0$ or $D = \ell_p$, we define the D-*direct sum* of an N-indexed sequence $(X_n)_{n\in N}$ of Banach spaces to be the set $\left(\bigoplus_{n\in N} X_n\right)_D$ consisting of N-indexed sequences $(x_n)_{n\in N}$ with $x_n \in X_n$ for each $n \in N$ such that

$$\begin{aligned} \|x_n\| \underset{N\ni n\to\infty}{\longrightarrow} 0 \quad &\text{for } D = c_0 \,, \\ \sum_{n\in N} \|x_n\|^p < \infty \quad &\text{for } D = \ell_p \,. \end{aligned}$$

Vector space operations on $\left(\bigoplus_{n\in N} X_n\right)_D$ are coordinatewise, and the norm is given by

$$\left\|(x_n)_{n\in N}\right\| = \begin{cases} \max_{n\in N}\|x_n\| & \text{for} \quad D = c_0 \,, \\ \left(\sum_{n\in N}\|x_n\|^p\right)^{\frac{1}{p}} & \text{for} \quad D = \ell_p \,. \end{cases}$$

Whenever a direct sum of Banach spaces is discussed without specifying the corresponding space D defining its structure, this is because the choice of D is not relevant and the discussion holds in full generality. Let Δ, Γ be subsets of $\mathbb{N}$, finite or infinite, and for each $\delta \in \Delta$, $\gamma \in \Gamma$, let X_δ and Y_γ be Banach spaces. Given an operator

$$T : X = \bigoplus_{\delta \in \Delta} X_\delta \to \bigoplus_{\gamma \in \Gamma} Y_\gamma = Y \tag{1.2.1}$$

between direct sums of Banach spaces, we associate with it the $|\Gamma| \times |\Delta|$ matrix $[T] := (T_{\gamma,\delta})_{\gamma \in \Gamma, \delta \in \Delta}$ whose $(\gamma, \delta)^{\text{th}}$ entry is defined as

$$T_{\gamma,\delta} = Q_\gamma T J_\delta \in \mathscr{B}(X_\delta, Y_\gamma) \ ,$$

where Q_γ and J_δ are the standard projection $Y \to Y_\gamma$ and inclusion $X_\delta \to X$ respectively.

The action of T on a sequence $x = (x_\delta)_{\delta \in \Delta}$ is given by the matrix-vector multiplication $[T](x_\delta)_{\delta \in \Delta}$, with the vector considered as a column in this setting. Frequently for us in Chapter 3 and Chapter 4, we will be concerned with 2 by 2 matrices describing operators on a direct sum of a pair of Banach spaces.

Fix a space $D \in \{\ell_p : 1 \leqslant p \leqslant \infty\} \cup \{c_0\}$ and a subset N of $\mathbb{N}$. Whenever we have an N-indexed sequence of operators $(T_i : X_i \to Y_i)_{i \in N}$ between Banach spaces X_i and Y_i for every $i \in N$, we may define the operator

$$\bigoplus_{i \in N} T_i : \left(\bigoplus_{i \in N} X_i \right)_D \to \left(\bigoplus_{i \in N} Y_i \right)_D \quad ; \quad (x_i)_{i \in N} \mapsto (T_i x_i)_{i \in N} \ .$$

The corresponding matrix to the above operator is diagonal, with entries T_i in position (i, i) for each $i \in N$, and is 0 elsewhere. It is easy to verify that its norm is $\sup\{\|T_i\| : i \in \mathbb{N}\}$.

1.3 Ideals and Quotients of $\mathscr{B}(X)$

An *operator ideal* (in the terminology of Pietsch [54]) is an assignment $\mathscr{I}$ which designates to each pair (X, Y) of Banach spaces a subspace $\mathscr{I}(X; Y)$ of $\mathscr{B}(X; Y)$ for which:

(i) there exists a pair (X, Y) of Banach spaces for which $\mathscr{I}(X; Y) \neq \{0\}$;

(ii) for any quadruple (W, X, Y, Z) of Banach spaces and any three operators $S \in \mathscr{B}(W; X)$, $T \in \mathscr{I}(X; Y)$, $U \in \mathscr{B}(Y; Z)$, we have that $UTS \in \mathscr{I}(W; Z)$.

We remark that the combination of the above two conditions implies the following third condition:

(iii) for every pair (X, Y) of Banach spaces, $\mathscr{I}(X; Y)$ contains every finite-rank operator $X \to Y$.

Write $\mathscr{I}(X)$ to abbreviate $\mathscr{I}(X; X)$. For any operator ideal $\mathscr{I}$, the map $\overline{\mathscr{I}}$ sending a pair of Banach spaces (X, Y) to the norm closure of $\mathscr{I}(X; Y)$ in $\mathscr{B}(X; Y)$ is also an operator ideal. If $\mathscr{I} = \overline{\mathscr{I}}$, then we call $\mathscr{I}$ a *closed operator ideal.* Over the course of this thesis, we will require the following operator ideals:

(a) For a Banach space D, we write $\mathscr{G}_D$ for the assignment which designates to a pair (X, Y) of Banach spaces the subspace $\mathscr{G}_D(X; Y)$ of $\mathscr{B}(X; Y)$ consisting of operators T *factoring through* D, in the sense that $T = UV$ for some operators $U \in \mathscr{B}(D; X)$, $V \in \mathscr{B}(X; D)$. The assignment $\mathscr{G}_D$ is an operator ideal when $D \cong D \oplus D$ (see e.g. the discussion after [40, Definition 3.6] for a proof of this), which we state here for future reference holds true for the spaces $D = c_0$ and $D = \ell_1$.

(b) We use the notation $\mathscr{K}$ to denote the assignment which maps every pair (X, Y) of Banach spaces to the subspace $\mathscr{K}(X; Y)$ of $\mathscr{B}(X; Y)$ consisting of compact operators from X to Y, which are those operators $T \in \mathscr{B}(X; Y)$ for which $\overline{T(B_X)}$ is compact. For an operator $T \in \mathscr{B}(X; Y)$, the quantity

$$\|T\|_e := \|T + \mathscr{K}(X; Y)\|$$

is the *essential norm* of T. For a Banach space X, the quotient $\mathscr{B}(X)/\mathscr{K}(X)$ is colloquially the *Calkin algebra* of X.

(c) Let κ be an infinite cardinal. An operator $T \in \mathscr{B}(X; Y)$ is *κ-compact* if for each $\epsilon > 0$, the closed unit ball B_X of X contains a subset X_ϵ with $|X_\epsilon| < \kappa$ such that

$$\inf\{\|T(x - y)\| : y \in X_\epsilon\} \leqslant \epsilon$$

for every $x \in B_X$. Writing $\mathscr{K}_\kappa(X;Y)$ for the set of κ-compact operators from X to Y, we obtain a closed operator ideal $\mathscr{K}_\kappa$. Notice that the definition of $\aleph_0$-compactness is that of compactness, so κ-compactness is indeed a generalisation of compactness. Naturally, we refer to the quotient $\mathscr{B}(X)/\mathscr{K}_\kappa(X)$ as the κ-*Calkin* algebra of X.

(d) An operator $T \in \mathscr{B}(X;Y)$ is *strictly singular* if there exists no infinite dimensional subspace of X on which T is bounded below. The assignment $\mathscr{S}$ which designates to two Banach spaces X and Y the space $\mathscr{S}(X;Y)$ of all strictly singular operators $X \to Y$, is a closed operator ideal. We note that for every pair (X,Y) of Banach spaces, we have that $\mathscr{K}(X;Y) \subseteq \mathscr{S}(X;Y)$, with it possible for this inclusion to be strict.

1.4 Transfinite Banach sequence spaces

Here, we define some generalisations of the classical Banach sequence spaces c_0 and ℓ_p for $1 \leqslant p \leqslant \infty$. For this section, let Γ be an arbitrary indexing set.

(a) Define $\ell_\infty(\Gamma)$ to be the set of all bounded functions $\Gamma \to \mathbb{K}$, which is a Banach space with the supremum norm.

(b) Define the space $c_0(\Gamma)$ of functions $f : \Gamma \to \mathbb{K}$ for which $\{\gamma \in \Gamma \ ; \ |f(\gamma)| > \epsilon\}$ is finite for all $\epsilon > 0$. This is also a Banach space which we equip with the supremum norm.

(c) For $1 \leqslant p < \infty$, we define the space $\ell_p(\Gamma)$ to be the set of functions $f : \Gamma \to \mathbb{K}$ which satisfy $\sum_{\gamma\in\Gamma} |f(\gamma)|^p < \infty$. This is a Banach space with respect to the norm $\|f\| = (\sum_{\gamma\in\Gamma} |f(\gamma)|^p)^{\frac{1}{p}}$.

We consider elements of these spaces as sequences of scalars indexed by Γ, and write ℓ_∞, c_0, and ℓ_p respectively whenever $\Gamma = \mathbb{N}$. Further, when $\Gamma = \{1, \dots, n\}$, we write $\ell_\infty(\Gamma) = c_0(\Gamma) = \ell_\infty^n$, and $\ell_p(\Gamma) = \ell_p^n$.

Let X be one of the spaces in the list above. We write $(e_\gamma)_{\gamma\in\Gamma}$ for the set of *standard unit vectors* of X, that is, e_γ has value 1 in position $\gamma \in \Gamma$ and 0 elsewhere. We notice here that the notation e_γ does not include reference to which sequence space it belongs to, and remark that this is unlikely to cause confusion. The *support* of a (possibly transfinite) sequence x in X is then defined as

$$\operatorname{supp}(x) := \{\gamma \in \Gamma : x_\gamma \neq 0\} .$$

It is inherent in the above definitions that the support of any element of $c_0(\Gamma)$ or $\ell_p(\Gamma)$ for $1 \leqslant p < \infty$ must be at most countable, regardless of the cardinality of Γ. This can not be said for $\ell_\infty(\Gamma)$.

The spaces $c_0(\Gamma)$ and $c_0(\Delta)$ (respectively $\ell_p(\Gamma)$ and $\ell_p(\Delta)$ for $1 \leqslant p \leqslant \infty$) are isometrically isomorphic whenever $|\Gamma| = |\Delta|$. Thus for these Banach spaces we may freely replace the indexing set Γ with the cardinal $|\Gamma|$, or with any other set of cardinality $|\Gamma|$ whenever it is convenient to do so.

We have that $c_0(\Gamma)^* \equiv \ell_1(\Gamma)$, $\ell_1(\Gamma)^* \equiv \ell_\infty(\Gamma)$, and for $1 < p < \infty$, $\ell_p(\Gamma)^* \equiv \ell_q(\Gamma)$, where q is the *conjugate exponent* of p, i.e. the unique real number that solves $p^{-1} + q^{-1} = 1$. As is standard practice, we make identifications of these spaces, replacing all of these isometries with equalities. We have that for $1 < p < \infty$, the canonical embedding of $\ell_p(\Gamma)$ in $\ell_p(\Gamma)^{**}$ is an isometric isomorphism, and hence the space $\ell_p(\Gamma)$ is *reflexive*.

Daws [12, Theorem 7.4] has shown that for $X = c_0(\Gamma)$ or $X = \ell_p(\Gamma)$, where Γ is an infinite cardinal and $1 \leqslant p < \infty$, the lattice of closed ideals of $\mathscr{B}(X)$ is

$$\begin{aligned} \{0\} \subsetneq \mathscr{K}(X) \subsetneq \mathscr{K}_{\aleph_1}(X) \subsetneq \cdots \subsetneq \mathscr{K}_\kappa(X) \subsetneq \mathscr{K}_{\kappa^+}(X) \subsetneq \cdots \\ \cdots \subsetneq \mathscr{K}_\Gamma(X) \subsetneq \mathscr{K}_{\Gamma^+}(X) = \mathscr{B}(X) , \end{aligned} \tag{1.4.1}$$

where κ^+ denotes the cardinal successor of κ.

An alternative description of the closed ideals of $\mathscr{B}(X)$ is given in [30, Theorem 1.5], where it is shown that the non-zero closed ideals of $\mathscr{B}(X)$ take the form $\mathscr{S}_{X_\kappa}(X)$, where $X_\kappa = c_0(\kappa)$ in the case of $X = c_0(\Gamma)$, and $X_\kappa = \ell_p(\kappa)$ in the case that $X = \ell_p(\Gamma)$, and

$$\mathscr{S}_{X_\kappa}(X) := \{T \in \mathscr{B}(X) : T \text{ is not bounded below on any copy of } X_\kappa\} .$$

Comparing this result with Daws', we can identify the ideals $\mathscr{K}_\kappa(X)$ and $\mathscr{S}_{X_\kappa}(X)$ in this setting. This lattice of closed ideals will be of key importance to us in Chapter 3 and Chapter 4.

1.5 The spaces $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$ for $D \in \{c_0, \ell_1\}$

Let X be either of the spaces $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$ for $D = c_0$ or $D = \ell_1$, with the latter being the dual space of the former. Then it was shown in [40] by Laustsen, Loy and Read, and in [41] by Laustsen, Schlumprecht and Zsak respectively, that the lattice of closed ideals of $\mathscr{B}(X)$ is

$$\{0\} \subsetneq \mathscr{K}(X) \subsetneq \overline{\mathscr{G}_D}(X) \subsetneq \mathscr{B}(X)\,. \tag{1.5.1}$$

Again, we will be focusing heavily on this lattice of ideals in Chapter 3 and Chapter 4. The following result pertains to the middle inclusion of the lattice (1.5.1). We record it here for future reference, and to highlight the importance of 'factorisation through idempotent'-style results in the domain of ideal classification.

Proposition 1.5.1. *Let $D = c_0$ or $D = \ell_1$, and let $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$. For every operator $T \in \mathscr{B}(X) \setminus \mathscr{K}(X)$, there exist operators $A \in \mathscr{B}(X; D), B \in \mathscr{B}(D; X)$ for which $I_D = ATB$.*

Proof. [40, Theorem 3.2] is a much more general version of this result, and [40, Example 3.9] explains why the result is applicable to the spaces c_0 and ℓ_1. □

The purpose of the above proposition in the ideal classification (1.5.1) is rather simple. It proves immediately that there could not possibly be a closed ideal lying strictly in between $\mathscr{K}(X)$ and $\overline{\mathscr{G}_D}(X)$. Indeed, if $\mathscr{J}$ is a closed ideal of $\mathscr{B}(X)$ containing $\mathscr{K}(X)$ strictly, then it contains some noncompact operator T. Let $R \in \mathscr{G}_D(X)$ and decompose as $R = UV$ for $U \in \mathscr{B}(D; X), V \in \mathscr{B}(X; D)$. Taking $A \in \mathscr{B}(X; D), B \in \mathscr{B}(D; X)$ with $I_D = ATB$, we then see that $R = (UA)T(BV)$, proving that $R \in \mathscr{J}$. It follows that $\mathscr{G}_D(X) \subseteq \mathscr{J}$, so because $\mathscr{J}$ is closed, we have that $\overline{\mathscr{G}_D}(X) \subseteq \mathscr{J}$ as claimed.

We shall sometimes make reference to the standard basis of $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$. This basis is simply the sequence formed by concatenating the finite sequences of standard basis vectors of the Hilbert spaces ℓ_2^n for $n \in \mathbb{N}$, in ascending order of n.

1.6 Almost disjoint families and Mrówka spaces

The following century-old combinatorial result originated from Sierpiński in [58]. We say that a family $\mathcal{A}$ of sets is *almost disjoint* if any distinct sets $A, B \in \mathcal{A}$ have finite intersection.

Theorem 1.6.1. *Let S be a countably infinite set. There exists an almost disjoint family $\mathcal{A}$ of infinite subsets of S of cardinality $2^{\aleph_0}$.*

Proof. Let $j : S \to \mathbb{Q}$ be a bijection. To each $r \in \mathbb{R} \setminus \mathbb{Q}$, assign a sequence $(s_n^r)_{n\in\mathbb{N}}$ in S such that $j(s_n^r) \to r$. Then

$$\mathcal{A} := \{\{s_n^r : n \in \mathbb{N}\} : r \in \mathbb{R} \setminus \mathbb{Q}\}$$

suffices. $\square$

We use this result to define a topological space as follows. Let $[\mathbb{N}]^{<\omega}$ and $[\mathbb{N}]^{\omega}$ denote the sets of all finite and infinite subsets of $\mathbb{N}$, respectively.

Definition 1.6.2. Consider a family $\mathcal{A} \subseteq [\mathbb{N}]^{\omega}$ which is almost disjoint. Define a set

$$K_{\mathcal{A}} := \{x_n : n \in \mathbb{N}\} \cup \{y_A : A \in \mathcal{A}\} ,$$

with topology such that for each $n \in \mathbb{N}$, x_n is isolated in $K_{\mathcal{A}}$, and for each $A \in \mathcal{A}$, the collection of sets of the form

$$\{y_A\} \cup \{x_n : n \in A \setminus F\} \text{ where } F \subset \mathbb{N} \text{ is finite}$$

is a neighbourhood basis of y_A.

The topological space $K_{\mathcal{A}}$ is the *Mrówka space* associated with $\mathcal{A}$, also known as the *Alexandroff–Urysohn space*, the Ψ*-space*, and the *Isbell–Mrówka space*.

Some important topological properties about Mrówka spaces are the following. Mrówka spaces $K_{\mathcal{A}}$ are *scattered* as topological spaces, meaning that each non-empty subset X of $K_{\mathcal{A}}$ has a point which is isolated in X. They are also locally compact, but non-compact whenever $\mathcal{A}$ is infinite.

Making minor appearances throughout this thesis until they are a main point of focus in Section 4.4 are the spaces $C_0(K_{\mathcal{A}})$ of continuous functions vanishing at infinity on Mrówka spaces. We remark here that there is a helpful way to construct $C_0(K_{\mathcal{A}})$ from the almost disjoint family $\mathcal{A}$ without invoking Definition 1.6.2.

Let $\mathcal{A}$ be an almost disjoint family of subsets of $\mathbb{N}$ and consider the closure $X_{\mathcal{A}}$ of the subspace of ℓ_∞ spanned by the indicator functions $\mathbb{1}_A$ for $A \in \mathcal{A}$ and $\mathbb{1}_{\{n\}}$ for $n \in \mathbb{N}$. Then $X_{\mathcal{A}}$ is a self-adjoint subalgebra of ℓ_∞. Hence, assuming complex scalars, we have that $X_{\mathcal{A}}$ is isometrically isomorphic to $C_0(K)$ for a locally compact Hausdorff space K. The space K turns out to be the Mrówka space $K_{\mathcal{A}}$ associated with $\mathcal{A}$.

In the case of Mrówka spaces $K_{\mathcal{A}}$, the spaces $C(\alpha(K_{\mathcal{A}}))$ and $C_0(K_{\mathcal{A}})$ are isomorphic, where $\alpha(K_{\mathcal{A}})$ is the *one point compactification* of $K_{\mathcal{A}}$. Indeed, let K be a scattered, compact Hausdorff space. Then $C(K)$ contains a complemented subspace isomorphic to c_0, so $C(K)$ is isomorphic to its hyperplanes (i.e. its closed, codimension one subspaces). As an application of this observation, suppose instead that K is a scattered, *locally* compact Hausdorff space. Then its one-point compactification αK is scattered. Since $C_0(K)$ is a hyperplane in $C(\alpha K)$, these two Banach spaces are isomorphic.

Koszmider in [37] defined a particular almost disjoint family $\mathcal{A}$ of subsets of $\mathbb{N}$ for which the space $C_0(K_{\mathcal{A}})$ of continuous functions vanishing at infinity on the resultant Mrówka space $K_{\mathcal{A}}$ has *few operators*, in the sense that each $T \in \mathscr{B}(C_0(K_{\mathcal{A}}))$ is the sum of a scalar multiple of the identity operator and an operator with separable range. The original construction of $\mathcal{A}$ in [37] required the Continuum Hypothesis. A construction solely within ZFC is given in [38]. The lattice of closed ideals of $\mathscr{B}(C_0(K_{\mathcal{A}}))$, as shown originally by Kania and Kochanek in [35, Theorem 5.5], is

$$\{0\} \subsetneq \mathscr{K}(C_0(K_{\mathcal{A}})) \subsetneq \mathscr{X}(C_0(K_{\mathcal{A}})) \subsetneq \mathscr{B}(C_0(K_{\mathcal{A}}))\ ,$$

where for a Banach space Y, we write $\mathscr{X}(Y)$ for the set of operators on Y which have separable range. This result was also obtained independently by Brooker (unpublished).

Notice that the 'few operators' property of $C(K)$ tells us that the closed ideal $\mathscr{X}(C(K))$ has codimension one in $\mathscr{B}(C(K))$ and is therefore maximal.

Chapter 2

The kernel problem for a selection of transfinite Banach sequence spaces

2.1 Background results and the statement of the kernel problem

In 2018, Laustsen and White published a paper [42] investigating for a given Banach space X, which closed subspaces of X are able to be realised as a kernel of a bounded operator on X. The key result of their findings is the following theorem.

Theorem 2.1.1. *There exists a reflexive Banach space E_W^* containing a closed subspace Y, such that $Y \neq \ker T$ for every operator $T \in \mathscr{B}(E_W^*)$.*

The space E_W^* is the dual of a certain reflexive and non-separable Banach space E_W, constructed by Wark in [64] and [65], which has the property that every $T \in \mathscr{B}(E_W)$ is the sum of an operator with separable range and a scalar multiple of the identity on E_W. The space E_W has in this sense *few operators*, suggesting that it has (informally speaking) few subspaces which are closures of images of bounded operators. By reflexivity, its dual E_W^* therefore has few closed subspaces which are kernels of bounded operators (see Lemma 2.2.10 below for more information).

However, the non-separability of E_W^* endows it with a certain 'largeness', suggesting that it has many closed subspaces. This apparent discrepancy highlighted E_W^* as a candidate for a space which may have a closed subspace not equal to the kernel of any bounded operator on E_W^*, which motivated the research.

This chapter can be seen as a continuation of the line of research started in the aforementioned paper [42] of Laustsen and White, where we shall be attempting to answer the following question for a selection of transfinite Banach sequence spaces.

Question 2.1.2 (The kernel problem). *Let X be a Banach space. Is it true that every closed subspace of X is equal to* $\ker T$ *for some $T \in \mathscr{B}(X)$?*

In the statement of the kernel problem, we are only interested in whether closed subspaces Y of X are kernels of some operator $T \in \mathscr{B}(X)$, and not whether they are kernels of some operator $R \in \mathscr{B}(X; Z)$ for some arbitrary Banach space Z. This is because the standard quotient map $X \to X/Y$ has kernel Y, so the question is trivial in this setting. Also, since kernels of bounded operators are always closed, the kernel problem becomes obviously trivial if we also consider also non-closed subspaces of X.

Our research on the kernel problem has culminated in the following theorem; this chapter is dedicated to its proof.

Theorem 2.1.3. *Let Γ be any uncountable set.*

(i) Let $1 < p < \infty$. Let $Y = \ell_p(\Gamma)$ or $Y = c_0(\Gamma)$, and let X be a closed subspace of Y. There exists $T \in \mathscr{B}(Y)$ with $\ker(T) = X$.

(ii) There exists a closed subspace X of $\ell_1(\Gamma)$ for which $X \neq \ker T$ for all $T \in \mathscr{B}(\ell_1(\Gamma))$.

The ultimate reason that Laustsen and White wanted to prove Theorem 2.1.1 was that White [66, Theorem 3.6.12] had proved that if E is a reflexive Banach space which contains a closed subspace not equal to the kernel of any $T \in \mathscr{B}(E)$, then $\mathscr{B}(E)$ as a dual Banach algebra fails to be 'weak-$*$ topologically left-Noetherian'. The space E_W^* provided them with an example of such a space. Theorem 2.1.3(i) shows us that taking the reflexive Banach space $E = \ell_p(\Gamma)$ for $1 < p < \infty$ and Γ uncountable would not have been suitable for their purposes.

Before giving our proof of Theorem 2.1.3, we will give an overview of the Banach spaces other than E_W^* which are known to answer Question 2.1.2 either positively or negatively. The first work pertaining to the kernel problem appears to be the following result of Kalton.

Proposition 2.1.4. *[34, Proposition 4] Let $T \in \mathscr{B}(\ell_\infty)$, let (e_n) be the standard unit vectors in ℓ_∞, and suppose that $Te_n = 0$ for all $n \in \mathbb{N}$. There exists an infinite subset $M \subset \mathbb{N}$ for which $Tx = 0$ for every x belonging to the set*

$$\ell_\infty(M) := \{t \in \ell_\infty : \operatorname{supp}(t) \subset M\} .$$

In the statement of the above theorem, it is clear that the canonical copy of c_0 in ℓ_∞ has as a basis the sequence of vectors (e_n). Also clear is that $\ell_\infty(M) \equiv \ell_\infty$ cannot be contained in c_0. What follows is a negative answer to Question 2.1.2 in the case of ℓ_∞.

Corollary 2.1.5. *Let $T \in \mathscr{B}(\ell_\infty)$. Then $c_0 \neq \ker T$.*

Again due to Laustsen and White in [42], our next result tells us that when examining the kernel problem, we need only consider closed subspaces which give non-separable quotients by their superspace as potential candidates for 'non-kernel' subspaces.

Proposition 2.1.6. *[42, Proposition 2.1] Let X be a Banach space, and let Y be a closed subspace of X for which X/Y is separable. Then $Y = \ker T$ for some $T \in \mathscr{B}(X)$.*

Because quotients of Banach spaces by their closed subspaces can never have density character exceeding that of the original space, an immediate, obvious consequence of Proposition 2.1.6 is the following, which explains why we are interested in only the transfinite, non-separable analogues of the classical Banach sequence spaces c_0 and ℓ_p for $1 \leqslant p < \infty$ in this section.

Corollary 2.1.7. *Let X be a separable Banach space. Every closed subspace of X is the kernel of some operator $T \in \mathscr{B}(X)$.*

A Banach space X is said to have the *separable complementation property* if every separable subspace of X is contained within a separable, complemented subspace of X. Another key result of Laustsen and White is given here, which again has the above corollary as a trivial consequence.

Proposition 2.1.8. *[42, Proposition 2.2] Let X be a Banach space with the separable complementation property. Every closed, separable subspace of X is the kernel of some operator $T \in \mathscr{B}(X)$.*

Finally, we have some very recent relevant results, which the author believes completes the collection of known results on Question 2.1.2. A preprint of Horváth and Laustsen [27] generalises Proposition 2.1.4 in the following way.

Theorem 2.1.9. *Let Γ be infinite, and let $\mathcal{A}$ be an almost disjoint family of subsets of Γ. Let $Y_{\mathcal{A}}$ denote the closed subspace of $\ell_\infty(\Gamma)$ spanned by the indicator functions $\mathbb{1}_{\bigcap_{j=1}^n A_j}$, where $A_1, \ldots, A_n \in \mathcal{A}$. If $\mathcal{A}$ has cardinality greater than $|\Gamma|$, then $Y_{\mathcal{A}}$ contains subspaces which are not expressible as the kernel of any operator $T \in \mathscr{B}(Y_{\mathcal{A}}; \ell_\infty(\Gamma))$.*

This theorem not only implies negative answers to the kernel problem for $\ell_\infty(\Gamma)$ for every infinite set Γ, but also for the exotic Banach spaces $C_0(K_{\mathcal{A}})$ of continuous functions vanishing at infinity on Mrówka topological spaces (see Section 1.6 for the definition). To summarise in terms of the kernel problem, we have the following.

Corollary 2.1.10. *Let $X = \ell_\infty(\Gamma)$ for some uncountable set Γ, or let $X = C_0(K_{\mathcal{A}})$: the space of continuous functions vanishing at infinity on a Mrówka topological space defined via an uncountable, almost disjoint family $\mathcal{A}$ of subsets of $\mathbb{N}$.*

There exists a closed subspace of X which is not equal to $\ker(T)$ for any $T \in \mathscr{B}(X)$.

Theorem 2.1.3(i) and (ii) combine to tell us that it is possible for a Banach space $X(= c_0(\Gamma))$ to answer Question 2.1.2 positively, whilst its dual $X^*(= \ell_1(\Gamma))$ answers Question 2.1.2 negatively. The dual E_W^* of Wark's space E_W remains the only reflexive Banach space known to have a closed subspace that is not the kernel of a bounded operator on itself, with the answer unknown for E_W. With this in mind, an interesting open question for future research on the kernel problem might be the following.

Question 2.1.11. Does there exist a Banach space X answering Question 2.1.2 negatively, whilst X^* answers Question 2.1.2 positively?

Particularly intriguing considering the partial answer obtained for E_W^* is the reflexive case for the above question:

Question 2.1.12. Does there exist a reflexive Banach space X for which X and X^* give opposite answers to Question 2.1.2?

2.2 The proof of Theorem 2.1.3

Owing to Proposition 2.1.6, for the rest of this chapter we may restrict our attention to non-separable Banach spaces, and the closed subspaces of them which yield non-separable quotient spaces. The definitions of the transfinite Banach sequence spaces that we shall be working with are given in Section 1.4. Our main strategy for answering Question 2.1.2 for our selected spaces will be to check them against the following simple lemma.

Lemma 2.2.1. *Let X be a Banach space, and let Y be a closed subspace of X. Then $Y = \ker T$ for some $T \in \mathscr{B}(X)$ if and only if there exists a bounded linear injection $X/Y \to X$.*

Proof. Suppose that $Y = \ker T$ for some $T \in \mathscr{B}(X)$. The fundamental isomorphism theorem states that there exists a bounded linear injection $X/Y \to \mathrm{Im}(T) \subset X$.

On the other hand, if there exists a bounded linear injection $R : X/Y \to X$, then $RQ \in \mathscr{B}(X)$ and $Y = \ker(RQ)$, where Q is the standard quotient map $X \to X/Y$. $\square$

2.2.1 The case $c_0(\Gamma)$ for Γ uncountable.

Our answer to the kernel problem in the case of $c_0(\Gamma)$ will be based around the concept of weak compact generation of Banach spaces.

Definition 2.2.2. Let X be a Banach space. We say that X is *weakly compactly generated* (WCG) if it contains a set A, which is compact in the weak topology, such that $\overline{\mathrm{span}} A = X$.

Separable Banach spaces and reflexive Banach spaces are always WCG. Perhaps the quintessential examples of WCG Banach spaces are the spaces $c_0(\Gamma)$ for Γ an arbitrary set. To verify that these spaces are indeed WCG, notice that the set $\{e_\gamma : \gamma \in \Gamma\} \cup \{0\}$ is weakly compact in $c_0(\Gamma)$ with dense span. A famous theorem of Amir and Lindenstrauss [2] is as follows.

Theorem 2.2.3 ([2, Main Theorem]). *Let X be a WCG Banach space. There exists a cardinal Δ and a bounded linear injection $X \to c_0(\Delta)$.*

Proof. See [2], or see e.g. [18, Corollary 13.17]. □

For our purposes, we require a slight strengthening of the above theorem, allowing us to take Δ to be the density character of X.

Lemma 2.2.4. *Let X be a WCG Banach space with density character Γ. There exists a bounded linear injection $X \to c_0(\Gamma)$.*

Proof. The case for $X = \{0\}$ is trivial so we may suppose that X is non-zero, meaning that Γ is an infinite cardinal.

