Lancaster University
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Arens Regularity of Banach Function Algebras and Decomposable Blaschke Products whose Degree is a Power of 2

M.Eugenia Celorrio Ramirez

This thesis is submitted for the degree of
Doctor of Philosophy

June 2023

A mi yo del pasado.

Lo conseguimos.

Abstract

of the thesis

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Submitted for the degree of Doctor of Philosophy

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This thesis presents three pieces of work.

Within the first two thirds of the thesis, we study Arens regularity of Banach algebras. We first study Arens regularity of weighted semigroup algebras that arise from totally ordered semilattices. This is a natural continuation of [24], where they focus on studying Arens regularity of the unweighted case. We provide a sufficient condition for when a weighted semigroup algebra is not strongly Arens irregular and a characterization of Arens regularity of the weighted semigroup algebra. We then focus on three specific totally ordered semilattices, $\mathbb{N}_\wedge$, $\mathbb{N}_\vee$ and $\mathbb{Z}_\vee$ to obtain stronger results than those obtained for a generic totally ordered semilattice.

Later on, we focus on two different Banach sequence algebras, the James p^{th} algebra and the Feinstein algebra. Amongst other properties, we prove that the Feinstein algebra is Arens regular, which provides a second example of an Arens regular natural Banach sequence algebra that is not an ideal in its bidual, the first one being the remarkable example obtained in [7]. We study whether J_p is a BSE algebra with a BSE norm, for $1 < p < \infty$. We finish this part by studying the Arens regularity of the tensor products of some of the algebras studied in this thesis.

In the final part of the thesis, we focus on Blaschke products. We study the decomposability of a finite Blaschke product B of degree 2^n into n degree-2 Blaschke products, examining the connections between Blaschke products, the elliptical range theorem, Poncelet theorem, and the monodromy group. We show that if the numerical range of the compression of the shift operator, $W(S_B)$, with B a Blaschke product of degree n, is an ellipse, then B can be written as a composition of lower-degree Blaschke products that correspond to a factorization of the integer n. We also show that a Blaschke product of degree 2^n with an elliptical Blaschke curve has at most n distinct critical values, and we use this to examine the monodromy group associated with a regularized Blaschke product B. We prove that if B can be decomposed into n degree-2 Blaschke products, then the monodromy group associated with B is the wreath product of n cyclic groups of order 2. Lastly, we study the group of invariants of a Blaschke product B of order 2^n when B is a composition of n Blaschke products of order 2.

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Sometimes, the walks of life of two people are parallel for some time, which makes walking easier and, if you are lucky, can end up in a long lasting friendship. I count myself extremely lucky, since on the very first day of my PhD I met Nikoletta Alexandri, who was also starting hers, and ever since we have shared our way and been part of each other's family in the UK. Part of this family are also Martina Arioli, Marta Riquelme and Neha Jindal, and I couldn't be happier about that.

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Gracias

Declaration

This thesis is my own work and contains nothing which is the outcome of work done in collaboration with others, except:

- Section 4.2.1 is joint work with Yemon Choi.
- Chapter 5 is joint work with Asuman Güven Aksoy, Francesca Arici and Pamela Gorkin.

This thesis has not been submitted in substantially the same form for the award of any other degree or qualification.

This thesis does not exceed the permitted maximum of 80,000 words.

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CHAPTER 1

Introduction

This thesis deals with two seemingly distant topics. Within the first two thirds of the thesis, we are concerned with Banach algebras, and more specifically with Arens regularity of Banach algebras. During the study of some specific Banach algebras, we run into the Hardy space, and in the literature regarding this space there is a notable amount of discussion about the relation between the compressions of the shift operator and Blaschke products. This led to the interest in the research reflected in the last third of the thesis, where we study some properties of Blaschke products.

1.1. Arens regularity of Banach algebras

More than 70 years ago, in [3] Richard Arens proved that, given a Banach algebra A, its product can be naturally extended to its bidual A'' in a way that A is a closed subalgebra of A''. This extension can be done in two completely symmetrical ways. From these, the first ($\Box$) and second ($\Diamond$) Arens products arise. These products are one-sided $\sigma(A'', A')$ continuous (each one on a different side). These two products can be one in disguise, or they can be different products in A'', which sparked the need to know which algebras belonged to each of these groups. Arens proved in [4] that the algebra $\ell^1(\mathbb{N})$ with pointwise product is Arens regular, but that $\ell^1(\mathbb{N})$ with the convolution product is not Arens regular. In [11], it is shown that all C^*-algebras are Arens regular and in [79] Young proved that $L^1(G)$ is not Arens regular for any infinite G.

A natural question to ask when we are dealing with non-Arens regular algebras is to what extent these products can differ from one another. The biggest set on which these two products can agree is the whole bidual, while the smallest set in which they have to be equal is A. This sparked the definition of Arens regularity, when the two multiplications coincide in the whole bidual, and, later on, strong Arens irregularity in [21], when both products are as different as possible. This leads naturally to the definition of the left and right topological centres, introduced in [59], which we shall see in Chapter 2.

There is an important fact about Banach algebras that leads to a different way of meassuring non-Arens regularity: The space $\mathrm{WAP}(A)$ of weakly almost periodic functionals is precisely the subspace of A' where the two Arens products coincide (i.e. for $\lambda \in \mathrm{WAP}(A)$, $\langle M\Box N, \lambda\rangle = \langle M\Diamond N, \lambda\rangle$ $(M, N \in A'')$). This follows from Grothendieck's double limit criterion (see for example [67]). Thus, when every functional in A' is weakly almost periodic, we know that A is Arens regular. With this characterization of Arens regularity, it follows naturally that is possible to measure the degree of non-Arens regularity of an algebra A by measuring the size of the quotient space $A'/\mathrm{WAP}(A)$, i.e., by measuring the size of the space $\mathrm{WAP}(A)$ against the size of A'. So, when A is Arens regular, $A'/\mathrm{WAP}(A)$ is trivial. The other extreme case is when $A'/\mathrm{WAP}(A)$ is as large as A'. In [48], Granirer coined the term extreme non-Arens regular for these algebras.

Strong Arens irregularity and extreme non-Arens regularity are not equivalent properties. In fact, neither of them implies the other, as it was seen in [52]. In [34] natural examples of algebras that are extreme non-Arens regular but not strongly Arens irregular can be found. In this thesis, however, we shall not consider extreme non-Arens irregularity.

As it has been pointed out before, the link between Arens regularity and convolution algebras was identified at the birth of the two Arens products, as in [4] Arens already discussed that $\ell^1(\mathbb{N})$ with the convolution product is not Arens regular. This just planted the seed for a very prolific research area.

The research about Arens regularity of (weighted) semigroup algebras is somehow divided into studying the unweighted and the weighted cases. They are obviously intertwined, but sometimes (as we shall see later on in this thesis), the results obtained for one cannot be extrapolated to the other.

For example, in [22], they study the structure of the convolution algebra and of its second dual. In order to do so, they use properties of the semigroup $(\beta S, \Box)$ and, due to the interweaving of mathematical concepts, some interesting questions about this semigroup arise from the study of Banach algebras. The recently published paper [24], which inspired some of the research of this thesis, considers totally ordered semilattices (see definition in Chapter 2) and determines the centres of the bidual of the unweighted convolution algebra, as well as the centres of βS.

A seminal paper on Arens regularity of weighted semigroup algebras is that of Craw and Young [13], where they introduced a criterion to know when a weighted semigroup algebra is Arens regular, giving a characterization in the case that the semigroup is (weakly) cancellative. In [20] they consider weighted convolution algebras on discrete groups and semigroups, concentrating on the rationals with addition and analogous semigroups in the real line. Although the main focus of [35] are weighted group convolution algebras, they also provide some examples about weakly cancellative and right cancellative discrete semigroups.

In [28], Daws provides some results regarding Arens regularity of weighted semigroup algebras, although the main focus is dual Banach algebras as a way to progress on the understanding of Connes-amenability. Duality of Banach algebras also provides a natural context for the study of Arens regularity, since Banach algebras that have the extra property of being dual Banach algebras can be Arens regular under certain conditions. For example, when an algebra is a dual Banach algebra and it is also an ideal in its bidual, it is immediately Arens regular. This is why in this thesis we also study the duality of some of the Banach algebras. A dual Banach algebra is an algebra that is not just dual as a Banach space, but also the predual satisfies certain conditions related to the algebraic structure (see the definition in Chapter 2). For example, von Neumann algebras are the only C^*-algebras that are dual Banach algebras. In the case of semigroup convolution algebras, it was seen in [22] that the unweighted convolution algebra is a dual Banach algebra if and only if the semigroup is weakly cancellative. As we shall see later in the thesis, this story is a bit more complicated when we add a weight.

Finally, other properties that can be linked with Arens regularity, and that will be a focus of study for this thesis, are the notions of BSE algebra and BSE norm, which were introduced in 1990 by Takahasi and Hatori in [74] as an abstraction of the Bochner-Schoenberg-Eberlein theorem. Some documents that focus on these properties in relation to Banach algebras are [25] and [57].

1.2. Blaschke products

Blaschke products are important functions for the study of bounded analytic functions. They play the same role hyperbolically on the unit disc as polynomials play in the Euclidean sense on the plane. Finite Blaschke products are n to 1 maps of the open unit disk into itself, mapping the unit circle to itself. They are holomorphic on an open set containing the closed unit disk and have finitely many zeros in the open unit disk.

More than 200 years ago J. V. Poncelet discovered that if there exists a polygon of n-sides that is inscribed on a given conic and circumscribed about another conic, then infinitely many such polygons exist. This theorem, which is sometimes called Poncelet's porism or Poncelet's closure theorem, has been studied in several settings (see, for example, [29, 32, 37, 49, 63], among others). Later, Darboux found a new proof of the Poncelet closure theorem based on the properties of certain curves, known as Poncelet–Darboux curves, [31]. These are the curves of degree n passing through the intersection points of $n+1$ tangents to a given conic.

If A is a $n \times n$ matrix, then its numerical range $W(A)$ is a convex subset of $\mathbb{C}$ which contains the spectrum of A. Surprisingly, Blaschke products, the numerical range, and Poncelet's theorem are all connected. Roughly speaking, this connection can be described by noting that the convex hull of each of the circumscribing polygons with vertices on $\mathbb{T}$ represents the numerical range of a certain unitary matrix, which is related to a Blaschke product via an operator that is a compression of the shift operator. See, for example, [17] for an overview of these connections.

A Blaschke product is decomposable if it can be written as the composition of two (or more) Blaschke products of degree greater than one (see Chapter 5 for the formal definition). Whether or not a Blaschke product is decomposable is a topic that has strongly drawn the attention of specialists. Amongst other things, this is due to the link between decomposability and some other interesting properties of Blaschke products. For example, in [77] and [14], a visual representation of composition is discussed, and in [15] both algorithmic and geometric arguments are presented, and the relationship between decomposable Blaschke products and curves with the Poncelet property are examined. In [40, 41], Fujimura considered geometric properties of Blaschke products of degree n and the line segments that are tangent to the unit circle at the points B maps to λ on the unit circle, as well as the line segments joining successive points. For degree-4 Blaschke products, she considered those for which the trace of these lines is an ellipse. She showed that an ellipse is inscribed in a quadrilateral that is inscribed in $\mathbb{T}$ if and only if the Blaschke product is a composition of two degree-2 Blaschke products. In [47], the authors gave an operator-theoretic proof of this result. They showed that an ellipse is a Poncelet ellipse (see Chapter 5 for a definition) inscribed in a quadrilateral inscribed in $\mathbb{T}$ if and only if the ellipse is the Blaschke curve associated with a Blaschke product $\widehat{B}(z) = zB(z)$ and the compression of the shift operator associated with the Blaschke product B has elliptical numerical range. Furthermore, the compression of the shift operator associated with a Blaschke product B has elliptical numerical range if and only if $\widehat{B}$ is a composition of two degree-2 Blaschke products. In [53] the authors obtain a similar result for Blaschke products of degree 6. In [68], Ritt classifies decomposability of B in terms of the monodromy group associated with B. See also [12] and [77] for more recent developments.

Finally, the last notion that will be the concern of this thesis is the group of invariants of a finite Blaschke product B. In [8] Cassier and Chalendar showed that the group of invariants of a Blaschke product of degree n is a cyclic group of order n. The group of invariants for infinite Blaschke products with finitely many singularities was considered in [10].

1.3. Thesis outline and published work

The rest of this thesis consists of 4 chapters.

In Chapter 2, we start by discussing the relevant background that will be useful during Chapter 3 and Chapter 4. This is divided in three sections: Preliminaries from Banach spaces, preliminaries from Banach algebras and preliminaries from weighted semigroup algebras.

Chapter 3 is based on a paper that, at the moment of submitting this thesis is under review by the journal. In it, we study Arens regularity of weighted semigroup algebras that arise from totally ordered semilattices. The restriction to cancellative or weakly cancellative semigroups when studying Arens regularity of weighted semigroup algebras is very common. However, the family of semilattices introduced in this chapter includes examples that are not necessarily weakly cancellative. So, our study includes a more general set-up in the study of Arens regularity. The inspiration to consider this specific family of semigroups came from [24], where they focus on studying Arens regularity of the unweighted case. Their main result is a characterization of the property of strong Arens irregularity in terms of some properties of S, a totally ordered semilattice. However, we shall see that this characterization cannot be translated to weighted semigroup algebras. We provide a sufficient condition for when a weighted semigroup algebra is not strongly Arens irregular. In contrast with strong Arens irregularity, we provide a characterization of Arens regularity of the weighted semigroup algebra that depends solely on the properties of the weight. We shall focus afterwards on three specific totally ordered semilattices, $\mathbb{N}_\wedge$, $\mathbb{N}_\vee$ and $\mathbb{Z}_\vee$ to obtain stronger results than those obtained for a generic totally ordered semilattice. For these, we shall also study when they are BSE-algebras and when they have a BSE norm.

Within Chapter 4, Section 4.2.1 on Arens regularity of the Feinstein algebra is joint work with Yemon Choi. We start Chapter 4 by proving some results about mixed identities that will be useful during the rest of the chapter. Later in this chapter, we shall focus on two different Banach sequence algebras, the James p^{th} algebra, and what we call the Feinstein algebra. In this chapter, we study whether J_p is a BSE algebra with a BSE norm. On the other hand, the Feinstein algebra A appears in [19, Example 4.1.46], although it was introduced in a lecture by Joel Feinstein. This algebra has very interesting properties. For example, it is a self-adjoint Banach sequence algebra that is not separable, not Tauberian, not an ideal in its bidual and without an approximate identity ([19, Example 4.1.46]). We shall see that it is not weakly sequentially complete, and we shall also study whether it is a BSE algebra and whether it has a BSE norm. However, the most interesting result obtained is that A is Arens regular. This provides a second example of an Arens regular natural Banach sequence algebra that is not an ideal in its bidual, the first one being the remarkable example obtained in [7]. The method used to prove Arens regularity can be extended to a more general setting, which is something we are working on at the moment. We finish this chapter by studying the Arens regularity of the tensor products of some of the algebras studied in this thesis.

Chapter 5 is based on the paper [1], which is joint work with Asuman Güven Aksoy, Francesca Arici and Pamela Gorkin. We start by discussing the relevant background in Section 5.1 and Section 5.2. Then, in Section 5.3, we study the decomposability of a finite Blaschke product B of degree 2^n into n degree-2 Blaschke products, and we study how this is linked to the elliptical range theorem and Poncelet theorem. We show that if the numerical range of the compression of the shift operator, $W(S_B)$, with B a Blaschke product of degree n, is an ellipse, then B can be written as a composition of lower-degree Blaschke products that correspond to a factorization of the integer n. We provide some examples of Blaschke products that enlighten the result obtained. In Section 5.4, we see that a Blaschke product of degree 2^n with an elliptical Blaschke curve has at most n distinct critical values, and we use this to study the monodromy group associated with a regularized Blaschke product B, in Section 5.5. We prove that if B can be decomposed into n degree-2 Blaschke products, then the monodromy group associated with B is the wreath product of n cyclic groups of order 2. Finally, in Section 5.6, we study the group of invariants of a Blaschke product B of order 2^n when B is a composition of n Blaschke products of order 2.

CHAPTER 2

Preliminaries

In this chapter, we shall introduce the background necessary for the rest of this thesis. The content of this section is not original. We aim to introduce the notation that we shall use during the rest of the thesis as well as give enough background to make this thesis self-contained. In some places we shall introduce some results that are not necessary for the thesis, but that would offer some more context about the material.

2.1. Frequently used notation and definitions

We shall denote by $\mathbb{N}$ the set of the natural numbers and $\mathbb{N}_n := \{k \in \mathbb{N} : k \leq n\}$. By $\mathbb{N}_\infty$ we mean $\mathbb{N} \cup \{+\infty\}$. We consider that $0 \notin \mathbb{N}$. We denote by $\mathbb{Z}$ the set of integers, and similarly, we write $\mathbb{Z}_\infty := \mathbb{Z} \cup \{+\infty\}$ and $\mathbb{Z}_m^n := \{k \in \mathbb{Z} : m \leq k \leq n\}$. For us $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Q}^{+\bullet} := \{q \in \mathbb{Q} : q > 0\}$, $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ the set of complex numbers.

Let X be a set, and let $Y = \mathbb{R}$ or $Y = \mathbb{C}$. If $f : X \to Y$ is a function, we write $\operatorname{supp} f$ for the support of f.

Given an element $x \in X$, we denote by δ_x the characteristic function of x. For $n \in \mathbb{N}$, $\Delta_n \in c_{00}$ is the element such that $\Delta_n(k) = 1$, for $k \leq n$, and $\Delta_n(k) = 0$, for $k > n$.

Finally, we shall write as $\mathfrak{c}$ the cardinality of the continuum.

2.2. Preliminaries from Banach spaces

Let E be a Banach space. We shall denote the closed unit ball of E by $E_{[1]}$. During this thesis we shall write E' for the dual space, and E'' for the second dual, or the *bidual* . The map $\kappa_E : E \to E''$ denotes the canonical embedding. For a functional $\lambda \in E'$ and $x \in E$, $\langle x, \lambda \rangle_{(E,E')}$ denotes the value of λ applied to x. Whenever the context is clear we omit the subscript and we shall write $\langle x, \lambda \rangle$.

Let E, F be two Banach spaces. The set of bounded linear maps from E to F is denoted by $\mathcal{B}(E, F)$, and $\mathcal{B}(E) := \mathcal{B}(E, E)$. For $T \in \mathcal{B}(E, F)$, we write T' for the dual map from F' to E' and $T'' := (T')'$. The set of compact operators from E to F is denoted by $\mathcal{K}(E,F)$ and the set of finite-rank operators by $\mathcal{F}(E,F)$. We denote by $\mathcal{W}(E,F)$ the set of weakly compact operators.

The following theorem is due to Mazur [61]:

THEOREM 2.2.1. *The closure and weak closure of a convex subset of a normed space are the same. In particular, a convex subset of a normed space is closed if and only if it is weakly closed.* $\square$

In 1927, J. Schauder introduced the concept of a Schauder basis:

DEFINITION 2.2.2. Let E be a Banach space. A sequence (x_n) in E is a *Schauder basis* for E if for each $x \in E$ there is a unique sequence (α_n) of scalars such that $x = \sum_{n\in\mathbb{N}} \alpha_n x_n$.

A property that will be useful in some sections is the Radon-Nikodým property.

DEFINITION 2.2.3. A Banach space E has the *Radon-Nikodým property* (RNP) if every closed, bounded subset of E is dentable.

We recall that a bounded subset B of E is dentable if, for each $\varepsilon > 0$ there exists $x \in B$ such that $x \notin \bar{co}(B \setminus B_\varepsilon(x))$. For more about dentable subsets, we recommend to have a look at [26, Section 1].

2.2.1. Tensor products. Let E and F be linear spaces. Then the tensor product of E and F is denoted by $E \otimes F$. For $z \in E \otimes F$, there exist $n \in \mathbb{N}$, $x_1, \cdots, x_n \in E$ and $y_1, \cdots, y_n \in F$ such that

$$z = \sum_{i=1}^{n} x_i \otimes y_i.$$

When $z \neq 0$ we can suppose that $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$ are linearly independent.

When E and F are normed spaces, we can consider the projective tensor norm

$$\|z\|_\pi = \inf\left\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : z = \sum_{i=1}^{n} x_i \otimes y_i, n \in \mathbb{N}\right\} \quad (z \in E \otimes F),$$

where the infimum is taken over all representations of z as an element of $E \otimes F$. Then $(E \otimes F, \|\cdot\|_\pi)$ is a normed space and it is complete if and only if either E or F is finite-dimensional. This leads to the following definition.

DEFINITION 2.2.4. Let E, F be normed spaces. Then the *projective tensor product* of E and F is the completion of $(E \otimes F, \|\cdot\|_\pi)$. We denote it by $E \hat{\otimes} F$.

2.2.2. Schur property. We shall talk about some properties of Banach spaces that will be useful throughout this thesis.

DEFINITION 2.2.5. Let E be a Banach space. We say that E is *weakly sequentially complete* if every weakly Cauchy sequence in E is weakly convergent in E. We say that E has the *Schur property* if given a sequence $\{x_n\}$ that converges weakly to x in E implies that $\{x_n\}$ converges to x in norm in E.

Note that whenever a Banach space has the Schur property, then it is weakly sequentially complete.

The main example of a Banach space with the Schur property is ℓ^1. In fact, any space with the Schur property contains a copy of ℓ^1.

2.3. Preliminaries from Banach algebras

2.3.1. Approximate identities.

DEFINITION 2.3.1. Let A be a Banach algebra. A *left approximate identity* for A is a net (e_ν) in A such that

$$\lim_\nu e_\nu a = a \quad (a \in A).$$

Symmetrically, a *right approximate identity for* A is a net (e_ν) in A such that

$$\lim_\nu a e_\nu = a \quad (a \in A).$$

A net (e_ν) in A is an *approximate identity* if it is both a left and right approximate identity.

Let (e_ν) be a left (respectively right) approximate identity. When (e_ν) is a sequence indexed by $\mathbb{N}$ we say that is a *left* (respectively *right*) *sequential approximate identity.* If there exists $M > 0$ such that $\sup_\nu \|e_\nu\| \leq M$, then we say that (e_ν) is a *left* (respectively *right*) *bounded approximate identity.*

2.3.2. Multiplier algebra. We proceed now to introduce the multiplier algebra of an algebra A. This algebra was originally introduced by Hochschild in [51]. Let A be an algebra. Then a *left multiplier* on A is a linear map $L : A \longrightarrow A$ such that

$$L(ab) = L(a)b \quad (a, b \in A).$$

Similarly, a *right multiplier* on A is a linear map $R : A \longrightarrow A$ such that

$$R(ab) = aR(b) \quad (a, b \in A).$$

A *multiplier* is a pair (L, R) where L, R are left and right multipliers and

$$aL(b) = R(a)b \quad (a, b \in A).$$

The set of all left multipliers is called the *multiplier algebra* of A, it is denoted by $\mathcal{M}(A)$ and it is a unital subalgebra of $\mathcal{L}(A)$, where $\mathcal{L}(A)$ is the algebra of linear maps from A to A.

An ideal I in A is said to be *left faithful in* A (respectively, *right faithful in* A) if, for every $a \in A$, $aI = 0$ implies that $a = 0$ (respectively, $Ia = 0$ implies that $a = 0$). An ideal I is *faithful* if it is both left and right faithful. A is said to be faithful if it is faithful as an ideal in itself. In the case where A is faithful and commutative, every left multiplier is also a right multiplier and vice versa, and so $\mathcal{M}(A)$ is the set of all multipliers of A and it is also a commutative subalgebra of $\mathcal{L}(A)$.

Let A be an algebra. For every $a \in A$ consider the linear map

$$\begin{aligned} L_a : A &\longrightarrow A \\ b &\longmapsto ab. \end{aligned}$$

The map

$$\begin{aligned} L : A &\longrightarrow \mathcal{M}(A) \\ a &\longmapsto L_a \end{aligned}$$

is an embedding that identifies A as a subalgebra of $\mathcal{M}(A)$.

When A is a faithful, commutative Banach algebra, $\mathcal{M}(A) \subset \mathcal{B}(A)$; we denote the relative operator norm on $\mathcal{M}(A)$ as $\| \cdot \|_{op}$ and we set $\|a\|_{op} = \|L_a\|_{op}$ for every $a \in A$.

Note that $A = \mathcal{M}(A)$ if and only if A has an identity.

We conclude this subsection with a definition that will be useful later on:

DEFINITION 2.3.2. Let A be a Banach algebra. Then A is *compact* if the maps L_a and R_a are compact operators for each $a \in A$.