Let W be a dense subset of X of cardinality Γ and define using Theorem 2.2.3 a bounded linear injection $\phi : X \to c_0(\Delta)$ for some cardinal Δ. Furthermore, define the set

$$A = \bigcup_{w \in W} \operatorname{supp} \phi(w) \subseteq \Delta \, .$$

Elements of $c_0(\Delta)$ have finite or countably infinite support, hence we have that $|A| = |W| \times \aleph_0 = \Gamma$.

Let $x \in X$, and let (w_n) be a sequence in W converging to x. If $\delta \in \Delta \setminus A$, then

$$\phi(x)(\delta) = \lim_{n\to\infty} \phi(w_n)(\delta) = \lim_{n\to\infty} 0 = 0 \, ,$$

hence $\operatorname{supp}(\phi(x)) \subseteq A$. It follows that the image of ϕ is contained within the set $\overline{\operatorname{span}}\{e_\gamma : \gamma \in A\} \equiv c_0(\Gamma)$. The result follows. □

We are now ready to answer the kernel problem for the spaces $c_0(\Gamma)$.

Theorem 2.2.5. *Let Y be a closed subspace of $c_0(\Gamma)$. There exists an operator $T \in \mathscr{B}(c_0(\Gamma))$ for which $Y = \ker T$.*

Proof. For any infinite cardinal Γ, the space $c_0(\Gamma)$ is WCG, with density character Γ. If Z is a quotient of $c_0(\Gamma)$ by one of its closed subspaces, then the density character Δ

of Z is at most Γ. Further, since bounded linear maps are always weakly continuous, and quotient maps are surjective, we have that Z is WCG.

Lemma 2.2.4 tells us that there is a bounded linear injection from Z to $c_0(\Delta)$, which embeds naturally into $c_0(\Gamma)$. The result follows from Lemma 2.2.1. $\square$

2.2.2 The case $\ell_1(\Gamma)$ for Γ uncountable.

A classical result of Pitt tells us that whenever $1 \leqslant p < q < \infty$, an operator from a closed subspace of ℓ_q to ℓ_p must be compact. Rosenthal proved a generalised version, which allows us to use the result on transfinite sequence spaces also. We state it here.

Theorem 2.2.6 (Generalised Pitt's Theorem). *Let $1 \leqslant p < q < \infty$, let Γ be any infinite set, and let X be a closed subspace of $\ell_q(\Gamma)$. Then $\mathscr{B}(X;\ell_p(\Gamma)) = \mathscr{K}(X;\ell_p(\Gamma))$.*

Proof. See [55, Theorem A2]. $\square$

Corollary 2.2.7. *Let $1 \leqslant p < q < \infty$, and let Γ be an uncountable set. Let $T \in \mathscr{B}(\ell_q(\Gamma);\ell_p(\Gamma))$. Then T is not injective.*

Proof. Theorem 2.2.6 tells us that the operator T is compact, and therefore has separable range. Hence, there exists a countable set A with $\overline{A} = \operatorname{im}(T)$. Enumerate A as $A = \{a_n : n \in \mathbb{N}\}$. Due to their belonging to $\ell_p(\Gamma)$, for each $n \in \mathbb{N}$, a_n has countable support. Hence the set $\Delta := \bigcup_{n\in\mathbb{N}} \operatorname{supp}(a_n)$ is a countable subset of Γ. It follows (similarly as to in the proof of Lemma 2.2.4) that $\operatorname{supp}(x) \subset \Delta$ for all $x \in \operatorname{im}(T)$.

Now, for each $\delta \in \Delta$ and $k \in \mathbb{N}$, set

$$A_\delta^k = \{\gamma \in \Gamma : |(Te_\gamma)(\delta)| \geqslant \frac{1}{k}\} .$$

Suppose that T is injective. We must have that

$$\bigcup_{\delta\in\Delta, k\in\mathbb{N}} A_\delta^k = \Gamma$$

because $Te_\gamma \neq 0$ for each $\gamma \in \Gamma$. Since Δ is countable and Γ is not, there is some $\delta \in \Delta$ and $k \in \mathbb{N}$ for which A_δ^k is uncountable.

Choose a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of distinct elements of A_δ^k, and for each $n \in \mathbb{N}$, pick norm one scalars $\sigma_n \in \mathbb{K}$ for which $\sigma_n(Te_{\alpha_n})(\delta) > 0$. Then define $x = \sum_{n\in\mathbb{N}} \frac{\sigma_n}{n} e_{\alpha_n}$.

Because $q > 1$, we have that $\sum_{n\in\mathbb{N}} \frac{1}{n^q} < \infty$, and hence $x \in \ell_q(\Gamma)$. However,

$$(Tx)(\delta) = \sum_{n\in\mathbb{N}} \frac{\sigma_n}{n}(Te_{\alpha_n})(\delta) = \sum_{n\in\mathbb{N}} |(Te_{\alpha_n})(\delta)| \geqslant \frac{1}{k}\sum_{n\in\mathbb{N}} \frac{1}{n} = \infty \ ,$$

a contradiction. □

The final result we need about the spaces $\ell_1(\Gamma)$ is the *lifting property* - a powerful and versatile result which we encounter many times throughout this thesis.

Proposition 2.2.8 (The lifting property). *Let X be a Banach space of density character Γ. Then X is isometrically isomorphic to a quotient space of $\ell_1(\Gamma)$ by one of its closed subspaces.*

Proof. See e.g. [25, Theorem 5.1]. □

We are now ready to answer the kernel problem for the spaces $\ell_1(\Gamma)$.

Theorem 2.2.9. *For any uncountable cardinal Γ, there exists a closed subspace of $\ell_1(\Gamma)$ which is not the kernel of any $T \in \mathscr{B}(\ell_1(\Gamma))$.*

Proof. Let $1 < p < \infty$. Then $\ell_p(\Gamma)$ is isometrically isomorphic to a quotient of $\ell_1(\Gamma)$ by one of its closed subspaces by Proposition 2.2.8, yet Corollary 2.2.7 tells us that there are no bounded linear injections $\ell_p(\Gamma) \to \ell_1(\Gamma)$. The result follows from Lemma 2.2.1. □

2.2.3 The case $\ell_p(\Gamma)$ for $1 < p < \infty$ and Γ uncountable.

For this section, let Γ be any infinite set, and let $1 < p < \infty$. Let q be the *conjugate exponent* of p, i.e. the unique real number satisfying the equation $p^{-1} + q^{-1} = 1$.

We first remark that the answer to Question 2.1.2 is trivially 'yes' in the setting of Hilbert spaces. To see this, let X be a closed subspace of any Hilbert space. Then X is the kernel of the orthogonal projection onto the complement of X in said

Hilbert space. So whilst the methods in this subsection work for the space $\ell_2(\Gamma)$, the cases of real interest to us are those of $\ell_p(\Gamma)$ for $p \in (1,\infty) \setminus \{2\}$.

Since $\ell_p(\Gamma)$ is a reflexive space, it has the separable complementation property. So, as discussed before, Proposition 2.1.8 tells us that every closed, separable subspace of $\ell_p(\Gamma)$ is the kernel of some $T \in \mathscr{B}(\ell_p(\Gamma))$. So for the rest of this section we need only consider non-separable subspaces of $\ell_p(\Gamma)$ as candidates for 'non-kernel' subspaces.

The following lemma outlines the method that we shall use for answering the kernel problem for the spaces $\ell_p(\Gamma)$ with $p \in (1,\infty)$. For a subspace Y of a Banach space X, the *annihilator* of Y is the subspace

$$Y^{\perp} = \{f \in X^* : \forall x \in Y,\ f(x) = 0\}$$

of X^*. Note that this definition depends on X as well as Y. Simple direct calculations tell us that the annihilator of a subset of X is always closed in X^*, and that $\overline{\mathrm{im}}(T^*) = \ker(T)^{\perp}$ whenever $T \in \mathscr{B}(X)$.

For a reflexive space X, we may by identifying X and X^{**} consider the annihilator of a subset of X^* to be a subset of X. In this case, we may also utilise the well-known fact that $(Y^{\perp})^{\perp} = \overline{Y}$ for any subspace Y of X.

Lemma 2.2.10. *Let X be a reflexive Banach space. Every closed subspace of X is the kernel of some $T \in \mathscr{B}(X)$ if and only if every closed subspace of X^* is the norm closure of the image of some $S \in \mathscr{B}(X^*)$.*

Proof. Suppose that every closed subspace of X is the kernel of some operator $T \in \mathscr{B}(X)$. Let Z be a closed subspace of X^*, and let $T \in \mathscr{B}(X)$ be such that $Z^{\perp} = \ker T$. Then

$$Z = \overline{Z} = (Z^{\perp})^{\perp} = (\ker T)^{\perp} = \overline{\mathrm{im}}(T^*)\ .$$

On the other hand, suppose that every closed subspace of X^* is the closure of the image of some $S \in \mathscr{B}(X^*)$, and let Y be a closed subspace of X. Let $S \in \mathscr{B}(X^*)$ be such that $\overline{\mathrm{im}}(S) = Y^{\perp}$. Using the fact that that $S^{**} = S$, we have

$$Y = \overline{Y} = (Y^{\perp})^{\perp} = \overline{\mathrm{im}}(S)^{\perp} = ((\ker S^*)^{\perp})^{\perp} = \ker S^*\ . \qquad \square$$

Solving the kernel problem positively for $\ell_p(\Gamma)$ therefore equates to proving that each closed subspace of $\ell_q(\Gamma)$ is the closure of the image of some $T \in \mathscr{B}(\ell_q(\Gamma))$. It is due to the similarity of structure between transfinite ℓ_p sequence spaces and their dual spaces that we can now shift our attention to subspaces which are closures of images of bounded operators rather than kernels thereof. The following result characterises such subspaces in terms of the supports of transfinite sequences in their possible dense spanning sets.

Lemma 2.2.11. *Let Γ be an uncountable cardinal, let $1 < p < \infty$, and let Y be any closed subspace of $\ell_p(\Gamma)$. The following conditions are equivalent.*

(a) *There exists an operator $T \in \mathscr{B}(\ell_p(\Gamma))$ with $\overline{\mathrm{im}}T = Y$.*

(b) *There exists a sequence $(D_n)_{n\in\mathbb{N}}$ of subsets of Y for which $\overline{\mathrm{span}}\bigcup_{n\in\mathbb{N}} D_n = Y$ and*

$$\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$$

for every $n \in \mathbb{N}$ and any $x, y \in D_n$ with $x \neq y$.

(c) *There exists a subset D of Y such that $\overline{\mathrm{span}}D = Y$ and the set*

$$\{d \in D : d(\alpha) \neq 0\}$$

is countable for every $\alpha \in \Gamma$.

Proof. (a)$\Rightarrow$(c). Suppose that $T \in \mathscr{B}(\ell_p(\Gamma))$ is an operator with $\overline{\mathrm{im}\,T} = Y$. We shall show that the set $D = \{Te_\beta : \beta < \Gamma\}$ satisfies (c). We have $\overline{\mathrm{span}}D = Y$ because the span of $\{e_\beta : \beta < \Gamma\}$ is dense in $\ell_p(\Gamma)$.

Let $\alpha < \Gamma$. Since $(Te_\beta)(\alpha) = \langle Te_\beta, e_\alpha^* \rangle = \langle e_\beta, T^* e_\alpha^* \rangle = (T^* e_\alpha^*)(\beta)$ for every $\beta < \Gamma$, we have that

$$\{d \in D : d(\alpha) \neq 0\} = \{Te_\beta : \beta < \Gamma,\, (Te_\beta)(\alpha) \neq 0\} = \{Te_\beta : \beta \in \mathrm{supp}(T^* e_\alpha^*)\}\,,$$

which is countable because $T^* e_\alpha^* \in \ell_q(\Gamma)$ has countable support.

(c)$\Rightarrow$(b) Let D satisfy the conditions of (c). Without loss of generality, we may suppose that $0 \notin D$. Define the functions

$$q : \mathcal{P}(D) \to \mathcal{P}(D)\ ;\ B \mapsto \{d \in D : \exists b \in B\ ;\ \operatorname{supp}(d) \cap \operatorname{supp}(b) \neq \emptyset\}\ ,$$

and

$$r : D \to \mathcal{P}(D)\ ;\ d \mapsto \bigcup_{n \in \mathbb{N}} q^n(\{d\})\ ,$$

where $\mathcal{P}(D)$ denotes the power set of D.

By hypothesis, the set $\{d \in D : d(\gamma) \neq 0\}$ is countable for every $\gamma \in \Gamma$. Combining this with the identity

$$q(B) = \bigcup_{b \in B} \bigcup_{\gamma \in \operatorname{supp} b} \{d \in D : d(\gamma) \neq 0\}$$

and the fact that vectors in $\ell_p(\Gamma)$ have countable support, we deduce that $q(B)$ is countable for every countable subset B of D. It follows by induction that $q^n(B)$ must be countable for every countable set B and every $n \in \mathbb{N}$. Then $r(d)$ is a countable union of at most countable sets for any $d \in D$, so we must have that $|r(d)| \leqslant \aleph_0$.

Let the relation $\sim$ on D be defined by $c \sim d$ if and only if there are $n \in \mathbb{N}$ and elements $b_1, \ldots, b_n \in D$ such that $c = b_1$, $d = b_n$ and $\operatorname{supp} b_j \cap \operatorname{supp} b_{j+1} \neq \emptyset$ for each $j \in \{1, \ldots, n-1\}$. It is easily checked that $\sim$ is an equivalence relation on D, where the fact that $0 \notin D$ ensures that we have reflexivity. The equivalence classes of $D/\sim$ are countable, as they are exactly the sets $r(d)$ for $d \in D$.

Express the quotient $D/\sim$ as $D/\sim\ = \{[c^\gamma] : \gamma \in \kappa\}$ for some indexing set κ, and further enumerate each equivalence class in $D/\sim$ as $[c^\gamma] = \{c^\gamma_n : n \in \mathbb{N}\}$. Two elements of D belonging to different equivalence classes of $D/\sim$ must have disjoint support by the definition of $\sim$, thus the sequence of sets (D_n) defined by $D_n = \{c^\gamma_n : \gamma \in \kappa\}$ for each $n \in \mathbb{N}$ satisfy (b).

(b)$\Rightarrow$(a). Let $(D_n)_{n \in \mathbb{N}}$ be a sequence of subsets of Y such that $\overline{\operatorname{span}} \bigcup_{n \in \mathbb{N}} D_n = Y$ and $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$ for every $n \in \mathbb{N}$ and every pair of distinct elements $x, y \in D_n$. Because the density character of Y can be at most Γ, we may suppose that $|\bigcup_{n \in \mathbb{N}} D_n| \leqslant \Gamma$.

For each $n \in \mathbb{N}$, normalise the elements of D_n to be unit vectors. Since the sets D_n consist of disjointly supported elements, there is an isometric isomorphism $U_n\colon \ell_p(D_n) \to \overline{\operatorname{span}}(D_n)$ determined by $U_n(e_d) = d$ for every $d \in D_n$, where $(e_d)_{d\in D_n}$ denotes the standard unit vector basis of $\ell_p(D_n)$. Let κ be the cardinal for which there is an isometric isomorphism $S\colon \ell_p(\kappa) \to \big(\bigoplus_{n\in\mathbb{N}} \ell_p(D_n)\big)_{\ell_p}$. Since $|\bigcup_{n\in\mathbb{N}} D_n| \leqslant \Gamma$, we must have that $\dim\big(\big(\bigoplus_{n\in\mathbb{N}} \ell_p(D_n)\big)_{\ell_p}\big) \leqslant \Gamma$. Consequently, we have that $\kappa \leqslant \Gamma$.

Now define an operator $U\colon \big(\bigoplus_{n\in\mathbb{N}} \ell_p(D_n)\big)_{\ell_p} \to Y$ by

$$U(x_n)_{n\in\mathbb{N}} = \sum_{n=1}^{\infty} \frac{U_n x_n}{2^n}\ .$$

Since $D_n \subseteq \operatorname{im} U_n$ for each $n \in \mathbb{N}$, we conclude that U has dense image in Y, and therefore the same is true for the composite operator $T = JUSP_\kappa \in \mathscr{B}(\ell_p(\Gamma))$, where $J\colon Y \hookrightarrow \ell_p(\Gamma)$ denotes the inclusion map and $P_\kappa : \ell_p(\Gamma) \to \ell_p(\kappa)$ denotes the canonical projection (considering κ as an initial segment of Γ). $\square$

Remark 2.2.12. We include here an alternative proof of (c)$\Rightarrow$(b) in the above theorem using graph theoretic principles, in the hope that the reader might find the method novel or illuminating in some respect. The terminology used, given below, is all standard and elementary to graph theorists. Notice that our definition of edges specifically excludes loops.

Definition 2.2.13.

- A *graph* is an ordered pair $G = (V, E)$, where V is the set of *vertices* of G, and E is a set of unordered pairs of distinct vertices called the set of *edges* of G. A *subgraph* $G' = (V', E')$ of G is a graph for which $V' \subset V$ and $E' \subset E$.

- The *degree* of a vertex $v \in V$ is the cardinality of the set $\{e \in E : v \in e\}$.

- A *colouring* of G is a function $f : V \to C$ for some set C of *colours*. A colouring of G is *proper* if $f(x) \neq f(y)$ whenever $\{x, y\} \in E$.

- A *path* in G beginning at a vertex $v_1 \in V$ and ending at $v_n \in V$ is a finite sequence of vertices $(v_1, v_2, \ldots, v_n)$ for some $n \in \mathbb{N}$ such that $\{v_j, v_{j+1}\} \in E$ for every $j \in \{1, \ldots, n-1\}$. Two vertices in V are *connected* if they belong to a common path, and a graph is *connected* if any two vertices within it are connected.

- A *connected component* of G is a connected subgraph of G which is not a subgraph of any strictly larger connected subgraph of G.

Alternative proof of (c)$\Rightarrow$(b). Define the graph $G = (V, E)$, with vertices $V = D$ and edges

$$E = \{\{b, d\} \subset V : [b \neq d] \wedge [\operatorname{supp}(b) \cap \operatorname{supp}(d) \neq \emptyset]\} .$$

Being p-summable, each of the vectors in V must have at most countable support, and by (c) there are only at most countably many vectors in V that may be supported at any given coordinate in Γ. As such, the degree of each vertex $d \in V$ must be at most $\aleph_0$.

It follows that for any given $d \in V$, there are at most $\aleph_0 \times n = \aleph_0$ vectors connected to d via a path of length n, so there are at most $|[\aleph_0]^{<\infty}| = \aleph_0$-many vectors which may be contained in a path beginning at d. This means that the number of vertices in the connected component of G containing d is at most countable.

We can therefore define a proper colouring $f : V \to \mathbb{N}$ of G with $\aleph_0$-many colours, simply using each colour at most once per connected component. Then, the sets

$$D_n := \{v \in V : f(v) = n\}$$

for $n \in \mathbb{N}$ partition D and satisfy the conditions of (b). $\square$

We finish this section by proving that every closed subspace of $\ell_p(\Gamma)$ must have a spanning set of the form described in Lemma 2.2.11(c), thus solving the kernel problem in this setting.

Definition 2.2.14. Let X be a Banach space.

- Let Γ be an arbitrary index set, and let $(x_\gamma, f_\gamma)_{\gamma\in\Gamma} \subseteq X \times X^*$. We say that $(x_\gamma, f_\gamma)_{\gamma\in\Gamma}$ is a *biorthogonal system on* X if for each $\gamma \in \Gamma$ we have

$$f_\gamma(x_\delta) = \begin{cases} 1 & \text{if } \gamma = \delta\ , \\ 0 & \text{if } \gamma \neq \delta\ . \end{cases}$$

- Let $K \subseteq X$, and $F \subseteq X^*$ both be non-empty. We say that F *separates the points of* K if for every $x, y \in K$ with $x \neq y$, there exists some $f \in F$ for which $f(x) \neq f(y)$.

- A *Markushevich basis* of X is a biorthogonal system $(x_\gamma, f_\gamma)_{\gamma\in\Gamma}$ on X such that $\overline{\mathrm{span}}\{x_\gamma : \gamma \in \Gamma\} = X$, and $\{f_\gamma : \gamma \in \Gamma\}$ separates the points of X.

A popular way to characterise the second bullet point in the above definition when K is a subspace of X is the following. If X is a Banach space, K is a subspace of X, and $F \subseteq X^*$, then F separates the points of K if and only if whenever $k \in K$ and $\langle k, f\rangle = 0$ for every $f \in F$, we must have that $k = 0$. This fact is easy to show from the definition, and will be used to prove our next result.

Lemma 2.2.15. *Let X be a closed subspace of $\ell_p(\Gamma)$ for some $p \in (1,\infty)$ and some infinite set Γ. Then X contains a subset D such that $\overline{\mathrm{span}}D = X$, where the set $\{d \in D : d(\gamma) \neq 0\}$ is countable for each $\gamma \in \Gamma$.*

Proof. Being a closed subspace of the reflexive Banach space $\ell_p(\Gamma)$, X is reflexive, and therefore has a Markushevich basis $(x_j, f_j)_{j\in\kappa}$ (see, e.g., [25, Theorem 5.1]). We may suppose that the vectors x_j all have norm one. Define the set $D = \{x_j : j \in \kappa\}$. Then $\overline{\mathrm{span}}D = X$ by the definition of a Markushevich basis. Let $\gamma \in \Gamma$ and $n \in \mathbb{N}$. We claim that the set

$$A_\gamma^n = \{j \in \kappa : |x_j(\gamma)| \geqslant 1/n\}$$

must be finite. Assume the contrary. The Banach–Alaoglu theorem and reflexivity imply that we can find a net $(x_{j_\lambda})_{\lambda\in\Lambda}$ which converges weakly to some $x \in X$, where $j_\lambda \in A_\gamma^n$ and $j_\lambda \neq j_\mu$ for $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$.

Then, on the one hand, for each $k \in \kappa$, $\langle x_{j_\lambda}, f_k\rangle \underset{\lambda}{\longrightarrow} \langle x, f_k\rangle$, where all but at most one term on the left-hand side vanish, so $\langle x, f_k\rangle$=0. Since $(f_k)_{k\in\kappa}$ separates the points of X, we conclude that $x = 0$. On the other hand,

$$\frac{1}{n} \leqslant |\langle x_{j_\lambda}, e_\gamma^*\rangle| \underset{\lambda}{\longrightarrow} |\langle x, e_\gamma^*\rangle| \ ,$$

so $|\langle x, e_\gamma^*\rangle| \geqslant 1/n > 0$, a contradiction.

Hence A_γ^n is finite for each $\gamma \in \Gamma$ and $n \in \mathbb{N}$, so the union

$$\bigcup_{n\in\mathbb{N}} A_\gamma^n = \{j \in \kappa : x_j(\gamma) \neq 0\}$$

is at most countable. This tells us that for each $\gamma \in \Gamma$ we have

$$|\{d \in D : d(\gamma) \neq 0\}| = |\{j \in \kappa : x_j(\gamma) \neq 0\}| \leqslant \aleph_0 \ . \qquad \square$$

Theorem 2.2.16. *Let X be a closed subspace of $\ell_p(\Gamma)$ for some $1 < p < \infty$. Then $X = \ker T$ for some $T \in \mathscr{B}(\ell_p(\Gamma))$.*

Proof. By Lemma 2.2.15, closed subspaces of $\ell_q(\Gamma)$ satisfy the conditions of Lemma 2.2.11(c). The result follows. $\square$

Chapter 3

The lattice of closed ideals of bounded operators on a direct sum of classical Banach spaces

3.1 Background

Very few Banach spaces X are known for which the lattice of closed ideals of the Banach algebra $\mathscr{B}(X)$ is fully understood. When X is finite-dimensional, $\mathscr{B}(X)$ is simple, meaning that it contains no non-zero, proper ideals, so for this chapter we shall henceforth discuss infinite-dimensional Banach spaces only.

Our focus is on the 'classical' case, that is, Banach spaces that can be defined by elementary means and/or were known to Banach and his contemporaries. We begin this chapter with an overview of the classical Banach spaces whose lattices of closed operator ideals are fully understood.

(i) Calkin [8] was the first to prove a result of this kind, showing that only non-trivial closed ideal of $\mathscr{B}(\ell_2)$ is $\mathscr{K}(\ell_2)$.

(ii) Gohberg, Markus and Feldman [22] improved on Calkin's result by showing that for $X = c_0$ or $X = \ell_p$, $1 \leqslant p < \infty$, the only non-trivial closed ideal of $\mathscr{B}(X)$ is $\mathscr{K}(X)$.

(iii) Gramsch [23] and Luft [45] independently found the lattice of closed ideals of $\mathscr{B}(\ell_2(\Gamma))$ for uncountable cardinals Γ, showing that the non-trivial closed ideals of $\mathscr{B}(\ell_2(\Gamma))$ are given by the sets of κ-compact operators on $\ell_2(\Gamma)$ (see page 6 for the precise definition), where $\aleph_0 \leqslant \kappa \leqslant \Gamma$.

(iv) Unifying and generalising the results (i), (ii), and (iii), Daws [12, Theorem 7.4] found the lattice of closed ideals of $X = c_0(\Gamma)$ and $X = \ell_p(\Gamma)$, where Γ is an infinite cardinal and $1 \leqslant p < \infty$. This was discussed in Section 1.4.

(v) Let $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$ for $D = c_0$ or $D = \ell_1$. As discussed in Section 1.5, the lattice of closed ideals of $\mathscr{B}(X)$ was found by Laustsen, Loy and Read, and by Laustsen, Schlumprecht and Zsak respectively.

Our goal for this chapter is to combine the results (iv) and (v) to obtain two new 'hybrid' families of Banach spaces, namely $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{c_0} \oplus c_0(\Gamma)$ and its dual space $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{\ell_1} \oplus \ell_1(\Gamma)$, for any uncountable cardinal Γ, whose closed ideals of operators we classify. The precise statement is as follows.

Theorem 3.1.1. *Let $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ or $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ for an uncountable cardinal Γ, and set $E = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$ and $X = E \oplus D_\Gamma$. Then the lattice of closed ideals of $\mathscr{B}(X)$ is*

$$\begin{tikzcd}
& \{0\} \arrow[d] & \\
& \mathscr{K}(X) \arrow[d] & \\
& \overline{\mathscr{G}_D}(X) \arrow[dl] \arrow[dr] & \\
\mathscr{K}_{\aleph_1}(X) \arrow[d] & & \mathscr{J}_{\aleph_2}(X) \arrow[d] \arrow[dll] \\
\mathscr{K}_{\aleph_2}(X) \arrow[d] & & \mathscr{J}_{\aleph_3}(X) \arrow[d] \arrow[dll] \\
\mathscr{K}_{\aleph_3}(X) \arrow[d] & & \mathscr{J}_{\aleph_4}(X) \arrow[d] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
\mathscr{K}_{\Gamma}(X) \arrow[dr] & & \mathscr{J}_{\Gamma^+}(X) \arrow[dl] \\
& \mathscr{B}(X)\,, &
\end{tikzcd}$$

where

$$\mathscr{J}_\kappa(X) = \left\{ \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \in \mathscr{B}(X) \; ; \; T_{1,1} \in \overline{\mathscr{G}_D}(E),\, T_{1,2} \in \mathscr{B}(D_\Gamma; E), \right.$$
$$\left. T_{2,1} \in \mathscr{B}(E; D_\Gamma),\, T_{2,2} \in \mathscr{K}_\kappa(D_\Gamma) \right\} \tag{3.1.1}$$

for each cardinal $\aleph_2 \leqslant \kappa \leqslant \Gamma^+$, *with an arrow from an ideal* $\mathscr{I}$ *pointing to an ideal* $\mathscr{J}$ *denoting that* $\mathscr{I} \subsetneq \mathscr{J}$ *and there are no closed ideals of* $\mathscr{B}(X)$ *strictly contained in between* $\mathscr{I}$ *and* $\mathscr{J}$.

The 2×2 matrices of operators in the definition of $\mathscr{J}_\kappa$ should be interpreted according to the prescription in Section 1.2.

Remark 3.1.2. In addition to the 'classical' Banach spaces listed above, there are a number of 'exotic', or purpose-built, Banach spaces whose closed ideals of operators can be classified. We list them here. They belong to two main classes.

The first class of these Banach spaces consists of Argyros and Haydon's construction of a Banach space solving the scalar-plus-compact problem, and some variants of it. The second such class consists of spaces of continuous functions on certain Mrówka topological spaces.

(i) Argyros and Haydon [3] constructed a Banach space X_{AH} which is hereditarily indecomposable, and has *few operators*, in the sense that each $T \in \mathscr{B}(X_{AH})$ is the sum of a scalar multiple of the identity and a compact operator. This space was the solution to the longstanding 'scalar-plus-compact problem'. The lattice of closed ideals of $\mathscr{B}(X_{AH})$ is given as

$$\{0\} \subsetneq \mathscr{K}(X_{AH}) \subsetneq \mathscr{B}(X_{AH}) \,,$$

with the 'few-operators' property of X_{AH} necessitating that $\mathscr{K}(X_{AH})$ is a maximal ideal since it has codimension 1 in $\mathscr{B}(X_{AH})$, and the approximation property giving that $\mathscr{K}(X_{AH})$ is minimal.

(ii) Tarbard in [60] defines for each $k \in \mathbb{N}$ with $k \geqslant 2$, a separable, hereditarily indecomposable space X_k with a k-dimensional Calkin algebra. On this space, there exists a non-compact, strictly singular operator S for which $S^k = 0$, and $S^j \neq 0$ for all $0 \leqslant j < k$, with the set $\{S^0, S, S^2, \ldots, S^{k-1}\}$ linearly independent. Every bounded operator T on X_k is expressible as

$$T = \sum_{i=0}^{k-1} \lambda_i S^i + K$$

for some scalars λ_i for $i \in \{0, \ldots, k-1\}$ and some compact operator K on X_k. As [60, Lemma 7.1], he displays the folowing closed ideal structure of $\mathscr{B}(X_k)$.

$$\{0\} \subsetneq \mathscr{K}(X_k) \subsetneq \langle S^{k-1} \rangle \subsetneq \langle S^{k-2} \rangle \subsetneq \cdots \subsetneq \langle S \rangle \subsetneq \mathscr{B}(X_k) ,$$

where for operators $T_1, T_2, \ldots T_n \in \mathscr{B}(Y)$ on a Banach space Y, $\langle T_1, T_2, \ldots, T_n \rangle$ is the closed ideal *generated* by $T_1, \ldots, T_n$, i.e. the norm closure of the set

$$\left\{ \sum_{j=1}^{n} A_j T_j B_j : A_j, B_j \in \mathscr{B}(Y) \; \forall j \in \{1, 2, \ldots, n\} \right\}$$

in $\mathscr{B}(Y)$.

(iii) Kania and Laustsen in [36] found a proper closed subspace Y of X_{AH} for which every operator $T \in \mathscr{B}(Y; X_{AH})$ is the sum of a scalar multiple of the inclusion $\iota : Y \hookrightarrow X_{AH}$ and a compact operator. They then considered the space $Z = X_{AH} \oplus Y$, and showed that $\mathscr{B}(Z)$ has this lattice of closed ideals:

$$\{0\} \subsetneq \mathscr{K}(Z) \subsetneq \mathcal{E}(Z) = \mathcal{M}_1 \cap \mathcal{M}_2 \begin{array}{c} \subsetneq \mathcal{M}_1 \subsetneq \\ \subsetneq \mathcal{M}_2 \subsetneq \end{array} \mathscr{B}(Z) ,$$

where $\mathcal{E}(Z)$ denotes the set of *inessential* operators on Z, which are those operators for which $I_Z - VT$ has finite-dimensional kernel and cofinite-dimensional image for every operator $V \in \mathscr{B}(Z)$.