2.3.3. Arens products. We proceed to introduce the two Arens products, $\Box$ and $\Diamond$. These products provide the bidual of a Banach algebra A with a Banach algebra structure in such a way that, when A is view as a subspace of A'' via the canonical embedding, the original multiplication on A coincides with the restriction of the new multiplication provided to the bidual.

There are two ways of defining the two Arens products. Originally, they were introduce in 1951 by Arens in [4] and [3] and the definition was as follows:

Let A be a Banach algebra. The dual space A' is a Banach A-bimodule. Then, for $a \in A, \lambda \in A'$, $a \cdot \lambda, \lambda \cdot a \in A'$ such that $\langle b, a \cdot \lambda \rangle = \langle ba, \lambda \rangle$ and $\langle b, \lambda \cdot a \rangle = \langle ab, \lambda \rangle$ $(b \in A)$. Also for $a \in A$ and $M \in A''$, we have that $a \cdot M, M \cdot a \in A''$ with $\langle a \cdot M, \lambda \rangle = \langle M, \lambda \cdot a \rangle$ and $\langle M \cdot a, \lambda \rangle = \langle M, a \cdot \lambda \rangle$ $(\lambda \in A')$. Now, for $\lambda \in A'$ and $M \in A''$, we can define $\lambda \cdot M$ and $M \cdot \lambda$ in A' in the following way

$$\langle a, \lambda \cdot M \rangle = \langle M, a \cdot \lambda \rangle, \quad \langle a, M \cdot \lambda \rangle = \langle M, \lambda \cdot a \rangle.$$

With this definition, we have that $\|\lambda \cdot M\| \leq \|M\| \, \|\lambda\|$ and $\|M \cdot \lambda\| \leq \|M\| \, \|\lambda\|$. If $M \in A$ these new definitions agree with the original ones. The final step towards defining the two Arens products is as follows. For $M, N \in A''$, we define $M \Box N$ and $M \Diamond N$ in A'' by

$$\langle M \Box N, \lambda \rangle = \langle M, N \cdot \lambda \rangle, \quad \langle M \Diamond N, \lambda \rangle = \langle N, \lambda \cdot M \rangle \quad (\lambda \in A'),$$

and we have that $\|M \Box N\| \leq \|M\| \, \|N\|$ and $\|M \Diamond N\| \leq \|M\| \, \|N\|$ for $M, N \in A''$. When M or N is in A, these new definitions agree with the already existing ones. It can be seen that this definition is the same as the following:

Let $M, N \in A''$, and take $(a_\alpha), (b_\beta)$ nets in A such that $\lim_\alpha a_\alpha = M$ and $\lim_\beta b_\beta = N$ in the weak-$*$ topology. Then the two Arens product are

$$M \Box N = \lim_\alpha \lim_\beta a_\alpha b_\beta, \quad M \Diamond N = \lim_\beta \lim_\alpha a_\alpha b_\beta,$$

where the limits are again in the weak-$*$ topology on A''.

Following the notation in [26], for each $N \in A''$, consider the maps

$$R_N : M \mapsto M \Box N, \quad A'' \to A'',$$
$$L_N : M \mapsto M \Diamond N, \quad A'' \to A''.$$

It follows that, for every $N \in A''$, R_N is weak-$*$ continuous on $(A'', \Box)$ while L_N is weak-$*$ continuous on $(A'', \Diamond)$. Throughout the rest of this thesis, unless specified otherwise, whenever we talk about the bidual of a Banach algebra we are implicitly talking about $(A'', \Box)$.

Definition 2.3.3. Let A be a Banach algebra. Then A is *an ideal in its bidual A''* whenever A is a closed ideal of $(A'', \Box)$.

When A is an ideal in its bidual, for every $M \in A''$, the maps $L_M : a \mapsto M \cdot a$ and $R_M : a \mapsto a \cdot M$ are bounded linear operators on A, and so their duals are bounded linear operators on A'.

The following theorem is due to Watanabe. For example, it can be found in [26].

THEOREM 2.3.4. *Let A be a Banach algebra. Then A is an ideal in its bidual if and only if, for any $a \in A$, the maps L_a and R_a are weakly compact operators in $\mathcal{B}(A)$.* $\square$

We proceed to define the left and right topological centres of a Banach algebra.

DEFINITION 2.3.5. Let A be a Banach algebra. The *left topological centre* of A'' is

$$\mathfrak{Z}^{(\ell)}(A'') = \{M \in A'' : M \Box N = M \Diamond N \quad (N \in A'')\}.$$

Symmetrically, we define the *right topological centre* of A'' as

$$\mathfrak{Z}^{(r)}(A'') = \{M \in A'' : N \Box M = N \Diamond M \quad (N \in A'')\}.$$

These two topological centres might be different. However, in the case where A is commutative, the right and the left topological centres are the same and we can speak about the *topological centre* of A'', which is

$$\mathfrak{Z}(A'') = \{M \in A'' : M \Box N = M \Diamond N \quad (N \in A'')\}.$$

Since $N \Box a = N \Diamond a = N \cdot a$ and $a \Box N = a \Diamond N = a \cdot N$, for $a \in A$ and $N \in A''$, then $A \subset \mathfrak{Z}^{(\ell)}(A'') \subset A''$ and $A \subset \mathfrak{Z}^{(r)}(A'') \subset A''$. It might be that the topological centres are neither A nor A'', which leads to the following two definitions:

DEFINITION 2.3.6. Let A be a Banach algebra. We say A is *Arens regular* when

$$\mathfrak{Z}^{(\ell)}(A'') = \mathfrak{Z}^{(r)}(A'') = A''.$$

A Banach algebra A is said to be *strongly Arens irregular* if

$$\mathfrak{Z}^{(\ell)}(A'') = \mathfrak{Z}^{(r)}(A'') = A.$$

In the special case when A is a commutative Banach algebra, A is *Arens regular* if and only if $\mathfrak{Z}(A'') = A''$ and *strongly Arens irregular* if and only if $\mathfrak{Z}(A'') = A$.

Thus, a commutative Banach algebra is Arens regular if and only if $(A'', \Box)$ is commutative.

Note that with this definitions any reflexive Banach algebra will be Arens regular and strongly Arens irregular at the same time.

Let A be a Banach algebra. Given $\lambda \in A'$ we say that λ is *weakly almost periodic* if

$$R_\lambda : A \to A'$$
$$a \mapsto a \cdot \lambda$$

is weakly compact. Following [26] we write $\mathrm{WAP}(A)$ for the space of weakly almost periodic functionals on A.

It is standard that A is Arens regular if and only if $\mathrm{WAP}(A) = A'$. See [60].

Definition 2.3.7. Let A be a Banach algebra. Then a subset V of A'' is *determining for the left topological centre* (a DLTC set) of A'' if, given $M \in A''$ such that $M \Box N = M \Diamond N$ $(N \in V)$, then $M \in A$. When the algebra A is commutative, we use *determining for the topological centre* (a DTC set).

Whenever A'' has a DTC set, A is strongly Arens irregular. We shall be interested in finding small DTC sets of strongly Arens irregular Banach algebras.

The following results are known to specialists, but we also add the proof below for completion of the text.

Lemma 2.3.8. *Let A be a Banach algebra and B a closed subalgebra of A with finite codimension. Then the Arens regularity of A is the same as the Arens regularity of B.*

Proof. Since B has finite codimension, then we can write

$$A = F \oplus B,$$

where F is finite dimensional subspace and we are writing the sum as Banach spaces. Hence $A'' = F \oplus B''$. Thus, for $M, N \in A''$, there exist $a, b \in F$ and $P, Q \in B''$ such that $M = a + P$, $N = b + Q$. Hence

$$M \Box N = (a + P)\Box(b + Q) = ab + a \cdot Q + b \cdot P + P \Box Q$$
$$M \Diamond N = (a + P)\Diamond(b + Q) = ab + a \cdot Q + b \cdot P + P \Diamond Q.$$

Thus $\mathfrak{Z}^{(\ell)}(A'') = F \oplus \mathfrak{Z}^{(\ell)}(B'')$ and $\mathfrak{Z}^{(r)}(A'') = F \oplus \mathfrak{Z}^{(r)}(B'')$. $\square$

Theorem 2.3.9. *Let A be a strongly irregular Banach algebra. Let B be a closed subalgebra and I a closed ideal of A such that $A = B \ltimes I$. Let V_B be a DLTC set for B'' and V_I a DLTC set for I''. Then $V = V_B \cup V_I$ is a DTC set for A''.*

PROOF. For every $M \in A''$ there exist $M_B \in B$, $M_I \in I$ such that $M = M_B + M_I$. If M is such that $M \Box N = M \Diamond N$ for every $N \in V$, then $M_B \in B$ and $M_I \in I$, and so $M \in A$. Hence V is a DLTC set for A''. $\square$

Finally, we end this section by adding two results that will be very useful during the rest of the thesis.

The following can be found in [5, Theorem 2.1 iii)]

THEOREM 2.3.10. *Let A be a Banach algebra that is an ideal in its bidual, is weakly sequentially complete and has a bounded approximate identity. Then A is strongly Arens irregular.* $\square$

The following theorem can be found in [26, Theorem 2.3.48]:

THEOREM 2.3.11. *Let A be a Banach algebra such that $\mathcal{B}(A, A') = \mathcal{W}(A, A')$. Then A is Arens regular. In particular, A is Arens regular whenever A' has the Schur property.* $\square$

The following can be found in [26, Corollary 6.1.7]:

COROLLARY 2.3.12. *Let $(A, \|\cdot\|)$ be a Banach function algebra such that A is strongly Arens irregular and has the Schur property. Then $\|\cdot\|_{op}$ and $\|\cdot\|$ are equivalent on A.* $\square$

The following can be found in [26, Corollary 6.2.7 (ii)]:

COROLLARY 2.3.13. *Let A and B be Arens regular Banach algebras. Suppose that A'' is a compact algebra, then $A \hat{\otimes} B$ is Arens regular.* $\square$

2.3.4. Dual Banach algebras. Banach algebras that have the extra property of being dual Banach algebras can be Arens regular under certain conditions. Hence it is natural to study this property. We proceed to introduce the notions of Banach-algebra predual and dual Banach algebra. For that, we shall start by the following:

DEFINITION 2.3.14. Let E be a Banach space. Then a *predual* of E is a pair (F, T) where F is a Banach space and $T : E \to F'$ is a linear homeomorphism. Whenever T is an isometry, we say that F is an *isometric predual*. A *concrete predual* of E is a closed subspace F of E' such that the map $T_F : E \to F'$ defined by

$$(T_F x)(\lambda) = \langle x, \lambda \rangle_{E,E'} \quad (x \in E, \lambda \in F), \tag{2.3.1}$$

is a linear homeomorphism.

It is standard that, for a Banach space E and a concrete predual F, we can write $E'' = E \oplus F^{\perp}$.

Let A be a Banach algebra, with dual module A', and let F be a concrete predual as in the definition above. Hence the contraction $T_F : A \to F'$ of equation (2.3.1) is a linear homeomorphism such that $T'_F|F : F \to A'$ is the identity map. Whenever F is a submodule of A', T_F is a module homomorphism.

DEFINITION 2.3.15. Let A be a Banach algebra. A *Banach-algebra predual* for A is a closed linear subspace F of A' that is a concrete predual of A and a sub-bimodule of A'. We say that A is a *dual Banach algebra* if it has a Banach-algebra predual. A Banach-algebra predual is *unique* if it is the only closed submodule of A' with respect to which A is a dual Banach algebra.

The following result can be found in [26, Theorem 2.4.4]:

THEOREM 2.3.16. *Let A be a Banach algebra, and let F be a concrete predual of A. Then F is a closed submodule of A' if and only if the product in A is separately $\sigma(A, F)$-continuous.* □

The following proposition can be found in [26, Proposition 1.3.25] and it will be useful in subsequent sections:

PROPOSITION 2.3.17. *Let E be a Banach space, and suppose that F and G are concrete preduals of E such that $F \subset G$. Then $F = G$.* □

2.3.5. Banach function algebras.

DEFINITION 2.3.18. Let K be a non-empty, locally compact space. A *Banach function algebra on K* is a function algebra on K with a norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is a Banach algebra. Where a *function algebra* on K is a non-zero subalgebra A of $C^b(K)$ that separates strongly the points of K, in the sense that, for each $x, y \in K$ with $x \neq y$ there exists $f \in A$ such that $f(x) = 0$ and $f(y) = 1$.

For a Banach function algebra A, we recall that Φ_A denotes the character space of A and that the space of all continuous, complex-valued functions on Φ_A that are bounded is denoted by $C^b(\Phi_A)$. We recall that $L(A) = \text{lin}\, \Phi_A$ and we see it as a linear subspace of A'. If the reader wants more details about φ_A, please refer to Section 3.1.

When we say that A is a Banach function algebra without specifying K, we are assuming that A is a Banach function algebra defined on Φ_A.

DEFINITION 2.3.19. Let A be a Banach function algebra on K, and let $x \in K$. The *evaluation character at* x is the map

$$\varepsilon_x(f) = f(x) \quad (f \in A).$$

We can regard K as a subset of Φ_A by considering the inclusion map

$$x \mapsto \varepsilon_x, \quad K \to \Phi_A.$$

DEFINITION 2.3.20. Let A be a Banach function algebra defined on K. We say that A is *natural* if $K = \Phi_A$.

Every Banach function algebra is a commutative, semisimple Banach algebra. Conversely, every commutative, semisimple Banach algebra A can be identified with a Banach function algebra via the Gel'fand transform.

Let A be a Banach function algebra on K. We denote by $J_\infty = A \cap C_{00}(K)$ the functions in A of compact support. For $x \in K$, we set

$$J_x = \{f \in J_\infty : x \notin \operatorname{supp} f\}.$$

Notice that J_x is an ideal for every $x \in K$.

DEFINITION 2.3.21. Let A be a Banach function algebra on K. We say that A is *Tauberian* if $\overline{J_\infty} = A$.

DEFINITION 2.3.22. Let S be a non-empty set, and consider the discrete topology on S. A *Banach sequence algebra on* S is a Banach function algebra A on S such that

$$c_{00}(S) \subset A \subset \ell^\infty(S),$$

where $c_{00}(S)$ is the algebra of all functions on S with finite support and $\ell^\infty(S)$ is the Banach space of bounded functions on S with the uniform norm on S. We shall denote the uniform norm on S as $|\cdot|_S$.

We recall that a Banach function algebra A on a non-empty, locally compact space K is *natural* if each character on A is an evaluation character, so, in the particular case when A is a natural Banach sequence algebra on a non-empty set S, then A is contained in $c_0(S)$, where $c_0(S)$ is the closure of $c_{00}(S)$ in $\ell^\infty(S)$. Also, a Banach sequence algebra A on a non-empty set S is Tauberian if and only if $c_{00}(S)$ is dense in A. Finally, if A is a Tauberian Banach sequence algebra, then it is also natural and an ideal in its bidual.

The following can be found in [26, Corollary 3.2.4]:

Corollary 2.3.23. *Let A be a Tauberian Banach sequence algebra that is a dual Banach algebra. Then A is Arens regular, and A'' is a compact algebra.* □

2.3.6. BSE norm and BSE algebras. For this section we shall follow the conventions of [26].

Before introducing the notion of BSE algebra, we need to talk about the quotient algebra $Q(A)$. For an introduction to this algebra, we recommend [25] and [26, §5.1].

Recall from the previous section that, for a Banach function algebra A, $L(A) = \operatorname{lin} \Phi_A$. Thus, we can see in [26, Theorem 5.1.3]:

Theorem 2.3.24. *Let A be a Banach function algebra. Then $L(A)^\perp$ is a closed ideal in A'', and the quotient space*

$$Q(A) := A''/L(A)^\perp \equiv L(A)'$$

is a commutative, semisimple Banach algebra.

It is seen in [26, Theorem 5.1.9] that $Q(A)$ is an isometric dual Banach function algebra, with Banach-algebra predual $\overline{L(A)}$.

Let A be a Banach function algebra and take $f \in C^b(\Phi_A)$. Then, f can be associated to a linear functional τ_f on $L(A)$, where

$$\langle \tau_f, \lambda \rangle = \sum_i^n \alpha_i f(\varphi_i) \quad \left(\lambda = \sum_i^n \alpha_i \varphi_i \in L(A), \quad (\varphi_i \text{ distinct in } L(A)) \right).$$

We define

$$\|\tau_f\| = \sup\{|\langle f, \lambda \rangle| : \lambda \in L(A)_{[1]}\} \quad (f \in C^b(\Phi_A)).$$

Then $C_{BSE}(A)$ is the set of bounded, continuous functions $f \in C^b(\Phi_A)$ such that $\|\tau_f\| < \infty$. For $f \in C_{BSE}(A)$,

$$\|f\|_{BSE} = \|\tau_f\|.$$

As it is seen in [26, § 5], $(C_{BSE}(A), \|\cdot\|_{BSE})$ is a Banach function algebra on Φ_A that contains A as a subalgebra. We shall use the following characterization that can be found in [26, Theorem 5.2.9]:

THEOREM 2.3.25. *Let A be a Banach function algebra. Then $C_{BSE}(A)$ is the set of functions $f \in C^b(\Phi_A)$ for which there is a bounded net (f_ν) in A converging to f pointwise in $C^b(\Phi_A)$. For $f \in C_{BSE}(A)$ the infimum of the bounds of such nets is equal to $\|f\|_{BSE}$. Further, for each $f \in C_{BSE}(A)$, there is a net (f_ν) in A with $\lim_\nu f_\nu = f$ pointwise in $C^b(\Phi_A)$ such that*

$$\lim_\nu \|f_\nu\| = \lim_\nu \|f_\nu\|_{BSE} = \|f\|_{BSE}.$$

□

As seen in [26, § 5], we have that $A \subset C_{BSE}(A)$. Also, $C_{BSE}(A)$ is a subalgebra of $Q(A)$.

DEFINITION 2.3.26. A Banach function algebra A is said to be a *BSE algebra* when $\mathcal{M}(A) = C_{BSE}(A)$. We say that A has a *BSE norm* when the norms $\|\cdot\|$ and $\|\cdot\|_{BSE}$ are equivalent on A.

The following result is within [26, Corollary 5.5.10]:

COROLLARY 2.3.27. *Let A be a Banach function algebra that is an ideal in its bidual. Then the following are equivalent:*

(a) A is a BSE algebra;

(b) A has a bounded approximate identity.

In this case, A has a BSE norm. □

The following Proposition can be found in [26, Proposition 5.2.38]:

PROPOSITION 2.3.28. *Let A be a Banach algebra that is an ideal in its bidual or a natural Banach sequence algebra. Then $C_{BSE}(A) = Q(A)$.* □

The following Proposition is [26, Proposition 5.2.39]:

PROPOSITION 2.3.29. *Let A be a BSE algebra. Then A has a BSE norm if and only if A is closed in $(\mathcal{M}(A), \|\cdot\|_{op})$.* □

The following is [26, Proposition 5.1.17]:

PROPOSITION 2.3.30. *Let A be a natural Banach sequence algebra on $\mathbb{N}$, and let us denote $\overline{J_\infty}$ by A_0. Suppose that A_0'' has an identity E. Then*

$$A'' = A_0'' \ltimes L(A)^\perp, \quad E\Box A'' = A_0'' \quad \text{and} \quad Q(A) = A_0''. \qquad \Box$$

The following can be found in [26, Proposition 5.2.29]:

PROPOSITION 2.3.31. *Let A be a Banach function algebra. Then the following conditions on A are equivalent:*

(a) A has a BSE norm;

(b) A is closed as a subalgebra of $C_{BSE}(A)$. $\Box$

The following can be found in [26, Corollary 5.5.5]:

COROLLARY 2.3.32. *Let A be a Tauberian Banach sequence algebra with a multiplier-bounded approximate identity. Then A and $A^\#$ have BSE norms.* $\Box$

Where $A^\#$ is the unitization of A

The following Corollary can be found in [26, Corollary 5.5.4]:

COROLLARY 2.3.33. *Let A be a Banach function algebra that is an ideal in its bidual. Then $A = C_{BSE}(A)$ if and only if A is a dual Banach algebra.* $\Box$

The following can be found in [26, Corollary 5.2.27]:

COROLLARY 2.3.34. *Let B be a Banach function algebra with a BSE norm, and suppose that A is a closed subalgebra of B. Then A has a BSE norm.* $\Box$

The following can be found within [26, Theorem 5.2.16]:

THEOREM 2.3.35. *Let A be a Banach function algebra and take $m \geq 1$. Then the following are equivalent:*

(a) $C_{BSE}(A)$ has an identity 1 with $\|1\|_{BSE} \leq m$;

(b) $\mathcal{M}(A)_{[1]} \subset C_{BSE}(A)_{[m]}$;

(c) $\|f\|_{BSE} \leq m\|f\|_{op}$ $(f \in \mathcal{M}(A))$. $\Box$

For more details about BSE algebras and BSE norms see [26, Chapter 5].

2.4. Preliminaries from weighted semigroup algebras

We start this section by giving some definitions related to semigroups that will be useful throughout this thesis. We shall continue with the definition of a semigroup algebra and known results about them.

2.4.1. Semigroups. Given S a semigroup, we denote the semigroup operation by juxtaposition, unless stated otherwise.

Definition 2.4.1. We say that S is *right cancellative* (respectively, *left cancellative*) if, for all $a, s, t \in S$, $sa = ta$ (respectively, $as = at$) implies that $s = t$. When S is both right and left cancellative we call it *cancellative*.

We say S is *weakly right cancellative* (respectively, *weakly left cancellative*) if, for all $s, t \in S$, the set $\{u \in S : us = t\}$ (respectively, $\{u \in S : su = t\}$) is finite. If S is both right and left weakly cancellative, we say that S is *weakly cancellative*.

An element $p \in S$ is an *idempotent* if $p^2 = p$. We say that S is an *idempotent semigroup* if every element of S is idempotent.

We say that a semigroup S is *separating* if $s = t$ whenever $s, t \in S$ such that $st = s^2 = t^2$.

Whenever S is cancellative or idempotent, then S is separating.

In grouo theory, a semilattice is a partially ordered set such that every nonempty finite subset has a greatest lower bound. In our context we have the following:

Definition 2.4.2. Let S be an abelian idempotent semigroup. Then we can define a partial order $\leq$ in S by setting

$$s \leq t \iff st = s \quad (s, t \in S).$$

For every $s, t \in S$, it can be seen that st is a greatest lower bound for $\{s, t\}$. Hence, $(S, \leq)$ is a semilattice. Symmetrically, when we have a semilattice $(S, \leq)$, we can define a semigroup operation by setting st as the greatest lower bound of $\{s, t\}$ $(s, t \in S)$. Hence, from now on, we shall say that S is a *semilattice* when S is an abelian idempotent semigroup.

We shall see now some examples of semigroups. In the cases where it makes sense we shall also talk about the order induced.

Example 2.4.3. Let $S = \mathbb{N}$ together with the operation

$$m \wedge n = \min\{m, n\} \quad (m, n \in \mathbb{N}),$$

Then $\mathbb{N}_\wedge$ is a semigroup and it is not weakly cancellative. However, it is an abelian idempotent semigroup. Thus, it is a separating semigroup.

The partial order defined as in Definition 2.4.2 is the standard $\leq$ in $\mathbb{N}$. We can see $\mathbb{N}_\wedge$ as a semilattice in the following way:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \quad \ldots \quad \infty$$

And $\sup \mathbb{N}_\wedge = \infty$.

More generally, let S be an infinite subset of $\mathbb{R}$, and consider the operation $\wedge$ defined as above, i.e.,

$$s \wedge t = \min\{s, t\} \quad (s, t \in S).$$

We have that $S_\wedge$ is an abelian idempotent semigroup and we can see it as a semilattice in the same way as we did with $\mathbb{N}_\wedge$.

Example 2.4.4. Let $S = \mathbb{N}$ together with the operation

$$m \vee n = \max\{m, n\} \quad (m, n \in \mathbb{N}).$$

Then $\mathbb{N}_\vee$ is an idempotent, weakly cancellative semigroup. Hence it is separating.

In this case, the partial order that arises from Definition 2.4.2 is $\geq$. So, we see $\mathbb{N}_\vee$ as a semilattice in the following way:

$$\infty \quad \ldots \quad 4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1$$

In this case, $\sup \mathbb{N}_\vee = 1$.

2.4.2. Weighted semigroup algebras.

In order to talk about weighted semigroup algebras, we first need to introduce the notion of a weight on a semigroup.

Definition 2.4.5. Let S be a semigroup. A function $\omega : S \longrightarrow (0, \infty)$ is a *weight* on S if it is submultiplicative, in the sense that

$$\omega(st) \leq \omega(s)\omega(t) \quad (s, t \in S).$$

Whenever S has an identity e, we shall suppose that $\omega(e) = 1$, unless we specify otherwise.

Example 2.4.6. Let us consider the semilattice $S = \mathbb{N}_\wedge$ defined as in Example 2.4.3. Then any sequence $\omega : \mathbb{N} \longrightarrow [1, \infty)$ is a weight on S.

More generally, for every semilattice S, any $\omega : S \to (0, \infty)$ is a weight on S if and only if $\omega(s) \geq 1$ $(s \in S)$.

Definition 2.4.7. Let S be a semigroup, and let ω be a weight on S. We shall denote by δ_s the characteristic function of an element $s \in S$. Then we define the *weighted semigroup algebra* of S as the Banach space

$$\mathcal{A}_\omega := \ell^1(S,\omega) = \left\{\alpha = \sum_{s\in S}\alpha(s)\delta_s : \|\alpha\|_\omega = \sum_{s\in S}|\alpha(s)|\omega(s) < \infty\right\}$$

together with the convolution multiplication specified by

$$\delta_s \star \delta_t = \delta_{st} \quad (s,t\in S).$$

For $\omega \equiv 1$, this is the usual convolution algebra. We shall refer to this specific situation as the *unweighted case*.

Weighted semigroup algebras have been broadly studied. Some of the main references used for this document are [20], [19], [21], [22] and [23].

The dual of $\mathcal{A}_\omega$ as a Banach space is

$$\mathcal{A}'_\omega := \ell^\infty(S,1/\omega) = \left\{\lambda\in\mathbb{C}^S : \sup\{|\lambda(s)|/\omega(s) : s\in S\} < \infty\right\},$$

with the norm denoted by $\|\cdot\|'_\omega$ so that

$$\|\lambda\|'_\omega := \sup\{|\lambda(s)|/\omega(s) : s\in S\} \quad (\lambda\in\ell^\infty(S,1/\omega)).$$

The duality $\langle\cdot,\cdot\rangle_\omega$ between $\mathcal{A}_\omega$ and $\mathcal{A}'_\omega$ is given by

$$\langle f,\lambda\rangle_\omega = \sum_{s\in S} f(s)\lambda(s),$$

where $f = \sum_{s\in S} f(s)\delta_s \in \mathcal{A}_\omega$ and $\lambda\in\mathcal{A}'_\omega$. The space

$$E_\omega := c_0(S,1/\omega)$$

where $c_0(S,1/\omega)$ is the closure of $c_{00}(S,1/\omega)$ in $\ell^\infty(S,1/\omega)$. We have that E_ω a concrete predual of $\mathcal{A}_\omega$.

In the following section we shall see that E_ω is not necessarily a Banach-algebra predual for $\mathcal{A}_\omega$, and we shall study in which cases it is.

For $\lambda = (\lambda(s))\in\ell^\infty(S,1/\omega)$ and $\delta_s\in\mathcal{A}_\omega$, the module operation is specified by

$$(\delta_s\cdot\lambda)(t) = \lambda(ts), \quad (\lambda\cdot\delta_s)(t) = \lambda(st) \quad (t\in S).$$

DEFINITION 2.4.8. Let S be a semigroup, and let ω be a weight on S. For $s\in S$, the *normalised point mass* at s is denoted by $\tilde{\delta_s}$ and it is defined as

$$\tilde{\delta_s} = \delta_s/\omega(s).$$

The following lemma is straightforward, and we omit the proof. We add it here to facilitate the reading of the document:

LEMMA 2.4.9. *Let S be a semigroup, and let ω be a weight on S. Let*

$$\theta_\omega : \alpha \mapsto \alpha/\omega,\ \ell^1(S) \longrightarrow \mathcal{A}_\omega\,.$$

Then θ_ω is an isometric isomorphism of Banach spaces. Also, for every $s \in S$, $\theta_\omega(\delta_s) = \tilde{\delta}_s$. □

COROLLARY 2.4.10. *Let S be a semigroup, and consider ω a weight on S. Then $\ell^1(S,\omega)$ has the Schur property.* □

2.4.3. Stone-Čech compactification. We denote by βS the Stone-Čech compactification of S, where S is a set with the discrete topology. We denote by S^* the growth of S, which is defined to be $\beta S \setminus S$. Given a weight $\omega : S \longrightarrow (0,\infty)$, we shall denote the weak-$*$ closure of $\{\tilde{\delta}_s : s \in S\}$ in $\mathcal{A}_\omega''$ by βS_ω, and so βS_ω is a closed subset of the unit ball of $\mathcal{A}_\omega''$ with respect to the weak-$*$ topology. We regard S as a subset of βS_ω via the map $s \mapsto \delta_s$, and we set $S_\omega^* = \beta S_\omega \setminus S$. For more details about βS_ω see [20, §3].

Let S be a semigroup. For each $s \in S$, the map

$$L_s : t \mapsto st, \quad S \to S \subset \beta S,$$

has a continuous extension $L_s : \beta S \to \beta S$. For each $u \in \beta S$, we define $s\Box u = L_s(u)$. Now, for $u \in \beta S$, let us consider

$$R_u : s \mapsto s\Box u, \quad S \to \beta S,$$

which has a continuous extension $R_u : \beta S \to \beta S$. We then set

$$u\Box v = R_v(u) \quad (u, v \in \beta S).$$

The binary operation $\Box$ is such that the restriction to $S \times S$ is the original product in S. For every $u, v \in \beta S$, there are nets (s_α), (t_β) in S converging to u, v, respectively. We can see that

$$u\Box v = \lim_\alpha \lim_\beta s_\alpha t_\beta.$$

Symmetrically, we can define an operation $\Diamond$ such that

$$u\Diamond v = \lim_\beta \lim_\alpha s_\alpha t_\beta.$$

We shall speak about $(\beta S, \Box)$, although symmetrical results are true for $(\beta S, \Diamond)$. We have the following result, that can be found in [22, Theorem 6.1]:

THEOREM 2.4.11. *Let S be a semigroup. Then $(\beta S, \Box)$ and $(\beta S, \Diamond)$ are semigroups containing S as a subsemigroup. Further:*

(i) for each $v \in \beta S$, the map $R_v : u \mapsto u \Box v$ is continuous, and $(\beta S, \Box)$ is a compact, right topological semigroup;

(ii) for each $s \in S$, the map $L_s : u \mapsto s \Box u$ is continuous. $\Box$

For more details of the semigroup $(\beta S, \Box)$, we recommend [22, Chapter 6].

The following definition can be found in [26]:

DEFINITION 2.4.12. Let S be a semigroup. Then the *left topological centre* of βS is

$$\mathfrak{Z}_t^{(l)}(\beta S) = \{u \in \beta S : u \Box v = u \Diamond v \ (v \in \beta S)\}.$$

Similarly, we define the *right topological centre* of βS as

$$\mathfrak{Z}_t^{(r)}(\beta S) = \{u \in \beta S : v \Box u = v \Diamond u \ (v \in \beta S)\}.$$

We say S is *Arens regular* when $\mathfrak{Z}_t^{(l)}(\beta S) = \mathfrak{Z}_t^{(r)}(\beta S) = \beta S$, *left* (respectively, *right*) *strongly Arens irregular* when $\mathfrak{Z}_t^{(l)}(\beta S) = S$ (respectively, $\mathfrak{Z}_t^{(r)}(\beta S) = S$) and *strongly Arens irregular* when $\mathfrak{Z}_t^{(l)}(\beta S) = \mathfrak{Z}_t^{(r)}(\beta S) = S$.

Let S be an infinite semigroup. A subset V of S^* is *determining for the left topological centre* (a DLTC set) of $M(\beta S)$ if $u \in S$ whenever $u \Box v = u \Diamond v \ (v \in V)$.

Hence, a subset V of S^* is determining for the left topological centre of βS if there are no elements $u \in S^*$ such that $u \Box v = u \Diamond v \ (v \in V)$.

CHAPTER 3

Semigroup algebras

3.1. Initial results

This chapter is concerned with the study of weighted semigroup algebras. We shall give some results regarding generic semigroups, but we shall mainly focus on totally ordered semilattices. With this in mind, we give the following definition.

Definition 3.1.1. Let S be an infinite set. Let $f : S \longrightarrow \mathbb{R}$. Given $C \in \mathbb{R}$, we write

$$\mathrm{Lim}_s\, f(s) = C$$

if, for each $\varepsilon > 0$, there is a finite set F of S such that

$$|f(s) - C| < \varepsilon \quad (s \in S \setminus F).$$

We write

$$\mathrm{Lim}_s\, f(s) = \infty$$

if, for each $M > 0$, there is a finite set F of S such that

$$f(s) > M \quad (s \in S \setminus F).$$

We write

$$\mathrm{Lim\ inf}_s\, f(s) < \infty$$

if and only if it is not true that $\mathrm{Lim}_s\, f(s) = \infty$, i.e., there exists $M > 0$ such that the set $\{s \in S : f(s) < M\}$ is infinite. We write Lim inf f for the infimum of these constants M.

Example 3.1.2 (Example 2.4.6 revisited). Let us consider the semigroup $\mathbb{N}_\wedge$ as in Example 2.4.6. We see that $\mathrm{Lim}_{n\to\infty}\, \omega(n) = C$ (respectively, $\mathrm{Lim}_{n\to\infty}\, \omega(n) = \infty$, $\mathrm{Lim\ inf}_{n\to\infty}\, \omega(n) < \infty$) whenever $\lim\limits_{n\to\infty} \omega(n) = C$ (respectively, $\lim\limits_{n\to\infty} \omega(n) = \infty$, $\liminf\limits_{n\to\infty} \omega(n) < \infty$).

Example 3.1.3. Let $S = \mathbb{Q}^{+\bullet} = \{s \in \mathbb{Q} : s > 0\}$ with the semigroup operation $\wedge$ defined as in Example 2.4.3. For clarity, during this example $p, q \in \mathbb{N}$ are coprime. Then we define

$$\omega(p/q) = p + q.$$

Hence $\mathrm{Lim}_s\, \omega(s) = \infty$ in the sense of Definition 3.1.1. Indeed, let $M > 0$. Then the set $\{p/q \in \mathbb{Q}^{+\bullet} : \omega(p/q) \leq 2M\}$ is strictly contained in

$$\{p/q : p, q \in \mathbb{N}, p \leq M, q \leq M\},$$

which is finite.

Example 3.1.4. Let $S = \mathbb{Q}^{+\bullet}_{\wedge}$ as above. Consider now $\omega : \mathbb{Q}^{+\bullet} \to [1, \infty)$ such that $\omega(s) = 1$ for $s \in (0, 1]$ and such that $\liminf_{s\to\infty} \omega(s) = \infty$ in the traditional sense. Then $\mathrm{Lim\ inf}_s\, \omega(s) < \infty$.

We proceed to introduce two definitions that are key in the study of Arens regularity of semigroup algebras.

Definition 3.1.5. Given a weight ω on S, we define Ω on $S \times S$ in the following way:

$$\Omega(s,t) = \frac{\omega(st)}{\omega(s)\omega(t)} \quad (s, t \in S).$$

Given a function $f : S \times S \to \mathbb{R}$, we say that f *clusters on* $S \times S$ if, for $(x_n), (y_m)$ sequences of distinct elements of S, then

$$\lim_{n\to\infty} \lim_{m\to\infty} f(x_n, y_m) = \lim_{m\to\infty} \lim_{n\to\infty} f(x_n, y_m)$$

whenever both iterated limits exist.

We say that f $0-$*clusters on* $S \times S$ if, for $(x_n), (y_m)$ sequences of distinct elements of S, then

$$\lim_{n\to\infty} \lim_{m\to\infty} f(x_n, y_m) = \lim_{m\to\infty} \lim_{n\to\infty} f(x_n, y_m) = 0$$

whenever both iterated limits exist.

The following result will be useful in the following section. It can be found in [26, Proposition 6.3.23]:

Proposition 3.1.6. *Let ω be a weight on a semigroup S, and let U and V be infinite subsemigroups of S. Suppose that Ω $0-$clusters on $U \times V$. Then $M\Box N = M\Diamond N = 0$ whenever $M \in \ell^1(U,\omega)'' \cap E_\omega^\perp$ and $N \in \ell^1(V,\omega)'' \cap E_\omega^\perp$.* $\square$

As we shall see in the two results below, the case when S is a weakly cancellative semigroup has been well studied.

It was proven in [22, Theorem 12.15] that whenever S is weakly cancellative and nearly right cancellative, then the semigroup algebra is strongly Arens irregular. In addition, they characterized the DTC of $(\ell^1(S)'', \Box)$. Due to the importance of this theorem, we copy it below:

THEOREM 3.1.7 (Dales-Lau-Strauss, 2010). *Let S be an infinite semigroup such that S is weakly cancellative and nearly right cancellative. Then there exists a and b in S^* that are right cancellable in $(\beta S, \Box)$ and such that the two-element set $\{a, b\}$ is determining for the left topological centre of $M(\beta S)$.*

Where a semigroup S is said to be *nearly right cancellative* if there is a subset X of S such that $|X| = |S|$ and such that the set $\{x \in X : sx = tx\}$ is finite for every $s, t \in S$ such that $s \neq t$. This property is essential in the theorem. In [22, Example 12.21] provide an example of an infinite, countable, weakly cancellative semigroup for which $\ell^1(S)$ is not strongly Arens irregular.

In 1974 Craw and Young already studied Arens regularity of weighted semigroup algebras in [13]. We provide a new proof of their main theorem ([13, Theorem 1]) below. The proof has been modified to match the terminology used here, which resulted in a simpler version of the necessary part. A similar observation was made in [28], however, we provide full details here.

THEOREM 3.1.8 (Craw-Young, 1974). *Let S be an infinite semigroup and ω a weight on S. Then:*

(a) If Ω $0-$clusters on $S \times S$, then $\mathcal{A}_\omega$ is Arens regular.

(b) If S is a weakly cancellative semigroup, then the Arens regularity of $\mathcal{A}_\omega$ implies that Ω $0-$clusters.

PROOF. (a) Let $\lambda \in \mathcal{A}_\omega{}'$. Then $\lambda \in \mathrm{WAP}(\mathcal{A}_\omega)$ if and only if

$$(i, j) \mapsto \langle R_\lambda(\tilde{\delta}_i), \tilde{\delta}_j\rangle, \quad t_\omega : S \times S \to \mathbb{C}.$$

clusters. This follows from Grothendieck's double limit criterion and can be seen in [28]. Since $\lambda \in \mathcal{A}_\omega{}'$, there exists $\lambda_0 \in \mathcal{A}'$ such that $\lambda = \omega\lambda_0$. The above then translates to the fact that

$$t_\omega(i, j) = \frac{\langle \delta_i \cdot \lambda, \delta_j\rangle}{\omega(i)\omega(j)} = \Omega(i, j)\langle \lambda_0, \delta_j \star \delta_i\rangle$$

clusters.
In particular, when Ω $0-$clusters, t_ω also 0-clusters. Thus, for $\lambda \in \mathcal{A}_\omega{}'$, $\lambda \in \mathrm{WAP}(\mathcal{A}_\omega)$. Hence, $\mathcal{A}_\omega$ is Arens regular.

(b) Suppose that S is weakly cancellative. The following argument is very similar to the one followed in [13, Theorem 1] but we provide full details for completeness. Suppose that there exist (s_n) and (t_m) sequences of distinct elements of S and $\varepsilon > 0$ such that

$$\lim_{n\to\infty}\lim_{m\to\infty}\Omega(s_n,t_m) > \varepsilon.$$

We shall see that there are two elements of the bidual such that the two Arens products are different. We may suppose that

$$\lim_{m\to\infty}\Omega(s_n,t_m) > \varepsilon > 0 \quad (n\in\mathbb{N}).$$

Let us choose two subsequences (s'_n) and (t'_m) of (s_n) and (t_m), respectively, such that $\Omega(s'_n,t'_m) > \varepsilon$ for $n \le m$. Indeed, take $s'_1 = s_1$ and take t'_1 to be the first element t_m such that $\Omega(s_1,t_m) > \varepsilon$. Let us suppose that we already have the first k elements $s'_1,...,s'_{k-1}$ and $t'_1,...,t'_{k-1}$. Since S is weakly cancellative the set

$$F = \{u \in S : ut'_l = s'_it'_j, 1 \le l,i,j < k\}$$

is finite. Hence we can chose as s'_k the first element s_n such that $s_n \notin F$. Following the same line of reasoning the set

$$E = \{u \in S : s'_lu = s'_it'_j, 1 \le j < k, 1 \le i,l \le k\}$$

is finite, and so we can chose t'_k as the first element t_m such that

$$\Omega(s'_i,t_m) > \varepsilon \quad (1\le i\le k), \quad t_m \notin E.$$

These subsequences are such that $\Omega(s'_n,t'_m) > \varepsilon$ for $n \le m$ and such that the elements $s'_nt'_m$ are distinct for $m,n\in\mathbb{N}$. Let α_n and β_m be the normalized point masses at s'_n and t'_m, respectively, and $\chi \in \ell^\infty(S)$ the characteristic function of the set $\{s'_nt'_m : n \le m\}$. Then

$$\langle \alpha_n \star \beta_m, \omega\chi\rangle = \Omega(s'_n,t'_m) > \varepsilon \quad (n\le m),$$
$$\langle \alpha_n \star \beta_m, \omega\chi\rangle = 0 \quad (n > m).$$

Let $M,N \in \mathcal{A}_\omega{}''$ be $\sigma(\mathcal{A}_\omega{}'',\mathcal{A}_\omega{}')$-accumulation points of (α_n) and (β_n), respectively. By construction, $\langle M\Box N, \omega\chi\rangle \ge \varepsilon$ and $\langle M\Diamond N, \omega\chi\rangle = 0$. Thus, $\mathcal{A}_\omega$ is not Arens regular, as desired.

When (s_n) and (t_m) are sequences of distinct elements of S and $\varepsilon > 0$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty}\Omega(s_n, t_m) > \varepsilon,$$

the argument is symmetrical. $\square$

Consider $\Omega : S \times S \longrightarrow \mathbb{R}$ defined as above. Recall that, for $s, t \in S$, we have $0 \leq \Omega(s,t) \leq 1$. Let $u, v \in \beta S$. Then there are nets $(s_\alpha), (t_\beta)$ in S such that $u = \lim_\alpha s_\alpha$, $v = \lim_\beta t_\beta$. We define

$$\Omega_\Box(u,v) = \lim_\alpha \lim_\beta \Omega(s_\alpha, t_\beta), \quad \Omega_\Diamond(u,v) = \lim_\beta \lim_\alpha \Omega(s_\alpha, t_\beta).$$

The following result can be found in [21, Proposition 3.1]. We write it here for completeness.

PROPOSITION 3.1.9. *Let S and T be non-empty sets, and let*

$$f : S \times T \to \mathbb{C}$$

be a function. Suppose that (s_α) and (t_β) are nets in S and T, respectively, such that $a = \lim_\alpha \lim_\beta f(s_\alpha, t_\beta)$ and $b = \lim_\beta \lim_\alpha f(s_\alpha, t_\beta)$ both exist. Then there are subsequences (s_{α_m}) and (t_{β_m}) of the nets (s_α) and (t_β), respectively, such that $a = \lim_m \lim_n f(s_{\alpha_m}, t_{\beta_n})$ and $b = \lim_\beta \lim_\alpha f(s_{\alpha_m}, t_{\beta_n})$. $\square$

We can see that if we apply this Proposition to $\Omega_\Box$ and $\Omega_\Diamond$ we can define them in terms of sequences instead of nets.

When the situation requires it, we shall write $\Box_\omega$ and $\Diamond_\omega$ to specify that we are in $\mathcal{A}_\omega''$.

COROLLARY 3.1.10. *Let S be a semigroup, and let ω be a weight on S. Consider $\Omega : S \times S \longrightarrow \mathbb{R}$ defined as above, and let $u, v \in \beta S$. Let $(s_\alpha), (t_\beta)$ be nets in S such that $u = \lim_\alpha s_\alpha$, $v = \lim_\beta t_\beta$. Then there are subsequences (s_m) and (t_m) of (s_α) and (t_β), respectively, such that*

$$\Omega_\Box(u,v) = \lim_m \lim_n \Omega(s_m, t_n) \quad \textit{and} \quad \Omega_\Diamond(u,v) = \lim_n \lim_m \Omega(s_m, t_n). \qquad \square$$

Let S be a semigroup, and let ω be a weight on S. Let us consider θ_ω as in Lemma 2.4.9. Let $u, v \in S^*$. Then there are nets (s_α) and (t_β) in S such that $u = \lim_\alpha s_\alpha$ and $v = \lim_\beta t_\beta$. Thus, for $\lambda \in \ell^\infty(S)$, we have

$$\langle \theta_\omega''(\delta_u) \Box_\omega \theta_\omega''(\delta_v), \omega\lambda \rangle = \lim_\alpha \lim_\beta \langle \theta_\omega(\delta_{s_\alpha}) \star \theta_\omega(\delta_{t_\beta}), \omega\lambda \rangle$$

$$= \lim_{\alpha} \lim_{\beta} \Omega(s_\alpha, t_\beta) \langle \delta_{s_\alpha} \star \delta_{t_\beta}, \lambda \rangle$$

$$= \Omega_{\Box}(u, v) \langle \delta_u \Box \delta_v, \lambda \rangle. \tag{3.1.1}$$

Symmetrically we obtain that

$$\langle \theta''_\omega(\delta_u) \Diamond_\omega \theta''_\omega(\delta_v), \omega\lambda \rangle = \Omega_{\Diamond}(u, v) \langle \delta_u \Diamond \delta_v, \lambda \rangle. \tag{3.1.2}$$

The following result can be found in [20, Proposition 4.8]:

THEOREM 3.1.11. *Let S an abelian semigroup, and let ω be a weight on S. For $s \in S$, set*

$$\nu_s = \inf\{\omega(s^n)^{1/n} : n \in \mathbb{N}\}.$$

Then $\ell^1(S, \omega)$ is semisimple if and only if S is separating and $\nu_s > 0$ $(s \in S)$. $\Box$

For the notions of *character* and *semicharacter* on S, we shall follow the conventions of [22].

DEFINITION 3.1.12. Let S be a semigroup. A *semicharacter* on S is a map $\theta : S \longrightarrow \bar{\mathbb{D}}$ such that $\theta \neq 0$ and

$$\theta(st) = \theta(s)\theta(t) \quad (s, t \in S).$$

We denote by Φ_S the space of semicharacters on S.

A *character* on S is a map $\theta : S \longrightarrow \mathbb{T}$ such that $\theta \neq 0$ and

$$\theta(st) = \theta(s)\theta(t) \quad (s, t \in S).$$

We denote by Ψ_S the space of characters on S. We have that $\Psi_S \subset \Phi_S$.

There is always at least one semicharacter on S, the *augmentation character*

$$1 : s \mapsto 1, \quad S \to \mathbb{T}.$$

It is possible that the augmentation character is the only semicharacter on a semigroup. The space $\Phi_S \cup \{0\}$ is a compact space with respect to the topology of pointwise convergence on S.

Let S be a semigroup and ω a weight on S. The character space of the weighted semigroup algebra $\mathcal{A}_\omega$ is denoted by Φ_ω. Thus, we have that $\Phi_S \subset \Phi_\omega$ as a closed semigroup.

Definition 3.1.13. The character in Φ_ω associated to the augmentation character on S is the *augmentation character* φ_S, where

$$\varphi_S : \sum_{s\in S} \alpha(s)\delta_s \mapsto \sum_{s\in S} \alpha(s)$$

It is seen in [22, §6] that, given a semigroup S, the character space of the semigroup algebra $\ell^1(S)$ can be identified with the space of semicharacters of S. Let ω be a weight on S such that $\omega(st) = \omega(s)\omega(t)$ for every $s, t \in S$. Then the above result can be extended to $\mathcal{A}_\omega$. Indeed, it is enough to consider that, for $\theta \in \Phi_S$, the map

$$\sum_{s\in S} \alpha(s)\delta_s \mapsto \sum_{s\in S} \omega(s)\alpha(s)\theta(s), \quad \mathcal{A}_\omega \longrightarrow \mathbb{C}$$

is a character on $\mathcal{A}_\omega$. Conversely, given a character φ on $\mathcal{A}_\omega$, then $\theta(s) = \varphi(\tilde{\delta_s})$ is a semicharacter on S. The topology of pointwise convergence on Φ_S coincides with the Gel'fand topology when Φ_S is viewed as the character space of $\mathcal{A}_\omega$.