The maximal ideals $\mathcal{M}_1$ and $\mathcal{M}_2$ are of codimension 1 in $\mathscr{B}(Z)$. As a left ideal, $\mathcal{M}_1$ can be generated by just two given elements in the following way:

$$\mathcal{M}_1 = \left\{ A \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + B \begin{pmatrix} 0 & \iota \\ 0 & 0 \end{pmatrix} : A, B \in \mathscr{B}(Z) \right\} .$$

The maximal ideal $\mathcal{M}_2$ cannot be generated as a left ideal by any finite set of operators, and is defined as

$$\mathcal{M}_2 = \left\{ \begin{pmatrix} 0 & \alpha\iota \\ 0 & \beta I_Y \end{pmatrix} + K : \alpha, \beta \in \mathbb{K}, K \in \mathscr{K}(Z) \right\} .$$

(iv) Giving constructions which also belong to the class of descendants of X_{AH}, Motakis, Puglisi, and Zisimopoulou [49] found that for each countable, compact metric space K, there exists a Banach space X_{MPZ} which has Calkin algebra isomorphic to $C(K)$.

There exists a natural order-preserving bijection (i.e. an *order isomorphism*) between the closed subsets of K (ordered by $\subsetneq$) and the non-zero closed ideals of $\mathscr{B}(X_{MPZ})$ (also ordered by $\subsetneq$) (see [36, Remark 1.5]). This bijection defines completely the desired lattice structure.

(v) The other class of exotic Banach spaces for which we know the entire lattice of closed ideals of $\mathscr{B}(X)$ consists of the Banach space $C_0(K_{\mathcal{A}})$ of continuous, scalar-valued functions vanishing at infinity defined on Koszmider's Mrówka space $K_{\mathcal{A}}$ (defined in Section 1.6), for which $C_0(K_{\mathcal{A}})$ has *few operators*. We clarify here that such spaces have few operators in a different sense to X_{AH}; their bounded operators are all scalar multiples of the identity plus some operator of separable range as opposed to scalar multiples of the identity plus some compact operator. The space $C_0(K_{\mathcal{A}})$ and the closed ideals of its space of bounded operators will be central to Section 4.4.

A possible explanation for the scarcity of Banach spaces X whose closed ideals of operators have been classified, especially among classical spaces, is that recent research has shown that in many cases $\mathscr{B}(X)$ has $2^{\mathfrak{c}}$-many closed ideals, where $\mathfrak{c}$ denotes the *cardinality of the continuum* (i.e. $\mathfrak{c} = 2^{\aleph_0}$). Note that this is the largest possible number of closed ideals of $\mathscr{B}(X)$ for a separable Banach space X, since it coincides with the maximum number of possible closed subsets of $\mathscr{B}(X)$.

Spaces for which $\mathscr{B}(X)$ has $2^{\mathfrak{c}}$ closed ideals include $X = L_p[0,1]$ for $p \in (1,\infty) \setminus \{2\}$ (see [33]), $X = \ell_p \oplus \ell_q$ for $1 \leqslant p < q \leqslant \infty$ with $(p,q) \neq (1,\infty)$ and $X = \ell_p \oplus c_0$ for $1 < p < \infty$ (see [19, 20]). Given in [33, Corollary 1] is a criterion for Banach spaces X that have 1-unconditional bases, which if fulfilled implies that $\mathscr{B}(X)$ has

$2^{\mathfrak{c}}$-many closed ideals.

For several other spaces X, it is known that $\mathscr{B}(X)$ contains at least continuum many closed ideals. This includes $X = L_1[0,1]$, $X = C[0,1]$ and $X = L_\infty[0,1]$ (see [32]; note that these results also cover $X = \ell_\infty$ because ℓ_∞ and $L_\infty[0,1]$ are isomorphic as Banach spaces by [52]), as well as the Tsirelson space and the Schreier space of order $n \in \mathbb{N}$ (see [7]). For $X = \ell_1 \oplus c_0$, the best known result is that $\mathscr{B}(X)$ has at least $\aleph_1$-many closed ideals (see [59]).

3.2 The proof of Theorem 3.1.1

To aid the presentation, we split the proof of Theorem 3.1.1 into a series of lemmas. The proof itself requires results about the transfinite sequence spaces $c_0(\Gamma)$ and $\ell_1(\Gamma)$ only, not $\ell_p(\Gamma)$ for $1 < p < \infty$. However, our first few results hold true also for the latter spaces and with identical proofs, so we give these more general results.

For the transfinite Banach sequence spaces D_Γ in question, with standard unit vector basis $(e_\gamma)_{\gamma\in\Gamma}$, we will use $P_\Delta \in \mathscr{B}(D_\Gamma)$ to denote the standard basis projection $\sum_{\gamma\in\Gamma} \lambda_\gamma e_\gamma \mapsto \sum_{\gamma\in\Delta} \lambda_\gamma e_\gamma$ whenever $\Delta \subseteq \Gamma$.

Lemma 3.2.1. *Let $D_\Gamma = c_0(\Gamma)$ or $D_\Gamma = \ell_p(\Gamma)$ for some $p \in [1,\infty)$ and some set $\Gamma \neq \emptyset$.*

(i) *Every separable subspace of D_Γ is contained in the image of the basis projection P_Δ for some countable subset Δ of Γ.*

(ii) *Suppose that $D_\Gamma \neq \ell_1(\Gamma)$. Then, for every Banach space E and every operator $T\colon D_\Gamma \to E$ for which there exists an injective operator from the image of T into ℓ_∞, there is a countable subset Δ of Γ such that $T = TP_\Delta$.*

Proof. (i). By definition, every separable subspace E of D_Γ has the form $E = \overline{W}$ for some countable subset W of D_Γ. Define $\Delta = \bigcup_{w\in W} \operatorname{supp} w$, which is a countable union of countable sets and is thus countable. The continuity of the projection P_Δ implies that $x = P_\Delta x$ for every $x \in E$. Hence the image of P_Δ contains E.

(ii). Let $U\colon T(D_\Gamma) \to \ell_\infty$ be an injective operator. Assume towards a contradiction that the set

$$\Delta_{k,m} = \left\{\gamma \in \Gamma : |UTe_\gamma(m)| \geqslant \frac{1}{k}\right\}$$

is infinite for some $k, m \in \mathbb{N}$, so that it contains an infinite sequence $(\gamma_n)_{n\in\mathbb{N}}$ of distinct elements. For each $n \in \mathbb{N}$, take a scalar σ_n of modulus one such that $\sigma_n \cdot (UTe_{\gamma_n})(m) \geqslant 1/k$. The sequence $(|\frac{\sigma_n}{n}|)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}}$ belongs to c_0 and ℓ_p for $1 < p < \infty$, so because we assumed that $D_\Gamma \neq \ell_1(\Gamma)$, we have $x = \sum_{n\in\mathbb{N}} \frac{\sigma_n}{n} e_{\gamma_n} \in D_\Gamma$. But

$$(UTx)(m) = \sum_{n\in\mathbb{N}} \frac{\sigma_n}{n}(UTe_{\gamma_n})(m) \geqslant \frac{1}{k}\sum_{n\in\mathbb{N}} \frac{1}{n} = \infty \ ,$$

a contradiction. Hence $\Delta_{k,m}$ is finite for each $k, m \in \mathbb{N}$, and thus the union $\Delta = \bigcup_{k,m\in\mathbb{N}} \Delta_{k,m}$ must be countable. For each $\gamma \in \Gamma \setminus \Delta$, we have that $UTe_\gamma = 0$, so $Te_\gamma = 0$ by the injectivity of U, and therefore $TP_\Delta = T$. □

Remark 3.2.2. The case $D_\Gamma = \ell_1(\Gamma)$ must be excluded in Lemma 3.2.1(ii) because the statement is false for $\ell_1(\Gamma)$, unless $E = \{0\}$ or Γ is countable. Indeed, take $y \in E$, and consider the summation operator

$$S_y \colon \ \ell_1(\Gamma) \to E \ ; \qquad x \mapsto \sum_{\gamma\in\Gamma} x(\gamma)\, y \ .$$

Since $S_y(e_\gamma) = y$ for every $\gamma \in \Gamma$, we see that $S_y = S_y P_\Delta$ if and only if $y = 0$ or $\Delta = \Gamma$.

Corollary 3.2.3. *Let $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ or $(D, D_\Gamma) = (\ell_p, \ell_p(\Gamma))$ for some $p \in [1,\infty)$ and some uncountable set Γ, and let E be any separable Banach space. Then*

$$\mathscr{B}(D_\Gamma; E) = \mathscr{G}_D(D_\Gamma; E) \qquad \textit{and} \qquad \mathscr{B}(E; D_\Gamma) = \mathscr{G}_D(E; D_\Gamma) \ .$$

Proof. The first identity for $D_\Gamma \neq \ell_1(\Gamma)$, and the second identity in full generality, both follow easily from Lemma 3.2.1 because the image of the projection P_Δ for Δ countable is either finite-dimensional or isomorphic to D.

It remains to show that every operator $T\colon \ell_1(\Gamma) \to E$ factors through ℓ_1. We use the lifting property of ℓ_1 (Proposition 2.2.8) to verify this. Let $\epsilon > 0$. Since E is separable, we can use the lifting property of ℓ_1 to find a quotient map $Q\colon \ell_1 \to E$. We have that for every $y \in E$, there is $x \in \ell_1$ with $Qx = y$ and $\|x\| \leqslant (1+\epsilon)\|y\|$.

Hence, for each $\gamma \in \Gamma$, we can find $x_\gamma \in \ell_1$ such that $Qx_\gamma = Te_\gamma$ and

$$\|x_\gamma\| \leqslant (1+\epsilon)\|Te_\gamma\| \leqslant (1+\epsilon)\|T\| \ .$$

Define an operator $R\colon \ell_1(\Gamma) \to \ell_1$ by $Re_\gamma = x_\gamma$ for each $\gamma \in \Gamma$. To check that R is bounded, let $z = \sum_{\gamma\in\Gamma} \lambda_\gamma e_\gamma \in \ell_1(\Gamma)$ have norm 1. Then

$$\|R(z)\| = \|R \sum_{\gamma\in\Gamma} \lambda_\gamma e_\gamma\| = \| \sum_{\gamma\in\Gamma} \lambda_\gamma x_\gamma\| \leqslant (1+\epsilon)\|T\| \sum_{\gamma\in\Gamma} |\lambda_\gamma| \leqslant (1+\epsilon)\|T\| \ .$$

It is clear that $T = QR$, proving the claim. □

Lemma 3.2.4. *Let $D = c_0$ or $D = \ell_p$ for some $p \in [1,\infty)$, and let $(E_n)_{n\in\mathbb{N}}$ be a sequence of non-zero Banach spaces. Then there are operators $R\colon D \to \left(\bigoplus_{n\in\mathbb{N}} E_n\right)_D$ and $S\colon \left(\bigoplus_{n\in\mathbb{N}} E_n\right)_D \to D$ such that $SR = I_D$.*

Proof. For each $n \in \mathbb{N}$, choose $y_n \in E_n$ and $f_n \in E_n^*$ with

$$\|y_n\| = \|f_n\| = \langle y_n, f_n\rangle = 1 \ ,$$

and define operators $R\colon (\lambda_n) \mapsto (\lambda_n y_n)$ for $(\lambda_n) \in D$ and $S\colon (x_n) \mapsto (\langle x_n, f_n\rangle)$ for $(x_n) \in \left(\bigoplus_{n\in\mathbb{N}} E_n\right)_D$. □

To prove Lemma 3.2.7, we will require the following result about isomorphic embeddings of finite dimensional Hilbert spaces. The result can be stated in several different ways, and we will be using the statement as in [18, Theorem 6.15]. If X and Y are isomorphic Banach space, the *Banach-Mazur distance* $d(X,Y)$ between X and Y is the infimum of all quantities $\|T\| \cdot \|T^{-1}\|$ ranging over all isomorphisms $T : X \to Y$.

Theorem 3.2.5 (Dvoretzky's Theorem)**.** *For every $\epsilon > 0$, there exists a constant $\lambda > 0$ with the following property. Let X be an n-dimensional Banach space. There exists an $N(n)$-dimensional subspace Y of X such that $d(Y, \ell_2^{N(n)}) \leqslant 1+\epsilon$, where $N(n)$ is the largest integer less than or equal to $\lambda \log(n)$.*

For our purposes, it helps to reinterpret Dvoretzky's theorem as follows.

Corollary 3.2.6. *Let $\epsilon > 0$. There exists a constant $\lambda > 0$ with the following property. For every $n \in \mathbb{N}$, if X is a Banach space with dimension $N(n) \geqslant \exp(\frac{n}{\lambda})$, there exists an n-dimensional subspace Y of X such that $d(Y, \ell_2^n) \leqslant 1+\epsilon$.*

Recall the definition of a diagonal operator between direct sums of Banach spaces (Section 1.2).

Lemma 3.2.7. *Let $D = c_0$ or $D = \ell_p$ for $1 \leqslant p < \infty$, and let $C > 1$. There exists an operator $U : \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D \to D$ for which $\|x\| \leqslant \|Ux\| \leqslant C\|x\|$ for all $x \in \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$.*

Proof. For every $n \in \mathbb{N}$, there exists by Corollary 3.2.6 some $N(n) \in \mathbb{N}$ and a linear injection $U_n : \ell_2^n \to D_{N(n)}$ for which

$$\|x\| \leqslant \|U_n x\| \leqslant C\|x\| \;,\; (\forall x \in \ell_2^n) \;, \tag{3.2.1}$$

where $D_{N(n)}$ denotes the canonical $N(n)$-dimensional version of D. Identify the spaces D and $\left(\bigoplus_{n\in\mathbb{N}} D_{N(n)}\right)_D$. We may then define the diagonal operator

$$U = \left(\bigoplus_{n\in\mathbb{N}} U_n\right) : \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D \to \left(\bigoplus_{n\in\mathbb{N}} D_{N(n)}\right)_D \,. \tag{3.2.2}$$

That U satisfies $\|x\| \leqslant \|Ux\| \leqslant C\|x\|$ for every $x \in \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$ follows from (3.2.1), the definition of U, and the fact that $\|U\| = \sup_{n\in\mathbb{N}} \|U_n\|$. □

We will be using the above lemma again in Chapter 4 where we require some technical details about the operator U. We note here that isomorphic copies of $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$ are never complemented in D if $D \neq \ell_2$.

Remark 3.2.8. For the purposes of this chapter, and its published version [4], only a non-quantitative version of Lemma 3.2.15 needs proving. We shall prove a stronger version where upper bounds are calculated on the norms of the operators involved. This strengthening is necessary in Section 4.3 for showing that the Calkin algebras of the spaces $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D \oplus D_\Gamma$ have unique algebra norms, where $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ or $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ for some infinite set Γ. To this end, we require several additional steps, and thus the content from here until the end of Lemma 3.2.15 deviates from that of [4].

Whenever we have a Banach space Y with unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$ and a subset A of Γ, we use the notation $P_A \in \mathscr{B}(Y)$ to denote the basis projection onto the coordinates indexed by the set A. If $\Gamma = \mathbb{N}$, we write P_n for $P_{\{1,\dots,n\}}$. The lack of reference to the space Y in the notation P_A is unlikely to cause confusion. By convention we let $P_\emptyset = 0$.

The basis $(e_\gamma)_{\gamma\in\Gamma}$ being unconditional implies the existence of some constant $C \geqslant 1$ for which $\|P_A\| \leqslant C$ and $\|I - P_A\| \leqslant C$ for every subset A of Γ. The least constant C for which this is true is the *basis constant* of $(e_\gamma)_{\gamma\in\Gamma}$, and we call the basis C-unconditional.

The following lemma will prove helpful for us in future calculations of quotient norms. It is an adapted version of a special case of [12, Proposition 5.1], and its proof is similar.

Lemma 3.2.9. *Let Y be a Banach space with a 1-unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$. Let $T \in \mathscr{B}(Z;Y)$ for some Banach space Z. Then*

$$\|T\|_e = \inf\{\|(I - P_A)T\| : A \subseteq \Gamma, |A| < \infty\} .$$

Proof. Let A be a finite subset of Γ. Then $\|(I - P_A)T\| \geqslant \|T\|_e$ holds because $P_A T$ is compact.

Suppose towards a contradiction that there is some $\epsilon > 0$ and some compact operator $K \in \mathscr{K}(Z;Y)$ for which $\|T + K\| \leqslant \|(I - P_A)T\| - \epsilon$ for every finite subset A of Γ.

Because K is compact, we can find a number $n \in \mathbb{N}$ for which $K(B_Z)$ is contained in the union of n-many open balls of radius $\frac{\epsilon}{4}$. Let the centres of these balls be denoted $y_1, y_2, \ldots, y_n$. Take a finite subset A of Γ for which $\|(I - P_A)y_j\| \leqslant \frac{\epsilon}{4}$ for every $j \in \{1, \ldots, n\}$.

Let $z \in B_Z$, and choose $j \in \{1, \ldots, n\}$ with $\|Kz - y_j\| \leqslant \frac{\epsilon}{4}$. Then

$$\|(I - P_A)Kz\| \leqslant \|(I - P_A)(Kz - y_j)\| + \|(I - P_A)y_j\| \leqslant \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} .$$

So $\|(I - P_A)K\| \leqslant \frac{\epsilon}{2}$, and we have that

$$\begin{aligned}\|(I - P_A)T\| &= \|(I - P_A)(T + P_A K)\| \leqslant \|T + P_A K\| \\ &\leqslant \|T + K\| + \|P_A K - K\| \leqslant \|(I - P_A)T\| - \epsilon + \frac{\epsilon}{2} < \|(I - P_A)T\| ,\end{aligned}$$

a contradiction. Thus

$$\|T\|_e \geqslant \inf\{\|(I - P_A)T\| : A \subset \Gamma, |A| < \infty\} ,$$

completing the proof. □

Lemma 3.2.10. *Let Y be a Banach space with a 1-unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$ for some infinite set Γ. Let $T \in \mathscr{B}(Y)$ be such that $\|T\|_e = 1$.*

(i) For all $\epsilon > 0$, there is a strictly increasing sequence $\emptyset = B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \dots$ of finite subsets of Γ, and a disjointly supported sequence (x_n) of norm-1 vectors in $Y_{00} := \operatorname{span}(e_\gamma)_{\gamma\in\Gamma}$ such that for each $n \in \mathbb{N}$, we have

$$\|Tx_n\| \geqslant \|(I - P_{B_{n-1}})Tx_n\| > 1 - \epsilon \quad \text{and} \quad \|(I - P_{B_n})Tx_n\| < \epsilon\ .$$

Furthermore, if $\Gamma = \mathbb{N}$, the sequence (x_n) can be defined to be consecutively supported.

(ii) Let W and Z be Banach spaces, and suppose that there is a quotient map $Q \in \mathscr{B}(W;Z)$. Let $T \in \mathscr{B}(Z;Y)$. Then

$$\|TQ\|_e = \|T\|_e\ . \tag{3.2.3}$$

Proof. (i) We construct the desired sequences by induction. Let $B_0 = \emptyset$. Since $1 = \|T + \mathscr{K}(Y)\| \leqslant \|T\|$, and Y_{00} is dense in Y, we can take $x_1 \in Y_{00}$ with norm 1 such that $\|Tx_1\| = \|(I - P_{B_0})Tx_1\| > 1 - \epsilon$. Let $B_1 \subset \operatorname{supp}(Tx_1)$ be finite such that $\|(I - P_{B_1})Tx_1\| < \epsilon$.

Suppose that for some $n \in \mathbb{N}$ we have chosen disjointly supported vectors $x_1, x_2, \dots, x_n \in Y_{00}$ of norm 1, along with finite subsets $B_0 \subsetneq B_1 \subsetneq B_2 \cdots \subsetneq B_n$ of Γ for which $\|Tx_j\| \geqslant \|(I - P_{B_{j-1}})Tx_j\| > 1 - \epsilon$ and $\|(I - P_{B_j})x_j\| < \epsilon$ for each $1 \leqslant j \leqslant n$. In case $\Gamma = \mathbb{N}$, suppose also that the vectors $x_1, \dots, x_n$ are consecutively supported. Set

$$A_n = \begin{cases} \bigcup_{j=1}^n \operatorname{supp}(x_j) & \text{if } \Gamma \neq \mathbb{N}\ , \\ \{1, 2, \dots, \max\operatorname{supp}(x_n)\} & \text{if } \Gamma = \mathbb{N}\ . \end{cases}$$

Since $\|(I - P_{B_n})T(I - P_{A_n})\| \geqslant \|T\|_e = 1$, we can choose a unit vector $x'_{n+1} \in$

$\mathrm{span}(e_\gamma)_{\gamma\in\Gamma}$ for which $\|(I-P_{B_n})T(I-P_{A_n})x'_{n+1}\| > 1-\epsilon$. Let

$$x_{n+1} = \frac{(I-P_{A_n})x'_{n+1}}{\|(I-P_{A_n})x'_{n+1}\|} .$$

Then $\|x_{n+1}\| = 1$, and since $\|(I-P_{A_n})x'_{n+1}\| \leqslant 1$, we have that

$$\|Tx_{n+1}\| \geqslant \|(I-P_{B_n})Tx_{n+1}\| > 1-\epsilon . \tag{3.2.4}$$

Take a finite subset B'_{n+1} of Γ for which $\|(I-P_{B'_{n+1}})Tx_{n+1}\| < \epsilon$, noticing that necessarily $B'_{n+1}\setminus B_n$ is non-empty by the right-hand inequality of (3.2.4), and set $B_{n+1} = B_n \cup B'_{n+1}$. Then $B_n \subsetneq B_{n+1}$ and we have that $\|(I-P_{B_{n+1}})Tx_{n+1}\| < \epsilon$ as required. The induction is complete. The final claim referring to the case $\Gamma = \mathbb{N}$ follows from the chosen definition of the sets A_n.

(ii) The inequality '$\leqslant$' holds because $\|Q\| \leqslant 1$. In the other direction, suppose that $\|TQ\|_e < 1$. It suffices to show that $\|T\|_e < 1$. By Lemma 3.2.9, there exists some $\epsilon > 0$ and a finite subset A of Γ for which we have that $\|(I-P_A)TQ\| < 1-\epsilon$.

Take $z \in Z$ with $\|z\| < 1$. Because Q is a quotient map, we may choose $w \in W$ with $\|w\| < 1$ and $Qw = z$. We have that

$$\|(I-P_A)Tz\| = \|(I-P_A)TQw\| \leqslant \|(I-P_A)TQ\|\|w\| < 1-\epsilon .$$

Then

$$\|T\|_e \leqslant \|(I-P_A)T\| \leqslant 1-\epsilon < 1 ,$$

as required. □

Lemma 3.2.11. *Let $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ or $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ for some infinite cardinal Γ, and let $\epsilon > 0$. Let $T \in \mathscr{B}(D_\Gamma)$ be such that $\|T\|_e = 1$. There exist operators $U \in \mathscr{B}(D; D_\Gamma)$ and $V \in \mathscr{B}(D_\Gamma; D)$ for which $\|V\| = \|U\| = 1$ and $\|VTU\|_e \geqslant 1-\epsilon$.*

Proof. Define using Lemma 3.2.10(i) a strictly increasing sequence $\emptyset = B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \dots$ of finite subsets of Γ, and a disjointly supported sequence (x_n) of norm-1 vectors in $\mathrm{span}(e_\gamma)_{\gamma\in\Gamma}$ for which we have $\|Tx_n\| \geqslant \|(I-P_{B_{n-1}})Tx_n\| > 1-\frac{\epsilon}{2}$ and

$\|(I - P_{B_n})Tx_n\| < \frac{\epsilon}{2}$ for each $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ we can deduce that

$$\|(P_{B_n} - P_{B_{n-1}})Tx_n\| \geqslant \|(I - P_{B_{n-1}})Tx_n\| - \|(I - P_{B_n})Tx_n\| > 1 - \epsilon \ .$$

Let $A = \bigcup_{n\in\mathbb{N}} \operatorname{supp}(x_n)$ and $B = \bigcup_{n\in\mathbb{N}} B_n$, and define the operator $S := P_B T P_A J \in \mathscr{B}(P_A(D_\Gamma); P_B(D_\Gamma))$, where J denotes the inclusion of $P_A(D_\Gamma)$ into D_Γ, and we have considered P_A as an operator on D_Γ, but P_B as an operator $D_\Gamma \to P_B(D_\Gamma)$.

For each $n \in \mathbb{N}$, we have that $P_A J x_n = x_n$, and because $B_{n-1} \subsetneq B_n \subsetneq B$, we also have that $P_{B_n} P_B = P_{B_n}$, and $P_{B_n} P_{B_{n-1}} = P_{B_{n-1}}$. It follows that the operator S satisfies

$$\begin{aligned} \|(I - P_{B_{n-1}})S\| &\geqslant \|(I - P_{B_{n-1}})Sx_n\| = \|(P_B - P_{B_{n-1}})Tx_n\| \\ &\geqslant \|P_{B_n}(P_B - P_{B_{n-1}})Tx_n\| = \|(P_{B_n} - P_{B_{n-1}})Tx_n\| > 1 - \epsilon \ . \end{aligned} \tag{3.2.5}$$

Now, if B' is a finite subset of B, then there exists some $n \in \mathbb{N}_0$ for which $B' \subset B_n$. This fact combines with Lemma 3.2.9 to tell us that

$$\|S\|_e = \inf\{\|(I - P_{B_n})S\| : n \in \mathbb{N}_0\} \ .$$

The above expression together with (3.2.5) show that $\|S\|_e \geqslant 1 - \epsilon$. Because A and B are countably infinite, we can find isometric isomorphisms $R_A : D \to P_A(D_\Gamma)$ and $R_B : P_B(D_\Gamma) \to D$. Then

$$\|S\|_e = \|R_B S R_A\|_e = \|R_B P_B T P_A J R_A\|_e \geqslant 1 - \epsilon \ .$$

The result follows. □

To prove Lemma 3.2.15, we introduce a quantitative version of a celebrated theorem of Rosenthal, originally stated as the first remark following [56, Theorem 3.4]. This theorem will be revisited later in the chapter.

Theorem 3.2.12. *Let $T \in \mathscr{B}(c_0(\Gamma); Y)$, where Γ is an infinite set and Y is a Banach space, and suppose that $\delta := \inf\{\|Te_\gamma\| : \gamma \in \Gamma\} > 0$. Then, for any*

$\epsilon \in (0, \delta)$, *there is a subset* Γ' *of* Γ *of the same cardinality as* Γ *for which the restriction of* T *to the subspace* $\overline{\operatorname{span}}\{e_\gamma : \gamma \in \Gamma'\}$ *is bounded below by* ϵ.

Proof. Rosenthal stated the result without specifying for which values of $\epsilon > 0$ we can find a subset Γ' of the same cardinality as Γ such that the restriction of T to $\overline{\operatorname{span}}\{e_\gamma : \gamma \in \Gamma'\}$ is bounded below by ϵ. However, inspection of his proof shows that this is possible for every $\epsilon < \delta$. $\square$

Also for the proof of Lemma 3.2.15, we require a classical result of Sobczyk. This result is often only stated in the isometric case ($C = 1$), however the general form that we require can be shown to be true by e.g. the proof of [1, Corollary 2.5.9]. To state it precisely, we shall use the following notion.

Definition 3.2.13. Let $C \geqslant 1$, and let Y and Z be Banach spaces. An operator $T \in \mathscr{B}(Y; Z)$ is a *C-isomorphism* if it is an isomorphism with $\|T\| \, \|T^{-1}\| \leqslant C$. We say that X and Y are *C-isomorphic* if there exists a C-isomorphism of Y onto Z.

Theorem 3.2.14 (Sobczyk's Theorem)**.** *Let* Y *be a separable Banach space, and let* Z *be a subspace of* Y *which is* C*-isomorphic to* c_0 *for some* $C \geqslant 1$*. Then* Z *is complemented in* Y *by a projection with norm at most* $2C$*.*

Lemma 3.2.15. *Let* $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ *or* $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ *for some infinite cardinal* Γ*. Let* $X_1 = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$ *and* $X_2 = D_\Gamma$*, and let* $\epsilon > 0$*. For each pair* $i, j \in \{1, 2\}$*, and every operator* $T \in \mathscr{B}(X_i; X_j)$ *satisfying* $\|T\|_e = 1$*, there are operators* $U \in \mathscr{B}(X_j; D)$*,* $V \in \mathscr{B}(D; X_i)$ *for which we have* $I_D = UTV$ *and* $\|V\|\|U\| \leqslant 2 + \epsilon$*.*

Proof. We examine each of the four possibilities for the pair (i, j) individually, and further break down case $(i, j) = (1, 1)$ into two sub-cases for $D = c_0$ and $D = \ell_1$ respectively.

Notice that certain cases end by deferring to using another case on the composition of T with other operators. This is allowed since factoring I_D through such a composition is also a factorisation of I_D through T, and care has been taken to avoid circular arguments. Our norm calculations will make use of an arbitrary constant $\delta > 0$. The final bound of $2 + \epsilon$ in the statement of the lemma can be shown by taking δ small enough. Thus, let $\delta > 0$.

- $(i,j) = (2,2)$: In this case $T \in \mathscr{B}(D_\Gamma)$. Using Lemma 3.2.11, find operators $V \in \mathscr{B}(D_\Gamma; D), U \in \mathscr{B}(D; D_\Gamma)$ that satisfy $\|V\| = \|U\| = 1$ and

$$\|VTU\|_e > 1 - \delta \ .$$

We may now apply [63, Lemma 3.3.6] to $VTU \in \mathscr{B}(D)$ and obtain operators $S, R \in \mathscr{B}(D)$ for which $\|S\|\|R\| \leqslant (1-\delta)^{-1}$ and $SVTUR = I_D$. The result follows.

- $(i,j) = (1,1)$, $D = c_0$: In this case, $T \in \mathscr{B}(X_1)$. Using Lemma 3.2.9, we can find a finite subset A of $\mathbb{N}$ for which $\|(I - P_A)T\| < 1 + \delta$. Set $S = (I - P_A)T$, so that

$$1 = \|T\|_e = \|S\|_e \leqslant \|S\| < 1 + \delta \ .$$

Next, apply Lemma 3.2.10(i) with $Y = X_1$ to obtain a sequence (y_n) of normalised blocks (i.e. finitely, consecutively supported vectors) in X_1 for which $\|Sy_n\| > 1 - \delta$ for every $n \in \mathbb{N}$. Let (z_n) be a subsequence of (y_n) for which there is no $k \in \mathbb{N}$ such that the space ℓ_2^k contains support from more than one of the vectors z_n. The purpose of this is to ensure that for any $i \in \mathbb{N}$ and any set of scalars $\lambda_1, \dots, \lambda_i \in \mathbb{K}$, we obtain

$$\|\sum_{n=1}^{i} \lambda_n z_n\| = \sup\{\|\lambda_n\| : n \in \{1, \dots, i\}\} \ ,$$

which is to say that the sequence (z_n) is isometrically equivalent to the unit vector basis (e_n) of c_0. We can therefore define a function $F : c_0 \to X_1$ by action $e_n \mapsto z_n$ for every $n \in \mathbb{N}$, which is a linear isometry onto its range.

Now, $\|SFe_n\| > 1 - \delta$ for each $n \in \mathbb{N}$, so we apply Theorem 3.2.12 to the operator SF and get a subsequence (e_{n_k}) of (e_n) for which SF is bounded below on $\overline{\text{span}}(e_{n_k})$ by $1 - 2\delta$. Define the operator $J \in \mathscr{B}(c_0)$ to act on the standard basis by $e_k \mapsto e_{n_k}$ for every $k \in \mathbb{N}$. Then J is also an isometry, so we have that $\forall x \in c_0$, with $\|x\| = 1$,

$$1 - 2\delta \leqslant \|SFJx\| < 1 + \delta \ .$$

Then $\mathrm{im}(SFJ)$ is $(1+\delta)(1-2\delta)^{-1}$-isomorphic to c_0, so must be complemented in X_1 by a projection $R : X_1 \to \mathrm{im}(SFJ)$ with norm at most $2(1+\delta)(1-2\delta)^{-1}$ by Sobczyk's theorem.

Now, $RSFJ \in \mathscr{B}(c_0; \mathrm{im}(SFJ))$ is an isomorphism, which has inverse $H \in \mathscr{B}(\mathrm{im}(SFJ); c_0)$ having norm at most $(1-2\delta)^{-1}$. Then

$$I_{c_0} = HRSFJ = HR(I - P_A)TFJ \ ,$$

and

$$\|HR(I - P_A)\|\|FJ\| \leqslant \|H\|\|R\|\|(I - P_A)\|\|F\|\|J\| \leqslant 2(1 + \delta)(1 - 2\delta)^{-2} \ ,$$

completing the proof.

- $(i, j) = (2, 1)$: In this case, $T \in \mathscr{B}(D_\Gamma; X_1)$. Use Lemma 3.2.7 to define an operator $J \in \mathscr{B}(X_1; D)$ by equation (3.2.2) (where it is called U) for which

$$\|x\| \leqslant \|Jx\| \leqslant (1 + \delta)\|x\| \tag{3.2.6}$$

for all $x \in X_1$, defining the sequence $(N(n))_{n\in\mathbb{N}} \subset \mathbb{N}$ and identifying D with the space $\left(\bigoplus_{n\in\mathbb{N}} D_{N(n)}\right)_D$ as in the proof of Lemma 3.2.7. Here, $D_{N(n)} = \ell_\infty^{N(n)}$ in case $D = c_0$ and $D_{N(n)} = \ell_1^{N(n)}$ in case $D = \ell_1$.

For each $n \in \mathbb{N}$, define the projections $S_n \in \mathscr{B}(D)$ and $R_n \in \mathscr{B}(X_1)$ onto the spaces $\left(\bigoplus_{k=1}^n D_{N(k)}\right)_D$ and $\left(\bigoplus_{k=1}^n \ell_2^k\right)_D$ respectively. Take note of the relation

$$S_n J = JR_n \ , \tag{3.2.7}$$

which follows from the definition of the diagonal operator J. We claim that

$$1 \leqslant \|JT\|_e \leqslant (1 + \delta) \ . \tag{3.2.8}$$

The right-hand inequality holds true because $\|J\| \leqslant (1+\delta)$. We will prove the left hand inequality by contradiction. So, suppose that $\|JT\|_e < 1$. For every finite subset A of $\mathbb{N}$, we can find some $n \in \mathbb{N}$ for which we have $\mathrm{span}\{e_j : j \in$

$A\} \subseteq \operatorname{im} S_n$. It follows from Lemma 3.2.9 that

$$\|JT\|_e = \inf\{\|(I - S_n)JT\| : n \in \mathbb{N}\} .$$

Take $n \in \mathbb{N}$ to satisfy $\|(I - S_n)JT\| < 1$. Then by (3.2.6) and (3.2.7), we have that

$$\begin{aligned} \|T\|_e \leqslant \|(I - R_n)T\| \leqslant \|J(I - R_n)T\| &= \|JT - JR_nT\| \\ &= \|JT - S_nJT\| = \|(I - S_n)JT\| < 1 , \end{aligned}$$

a contradiction, proving claim (3.2.8).

Let $\{e_{\gamma_n} : n \in \mathbb{N}\}$ be a countably infinite subset of the standard basis of D_Γ, and define the embedding $L : D \hookrightarrow D_\Gamma$ by action $e_n \mapsto e_{\gamma_n}$ for every $n \in \mathbb{N}$. Further, let $P : D_\Gamma \to D$ be defined by

$$e_\gamma \mapsto \begin{cases} e_n & \text{if } \gamma = \gamma_n \text{ for some } n \in \mathbb{N} , \\ 0 & \text{otherwise} , \end{cases}$$

for every $\gamma \in \Gamma$. Because $\|L\| = 1$, we have that $\|LJT\|_e \leqslant \|JT\|_e$. Also, since $PL = I_D$ and $\|P\| = 1$, we obtain

$$\|LJT\|_e \geqslant \|PLJT\|_e = \|JT\|_e .$$

So $\|LJT\|_e = \|JT\|_e$. By (3.2.8), we then have that

$$1 \leqslant \|LJT\|_e \leqslant 1 + \delta .$$

We proceed by applying case $(i, j) = (2, 2)$ to LJT.

- $(i, j) = (1, 1)$, $D = \ell_1$: Using Lemma 3.2.10(ii) in conjunction with the lifting property of $\ell_1(\Gamma)$ (Proposition 2.2.8), we can find a quotient map $Q \in \mathscr{B}(\ell_1(\Gamma); X_1)$ with $\|TQ\|_e = 1$. Then $\|Q\| = 1$, and we can apply case $(i, j) = (1, 2)$ to the operator TQ.

- $(i, j) = (1, 2)$: In this case, $T \in \mathscr{B}(X_1; D_\Gamma)$. Using Lemma 3.2.1(ii), we have that there exists a countably infinite subset $\Delta := \{\gamma_n : n \in \mathbb{N}\}$ of Γ for which $T = P_\Delta T$. For each $n \in \mathbb{N}$, let $e'_n \in X_1$ be the first standard basis vector from the subspace ℓ_2^n of X_1. Define $P \in \mathscr{B}(D_\Gamma; X_1)$ by

$$e_\gamma \mapsto \begin{cases} e'_n & \text{if } \gamma = \gamma_n \text{ for some } n \in \mathbb{N} \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

for every $\gamma \in \Gamma$. Further, define $L \in \mathscr{B}(X_1; D_\Gamma)$ to map the basis vectors e'_n to e_{γ_n} for every $n \in \mathbb{N}$, and to map all other basis vectors to zero. Then $LP = P_\Delta$, so $LPT = T$. Since $\|L\| = 1$ and $\|P\| = 1$, we have that

$$1 = \|T\|_e = \|LPT\|_e \leqslant \|PT\|_e \leqslant \|P\|\|T\|_e = 1 \ .$$

So $\|PT\|_e = 1$, and we may apply case $(1,1)$ to the operator PT. □

Let X_1 and X_2 be Banach spaces. For the rest of this chapter, when considering operators on the direct sum $X_1 \oplus X_2$, for $m \in \{1,2\}$, let $J_m \colon X_m \to X_1 \oplus X_2$ and $Q_m \colon X_1 \oplus X_2 \to X_m$ denote the m^{th} coordinate embedding and projection, respectively. Recall that an operator T on $X_1 \oplus X_2$ can then be decomposed as

$$T = \sum_{i,j=1}^{2} J_i T_{i,j} Q_j \ , \tag{3.2.9}$$

which immediately tells us that

$$\max_{i,j \in \{1,2\}} \|T_{i,j}\| \leqslant \|T\| \leqslant \sum_{i,j=1}^{2} \|T_{i,j}\| \ . \tag{3.2.10}$$

Corollary 3.2.16. *Let $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D \oplus D_\Gamma$, where $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ or $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ for some infinite set Γ, and let $\mathscr{I}$ be an ideal of $\mathscr{B}(X)$. Then either $\mathscr{I} \subseteq \mathscr{K}(X)$ or $\mathscr{G}_D(X) \subseteq \mathscr{I}$.*

Proof. For notational convenience, write $X = X_1 \oplus X_2$, where $X_1 = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$ and $X_2 = D_\Gamma$. Suppose that $\mathscr{I} \nsubseteq \mathscr{K}(X)$, and choose $T \in \mathscr{I} \setminus \mathscr{K}(X)$. Then clearly $T_{i,j} \notin \mathscr{K}(X_j; X_i)$ for some $i, j \in \{1,2\}$. Lemma 3.2.15 implies that there are operators $U \colon D \to X_j$ and $V \colon X_i \to D$ such that $VT_{i,j}U = I_D$. Hence, for each

$S = R_2 R_1 \in \mathscr{G}_D(X)$, where $R_1 \in \mathscr{B}(X; D)$ and $R_2 \in \mathscr{B}(D; X)$, because $\mathscr{I}$ is an ideal of $\mathscr{B}(X)$, we have

$$S = R_2 V T_{i,j} U R_1 = (R_2 V Q_i) T (J_j U R_1) \in \mathscr{I} \ .$$

This shows that $\mathscr{G}_D(X) \subseteq \mathscr{I}$, as desired. $\square$

For a subset $\mathscr{I}$ of $\mathscr{B}(X_1 \oplus X_2)$ and $j, k \in \{1, 2\}$, we define the $(j, k)^{\text{th}}$ *quadrant* of $\mathscr{I}$ by

$$\mathscr{I}_{j,k} = \{Q_j T J_k : T \in \mathscr{I}\} \subseteq \mathscr{B}(X_k; X_j) \ .$$

On the other hand, given subsets $\mathscr{I}_{j,k}$ of $\mathscr{B}(X_k, X_j)$ for $j, k \in \{1, 2\}$, we define

$$\begin{pmatrix} \mathscr{I}_{1,1} & \mathscr{I}_{1,2} \\ \mathscr{I}_{2,1} & \mathscr{I}_{2,2} \end{pmatrix} = \left\{ \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} : T_{j,k} \in \mathscr{I}_{j,k} \ \big(j, k \in \{1, 2\}\big) \right\} \subseteq \mathscr{B}(X_1 \oplus X_2).$$

Using the fact that for each $T \in \mathscr{B}(X)$ we have the decomposition (3.2.9), we can deduce that for any pair X_1, X_2 of Banach spaces and any operator ideal $\mathscr{I}$,

$$T \in \mathscr{I}(X_1 \oplus X_2) \quad \iff \quad T_{i,j} \in \mathscr{I}(X_j; X_i) \text{ for } i, j \in \{1, 2\} \ . \tag{3.2.11}$$

The next lemma shows that we can freely decompose and recompose ideals of bounded operators on direct sums of Banach spaces.

Lemma 3.2.17. *Let $\mathscr{I}$ be an ideal of $\mathscr{B}(X_1 \oplus X_2)$ for some Banach spaces X_1 and X_2. Then*

$$\mathscr{I} = \begin{pmatrix} \mathscr{I}_{1,1} & \mathscr{I}_{1,2} \\ \mathscr{I}_{2,1} & \mathscr{I}_{2,2} \end{pmatrix}, \tag{3.2.12}$$

and $\mathscr{I}_{i,i}$ is an ideal of $\mathscr{B}(X_i)$ for $i \in \{1, 2\}$. Moreover, $\mathscr{I}_{i,j}$ is closed in $\mathscr{B}(X_j, X_i)$ for each $i, j \in \{1, 2\}$ if and only if $\mathscr{I}$ is closed in $\mathscr{B}(X_1 \oplus X_2)$.

Proof. The inclusion $\subseteq$ in (3.2.12) holds true by the involved definitions.

Conversely, suppose that $T = (T_{i,j})_{i,j=1}^2$ with $T_{i,j} \in \mathscr{I}_{i,j}$ for each $i, j \in \{1, 2\}$, say $T_{i,j} = Q_i S^{i,j} J_j$, where $S^{i,j} \in \mathscr{I}$. Then by decomposing T using (3.2.9), we have

$$T = \sum_{i,j=1}^{2} J_i T_{i,j} Q_j = \sum_{i,j=1}^{2} (J_i Q_i) S^{i,j} (J_j Q_j) \in \mathscr{I}$$

because $\mathscr{I}$ is an ideal of $\mathscr{B}(X_1 \oplus X_2)$ and $J_kQ_k \in \mathscr{B}(X_1 \oplus X_2)$ for $k \in \{1,2\}$.

Next, we verify that $\mathscr{I}_{i,i}$ is an ideal of $\mathscr{B}(X_i)$ for $i \in \{1,2\}$. It is clear that $\mathscr{I}_{i,i}$ is a subspace. Suppose that $S \in \mathscr{I}_{i,i}$ and $T \in \mathscr{B}(X_i)$, say $S = U_{i,i}$, where $U \in \mathscr{I}$. Then $UJ_iTQ_i \in \mathscr{I}$ because $\mathscr{I}$ is an ideal of $\mathscr{B}(X_1 \oplus X_2)$ and $J_iTQ_i \in \mathscr{B}(X_1 \oplus X_2)$, and hence

$$\mathscr{I}_{i,i} \ni (UJ_iTQ_i)_{i,i} = Q_iUJ_iTQ_iJ_i = ST\,.$$

The proof that $TS \in \mathscr{I}_{i,i}$ is similar.

The final clause follows easily from (3.2.10) and (3.2.12). □

For a Banach space X, define

$$\Xi(X) = \{\mathscr{I}\ ;\ \mathscr{I} \text{ is a closed ideal of } \mathscr{B}(X) \text{ and } \mathscr{I} \supsetneq \mathscr{K}(X)\}\,,$$

and order $\Xi(X)$ by inclusion. For a pair of Banach spaces X_1 and X_2, we endow the set $\Xi(X_1) \times \Xi(X_2)$ with the product order; that is,

$$(\mathscr{I}_1, \mathscr{I}_2) \leqslant (\mathscr{J}_1, \mathscr{J}_2) \iff [\mathscr{I}_1 \subseteq \mathscr{J}_1] \wedge [\mathscr{I}_2 \subseteq \mathscr{J}_2]\,.$$

Let A and B be ordered sets. A function $\phi : A \to B$ is an *order isomorphism* if it is bijective and satisfies $a < b$ if and only if $\phi(a) < \phi(b)$ for every $a, b \in A$.

Proposition 3.2.18. *Let $X = E \oplus D_\Gamma$, where $E = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$ and either $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ or $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ for some uncountable set Γ. The map*

$$\xi\colon\ \Xi(E) \times \Xi(D_\Gamma) \to \Xi(X)\,; \quad (\mathscr{I}, \mathscr{J}) \mapsto \begin{pmatrix} \mathscr{I} & \mathscr{B}(D_\Gamma; E) \\ \mathscr{B}(E; D_\Gamma) & \mathscr{J} \end{pmatrix}\,,$$

is an order isomorphism.

Proof. Recall from Corollary 3.2.3 that $\mathscr{B}(E; D_\Gamma) = \mathscr{G}_D(E; D_\Gamma)$ and $\mathscr{B}(D_\Gamma; E) = \mathscr{G}_D(D_\Gamma; E)$, and that $\mathscr{G}_D(E) \subseteq \mathscr{I}$ and $\mathscr{G}_D(D_\Gamma) \subseteq \mathscr{J}$ for every $(\mathscr{I}, \mathscr{J}) \in \Xi(E) \times \Xi(D_\Gamma)$ by the ideal classifications (1.5.1) and (1.4.1), respectively. Using these facts, one can easily verify that $\xi(\mathscr{I}, \mathscr{J})$ is an ideal of $\mathscr{B}(X)$ with $\mathscr{K}(X) \subsetneq \xi(\mathscr{I}, \mathscr{J})$. Moreover, $\xi(\mathscr{I}, \mathscr{J})$ is closed because each of its quadrants is, so it belongs to $\Xi(X)$.

To see that ξ is surjective, let $\mathscr{L} \in \Xi(X)$. Theorem 3.2.17 shows that

$$\mathscr{L} = \begin{pmatrix} \mathscr{L}_{1,1} & \mathscr{L}_{1,2} \\ \mathscr{L}_{2,1} & \mathscr{L}_{2,2} \end{pmatrix},$$

where $\mathscr{L}_{1,1}$ and $\mathscr{L}_{2,2}$ are closed ideals of $\mathscr{B}(E)$ and $\mathscr{B}(D_\Gamma)$, respectively. Moreover, Corollary 3.2.16 implies that $\mathscr{G}_D(X) \subseteq \mathscr{L}$, so by (3.2.11), we have:

- $\mathscr{L}_{1,1} \supseteq \mathscr{G}_D(E)$, so $\mathscr{L}_{1,1} \in \Xi(E)$;
- $\mathscr{L}_{1,2} \supseteq \mathscr{G}_D(D_\Gamma; E) = \mathscr{B}(D_\Gamma; E)$, so $\mathscr{L}_{1,2} = \mathscr{B}(D_\Gamma; E)$, and similarly $\mathscr{L}_{2,1} = \mathscr{B}(E; D_\Gamma)$;
- $\mathscr{L}_{2,2} \supseteq \mathscr{G}_D(D_\Gamma)$, so $\mathscr{L}_{2,2} \in \Xi(D_\Gamma)$.

This verifies that $\mathscr{L} = \xi(\mathscr{L}_{1,1}, \mathscr{L}_{2,2})$.

Finally, working straight from the definitions, we see that $(\mathscr{I}_1, \mathscr{J}_1) \leqslant (\mathscr{I}_2, \mathscr{J}_2)$ if and only if $\xi(\mathscr{I}_1, \mathscr{J}_1) \subseteq \xi(\mathscr{I}_2, \mathscr{J}_2)$ for $(\mathscr{I}_1, \mathscr{J}_1), (\mathscr{I}_2, \mathscr{J}_2) \in \Xi(E) \times \Xi(D_\Gamma)$. This shows first that ξ is injective and thus a bijection, and secondly that both ξ and its inverse are order-preserving. □

We can now prove Theorem 3.1.1 easily.

Proof of Theorem 3.1.1. Both E and D_Γ have the approximation property, so the same is true for their direct sum X. Therefore $\mathscr{K}(X)$ is the smallest non-zero closed ideal of $\mathscr{B}(X)$. Proposition 3.2.18 shows that any other non-zero closed ideal $\mathscr{L}$ of $\mathscr{B}(X)$ has the form $\mathscr{L} = \xi(\mathscr{I}, \mathscr{J})$ for unique closed ideals $\mathscr{I} \in \Xi(E)$ and $\mathscr{J} \in \Xi(D_\Gamma)$. By the ideal classifications (1.5.1) and (1.4.1), either $\mathscr{I} = \overline{\mathscr{G}_D}(E)$ or $\mathscr{I} = \mathscr{B}(E)$, while $\mathscr{J} = \mathscr{K}_\kappa(D_\Gamma)$ for a unique cardinal $\aleph_1 \leqslant \kappa \leqslant \Gamma^+$.

Suppose first that $\mathscr{I} = \overline{\mathscr{G}_D}(E)$. If $\kappa = \aleph_1$, then $\mathscr{J} = \overline{\mathscr{G}_D}(D_\Gamma)$, so $\mathscr{L} = \overline{\mathscr{G}_D}(X)$. Otherwise $\kappa \geqslant \aleph_2$ and $\mathscr{L} = \mathscr{J}_\kappa(X)$ in the notation of (3.1.1).

Next, suppose that $\mathscr{I} = \mathscr{B}(E)$, which is equal to $\mathscr{K}_\kappa(E)$ because E has density character $\aleph_0 < \kappa$. Hence we have $\mathscr{L} = \mathscr{K}_\kappa(X)$. (Note that this is equal to $\mathscr{B}(X)$ if $\kappa = \Gamma^+$.) □

Chapter 4

Uniqueness of norms of quotients of $\mathscr{B}(X)$

4.1 Background

Definition 4.1.1. Let $(\mathscr{A}, \|\cdot\|)$ be a normed algebra. We say that the given norm $\|\cdot\|$ on $\mathscr{A}$ is *maximal* if, for every algebra norm $|||\cdot|||$ on $\mathscr{A}$, there is a constant $C_1 > 0$ such that $|||a||| \leqslant C_1\|a\|$ for every $a \in \mathscr{A}$.

Analogously, we say that $\|\cdot\|$ is *minimal* if, for every algebra norm $|||\cdot|||$ on $\mathscr{A}$, there is a constant $C_2 > 0$ such that $\|a\| \leqslant C_2|||a|||$ for every $a \in \mathscr{A}$.

A pair $\|\cdot\|$ and $|||\cdot|||$ of norms on some normed algebra $\mathscr{A}$ are *equivalent* if there exist constants $C_3, C_4 > 0$ such that $C_3\|a\| \leqslant |||a||| \leqslant C_4\|a\|$ for every $a \in \mathcal{A}$.

We say that $\mathscr{A}$ has a *unique algebra norm* if the given norm is both maximal and minimal, or in other words if every algebra norm on $\mathscr{A}$ is equivalent to the given norm.

As the name suggests, equivalence of algebra norm on a normed algebra is an equivalence relation on the class of all norms on said algebra. In the case where $\mathscr{A}$ is a Banach algebra (which will be the case for us), the notion of uniqueness of algebra norm should not be confused with the weaker notion of having a unique *complete* algebra norm, which asserts that every complete algebra norm on $\mathscr{A}$ is equivalent to the given norm. The Banach Isomorphism Theorem implies that whenever the given norm on a Banach algebra $\mathscr{A}$ is either minimal or maximal, it is automatically equivalent to any other complete norm on $\mathscr{A}$, so $\mathscr{A}$ has unique complete algebra norm. Our aim for this final chapter is to prove the following theorem.

Theorem 4.1.2. *Every quotient of $\mathscr{B}(X)$ by one of its closed ideals has a unique algebra norm for each of the following Banach spaces X:*

(i) $X = \big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_{c_0}$ *and* $X = \big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_{\ell_1}$;

(ii) $X = \big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_{c_0} \oplus c_0(\Gamma)$ *and* $X = \big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_{\ell_1} \oplus \ell_1(\Gamma)$ *for an uncountable index set* Γ;

(iii) $X = C_0(K_{\mathcal{A}})$, *the Banach space of continuous functions vanishing at infinity on the Mrówka space $K_{\mathcal{A}}$ associated with an almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$, chosen such that $C_0(K_{\mathcal{A}})$ admits 'few operators'.*

We refer to Section 1.6 for details of the terminology used in (iii). One usually requires that an algebra norm on a unital algebra $\mathscr{A}$ must take the value 1 at the multiplicative identity $1_{\mathscr{A}}$. We shall not adhere to this convention. This will not cause any problems because, given any algebra norm $\|\cdot\|$ on $\mathscr{A}$,

$$|||a||| = \sup\big\{\|ab\| : b \in \mathscr{A},\ \|b\| \leqslant 1\big\} \qquad (a \in \mathscr{A})$$

always defines an equivalent algebra norm on $\mathscr{A}$ which satisfies $|||1_{\mathscr{A}}||| = 1$. Much more general results are given in [10, Proposition 2.1.9] or [51, Proposition 1.1.9].

One of the earliest results regarding equivalence of algebra norms of quotients of the Banach algebra $\mathscr{B}(X)$ of operators on a Banach space X (predating even the term 'Banach algebra') is due to Eidelheit, and is given below.

Theorem 4.1.3 ([15, Theorem 1])**.** *Let X be a Banach space. Then $\mathscr{B}(X)$ has a unique complete algebra norm.*

Building on Eidelheit's methods, one can deduce a stronger conclusion, which we shall state in Theorem 4.1.9 below, once we have established the necessary terminology.

Many related results have subsequently been obtained; [62, §0] contains an excellent survey of what was known around the turn of the millenium. We would like to highlight the following results, which provide the most important context for our work:

Example 4.1.4. (i) Meyer [46] proved that the Calkin algebra $\mathscr{B}(X)/\mathscr{K}(X)$ has a unique algebra norm for each of the classical sequence spaces $X = c_0$ and $X = \ell_p$, where $1 \leqslant p < \infty$. For later reference, we recall the classical result of Gohberg, Markus and Feldman [22] that the ideal $\mathscr{K}(X)$ of compact operators is the unique non-trivial closed ideal of $\mathscr{B}(X)$ for each of these Banach spaces X.

(ii) Astala and Tylli [6] gave the first known examples of Banach spaces X for which the Calkin algebra $\mathscr{B}(X)/\mathscr{K}(X)$ fails to have a unique algebra norm; more precisely, $\mathscr{B}(X)/\mathscr{K}(X)$ admits an algebra norm which is dominated by the quotient norm without being equivalent to it. Tylli [61] obtained similar examples for quotients of $\mathscr{B}(X)$ by closed ideals other than the compact operators. These results were in fact not originally proved to produce counterexamples to questions about the uniqueness of algebra norm, but after becoming aware of these applications, Tylli [62, §1] wrote a survey presenting this body of work from that perspective.

(iii) In his dissertation [63], Ware launched a systematic attack on the uniqueness-of-algebra-norm question for Calkin algebras, generalising the aforementioned results of Meyer by showing that the Calkin algebra $\mathscr{B}(X)/\mathscr{K}(X)$ has a unique algebra norm for a wide range of Banach spaces X, including the following [63, §§5.2–5.5]:

- every finite direct sum of spaces from the family $\{c_0\} \cup \{\ell_p : 1 \leqslant p < \infty\}$;
- the infinite direct sums $\left(\bigoplus_{n\in\mathbb{N}} \ell_p^n\right)_{c_0}$ and $\left(\bigoplus_{n\in\mathbb{N}} \ell_p^n\right)_{\ell_q}$ for $1 \leqslant p \leqslant \infty$ and $1 \leqslant q < \infty$;
- the Tsirelson space T and its dual T^*;
- the quasi-reflexive James spaces J_p for $1 < p < \infty$.

(iv) In a somewhat different direction, Ware [63, Section 6] generalised Meyer's results to the non-separable setting as follows. Take an uncountable index set Γ, and set $X = c_0(\Gamma)$ or $X = \ell_p(\Gamma)$ for some $1 \leqslant p < \infty$. Then the quotient $\mathscr{B}(X)/\mathscr{I}$ has a unique algebra norm for every closed ideal $\mathscr{I}$ of $\mathscr{B}(X)$. Daws' classification [12] of the closed ideals of $\mathscr{B}(X)$ for these Banach spaces X plays a key role in this work (see Page 8 for this lattice).

(v) More recently, Johnson, Phillips and Schechtman [31] have answered one of the main questions left open in Ware's dissertation [63] by showing that the Calkin algebra $\mathscr{B}(L_p[0,1])/\mathscr{K}(L_p[0,1])$ has a unique algebra norm whenever $1 < p < \infty$. Moreover, for $p \neq 2$, they have shown that $\mathscr{B}(L_p[0,1])$ contains a closed ideal $\mathscr{I}$ for which the quotient $\mathscr{B}(L_p[0,1])/\mathscr{I}$ admits at least two non-equivalent algebra norms.

Motivated by these results, especially (i), (iv) and (v), we asked ourselves the following question.

Question 4.1.5. For which Banach spaces X other than $c_0(\Gamma)$ and $\ell_p(\Gamma)$ for $1 \leqslant p < \infty$ and Γ infinite, is it true that every quotient of $\mathscr{B}(X)$ by a closed ideal has a unique algebra norm?

Our focus was naturally turned towards the relatively meagre list of Banach spaces X for which the lattice of closed ideals of $\mathscr{B}(X)$ is fully classified. The comprehensive list of such Banach spaces X was discussed in Remark 3.1.2. Hence, the spaces X of Theorem 4.1.2 were chosen.

It is well-known that maximality of the norm can be rephrased in terms of automatic continuity of homomorphisms, which is one of the oldest topics in the theory of Banach algebras. Here, and elsewhere, the term 'homomorphism' means a linear and multiplicative map between two algebras. There are many ways to express this relationship. We have chosen the following, which reflects the applications we have in mind. A much more comprehensive result can be found in [51, Theorem 6.1.5].

Proposition 4.1.6. *The following conditions are equivalent for a Banach algebra* $(\mathscr{A}, \|\cdot\|)$:

(a) *The given norm* $\|\cdot\|$ *on* $\mathscr{A}$ *is maximal.*

(b) *Every homomorphism from* $\mathscr{A}$ *into a normed algebra is continuous.*

(c) *Every injective homomorphism from* $\mathscr{A}$ *into a Banach algebra is continuous.*

(d) *The quotient norm on* $\mathscr{A}/\mathscr{I}$ *is maximal for every closed ideal* $\mathscr{I}$ *of* $\mathscr{A}$.

(e) *Every homomorphism from $\mathscr{A}/\mathscr{I}$ into a normed algebra is continuous for every closed ideal $\mathscr{I}$ of $\mathscr{A}$.*

Proof. This is standard. For example, it is easy to adapt the proof of [10, Proposition 2.1.7] to verify that conditions (a)–(c) are equivalent, and therefore conditions (d) and (e) are also equivalent. We note that this part of the proof does not require completeness of $\mathscr{A}$. Completeness is, however, required to prove that (b) implies (e); see [10, Proposition 2.1.5] for details. Finally, the implication (e)$\Rightarrow$(b) is trivial. $\square$

B. E. Johnson [29] proved what has become the classical automatic continuity result for homomorphisms from the Banach algebra $\mathscr{B}(X)$. We state it in a simplified non-technical form, as it will suffice for our purposes.

Theorem 4.1.7 (Johnson)**.** *Let X be a Banach space which is isomorphic to its square $X \oplus X$. Then every homomorphism from $\mathscr{B}(X)$ into a Banach algebra is continuous.*

Proposition 4.1.6 has a counterpart for minimality. Its statement involves the notion of an operator being bounded below (see Page 3 for the definition). We give it here, along with its straightforward proof.

Lemma 4.1.8. *Let $(\mathscr{A}, \|\cdot\|)$ be a normed algebra. Then its norm $\|\cdot\|$ is minimal if and only if every injective homomorphism from $\mathscr{A}$ into a normed algebra is bounded below.*

Proof. Let $\|\cdot\|$ be minimal on $\mathcal{A}$. Let ϕ be an injective homomorphism from $\mathcal{A}$ into any normed algebra. The function $|||\cdot||| : a \mapsto \|\phi(a)\|$ from $\mathcal{A}$ to $[0, \infty)$ defines a new algebra norm on $\mathcal{A}$. By minimality of $\|\cdot\|$, there is a constant $c > 0$ such that $|||a||| \geqslant c\|a\|$ for every $a \in \mathcal{A}$. The constant c is a lower bound of the homomorphism ϕ as desired.

On the other hand, suppose that every injective homomorphism from $\mathcal{A}$ into a normed algebra is bounded below. The identity map from $(\mathcal{A}, \|\cdot\|)$ into any renormed version of $\mathcal{A}$ defines an injective algebra homomorphism. The existence of a lower bound for this homomorphism proves that $\|\cdot\|$ is dominated by this new norm. $\square$

The subsequent result originates in the work of Eidelheit [15, Lemma 1 and Theorem 1], as already mentioned in this section. It can for instance be found in [10, Theorem 5.1.14] or [51, Theorem, page 107].