For a weight bounded below, we know that $\Phi_S \subset \Phi_\omega$, but it is not necessarily true that $\Phi_\omega = \Phi_S$. For example, consider $S = \mathbb{Z}$ with addition and $\omega(n) = \mathrm{e}^{|n|}$. However, as we shall see in the result below, for the semigroups in this chapter this problem does not arise.

Let S be a semigroup and ω a weight on S. Whenever $\Phi_S = \Phi_\omega$, the Gel'fand transform of an element $\alpha \in \mathcal{A}_\omega$ has the following form

$$\widehat{\alpha} = \left(\sum_{s\in S} \alpha(s)\theta(s)\right)_{\theta\in\Phi_S}.$$

Proposition 3.1.14. *Let S be a semilattice, and let Φ_S be the semicharacter space of S. Let $\omega : S \to [1, \infty)$ be a weight on S. Then $\Phi_\omega = \Phi_S$.*

Proof. We know that $\Phi_S \subset \Phi_\omega$. Now let φ be a character on $\mathcal{A}_\omega$ and define $\theta_\varphi(s) = \varphi(\delta_s)$. Since φ is a character,

$$\theta_\varphi(s) = \varphi(\delta_s) = \varphi(\delta_s \star \delta_s) = \varphi(\delta_s)\varphi(\delta_s) \quad (s \in S),$$

and so $\theta_\varphi(s) \in \{0, 1\}$. Also

$$\theta_\varphi(st) = \varphi(\delta_{st}) = \varphi(\delta_s \star \delta_t) = \varphi(\delta_s)\varphi(\delta_t) = \theta_\varphi(s)\theta_\varphi(t) \quad (s, t \in S).$$

Hence θ_φ is a semicharacter on S and so we can identify Φ_ω with Φ_S. □

3.2. Totally ordered semilattices

Let $(S, \leq)$ be a semilattice, as described in Definition 2.4.2. Suppose also that the order $\leq$ is a total order, meaning that, for any two elements $s, t \in S$, it is always true that either $s \leq t$ or $t \leq s$. We shall refer to $(S, \leq)$ with these characteristics as a *totally ordered semilattice*. Totally ordered semilattices are the object of study of this section. Conversely, any totally ordered set S becomes a semigroup if we take $st = \min\{s, t\}$; this semigroup is a semilattice and the partial order defined in Definition 2.4.2 coincides with the original order on S.

Some preliminaries about semigroups of the form of S are in [69].

Remark 3.2.1. Notice that the natural numbers with the usual order $(\mathbb{N}, \leq)$ belongs to this family. As it was seen in Example 2.4.3, this semilattice arises from the semigroup $\mathbb{N}$ with the minimum operation, which is not even weakly cancellative. Thus, the family studied comprise a wide variety of examples with characteristics that differ from those usually considered in the literature focused on weighted semigroup algebras. We shall focus on this example in the following section.

3.2.1. Arens regularity. We shall assume from now on that there exists an embedding from S into some infinite, semigroup T that contains S as a subsemigroup, with the following additional characteristics:

- T must be a totally ordered set that preserves the order in S;
- T has a minimum and a maximum, which we shall call 0 and ∞, respectively;
- T is complete in the sense that every non-empty subset of T has a supremum and an infimum;
- We consider the interval topology on T, in which case T is a compact topological semigroup.

Note that in this case every strictly increasing, respectively, strictly decreasing, net in S converges to its supremum, respectively, infimum.

In the following remark we shall see that given an infinite totally ordered semilattice S, we can always find such a T. This is well-known to specialists, but we add it to make it more accessible for the reader.

Remark 3.2.2. Since S is a semilattice, we know that Φ_S separates the points of S. Let Σ be a subset of Φ_S that separates the points of S, and let $\kappa = |\Sigma|$. Since every character of S maps into $\{0, 1\}$, $(S, \wedge)$ can be embedded as a semigroup in

$C := (\{0,1\}^\kappa, \wedge)$. Let T be the closure of S in C. In this case, T is a complete totally ordered lattice which is compact in its interval topology, as desired. For details, see [24, §2].

Let S be a totally ordered semilattice and T as above. Let U be a subset of S, we write as $\mathrm{cl}_T\, U$ and $\mathrm{cl}_{\beta S}\, U$ the closures of U in T and in βS, respectively. The continuous extension of the inclusion map of S into T is denoted by

$$\pi : \beta S \to T.$$

Thus $\pi(\beta S) = \mathrm{cl}_T\, S$. For $t \in \mathrm{cl}_T\, S$, we shall write F_t for the fibre $\{x \in \beta S : \pi(x) = t\}$ and $F_t^* = F_t \cap S^*$. Thus we have that

$$F_t^* = F_t \quad (t \in T \setminus S) \quad \text{and} \quad F_t^* = F_t \setminus \{t\} \quad (t \in S).$$

We shall denote by E the set of accumulation points of S in T. We have then that $E \neq \emptyset$. For $t \in T$, F_t^* is a closed, compact subset of βS and $F_t^* \neq \emptyset$ if and only if $t \in E$.

Consider Ω as in Definition 3.1.5. When S is a totally ordered semilattice, $\Omega(s,t) = 1/\omega(t)$ $(s \leq t)$. What is more, we can see that Ω $0-$clusters if and only if $\mathrm{Lim}_s\, \omega(s) = \infty$. Indeed $\mathrm{Lim}_s\, \omega(s) = \infty$ implies that Ω $0-$clusters, since, for every $\varepsilon > 0$, the set of elements $s \in S$ such that $\frac{1}{\omega(s)} > \varepsilon$ is finite.
Now suppose that $\mathrm{Lim\ inf}_s\, \omega(s) < \infty$. Then there exists $M > 0$ such that the set $\{s \in S : \omega(s) < M\}$ is infinite. Hence, we can then take two sequences $(s_m), (t_n)$ of distinct elements belonging to that set and so $\Omega(s_m, t_n) \geq 1/M^2 > 0$. Thus, Ω does not $0-$cluster.

Theorem 3.2.3. *Let $(S, \wedge)$ be an infinite, totally ordered semilattice. Let ω be a weight on S. Then the following conditions are equivalent:*

(a) the algebra $\mathcal{A}_\omega$ is Arens regular;

(b) $\mathrm{Lim}_s\, \omega(s) = \infty$;

(c) $M \Box N = M \Diamond N = 0$ $(M, N \in E_\omega^\perp)$.

Proof. $(a) \Rightarrow (b)$ Suppose that $\mathrm{Lim\ inf}_{s\to\infty}\, \omega(s) < \infty$, and let $M \geq \mathrm{Lim\ inf}\, \omega$. Let $U := \{s \in S : \omega(s) < M\}$. Take $t \in E \cap \mathrm{cl}_{\beta S}\, U$. By [24, Lemma 2.4], $|F_t^*| \geq 2^{\mathfrak{c}}$. Take $p \in F_t^*$. If $p \in F_t^* \cap \mathrm{cl}_{\beta S}(U \cap [0,t))$, there exists q in $F_t^* \cap \mathrm{cl}_{\beta S}(U \cap [0,t))$ with $\delta_p \notin \mathrm{lin}\{\delta_q\}$. Then

$$\delta_p \Box \delta_q = \delta_p \quad \text{and} \quad \delta_p \Diamond \delta_q = \delta_q.$$

Let us consider the isometric isomorphism θ_ω as in Lemma 2.4.9. For $\lambda \in C(\beta S)$, we have

$$\langle \theta_\omega''(\delta_p) \Box_\omega \theta_\omega''(\delta_q), \omega\lambda\rangle = \Omega_\Box(p,q)\langle \delta_p, \lambda\rangle, \tag{3.2.1}$$

and

$$\langle \theta_\omega''(\delta_p) \Diamond_\omega \theta_\omega''(\delta_q), \omega\lambda\rangle = \Omega_\Diamond(p,q)\langle \delta_q, \lambda\rangle. \tag{3.2.2}$$

Take (s_α), (t_β) nets in S converging to p and q, respectively. Observe that, since $1 \leq \omega(s_\alpha) \leq M$ and $1 \leq \omega(t_\beta) \leq M$ for all α, β, we have that

$$0 < 1/M^2 \leq \Omega_\Box(p,q) \leq M \quad \text{and} \quad 0 < 1/M^2 \leq \Omega_\Diamond(p,q) \leq M.$$

Hence the equations (3.2.1) and (3.2.2) from above are equal for every $\lambda \in C(\beta S)$ if and only if $\delta_p = \frac{\Omega_\Diamond(p,q)}{\Omega_\Box(p,q)}\delta_q$. But that is not possible as $\delta_p \notin \operatorname{lin}\{\delta_q\}$. Thus, $\mathcal{A}_\omega$ is not Arens regular.

If $p \notin F_t^* \cap \operatorname{cl}_{\beta S}(U \cap [0,t))$, as it is stated in [24], then $p \in F_t^* \cap \operatorname{cl}_{\beta S}(U \cap (t,\infty])$, and the argument is symmetrical.

$(b) \Rightarrow (c)$ follows from Proposition 3.1.6.

$(c) \Rightarrow (a)$ Since E_ω is a concrete predual of $\mathcal{A}_\omega$, we know that $\mathcal{A}_\omega'' = \mathcal{A}_\omega \oplus E_\omega^\perp$. Since $M \Box N = M \Diamond N = 0$ for every $M, N \in E_\omega^\perp$, the result follows. □

The next question would be which conditions on ω ensure that $\mathcal{A}_\omega$ is strongly Arens irregular. In [24], they proved that the semigroup algebra $(\ell^1(S), \star)$ is strongly Arens irregular if and only if $\operatorname{cl}_T S$ is scattered. We shall see below that when we add a weight the situation is more complex. We start by considering the simplest case. When S is a semigroup and ω is a bounded weight on S, then the inclusion map $\mathcal{A}_\omega \hookrightarrow \ell^1(S)$ is a Banach algebra isomorphism, and so $\mathcal{A}_\omega$ is strongly Arens irregular if and only if $\operatorname{cl}_T S$ is scattered.

However, we shall see below that if ω is not bounded we can have several different options. The next provides a sufficient condition for $\mathcal{A}_\omega$ to not be strongly Arens irregular.

Proposition 3.2.4. *Let S be a totally ordered semilattice, and let ω be a weight on S. Suppose that for every $p \in F_\infty^*$ and every net (s_α) in S such that $s_\alpha \to p$, the set $\{\omega(s_\alpha)\}$ is unbounded. Then $\mathcal{A}_\omega$ is not strongly Arens irregular.*

Proof. Let $p \in F^*_\infty$. We shall see that $\theta''_\omega(\delta_p) \in \mathfrak{Z}(\mathcal{A}_\omega{''})$. Take $q \in S^*$. If $q \in F^*_\infty$, then $\Omega_\Box(p,q) = \Omega_\Diamond(p,q) = 0$ and so

$$\theta''_\omega(\delta_p)\Box_\omega\theta''_\omega(\delta_q) = 0 = \theta''_\omega(\delta_p)\Diamond_\omega\theta''_\omega(\delta_q).$$

Take now $q \in S^*$ such that $q \notin F^*_\infty$, and let (s_α) be a net in S converging to p and (t_β) a net in S converging to q. Since $\pi(q) < \pi(p)$, this implies that, passing to a subnet if needed, we can suppose that $t_\beta < s_\alpha$ for every α, β. Hence we have that

$$\Omega(t_\beta, s_\alpha) = \frac{\omega(t_\beta \wedge s_\alpha)}{\omega(t_\beta)\omega(s_\alpha)} = \frac{1}{\omega(s_\alpha)}$$

and so $\Omega_\Box(p,q) = \Omega_\Diamond(p,q) = 0$ which gives us again that

$$\theta''_\omega(\delta_p)\Box_\omega\theta''_\omega(\delta_q) = 0 = \theta''_\omega(\delta_p)\Diamond_\omega\theta''_\omega(\delta_q).$$

We conclude then that $\theta''_\omega(\delta_p) \in \mathfrak{Z}(\mathcal{A}_\omega{''})$ but $\theta''_\omega(\delta_p) \notin \mathcal{A}_\omega$. Thus $\mathcal{A}_\omega$ is not strongly Arens irregular. $\Box$

This previous result, together with Theorem 3.2.3, allow us to obtain plenty of weighted semigroup algebras that are neither Arens regular nor strongly Arens irregular. The following two examples portray two different semilattices, one of them is such that $\mathrm{cl}_T\, S$ is scattered and the other one is such that $\mathrm{cl}_T\, S$ is not scattered. In contrast with the unweighted case, we shall see that both of them are neither Arens regular nor strongly Arens irregular.

Example 3.2.5. Let $S = \mathbb{Z}$, and $T = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$. Then, $\mathrm{cl}_T\, S = \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, which is scattered.

Consider ω a weight on S such that $\lim_{n\to\infty} \omega(n) = \infty$ and such that $\omega|(\mathbb{Z} \setminus \mathbb{N})$ is bounded. Then $\mathcal{A}_\omega$ is neither Arens regular nor strongly Arens irregular.

Example 3.2.6. Let $S = \mathbb{Q}^{+\bullet} = \{p \in \mathbb{Q} : p > 0\}$. Consider a weight $\omega : \mathbb{Q} \to [1,\infty)$ such that $\omega(p) = 1$ $(p \in [0,1] \cap S)$ and such that $\lim\limits_{p\to\infty} \omega(p) = \infty$. Then $\mathcal{A}_\omega$ is not Arens regular, by Theorem 3.2.3 and it is also not strongly Arens irregular. This follows from Proposition 3.2.4. However, in this case we shall find a concrete element $M \in \mathfrak{Z}(\mathcal{A}_\omega{''})$, but such that $M \notin \mathcal{A}_\omega$.

Since $\lim\limits_{p\to\infty} \omega(p) = \infty$, there exists (p_n) a strictly increasing sequence such that $\lim\limits_{n\to\infty} \omega(p_n) = \infty$. For clarity we will call that sequence P. Consider u in the growth of P, and let $v \in S^*$ a different element. By Proposition 3.1.9 there are two sequences (s_n), (t_n) of elements of S such that $\Omega_\Box(u,v) = \lim\limits_{m\to\infty}\lim\limits_{n\to\infty} \Omega(s_m, t_n)$ and

$\Omega_\Diamond(u,v) = \lim_{n\to\infty}\lim_{m\to\infty}\Omega(s_m,t_n)$. As (s_n) is unbounded, we have that $\Omega_\Diamond(u,v)=0$. Now, if (t_n) is also unbounded, then $\lim_{n\to\infty}\omega(t_n)=\infty$ too. Thus, $\Omega_\Box(u,v)=0$. If (t_n) is bounded, then, for every $n,m\in\mathbb{N}$ (except maybe a finite number), $s_m\geq t_n$, and so

$$\Omega_\Diamond(u,v) = \lim_{m\to\infty}\lim_{n\to\infty}\frac{\omega(s_m\wedge t_n)}{\omega(s_n)\omega(t_n)} = \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{\omega(s_n)} = 0.$$

Hence

$$\Omega_\Box(u,v) = \Omega_\Diamond(u,v) = 0. \tag{3.2.3}$$

So, by (3.2.1), (3.2.2) and (3.2.3), $\delta_u \in \mathfrak{Z}(\mathcal{A}_\omega{}'')$, but $\delta_u \notin \mathcal{A}_\omega$. Thus $\mathcal{A}_\omega$ is not strongly Arens irregular.

3.2.2. Approximate identities. The existence of approximate identities is an interesting characteristic of Banach algebras. We shall see in the following subsection that the existence of approximate identities is intrinsically linked to duality and to Arens regularity.

In the following result we refer to a sequence (s_n) of elements of S that tends to $\sup S$. This means that, for any element $r\in S$, there exists $N\in\mathbb{N}$ such that $r\leq s_n$ for any $n\geq N$.

Proposition 3.2.7. *Let $(S,\wedge)$ be an infinite, totally ordered semilattice and ω a weight on S. Then, the weighted semigroup algebra $\mathcal{A}_\omega$ has an approximate identity. The following are true:*

(a) $\mathcal{A}_\omega$ has a bounded approximate identity if and only if there exists a net (s_ν) tending to $\sup S$ such that the set $\{\omega(s_\nu):\nu\}$ is bounded.

(b) Suppose that for every strictly increasing net (t_ν) tending to $\sup S$, the set $\{\omega(t_\nu):\nu\}$ is unbounded. Then, $\mathcal{A}_\omega$ has a sequential approximate identity.

$\mathcal{A}_\omega$ always has a multiplier-bounded approximate identity.

Proof. Let $\omega : S\to[1,\infty)$, and let $\alpha\in\mathcal{A}_\omega$. Then, for $s\in S$, we have

$$\alpha\star\delta_s = \sum_{t<s}\alpha(t)\delta_t + \sum_{t\geq s}\alpha(t)\delta_s \tag{3.2.4}$$

and

$$\alpha - \alpha\star\delta_s = \sum_{t>s}\alpha(t)(\delta_t-\delta_s). \tag{3.2.5}$$

Suppose that there exists a strictly increasing net (s_ν) tending to $\sup S$ such that the set $\{\omega(s_\nu) : \nu\}$ is bounded by $M \geq 1$. Then, for $\alpha \in \mathcal{A}_\omega$, using (3.2.5) we see that

$$\|\alpha - \alpha \star \delta_{s_\nu}\|_\omega \leq \sum_{t>s_\nu} |\alpha(t)|\omega(t) + \left|\sum_{t>s_\nu} \alpha(t)\right| \omega(s_\nu) \leq (M+1) \sum_{t>s_\nu} |\alpha(t)|\omega(t),$$

which tends to zero since $\|\alpha\|_\omega < \infty$ and $s_\nu \to \sup S$. Thus, (s_ν) is an approximate identity. Since $\|\delta_{s_\nu}\|_\omega = \omega(s_\nu)$, then it is a bounded approximate identity, with bound M. From (3.2.4) we deduce that (s_ν) is a multiplier-bounded approximate identity.

On the other hand, suppose that for every strictly increasing net (t_ν) tending to $\sup S$, the set $\{\omega(t_\nu) : \nu\}$ is unbounded. In this case, we can choose a strictly increasing sequence (s_ν) tending to $\sup S$ and such that $\omega(s_\nu) = \inf\{\omega(t) : s_\nu \leq t\}$. To see this, consider the map $\tilde{\omega} : S \longrightarrow [1, \infty)$ such that $\tilde{\omega}(s) = \inf\{\omega(t) : s \leq t\}$. Since $\lim_\nu \omega(t_\nu) = \infty$ for every (t_ν) tending to $\sup S$, this infimum exists. Let s_1 such that $\omega(s_1) = \tilde{\omega}(1)$. Knowing s_n, take s_{n+1} with $s_n < s_{n+1}$ and such that $\omega(s_{n+1}) = \tilde{\omega}(s_n + 1)$. This sequence is such that $\omega(s_n) \leq w(t)$ for all $s_n \leq t$, and so, using (3.2.5), we have that

$$\|\alpha - \alpha \star \delta_{s_n}\|_\omega \leq \sum_{t>s_n} |\alpha(t)|\omega(t) + \left|\sum_{t>s_n} \alpha(t)\right| \omega(s_n) \leq 2 \sum_{t>s_n} |\alpha(t)|\omega(t).$$

Using the same reasoning as before, we see that (s_n) defined this way is an approximate identity, and it is again a multiplier-bounded approximate identity.

Assume now towards contradiction that there is a bounded approximate identity in $\mathcal{A}_\omega$. Then $\|\cdot\|_\omega$ and $\|\cdot\|_{op}$ are equivalent. However

$$\lim_{n\to\infty} \|\delta_{s_n}\|_\omega = \lim_{n\to\infty} \omega(s_n) = \infty$$

but $\|\delta_{s_n}\|_{op} \leq 1$, for all $n \in \mathbb{N}$. Thus, in this case $\mathcal{A}_\omega$ does not have a bounded approximate identity, as desired. $\square$

Note that [(b)] is not a characterization, since as long as we can take a sequence (s_n) tending to $\sup S$, we can have totally ordered semilattices with a bounded sequential approximate identity. We shall see examples of this in the following sections.

Notice that, when S is an infinite totally ordered semilattice such that the supremum of S, r, belongs to S, then $\delta_r/\omega(r)$ is an identity in $\mathcal{A}_\omega$. In fact we have a better result:

COROLLARY 3.2.8. *Let S be a totally ordered semilattice. Then $\mathcal{A}_\omega$ has an identity if and only if* $\sup S \in S$.

PROOF. As seen above, if $\sup S \in S$, then $\mathcal{A}_\omega$ has an identity. Now suppose that $r = \sup S \notin S$ and assume towards contradiction that $\mathcal{A}_\omega$ has an identity e. Let us consider (s_ν) a net of elements in S such that (δ_{s_ν}) is an approximate identity defined as in Proposition 3.2.7. Recall that we can choose (s_ν) tending to $\sup S$. Then we have that

$$e = \lim_\nu \delta_{s_\nu} \star e = \lim_\nu \delta_{s_\nu} = \delta_r$$

but we were supposing that $r \notin S$, and so $e = \delta_r \notin \mathcal{A}_\omega$. □

COROLLARY 3.2.9. *Let $(S, \wedge)$ be an infinite, totally ordered semilattice, and let $\omega : S \to [1, \infty)$ be a weight on S. Suppose that $\mathcal{A}_\omega$ has a sequential approximate identity. Then when we view $\mathcal{A}_\omega$ as a Banach function algebra on Φ_S, it is Tauberian.*

PROOF. This follows from the fact that the sequential approximate identity of $\mathcal{A}_\omega$ belongs to $c_{00}(S, 1/\omega)$. □

3.2.3. (Non)-existence of Banach algebra preduals. As we have seen in Section 2.4.2, the space E_ω is a concrete predual of $\mathcal{A}_\omega$. However, it is not true that it is always a Banach-algebra predual. We shall study below in which situations we can know that E_ω is a predual of $\mathcal{A}_\omega$ and when it is unique. Also we shall give characterizations of when $\mathcal{A}_\omega$ is not a dual Banach-algebra for any concrete predual.

We shall start with some results that refer to generic semigroups, and we shall focus on totally ordered semilattices afterwards.

The following result is an extension of [22, Theorem 4.6], where they work only with the unweighted case.

PROPOSITION 3.2.10. *Let S be an infinite semigroup, and let ω be a weight on S.*

(a) Suppose that S is weakly cancellative. Then E_ω is a Banach-algebra predual of $\mathcal{A}_\omega$.

(b) Suppose that S is not weakly cancellative. Suppose that there is a subset $U \subset S$ such that

(i) U is infinite;

(ii) $\omega|U$ is bounded;

(iii) there exist $t, u \in S$ such that $\{r \in U : rt = u\}$ is infinite.

Then E_ω is not the predual of $\mathcal{A}_\omega$ as a Banach algebra.

Proof. (a) Let S be a weakly cancellative semigroup. Let $E_\omega = c_0(S, 1/\omega)$, and let $\lambda = (\lambda(s)) \in E_\omega$ and $\tilde{\delta_s} \in \mathcal{A}_\omega$. Then $\tilde{\delta_s} \cdot \lambda \in E_\omega$. Indeed, for $\varepsilon > 0$ we have that there is a finite subset F of S such that

$$\left|\frac{\lambda(t)}{\omega(t)}\right| < \varepsilon \quad (t \in S \setminus F).$$

Hence, as S is weakly cancellative, the set $Fs^{-1} = \{r \in S : rs \in F\}$ is finite too. Thus

$$\left|\frac{(\tilde{\delta_s} \cdot \lambda)(r)}{\omega(r)}\right| = \left|\frac{\lambda(rs)}{\omega(r)\omega(s)}\right| \leq \left|\frac{\lambda(rs)}{\omega(rs)}\right| < \varepsilon \quad (r \in S \setminus Fs^{-1}).$$

The fact that $\lambda \cdot \tilde{\delta_s} \in E_\omega$ is symmetrical. And so E_ω is a submodule of $\mathcal{A}_\omega{}'$. Thus $\mathcal{A}_\omega$ is a dual Banach algebra as desired.