Theorem 4.1.9. *Let X be a Banach space, and let $\mathscr{A}$ be a subalgebra of $\mathscr{B}(X)$ which contains the finite-rank operators. Then the operator norm on $\mathscr{A}$ is minimal.*

In a forthcoming paper [31], W. B. Johnson, Phillips and Schechtman introduce and explore the concepts of incompressibility and uniform incompressibility.

Definition 4.1.10. A normed algebra $(\mathscr{A}, \|\cdot\|)$ is:

- *incompressible* if every continuous, injective homomorphism from $\mathscr{A}$ into a normed algebra is bounded below;
- *uniformly incompressible* if there is a function $f\colon (0,\infty) \to (0,\infty)$ such that every continuous, injective homomorphism φ from $\mathscr{A}$ into a normed algebra is bounded below by $f(\|\varphi\|)$.

Remark 4.1.11. Let $\mathscr{A}$ and $\mathscr{B}$ be isomorphic Banach algebras. Whilst it is simple to prove that $\mathscr{B}$ is incompressible whenever $\mathscr{A}$ is, it is not clear to us whether $\mathscr{B}$ must be *uniformly* incompressible whenever $\mathscr{A}$ is. This concern motivates us later on to introduce Definition 4.2.1 - an even stronger condition than uniform incompressibility, which is invariant under Banach algebra isomorphism.

It is clear then that uniform incompressibility is strictly stronger as a condition than incompressibility, and in view of Lemma 4.1.8, incompressibility is slightly weaker than minimality of the algebra norm, with incompressible normed algebras able to be the domain of an injective algebra homomorphism which is not bounded below, if that homomorphism were to be discontinuous.

The author thanks Yemon Choi for the following result and its proof, which is strictly a strengthening of Theorem 4.1.9 using the notion of uniform incompressibility. Recall the standard tensor notation for rank-one operators on a Banach space X: For $f \in X^*$ and $x \in X$, $f \otimes x$ denotes the operator on X defined by

$$(f \otimes x)y = \langle y, f\rangle x \qquad (y \in X)\ . \tag{4.1.1}$$

Proposition 4.1.12. *Let X be a Banach space, and let $\mathscr{A}$ be a subalgebra of $\mathscr{B}(X)$ which contains the finite-rank operators. Then $\mathscr{A}$ is uniformly incompressible*

Proof. Let $T \in \mathscr{A}$ with $\|T\| = 1$. For every $C > 1$, we can take a norm one vector $x \in X$ for which $\|Tx\| \geqslant 1/\sqrt{C}$, and thus also norm one functional $\phi \in X^*$ such that $|\phi(Tx)| \geqslant 1\sqrt{C}$. We can then let $\lambda \in X^*$ be a rescaling of ϕ for which $\|\lambda\| \leqslant C$ and $\lambda(Tx) = 1$. Now define the operator $V = \lambda \otimes x \in \mathcal{A}$, so that $\|V\| \leqslant \|x\|\|\lambda\| \leqslant C$, and

$$(TV)^2 = (\lambda \otimes Tx)(\lambda \otimes Tx) = \lambda \otimes Tx = TV \ .$$

The operator TV is therefore a non-zero idempotent in $\mathscr{A}$. We thus have that if ψ is a continuous homomorphism from $\mathscr{A}$ into a normed algebra, then

$$1 \leqslant \|\psi(TV)\| \leqslant \|\psi(T)\|\|\psi\|C \ .$$

Since this holds for all $C > 1$, we have that $\mathscr{A}$ is uniformly incompressible with the control function $f : (0, \infty) \to (0, \infty)$ which has action $f(\|\psi\|) = \|\psi\|^{-1}$. □

Lemma 4.1.8 shows that incompressibility is equivalent to minimality of the norm if every homomorphism from $\mathscr{A}$ into a normed algebra is continuous. More precisely, we have the next result.

Lemma 4.1.13. *A normed algebra $(\mathscr{A}, \|\cdot\|)$ has a unique algebra norm if and only if $\mathscr{A}$ is incompressible and the given norm $\|\cdot\|$ is maximal.*

Proof. This is immediate from Lemma 4.1.8 because, by Proposition 4.1.6, maximality of the norm $\|\cdot\|$ means that every homomorphism from $\mathscr{A}$ into a normed algebra is continuous. □

One difficulty with some of the notions discussed so far in this section is that they have similarities and interactions which are at times subtle. To their advantage however, they can be combined elegantly.

Corollary 4.1.14. *The following three conditions are equivalent for a Banach algebra $\mathscr{A}$:*

(a) *Every quotient algebra of $\mathscr{A}$ by a closed ideal has a unique algebra norm.*

(b) *Every homomorphism from $\mathscr{A}$ into a Banach algebra is continuous and has closed range.*

(c) *Every homomorphism from $\mathscr{A}$ into a Banach algebra is continuous, and $\mathscr{A}/\mathscr{I}$ is incompressible for every closed ideal $\mathscr{I}$ of $\mathscr{A}$.*

Proof. (a)$\Rightarrow$(b). Let $\varphi\colon \mathscr{A} \to \mathscr{B}$ be a homomorphism into a Banach algebra $\mathscr{B}$. Proposition 4.1.6 shows that φ is continuous because the norm on $\mathscr{A}$ is maximal. In particular, the ideal $\ker\varphi$ is closed in $\mathcal{A}$, and the quotient norm on $\mathscr{A}/\ker\varphi$ is therefore minimal. Let $\widetilde{\varphi}\colon \mathscr{A}/\ker\varphi \to \mathscr{B}$ be the induced homomorphism given by $\widetilde{\varphi}(a+\ker\varphi) = \varphi(a)$. It satisfies:

(i) $\widetilde{\varphi}[\mathscr{A}/\ker\varphi] = \varphi[\mathscr{A}]$,

(ii) $\widetilde{\varphi}$ is injective and therefore bounded below by Lemma 4.1.8,

(iii) $\widetilde{\varphi}$ is continuous, hence it has closed range by (ii), and therefore so does φ by (i).

(b)$\Rightarrow$(c). Let $\mathscr{I}$ be a closed ideal of $\mathscr{A}$, and consider some continuous, injective homomorphism $\varphi\colon \mathscr{A}/\mathscr{I} \to \mathscr{B}$ into a normed algebra $\mathscr{B}$. By hypothesis, the composite homomorphism $\iota\varphi\pi\colon \mathscr{A} \to \widehat{\mathscr{B}}$ has closed range, where $\pi\colon \mathscr{A} \to \mathscr{A}/\mathscr{I}$ is the quotient map and $\iota\colon \mathscr{B} \to \widehat{\mathscr{B}}$ is the isometric embedding of $\mathscr{B}$ into its completion $\widehat{\mathscr{B}}$. It follows that the injection $\iota\varphi$ has closed range because π is surjective, so $\iota\varphi$ is bounded below, and therefore the same is true for φ.

(c)$\Rightarrow$(a). This follows from Proposition 4.1.6 and Lemma 4.1.13. $\square$

We are now equipped to outline our strategy to prove Theorem 4.1.2. According to Corollary 4.1.14, we must show that each of the Banach spaces X listed in clauses (i)–(iii) of Theorem 4.1.2 satisfies:

(I) every homomorphism from $\mathscr{B}(X)$ into a Banach algebra is continuous;

(II) $\mathscr{B}(X)/\mathscr{I}$ is incompressible for every closed ideal $\mathscr{I}$ of $\mathscr{B}(X)$.

Conveniently for us, in each case (I) is already known to hold true. This follows from Johnson's result stated in Theorem 4.1.7 above for the Banach spaces listed in clauses (i)–(ii) because they are all isomorphic to their squares, while [38, Corollary 39] verifies it for the particular Banach space $X = C_0(K_{\mathcal{A}})$ considered in clause (iii), which fails to be isomorphic to its square.

When verifying (II), we may clearly suppose that the closed ideal $\mathscr{I}$ is proper because $\mathscr{B}(X)/\mathscr{B}(X) \equiv \{0\}$ is incompressible trivially. Also, we may suppose that $\mathscr{I}$ is non-zero because incompressibility of $\mathscr{B}(X)/\{0\} \equiv \mathscr{B}(X)$ follows directly from Theorem 4.1.9. Moreover, Ware [63] has already handled some cases, as previously mentioned. These results partially prove the following theorem, for certain spaces X and certain closed ideals $\mathscr{J}$ of $\mathscr{B}(X)$. In the rest of this chapter, we shall complete its proof, which will consequently complete the proof of Theorem 4.1.2.

Theorem 4.1.15. *Let X be any of the Banach spaces listed in Theorem 4.1.2, and let $\mathscr{J}$ be a closed ideal of $\mathscr{B}(X)$. Then $\mathscr{B}(X)/\mathscr{J}$ is uniformly incompressible.*

This will be achieved by showing that certain idempotent operators admit 'quantitative factorisations' through the operators in $\mathscr{B}(X)\setminus\mathscr{I}$ and then applying Lemma 4.1.16 below, which is an adaptation of [31, Lemma 0.2], specialised for our purposes. See also [47] for a related, more abstract version of this result.

Factorisations of idempotent operators play a key role in the classifications of the closed ideals of $\mathscr{B}(X)$ for each of the Banach spaces X we consider. A simple example illustrating why this is the case is given in the discussion immediately after Proposition 1.5.1.

Such factorisations are therefore natural tools to bring to bear on the problem at hand. In order to do so, we require norm bounds on the auxiliary operators used in the factorisations, as we specify in our next result, Lemma 4.1.16. The statement involves the standard definition of operator ideals, which is due to Pietsch [54] and is given in Section 1.3.

Lemma 4.1.16. *Let X be a Banach space and $\mathscr{J}$ a closed operator ideal. Suppose that there exists a constant $C \geqslant 1$ such that, for every $T + \mathscr{J}(X) \in \mathscr{B}(X)/\mathscr{J}(X)$ of norm 1, there exist a Banach space Y and operators $R \in \mathscr{B}(X;Y)$ and $S \in \mathscr{B}(Y;X)$ with $\|R\|\,\|S\| \leqslant C$ such that $RTS + \mathscr{J}(Y)$ is a non-zero idempotent in $\mathscr{B}(Y)/\mathscr{J}(Y)$. Then $\mathscr{B}(X)/\mathscr{J}(X)$ is uniformly incompressible.*

Proof. If $\mathscr{J}(X) = \mathscr{B}(X)$, there is nothing to prove, so suppose that the closed ideal $\mathscr{J}(X)$ is proper. Choose $C \geqslant 1$ as specified, and let φ be a continuous, injective homomorphism from $\mathscr{B}(X)/\mathscr{J}(X)$ into a normed algebra. Given an element $t = T+\mathscr{J}(X)$ of norm 1 in $\mathscr{B}(X)/\mathscr{J}(X)$, we can find a Banach space Y and operators $R \in \mathscr{B}(X,Y)$ and $S \in \mathscr{B}(Y,X)$ with $\|R\|\,\|S\| \leqslant C$ such that $P+\mathscr{J}(Y)$ is a non-zero idempotent, where $P = RTS \in \mathscr{B}(Y)$. Then $P \notin \mathscr{J}(Y)$, and we have $P^3 - P = (I_Y + P)(P^2 - P) \in \mathscr{J}(Y)$, so the operator $Q = (TSR)^2 \in \mathscr{B}(X)$ satisfies

$$RQTS = P^3 \notin \mathscr{J}(Y) \qquad \text{and} \qquad Q^2 - Q = TS(P^3 - P)R \in \mathscr{J}(X)\,.$$

It follows that $q = Q+\mathscr{J}(X)$ is a non-zero idempotent in $\mathscr{B}(X)/\mathscr{J}(X)$. Therefore its image under the injective homomorphism φ is also a non-zero idempotent, which implies that $\|\varphi(q)\| \geqslant 1$. Set $u = SR+\mathscr{J}(X) \in \mathscr{B}(X)/\mathscr{J}(X)$, and observe that $\|u\| \leqslant \|S\|\,\|R\| \leqslant C$ and $q = (tu)^2$. Combining these facts, we conclude that

$$1 \leqslant \|\varphi(q)\| = \big\|\big(\varphi(t)\varphi(u)\big)^2\big\| \leqslant \|\varphi(t)\|^2\|\varphi(u)\|^2 \leqslant \|\varphi(t)\|^2\|\varphi\|^2C^2\,,$$

which shows that the definition of uniform incompressibility is satisfied with respect to the function $f\colon (0,\infty) \to (0,\infty)$ given by $f(x) = (Cx)^{-1}$ for every $x \in (0,\infty)$. $\square$

Remark 4.1.17. Recall from Remark 3.1.2 that the closed ideals of $\mathscr{B}(X)$ have been classified for several Banach spaces X other than those mentioned in Theorem 4.1.2. These spaces are all descendants of the famous Argyros–Haydon space X_{AH} which solved the scalar-plus-compact problem [3]. We do not know whether $\mathscr{B}(X_{\mathrm{AH}})$ has a unique algebra norm because we do not know whether every homomorphism from $\mathscr{B}(X_{\mathrm{AH}})$ into a Banach algebra is continuous. This question is also open for the variants of X_{AH} whose closed operator ideals have been classified in [60, Theorem 2.1] and [36, Theorem 1.4], respectively. (We shall consider the latter Banach space in much more detail in Example 4.1.18 below.)

The situation is even more intriguing for the family of Banach spaces X_{MPZ} of Motakis, Puglisi and Zizimopoulou [49], each defined by a compact, countable space K, with the key property of the space X_{MPZ} being that its Calkin algebra is isometrically isomorphic to the Banach algebra $C(K)$ of continuous, scalar-valued functions on K. It is undecidable in ZFC whether every homomorphism from $C(K)$ into a Banach algebra is continuous; more precisely, Dales [9] and Esterle [17] independently proved that discontinuous homomorphisms from $C(K)$ exist if we assume

ZFC and the Continuum Hypothesis, whereas Solovay and Woodin constructed a different model of ZFC in which all such homomorphisms are continuous; see [11] for an exposition of this result.

Therefore, it is also undecidable in ZFC whether every homomorphism from $\mathscr{B}(X_{MPZ})$ into a Banach algebra is continuous. Note that we have a complete classification of the closed ideals of $\mathscr{B}(X_{MPZ})$, as discussed in Item (iv).

To justify within the context of quotients of $\mathscr{B}(X)$ the differences between incompressibility and uniform incompressibility, we conclude this section with an example of a Calkin algebra which is incompressible, but not uniformly incompressible.

Example 4.1.18. This example is based on the Banach space Z studied in [36], and described in Remark 3.1.2. It is given by

$$Z = X_{\mathrm{AH}} \oplus_\infty Y \ , \tag{4.1.2}$$

where X_{AH} denotes the Banach space of Argyros and Haydon already mentioned in Remark 4.1.17, Y is a certain closed, infinite-dimensional subspace of X_{AH} constructed in [36, Theorem 1.2], and the subscript ∞ indicates that we equip Z with the norm

$$\|(x,y)\| = \max\{\|x\|, \|y\|\} \qquad (x \in X_{\mathrm{AH}},\ y \in Y)\ .$$

The key property of Z that we require is that, according to [36, Equation (1.2)], every operator $T \in \mathscr{B}(Z)$ has the form

$$T = \begin{pmatrix} \alpha_{1,1} I_{X_{\mathrm{AH}}} & \alpha_{1,2} J \\ 0 & \alpha_{2,2} I_Y \end{pmatrix} + K \ , \tag{4.1.3}$$

where $\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,2} \in \mathbb{K}$, $J\colon Y \to X_{\mathrm{AH}}$ denotes the inclusion map and $K \in \mathscr{K}(Z)$. (We note in passing that Z has a basis, so the ideal of finite-rank operators is dense in $\mathscr{K}(Z)$, which proves that $\mathscr{K}(Z)$ is the minimal closed ideal in $\mathscr{B}(Z)$.)

Johnson, Phillips and Schechtman [31] observed that the subalgebra

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{K} \right\}$$

of the algebra $M_2(\mathbb{K})$ of scalar-valued 2×2 matrices, equipped with the spectral norm, is incompressible, but not uniformly incompressible. Building on this example, we shall show that the same conclusion holds true for the Calkin algebra $\mathscr{B}(Z)/\mathscr{K}(Z)$.

First, (4.1.3) shows that $\mathscr{B}(Z)/\mathscr{K}(Z)$ is finite-dimensional. Trivially, it is therefore incompressible.

Second, we observe that the scalars $\alpha_{1,1}$, $\alpha_{1,2}$ and $\alpha_{2,2}$ in (4.1.3) are uniquely determined by T because the operators $I_{X_{\mathrm{AH}}}$, J and I_Y are non-compact, so for each $\delta \in (0,1)$, we can define a map $\phi_\delta : \mathscr{B}(Z) \to M_2(\mathbb{K})$ by

$$\phi_\delta(T) = \begin{pmatrix} \alpha_{1,1} & \delta\alpha_{1,2} \\ 0 & \alpha_{2,2} \end{pmatrix} .$$

Straightforward calculations show that ϕ_δ is a unital algebra homomorphism. Moreover, ϕ_δ is continuous with norm 1 provided that we equip $M_2(\mathbb{K})$ with the norm induced by identifying it with the Banach algebra $\mathscr{B}(\ell^2_\infty)$, that is,

$$\left\| \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \right\| = \max\bigl\{|\alpha_{1,1}| + |\alpha_{1,2}|, |\alpha_{2,1}| + |\alpha_{2,2}|\bigr\} .$$

Since $\ker\phi_\delta = \mathscr{K}(Z)$, the Fundamental Isomorphism Theorem implies that ϕ_δ induces a continuous, injective, unital algebra homomorphism $\widetilde{\phi}_\delta \colon \mathscr{B}(Z)/\mathscr{K}(Z) \to M_2(\mathbb{K})$, also of norm 1, by the formula $\widetilde{\phi}_\delta(T + \mathscr{K}(Z)) = \phi_\delta(T)$.

Consequently, if $\mathscr{B}(Z)/\mathscr{K}(Z)$ were to be uniformly incompressible, there would be a constant $c > 0$ such that $\widetilde{\phi}_\delta$ is bounded below by c for every $\delta \in (0,1)$. However, it was shown in [36, Lemma 4.2] that the operator J has distance 1 to the subspace $\mathscr{K}(Y; X_{\mathrm{AH}})$ of compact operators, so

$$\left\| \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} + \mathscr{K}(Z) \right\| = 1 .$$

This implies that, for every $\delta \in (0,1)$,

$$c \leqslant \left\| \widetilde{\phi}_\delta \left(\begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} + \mathscr{K}(Z) \right) \right\| = \left\| \begin{pmatrix} 0 & \delta \\ 0 & 0 \end{pmatrix} \right\| = \delta ,$$

which contradicts that the same constant $c > 0$ should work for every $\delta \in (0,1)$. Hence we conclude that the Calkin algebra $\mathscr{B}(Z)/\mathscr{K}(Z)$ fails to be uniformly incompressible.

4.2 The proof of Theorem 4.1.15 for $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{c_0}$ and $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{\ell_1}$

Throughout this section, set $D = c_0$ or $D = \ell_1$ so that D is isomorphic to its square and hence $\overline{\mathscr{G}}_D$, defined on Page 6, is indeed a closed operator ideal. Let X denote either of the Banach spaces $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$, the latter being the dual space of the former. The lattice of closed ideals of $\mathscr{B}(X)$, as previously discussed on Page 9, is

$$\{0\} \subsetneq \mathscr{K}(X) \subsetneq \overline{\mathscr{G}}_D(X) \subsetneq \mathscr{B}(X)\ . \tag{4.2.1}$$

A result of Ware [63, Theorem 5.3.1], already mentioned in the second bullet point of clause (iii) on page 53, tells us that the Calkin algebra $\mathscr{B}(X)/\mathscr{K}(X)$ of X has a unique algebra norm for both of these Banach spaces X. In particular, this tells us that it is incompressible. We remark that Lemma 4.1.16 in conjunction with case $(i,j) = (1,1)$ of Lemma 3.2.15 proves that it is uniformly incompressible too. Before proceeding, we introduce the following notion.

Definition 4.2.1. Let $C \geqslant 1$. A Banach algebra $\mathscr{A}$ has the *idempotent factorisation property with constant* C (abbreviated C-IFP) if, for every $a \in \mathscr{A}$ of norm 1, there are $b, c \in \mathscr{A}$ with $\|b\|\,\|c\| \leqslant C$ such that bac is a non-zero idempotent.

To see why the C-IFP is relevant for our purposes, observe the following result, the proof of which is similar to the latter part of the proof of Lemma 4.1.16.

Lemma 4.2.2. *Every Banach algebra which has the C-IFP for some constant $C \geqslant 1$ is uniformly incompressible.*

Proof. Let $C \geqslant 1$ and let $\mathcal{A}$ be a Banach algebra with the C-IFP. Let ϕ be a continuous, injective homomorphism from $\mathcal{A}$ into some normed algebra. Let $a \in \mathcal{A}$ have norm one, and using the C-IFP of $\mathcal{A}$, take $b, c \in \mathcal{A}$ such that bac is a nonzero idempotent and $\|b\|\|c\| \leqslant C$.

Since ϕ is an injective homomorphism, we have that $\phi(bac)$ is a nonzero idempotent, so that $\|\phi(bac)\| \geqslant 1$. This tells us that

$$1 \leqslant \|\phi(bac)\| \leqslant \|\phi(b)\|\|\phi(a)\|\|\phi(c)\| \leqslant \|\phi(a)\|\|\phi\|^2\|b\|\|c\| \leqslant \|\phi(a)\|\|\phi\|^2 C\ .$$

Hence ϕ is bounded below by $(\|\phi\|^2 C)^{-1}$. The result follows. □

In view of our remarks on page 58, it will suffice to prove the following result in order to establish Theorem 4.1.2(i).

Theorem 4.2.3. *Let $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$, where $D = c_0$ or $D = \ell_1$. Then the quotient algebra $\mathscr{B}(X)/\overline{\mathscr{G}}_D(X)$has the C-IFP for every $C > 1$ and thus is uniformly incompressible.*

Thus, we shall in this section show that the quotients $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$, where $D = c_0$ or $D = \ell_1$ have the C-IFP for some constant $C \geqslant 1$.

For this section, let N be a countable subset of $\mathbb{N}$. Recall from Section 1.2 the definition of a D-direct sum of an N-indexed sequence $(X_n)_{n\in N}$ of Banach spaces X_n. For $m \in N$, we reserve the symbols $J_m\colon X_m \to \left(\bigoplus_{n\in N} X_n\right)_D$ and $Q_m\colon \left(\bigoplus_{n\in N} X_n\right)_D \to X_m$ for the canonical m^{th} coordinate embedding and projection, respectively.

Suppose that N is infinite, let $T \in \mathscr{B}(\left(\bigoplus_{n\in N} X_n\right)_D)$, and identify T with its canonical matrix expression (see Section 1.2 for details on this correspondence). Following [40], we say that the operator T has *finite rows with respect to N* if the set $\{k \in N : T_{j,k} \neq 0\}$ is finite for each $j \in N$; and analogously T has *finite columns with respect to N* if the set $\{j \in N : T_{j,k} \neq 0\}$ is finite for each $k \in N$. An operator which has both finite rows and finite columns with respect to N has *locally finite matrix with respect to N*. Indeed, when examining the matrix representation $[T]$ of T, the concept of T having finite rows/columns with respect to N corresponds to $[T]$ having finitely supported rows/columns as expected.

The case $D = c_0$.

Throughout this subsection, we consider the Banach space

$$X = \Big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\Big)_{c_0}.$$

The following index $m_\epsilon(T)$, which was originally introduced in [40, Definition 5.2(ii)], played a key role in the classification (4.2.1) of the closed ideals of $\mathscr{B}(X)$, and it will also be an essential ingredient in our proof of Theorem 4.2.3.

Definition 4.2.4. For $\epsilon > 0$ and an operator $T \in \mathscr{B}\big(\big(\bigoplus_{n\in N} H_n\big)_{c_0}; E\big)$, where $N \subseteq \mathbb{N}$ is finite, H_n is a Hilbert space for each $n \in N$ and E is a Banach space, we define

$$m_\epsilon(T) = \sup\Big\{ m \in \mathbb{N}_0 : \Big\| T \circ \bigoplus_{n\in N} (I_{H_n} - P_{G_n}) \Big\| > \epsilon \text{ for every subspace } G_n \text{ of } H_n \text{ with } \dim G_n \leqslant m \text{ for each } n \in N \Big\} \in \mathbb{N}_0 \cup \{\pm\infty\},$$

where P_{G_n} denotes the orthogonal projection of H_n onto the subspace G_n.

Loosely speaking, $m_\epsilon(T)$ is the largest number of dimensions that we can remove from each of the Hilbert spaces in the domain of T and still obtain an operator with norm greater than ϵ.

Suppose that $T \in \mathscr{B}(X)$ is an operator with finite rows with respect to $\mathbb{N}$. The operator $Q_jT : X \to \ell_2^j$ acts trivially on all but finitely many of the spaces ℓ_2^n in its domain X. As described in [40, Remark 5.4], the above definition of m_ϵ can therefore be applied to define $m_\epsilon(Q_jT)$ for every $j \in \mathbb{N}$ in a natural way by ignoring the cofinite number of Hilbert spaces in the domain of Q_jT on which it acts trivially.

The following lemma explains why the quantities $m_\epsilon(Q_jT)$ for $j \in \mathbb{N}$ are relevant in our investigation. It is proved in [40, Theorem 5.5(iii)]. The bounds on the norms of the operators R and S are not stated explicitly in [40], but can be found easily by examining the proof of the result.

Lemma 4.2.5. *Let $T \in \mathscr{B}(X)$ be an operator with locally finite matrix with respect to $\mathbb{N}$, and suppose that the set $\{m_\epsilon(Q_jT) : j \in \mathbb{N}\}$ is unbounded for some $\epsilon > 0$. Then there are operators $R, S \in \mathscr{B}(X)$ with $\|R\| \leqslant 1$ and $\|S\| \leqslant 1/\epsilon$ such that $STR = I_X$.*

In order to apply this result, we require specific values of $\epsilon > 0$ for which the set $\{m_\epsilon(Q_jT) : j \in \mathbb{N}\}$ is unbounded. The following lemma addresses this point.

Lemma 4.2.6. *Let $T \in \mathscr{B}(X) \setminus \overline{\mathscr{G}}_{c_0}(X)$ be an operator with locally finite matrix with respect to $\mathbb{N}$, and suppose that the set $\{m_\epsilon(Q_jT) : j \in \mathbb{N}\}$ is bounded for some $\epsilon > 0$. Then $\|T + \overline{\mathscr{G}}_{c_0}(X)\| \leqslant \epsilon$.*

Consequently the set $\{m_\epsilon(Q_jT) : j \in \mathbb{N}\}$ is unbounded whenever we have that $0 < \epsilon < \|T + \overline{\mathscr{G}}_{c_0}(X)\|$.

Proof. Set $m = \sup\{m_\epsilon(Q_jT) : j \in \mathbb{N}\} \in \mathbb{N}_0$. Our proof uses the machinery from [40, Construction 4.2], so we begin by introducing the necessary notation. For every $j \in \mathbb{N}$, define $N_j = \{k \in \mathbb{N} : T_{j,k} \neq 0\}$, which is finite, let $B_j = \big(\bigoplus_{k\in N_j} \ell_2^k\big)_{c_0}$, and let $L_j \colon B_j \to X$ be the natural inclusion map. Moreover, set $B = \big(\bigoplus_{j\in\mathbb{N}} B_j\big)_{c_0}$ and

$$\widetilde{T} = \bigoplus_{j\in\mathbb{N}} Q_jTL_j \in \mathscr{B}(B;X)\ .$$

The operator $V \in \mathscr{B}(X;B)$ with $\|V\| = 1$ defined with matrix coordinates

$$V_{m,n} := \begin{cases} J_n^{B_m} & \text{if } N_m \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \in \mathscr{B}(\ell_2^n, B_m)\ ,$$

for every $m, n \in \mathbb{N}$, where $J_n^{B_m}$ donotes the natural embedding of ℓ_2^n into B_m, (as found in [40, Construction 4.2]), satisfies $T = \widetilde{T}V$. The convention described after Definition 4.2.4 means that

$$m_\epsilon(Q_jTL_j) = m_\epsilon(Q_jT) \leqslant m \ \text{ for every } \ j \in \mathbb{N}\ .$$

Therefore we can find orthogonal projections $P_{j,k} \in \mathscr{B}(\ell_2^k)$ for $k \in N_j$, each having rank at most $m+1$, such that

$$\Big\| Q_jTL_j \circ \bigoplus_{k\in N_j} (I_{\ell_2^k} - P_{j,k}) \Big\| \leqslant \epsilon\ . \tag{4.2.2}$$

Set $R_j = \bigoplus_{k\in N_j} P_{j,k} \in \mathscr{B}(B_j)$ and $R = \bigoplus_{j\in\mathbb{N}} R_j \in \mathscr{B}(B)$. Since each of the projections $P_{j,k}$ has rank at most $m+1$, the image of R embeds isometrically into a direct sum of the form $(\ell_2^{m+1} \oplus \ell_2^{m+1} \oplus \cdots)_{c_0}$, which in turn is isomorphic to

$(\ell_\infty^{m+1} \oplus \ell_\infty^{m+1} \oplus \cdots)_{c_0} \equiv c_0$. This shows that R factors through c_0, and therefore the same is true for the composite operator $\widetilde{T}RV$. Hence we have

$$\begin{aligned}\|T + \overline{\mathscr{G}}_{c_0}(X)\| &\leqslant \|T - \widetilde{T}RV\| = \|\widetilde{T}(I_B - R)V\| \\ &\leqslant \Big\| \bigoplus_{j\in\mathbb{N}} Q_j T L_j (I_{B_j} - R_j) \Big\| \, \|V\| \leqslant \epsilon \ ,\end{aligned}$$

where the final estimate follows because $\|V\| = 1$ and $\|Q_j T L_j (I_{B_j} - R_j)\| \leqslant \epsilon$ for every $j \in \mathbb{N}$ by (4.2.2). The final clause is immediate. □

We now have everything required to prove that the quotient Banach algebra $\mathscr{B}(X)/\overline{\mathscr{G}}_{c_0}(X)$ has the C-IFP for every $C > 1$, and thus is uniformly incompressible by Lemma 4.2.2.

Proof of Theorem 4.2.3 *for* $D = c_0$. Fix a constant $C > 1$, and take an operator $T \in \mathscr{B}(X)$ with $\|T + \overline{\mathscr{G}}_{c_0}(X)\| = 1$. By [40, Lemma 2.7(iii)], we can find a compact operator $K \in \mathscr{K}(X)$ such that $T - K$ has locally finite matrix with respect to $\mathbb{N}$. Noting that compact operators on X necessarily factor approximately through c_0, we see that

$$1/C < 1 = \|T + \overline{\mathscr{G}}_{c_0}(X)\| = \|T - K + \overline{\mathscr{G}}_{c_0}(X)\| \ ,$$

so Lemma 4.2.6 shows that the set $\{m_{1/C}(Q_j(T - K)) : j \in \mathbb{N}\}$ is unbounded. Consequently there are operators $R, S \in \mathscr{B}(X)$ with $\|R\| \leqslant 1$ and $\|S\| \leqslant C$ such that $S(T - K)R = I_X$ by Lemma 4.2.5. The latter identity implies that

$$STR + \overline{\mathscr{G}}_{c_0}(X) = STR - SKR + \overline{\mathscr{G}}_{c_0}(X) = I_X + \overline{\mathscr{G}}_{c_0}(X) \ ,$$

which is a non-zero idempotent in $\mathscr{B}(X)/\overline{\mathscr{G}}_{c_0}(X)$. Hence the conditions for applying Lemma 4.1.16 are satisfied for any constant $C > 1$ (and using the Banach space $Y = X$ for every operator T). The conclusion follows. □

The case $D = \ell_1$.