(b) Suppose now that S is not weakly cancellative and that $U \subset S$ satisfies (i), (ii) and (iii). Take $s_1, s_2, ... \in U$ with $s_n t = u$ for every $n \in \mathbb{N}$.
Consider $\alpha^{(n)} = \frac{1}{n}\left(\tilde{\delta}_{s_1} + \cdots + \tilde{\delta}_{s_n}\right)$ for $n \in \mathbb{N}$. Then $(\alpha^{(n)})$ is a sequence in $\mathcal{A}_\omega$ that tends to zero in $\sigma(\mathcal{A}_\omega, E_\omega)$. Indeed, for $\lambda \in E_\omega$ and for $\varepsilon > 0$ there exists a finite subset F of S such that $\left|\frac{\lambda(s)}{\omega(s)}\right| < \varepsilon$ for $s \in S \setminus F$. Thus $V = \{s_1, s_2, ...\} \cap F$ is finite and so $\sum_{s \in V} \left|\frac{\lambda(s)}{\omega(s)}\right|$ is bounded by $K > 0$. Then

$$\left|\langle \alpha^{(n)}, \lambda \rangle\right| \leq \frac{1}{n} \sum_{i=1}^{n} \left|\frac{\lambda(s_i)}{\omega(s_i)}\right| \leq \frac{K}{n} + \varepsilon.$$

Thus $(\alpha^{(n)})$ tends to zero in $\sigma(\mathcal{A}_\omega, E_\omega)$.

Let $M < \infty$ be a bound for $\omega(s)$ $(s \in U)$, and take $\lambda \in E_\omega$ such that $\langle \delta_u, \lambda \rangle \neq 0$. Then

$$\left|\langle \alpha^{(n)} \star \delta_t, \lambda \rangle\right| \geq \frac{1}{M} \left|\langle \delta_u, \lambda \rangle\right| > 0.$$

Thus the multiplication in $\mathcal{A}_\omega$ is not separately $\sigma(\mathcal{A}_\omega, E_\omega)$-continuous, and so, by Theorem 2.3.16, E_ω is not a submodule of $\mathcal{A}_\omega{}'$. □

As we have pointed out before, even some of the simplest examples of totally ordered semilattices are not weakly cancellative, so they are not covered by the former result. In the rest of this section we shall study what can be said for totally ordered semilattices.

PROPOSITION 3.2.11. *Let $(S, \wedge)$ be an infinite, totally ordered semilattice, and let $\omega : S \longrightarrow [1, \infty)$ be a weight on S. Suppose that* $\operatorname{Lim}_s \omega(s) = \infty$. *Then $\mathcal{A}_\omega$ is a dual Banach algebra with Banach-algebra predual E_ω.*

PROOF. Let $\lambda = (\lambda(s)) \in E_\omega$ such that $\|\lambda\|'_\omega = 1$, and let $\tilde{\delta}_t \in \mathcal{A}_\omega$ be the normalised point mass at $t \in S$ defined as above. Then, for $s \geq t$, we have

$$\frac{|(\tilde{\delta}_t \cdot \lambda)(s)|}{\omega(s)} = \frac{|\lambda(t)|}{\omega(s)\omega(t)} \leq \frac{1}{\omega(s)}.$$

As in Definition 3.1.1, for every $\varepsilon > 0$, there is a finite subset U_1 of S such that $\omega(s) > 1/\varepsilon$ $(s \in S \setminus U_1)$. Hence in particular we have that $|(\tilde{\delta}_t \cdot \lambda)(s)|/\omega(s) < \varepsilon$ $(s \in S \setminus U_1)$. Now, for $s < t$, we have

$$\frac{|(\tilde{\delta}_t \cdot \lambda)(s)|}{\omega(s)} = \frac{|\lambda(s)|}{\omega(s)\omega(t)} \leq \frac{|\lambda(s)|}{\omega(s)}.$$

Since $\lambda \in E_\omega$, there exists U_2 a finite subset of S such that $|\lambda(s)|/\omega(s) < \varepsilon$ for every $s \in S \setminus U_2$. Thus $|(\tilde{\delta}_t \cdot \lambda)(s)|/\omega(s) < \varepsilon$ for every $s \in S \setminus (U_1 \cup U_2)$. It follows that $(\tilde{\delta}_s \cdot \lambda) \in E_\omega$. Since

$$\langle \tilde{\delta}_s \cdot \lambda, \delta_t \rangle = \frac{\langle \lambda, \delta_{t \wedge s} \rangle}{\omega(s)} = \frac{\langle \lambda, \delta_{s \wedge t} \rangle}{\omega(s)} = \langle \lambda \cdot \tilde{\delta}_s, \delta_t \rangle \quad (s, t \in S),$$

we have that $(\lambda \cdot \tilde{\delta}_s) \in E_\omega$ too, and so E_ω is a closed submodule of $\mathcal{A}_\omega{}'$. Hence $\mathcal{A}_\omega$ is a dual Banach algebra as desired. □

For an infinite semilattice (not necessarily totally ordered), the following result give us a condition so that E_ω is unique as a Banach-algebra predual. Here we make use of previous results on the character space of $\mathcal{A}_\omega$. We remind the reader that, for a Banach algebra A, $L(A) = \operatorname{lin} \Phi_A$ and we see it as a linear subspace of A'.

PROPOSITION 3.2.12. *Let S be an infinite semilattice and let ω be a weight on S. Suppose that for all $x \in S$ the set $xS = \{xs : s \in S\}$ is finite and that* $\operatorname{Lim}_{t \to \infty} \omega(t) = \infty$. *Then:*

(a) For every $x \in S$, $L_{\delta_x} : \mathcal{A}_\omega \to \mathcal{A}_\omega$ is compact.

(b) Suppose that $E_\omega = \overline{L(\mathcal{A}_\omega)}$. Then E_ω is the unique Banach-algebra predual of $\mathcal{A}_\omega$.

PROOF. (*a*) Let $x \in S$, and let $\varepsilon > 0$. Let $M = \max\{\omega(xt) : t \in S\}$, which is well defined since xS is finite. Since $\operatorname{Lim}_{t \to \infty} \omega(t) = \infty$, there exists G a finite subset of

S such that $M/\omega(t) < \varepsilon$, $(t \in S \setminus G)$. Let $F = xG$, then

$$\{t \in S : xt \notin F\} \subseteq \{t \in S : t \notin G\}.$$

Let $f \in \mathcal{A}_\omega$. Then,

$$\begin{aligned} \|\pi_F\left(L_{\delta_x}(f)\right) - L_{\delta_x}(f)\|_\omega &= \sum_{u \in S \setminus F} \left| \sum_{t \in S : xt = u} f(t) \right| \omega(u) \\ &\leq \sum_{u \in S \setminus F} \sum_{t \in S : xt = u} |f(t)|\omega(u) = \sum_{t \in S : xt \notin F} |f(t)|\omega(xt) \\ &\leq \|f\|_\omega \sup\{\frac{M}{\omega(t)} : xt \notin F\} \leq \varepsilon \|f\|_\omega, \end{aligned}$$

where π_F denotes the projection of $\ell^1(S,\omega)$ onto $span\{\delta_s : s \in F\}$. Thus, since $\pi_F \circ L_{\delta_x}$ is finite rank and therefore compact, we obtain that L_{δ_x} is also compact.

(b) Let W be a Banach-algebra predual of $\mathcal{A}_\omega$. Assume towards contradiction that there exists $\varphi \in \Phi_\omega \setminus W$. Then there exists $M \in {\mathcal{A}_\omega}''$ with $\|M\|_\omega \leq 1$ and such that $\langle M, \varphi\rangle = 1$ and $\langle M, \lambda\rangle = 0$ $(\lambda \in W)$. Thus there is a net (f_α) in $\mathcal{A}_\omega$ with norm bounded by 1 such that $\lim_\alpha f_\alpha(\varphi) = 1$ and $\lim_\alpha \langle f_\alpha, \lambda\rangle = 0$ $(\lambda \in W)$. Since W is a Banach-algebra predual of $\mathcal{A}_\omega$, we may suppose that there exists $f \in \mathcal{A}_\omega$ such that $\lim_\alpha f_\alpha = f$ in $\sigma(\mathcal{A}_\omega, W)$. Thus $\langle f, \lambda\rangle = 0$ $(\lambda \in W)$. Hence $f = 0$. Let $s \in S$ such that $\delta_s(\varphi) = 1$, which exists by Proposition 3.1.14. By part [(a)], L_{δ_s} is compact and so δ_s is compact. Thus, $\lim_\alpha \delta_s \star f_\alpha = \delta_s \star f$ in $\sigma(W', W'')$. Since we assumed that W is a Banach-algebra predual of $\mathcal{A}_\omega$, $\sigma(W', W'') = \sigma(\mathcal{A}_\omega, {\mathcal{A}_\omega}')$. Hence, $1 = \lim_\alpha f_\alpha(\varphi) = f(\varphi)$, which is a contradiction, since $f \equiv 0$. Thus, $\Phi_\omega \subset W$. But this implies that $\overline{L(\mathcal{A}_\omega)} \subset W$. Hence, by Proposition 2.3.17, $W = \overline{L(\mathcal{A}_\omega)} = E_\omega$ and so E_ω is the unique Banach-algebra predual of $\mathcal{A}_\omega$. $\square$

In Proposition 3.3.4 we shall see that $\mathbb{N}$ with the minimum operation is in this situation and so we shall be able to identify when the Banach-algebra predual E_ω is unique.

Under certain conditions, $\mathcal{A}_\omega$ is not a dual Banach algebra for any predual.

Proposition 3.2.13. *Let $(S, \wedge)$ be an infinite, totally ordered semilattice. Suppose that* $\operatorname{Lim\,inf}_s \omega(s) < \infty$. *Suppose there exists an embedding of S in T as specified in Section 3.2.1, and that the set $\{s \in S : \omega(s) \leq \operatorname{Lim\,inf} \omega\}$ has an accumulation point $r \in T \setminus S$. Then the Banach algebra $\mathcal{A}_\omega$ is not a dual Banach algebra with respect to any predual.*

Proof. Let $U = \{s \in S : \omega(s) \leq \operatorname{Lim\,inf} \omega\}$, and let us assume towards a contradiction that there exists a Banach-algebra predual W for $\mathcal{A}_\omega$.

Suppose that $r = \sup S$. Then, by Proposition 3.2.7, $\mathcal{A}_\omega$ has a bounded approximate identity (δ_{s_ν}), with (s_ν) tending to r. Since $\mathcal{A}_\omega$ is a dual Banach algebra, there exists a subnet (δ_α) converging in the topology $\sigma(\mathcal{A}_\omega, W)$ to an identity $e \in \mathcal{A}_\omega$, but this is a contradiction with Corollary 3.2.8.

Suppose that $r \neq \sup S$. Then there exists a net (s_β) in U monotone decreasing or increasing converging to r. Suppose that (s_β) is decreasing. Since $\sup\{\|\delta_{s_\beta}\|_\omega : \beta\}$ is bounded, there exists a subnet (δ_α) that converges to an element $f \in \mathcal{A}_\omega$ in the topology $\sigma(\mathcal{A}_\omega, W)$. Let $s \in S$ such that $s > r$. Then $\delta_s \star \delta_\alpha = \delta_\alpha$ for every α large enough. Since the multiplication is separately $\sigma(\mathcal{A}_\omega, W)$-continuous, $\delta_s \star f = f$. This implies that $\operatorname{supp} f \subset (0, s] \cap S$ for every $s > r$. Thus, $\operatorname{supp} f \subset (0, r] \cap S$. Now let $s \in S$ such that $s < r$. Then $\delta_s \star \delta_\alpha = \delta_s$. Thus $\delta_s \star f = \delta_s$, which implies that f is not zero and that $\operatorname{supp} f \subset [s, \sup S) \cap S$. Since this is true for every $s < r$, then $\operatorname{supp} f \subset [r, \sup S) \cap S$. We conclude that $\operatorname{supp} f \subset \{r\} \cap S$. But $\{r\} \cap S = \emptyset$, and so there is no such $f \in \mathcal{A}_\omega$. The case where (s_β) is increasing is symmetrical. Thus $\mathcal{A}_\omega$ is not a dual Banach algebra. $\square$

Example 3.2.14. Let us consider again the semigroup and weight defined in Example 3.2.6, $\mathbb{Q}_\wedge^{+\bullet}$. In this case, $T = \mathbb{R}^{+\bullet} \cup \{\infty\}$, and $\operatorname{cl}_T S = \mathbb{R}^{+\bullet} \cup \{\infty\}$. Thus, by Proposition 3.2.13, $\mathcal{A}_\omega$ is not a dual Banach algebra.

3.3. Semigroup $S = \mathbb{N}_\wedge$

3.3.1. Arens Regularity and strong Arens irregularity, DTC sets. Consider the semigroup $S := \mathbb{N}$ with the semigroup operation

$$m \wedge n = \min\{m, n\} \quad (m, n \in \mathbb{N}),$$

which is a particular case of the above. Throughout this section we shall write $D_\omega = \ell^1(\mathbb{N}_\wedge, \omega)$.

We shall give below some results that improve what we have obtained in the more generic case.

In the previous section we have seen when a weighted semigroup algebra is not strongly Arens irregular, however we do not have a characterization of when it is strongly Arens irregular. In the following result we shall see that for $S = \mathbb{N}_\wedge$, not only we can see when D_ω is strongly Arens irregular, but we can also determine the smallest DTC set. In [22, Example 7.33] they work with the unweighted case and they prove that it is strongly Arens irregular, which also follows from [24, Theorem 2.14]. The argument followed here is very similar to the one followed in [22]. Note that smallest might suggest that it is unique, however, we mean that the size is as small as possible, but there might be several DTC sets with the same size (as small as possible). This terminology follows that of [22].

PROPOSITION 3.3.1. *Let $S = \mathbb{N}$ with the semigroup operation $\wedge$ defined as above. Then whenever $\liminf_{n\to\infty} \omega(n) < \infty$ the Banach algebra D_ω is strongly Arens irregular. Furthermore, there is a two-point DTC set of D''_ω.*

PROOF. Consider the isometric isomorphism $\theta_\omega : \ell^1(\mathbb{N}) \to D_\omega$ defined in Lemma 2.4.9.

We have that

$$\delta_u \Box \delta_v = \delta_u, \quad (u \in \beta\mathbb{N}, v \in \mathbb{N}^*);$$

$$\delta_u \Diamond \delta_v = \delta_v, \quad (u \in \mathbb{N}^*, v \in \beta\mathbb{N}).$$

Let $u, v \in \beta\mathbb{N}$ and (s_α), (t_β) nets in $\mathbb{N}$ such that $u = \lim_\alpha s_\alpha$ and $v = \lim_\beta t_\beta$. Thus, for $\lambda \in C(\beta\mathbb{N})$, we have

$$\begin{aligned}\langle \theta''_\omega(\delta_u) \Box_\omega \theta''_\omega(\delta_v), \omega\lambda\rangle &= \lim_\alpha \lim_\beta \langle \theta''_\omega(\delta_{s_\alpha}) \Box_\omega \theta''_\omega(\delta_{t_\beta}), \omega\lambda\rangle \\ &= \Omega_\Box(u, v)\langle \delta_u \Box \delta_v, \lambda\rangle = \Omega_\Box(u, v)\langle \delta_u, \lambda\rangle.\end{aligned}$$

Symmetrically we obtain that

$$\langle \theta''_\omega(\delta_u) \Diamond_\omega \theta''_\omega(\delta_v), \omega\lambda\rangle = \Omega_\Diamond(u, v)\langle \delta_v, \lambda\rangle.$$

Consider $\psi_\omega \in \ell^\infty(\mathbb{N})$ defined as $\psi_\omega(n) = 1/\omega(n)$, $(n \in \mathbb{N})$. Let $u \in \mathbb{N}^*$, $v \in \beta S$ and say $u = \lim_\alpha s_\alpha$, $v = \lim_\alpha t_\beta$ where (s_α), (t_β) are nets in $\mathbb{N}$. Then,

$$\Omega_\Box(u, v) = \lim_\beta 1/\omega(t_\beta) = \lim_\alpha \langle \delta_{t_\beta}, \psi_\omega\rangle = \langle \delta_v, \psi_\omega\rangle.$$

Let $v \in \beta\mathbb{N}$, $g \in \ell^1(\mathbb{N}^*)$ and $\lambda \in C(\beta\mathbb{N})$. Then

$$\begin{aligned}\langle \theta''_\omega(\delta_v) \Box_\omega \theta''_\omega(g), \omega\lambda\rangle &= \langle \sum_{u\in\mathbb{N}^*} g(u)(\theta''_\omega(\delta_v) \Box_\omega \theta''_\omega(\delta_u), \omega\lambda\rangle \\ &= \langle \sum_{u\in\mathbb{N}^*} \Omega_\Box(v, u) g(u)\delta_v, \lambda\rangle = \langle g, \psi_\omega\rangle\langle \delta_v, \lambda\rangle.\end{aligned}$$

Thus, for $\mu \in M(\beta\mathbb{N}), \nu \in M(\mathbb{N}^*)$, we have

$$\theta''_\omega(\mu) \Box_\omega \theta''_\omega(\nu) = \langle \nu, \psi_\omega\rangle\mu. \tag{3.3.1}$$

Symmetrically, we obtain

$$\theta''_\omega(\nu)\Diamond_\omega\theta''_\omega(\mu) = \langle \mu, \psi_\omega\rangle \nu \quad (\nu \in M(\beta\mathbb{N}), \mu \in M(\mathbb{N}^*)) \tag{3.3.2}$$

Hence $\theta''_\omega(M(\mathbb{N}^*))$ is a closed subalgebra of D''_ω. Also for $\mu \in M(\beta\mathbb{N})$ and $f \in \ell^1(\mathbb{N})$, we have

$$\theta_\omega(f) \cdot \theta''_\omega(\mu) = \theta''_\omega(\mu) \cdot \theta_\omega(f) = \langle \mu, \psi_\omega\rangle f.$$

Hence D_ω is an ideal in its bidual and we can write

$$D''_\omega = \theta''_\omega(M(\mathbb{N}^*)) \ltimes D_\omega.$$

Let (s_n) be a sequence in $\mathbb{N}$ such that $\sup\{\omega(s_n) : n \in \mathbb{N}\} < \infty$ and let a be in the growth of this sequence, so that $\langle \delta_a, \psi_\omega\rangle$ is not zero. Let $b \in \mathbb{N}^*$ different from a. Take $\mu \in M(\mathbb{N}^*)$ such that

$$\theta''_\omega(\mu)\Box_\omega\theta''_\omega(\delta_a) = \theta''_\omega(\mu)\Diamond_\omega\theta''_\omega(\delta_a) \text{ and } \theta''_\omega(\mu)\Box_\omega\theta''_\omega(\delta_b) = \theta''_\omega(\mu)\Diamond_\omega\theta''_\omega(\delta_b).$$

By (3.3.1),

$$\theta''_\omega(\mu)\Box_\omega\theta''_\omega(\delta_a) = \langle \delta_a, \psi_\omega\rangle \mu$$

and by (3.3.2), we have that

$$\theta''_\omega(\mu)\Diamond_\omega\theta''_\omega(\delta_a) = \langle \mu, \psi_\omega\rangle \delta_a. \tag{3.3.3}$$

Since $\theta''_\omega(\mu)\Box_\omega\theta''_\omega(\delta_a) = \theta''_\omega(\mu)\Diamond_\omega\theta''_\omega(\delta_a)$, we have

$$\langle \delta_a, \psi_\omega\rangle \mu = \langle \mu, \psi_\omega\rangle \delta_a.$$

Repeating exactly the same calculations for μ and δ_b we obtain

$$\langle \delta_b, \psi_\omega\rangle \mu = \langle \mu, \psi_\omega\rangle \delta_b.$$

Hence $\langle \delta_b, \psi_\omega\rangle\langle \mu, \psi_\omega\rangle \delta_a = \langle \delta_a, \psi_\omega\rangle\langle \mu, \psi_\omega\rangle \delta_b$. Since $\langle \delta_a, \psi_\omega\rangle \neq 0$, we must have $\langle \mu, \psi_\omega\rangle = 0$. Thus, by substituting this in (3.3.3) we obtain that $\mu = 0$ and so $\theta''_\omega(\mu) = 0$. Thus D_ω is strongly Arens irregular and

$$V = \{\theta''_\omega(\delta_a), \theta''_\omega(\delta_b)\}$$

is a DTC set for D''_ω. $\square$

REMARK 3.3.2. In [20, Example 9.13] it is observed that D_ω is Arens regular for any weight ω such that $\omega(n) \to \infty$ when $n \to \infty$. However, the authors justify this by appealing to a theorem only stated for cancellative semigroups, while $\mathbb{N}_\wedge$ is not even weakly cancellative. By applying our Theorem 3.2.3 the desired result follows.

Thus we obtain the following characterization of Arens regularity for D_ω:

THEOREM 3.3.3. *Let $\omega : \mathbb{N} \to [1, \infty)$. Then:*

(a) D_ω is Arens regular if and only if $\liminf_{n\to\infty} \omega(n) = \infty$;

(b) D_ω is strongly Arens irregular if and only if $\liminf_{n\to\infty} \omega(n) < \infty$. □

3.3.2. Duality of D_ω. We shall now look at the duality of D_ω.

PROPOSITION 3.3.4. *Let $\omega : \mathbb{N} \to [1, \infty)$. Then D_ω is a dual Banach algebra if and only if $\lim_{n\to\infty} \omega(n) = \infty$. In this case, E_ω is the unique Banach-algebra predual.*

PROOF. When $\liminf\limits_{n\to\infty} \omega(n) < \infty$ the Banach algebra D_ω is not a dual Banach algebra, as follows from Proposition 3.2.13. The fact that D_ω is a dual Banach algebra with predual E_ω when $\lim\limits_{n\to\infty} \omega(n) = \infty$ follows from Proposition 3.2.11.

Let us see now that E_ω is unique. Let $\varphi \in \Phi_\omega$. Then there exists $k \in \mathbb{N}$ such that $\varphi = \varphi_k$ where

$$\varphi_k(\alpha) = \sum_{n=k}^{\infty} \alpha(n) \quad (\alpha = (\alpha(n)) \in D_\omega). \tag{3.3.4}$$

When $\lim_{n\to\infty} \omega(n) = \infty$, we have that $\varphi \in E_\omega$ and so $\Phi_\omega \subset E_\omega$. Hence $\overline{L(D_\omega)} \subset E_\omega$. Let us see now that $E_\omega \subset \overline{L(D_\omega)}$. Since E_ω is the closure of c_{00} in $\ell^\infty(\mathbb{N}, 1/\omega)$, it is enough to see that $c_{00} \subset \overline{L(D_\omega)}$. For $k \in \mathbb{N}$, consider $\rho^{(k)}$ defined as follows:

$$\rho^{(k)} = \varphi_k - \varphi_{k+1},$$

where φ_k is defined in (3.3.4). Then $c_{00} = span\{\rho^{(k)} : k \in \mathbb{N}\}$. Also, we have that $\rho^{(k)} \in \Phi_\omega$ and so, $c_{00} \subset \overline{L(D_\omega)}$ as needed. We conclude then that $E_\omega \subset \overline{L(D_\omega)}$ as desired.

Finally, we see that, for $s \in \mathbb{N}$,

$$s\mathbb{N} = \{s \wedge n : n \in \mathbb{N}\} = \{n \in \mathbb{N} : n \leq s\},$$

which is finite. So, we can apply Proposition 3.2.12 to obtain that E_ω is the unique Banach-algebra predual of D_ω. □

To conclude the study of D_ω, notice that by Proposition 3.2.7, we obtain that D_ω has a bounded approximate identity if and only if $\liminf_{n\to\infty} \omega(n) < \infty$ and that it always has a multiplier-bounded approximate identity.

3.3.3. Weighted bounded variation algebras.

These algebras were first studied in [33]; for an account see [26, Example 3.2.12]. Let $\omega = (\omega(i))$ be a sequence in $[1, \infty)$, and, for $n \in \mathbb{N}$, $\alpha \in \mathbb{C}^{\mathbb{N}}$, define

$$p_\omega^n(\alpha) = \sum_{i=1}^{n} \omega(i)|\alpha(i+1) - \alpha(i)|,$$

so $(p_\omega^n(\alpha) : n \in \mathbb{N})$ is increasing.

Set $p_\omega(\alpha) = \lim_{n\to\infty} p_\omega^n(\alpha)$, and consider

$$B_\omega = \{\alpha \in \ell^\infty : p_\omega(\alpha) < \infty\}$$

with the norm in B_ω defined as follows:

$$|||\alpha|||_\omega = |\alpha|_{\mathbb{N}} + p_\omega(\alpha) \quad (\alpha \in B_\omega).$$

Then $(B_\omega, |||\cdot|||_\omega)$ is a self-adjoint Banach sequence algebra on $\mathbb{N}$, and it is natural on $\mathbb{N}_\infty$. In the case where $\omega(n) = 1$ for all $n \in \mathbb{N}$, the elements of B_ω are the sequences of *bounded variation*, and we denote $(B_\omega, |||\cdot|||_\omega)$ by $(bv, |||\cdot|||_{bv})$ and p_ω as p_{bv}. As it is seen in [26, Example 3.2.12] for any sequence $\omega : \mathbb{N} \longrightarrow [1, \infty)$ we have that $B_\omega \subset bv$ and $|||\alpha|||_{bv} \le |||\alpha|||_\omega$. If ω is bounded by $c \ge 1$, then, $bv = B_\omega$ and the norms $|||\cdot|||_\omega$ and $|||\cdot|||_{bv}$ are equivalent in B_ω as $|||\alpha|||_\omega = |\alpha|_{\mathbb{N}} + p_\omega(\alpha) \le |\alpha|_{\mathbb{N}} + c p_{bv}(\alpha) \le c|||\alpha|||_{bv}$.

Set $M_\omega = B_\omega \cap c_0$. Then M_ω is a maximal ideal in B_ω, as it is the kernel of a character, namely the evaluation at ∞. Hence, M_ω is a natural Banach sequence algebra on $\mathbb{N}$. In particular, $bv_0 = bv \cap c_0$ is a natural Banach sequence algebra on $\mathbb{N}$.

The following result can be found in [23, § 3.2]:

PROPOSITION 3.3.5. *Let ω be a sequence such that $\omega \ge 1$. Then the algebra M_ω is the Gel'fand transform of D_ω.*

PROOF. By Proposition 3.1.14 the character space of D_ω is $\Phi_\omega = \mathbb{N}$. The Gel'fand transform $\widehat{\alpha}$ of an element $\alpha = (\alpha(n)) \in D_\omega$ is a sequence such that

$$\widehat{\alpha}(n) = \sum_{i=n}^{\infty} \alpha(i) \quad (n \in \mathbb{N}).$$

Take $\widehat{\alpha}$ defined in this way. Then, we have that

$$\sum_{i=1}^{n} \omega(i)|\widehat{\alpha}(i+1) - \widehat{\alpha}(i)| = \sum_{i=1}^{n} \omega(i)|\alpha(i)| \le \|\alpha\|_\omega.$$

and so $p_\omega(\widehat{\alpha})$ is bounded. Also,

$$|\widehat{\alpha}(n)| \leq \sum_{i=n}^{\infty} |\alpha(i)| \to 0 \quad (n \to \infty).$$

Thus, $\widehat{\alpha} \in M_\omega$.

Now, given an element $\beta = (\beta(n)) \in \widehat{D_\omega}$, the element $\alpha \in D_\omega$ such that $\widehat{\alpha} = \beta$ is defined as

$$\alpha(n) = \beta(n+1) - \beta(n) \quad (n \in \mathbb{N}).$$

Then we have that

$$\widehat{D_\omega} = \{\alpha \in c_0 : \sum_{n=1}^{\infty} |\alpha(n+1) - \alpha(n)|\omega(n) < \infty\} = M_\omega,$$

together with the norm $\|\alpha\| = \sum_{n=1}^{\infty} |\alpha(n+1) - \alpha(n)|\omega(n) < \infty$.
Finally, we see that $\|\alpha\| \leq |||\alpha|||_\omega \leq 2\|\alpha\|$ for every $\alpha \in M_\omega$. □

From this result, and taking into account the results from last section, we can obtain some conclusions for M_ω. For example, we see that it has a bounded approximate identity if and only if $\liminf \omega(n) < \infty$. It always has a multiplier-bounded approximate identity. Consider $\widehat{\delta_n} = \Delta_n$. We can find a subsequence that is an approximate identity for M_ω, and so we deduce that M_ω is Tauberian.

Regarding Arens regularity, when $\liminf \omega(n) = \infty$ we see that M_ω is Arens regular; and M_ω is strongly Arens irregular when $\liminf_{n\to\infty} \omega(n) < \infty$. This follows from Proposition 3.3.1 and Remark 3.3.2.

We proceed to study now whether M_ω is a BSE algebra or not and whether it has a BSE norm.

Proposition 3.3.6. *Let $\omega : \mathbb{N} \to [1, \infty)$ be a sequence, and consider M_ω defined as above. Then:*

(a) Suppose that $\liminf_{n\to\infty} \omega(n) < \infty$. Then $B_\omega = C_{BSE}(M_\omega) = \mathcal{M}(M_\omega)$, and so M_ω is a BSE algebra.

(b) Suppose $\liminf_{n\to\infty} \omega(n) = \infty$, then $M_\omega = C_{BSE}(M_\omega)$, and so M_ω is not a BSE algebra.

In these cases M_ω has a BSE norm.

Proof. Let $\omega : \mathbb{N} \to [1, \infty)$ be a sequence, then $C_{BSE}(M_\omega) \subset B_\omega$. Indeed, for $\beta \in C_{BSE}(M_\omega)$ there exists a net $(\alpha^{(\nu)})$ in M_ω converging to β pointwise with $\left|\left|\left|\alpha^{(\nu)}\right|\right|\right|_\omega \leq C$ for a given C. Let $n \in \mathbb{N}$ be fixed. Then $|\alpha^{(\nu)}(n)| + p_\omega^n(\alpha^{(\nu)}) \leq C$ for all

ν. Taking limits when $\nu \to \infty$ we have that $|\beta(n)| + p_\omega^n(\beta) \leq C$, and so we have that $\beta \in B_\omega$ with $|||\beta|||_\omega \leq C$. We conclude that $|||\beta|||_\omega = \|\beta\|_{BSE}$. Hence we have that $M_\omega \subset C_{BSE}(M_\omega) \subset B_\omega$. As M_ω is a maximal ideal in B_ω, either $M_\omega = C_{BSE}(M_\omega)$ or $B_\omega = C_{BSE}(M_\omega)$.

Since M_ω is a Tauberian banach sequence algebra (recall $(\Delta - n)$ is an approximate identity in M_ω), and so an ideal in its bidual, we can apply Corollary 2.3.33, and so $M_\omega = C_{BSE}(M_\omega)$ if and only if it is dual. We know that M_ω is a dual Banach algebra if and only if $\lim \omega(n) = \infty$. So, we conclude that $M_\omega = C_{BSE}(M_\omega)$ if and only if $\lim \omega(n) = \infty$. In this case, M_ω is not a BSE algebra.

Now suppose that $\liminf_{n\to\infty} \omega(n) \leq c$ for $c \geq 1$, then $\mathcal{M}(M_\omega) \subset B_\omega$. Indeed, for $\beta = (\beta_n)$ in $\mathcal{M}(M_\omega)$ we have that

$$\begin{aligned}
&\max\{|\beta(i)| : i \in \mathbb{N}_{n_k}\} + \sum_{i=1}^{n_k - 1} \omega(i)|\beta(i+1) - \beta(i)| + \omega(n_k)|\beta(n_k)| \\
&= |||\Delta_{n_k}\beta|||_\omega \leq |||\Delta_{n_k}|||_\omega \|\beta\|_{op} \leq (1+c)\|\beta\|_{op}.
\end{aligned}$$

Hence, $|||\beta|||_\omega \leq (1+c)\|\beta\|_{op}$, and so for ω bounded we have that $\mathcal{M}(M_\omega) = B_\omega$ and so M_ω is a BSE algebra.

Since for any ω, M_ω as a multiplier-bounded approximate identity, by Corollary 2.3.32, M_ω has a BSE norm. □

Remark 3.3.7. Note that, since $B_\omega = M_\omega^{\#}$, by Corollary 2.3.32, B_ω also has a BSE norm.

3.4. Semigroup $\mathbb{N}_\vee$

During this section we shall study the weighted semigroup algebra that arises from the semigroup $\mathbb{N}_\vee$. This semigroup is very similar to the one studied in the previous section. However, there are important differences between both semigroups, as we shall see below, and so the reasonings used, although similar, are not identical.

3.4.1. BSE algebras and BSE norms. Consider the semigroup $S := \mathbb{N}$ with the semigroup operation

$$m \vee n = \max\{m, n\} \quad (m, n \in \mathbb{N}).$$

Every sequence $\omega : \mathbb{N} \longrightarrow [1, \infty)$ is a weight on $\mathbb{N}_\vee$. During this section we shall denote by C_ω the semigroup algebra $\ell^1(\mathbb{N}_\vee, \omega)$.

As it was seen in Example 2.4.4, $\mathbb{N}_\vee$ is a semilattice as in Definition 2.4.2, and so, results from Section 3.2 apply to it. However, as it was the case with $\mathbb{N}_\wedge$, we can provide stronger results in this situation.

The Banach algebra C_ω has an identity, namely δ_1. Note that this agrees with Corollary 3.2.8, since when we consider the semilattice structure that follows from this multiplication the supremum of the semilattice $\mathbb{N}_\vee$ is 1.

The main difference with the case $\mathbb{N}_\wedge$ is that the semigroup $\mathbb{N}_\vee$ is weakly cancellative. We shall see that this difference creates some variations in the characteristics of C_ω and D_ω. However, some properties, like Arens regularity, are very similar.

For example, since $\mathbb{N}_\vee$ is weakly cancellative we obtain the following result, in contrast with the one obtained for D_ω.

Corollary 3.4.1. *Let $\omega : \mathbb{N} \longrightarrow [1, \infty)$. Then C_ω is a dual Banach algebra with predual E_ω.*

Proof. This follows from Proposition 3.2.10. $\square$

Note that since, for $x \in \mathbb{N}$, $x\mathbb{N} = \{xn : n \in \mathbb{N}\} = \{n \geq s\}$ is infinite, we cannot apply Proposition 3.2.12.

We shall continue studying whether C_ω is a BSE algebra and whether it has a BSE norm.

Proposition 3.4.2. *Let ω be a weight on S. The Gel'fand transform of C_ω is a BSE algebra with a BSE norm, with $C_{BSE}(\widehat{C_\omega}) = \widehat{C_\omega}$.*

Proof. Given $\theta \in \Phi_{\mathbb{N}} \setminus \{1\}$, there exists $k \in \mathbb{N}$ such that $\theta(n) = 1$ $(n \leq k)$ and $\theta(n) = 0$ $(n > k)$. As above, the augmentation character on C_ω is

$$(\alpha(n)) \mapsto \sum_{n=1}^{\infty} \alpha(n), \quad C_\omega \to \mathbb{C}.$$

Hence, we can identify $\Phi_{\mathbb{N}}$ with $\mathbb{N}_\infty$. So, by Proposition 3.1.14, $\Phi_\omega = \mathbb{N}_\infty$ and the Gel'fand transform is the map

$$\alpha = (\alpha(n)) \mapsto \widehat{\alpha} = (\widehat{\alpha}(k)) = \left(\sum_{n=1}^{k} \alpha(n) : k \in \mathbb{N}_\infty \right), \quad C_\omega \to C(\mathbb{N}_\infty).$$

Also, given an element $\beta = (\beta(n)) \in \widehat{C_\omega}$, then the element $\alpha \in C_\omega$ such that $\widehat{\alpha} = \beta$ is defined as

$$\alpha(1) = \beta(1), \quad \alpha(n+1) = \beta(n+1) - \beta(n) \quad (n \geq 1).$$

Then we have that

$$\widehat{C_\omega} = \{\alpha \in C(\mathbb{N}_\infty) : \sum_{n=1}^{\infty} |\alpha(n+1) - \alpha(n)|\omega(n+1) < \infty\}$$

together with the norm $\|\alpha\|_\omega = |\alpha(1)|\omega(1) + \sum_{n=1}^{\infty} |\alpha(n+1) - \alpha(n)|\omega(n+1)$.

Consider now $\beta \in C_{BSE}(\widehat{C_\omega})$. Then there is a net $(\alpha^{(\nu)})$ in $\widehat{C_\omega}$ converging to β pointwise. Also $\|\alpha^{(\nu)}\|_\omega \leq C$ for a given C. Let $n \in \mathbb{N}$ be fixed. Then $\omega(1)|\alpha^{(\nu)}(1)| + \sum_{k=1}^{n} |\alpha^{(\nu)}(k+1) - \alpha^{(\nu)}(k)|\omega(k+1) \leq C$ for all ν and hence, by taking limits,

$$|\beta(1)|\omega(1) + \sum_{k=1}^{n} |\beta(k+1) - \beta(k)|\omega(k+1) \leq C. \tag{3.4.1}$$

Since β verifies (3.4.1), then it belongs to $\widehat{C_\omega}$ and $\|\beta\|_\omega = \|\beta\|_{BSE}$. So, $\widehat{C_\omega}$ is a BSE algebra, and by Proposition 2.3.31, it also has a BSE norm. □

3.4.2. Arens regularity and strongly Arens irregularity, DTC sets. We proceed now to study the Arens regularity of C_ω. The case where $\omega = 1$ has been studied before, see for example [22, Example 7.32]. We study here the Arens regularity of C_ω when ω is a generic weight on $\mathbb{N}$. Similarly to the previous section, we will have to differentiate between $\liminf_{n\to\infty} \omega(n) < \infty$ or $\lim_{n\to\infty} \omega(n) = \infty$. The following result is a generalisation of the procedure followed in [22, Example 7.32]. It also has some similarities with Proposition 3.3.1, although since the structure of C_ω and D_ω is not the same, it is not possible to follow exactly the same reasoning.

Proposition 3.4.3. *Let ω be a weight on $\mathbb{N}_\vee$ such that $\liminf_{n\to\infty} \omega(n) < \infty$. Then C_ω is strongly Arens irregular. Moreover, there is a two-point DTC set of C_ω'''.*

Proof. Let $\theta_\omega : \alpha \mapsto \alpha/\omega, \quad \ell^1(\mathbb{N}) \longrightarrow C_\omega$ as in Lemma 2.4.9.

We have that

$$\delta_u \Box \delta_v = \delta_v \quad (u \in \beta\mathbb{N}, v \in \mathbb{N}^*) \quad \text{and} \quad \delta_u \Diamond \delta_v = \delta_u \quad (u \in \mathbb{N}^*, v \in \beta\mathbb{N}).$$

Let $u \in \beta\mathbb{N}$, $v \in \mathbb{N}^*$. Then there are nets (s_α) and (t_β) in $\mathbb{N}$ such that $u = \lim_\alpha s_\alpha$ and $v = \lim_\beta t_\beta$. Thus, for $\lambda \in C(\beta\mathbb{N})$, we have

$$\begin{aligned}\langle \theta_\omega''(\delta_u) \Box_\omega \theta_\omega''(\delta_v), \omega\lambda \rangle &= \lim_\alpha \lim_\beta \langle \theta_\omega(\delta_{s_\alpha}) \star \theta_\omega(\delta_{t_\beta}), \omega\lambda \rangle \\ &= \lim_\alpha \lim_\beta \Omega(s_\alpha, t_\beta) \langle \delta_{s_\alpha} \star \delta_{t_\beta}, \lambda \rangle \\ &= \Omega_\Box(u, v) \langle \delta_u \Box \delta_v, \lambda \rangle = \Omega_\Box(u, v) \langle \delta_v, \lambda \rangle.\end{aligned}$$

Symmetrically, for $u \in \mathbb{N}^*$, $v \in \beta\mathbb{N}$, we obtain that

$$\langle \theta''_\omega(\delta_u) \Diamond_\omega \theta''_\omega(\delta_v), \omega\lambda \rangle = \Omega_\Diamond(u,v)\langle \delta_u, \lambda \rangle.$$

Consider $\psi_\omega \in \ell^\infty(\mathbb{N})$ defined as $\psi_\omega(n) = 1/\omega(n)$, $(n \in \mathbb{N})$. Let $u \in \beta\mathbb{N}$, $v \in \mathbb{N}^*$, and say $u = \lim_\alpha s_\alpha$, $v = \lim_\beta t_\beta$. Then we have

$$\Omega_\Box(u,v) = \lim_\alpha \lim_\beta \Omega(s_\alpha, t_\beta) = \lim 1/\omega(s_\alpha) = \lim_\alpha \langle \delta_{s_\alpha}, \psi_\omega \rangle = \langle \delta_u, \psi_\omega \rangle.$$

Let $\mu \in M(\beta\mathbb{N})$, $v \in \mathbb{N}^*$ and $\lambda \in C(\beta\mathbb{N})$. Then

$$\begin{aligned} \langle \theta''_\omega(\mu) \Box_\omega \theta''_\omega(\delta_v), \omega\lambda \rangle &= \langle \sum_{u \in \beta\mathbb{N}} \mu(u) \theta''_\omega(\delta_u) \Box_\omega \theta''_\omega(\delta_v), \omega\lambda \rangle \\ &= \langle \sum_{u \in \beta\mathbb{N}} \Omega_\Box(u,v)\mu(u)\delta_v, \lambda \rangle = \langle \mu, \psi_\omega \rangle \langle \delta_v, \lambda \rangle. \end{aligned}$$

Hence, for $\mu \in M(\beta\mathbb{N}), \nu \in M(\mathbb{N}^*)$. Then

$$\theta''_\omega(\mu) \Box_\omega \theta''_\omega(\nu) = \langle \mu, \psi_\omega \rangle \nu. \tag{3.4.2}$$

Symmetrically, we obtain

$$\theta''_\omega(\mu) \Diamond_\omega \theta''_\omega(\nu) = \langle \nu, \psi_\omega \rangle \mu \quad (\mu \in M(\mathbb{N}^*), \nu \in M(\beta\mathbb{N})). \tag{3.4.3}$$

Thus $\theta''_\omega(M(\mathbb{N}^*))$ is a closed ideal of C''_ω such that $C_\omega \cap \theta''_\omega(M(\mathbb{N}^*)) = \{0\}$, and so

$$C''_\omega = C_\omega \ltimes \theta''_\omega(M(\mathbb{N}^*)).$$

Let (s_n) be a sequence in $\mathbb{N}$ such that $\lim_{n\to\infty} \omega(s_n) < \infty$ and let a in the growth of this sequence, so that $\langle \delta_a, \psi_\omega \rangle$ is not zero. Let $b \in \mathbb{N}^*$ different from a. Let us take $\mu \in M(\mathbb{N}^*)$ such that

$$\theta''_\omega(\mu) \Box_\omega \theta''_\omega(\delta_a) = \theta''_\omega(\mu) \Diamond_\omega \theta''_\omega(\delta_a), \quad \text{and} \quad \theta''_\omega(\mu) \Box_\omega \theta''_\omega(\delta_b) = \theta''_\omega(\mu) \Diamond_\omega \theta''_\omega(\delta_b).$$

Thus , by (3.4.2)

$$\theta''_\omega(\mu) \Box_\omega \theta''_\omega(\delta_a) = \langle \mu, \psi_\omega \rangle \delta_a$$

and by (3.4.3)

$$\theta''_\omega(\mu) \Diamond_\omega \theta''_\omega(\delta_a) = \langle \delta_a, \psi_\omega \rangle \mu.$$

Since we are assuming that $\theta''_\omega(\mu) \Box_\omega \theta''_\omega(\delta_a) = \theta''_\omega(\mu) \Diamond_\omega \theta''_\omega(\delta_a)$, we have

$$\langle \mu, \psi_\omega \rangle \delta_a = \langle \delta_a, \psi_\omega \rangle \mu. \tag{3.4.4}$$

Identically for δ_b we obtain

$$\langle \mu, \psi_\omega \rangle \delta_b = \langle \delta_b, \psi_\omega \rangle \mu.$$

Hence $\langle \delta_b, \psi_\omega \rangle \langle \mu, \psi_\omega \rangle \delta_a = \langle \delta_a, \psi_\omega \rangle \langle \mu, \psi_\omega \rangle \delta_b$. Since $\langle \delta_a, \psi_\omega \rangle \neq 0$, we must have $\langle \mu, \psi_\omega \rangle = 0$. By substituting this in (3.4.4), we see that $\theta''_\omega(\mu) = 0$. Thus $\mathfrak{Z}(C''_\omega) = C_\omega$, C_ω is strongly Arens irregular and

$$V = \{\theta''_\omega(\delta_a), \theta''_\omega(\delta_b)\}$$

is a DTC set for it. □

Corollary 3.4.4. *Let ω be a weight on $\mathbb{N}$ such that $\lim_{n\to\infty} \omega(n) = \infty$. Then C_ω is Arens regular.*

Proof. Let $(x_n), (y_m)$ be sequences of distinct elements of $\mathbb{N}$. We have that both sequences are unbounded. Then, for $x_n \leq y_m$ we have

$$\Omega(x_n, y_m) = \frac{\omega(x_n \vee y_m)}{\omega(x_n)\omega(y_m)} = \frac{\omega(y_m)}{\omega(x_n)\omega(y_m)} = \frac{1}{\omega(x_n)},$$

and so

$$\lim_{n\to\infty} \lim_{m\to\infty} \Omega(x_n, y_m) = 0.$$

Similarly,

$$\lim_{m\to\infty} \lim_{n\to\infty} \Omega(x_n, y_m) = 0.$$

Thus, by Theorem 3.1.8, C_ω is Arens regular. □

3.5. Semigroup $\mathbb{Z}_\vee$

We finish this chapter by studying the semigroup $\mathbb{Z}_\vee$. In contrast with the semigroup $\mathbb{N}_\vee$ previously studied, $\mathbb{Z}_\vee$ does not have an identity and it is not weakly cancellative.

3.5.1. Character space, Gel'fand transform and approximate identities.

Consider the semigroup $S := \mathbb{Z}$ with the semigroup operation

$$m \vee n = \max\{m, n\} \quad (m, n \in \mathbb{Z}).$$

Every sequence $\omega : \mathbb{Z} \to [1, \infty)$ is a weight on $\mathbb{Z}_\vee$. During this section we shall denote $Z_\omega = \ell^1(\mathbb{Z}_\vee, \omega)$. As opposite to the case $\mathbb{N}_\vee$, this semigroup is not weakly cancellative and the Banach algebra $\mathbb{Z}_\vee$ does not have an identity.

When we regard $\mathbb{Z}_\vee$ as a semilattice as in Definition 2.4.2, the order obtained is $\geq$, $\sup \mathbb{Z}_\vee = -\infty$ and $\inf \mathbb{Z}_\vee = \infty$. As a graphic representation:

$$\infty\ldots \quad 2 \longrightarrow 1 \longrightarrow 0 \longrightarrow -1 \longrightarrow -2\ldots \quad -\infty$$

So, we can apply some of the results obtained in Section 3.2. However, we shall provide some stronger results in this case.

LEMMA 3.5.1. *Let $S = \mathbb{Z}_\vee$ described as above. Consider the semigroup algebra $Z_1 = \ell^1(S)$. Then $\Phi_{Z_1} = \Phi_S = \mathbb{Z}_\infty$. Furthermore, for $m \in \mathbb{Z}$, consider*

$$(m, \infty] = \{n \in \mathbb{Z} : n > m\} \cup \{\infty\}.$$

Then the Gel'fand topology of Φ_S has a basis the singletons $\{m\}$ for every $m \in \mathbb{Z}$ together with the intervals $(m, \infty]$ for every $m \in \mathbb{Z}$.

Furthermore, for $\omega : \mathbb{Z} \to [1, \infty)$ a weight on $\mathbb{Z}$, $\Phi_\omega = \Phi_S = \mathbb{Z}_\infty$.

PROOF. For $k \in \mathbb{Z}$, we define

$$\chi_k(m) = 1 \quad (m \in \mathbb{Z}, m \leq k), \quad \chi_k(m) = 0 \quad (m \in \mathbb{Z}, m > k). \tag{3.5.1}$$

Also we take $\chi_\infty(m) = 1$ for every $m \in \mathbb{Z}$. It is a routine check that all these functions are semicharacters on S.

Now let $\theta \in \Phi_S$. Since S is idempotent we have that $\theta(m) \in \{0, 1\}$ for every $m \in \mathbb{Z}$. If $\theta(m) = 1$ for all $m \in \mathbb{Z}$, then $\theta = \chi_\infty$. Suppose now that there exists $m \in \mathbb{Z}$ such that $\theta(m) = 0$. Then, for $n \geq m$, we have that

$$\theta(m)\theta(n) = \theta(m \vee n) = \theta(n).$$

Thus $\theta(n) = 0$ whenever $n \geq m$. Since $\theta \neq 0$ there must be k such that $\theta(k) = 1$, and so by a similar reasoning of the above $\theta(n) = 1$ for every $n \leq k$. Hence, there is $k \in \mathbb{Z}$ such that $\theta = \chi_k$. Thus, by Proposition 3.1.14, $\Phi_{Z_1} = \Phi_S = \mathbb{Z}_\infty$.