We now turn our attention to the Banach space $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{\ell_1}$. It identifies naturally with the dual space of the Banach space $\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{c_0}$ that we considered in the previous subsection. Our proof is essentially a dual version of the proof we have just completed. As observed in [41, Remark 2.13], the dualisation is not straightforward because we can no longer perturb operators by compact operators to arrange that they have locally finite matrix with respect to $\mathbb{N}$; only finite columns with respect to $\mathbb{N}$ can be achieved. Fortunately, that will suffice to carry out the necessary steps, beginning with the following index originally introduced in [41, Definition 2.4], which is the dual version of Definition 4.2.4 above.

Definition 4.2.7. For $\epsilon > 0$ and an operator $T \in \mathscr{B}\big(E; \big(\bigoplus_{j\in M} H_j\big)_{\ell_1}\big)$, where E is a Banach space, $M \subseteq \mathbb{N}$ is finite and H_j is a Hilbert space for each $j \in M$, we define

$$n_\epsilon(T) = \sup\bigg\{n \in \mathbb{N}_0 : \Big\|\Big(\bigoplus_{j\in M}(I_{H_j} - P_{G_j})\Big)T\Big\| > \epsilon \text{ for every subspace } G_j \text{ of } H_j \text{ with } \dim G_j \leqslant n \text{ for each } j \in M\bigg\} \in \mathbb{N}_0 \cup \{\pm\infty\}\,,$$

where P_{G_j} denotes the orthogonal projection of H_j onto the subspace G_j, as before.

Hence $n_\epsilon(T)$ is loosely speaking the largest number of dimensions that we can remove from each of the Hilbert spaces in the codomain of T and still obtain an operator with norm greater than ϵ.

When $T \in \mathscr{B}(X)$ has finite columns with respect to $\mathbb{N}$, for each $k \in \mathbb{N}$ we have that the image of the operator TJ_k is contained in a finite direct sum $\ell_2^{n_1} \oplus \cdots \oplus \ell_2^{n_j}$ for some $n_1 < \cdots < n_j$, so we can apply Definition 4.2.7 to the operator TJ_k by simply ignoring the Hilbert spaces ℓ_2^m in the codomain of TJ_k corresponding to the cofinite set of indices $m \in \mathbb{N}$ for which $Q_mTJ_k = 0$. Then, as one would hope, the quantities $n_\epsilon(TJ_k)$ for $k \in \mathbb{N}$ measure how close the identity operator on X is to factoring through T. More precisely, we have the following counterpart of Lemma 4.2.5, originally proved as part of [41, Proposition 2.11]. The norm bounds on U and V are not stated explicitly in [41], but follow easily from the proof of the implication (ii)$\Rightarrow$(iii) therein.

Lemma 4.2.8. *Let $T \in \mathscr{B}(X)$ be an operator with finite columns with respect to $\mathbb{N}$, and suppose that the set $\{n_\epsilon(TJ_k) : k \in \mathbb{N}\}$ is unbounded for some $\epsilon > 0$. Then there are operators $U, V \in \mathscr{B}(X)$ with $\|U\| \leqslant 1/\epsilon$ and $\|V\| \leqslant 1$ such that $VTU = I_X$.*

Our next lemma serves an analogous purpose as Theorem 4.2.6, and its proof is similar.

Lemma 4.2.9. *Let $T \in \mathscr{B}(X) \setminus \overline{\mathscr{G}}_{\ell_1}(X)$ be an operator with finite column with respect to $\mathbb{N}$, and suppose that $\sup\{n_\epsilon(TJ_k) : k \in \mathbb{N}\}$ is finite for some $\epsilon > 0$. Then $\|T + \overline{\mathscr{G}}_{\ell_1}(X)\| \leqslant \epsilon$.*

Consequently the set $\{n_\epsilon(TJ_k) : k \in \mathbb{N}\}$ is unbounded whenever we have that $0 < \epsilon < \|T + \overline{\mathscr{G}}_{\ell_1}(X)\|$.

Proof. Set $n = \sup\{n_\epsilon(TJ_k) : k \in \mathbb{N}\} \in \mathbb{N}_0$. For each $k \in \mathbb{N}$, we see that the set $M_k = \{j \in \mathbb{N} : T_{j,k} \neq 0\}$ is finite because T has finite columns with respect to $\mathbb{N}$. Define $Y_k = \big(\bigoplus_{j\in M_k} \ell_2^j\big)_{\ell_1}$, and let $L_k\colon Y_k \to X$ and $S_k\colon X \to Y_k$ denote the natural inclusion map and projection, respectively. Further, define $Y = \big(\bigoplus_{k\in\mathbb{N}} Y_k\big)_{\ell_1}$ and let

$$\widetilde{T} = \bigoplus_{k\in\mathbb{N}} S_k T J_k \in \mathscr{B}(X;Y)\ .$$

Then, for each element $(y_k) \in Y$, the series $\sum_{k\in\mathbb{N}} L_k y_k$ converges absolutely in X, and the operator $W \in \mathscr{B}(Y;X)$ given by $W(y_k) = \sum_{k\in\mathbb{N}} L_k y_k$ satisfies $\|W\| = 1$ and $T = W\widetilde{T}$.

For every $k \in \mathbb{N}$, we have $n \geqslant n_\epsilon(TJ_k) = n_\epsilon(S_kTJ_k)$, so we can find orthogonal projections $P_{j,k} \in \mathscr{B}(\ell_2^j)$ for $j \in M_k$, each having rank at most $n+1$, such that

$$\Big\|\Big(\bigoplus_{j\in M_k} (I_{\ell_2^j} - P_{j,k})\Big) S_k T J_k\Big\| \leqslant \epsilon\ . \tag{4.2.3}$$

Set $R_k = \bigoplus_{j\in M_k} P_{j,k} \in \mathscr{B}(Y_k)$ and $R = \bigoplus_{k\in\mathbb{N}} R_k \in \mathscr{B}(Y)$, and observe that R factors through ℓ_1 because the image of R embeds isometrically into $(\ell_2^{n+1} \oplus \ell_2^{n+1} \oplus \cdots)_{\ell_1}$, which is isomorphic to $(\ell_1^{n+1} \oplus \ell_1^{n+1} \oplus \cdots)_{\ell_1} \equiv \ell_1$. Consequently we have

$$\|T + \overline{\mathscr{G}}_{\ell_1}(X)\| \leqslant \|T - WR\widetilde{T}\| = \|W(I_Y - R)\widetilde{T}\| \leqslant \|W\| \Big\|\bigoplus_{k\in\mathbb{N}} (I_{Y_k} - R_k) S_k T J_k\Big\| \leqslant \epsilon$$

by (4.2.3), as required. The final clause is immediate. □

We now have everything we need to conclude this section by proving that the quotient Banach algebra $\mathscr{B}(X)/\overline{\mathscr{G}}_{\ell_1}(X)$ has the C-IFP for every $C > 1$, and thus is uniformly incompressible by Lemma 4.2.2.

Proof of Theorem 4.2.3 *for $D = \ell_1$.* Take any constant $C > 1$ and let $T \in \mathscr{B}(X)$ with $\|T + \overline{\mathscr{G}}_{\ell_1}(X)\| = 1$. Apply [40, Lemma 2.7(i)] to find a compact operator

$K \in \mathscr{K}(X)$ such that $T - K$ has finite columns with respect to $\mathbb{N}$. Then

$$1/C < 1 = \|T + \overline{\mathscr{G}}_{\ell_1}(X)\| = \|T - K + \overline{\mathscr{G}}_{\ell_1}(X)\| \, .$$

We have that the set $\{n_{1/C}(Q_j(T-K)) : j \in \mathbb{N}\}$ is unbounded by Lemma 4.2.9, so we can use Lemma 4.2.8 to obtain operators $U, V \in \mathscr{B}(X)$ with $\|U\| \leqslant C$ and $\|V\| \leqslant 1$ for which $V(T-K)U = I_X$, so

$$VTU + \overline{\mathscr{G}}_{\ell_1}(X) = VTU - VKU + \overline{\mathscr{G}}_{\ell_1}(X) = I_X + \overline{\mathscr{G}}_{\ell_1}(X)$$

is a nonzero idempotent in $\mathscr{B}(X)/\overline{\mathscr{G}}_{\ell_1}(X)$. Lemma 4.1.16 therefore applies and the result follows. □

4.3 The proof of Theorem 4.1.15 for $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D \oplus D_\Gamma$

The aim of this section is to show that every quotient algebra of $\mathscr{B}(X)$ by one of its closed ideals has a unique algebra norm for the direct sums

$$X = \Big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\Big)_{c_0} \oplus c_0(\Gamma) \text{ and } X = \Big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\Big)_{\ell_1} \oplus \ell_1(\Gamma) \, ,$$

where the index set Γ is an uncountable cardinal number. To enable us to discuss the two cases simultaneously, we reintroduce from Chapter 3 the notation $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ or $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$, meaning that in either case we can simply use the notation

$$X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D \oplus D_\Gamma \, , \tag{4.3.1}$$

which we fix for the rest of this section.

We first give a fairly simple general result, Lemma 4.3.1 below, which allows us to combine IFP-style results for direct sums of Banach algebras.

Lemma 4.3.1. *Let $\mathscr{A}_1, \ldots, \mathscr{A}_n$ be Banach algebras, where $n \in \mathbb{N}$, and suppose that for every $i \in \{1, \ldots, n\}$, we have that $\mathscr{A}_i$ has the C_i-IFP for some constant $C_i \geqslant 1$. The direct sum $\mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_n$ has the C-IFP for $C = \max\{C_1, \ldots, C_n\}$.*

Proof. Take $a \in \mathscr{A}$ with $\|a\| = 1$, and write $a = a_1 + \cdots + a_n$, where $a_i \in \mathscr{A}_i$ for every $i \in \{1, \ldots, n\}$. Let $j \in \{1, \ldots, n\}$ be such that $\|a_j\| = 1$. Then, because $\mathscr{A}_j$ has the C_j-IFP, we can find $b, c \in \mathscr{A}_j$ for which ba_jc is a non-zero idempotent in $\mathscr{A}_j$ with $\|b\|\|c\| \leqslant C_j$.

Now, the canonical embedding $J_j : \mathscr{A}_j \to \mathscr{A}$ and projection $P_j : \mathscr{A} \to \mathscr{A}_j$ both have norm 1, and $a_j = P_j a$, meaning that $(J_j b P_j)a(J_j c P_j)$ is a non-zero idempotent in $\mathscr{A}$, with $\|(J_j b P_j)\|\|(J_j c P_j)\| \leqslant C_j$. The result follows. □

Fix $n \in \mathbb{N}$ and Banach spaces $X_1, X_2, \ldots, X_n$. For a subset $\mathscr{I}$ of $\mathscr{B}(X_1 \oplus X_2 \oplus \cdots \oplus X_n)_\infty$ and $j, k \in \{1, 2, \ldots, n\}$, we define

$$\mathscr{I}_{j,k} = \{Q_j T J_k : T \in \mathscr{I}\} \subseteq \mathscr{B}(X_k; X_j)\ .$$

This notion generalises the idea of 'quadrants' as seen in Chapter 3.

The following result makes clear how Lemma 4.3.1 can be used in our context.

Proposition 4.3.2. *Let $\mathscr{J}$ be a closed ideal in $\mathscr{B}(E_D \oplus D_\Gamma)$ which properly contains $\mathscr{K}(E_D \oplus D_\Gamma)$. Then, there are closed ideals $\mathscr{J}_1$ of $\mathscr{B}(E_D)$ and $\mathscr{J}_2$ of $\mathscr{B}(D_\Gamma)$, both of which strictly contain the set of compact operators on their respective spaces, such that*

$$\mathscr{J} \equiv (\mathscr{B}(E_D)/\mathscr{J}_1) \oplus_\infty (\mathscr{B}(D_\Gamma)/\mathscr{J}_2)\ .$$

Proof. Since $\mathscr{I}$ is a closed ideal of $\mathscr{B}(X)$ and $\mathscr{K}(X) \subsetneq \mathscr{I}$, Proposition 3.2.18 tells us that $\mathscr{J}_{1,1}$ and $\mathscr{J}_{2,2}$ must be closed ideals of E_D and D_Γ strictly containing the compact operators, and that the quadrants $\mathscr{I}_{2,1}$ and $\mathscr{I}_{1,2}$ of $\mathscr{I}$ are their entire respective spaces of bounded operators $\mathscr{B}(X_1; X_2)$ and $\mathscr{B}(X_2; X_1)$. It follows that

$$\|T + \mathscr{J}\| = \max\{\|T_{1,1} + \mathscr{J}_{1,1}\|, \|T_{2,2} + \mathscr{J}_{2,2}\|\}\ ,$$

from which the result follows easily. □

Given that the hypothesis of the above proposition necessitates that $\mathscr{J}$ contains strictly the space of compact operators, we must consider the Calkin algebra of X as a separate case, and hence the finale of the proof of Theorem 4.1.2(ii) needs to be split into two parts.

The Calkin algebra of X is uniformly incompressible

We first tackle the case for the Calkin algebra of X. The desired result, Proposition 4.3.3 below, is a simple consequence of Lemma 3.2.15.

Proposition 4.3.3. *Let $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ or $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ for some infinite cardinal Γ. Let $X_1 = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$, $X_2 = D_\Gamma$, and $X = X_1 \oplus X_2$. The Calkin algebra of X is uniformly incompressible.*

Proof. Let $T \in \mathscr{B}(X)$ be such that $\|T\|_e = 1$, and let $\epsilon > 0$. As usual, for each $i, j \in \{1, 2\}$, let J_i and Q_j denote the natural inclusion and projection $X_i \to X$ and $X \to X_j$ respectively.

We may decompose the operator T as $T = \sum_{i,j=1}^{2} J_i T_{i,j} Q_j$, for operators $T_{i,j} \in \mathscr{B}(X_j; X_i)$, $i, j \in \{1, 2\}$. It follows that there is a pair $i, j \in \{1, 2\}$ for which

$$\|T_{i,j}\|_e = \|J_i T_{i,j} Q_j\|_e \geqslant \frac{1}{4} .$$

Now, Lemma 3.2.15 tells us that I_D factors through $T_{i,j}$ as $UT_{i,j}V = I_D$ for some operators $U \in \mathscr{B}(X_i; D), V \in \mathscr{B}(D; X_j)$, with $\|U\|\|V\| \leqslant 8(1 + \epsilon)$. Then

$$I_D = UT_{i,j}V = UQ_iTJ_jV .$$

The result follows from Lemma 4.1.16. □

Quotients of $\mathscr{B}(X)$ by large closed ideals are uniformly incompressible

Here, we will show that all of the quotients of $\mathscr{B}(X)$ by a closed ideal strictly containing $\mathscr{K}(X)$ have the C-IFP for every $C > 1$, and hence are uniformly incompressible by Lemma 4.2.2. The proof of this part of the result has four main ingredients:

(i) The fact that each quotient of $\mathscr{B}(\left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D)$ by one of its closed ideals has the C-IFP for every $C > 1$, which was established previously in this chapter.

(ii) The fact that each quotient of $\mathscr{B}(D_\Gamma)$ by one of its closed ideals has the C-IFP for every $C > 1$, which Ware proved in his dissertation [63].

(iii) The classification of the closed ideals of $\mathscr{B}(X)$, which is displayed on Page 30, and proved in Chapter 3.

(iv) Lemma 4.3.1 and Proposition 4.3.2 above, which allow us to combine the results for the two summands in (i) and (ii) to draw the desired conclusion for their direct sum (4.3.1).

Given the above conditions and the statement of Lemma 4.3.1, the claim in Lemma 4.2.3 that for $Y = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_D$, where $D = c_0$ or $D = \ell_1$, the quotient algebra $\mathscr{B}(Y)/\overline{\mathscr{G}}_D(Y)$ has the C-IFP for every constant $C > 1$, plays a key role going forwards.

As displayed on Page 8, the lattice of closed ideals of $\mathscr{B}(D_\Gamma)$ was classified by Daws in [12], and the non-trivial closed ideals of $\mathscr{B}(D_\Gamma)$ are the ideals of κ-compact operators on D_Γ for the cardinals κ with $\aleph_0 \leqslant \kappa \leqslant \Gamma$.

Ware [63, Propositions 6.3.1–6.3.2] observed that Daws' proofs of [12, Theorems 6.2 and 7.3] establish the following conclusions, stated here using our proprietary terminology for this subsection.

Lemma 4.3.4. *Let Γ and κ be uncountable cardinal numbers with $\kappa \leqslant \Gamma$. Then:*

(i) *The κ-Calkin algebras of $c_0(\Gamma)$ and $\ell_p(\Gamma)$ for $1 < p < \infty$ have the 4-IFP.*

(ii) *The κ-Calkin algebra of $\ell_1(\Gamma)$ has the C-IFP for every constant $C > 1$.*

Proposition 4.3.5. *Let $\mathscr{J}$ be a closed ideal of $\mathscr{B}(X)$ for which $\mathscr{K}(X) \subsetneq \mathscr{J}$. The quotient algebra $\mathscr{B}(X)/\mathscr{J}$ is uniformly incompressible.*

Proof. By Proposition 4.3.2, we have that $\mathscr{B}(X)/\mathscr{J}$ is isometrically isomorphic to $(\mathscr{B}(X_1)/\mathscr{J}_{1,1}) \oplus_\infty (\mathscr{B}(X_2)/\mathscr{J}_{2,2})$ where $\mathscr{J}_{1,1}$ and $\mathscr{J}_{2,2}$ are closed ideals of $\mathscr{B}(X_1)$ and $\mathscr{B}(X_2)$ respectively, strictly containing their entire sets of compact operators.

Moreover, using the ideal classifications (4.2.1) and (1.4.1) for $\mathscr{B}(X_1)$ and $\mathscr{B}(X_2)$, we find

$$\mathscr{I}_{1,1} = \begin{cases} \overline{\mathscr{G}}_D(X_1) & \text{for } \mathscr{I} = \mathscr{J}_\kappa(X), \text{ where } \aleph_1 \leqslant \kappa \leqslant \Gamma^+ \ , \\ \mathscr{B}(X_1) & \text{for } \mathscr{I} = \mathscr{K}_\kappa(X)\ , \text{ where } \aleph_1 \leqslant \kappa \leqslant \Gamma, \end{cases}$$

and

$$\mathscr{I}_{2,2} = \begin{cases} \mathscr{K}_\kappa(X_2) & \text{for } \mathscr{I} = \mathscr{K}_\kappa(X) \text{ or } \mathscr{I} = \mathscr{J}_\kappa(X), \text{ where } \aleph_1 \leqslant \kappa \leqslant \Gamma\ , \\ \mathscr{B}(X_2) & \text{for } \mathscr{I} = \mathscr{J}_{\Gamma^+}(X)\ . \end{cases}$$

Consequently, Lemmas 4.2.3 and 4.3.4 imply that $\mathscr{B}(X_1)/\mathscr{I}_{1,1}$ and $\mathscr{B}(X_2)/\mathscr{I}_{2,2}$ have the 4-IFP whenever they are non-zero. If they are both non-zero, then Lemma 4.3.1 shows that $\mathscr{B}(X)/\mathscr{I}$ has the 4-IFP. Otherwise we have $\mathscr{I}_{j,j} = \mathscr{B}(X_j)$ for one $j \in \{1,2\}$, and $\mathscr{B}(X)/\mathscr{I} \cong \mathscr{B}(X_k)/\mathscr{I}_{k,k}$, where $k = 2$ if $j = 1$ and $k = 1$ if $j = 2$, so $\mathscr{B}(X)/\mathscr{I}$ also has the 4-IFP in this case. It is therefore uniformly incompressible by Lemma 4.2.2, as desired. □

4.4 The proof of Theorem 4.1.15 for the space $X = C_0(K_{\mathcal{A}})$ of continuous functions vanishing at infinity on Koszmider's Mrówka space

We begin this section with two definitions and two classical results, positioned here ahead of the main body of work.

Definition 4.4.1. A Banach space X is an *Asplund space* if every separable subspace of X has a separable dual space.

Definition 4.4.2. A Banach space X has the *Schur property* if every weakly convergent sequence in X converges in norm.

Theorem 4.4.3 (Gantmacher's Theorem)**.** *A bounded operator between Banach spaces is weakly compact if and only if its adjoint operator is.*

Theorem 4.4.4 (Schauder's Theorem)**.** *A bounded operator between Banach spaces is compact if and only if its adjoint operator is.*

4.4.1 Incompressibility of the Calkin algebra of the Banach space of continuous functions on a scattered, locally compact space

Let K be a locally compact Hausdorff space. Recall that a topological space is scattered if each subspace of it contains an isolated point, and recall from Page 2 the definition of the Banach space $C_0(K)$.

The aim of this section is to prove the following theorem which, broadly speaking, says that the idempotent factorisation property considered in Lemma 4.1.16 is not only sufficient, but also necessary for the (uniform) incompressibility of the Calkin algebra of $C_0(K)$. Moreover, it provides an equivalent condition (a) which may be easier to verify in applications. Section 4.4.2 provides an example of this method in action.

Theorem 4.4.5. *Let K be a scattered, locally compact Hausdorff space. Then the following conditions are equivalent:*

(a) *There exists a constant $\delta \in (0,1)$ such that, for every non-compact operator $T \in \mathscr{B}(C_0(K))$ with $\|T\|_e = 1$, $C_0(K)$ contains a sequence $(f_n)_{n\in\mathbb{N}}$ with*

$$\sup_{k\in K} \sum_{n=1}^{\infty} |f_n(k)| \leqslant 1 \qquad \text{and} \qquad \inf_{n\in\mathbb{N}} \|Tf_n\| > \delta\,. \tag{4.4.1}$$

(b) *There is a constant $C > 1$ such that, for every non-compact operator $T \in \mathscr{B}(C_0(K))$ with $\|T\|_e = 1$, there are operators $U \in \mathscr{B}(C_0(K); c_0)$ and $V \in \mathscr{B}(c_0; C_0(K))$ with $\|U\|\,\|V\| < C$ such that $UTV = I_{c_0}$.*

(c) *The Calkin algebra $\mathscr{B}(C_0(K))/\mathscr{K}(C_0(K))$ is uniformly incompressible.*

(d) *The Calkin algebra $\mathscr{B}(C_0(K))/\mathscr{K}(C_0(K))$ is incompressible.*

There are currently no known spaces K for which the conditions of Theorem 4.4.5 are not satisfied. Take K_n to be the ordinal interval $[0, \omega^n)$ equipped with the order topology, and for each $n \in \mathbb{N}$, let C_n be a constant satisfying Theorem 4.4.5(b) for $K = K_n$ (The fact that $c_0 \cong C_0(K_n)$ for every $n \in \mathbb{N}$ proves that such C_n exist). Although we were not able to prove this concretely, the author suspects that the sequence $(C_n)_{n\in\mathbb{N}}$ must tend to infinity. With this in mind, we leave the following open question which highlights a promising candidate for such a space K.

Question 4.4.6. Is it true that $K = [0, \omega^\omega)$ fails to satisfy the conditions of Theorem 4.4.5?

Remark 4.4.7. Let K be a scattered, locally compact Hausdorff space. Recall from Page 11 that $C_0(K)$ is a hyperplane in $C(\alpha K)$, and hence the two Banach spaces are isomorphic.

Consequently, in view of the of the isomorphic invariance of Theorem 4.4.5, we could have stated it only for compact K without formally losing any generality. We have nevertheless chosen to state it in the locally compact case for the following reasons:

(i) The isomorphism between $C_0(K)$ and $C(\alpha K)$ is not isometric (for an example, consider c_0 and c), and the constants δ and C in conditions (a) and (b) may therefore change when passing from one space to the other. We consider the values of these constants to be of some interest.

(ii) The present version includes the case $K = \mathbb{N}$ (or in other words $C_0(K) = c_0$) explicitly, which seems natural given the role c_0 plays in the result. Perhaps more importantly, in the proof of Theorem 4.4.5, we shall apply various lemmas to c_0 as well as $C_0(K)$. If the latter class did not contain c_0, some statements would become more complicated.

(iii) In Subsection 4.4.2 we shall apply Theorem 4.4.5 in the locally compact case, where the 'vanishing at infinity' property will be convenient for us.

To help the presentation of the proof of Theorem 4.4.5, we have split it into a number of separate statements, some of which may also be of independent interest. The first of these will be essential when showing that conditions (a) and (b) are equivalent.

Lemma 4.4.8. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C_0(K)$ for some locally compact Hausdorff space K. Then there is an operator $V \in \mathscr{B}(c_0; C_0(K))$ such that $Ve_n = f_n$ for every $n \in \mathbb{N}$ if and only if*

$$\sup_{k\in K} \sum_{n=1}^{\infty} |f_n(k)| < \infty . \tag{4.4.2}$$

If one, and hence both, of these conditions are satisfied, the norm of the operator V is equal to the supremum (4.4.2).

Proof. Let $C \in [0, \infty]$ denote the supremum in (4.4.2), and observe that

$$C = \sup\left\{ \sum_{n=1}^{m} |f_n(k)| : k \in K,\, m \in \mathbb{N} \right\} .$$

$\Rightarrow$. Suppose that $V \in \mathscr{B}(c_0; C_0(K))$ is an operator with $Ve_n = f_n$ for every $n \in \mathbb{N}$, and take $m \in \mathbb{N}$ and $k \in K$. For each $n \in \{1, \ldots, m\}$, choose $\sigma_n \in \mathbb{K}$ such that $|\sigma_n| = 1$ and $\sigma_n f_n(k) \geqslant 0$. Then $x = \sum_{n=1}^m \sigma_n e_n \in c_0$ has norm 1, so

$$\|V\| \geqslant \|Vx\|_\infty = \Big\|\sum_{n=1}^m \sigma_n f_n\Big\|_\infty \geqslant \Big|\sum_{n=1}^m \sigma_n f_n(k)\Big| = \sum_{n=1}^m \sigma_n f_n(k) = \sum_{n=1}^m |f_n(k)| \, .$$

This proves that $C \leqslant \|V\| < \infty$.

$\Leftarrow$. Suppose that the supremum C is finite. Since c_{00} is dense in c_0, it suffices to show that the linear map $V\colon c_{00} \to C_0(K)$ given by $Ve_n = f_n$ for every $n \in \mathbb{N}$ is bounded. To verify this, we observe that, for $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$ with $\max_{1\leqslant n\leqslant m}|\alpha_n| \leqslant 1$,

$$\Big\|V\Big(\sum_{n=1}^m \alpha_n e_n\Big)\Big\|_\infty = \Big\|\sum_{n=1}^m \alpha_n f_n\Big\|_\infty = \sup_{k\in K}\Big|\sum_{n=1}^m \alpha_n f_n(k)\Big| \leqslant \sup_{k\in K}\sum_{n=1}^m |\alpha_n|\,|f_n(k)| \leqslant C \, .$$

Hence V is bounded with $\|V\| \leqslant C$. The final clause follows by combining the estimates obtained in the two parts of the proof. $\square$

Lemma 4.4.9. *Let X be a Banach space with a normalised, bimonotone basis $(b_n)_{n\in\mathbb{N}}$, and suppose that*

$$\Big\|\sum_{n=1}^m \alpha_n b_n\Big\| = \Big\|\sum_{n=1}^m \alpha_n b_{n+1}\Big\| \qquad (m \in \mathbb{N},\ \alpha_1, \ldots, \alpha_m \in \mathbb{K}) \, . \tag{4.4.3}$$

Then, for every $C > 1$ and every operator $T \in \mathscr{B}(X)$ such that $I_X - T \in \mathscr{K}(X)$, there are operators $U, V \in \mathscr{B}(X)$ with $\|U\|\,\|V\| < C$ such that $UTV = I_X$.

Proof. Equation (4.4.3) shows that the linear map R given by $Rb_n = b_{n+1}$ for $n \in \mathbb{N}$ is an isometry. Now consider the linear map L given by $Lb_1 = 0$ and $Lb_{n+1} = b_n$ for $n \in \mathbb{N}$. For each $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_{m+1} \in \mathbb{K}$, it satisfies

$$\begin{aligned}\Big\|L\Big(\sum_{n=1}^{m+1} \alpha_n b_n\Big)\Big\| &= \Big\|\sum_{n=1}^m \alpha_{n+1} b_n\Big\| = \Big\|R\Big(\sum_{n=1}^m \alpha_{n+1} b_n\Big)\Big\| = \Big\|\sum_{n=1}^m \alpha_{n+1} b_{n+1}\Big\| \\ &= \Big\|(I_X - P_1)\Big(\sum_{n=1}^{m+1} \alpha_n b_n\Big)\Big\| \leqslant \Big\|\sum_{n=1}^{m+1} \alpha_n b_n\Big\|\end{aligned}$$

by bimonotonicity. Hence X admits shift operators $R, L \in \mathscr{B}(X)$, both having norm 1, and

$$L^nR^n = I_X \qquad \text{and} \qquad R^nL^n = I_X - P_n \qquad (n \in \mathbb{N})\,. \tag{4.4.4}$$

Let $C > 1$, and suppose that $S := I_X - T \in \mathscr{K}(X)$. Then $P_nS \to S$ as $n \to \infty$, so we can find $n \in \mathbb{N}$ such that $\|(I_X - P_n)S\| \leqslant 1 - \frac{1}{C}$. Combining this with (4.4.4), we obtain

$$\begin{aligned}\|I_X - L^nTR^n\| &= \|L^n(I_X - T)R^n\| = \|L^nSR^n\| \\ &\leqslant \|L^nS\| = \|R^nL^nS\| = \|(I_X - P_n)S\| \leqslant 1 - \frac{1}{C}\,.\end{aligned}$$

This implies that the operator L^nTR^n is invertible by the C. Neumann series, and its inverse has norm at most C. Consequently we can define operators $U = (L^nTR^n)^{-1}L^n \in \mathscr{B}(X)$ and $V = R^n \in \mathscr{B}(X)$ with $\|U\| \leqslant C$ and $\|V\| = 1$, and $UTV = I_X$ by definition. □

We now come to what is arguably the most important ingredient in the proof of Theorem 4.4.5, namely a generalisation of a theorem of Dowling, Randrianantoanina, and Turett [14, Theorem 6].

Theorem 4.4.10. *Let W be a closed subspace of a Banach space X for which the unit ball of X^* is weak* sequentially compact, and suppose that W contains a closed subspace which is isomorphic to c_0. Then, for every $C > 1$, there is a projection $P \in \mathscr{B}(X)$ with $\|P\| \leqslant C$ such that $P[X]$ is contained in W and C-isomorphic to c_0.*

Proof. For $X = W$, this is precisely the result which Dowling, Randrianantoanina, and Turett proved in [14, Theorem 6]. In order to adapt their proof to the setting where $W \subsetneq X$, it suffices to observe that they define the projection P using Hahn–Banach extensions of the coordinate functionals corresponding to a $(1-\delta)^{-1}$-isomorphic copy of c_0 inside W, for a suitably defined $\delta \in (0, 1)$. By extending these coordinate functionals to all of X, rather than only to W, we can follow the rest of their proof verbatim to obtain the stated result. □

Remark 4.4.11. In the case of real scalars, Galego and Plichko [21, Theorem 4.3] have proved a result similar to Theorem 4.4.10 under the weaker hypothesis that X

does not contain a copy of ℓ_1. However, it is not clear to us whether this result carries over to complex scalars. Therefore we have opted for the above version, which will suffice for our purposes.

We note in passing that Galego and Plichko do not state explicitly that the C-complemented copy of c_0 they construct inside W is C-isomorphic to c_0. However, a close inspection of their proof reveals that it is.

Lemma 4.4.12. *Let (n_k) be a strictly increasing sequence of natural numbers, and let (w_k) be a sequence of unit vectors in c_0 for which $\operatorname{supp}(w_k) \subseteq [n_k, n_{k+1})$ for all $k \in \mathbb{N}$. The operator on c_0 that sends e_k to w_k for each $k \in \mathbb{N}$ is an isometry, and its image is complemented in c_0 by a projection of norm 1.*

Proof. That the operator in question is an isometry is easy to see. For the remainder of the proof see e.g. [43, Proposition 2.a.1]. □

A basis $(e_n)_{n\in\mathbb{N}}$ of a Banach space Z is *shrinking* if the sequence $(e_n^*)_{n\in\mathbb{N}} \subset Z^*$ of its coordinate functionals forms a basis of Z^*. This property clearly holds for the standard basis of c_0 whose coordinate functionals give a basis to ℓ_1.

Lemma 4.4.13. *Let $T \in \mathscr{K}(X;Y)$ for some Banach spaces X and Y for which X has a shrinking basis. For each $n \in \mathbb{N}$, let $P_n \in \mathscr{B}(X)$ be the canonical projection onto the first n coordinates in said basis. Then $TP_n \to T$ as $n \to \infty$.*

Proof. Suppose towards a contradiction TP_n does not converge to T as $n \to \infty$. It follows that there is some $\epsilon > 0$ such that for every $N \in \mathbb{N}$, there is some $n > N$ for which $\|TP_n - T\| \geqslant \epsilon$. Thus, we can find a subsequence $(m_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ such that $\|TP_{m_n} - T\| \geqslant \epsilon$ for every $n \in \mathbb{N}$, and hence a sequence $(x_n)_{n\in\mathbb{N}} \in B_X$ for which

$$\|T(I - P_{m_n})x_n\| \geqslant \epsilon \text{ for every } n \in \mathbb{N}\,. \tag{4.4.5}$$

Now, for each $n \in \mathbb{N}$, define $y_n = (I - P_{m_n})x_n \in B_X$. Because T is compact, we can (by replacing with a subsequence if necessary) suppose that $(Ty_n)_{n\in\mathbb{N}}$ is convergent with some limit $z \in Y$, where (4.4.5) tells us that $z \neq 0$.

The sequence $(Ty_n)_{n\in\mathbb{N}} \subseteq Y$ is also weakly convergent to z, hence for all $f \in Y^*$, we have that

$$\langle x_n, (I - P_{m_n})^* T^* f\rangle = \langle y_n, T^* f\rangle = \langle Ty_n, f\rangle \to \langle z, f\rangle \tag{4.4.6}$$

as $n \to \infty$. Now, the operator $(I - P_{m_n})^* \in \mathscr{B}(X^*)$ is simply the standard projection onto the coordinates $\mathbb{N} \setminus [1, \ldots, m_n]$, and so because the basis of X is shrinking, we must have that $(I - P_{m_n})^* T^* f \to 0$ as $n \to \infty$ for any $f \in Y^*$. Hence

$$|\langle x_n, (I - P_{m_n})^* T^* f \rangle| \leqslant \|x_n\| \|(I - P_{m_n})^* T^* f\| \to 0$$

as $n \to \infty$ for every $f \in Y^*$, so by (4.4.6), we have that $z = 0$. This contradiction proves the claim. $\square$

Lemma 4.4.14. *Let $R \in \mathscr{B}(X; Y)$ be a non-compact operator between Banach spaces X and Y, where X is C-isomorphic to c_0 for some constant $C \geqslant 1$. Then, for each $\eta \in (0, 1)$, X contains a C-complemented subspace W which is C-isomorphic to c_0 and satisfies*

$$\frac{1-\eta}{C} \|R\|_e \|w\| \leqslant \|Rw\| \leqslant (1+\eta) C \|R\|_e \|w\| \qquad (w \in W) . \tag{4.4.7}$$

Proof. Take an isomorphism $U \in \mathscr{B}(c_0; X)$ with $\|U\| \, \|U^{-1}\| \leqslant C$, and set $S = RU \in \mathscr{B}(c_0; Y)$. Choose $T \in \mathscr{K}(c_0; Y)$ such that $\|S + T\| \leqslant (1 + \frac{\eta}{2}) \|S\|_e$. Using Lemma 4.4.13, we have $TP_n \to T$ as $n \to \infty$, so $\|T(I_{c_0} - P_{m_0})\| \leqslant \frac{\eta}{2} \|S\|_e$ for some $m_0 \in \mathbb{N}$.

On the other hand, $\|S(I_{c_0} - P_m)\| > (1 - \frac{\eta}{2}) \|S\|_e$ for each $m \in \mathbb{N}$. Consequently we can find an integer $k > m$ and a unit vector $w \in \operatorname{span}\{e_j : m < j \leqslant k\}$ such that $\|Sw\| > (1 - \frac{\eta}{2}) \|S\|_e$. Using this, we can recursively choose integers $m_0 < m_1 < m_2 < \cdots$ and unit vectors $w_n \in \operatorname{span}\{e_j : m_{n-1} < j \leqslant m_n\}$ such that

$$\|Sw_n\| > \left(1 - \frac{\eta}{2}\right) \|S\|_e \qquad (n \in \mathbb{N}) .$$

Using Lemma 4.4.12, the linear map on c_0 determined by $e_n \mapsto w_n$ for each $n \in \mathbb{N}$ is an isometry, and Theorem 3.2.12 implies that we can find a subsequence (w_{n_j}) of (w_n) such that the restriction of S to the subspace $W_0 = \overline{\operatorname{span}}\{w_{n_j} : j \in \mathbb{N}\}$ is bounded below by $(1 - \eta) \|S\|_e$. Furthermore, (w_{n_j}) is a normalised block basic sequence of (e_n), so W_0 is isometrically isomorphic to c_0, and we can take a projection Q of c_0 onto W_0 with $\|Q\| = 1$. Consequently the subspace $W = U[W_0]$ is C-isomorphic to c_0 and complemented in X via the projection UQU^{-1}, which has norm at most C.

Finally, to verify (4.4.7), take $w \in W$, and set $w_0 = U^{-1}w \in W_0$. Then $Rw = Sw_0$ and $w_0 = (I_{c_0} - P_{m_0})w_0$, so

$$\begin{aligned}\|Rw\| &\leqslant \|(S+T)w_0\| + \|T(I_{c_0} - P_{m_0})w_0\| \leqslant (1+\eta)\|S\|_e\|w_0\| \\ &\leqslant (1+\eta)\|R\|_e\|U\|\,\|U^{-1}\|\,\|w\| \leqslant (1+\eta)C\|R\|_e\|w\|\end{aligned}$$

and

$$\|Rw\| \geqslant (1-\eta)\|S\|_e\|w_0\| \geqslant (1-\eta)\frac{\|SU^{-1}\|_e}{\|U^{-1}\|}\frac{\|Uw_0\|}{\|U\|} \geqslant \frac{1-\eta}{C}\|R\|_e\|w\| \,. \square$$

Definition 4.4.15. Let W, X and Y be Banach spaces. An operator $R \in \mathscr{B}(X;Y)$ *fixes a copy of* c_0 if there is a subspace W of X with $W \cong c_0$ for which $R|_W$ is an isomorphism onto its range.

Corollary 4.4.16. *Let X and Y be Banach spaces for which the unit ball of Y^* is weak* sequentially compact, and let $R \in \mathscr{B}(X;Y)$ be an operator which fixes a copy of c_0. Then, for every $\epsilon \in (0,1)$, there exist a constant $C > 0$ and a closed, infinite-dimensional subspace W of X such that*

$$(1+\epsilon)C\|w\| \geqslant \|Rw\| \geqslant (1-\epsilon)C\|w\| \qquad (w \in W)\,, \tag{4.4.8}$$

and the subspace $R[W]$ is $(1+\epsilon)$-complemented in Y and $(1+\epsilon)$-isomorphic to c_0. Consequently there are operators $U \in \mathscr{B}(Y;c_0)$ and $V \in \mathscr{B}(c_0;X)$ such that

$$URV = I_{c_0} \qquad \text{and} \qquad \|U\|\,\|V\| \leqslant \frac{(1+\epsilon)^2}{(1-\epsilon)C}\,.$$

Proof. Given $\epsilon \in (0,1)$, choose $\eta > 0$ such that $\eta^2 + 2\eta \leqslant \epsilon$. Since R fixes a copy of c_0, we can take a closed subspace X_1 of X such that X_1 is isomorphic to c_0 and the restriction of R to X_1 is bounded below. By James' Distortion Theorem [28], X_1 contains a closed subspace X_2 which is $(1+\eta)$-isomorphic to c_0. The restriction of R to X_2 is non-compact because it is bounded below, so we can apply Lemma 4.4.14

to obtain a closed subspace X_3 of X_2 such that X_3 is isomorphic to c_0 and

$$(1+\eta)^2\|R|_{X_2}\|_e\|w\| \geqslant \|Rw\| \geqslant \frac{1-\eta}{1+\eta}\|R|_{X_2}\|_e\|w\| \qquad (w \in X_3)\,. \tag{4.4.9}$$

In particular, $R[X_3]$ is also isomorphic to c_0. Since the unit ball of Y^* is weak* sequentially compact, Theorem 4.4.10 implies that $R[X_3]$ contains a closed subspace Z which is $(1+\epsilon)$-complemented in Y and $(1+\epsilon)$-isomorphic to c_0. Set $W = R^{-1}[Z]\cap X_3$ and $C = \|R|_{X_2}\|_e$. Then $R[W] = Z$, and (4.4.8) follows from (4.4.9) because the choice of η implies that $1+\epsilon \geqslant (1+\eta)^2$ and $1-\epsilon \leqslant (1-\eta)/(1+\eta)$.

To prove the final clause, take a projection Q of Y onto Z with $\|Q\| \leqslant 1+\epsilon$ and an isomorphism $S \in \mathscr{B}(c_0; Z)$ with $\|S\|\,\|S^{-1}\| \leqslant 1+\epsilon$, and let $\widetilde{R} \in \mathscr{B}(W; Z)$ denote the restriction of R, which is an isomorphism with $\|\widetilde{R}^{-1}\| \leqslant \frac{1}{(1-\epsilon)C}$ because R is bounded below by $(1-\epsilon)C$ on W. Then we have a commutative diagram:

$$\begin{tikzcd}
c_0 \arrow[rr, "I_{c_0}"] \arrow[d, "S"'] & & c_0 \\
Z \arrow[r, "\widetilde{R}^{-1}"] & W \arrow[r, "\widetilde{R}"] \arrow[d, "J"'] & Z \arrow[u, "S^{-1}"'] \\
& X \arrow[r, "R"] & Y \arrow[u, "Q"']
\end{tikzcd} \tag{4.4.10}$$

where J is the inclusion map. Consequently the operators $U = S^{-1}Q \in \mathscr{B}(Y; c_0)$ and $V = J\widetilde{R}^{-1}S \in \mathscr{B}(c_0; X)$ satisfy $URV = I_{c_0}$, and

$$\|U\|\,\|V\| \leqslant \|S^{-1}\|\,\|Q\|\,\|J\|\,\|\widetilde{R}^{-1}\|\,\|S\| \leqslant \frac{(1+\epsilon)^2}{(1-\epsilon)C}\,.$$

□

Our next result shows that Theorem 4.4.10 and Corollary 4.4.16 apply to $C_0(K)$, justifying their inclusion in our work.

Theorem 4.4.17. *Let K be a scattered, locally compact Hausdorff space.*

(i) *The unit ball of $C_0(K)^*$ is weak* sequentially compact.*

(ii) *An operator from $C_0(K)$ into a Banach space is compact if and only if it is weakly compact, if and only if it does not fix a copy of c_0.*

Proof. In view of Remark 4.4.7 and the isomorphic nature of both parts of this result, we may suppose that K is compact by replacing it with its one-point compactification if necessary.

(i). This follows by combining two famous results: First, Hagler and Johnson [24, Theorem 1(b)] have shown that the unit ball of X^* is weak* sequentially compact for every Asplund space X, and second Namioka and Phelps [50, Theorem 18] have shown that $C(K)$ is an Asplund space whenever K is a scattered compact space.

(ii). A classical result of Pełczyński [53] states that an operator defined on a $C(K)$-space is weakly compact if and only if it does not fix a copy of c_0, and of course every compact operator is weakly compact.

To complete this proof, suppose that R is a weakly compact operator defined on $C(K)$. By Gantmacher's Theorem, its adjoint R^* is also weakly compact. This implies that R^* is compact because a famous result of Rudin [57] states that its codomain $C(K)^*$ is isomorphic to $\ell_1(K)$, which has the Schur property. Hence R is compact by Schauder's Theorem. $\square$

With these results at hand, we can establish the first of the two main 'building blocks' that the proof of Theorem 4.4.5 will rely on; specifically, the equivalence of conditions (a) and (b) will be an easy consequence of the following proposition.

Proposition 4.4.18. *Let $T \in \mathscr{B}(C_0(K); Y)$ be a non-compact operator, where K is a scattered, locally compact Hausdorff space and Y is a Banach space for which the unit ball of Y^* is weak* sequentially compact, and let $C > 1$. Then $C_0(K)$ contains a sequence $(f_n)_{n\in\mathbb{N}}$ for which*

$$\sup_{k\in K}\sum_{n=1}^{\infty}|f_n(k)| \leqslant 1 \qquad \textit{and} \qquad \inf_{n\in\mathbb{N}}\|Tf_n\| > \frac{1}{C} \tag{4.4.11}$$

if and only if there are operators $U \in \mathscr{B}(Y; c_0)$ and $V \in \mathscr{B}(c_0; C_0(K))$ such that

$$UTV = I_{c_0} \qquad \textit{and} \qquad \|U\|\,\|V\| < C\,. \tag{4.4.12}$$

Proof. To prove the implication $\Rightarrow$, take a sequence (f_n) in $C_0(K)$ which satisfies (4.4.11). Lemma 4.4.8 shows that we can define an operator $V_0 \in \mathscr{B}(c_0; C_0(K))$ of norm at most 1 by $V_0 e_n = f_n$ for each $n \in \mathbb{N}$. Choose $\eta > 1$ such that

$\inf_{n\in\mathbb{N}}\|Tf_n\| > \eta^3/C$. Then $\inf_{n\in\mathbb{N}}\|TV_0e_n\| > \eta^3/C$, so Theorem 3.2.12 implies that $\mathbb{N}$ contains an infinite subset N such that the restriction of TV_0 to the subspace $X = \overline{\operatorname{span}}\{e_n : n \in N\}$ is bounded below by η^3/C.

Set $R = TV_0|_X \in \mathscr{B}(X;Y)$. We can now establish (4.4.12) by essentially repeating the arguments from the last part of the proof of Corollary 4.4.16, beginning just below (4.4.9). Indeed, since $R[X]$ is isomorphic to X, which in turn is isomorphic to c_0, and the unit ball of Y^* is weak* sequentially compact, we can apply Theorem 4.4.10 to find a closed subspace Z of $R[X]$ for which there are a projection Q of Y onto Z with $\|Q\| \leqslant \eta$ and an isomorphism $S \in \mathscr{B}(c_0;Z)$ with $\|S\|\,\|S^{-1}\| \leqslant \eta$. The fact that R is bounded below by η^3/C implies that we can regard its restriction $\widetilde{R}$ to $W = R^{-1}[Z]$ as an isomorphism onto Z, and $\|\widetilde{R}^{-1}\| \leqslant C/\eta^3$.

In this way we obtain the same commutative diagram (4.4.10) as in the proof of Corollary 4.4.16. Since $R = TV_0|_X$, it follows that the operators $U = S^{-1}Q \in \mathscr{B}(Y;c_0)$ and $V = V_0J_0\widetilde{R}^{-1}S \in \mathscr{B}(c_0;C_0(K))$, where $J_0\colon W \to c_0$ is the inclusion map, satisfy (4.4.12); that is, $UTV = I_{c_0}$ and

$$\|U\|\,\|V\| \leqslant \|S^{-1}\|\,\|Q\|\,\|V_0\|\,\|J_0\|\,\|\widetilde{R}^{-1}\|\,\|S\| \leqslant \frac{\eta^2 C}{\eta^3} = \frac{C}{\eta} < C\,.$$

Conversely, suppose that $U \in \mathscr{B}(Y;c_0)$ and $V \in \mathscr{B}(c_0;C_0(K))$ are operators satisfying (4.4.12), and set $f_n = \frac{1}{\|V\|}Ve_n \in C_0(K)$ for each $n \in \mathbb{N}$. Then Lemma 4.4.8 shows that $\sup_{k\in K}\sum_{n=1}^{\infty}|f_n(k)| = 1$ because the operator $V/\|V\|$ has norm 1. Furthermore, since $UTf_n = \frac{1}{\|V\|}e_n$ for each $n \in \mathbb{N}$, we have

$$\inf_{n\in\mathbb{N}}\|Tf_n\| \geqslant \frac{1}{\|U\|\,\|V\|} > \frac{1}{C}\,,$$

so the sequence (f_n) satisfies (4.4.11). □

The other major ‘building block’ we require in the proof of Theorem 4.4.5 (specifically to verify that (d) implies (b)) is an alternative algebra norm ν on the Calkin algebra of $C_0(K)$ for a scattered, locally compact Hausdorff space K. To define this norm, note that according to the proof of [35, Proposition 5.4(ii)], the identity operator on c_0 factors through every non-compact operator on $C_0(K)$. (This is also a consequence of Corollary 4.4.16 and Theorem 4.4.17, as we shall explain in detail at the beginning of the proof of Proposition 4.4.19 below.) Consequently we can define a map $\nu\colon \mathscr{B}(C_0(K))/\mathscr{K}(C_0(K)) \to [0,\infty)$ by

$$\nu(T+\mathscr{K}(C_0(K))) := \begin{cases} 0 & \text{if } T \in \mathscr{K}(C_0(K)), \\ \sup\left\{ \dfrac{1}{\|U\|\,\|V\|} : I_{c_0} - UTV \in \mathscr{K}(c_0) \right\} & \text{otherwise.} \end{cases} \tag{4.4.13}$$

As already indicated, our aim is to establish the following result.

Proposition 4.4.19. *Let K be a scattered, locally compact Hausdorff space. Then ν defined by* (4.4.13) *is an algebra norm on $\mathscr{B}(C_0(K))/\mathscr{K}(C_0(K))$, and*

$$\nu(T + \mathscr{K}(C_0(K))) \leqslant \|T\|_e \qquad (T \in \mathscr{B}(C_0(K)))\,. \tag{4.4.14}$$

Before engaging with the proof, let us point out that Proposition 4.4.19 is heavily inspired by a similar result in [31] concerning operators that the identity operator on ℓ_2 factors through. Our version, which involves replacing ℓ_2 with c_0, is somewhat harder to prove because c_0 has a much richer subspace structure than ℓ_2.

The following elementary lemma will help us shorten a couple of steps in the proof of Proposition 4.4.19.

Lemma 4.4.20. *Let X, Y and Z be infinite-dimensional Banach spaces, and suppose that the operators $R \in \mathscr{B}(X;Y)$, $T \in \mathscr{B}(Y;Z)$ and $U \in \mathscr{B}(Z;X)$ satisfy*

$$I_X - UTR \in \mathscr{K}(X)\,. \tag{4.4.15}$$

Then $R[X]$ is closed and isomorphic to a closed subspace of finite codimension in X, and the restriction of T to any closed, infinite-dimensional subspace of $R[X]$ is non-compact.

Proof. It follows from (4.4.15) that UTR is a Fredholm operator. This implies that R is an upper semi-Fredholm operator (see e.g. [39, Proposition 3.3.2 (v)]), so it has closed range, and $\ker R$ is finite-dimensional. Let W be a closed, complementary subspace of $\ker R$; that is, $W + \ker R = X$ and $W \cap \ker R = \{0\}$. Then the restriction of R to W is an isomorphism onto $R[X]$.

Suppose that Y_0 is a closed subspace of $R[X]$ such that $T|_{Y_0}$ is compact, and set $X_0 = R^{-1}[Y_0]$. Then we may regard the restriction $\widetilde{R}$ of R to X_0 as an operator into Y_0; it satisfies $UTR|_{X_0} = UT|_{Y_0}\widetilde{R}$, which implies that $UTR|_{X_0}$ is compact. Hence X_0 is finite-dimensional by (4.4.15), so $R[X_0] = Y_0$ is also finite-dimensional. $\square$

Proof of Proposition 4.4.19. We shall use Corollary 4.4.16 several times in this proof, in each case for a non-compact operator R whose domain and codomain are (isomorphic to) either c_0 or $C_0(K)$. To avoid repetition, let us once and for all state that Theorem 4.4.17 ensures that the hypotheses of Corollary 4.4.16 are satisfied in this case.

This observation shows in particular that the identity operator on c_0 factors through every non-compact operator on $C_0(K)$, so the definition of ν makes sense. Furthermore, it is clear that ν is faithful and absolutely homogeneous.

In the remainder of the proof, suppose that $T_1, T_2 \in \mathscr{B}(C_0(K))$.

Subadditivity. Let us begin by observing that the inequality

$$\nu(T_1 + T_2 + \mathscr{K}(C_0(K))) \leqslant \nu(T_1 + \mathscr{K}(C_0(K))) + \nu(T_2 + \mathscr{K}(C_0(K)))$$

is trivial if $T_1 + T_2$ is compact. Otherwise, for each $\epsilon \in (0, 1)$, we can find operators $U \in \mathscr{B}(C_0(K); c_0)$ and $V \in \mathscr{B}(c_0; C_0(K))$ such that

$$I_{c_0} - U(T_1 + T_2)V \in \mathscr{K}(c_0) \tag{4.4.16}$$

and

$$\frac{1}{\|U\|\,\|V\|} \geqslant (1 - \epsilon)\nu(T_1 + T_2 + \mathscr{K}(C_0(K)))\,.$$

Set $R_i = UT_iV \in \mathscr{B}(c_0)$ for $i = 1, 2$. By (4.4.16), at least one of these operators is non-compact, say R_1 (relabelling them if necessary). Therefore we can apply Corollary 4.4.16 to find a constant $C_1 > 0$ and a closed, infinite-dimensional subspace W_1 of c_0 such that

$$(1 + \epsilon)C_1\|w\| \geqslant \|R_1w\| \geqslant (1 - \epsilon)C_1\|w\| \qquad (w \in W_1) \tag{4.4.17}$$

and $I_{c_0} = U_1R_1V_1$ for some operators $U_1, V_1 \in \mathscr{B}(c_0)$ with

$$\|U_1\|\,\|V_1\| \leqslant \frac{(1+\epsilon)^2}{(1-\epsilon)C_1}\ .$$

Furthermore, Corollary 4.4.16 states that $R_1[W_1]$ is isomorphic to c_0, which implies that W_1 is isomorphic to c_0 because the restriction of R_1 to W_1 is an isomorphism onto $R_1[W_1]$ by (4.4.17).

Now we split in two cases, beginning with the case where the restriction of R_2 to W_1 is non-compact. Since W_1 is isomorphic to c_0, we can apply Corollary 4.4.16 once more, this time to the operator $R_2|_{W_1}$, to find a constant $C_2 > 0$ and a closed, infinite-dimensional subspace W_2 of W_1 such that

$$(1+\epsilon)C_2\|w\| \geqslant \|R_2w\| \geqslant (1-\epsilon)C_2\|w\| \qquad (w \in W_2) \tag{4.4.18}$$

and $I_{c_0} = U_2R_2V_2$ for some operators $U_2, V_2 \in \mathscr{B}(c_0)$ with $\|U_2\|\,\|V_2\| \leqslant \frac{(1+\epsilon)^2}{(1-\epsilon)C_2}$.

Then, for $i \in \{1,2\}$, we have $I_{c_0} = U_iR_iV_i = (U_iU)T_i(VV_i)$, so

$$\nu(T_i + \mathscr{K}(C_0(K))) \geqslant \frac{1}{\|U_i\|\,\|U\|\,\|V\|\,\|V_i\|} \geqslant \frac{(1-\epsilon)^2C_i}{(1+\epsilon)^2}\nu(T_1+T_2+\mathscr{K}(C_0(K)))\ . \tag{4.4.19}$$

The left-hand inequalities in (4.4.17)–(4.4.18) and the fact that $W_2 \subseteq W_1$ imply that

$$(1+\epsilon)(C_1+C_2) \geqslant \|(R_1+R_2)|_{W_2}\| \geqslant \|(R_1+R_2)|_{W_2}\|_e = \|I_{c_0}|_{W_2}\|_e = 1\ , \tag{4.4.20}$$

where the penultimate equality follows from (4.4.16). Adding up the estimates (4.4.19) for $i = 1$ and $i = 2$ and substituting the lower bound on $C_1 + C_2$ from (4.4.20) into this sum, we obtain

$$\nu(T_1+\mathscr{K}(C_0(K)))+\nu(T_2+\mathscr{K}(C_0(K))) \geqslant \frac{(1-\epsilon)^2}{(1+\epsilon)^3}\nu(T_1+T_2+\mathscr{K}(C_0(K)))\ . \tag{4.4.21}$$

We claim that this inequality also holds true in the case where $R_2|_{W_1}$ is compact. Indeed, the estimate (4.4.19) remains valid for $i = 1$, while we can modify (4.4.20) in the following way:

$$(1+\epsilon)C_1 \geqslant \|R_1|_{W_1}\| \geqslant \|R_1|_{W_1}\|_e = \|(R_1+R_2)|_{W_1}\|_e = \|I_{c_0}|_{W_1}\|_e = 1\ .$$

Now (4.4.21) follows by using the trivial lower bound 0 on $\nu(T_2+\mathscr{K}(C_0(K)))$.

Since (4.4.21) holds true for arbitrary $\epsilon \in (0,1)$, we conclude that ν is subadditive.

Submultiplicativity. The desired inequality

$$\nu(T_2T_1+\mathscr{K}(C_0(K))) \leqslant \nu(T_2+\mathscr{K}(C_0(K)))\nu(T_1+\mathscr{K}(C_0(K)))$$

is trivial if T_2T_1 is compact. Otherwise, for each $\epsilon \in (0,1)$, we can find operators $U \in \mathscr{B}(C_0(K);c_0)$ and $V \in \mathscr{B}(c_0;C_0(K))$ such that

$$I_{c_0} - UT_2T_1V \in \mathscr{K}(c_0) \qquad \text{and} \qquad \frac{1}{\|U\|\,\|V\|} \geqslant (1-\epsilon)\nu(T_2T_1+\mathscr{K}(c_0))\ . \tag{4.4.22}$$

Applying Corollary 4.4.16 and Lemma 4.4.20 twice, we obtain constants $C_1, C_2 > 0$, closed subspaces $W_1 \subseteq V[c_0]$ and $W_2 \subseteq T_1[W_1]$ and operators $U_1, U_2 \in \mathscr{B}(C_0(K);c_0)$ and $V_1, V_2 \in \mathscr{B}(c_0;C_0(K))$ such that $T_1[W_1]$ and $T_2[W_2]$ are isomorphic to c_0 and satisfy

$$\begin{aligned} (1+\epsilon)C_i\|w\| \geqslant \|T_iw\| \geqslant (1-\epsilon)C_i\|w\|, \quad U_iT_iV_i = I_{c_0}, \\ \|U_i\|\,\|V_i\| \leqslant \frac{(1+\epsilon)^2}{(1-\epsilon)C_i} \end{aligned} \tag{4.4.23}$$

for every $w \in W_i$ and $i \in \{1,2\}$. To explain this construction in detail, for $i = 1$ we apply Corollary 4.4.16 to the operator $T_1|_{V[c_0]}$; this is justified because Lemma 4.4.20 and the first part of (4.4.22) show that the domain $V[c_0]$ is isomorphic to c_0 and $T_1|_{V[c_0]}$ is non-compact. Then, for $i = 2$, we apply Corollary 4.4.16 to the operator $T_2|_{T_1[W_1]}$; by construction its domain $T_1[W_1]$ is isomorphic to c_0, and since $T_1[W_1] \subseteq T_1V[c_0]$, we can apply Lemma 4.4.20 and (4.4.22) once more to deduce that $T_2|_{T_1[W_1]}$ is non-compact.

The second and third identity in (4.4.1) show that

$$\begin{aligned}\nu(T_2 + \mathscr{K}(C_0(K)))\nu(T_1 + \mathscr{K}(C_0(K))) &\geqslant \frac{1}{\|U_2\|\,\|V_2\|\,\|U_1\|\,\|V_1\|} \\ &\geqslant \frac{(1-\epsilon)^2 C_1 C_2}{(1+\epsilon)^4}\,. \end{aligned} \tag{4.4.24}$$

Set $W_0 = V^{-1}[T_1^{-1}[W_2] \cap W_1]$. This is a closed subspace of c_0 such that

$$V[W_0] = T_1^{-1}[W_2] \cap W_1 \qquad \text{and} \qquad T_1 V[W_0] = W_2\,.$$

The latter identity implies that W_0 is infinite-dimensional. Combining this with the first part of (4.4.22) and the left-hand inequality in (4.4.1), we find

$$1 = \|I_{c_0}|_{W_0}\|_e = \|UT_2T_1V|_{W_0}\|_e \leqslant \|U\|\,\|T_2|_{W_2}\|\,\|T_1|_{W_1}\|\,\|V\| \leqslant (1+\epsilon)^2 C_1C_2\|U\|\,\|V\|\,.$$

This produces a lower bound on C_1C_2, which we can substitute into (4.4.24) to obtain

$$\begin{aligned}\nu(T_2 + \mathscr{K}(C_0(K)))\nu(T_1 + \mathscr{K}(C_0(K))) &\geqslant \frac{(1-\epsilon)^2}{(1+\epsilon)^6\|U\|\,\|V\|} \\ &\geqslant \frac{(1-\epsilon)^3}{(1+\epsilon)^6}\nu(T_2T_1 + \mathscr{K}(C_0(K)))\,.\end{aligned}$$

Since this inequality holds true for arbitrary $\epsilon \in (0,1)$, we conclude that ν is submultiplicative.

Finally, the inequality (4.4.14) follows from the fact that if $I_{c_0} - UTV \in \mathscr{K}(c_0)$, then

$$1 = \|I_{c_0}\|_e = \|UTV\|_e \leqslant \|U\|\,\|T\|_e\,\|V\|\,. \qquad \square$$

Proof of Theorem 4.4.5. Proposition 4.4.18 shows that conditions (a) and (b) are equivalent. (Note that Proposition 4.4.18 applies because the unit ball of $C_0(K)^*$ is weak* sequentially compact by Theorem 4.4.17(i).)