Finally it is enough to remember that a basis of the Gel'fand topology is given by the sets defined as follows: For $\varphi_0 \in \Phi_{Z_1}$, $s_i \in S$ $(1 \leq i \leq n)$ and $\varepsilon > 0$,

$$V(\varphi_0; s_1, \cdots, s_n; \varepsilon) = \{\varphi \in \Phi_S : |\varphi(s_i) - \varphi_0(s_i)| < \varepsilon\}.$$

We just proved that $\Phi_S = \mathbb{Z}_\infty$, and so we can rewrite these sets in the following way. Let $n \in \mathbb{N}$, and take $m_0, s_1, ..., s_n \in \mathbb{Z}$ and $\varepsilon > 0$. Then

$$V(m_0; s_1, \cdots, s_n; \varepsilon) = \{m \in \mathbb{Z} : |\chi_m(s_i) - \chi_{m_0}(s_i)| < \varepsilon\}.$$

Whenever $\varepsilon > 1$, for any combination of integers $m_0, s_1, ...s_n \in \mathbb{Z}$ we have that $V(m_0; s_1, ..., s_n; \varepsilon) = \mathbb{Z}_\infty$.

Now let $\varepsilon \leq 1$. For $m_0, s_1 < ... < s_n \in \mathbb{Z}$, we can write

$$V(m_0; s_1, \cdots, s_n; \varepsilon) = \{m \in \mathbb{Z} : \chi_m(s_i) = \chi_{m_0}(s_i) \ (1 \leq i \leq n)\}.$$

Suppose there exists i such that $s_i \leq m_0 < s_{i+1}$. Then

$$V(m_0; s_1, ..., s_n; \varepsilon) = \bigcup_{s_i \leq m < s_{i+1}} \{m\},$$

in particular when $s_{i+1} = s_i + 1$ we have the singletons. When $s_1 > m_0$, $V(m_0; s_1, ..., s_n, \varepsilon) = (-\infty, s_1) \cap \mathbb{Z}$. But this can be written as the union of single points below s_1. Finally, whenever $m_0 \geq s_n$,

$$V(m_0; s_1, ..., s_n, \varepsilon) = (s_n, \infty].$$

The last part of the theorem now follows from Proposition 3.1.14. □

LEMMA 3.5.2. *Let* $\omega : \mathbb{Z} \to [1, \infty)$. *Then the weighted semigroup algebra* Z_ω *has a sequential approximate identity. It has a bounded approximate identity if and only if* $\liminf_{n \to -\infty} \omega(n) < \infty$. *It always has a multiplier-bounded approximate identity.*

PROOF. We can apply Proposition 3.2.7. Recall that $\sup \mathbb{Z}_\vee = -\infty$. Thus, there exists a sequence (s_n) tending to $\sup \mathbb{Z}_\vee$ with $\{\omega(s_n) : n \in \mathbb{N}\}$ bounded if and only if $\liminf_{n \to -\infty} \omega(n) < \infty$. □

3.5.2. Arens regularity. We proceed to study the Arens regularity of Z_ω.

COROLLARY 3.5.3. *Let* $\omega : \mathbb{Z} \to [1, \infty)$. *Then* Z_ω *is Arens regular whenever* $\liminf_{n \to \infty} \omega(n) = \infty$ *and* $\liminf_{n \to -\infty} \omega(n) = \infty$.

PROOF. It follows from Theorem 3.2.3, since we have that $\liminf_{n \to \infty} \omega(n) = \infty$ and $\liminf_{n \to -\infty} \omega(n) = \infty$ is equivalent to $\mathrm{Lim}_{s \to \infty} \omega(s) = \infty$ when $S = \mathbb{Z}_\vee$. □

We shall proceed now to study non-Arens regularity. This example provides more cases than those seen in previous sections. Consider

$$B = \{\alpha = (\alpha(n)) \in Z_\omega : \alpha(n) = 0 \ (n \in \mathbb{Z} \setminus \mathbb{N})\},$$

$$G = \{\alpha = (\alpha(n)) \in Z_\omega : \alpha(n) = 0 \ (n \in \mathbb{N})\},$$

respectively, a closed ideal and a closed subalgebra of Z_ω with the restriction of the norm of Z_ω such that $Z_\omega = B \oplus G$.

LEMMA 3.5.4. *Let $\omega : \mathbb{Z} \to [1, \infty)$. Consider B and G defined as above. Then for $M \in B''$ and $N \in G''$ we have that $M \Box N = M \Diamond N$.*

PROOF. Since $Z_\omega = B \oplus G$, we have that $Z_\omega'' = B'' \oplus G''$. Let $M \in B'' \setminus B$ and $N \in G'' \setminus G$. There exist a bounded net $(f^{(\alpha)})$ in B converging to M in the weak-$*$ topology and a bounded net $(g^{(\beta)})$ in G converging to N in the weak-$*$ topology. Then

$$f^{(\alpha)} \star g^{(\beta)} = \left(\sum_{n \in \mathbb{N}} f^{(\alpha)}(n) \delta_n \right) \star \left(\sum_{n \in \mathbb{Z} \setminus \mathbb{N}} g^{(\beta)}(n) \delta_n \right) = \left(\sum_{n \in \mathbb{Z} \setminus \mathbb{N}} g^{(\beta)}(n) \right) f^{(\alpha)}.$$

Let χ_0 be the semicharacter defined as in 3.5.1 and note that $\sum_{n \in \mathbb{Z} \setminus \mathbb{N}} g^{(\beta)}(n) = \chi_0(g^{(\beta)})$. Since $(g^{(\beta)}) \to N$ in the weak-$*$ topology, $\lim_\beta \chi_0(g^{(\beta)}) = N(\chi_0) = \zeta$ and so

$$M \Box N = \zeta M = M \Diamond N.$$

□

PROPOSITION 3.5.5. *Let $\omega : \mathbb{Z} \to [1, \infty)$, and let B and G as above.*

(a) *Suppose that $\liminf_{n \to -\infty} \omega(n) < \infty$ and $\liminf_{n \to \infty} \omega(n) = \infty$. Then $\mathfrak{Z}(Z_\omega'') = B'' \oplus G$.*

(b) *$\liminf_{n \to -\infty} \omega(n) = \infty$ and $\liminf_{n \to \infty} \omega(n) < \infty$. Then $\mathfrak{Z}(Z_\omega'') = B \oplus G''$.*

PROOF. Since $B \cap G = \{0\}$ and $M \Box N = M \Diamond N$, $(M \in B, N \in G)$, by Lemma 3.5.4, we have that $\mathfrak{Z}(Z_\omega'') = \mathfrak{Z}(B'') \oplus \mathfrak{Z}(G'')$.

(a) Consider ω_1 the weight on $\mathbb{N}_\vee$ such that $\omega_1(n) = \omega(n)$ $(n \in \mathbb{N})$. B is isometrically isomorphic as a Banach algebra to C_{ω_1}. Since $\liminf_{n \to \infty} \omega(n) = \infty$, then $\liminf_{n \to \infty} \omega_1(n) = \infty$. By Corollary 3.4.4, C_{ω_1} is Arens regular, and so is B. Similarly, consider ω_2 the weight on $\mathbb{N}_\wedge$ such that $\omega_2(n) = \omega(1 - n)$ $(n \in \mathbb{N})$. Then G is isomorphic to D_{ω_2}. Thus, by Proposition 3.3.1, $\mathfrak{Z}(G'') = G$, since $\liminf_{n \to -\infty} \omega(n) < \infty$ implies that $\liminf_{n \to \infty} \omega_2(n) < \infty$, and so the result follows.

(b) This case is symmetrical to (a). Since $\liminf_{n \to -\infty} \omega(n) = \infty$, $\mathfrak{Z}(B'') = B$ and $\liminf_{n \to \infty} \omega(n) < \infty$ implies that $\mathfrak{Z}(G'') = G''$. Thus, the desired result follows. □

Note that the fact that Z_ω is neither Arens regular nor strongly Arens irregular when $\omega : \mathbb{Z} \to [1, \infty)$ is such that $\liminf_{n \to -\infty} \omega(n) < \infty$ and $\liminf_{n \to \infty} \omega(n) = \infty$ also follows from Theorem 3.2.3 and Proposition 3.2.4.

Corollary 3.5.6. *Let $\omega : \mathbb{Z} \to [1, \infty)$ such that $\liminf_{n\to-\infty} \omega(n) < \infty$ and $\liminf_{n\to\infty} \omega(n) < \infty$. Then Z_ω is strongly Arens irregular with a four point DTC set.*

Proof. As above we can write it as $Z_\omega = B \oplus G$, and following a similar reasoning, we can see that B and C are isometrically isomorphic as Banach algebras to D_{ω_1} and C_{ω_2}, with $\liminf_{n\to\infty} \omega_1(n) < \infty$ and $\liminf_{n\to\infty} \omega_2(n) < \infty$. Hence, by Theorem 2.3.9, Proposition 3.3.1 and Proposition 3.4.3, Z_ω has a 4-point DTC set. $\square$

We present a table below to summarize the results about Arens regularity of Z_ω.

Conditions on ω	Arens regularity
$\liminf_{n\to\infty} \omega(n) = \infty$, $\liminf_{n\to-\infty} \omega(n) = \infty$	AR, Corollary 3.5.3
$\liminf_{n\to\infty} \omega(n) < \infty$, $\liminf_{n\to-\infty} \omega(n) = \infty$	Neither AR nor SAI, Proposition 3.5.5
$\liminf_{n\to\infty} \omega(n) = \infty$, $\liminf_{n\to-\infty} \omega(n) < \infty$	Neither AR nor SAI, Proposition 3.5.5
$\liminf_{n\to\infty} \omega(n) < \infty$, $\liminf_{n\to-\infty} \omega(n) < \infty$	SAI Corollary 3.5.6

3.5.3. BSE algebra, BSE norm and other properties. We shall study now when Z_ω is a BSE algebra and when it has a BSE norm. In order to do so, we shall introduce the following Banach algebra. Let $\omega : \mathbb{Z} \to [1, \infty)$. For $n \in \mathbb{N}$ and $\alpha \in \ell^\infty(\mathbb{Z})$, consider

$$p_\omega^n(\alpha) = \sum_{-n\leq i\leq n} \omega(i)|\alpha(i) - \alpha(i-1)|.$$

Set $p_\omega(\alpha) = \lim_{n\to\infty} p_n(\alpha)$, and consider

$$B_\omega = \{\alpha \in \ell^\infty(\mathbb{Z}) : p_\omega(\alpha) < \infty\}$$

with the norm on B_ω defined as follows:

$$\|\alpha\|_\omega = |\alpha|_\mathbb{Z} + p_\omega(\alpha) \quad (\alpha \in B_\omega).$$

Then, B_ω equipped with pointwise multiplication is a unital Banach sequence algebra on $\mathbb{Z}$. Also, for every $\alpha \in B_\omega$ and every $p \geq q \in \mathbb{N}$ we have that

$$|\alpha(p) - \alpha(q)| \leq p_\omega^p(\alpha) - p_\omega^q(\alpha), \quad |\alpha(-p) - \alpha(-q)| \leq p_\omega^p(\alpha) - p_\omega^q(\alpha).$$

Since $p_\omega(\alpha) < \infty$, this shows that for every $\alpha \in B_\omega$ the following two limits exist

$$\alpha(-\infty) = \lim_{n\to-\infty} \alpha(n), \quad \alpha(\infty) = \lim_{n\to\infty} \alpha(n).$$

Thus, the character space of B_ω is $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$.

Consider now $G_\omega = \{\alpha \in B_\omega : \alpha(-\infty) = 0\}$, which is a maximal ideal of B_ω with character space $\Phi_\omega = \mathbb{Z}_\infty$. Also, let $M_\omega = G_\omega \cap c_0(\mathbb{Z})$, a maximal ideal of G_ω. Recall that the norm is given by

$$\|\alpha\|_\omega = |\alpha|_\mathbb{Z} + p_\omega(\alpha), \quad (\alpha \in G_\omega, \text{respectively, } \alpha \in M_\omega).$$

We see that M_ω is a Banach sequence algebra on $\mathbb{Z}$. So it is an ideal of B_ω of codimension 2.

Proposition 3.5.7. *Let $\omega : \mathbb{Z} \to [1, \infty)$. Then G_ω is the Gel'fand transform of Z_ω. Furthermore, the Banach algebra M_ω is the Gel'fand transform of the maximal modular ideal $\{\alpha = (\alpha(n)) \in Z_\omega : \sum_{n\in\mathbb{Z}} \alpha(n) = 0\}$.*

Proof. Let $\alpha = \sum_{n\in\mathbb{Z}} \alpha(n)\delta_n \in Z_\omega$. The Gelf'and transform $\widehat{\alpha}$ of α is such that, for $m \in \mathbb{Z}$,

$$\widehat{\alpha}(m) = \sum_{n\leq m} \alpha(n), \tag{3.5.2}$$

and its value at ∞ is given by

$$\widehat{\alpha}(\infty) = \sum_{n\in\mathbb{Z}} \alpha(n).$$

Using 3.5.2, we can see that $p_\omega(\widehat{\alpha}) = \|\alpha\|_\omega < \infty$, and so $\widehat{\alpha} \in B_\omega$. Also, we see that $\lim_{n\to-\infty} \widehat{\alpha}(n) = 0$. Thus $\widehat{Z_\omega} \subset G_\omega$. Let us see now that $G_\omega \subset \widehat{Z_\omega}$. Given an element $\beta = (\beta(n))$ in the Gel'fand transform of Z_ω, the element $\alpha \in Z_\omega$ such that $\widehat{\alpha} = \beta$ is defined as

$$\alpha(n) = \beta(n) - \beta(n-1) \quad (n \in \mathbb{Z}).$$

Thus, the Gel'fand tranform of Z_ω is

$$\widehat{Z_\omega} = \{\beta \in \ell^\infty(\mathbb{Z}) : p_\omega(\beta) < \infty, \lim_{n\to-\infty} \beta(n) = 0\} = G_\omega,$$

with the norm $\|\beta\| = \sum_{n\in\mathbb{Z}} |\beta(n) - \beta(n-1)|\omega(n) < \infty$ $(\beta \in G_\omega)$. Finally, for $\beta \in G_\omega$, we have

$$\|\beta\| \leq \|\beta\|_\omega \leq 2\|\beta\|.$$

Whenever $\alpha \in M_\omega$ we have that $\lim_{n\to\infty} \alpha(n) = 0$ too. Thus the Gel'fand transform of $\{\alpha = (\alpha(n)) \in Z_\omega : \sum_{n\in\mathbb{Z}} \alpha(n) = 0\}$ is M_ω. $\square$

The following lemma will be useful for later results.

LEMMA 3.5.8. *Let $\omega : \mathbb{Z} \to [1, \infty)$. Then, we can always find two sequences of increasing positive integers (m_k) and (n_k) such that, for $\alpha \in M_\omega$,*

$$\omega(n_k)|\alpha(n_k)| \to 0 \quad \textit{and} \quad \omega(-m_k)|\alpha(-m_k)| \to 0 \quad (k \to \infty). \tag{3.5.3}$$

PROOF. Suppose that $\liminf_{n\to\infty} \omega(n) < \infty$. Then there exists (n_k) a strictly increasing sequence of positive integers such that

$$\lim_{k\to\infty} \omega(n_k) = \liminf_{n\to\infty} \omega(n).$$

In particular, this implies that there exists $M \geq 1$ such that $\omega(n_k) \leq M$ for every $k \in \mathbb{N}$. Hence, for $\alpha \in M_\omega$, we have

$$\omega(n_k)|\alpha(n_k)| \leq M|\alpha(n_k)| \to 0 \quad (k \to \infty). \tag{3.5.4}$$

Following a similar reasoning, when $\liminf_{n\to-\infty} \omega(n) < \infty$, we can take a strictly increasing sequence of positive integers (m_k) such that

$$\lim_{k\to\infty} \omega(-m_k) = \liminf_{n\to-\infty} \omega(n)$$

and so

$$\omega(-m_k)|\alpha(-m_k)| \to 0 \quad (k \to \infty). \tag{3.5.5}$$

Suppose now that $\liminf_{n\to\infty} \omega(n) = \infty$. In this case we can construct (n_k) a sequence of strictly increasing positive integers such that

$$\omega(n_k) \leq \omega(n) \quad (n \geq n_k).$$

Indeed, consider $\tilde{\omega}^+ : \mathbb{N} \to [1, \infty)$ defined as $\tilde{\omega}^+(n) = \inf\{\omega(k) : k \geq n\}$. Now let n_1 such that $\omega(n_1) = \tilde{\omega}^+(1)$. Knowing n_j define n_{j+1} such that $\omega(n_{j+1}) = \tilde{\omega}^+(n_j + 1)$. These infima exist since $\liminf_{n\to\infty} \omega(n) = \infty$. Thus, for $\alpha \in M_\omega$ we have that

$$\omega(n_k)|\alpha(n_k)| \leq \sum_{i>n_k} \omega(i)|\alpha(i) - \alpha(i-1)| \to 0. \tag{3.5.6}$$

Symmetrically,when $\liminf_{n\to-\infty} \omega(n) = \infty$, we can construct (m_k) a sequence of strictly increasing positive integers such that

$$\omega(-m_k) \leq \omega(-n) \quad (n \geq m_k).$$

We proceed as above, but in this case we consider $\tilde{\omega}^- : \mathbb{N} \to [1,\infty)$ defined as $\tilde{\omega}^-(n) = \inf\{\omega(-k) : k \geq n\}$. Thus we obtain a sequence (m_k) such that, for any $\alpha \in M$,

$$\omega(-m_k)|\alpha(-m_k)| \leq \sum_{i<-m_k} \omega(i)|\alpha(i) - \alpha(i-1)| \to 0. \tag{3.5.7}$$

□

PROPOSITION 3.5.9. *Let $\omega : \mathbb{Z} \to [1,\infty)$. The Banach algebra M_ω is a Tauberian Banach sequence algebra. It has a bounded approximate identity if and only if $\liminf_{n\to\infty} \omega(n) < \infty$ and $\liminf_{n\to-\infty} \omega(n) < \infty$ and it always has a multiplier-bounded approximate identity.*

PROOF. For $m, n \in \mathbb{N}$, consider $\xi^n_{-m} \in M_\omega$ such that $\xi^n_{-m}(k) = 1$, for $k \in \mathbb{Z}^n_{-m}$ and 0 otherwise. For $\alpha \in M_\omega$, we have

$$\begin{aligned} \|\alpha - \alpha\xi^n_{-m}\|_\omega &= \sup\{|\alpha(k)| : k \notin \mathbb{Z}^n_{-m}\} \\ &\quad + \sum_{k\notin\mathbb{Z}^n_{-m}} \omega(k)|\alpha(k) - \alpha(k-1)| + \omega(-m)|\alpha(-m)| + \omega(n)|\alpha(n)|. \end{aligned} \tag{3.5.8}$$

By Lemma 3.5.8, we can always find two sequences (n_k), (m_k) of increasing positive integers such that $\|\alpha - \xi^{n_k}_{-m_k}\alpha\|_\omega \to 0 \quad (k\to\infty)$, for every $\alpha \in M_\omega$. Thus, for this selection of (n_k) and (m_k), $(\xi^{n_k}_{-m_k} : k \in \mathbb{N})$ is an approximate identity. We conclude that M_ω is Tauberian for any weight ω.

By the uniform boundness theorem, $(\xi^{n_k}_{-m_k} : k \in \mathbb{N})$ is a multiplier-bounded approximate identity. When $\liminf_{n\to\infty} \omega(n) < \infty$ and $\liminf_{n\to-\infty} \omega(n) < \infty$, we can chose (n_k) and (m_k) so that $\omega(n_k)$ and $\omega(-m_k)$ are bounded by $M \geq 1$. and so $(\xi^{n_k}_{-m_k} : k \in \mathbb{N})$ is a bounded approximate identity with bound $1 + 2M$. Suppose now that $\liminf_{n\to\infty} \omega(n) = \infty$ or $\liminf_{n\to-\infty} \omega(n) = \infty$ and assume that there is a bounded approximate identity in M_ω. In this case $\|\cdot\|_{op}$ and $\|\cdot\|_\omega$ are equivalent, but $\|\xi^{n_k}_{-m_k}\|_\omega \to \infty$ at the same time as it is a multiplier-bounded approximate identity, which is a contradiction. Hence, M_ω does not have a bounded approximate identity. □

COROLLARY 3.5.10. *Let $\omega : \mathbb{Z} \to [1,\infty)$, and M_ω be the Banach sequence algebra defined as above. Then M_ω has a BSE norm. M_ω is a BSE algebra if and only if $\liminf_{n\to\infty} \omega(n) < \infty$ and $\liminf_{n\to-\infty} \omega(n) < \infty$.*

PROOF. Let $\omega : \mathbb{Z} \to [1,\infty)$. Since M_ω is a Tauberian Banach sequence algebra with a multiplier-bounded approximate identity, by Corollary 2.3.32, it has a BSE norm.

By Proposition 3.5.9, M_ω has a bounded approximate identity if and only if $\liminf_{n\to\infty}\omega(n)<\infty$ and $\liminf_{n\to-\infty}\omega(n)<\infty$. Thus by Corollary 2.3.27 this is a necessary and sufficient condition to be a BSE algebra. $\square$

PROPOSITION 3.5.11. *Let* $\omega:\mathbb{Z}\to[1,\infty)$ *such that* $\liminf_{n\to\infty}\omega(n)<\infty$ *and* $\liminf_{n\to-\infty}\omega(n)<\infty$. *Let* M_ω *and* B_ω *be the Banach sequence algebras defined as above. Then* $C_{BSE}(M_\omega)=\mathcal{M}(M_\omega)=B_\omega$.

PROOF. We have that $B_\omega\subset\mathcal{M}(M_\omega)$ since M_ω is an ideal in B_ω. Now let us take $\beta\in C_{BSE}(M_\omega)$. By Theorem 2.3.25 there exists a net $(\alpha^{(\nu)})$ of elements of M_ω that converges to β pointwise and such that

$$\lim_\nu\|\alpha^{(\nu)}\|_\omega=\|\beta\|_{BSE}.$$

Let $\varepsilon>0$. For $n\in\mathbb{N}$, there exists ν_0 such that for any $\nu\geq\nu_0$ we have

$$\max\{|\alpha^{(\nu)}(i)|:-n\leq i\leq n\}+\sum_{-n\leq i\leq n}\omega(i)|\alpha^{(\nu)}(i-1)-\alpha^{(\nu)}(i)|\leq\|\beta\|_{BSE}+\varepsilon.$$

By taking limits in ν we obtain that

$$\max\{|\beta(i)|:-n\leq i\leq n\}+\sum_{-n\leq i\leq n}\omega(i)|\beta(i-1)-\beta(i)|\leq\|\beta\|_{BSE}+\varepsilon.$$

Thus we can conclude that $\beta\in B_\omega$ and $\|\beta\|_{BSE}=\|\beta\|_\omega$. By Corollary 3.5.10, M_ω is a BSE algebra when $\liminf_{n\to\infty}\omega(n)<\infty$ and $\liminf_{n\to-\infty}\omega(n)<\infty$. Thus $\mathcal{M}(M_\omega)=C_{BSE}(M_\omega)=B_\omega$. $\square$

COROLLARY 3.5.12. *Let* $\omega:\mathbb{Z}\to[1,\infty)$, *and* G_ω *be the Banach sequence algebra defined as above.* G_ω *is a BSE algebra if and only if* $\liminf_{n\to-\infty}\omega(n)<\infty$.

PROOF. By Theorem 2.3.35, we know that $\mathcal{M}(G_\omega)\subset C_{BSE}(G_\omega)$ if and only if $\liminf_{n\to-\infty}\omega(n)<\infty$. Hence, repeating almost the same argument as above we obtain that $C_{BSE}(G_\omega)=\mathcal{M}(G_\omega)=B_\omega$, and so G_ω is a BSE algebra if and only if $\liminf_{n\to-\infty}\omega(n)<\infty$.

$\square$

CHAPTER 4

Banach sequence algebras

Some of the results of this chapter will appear in the forthcoming monograph [26].

4.1. Mixed identities

We shall now see some results about mixed identities that will be useful in the following sections. For more information concerning mixed identities, see [19], [66].

DEFINITION 4.1.1. Let A be a Banach algebra, and let $E \in A''$. We say E is a *mixed identity* for A if $E \neq 0$ and, for all $M \in A''$, we have

$$M \Box E = E \Diamond M = M.$$

The set of all mixed identities for A is denoted as $\mathcal{E}_A$.

Note that this is equivalent to

$$a \cdot E = E \cdot a = a \quad (a \in A).$$

The existence of mixed identities and Arens regularity have a strong connection. We shall see the link in the following results. The first one, is a direct conclusion from the definition of mixed identity and Arens regularity. It is known to specialists, but we add it here for completion.