We shall now complete the proof by showing that conditions (b), (c) and (d) are equivalent. Lemma 4.1.16 shows that (b) implies (c), which trivially implies (d), so it only remains to show that (d) implies (b).

To this end, suppose that $\mathscr{B}(C_0(K))/\mathscr{K}(C_0(K))$ is incompressible, and consider the identity map ι on $\mathscr{B}(C_0(K))/\mathscr{K}(C_0(K))$, which is obviously an algebra isomorphism. Proposition 4.4.19 shows that ι is continuous if we endow its domain with the essential norm and its codomain with the algebra norm ν given by (4.4.13). Consequently the hypothesis implies that ι is bounded below; that is, we can find a constant $\eta > 0$ such that

$$\nu(T + \mathscr{K}(C_0(K))) \geqslant \eta\|T\|_e \qquad (T \in \mathscr{B}(C_0(K))) .$$

Note that $\eta \leqslant 1$ by (4.4.14). We claim that (b) is satisfied for any constant $C > 1/\eta$. Indeed, let $T \in \mathscr{B}(C_0(K))$ be a non-compact operator with $\|T\|_e = 1$, and take $\xi \in (1/C, \eta)$. Then we have $\nu(T + \mathscr{K}(C_0(K))) \geqslant \eta > \xi$, so the definition of ν implies that there are operators $U_1 \in \mathscr{B}(C_0(K); c_0)$ and $V_1 \in \mathscr{B}(c_0; C_0(K))$ such that

$$I_{c_0} - U_1TV_1 \in \mathscr{K}(c_0) \qquad \text{and} \qquad \frac{1}{\|U_1\|\,\|V_1\|} > \xi .$$

Since $C\xi > 1$, Lemma 4.4.9 shows that $U_2(U_1TV_1)V_2 = I_{c_0}$ for some operators $U_2, V_2 \in \mathscr{B}(c_0)$ with $\|U_2\|\,\|V_2\| < C\xi$. Hence the operators $U = U_2U_1$ and $V = V_1V_2$ have the required properties. □

4.4.2 Application of Theorem 4.4.5 to certain $C_0(K)$ spaces.

Definition 4.4.21. Let K be a subset of a topological space. The *Cantor-Bendixson derivative* K' of K is defined as

$$K' = \{x \in K : x \in \overline{K \setminus \{x\}}\} .$$

In words, K' is the set of all points in K which are not isolated in K.

Definition 4.4.22. Let K be a topological space. We say that K is *zero-dimensional* if every $k \in K$ has a neighbourhood base consisting of clopen sets.

In this subsection, we apply Theorem 4.4.5 to $C_0(K)$ where K is a zero-dimensional, locally compact Hausdorff space for which $K'' = \emptyset$. Explicitly, the following theorem is our goal.

Theorem 4.4.23. *Let K be a zero-dimensional, locally compact, Hausdorff space K with $K'' = \emptyset$. The Calkin algebra $C_0(K)$ is uniformly incompressible.*

Before embarking on the proof of Theorem 4.4.23, let us show how we can apply it to establish our desired conclusion, Theorem 4.1.2(iii) (restated as Corollary 4.4.25 below). For the construction of Mrówka spaces, see Section 1.6.

The setting is as follows: Recall that in the context of an uncountable, almost disjoint family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$, we say that $C_0(K_{\mathcal{A}})$ has *few operators* if every bounded operator on $C_0(K_{\mathcal{A}})$ is the sum of a scalar multiple of the identity operator and an operator which has separable range.

The existence of almost disjoint families $\mathcal{A}$ of subsets of $\mathbb{N}$ for which $C_0(K_{\mathcal{A}})$ has few operators was discussed in Section 1.6, where we also described how when $C_0(K_{\mathcal{A}})$ has few operators, $\mathscr{B}(C_0(K_{\mathcal{A}}))$ contains only four closed ideals, namely

$$\{0\} \subsetneq \mathscr{K}(C_0(K_{\mathcal{A}})) \subsetneq \mathscr{X}(C_0(K_{\mathcal{A}})) \subsetneq \mathscr{B}(C_0(K_{\mathcal{A}})) \,. \tag{4.4.25}$$

We require the following lemma, which shows why at the end of the proof of Corollary 4.4.25, we may apply Theorem 4.4.23.

Lemma 4.4.24. *Let K be a locally compact, scattered, Hausdorff space. Then K is zero-dimensional.*

Proof. Let L be a non-empty, connected subset of K. Because K is scattered, L must contain some isolated point, l. Since L is connected and l is isolated in L, we must have that $L = \{l\}$. Thus, any subset of K containing more than one point is disconnected, i.e., K is totally disconnected. For locally compact spaces, the property of being totally disconnected implies zero-dimensionality (see e.g., [16, Theorem 6.2.9]). □

Corollary 4.4.25. *Let $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ be an uncountable, almost disjoint family for which $C_0(K_{\mathcal{A}})$ has few operators. Then every quotient of $\mathscr{B}(C_0(K_{\mathcal{A}}))$ by one of its closed ideals has a unique algebra norm.*

Proof. Every algebra homomorphism from $\mathscr{B}(C_0(K_{\mathcal{A}}))$ into a Banach algebra is continuous by [38, Corollary 39], so Proposition 4.1.6 shows that the quotient norm on $\mathscr{B}(C_0(K_{\mathcal{A}}))/\mathscr{J}$ is maximal for every closed ideal $\mathscr{J}$ of $\mathscr{B}(C_0(K_{\mathcal{A}}))$.

According to (4.4.25), $\mathscr{B}(C_0(K_{\mathcal{A}}))$ contains only four closed ideals $\mathscr{J}$. We check that the quotient norm on $\mathscr{B}(C_0(K_{\mathcal{A}}))/\mathscr{J}$ is minimal in each of these cases.

For $\mathscr{J} = \{0\}$, we have that $\mathscr{B}(X)/\mathscr{J} = \mathscr{B}(X)$, so the result follows from Theorem 4.1.9. The result is trivial for the case $\mathscr{J} = \mathscr{X}(C_0(K_{\mathcal{A}}))$ because the 'few operators' property of $C_0(K_{\mathcal{A}})$ tells us that $\mathscr{B}(C_0(K_{\mathcal{A}}))/\mathscr{X}(C_0(K_{\mathcal{A}})) \cong \mathbb{K}$. The result is also trivial for $\mathscr{J} = \mathscr{B}(C_0(K_{\mathcal{A}}))$ because $\mathscr{B}(C_0(K_{\mathcal{A}}))/\mathscr{B}(C_0(K_{\mathcal{A}})) = \{0\}$.

Finally, we remark that it is clear from the definition of $K_{\mathcal{A}}$ from Section 1.6 that $K_{\mathcal{A}}' = \{y_A : A \in \mathcal{A}\}$ and thus $K_{\mathcal{A}}'' = \emptyset$. Further, because $K_{\mathcal{A}}$ is scattered and locally compact, Lemma 4.4.24 shows that we can combine Theorem 4.4.23 with Lemma 4.1.13 to reach the conclusion for $\mathscr{J} = \mathscr{K}(C_0(K_{\mathcal{A}}))$. □

It remains to prove Theorem 4.4.23. To execute our strategy, we verify that Theorem 4.4.5(a) is satisfied for every locally compact, zero-dimensional, Hausdorff space K with $K'' = \emptyset$. To facilitate the presentation of this argument, we introduce the following list, in which K always denotes a locally compact, zero-dimensional, Hausdorff space with $K'' = \emptyset$.

- A (finite or infinite) sequence (f_n) of scalar-valued functions, all defined on the set K, is *disjoint* if $f_m^{-1}(0) \cup f_n^{-1}(0) = K$ whenever the indices m and n are distinct. Note that this is a slight weakening of the conventional notion of disjointly supported functions since the support of a function in a $C(K)$ space is defined as the closure of the set $\{k \in K : f(k) \neq 0\}$, however our definition is strong enough to ensure that a sequence $(f_n)_{n\in\mathbb{N}}$ of functions of norm 1 in $C_0(K)$ which is disjoint in this sense can satisfy (4.4.1).

- Let $\mathbb{1}_L \colon K \to \{0,1\}$ denote the indicator function of a subset L of K.

- Let $\mathcal{A}$ be a finite set of pairs (x, U) where U is compact and open in K, $x \in U$, and whenever (x, U) and (y, V) are distinct elements of $\mathcal{A}$, we have that $x \notin V$. Define the operator $P_{\mathcal{A}}$ on $C_0(K)$ by

$$P_{\mathcal{A}}(f) := \sum_{(x,U)\in\mathcal{A}} f(x)\mathbb{1}_U ,$$

  for every $f \in C_0(K)$.

- Because $K'' = \emptyset$, we have that K' is a discrete set. Let $k \in K$. Using the zero-dimensionality of K, take an open subset U_1 of K with $U_1 \cap K' = \{k\}$.

Now, using the fact that K is locally compact, take $U_2 \subset K$ to be a compact neighbourhood of k. Now, since $U_1 \cap U_2$ is a neighbourhood of K, by zero-dimensionality, we can take a clopen neighbourhood $U(k) \subset U_1 \cap U_2$ of K.

Then $U(k)$ is compact and open neighbourhood of k, for which $U(k) \cap K' = \{k\}$. Thus, if $l \in K'$ with $l \neq k$, we must have that $U(k) \cap U(l) \subset K \setminus K'$.

Let these sets $U(k)$ be fixed for the rest of the chapter.

The author would like to thank Richard Smith for his suggestion of notation for the projections $P_{\mathcal{A}}$, which considerably simplified the calculations in this subsection and has improved its readability.

Lemma 4.4.26. *Let K be a locally compact, zero-dimensional Hausdorff space with $K'' = \emptyset$, and for each $k \in K'$, let $U(k)$ be defined as above. Take*

$$f_1, \ldots, f_n \in \operatorname{span}\{\mathbb{1}_{U(k)}, \mathbb{1}_{\{l\}} : k \in K', l \in K \setminus K'\}$$

for some $n \in \mathbb{N}$. There exist finite subsets $L \subset K \setminus K'$ and $M \subset K'$ such that

$$f_1, \ldots, f_n \in \operatorname{span}\{\mathbb{1}_{U(k) \setminus L}, \mathbb{1}_{\{l\}} : k \in M, l \in L\} ,$$

and for which

$$U(k) \cap U(m) \subseteq L$$

whenever k and m are distinct elements of M.

Proof. For each $j \in \{1, \ldots, n\}$, define $A_j = \{k \in K' : f_j(k) \neq 0\}$, which must be finite. Further, there is a finite subset B_j of $K \setminus K'$ for which

$$f_j = \sum_{l \in B_j} \lambda_l^j \mathbb{1}_{\{l\}} + \sum_{k \in A_j} \lambda_k^j \mathbb{1}_{U(k)} ,$$

where λ_l^j is some scalar for every $l \in B_j \cup A_j$.

Define the set

$$D = \bigcup \left\{ U(k) \cap U(m) : \, k, m \in \bigcup_{j=1}^{n} A_j, k \neq m \right\} .$$

For any distinct $k, m \in K'$, since $U(k)$ may not contain m, we have that $U(k) \cap U(m)$ is a compact subset of the discrete set $K \setminus K'$. Thus $U(k) \cap U(m)$ is finite, and hence D is a finite union of finite sets so is also finite.

So, to complete our proof, we define the sets $L = \bigcup_{j=1}^{n} B_j \cup D$, and $M = \bigcup_{j=1}^{n} A_j$. The claim concerning the pairwise disjointness of the sets $U(k) \setminus L$ for $k \in M$ follows easily from the definition of D. □

Lemma 4.4.27. *Let K be a locally compact, zero-dimensional Hausdorff space with $K'' = \emptyset$, and let $\mathcal{A}$ be a finite set of pairs (x, U) where U is a compact and open subset of K, $x \in U$, and whenever (x, U), and (y, V) are distinct in $\mathcal{A}$, we have that $x \notin V$. The following properties of $P_{\mathcal{A}}$ hold.*

(i) $P_{\mathcal{A}}$ is a projection.

(ii) Suppose that $\mathcal{A}$ is non-empty and that the sets U for which $(x, U) \in \mathcal{A}$ for some $x \in K$, are disjoint from one another. Then $\|P_{\mathcal{A}}\| = 1$. Further, if $\mathcal{A}$ contains only pairs of the form $(x, \{x\})$ for $x \in K$, then $\|I - P_{\mathcal{A}}\| = 1$ also.

Proof. (i) Let $f \in C_0(K)$. Then

$$P_{\mathcal{A}}^2 f = \sum_{(x,U)\in\mathcal{A}} (P_{\mathcal{A}} f(x)) \mathbb{1}_U = \sum_{(x,U)\in\mathcal{A}} f(x) \mathbb{1}_U = P_{\mathcal{A}} f \ .$$

(ii) The first claim is easy to see from the definition of $P_{\mathcal{A}}$. For the second, we notice that if $\mathcal{A}$ is as specified and $f \in C_0(K)$, then $(I - P_{\mathcal{A}}) f$ acts identically to f on K except for on the set of $x \in K$ for which $(x, \{x\}) \in \mathcal{A}$, which it maps to zero. □

Proof of Theorem 4.4.23. As previously mentioned, our plan will be to verify that Theorem 4.4.5(a) is satisfied for the zero-dimensional, locally compact space K for which $K'' = \emptyset$. More precisely, fix $\delta \in (0, \frac{1}{3})$, and let $T \in \mathscr{B}(C_0(K))$ be any operator which satisfies $\|T\|_e = 1$. By recursion, we shall construct a disjoint sequence $(f_n)_{n\in\mathbb{N}}$ in $B_{C_0(K)}$ such that

$$\|T f_n\| \geqslant \delta \quad \text{for every} \quad n \in \mathbb{N} \ , \tag{4.4.26}$$

from which the conclusion will follow.

To facilitate this recursive construction, we consider two distinct cases, depending on the norms of the restrictions of a certain family of operators to the subspace

$$X_{00} := \operatorname{span}\{\mathbb{1}_{\{l\}} : l \in K \setminus K'\} .$$

Case 1: Suppose that

$$\|T(I - P_E)|_{X_{00}}\| > \delta \tag{4.4.27}$$

for every finite set E of pairs of the form $(x, \{x\})$ where $x \in K \setminus K'$. In this case, we shall construct the disjoint sequence $(f_n)_{n\in\mathbb{N}}$ satisfying (4.4.26) inside $B_{X_{00}}$.

To start the recursion, we apply (4.4.27) with $E = \emptyset$ to see that $\|T|_{X_{00}}\| > \delta$, so we can select $f_1 \in B_{X_{00}}$ such that $\|Tf_1\| > \delta$.

Now assume recursively that, for some $n \in \mathbb{N}$, we have selected a disjoint sequence of functions $(f_j)_{j=1}^n$ in $B_{X_{00}}$ for which $\|Tf_j\| > \delta$ for each $j \in \{1, \ldots, n\}$. By another application of (4.4.27), this time with E being the set of pairs of the form $(x, \{x\})$ where $x \in \bigcup_{j=1}^n \operatorname{supp}(f_j)$, we can find $g_{n+1} \in B_{X_{00}}$ such that

$$\|T(I - P_E)g_{n+1}\| > \delta .$$

Define $f_{n+1} = (I - P_E)g_{n+1} \in X_{00}$. Then we have that $\|Tf_{n+1}\| > \delta$, and that the support of f_{n+1} is contained entirely in $(K \setminus K') \setminus E$, making it disjoint from f_j for every $j \in \{1, \ldots, n\}$ by the definition of E. Further, $\|f_{n+1}\| \leqslant \|g_{n+1}\| \leqslant 1$ by Lemma 4.4.27(ii).

Case 2: Suppose that there exists a finite set E of pairs of the form $(x, \{x\})$ where $x \in K \setminus K'$ for which

$$\|T(I - P_E)|_{X_{00}}\| \leqslant \delta . \tag{4.4.28}$$

Set $S = T(I - P_E)$. Because P_E is finite rank, we have that $\|S\|_e = \|T\|_e = 1$. Let

$$Z_{00} = \operatorname{span}\{\mathbb{1}_{U(k)}, \mathbb{1}_{\{l\}} : k \in K', l \in K \setminus K'\}$$

and notice that Z_{00} is dense in $C_0(K)$.

We shall construct a disjoint sequence $(g_n)_{n\in\mathbb{N}}$ in $B_{Z_{00}}$ for which $\|Sg_n\| > \delta$ for all

$n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ we shall let $f_n = (I - P_E)g_n$, so that (f_n) is a disjoint sequence, which belongs to $B_{Z_{00}}$ by Lemma 4.4.27(ii), and $\|Tf_n\| = \|Sg_n\| > \delta$, completing the proof.

To begin, since $\|S\| \geqslant \|S\|_e > \delta$, we can simply take $g_1 \in B_{Z_{00}}$ for which $\|Sg_1\| > \delta$. Next, suppose that we have chosen disjoint functions $g_1, \dots, g_n \in B_{Z_{00}}$ such that $\|Sg_j\| > \delta$ for all $j \in \{1, \dots, n\}$.

Use Lemma 4.4.26 to find finite subsets $L \subset K \setminus K'$ and $M \subset K'$ for which

$$g_1, \dots, g_n \in \operatorname{span}\{\mathbb{1}_{U(k)\setminus L}, \mathbb{1}_{\{l\}} : k \in M, l \in L\} ,$$

such that the set $\{U(k) \setminus L : k \in M\}$ is pairwise disjoint. Now, set

$$\mathcal{A} = \{(l, \{l\}), (k, U(k) \setminus L) : l \in L, k \in M\} .$$

Since $P_{\mathcal{A}}$ is finite-rank, we have that $\|S(I - P_{\mathcal{A}})\| \geqslant \|S\|_e = 1$. Thus, using the density of Z_{00} in $C_0(K)$, we can take some $h_{n+1} \in B_{Z_{00}}$ for which

$$\|S(I - P_{\mathcal{A}})h_{n+1}\| \geqslant 1 > 3\delta .$$

For every $k \in K'$, if there exists some $j \in \{1, \dots, n\}$ with $g_j(k) \neq 0$, then we have $k \in M$ with $(k, U(k) \setminus L) \in \mathcal{A}$, from which we see that

$$\begin{aligned}(I - P_{\mathcal{A}})h_{n+1}(k) &= h_{n+1}(k) - \sum_{(x,U)\in\mathcal{A}} h_{n+1}(x)\mathbb{1}_U(k) \\ &= h_{n+1}(k) - h_{n+1}(k)\mathbb{1}_{U(k)\setminus L}(k) = 0 . \end{aligned} \tag{4.4.29}$$

We next apply Lemma 4.4.26 once more, this time to the functions

$$g_1, \dots, g_n, (I - P_{\mathcal{A}})h_{n+1} .$$

From this, we obtain a finite subset L_0 of $K \setminus K'$ and a finite subset M_0 of K' for which

$$g_1, \dots, g_n, (I - P_{\mathcal{A}})h_{n+1} \in \operatorname{span}\{\mathbb{1}_{U(k)\setminus L_0}, \mathbb{1}_{\{l\}} : k \in M_0, l \in L_0\} ,$$

with the sets $U(k) \setminus L_0$ for $k \in M_0$ pairwise disjoint. Define $\Delta = \{(l, \{l\}) : l \in L_0\}$, and let

$$g_{n+1} = (I - P_\Delta)(I - P_{\mathcal{A}})h_{n+1} \ .$$

We have that

$$\|Sg_{n+1}\| \geqslant \|S(I - P_{\mathcal{A}})h_{n+1}\| - \|SP_\Delta(I - P_{\mathcal{A}})h_{n+1}\| > 3\delta - 2\delta = \delta \ ,$$

where the second inequality holds by (4.4.28) since P_Δ has its image in X_{00} and $(I - P_{\mathcal{A}})$ can have norm at most 2.

We complete the proof by showing that the function g_{n+1} is disjoint from the previous functions $g_1, \ldots, g_n$. Indeed, if $k \in K'$, then $g_{n+1}(k) = (I - P_{\mathcal{A}})h_{n+1}(k)$, which we showed via (4.4.29) to be 0 whenever $g_j(k) = 0$ for any $j \in \{1, \ldots, n\}$. Moreover, if $l \in K \setminus K'$ with $g_{n+1}(l) \neq 0$, then $l \notin L_0$, so there must be some $k \in M_0$ for which $l \in U(k) \setminus L_0$, and this k must be unique because the sets $U(k) \setminus L_0$ are pairwise disjoint. Thus, if $j \in \{1, \ldots, n\}$, then $g_{n+1}(k) = g_{n+1}(l) \neq 0$. So, our preceding argument implies also that $g_j(k) = g_j(l) = 0$, as required.

□

Bibliography

[1] F. Albiac and N. J.Kalton, *Topics in Banach Space Theory, Second Edition*, Graduate Texts in Mathematics **92**, Springer-Verlag, 2016.

[2] D. Amir and J. Lindenstrauss, The structure of weakly compact sets in Banach spaces, *Ann. of Math.* **88** (1968), 35–46.

[3] S. A. Argyros and R. G. Haydon, A hereditarily indecomposable $\mathscr{L}_\infty$-space that solves the scalar-plus-compact problem, *Acta Math.* **206** (2011), 1–54.

[4] M. Arnott and N. J. Laustsen, Classifying the closed ideals of bounded operators on two families of non-separable classical Banach spaces, *J. Math. Anal. Appl.* (2021), 125105.

[5] M. Arnott and N. J. Laustsen, Uniqueness of algebra norm on quotients of the algebra of bounded operators on a Banach space, Work in progress.

[6] K. Astala and H.-O. Tylli, On the bounded compact approximation property and measures of noncompactness, *J. Funct. Anal.* **70** (1987), 388–401.

[7] K. Beanland, T. Kania and N. J. Laustsen, The lattices of closed operator ideals on the Tsirelson and Schreier spaces, *J. Funct. Anal.* **279** (2020), 108668.

[8] J. W. Calkin, Two-sided ideals and congruences in the ring of bounded operators in Hilbert space, *Ann. of Math.* **42** (1941), 839–873.

[9] H. G. Dales, A discontinuous homomorphism from $C(X)$, *Amer. J. Math.* **101** (1979), 647–734.

[10] H. G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monogr. Ser. 24, Clarendon Press, Oxford, 2000.

[11] H. G. Dales and W. H. Woodin, *An introduction to independence for analysts,* London Math. Soc. Lecture Note Series 115, Cambridge University Press, 1987.

[12] M. Daws, Closed ideals in the Banach algebra of operators on classical non-separable spaces, *Math. Proc. Cambridge Philos. Soc.* **140** (2006), 317–332.

[13] J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Mathematics **92**, Springer-Verlag, 1983.

[14] P. N. Dowling, N. Randrianantoanina, and B. Turett, Remarks on James's distortion theorems II, *Bull. Austral. Math. Soc.* **59** (1999), 515–522.

[15] M. Eidelheit, On isomorphisms of rings of linear operators, *Studia Math.* **9** (1940), 97–105.

[16] R. Engelking, *General Topology (Revised and Completed Edition)*, Heldermann Verlag, 1989.

[17] J. Esterle, Injection de semi-groupes divisibles dans des algèbres de convolution et construction d'homomorphismes discontinus de $\mathcal{C}(K)$, *Proc. London Math. Soc.* **36** (1978), 59–85.

[18] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, *Banach Space Theory; The Basis for Linear and Nonlinear Analysis*, Canadian Mathematical Society, (2011).

[19] D. Freeman, Th. Schlumprecht and A. Zsák, Addendum: Closed ideals of operators between the classical sequence spaces, *Bull. Lond. Math. Soc* **53** (2021), 593-595

[20] D. Freeman, Th. Schlumprecht and A. Zsák, Banach spaces for which the space of operators has $2^{\mathfrak{c}}$ closed ideals, *Forum Math. Sigma* **9** (2021), E27.

[21] E. M. Galego and A. Plichko, On Banach spaces containing complemented and uncomplemented subspaces isomorphic to c_0, *Ext. Math.* **18** (2003), 315–319.

[22] I. C. Gohberg, A. S. Markus, and I. A. Feldman, Normally solvable operators and ideals associated with them, *Amer. Math. Soc. Translat.* **61** (1967), 63–84, Russian original in *Bul. Akad. Štiince RSS Moldoven* **10** (76) (1960), 51–70.

[23] B. Gramsch, Eine Idealstruktur Banachscher Operatoralgebren, *J. Reine Angew. Math.* **225** (1967), 97–115.

[24] J. Hagler and W. B. Johnson, On Banach spaces whose dual balls are not weak* sequentially compact, *Israel J. Math.* **28** (1977), 325–330.

[25] P. Hájek, V. Montesinos Santalucía, J. Vanderwerff, and V. Zizler, *Biorthogonal systems in Banach spaces,* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC **26**, Springer-Verlag, New York, 2008.

[26] B. Horváth and T. Kania, Surjective homomorphisms from algebras of operators on long sequence spaces are automatically injective, preprint available at arXiv:2007.14112.

[27] B. Horváth and N. J. Laustsen, Kernels of operators on certain Banach spaces associated with almost disjoint families, preprint available at arXiv:2211.12795.

[28] R. C. James, Uniformly non-square Banach spaces, *Ann. of Math.* **80** (1964), 542–550.

[29] B. E. Johnson, Continuity of homomorphisms of algebras of operators, *J. London Math. Soc.* **42** (1967), 537–541.

[30] W. B. Johnson, T. Kania and G. Schechtman, Closed ideals of operators on and complemented subspaces of Banach spaces of functions with countable support, *Proc. Amer. Math. Soc.* **144** (2016), 4471–4485.

[31] W. B. Johnson, N. C. Phillips, and G. Schechtman, The IHABB property for quotients of $L(L^p)$, Work in progress.

[32] W. B. Johnson, G. Pisier and G. Schechtman, Ideals in $L(L_1)$, *Math. Ann.* **376** (2020), 693–705.

[33] W. B. Johnson and G. Schechtman, The number of closed ideals in $L(L_p)$, *Acta Math.* **227** (2021), 101-113

[34] N.Kalton, Spaces of compact operators, *Math. Ann.* **208** (1974), 267-278

[35] T. Kania and T. Kochanek, The ideal of weakly compactly generated operators acting on a Banach space, *J. Operator Th.* **71** (2014), 455–477.

[36] T. Kania and N. J. Laustsen, Ideal structure of the algebra of bounded operators acting on a Banach space, *Indiana Univ. Math. J.* **66** (2017), 1019–1043.

[37] P. Koszmider, On decompositions of Banach spaces of continuous functions on Mrówka's spaces, *Proc. Amer. Math. Soc.* **133** (2005), 2137–2146.

[38] P. Koszmider and N. J. Laustsen, A Banach space induced by an almost disjoint family, admitting only few operators and decompositions, *Adv. Math.* **381** (2021), 107613.

[39] N. J. Laustsen, MAGIC Lecture notes: Banach spaces and their operators.

[40] N. J. Laustsen, R. J. Loy, and C. J. Read, The lattice of closed ideals in the Banach algebra of operators on certain Banach spaces, *J. Funct. Anal.* **214** (2004), 106–131.

[41] N. J. Laustsen, Th. Schlumprecht, and A. Zsák, The lattice of closed ideals in the Banach algebra of operators on a certain dual Banach space, *J. Op. Th.* **56** (2006), 391–402.

[42] N. J. Laustsen and J. T. White, Subspaces that can and cannot be the kernel of a bounded operator on a Banach space, *Banach algebras and applications. Proceedings of the International Conference held at the University of Oulu, July 3-11, 2017* (editor: M. Filali), De Gruyter Proceedings in Mathematics (2020), 189–196.

[43] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, 1977.

[44] H. P. Lotz, N. T. Peck, and H. Porta, Semi-embeddings of Banach spaces, *Proc. Edinb. Math. Soc.* **22** (1979), 233–240.

[45] E. Luft, The two-sided closed ideals of the algebra of bounded linear operators of a Hilbert space, *Czechoslovak Math. J.* **18** (1968), 595–605.

[46] M. J. Meyer, On a topological property of certain Calkin algebras, *Bull. London Math. Soc.* **24** (1992), 591–598.

[47] M. J. Meyer, Lower bounds for norms on certain algebras, *Illinois J. Math.* **39** (1995), 567–575.

[48] P. Motakis, Separable spaces of continuous functions as Calkin algebras, *preprint* (2021), available at arXiv:2110.10868.

[49] P. Motakis, D. Puglisi and D. Zisimopoulou, A hierarchy of Banach spaces with $C(K)$ Calkin algebras, *Indiana Univ. Math. J.* **65** (2016), 39–67.

[50] I. Namioka and R. R. Phelps, Banach spaces which are Asplund spaces, *Duke Math. J.* **42** (1975), 735–750.

[51] T. W. Palmer, *Banach algebras and the general theory of* **-algebras,* Vol. I, Cambridge University Press, 1994.

[52] A. Pełczyński, On the isomorphism of the spaces m and M, *Bull. Acad. Pol. Sci.* **6** (1958), 695–696.

[53] A. Pełczyński, On strictly singular and strictly cosingular operators. I. Strictly singular and strictly cosingular operators in $C(S)$-spaces. *Bull. Acad. Pol. Sci.* **13** (1965), 31–37.

[54] A. Pietsch, *Operator ideals*, North Holland, 1980.

[55] H. P. Rosenthal, On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from $L^p(\mu)$ to $L^r(\nu)$, *J. Funct. Anal.* **4** (1969) 176–214

[56] H. P. Rosenthal, On relatively disjoint families of measures, with some applications to Banach space theory, *Studia Math.* **37** (1970), 13–36.

[57] W. Rudin, Continuous functions on compact spaces without perfect subsets, *Proc. Amer. Math. Soc.* **8** (1957), 39–42.

[58] W. Sierpiński, Sur une décomposition d'ensembles, *Monatsh. Math. Phys.* **35** (1928), 239–242.

[59] G. Sirotkin and B. Wallis, Sequence-singular operators, *J. Math. Anal. Appl.* **443** (2016), 1208–1219.

[60] M. Tarbard, Hereditarily indecomposable, separable $\mathscr{L}_\infty$ Banach spaces with ℓ_1 dual having few but not very few operators, *J. London Math. Soc.* **85** (2012), 737–764.

[61] H.-O. Tylli, The essential norm of an operator is not self-dual, *Israel J. Math.* **91** (1995), 93–110.

[62] H.-O. Tylli, Exotic algebra norms on quotient algebras of the bounded linear operators, *Proceedings of the* 3^{rd} *International Conference on Topological Algebra and Applications, Oulu 2–6 July, 2001,* (editors: M. Filali and J. Arhippainen), Oulu University Press, Oulu (2004), 210–224.

[63] G. K. Ware, *Uniqueness of norm properties of Calkin algebras,* PhD thesis, Australian National University, 2014; available online at https://openresearch-repository.anu.edu.au/handle/1885/14510

[64] H. M. Wark, A non-separable reflexive Banach space on which there are few operators, *J. London Math. Soc.* **64** (2001), 675–689.

[65] H. M. Wark, A non-separable uniformly convex Banach space on which there are few operators, *Studia Math.* **241** (2018), 241–256.

[66] J. T. White, *Banach algebras on groups and semigroups*, PhD thesis, Lancaster University, 2018; available online at https://doi.org/10.17635/lancaster/thesis/307

[67] B. Yood, Homomorphisms on normed algebras, *Pacific J. Math.* **8** (1958), 373–381.