PROPOSITION 4.1.2. *Let A be a Banach algebra with a mixed identity. A is Arens regular. Then $\mathcal{E}_A$ has a unique element.*

PROOF. Let $E, F \in \mathcal{E}_A$ and suppose that A is Arens regular. Since $E \in \mathcal{E}(A)$,

$$F \Box E = F.$$

Since $F \in \mathcal{E}_A$,

$$F \Diamond E = E.$$

Therefore, as A is Arens regular, $E = F$. $\square$

As we can imagine, approximate identities, in particular bounded ones, are also linked to mixed identities. When A has a bounded approximate identity, then $\mathcal{E}_A \neq \emptyset$. In fact, we have the following result, that can be found in [19, Corollary 2.9.15]:

Corollary 4.1.3. *Let A be a Banach algebra. Then A has a mixed identity $E \in \mathcal{E}_A$ if and only if it has a bounded approximate identity.* $\square$

Let us recall that we define the set $A \cdot A' := \{a \cdot \lambda : a \in A, \lambda \in A'\}$ and that AA' is the linear span of $A \cdot A'$. We define the sets $A' \cdot A$ and $A'A$ symmetrically.

Proposition 4.1.4. *Let A be a Banach algebra with a bounded approximate identity, and let E be a weak-$*$ accumulation point of it. Then*

$$\mathcal{E}_A = \{E\} + (AA')^\perp \cap (A'A)^\perp.$$

Proof. Let $F \in \mathcal{E}_A$ and define $M = F - E$. Then, for $a \in A$ and $\lambda \in A'$, we have

$$\langle M, a \cdot \lambda\rangle = \langle (F - E) \cdot a, \lambda\rangle = \langle (F \cdot a - E \cdot a), \lambda\rangle = 0.$$

Thus $M \in (AA')^\perp$. Symmetrically, $M \in (A'A)^\perp$.

Let $N \in (AA')^\perp \cap (A'A)^\perp$. For $a \in A$ and $\lambda \in A'$, $\langle a \cdot N, \lambda\rangle = \langle N, \lambda \cdot a\rangle = 0$, so $M \Box N = 0$ for all $M \in A''$. Thus $M \Box (E + N) = M$. Following a similar reasoning, we can see that $(E + N) \Diamond M = M$. Thus $E + N$ is a mixed identity for A. $\square$

Corollary 4.1.5. *Let A be a Banach algebra with a bounded approximate identity. Suppose that $AA' = A'A$. Let $E \in \mathcal{E}_A$. Then the following are equivalent:*

(a) $\mathcal{E}_A$ has a unique element;

(b) $A' \cdot A = A'$;

(c) E is an identity for $(A'', \Box)$ and for $(A'', \Diamond)$.

Proof. $(a) \Rightarrow (b)$ Let E be a weak-$*$ accumulation point of the bounded approximate identity. By Proposition 4.1.4, $\mathcal{E}_A = \{E\} + (AA')^\perp \cap (A'A)^\perp$. If E is the only element of $\mathcal{E}_A$, then $(AA')^\perp \cap (A'A)^\perp = \{0\}$. Since by hypothesis we have that $(AA')^\perp \cap (A'A)^\perp = (A'A)^\perp$, then $\overline{A'A} = A'$. But by Cohen's factorization Theorem $\overline{A'A} = A' \cdot A$. Hence $A' \cdot A = A'$.

$(b) \Rightarrow (c)$ Let $E \in \mathcal{E}_A$ and suppose it is not an identity for $(A'', \Box)$. Then there exists $M \in A''$ such that $E \Box M \neq M$. Consider $F = M - E \Box M \neq 0$. Then, for $a \in A$ and $\lambda \in A'$, we have

$$\langle F, \lambda \cdot a\rangle = \langle a \cdot M - a \cdot E \Box M, \lambda\rangle = \langle a \cdot M - a \cdot M, \lambda\rangle = 0.$$

Thus $F \in (A'A)^{\perp}$. But this implies that $A' \cdot A \neq A'$. Following a symmetric argument, we can see that if E is not an identity for $(A'', \Diamond)$, then $A' \cdot A' \neq A$ again.

$(c) \Rightarrow (a)$ Let $F \in \mathcal{E}_A$. As E is an identity for $(A'', \Box)$, we have that

$$F = E \Box F = E,$$

and so E is unique. $\square$

Finally, we shall apply all these results to a Banach algebra A that is an ideal in its bidual, since it will the case we shall focus on in the following subsections. The following theorem can be found in [26, Theorem 2.3.44] and will be key for that.

THEOREM 4.1.6. *Let A be a Banach algebra that is an ideal in its bidual. Suppose it has a bounded approximate identity . Then* $A \cdot A' = A' \cdot A = A \cdot A' \cdot A = \mathrm{WAP}(A)$. $\square$

We know that when a Banach algebra A is Arens regular $\mathrm{WAP}(A) = A'$, thus $\mathrm{WAP}(A) = A' = A \cdot A' = A' \cdot A = A \cdot A' \cdot A$ when A is also an ideal in its bidual, and so the following is an immediate conclusion of Corollary 4.1.5 and Proposition 4.1.2.

COROLLARY 4.1.7. *Let A be a Banach algebra with a bounded approximate identity and such that A is an ideal in its bidual. Let $E \in \mathcal{E}_A$. Then the following are equivalent:*

(a) $\mathcal{E}_A$ has a unique element;
(b) E is the identity of A'';
(c) $A' \cdot A = A'$;
(d) A is Arens regular. $\square$

4.2. The Feinstein algebra

We proceed to introduce now the *Feinstein algebra*. This algebra appears in [19, Example 4.1.46] and some of the initial results that we shall introduce now are proved there. Let $\alpha \in \mathbb{C}^{\mathbb{N}}$ and, for $n \in \mathbb{N}$, consider

$$p_n(\alpha) = \sum_{k=1}^{n} \frac{k}{n} |\alpha_{k+1} - \alpha_k|.$$

We define $A := \{\alpha \in c_0 : p(\alpha) < \infty\}$ where

$$p(\alpha) = \sup\{p_n(\alpha) : n \in \mathbb{N}\}.$$

Consider the following norm in A:

$$\|\alpha\| = |\alpha|_{\mathbb{N}} + p(\alpha) \quad (\alpha \in A).$$

Then $(A, \|\cdot\|)$ is a Banach space, and it is a Banach algebra when we consider the pointwise product. Following [19], we call $(A, \|\cdot\|)$ the Feinstein algebra, since it was introduced by Joel Feinstein in a lecture.

We shall introduce now a subalgebra of A that will be of interest. Let $A_0 := \overline{J_\infty(A)}$. It is seen in [19, Example 4.1.46] that

$$A_0 = \{\alpha \in c_0 : p_n(\alpha) \to 0\}.$$

Then A_0 is a Tauberian Banach sequence algebra and (Δ_n) is a bounded approximate identity for A_0 with $\|\Delta_n\| = 2$.

These two algebras present some very interesting characteristics. For example, it can be seen that $A^2 = A_0^2 = A_0$ and that A_0 is separable while A is not. Thus A^2 is a closed subspace of infinite codimension in A. Also, A is not Tauberian and it does not have an approximate identity.

4.2.1. Study of Arens regularity. We proceed to study the Arens regularity of A and A_0. In order to do so, we shall construct a new Banach algebra B which is Arens regular, and we shall see that A_0 can be seen as a closed subalgebra of B. This method can be extended to a more generic setting and this is something we are working on at the moment.

Let $j \in \mathbb{N}$. For $\alpha \in \mathbb{C}^{2^{j-1}+2}$ consider

$$g_j(\alpha) = \sum_{k=2}^{2^{j-1}+1} |\alpha_k - \alpha_{k+1}|.$$

Let $B^{(j)} = \mathbb{C}^{2^{j-1}+2}$ with the norm in $B^{(j)}$ defined as follows:

$$\|\alpha\|_j = \max\{|\alpha_1|, ..., |\alpha_{2^{j-1}+2}|\} + g_j(\alpha).$$

Lemma 4.2.1. *For every $j \in \mathbb{N}$, the space $(B^{(j)}, \|\cdot\|_j)$ is a unital Banach algebra with pointwise multiplication.*

Proof. For $j \in \mathbb{N}$, the space $(B^{(j)}, \|\cdot\|_j)$ is a Banach space. Let us take two elements $\alpha, \beta \in B^{(j)}$. Then $\alpha\beta \in B^{(j)}$ and

$$\begin{aligned}
\|\alpha\beta\|_j &= \max\{|\alpha_k\beta_k| : k \leq 2^{j-1}+2\} + g_j(\alpha\beta) \\
&\leq \max\{|\alpha_k| : k \leq 2^{j-1}+2\} \max\{|\beta_k| : k \leq 2^{j-1}+2\} \\
&+ \max\{|\beta_k| : k \leq 2^{j-1}+2\}\, g_j(\alpha) \\
&+ \max\{|\alpha_k| : k \leq 2^{j-1}+2\}\, g_j(\beta) \leq \|\alpha\|_j \|\beta\|_j,
\end{aligned}$$

and so $(B^{(j)}, \|\cdot\|_j)$ is a Banach algebra as desired. Since $\|1\|_j = 1$, we have that $B^{(j)}$ is unital. $\square$

Let us consider $B = c_0(B^{(j)})$. This is the space of sequences $\alpha = (\alpha^{(j)})$ where $\alpha^{(j)} \in B^{(j)}$, such that

$$|||\alpha||| = \sup\{\|\alpha^{(j)}\|_j : j \in \mathbb{N}\} < \infty \quad (\alpha = (\alpha^{(j)}) \in B)$$

and such that

$$\lim_{j\to\infty} \|\alpha^{(j)}\|_j = 0.$$

For $\alpha = (\alpha^{(j)}) \in B$, we define $g(\alpha) := \sup\{g_j(\alpha^{(j)}) : j \in \mathbb{N}\}$, which shall be useful for some calculations below. Note that $g(\alpha) \leq |||\alpha|||$, for $\alpha \in B$.

LEMMA 4.2.2. *The space $(B, |||\cdot|||)$ is a Banach algebra with pointwise multiplication.*

PROOF. Let $\alpha = (\alpha^{(j)}), \beta = (\beta^{(j)}) \in B$. Then, by the calculation above,

$$\|\alpha^{(j)}\beta^{(j)}\|_j \leq \|\alpha^{(j)}\|_j\|\beta^{(j)}\|_j \leq |||\alpha||| |||\beta|||.$$

Thus $|||\alpha\beta||| \leq |||\alpha||| |||\beta|||$. $\square$

Since $B^{(j)}$ is finite dimensional for every $j \in \mathbb{N}$, we know that the dual is E_j with $\dim E_j = \dim B^{(j)}$. Hence the following result follows.

LEMMA 4.2.3. *The dual space of B is $B' = \ell^1(E_j)$.* $\square$

The following lemma can be found in [26]:

LEMMA 4.2.4. *Let $(E_n, \|\cdot\|_n)$ be a finite-dimensional space for each $n \in \mathbb{N}$. Let $E = \ell^1(E_n)$. Then E has the Schur property.* $\square$

COROLLARY 4.2.5. *The Banach algebra B is Arens regular.*

PROOF. By Theorem 2.3.11, if B' has the Schur property, then B is Arens regular. It follows from Lemma 4.2.4 and Lemma 4.2.3 that the dual of B has the Schur property, and so the result follows. $\square$

We shall see now that A_0 is a closed subalgebra of B. In order to do so, we shall define an equivalent norm in A_0, which will facilitate the process.

Let $r \in (0, 1)$. For $n \in \mathbb{N}$, consider the set $H_n^{(r)} = (rn, n] \cap \mathbb{N}$ and define

$$h_n^{(r)}(\alpha) = \sum_{k\in H_n^{(r)}} |\alpha_k - \alpha_{k+1}|.$$

Consider the norm in A_0 defined as

$$\|\alpha\|_1^{(r)} = |\alpha|_{\mathbb{N}} + h^{(r)}(\alpha), \quad (\alpha \in A_0)$$

where $h^{(r)}(\alpha) = \sup\{h_n^{(r)}(\alpha) : n \in \mathbb{N}\}$.

LEMMA 4.2.6. *Let $r \in (0,1)$. Let $\alpha \in \mathbb{C}^{\mathbb{N}}$. If $p_n(\alpha) \to 0$ $(n \to \infty)$, we have that $h_n^{(r)}(\alpha) \to 0$ $(n \to \infty)$. In addition, the norms $\|\cdot\|$ and $\|\cdot\|_1^{(r)}$ are equivalent in A_0. In particular*

$$r\|\alpha\|_1^{(r)} \le \|\alpha\| \le \frac{1}{1-r}\|\alpha\|_1^{(r)}, \quad (\alpha \in A_0).$$

PROOF. Let $\alpha \in \mathbb{C}^{\mathbb{N}}$ such that $p_n(\alpha) \to 0$ when n tends to ∞. Let $n \in \mathbb{N}$ and take $m \in \mathbb{N}$ such that $m - 1 \le rn < m$. Then $H_n^{(r)} = \{m, \dots, n\}$, and

$$p_n(\alpha) \ge \sum_{k=m}^{n} \frac{k}{n}|\alpha_k - \alpha_{k+1}| \ge r \sum_{k=m}^{n} |\alpha_k - \alpha_{k+1}| = r\, h_n^{(r)}(\alpha).$$

Thus $h_n^{(r)}(\alpha)$ tends to zero when n tends to ∞. In particular, $h^{(r)}(\alpha)$ is finite and

$$p(\alpha) \ge r\, h^{(r)}(\alpha). \tag{4.2.1}$$

On the other hand, for $n \in \mathbb{N}$ we define $m \in \mathbb{N}$ to satisfy $m - 1 \le rn < m$. Then $H_n^{(r)} = \{m, \cdots, n\}$. Hence

$$\begin{aligned} p_n(\alpha) &= \sum_{k=1}^{m-1} \frac{k}{n}|\alpha_k - \alpha_{k+1}| + \sum_{k=m}^{n} \frac{k}{n}|\alpha_k - \alpha_{k+1}| \\ &\le \sum_{k=1}^{m-1} \frac{k}{n}|\alpha_k - \alpha_{k+1}| + h^{(r)}(\alpha). \end{aligned}$$

If $m = 1$, the first term on the right hand side vanishes, so it is bounded above by $rp(\alpha)$. If $m \ge 2$, then this term can be rewritten as $\frac{m-1}{n}p_{m-1}(\alpha)$, which once again is bounded above by $rp(\alpha)$. Thus $p_n(\alpha) \le rp(\alpha) + h^{(r)}(\alpha)$ for all $n \in \mathbb{N}$. Hence $p(\alpha) \le rp(\alpha) + h^{(r)}(\alpha)$ and so

$$(1 - r)p(\alpha) \le h^{(r)}(\alpha). \tag{4.2.2}$$

Combining (4.2.2) and (4.2.1) we see that, for $\alpha \in A_0$,

$$r\|\alpha\|_1^{(r)} \le \|\alpha\| \le \frac{1}{1-r}\|\alpha\|_1^{(r)},$$

as desired. ∎

PROPOSITION 4.2.7. *The algebra A_0 can be identified with a closed subalgebra of B.*

PROOF. Let $j \in \mathbb{N}$. Take $\alpha = (\alpha_n) \in A_0$, and consider $\theta^{(j)} : A_0 \longrightarrow B^{(j)}$, such that $\theta^{(1)}(\alpha) = (\alpha_1, \alpha_2, \alpha_3)$, $\theta^{(2)}(\alpha) = (\alpha_2, \alpha_3, \alpha_4, \alpha_5)$, $\theta^{(3)}(\alpha) = (\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9)$ and so on. So, for $j \in \mathbb{N}$, $\theta^{(j)}$ is such that the $k - th$ coordinate of $\theta^{(j)}(\alpha)$ is given by $\theta^{(j)}(\alpha)_k = \alpha_{2^{j-1}+k-1}$, for $k = 1, \cdots, 2^{j-1} + 2$. Note that $H_{2^j}^{(1/2)} = \{2^{j-1} + 1, \cdots, 2^j\}$, and so

$$\begin{aligned} g_j(\theta^{(j)}(\alpha)) &= \sum_{k=2}^{2^{j-1}+1} |\theta^{(j)}(\alpha)_k - \theta^{(j)}(\alpha)_{k+1}| \qquad (4.2.3)\\ &= \sum_{k=2}^{2^{j-1}+1} |\alpha_{2^{j-1}+k-1} - \alpha_{2^{j-1}+k}| = \sum_{k=2^{j-1}+1}^{2^j} |\alpha_k - \alpha_{k+1}| = h_{2^j}^{(1/2)}(\alpha). \end{aligned}$$

Hence, by Lemma 4.2.6, we have

$$\|\theta^{(j)}(\alpha)\|_j \leq \|\alpha\|_1^{(1/2)} \leq 2\|\alpha\|. \qquad (4.2.4)$$

Thus $\theta^{(j)} : A_0 \longrightarrow B^{(j)}$ is bounded for every j. Also $\theta^{(j)}(\alpha\beta) = \theta^{(j)}(\alpha)\theta^{(j)}(\beta)$, for $\alpha, \beta \in A_0$, and so $\theta^{(j)}$ is an algebra homomorphism, for every $j \in \mathbb{N}$.

Consider now the linear operator

$$\theta : \alpha \mapsto \theta(\alpha) = (\theta^{(j)}(\alpha)), \quad (\alpha \in A_0).$$

Since $\alpha \in A_0$, $\lim_{n\to\infty} |\alpha_n| = 0$ and so

$$\lim_{j\to\infty} \max\{|\alpha_k| : 2^{j-1} \leq k \leq 2^j + 2\} = 0.$$

In addition, since $p_n(\alpha)$ tends to zero when n tends to ∞, by Lemma 4.2.6, $h_{2^j}^{(1/2)}(\alpha)$ also tends to zero when j tends to ∞. Thus $\lim_{j\to\infty} \|\theta^{(j)}(\alpha)\|_j = 0$, and so $\theta(\alpha) \in B$. Thus $\theta(A_0) \subset B$ and, from (4.2.4), we obtain that

$$|||\theta(\alpha)||| \leq 2\|\alpha\|.$$

Hence θ is bounded.

For $n = 1$, since $H_1^{(1/2)} = \{1\}$ and $\theta^{(1)}(\alpha) = (\alpha_1, \alpha_2, \alpha_3)$, using (4.2.3) we see

$$h_1^{(1/2)}(\alpha) = |\alpha_1 - \alpha_2| \leq |\alpha_1| + |\alpha_3| + |\alpha_2 - \alpha_3| \leq 2\max\{|\alpha_1|, |\alpha_2|, |\alpha_3|\} + g_1(\theta^{(1)}(\alpha))$$

In general, given $n \in \mathbb{N}$ with $n \geq 2$, let $m \in \mathbb{N}$ satisfy $m - 1 \leq n/2 < m$. There exists $j \in \mathbb{N}$ such that $2^{j-1} \leq n/2$ and $n \leq 2^{j+1}$. We have

$$H_n^{(1/2)} = \{m, \cdots, n\} \subseteq \{2^{j-1} + 1, \cdots, 2^{j+1}\}$$

hence

$$\begin{aligned} {h_n}^{(1/2)}(\alpha) &\leq \sum_{k=2^{j-1}+1}^{2^j} |\alpha_k - \alpha_{k+1}| + \sum_{k=2^j+1}^{2^{j+1}} |\alpha_k - \alpha_{k+1}| \\ &= g_j(\theta^{(j)}(\alpha)) + g_{j+1}(\theta^{(j+1)}(\alpha)), \end{aligned}$$

where we have used (4.2.3). Combining these inequalities, we obtain

$$h^{(1/2)}(\alpha) \leq \sup\{{h_n}^{(1/2)}(\alpha) : n \in \mathbb{N}\} \leq 2 \sup\{g_j(\theta^{(j)}(\alpha)) : j \in \mathbb{N}\} = 2\, g(\theta(\alpha)).$$

Thus $\|\alpha\|_1^{(1/2)} \leq 2|||\theta(\alpha)|||$. Hence by the above and Lemma 4.2.6, we have that

$$\|\alpha\| \leq 4|||\theta(\alpha)|||.$$

Thus A_0 can be identified with a closed subalgebra of B, as desired. $\square$

THEOREM 4.2.8. *The algebra A_0 is Arens regular.*

PROOF. This follows from the fact that B is Arens regular, by Corollary 4.2.5, and that, by Proposition 4.2.7, A_0 can be identified with a closed subalgebra of B. $\square$

We know now that A_0 is Arens regular, and it has a bounded approximate identity. By Proposition 4.1.2 it follows that A_0'' has a unique mixed identity.

We proceed now to see that the Feinstein algebra A is Arens regular. For this, the following proposition will be key. This proposition can be found in [26, Proposition 2.3.4]

PROPOSITION 4.2.9. *Let A be a Banach algebra with a closed subalgebra B and suppose that $A^2 \subset B$. Then $A''\Box A'' \subset B''$.* $\square$

THEOREM 4.2.10. *The Feinstein algebra A is Arens regular*

PROOF. Let $E \in A_0''$ be the (unique) mixed identity of A_0''. We shall see first that $E \in \mathfrak{Z}(A'')$. Indeed, for $M \in A''$, by Proposition 4.2.9, we have that $E\Box M \in A_0''$, as well as $M\Box E \in A_0''$. Then

$$E\Box M = (E\Box M)\Box E = E\Box(M\Box E) = M\Box E.$$

Thus $E \in \mathfrak{Z}(A'')$ as desired. Now, let $M, N \in A''$. Since A_0 is commutative and Arens regular, A_0'' is commutative. Hence

$$\begin{aligned} M\Box N &= E\Box(M\Box N)\Box E = (E\Box M)\Box(N\Box E) \\ &= (N\Box E)\Box(E\Box M) = N\Box E\Box M = E\Box(N\Box M) = N\Box M, \end{aligned}$$

where we have used again that $A''\Box A'' \subset A_0''$. Thus A is Arens regular as desired. $\square$

4.2.2. Study of other properties. We proceed now to study some interesting properties of these algebras. We shall firstly focus on A_0, since some of the properties of A can be deduced from those of A_0.

PROPOSITION 4.2.11. *There exists a Banach-algebra isomorphism from c_0 onto a closed subalgebra of A_0.*

PROOF. Let (n_j) be a sequence in $\mathbb{N}$ such that $n_{j+1} \geq 2n_j + 1$ $(j \in \mathbb{N})$, and consider the following linear operator:

$$T : (\alpha_n) \longmapsto \sum_{j=1}^{\infty} \alpha_j \delta_{n_j} \quad (\alpha = (\alpha_n) \in c_0).$$

We see that $T(\alpha\beta) = T(\alpha)T(\beta)$.

Let us verify that the range of T is contained in A_0. Let $\alpha \in c_0$. For $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $|\alpha_j| < \varepsilon$ whenever $j \leq k$. Consider

$$K = \sum_{j=1}^{k} j|\alpha_{j+1} - \alpha_j|.$$

For $n \geq n_k$, there exists $q \geq k$ such that $n_q \leq n < n_{q+1}$, and so

$$p_n(T(\alpha)) \leq \frac{K}{n} + \frac{1}{n_q}\sum_{j=k}^{q} 2n_j|\alpha_{n_j}| \leq \frac{K}{n} + 4\varepsilon.$$

Hence $p_n(T(\alpha))$ tends to zero when n tends to infinity. Thus the image of c_0 by T is contained in A_0.

Let us see now that T is bounded. Let $n \in \mathbb{N}$, and consider $k \in \mathbb{N}$ such that $n_k \leq n < n_{k+1}$. Then

$$p_n(T(\alpha)) \leq \frac{1}{n_k}\sum_{j=1}^{k} 2n_j|\alpha_{n_j}| \leq 2|\alpha|_{\mathbb{N}} \sum_{j=1}^{k} \frac{n_j}{n_k}$$
$$\leq 2|\alpha|_{\mathbb{N}} \sum_{j=1}^{k} \left(\frac{1}{2^j}\right) \leq 4|\alpha|_{\mathbb{N}},$$

and so $\|T(\alpha)\| \leq 5|\alpha|_{\mathbb{N}}$.

Since $|\alpha|_{\mathbb{N}} = |T(\alpha)|_{\mathbb{N}} \leq \|T(\alpha)\|$, the image of c_0 by T is injective with closed range. $\square$

COROLLARY 4.2.12. *The space A_0 is not weakly sequentially complete. The Feinstein algebra A is not weakly sequentially complete.* $\square$

Consider now

$$C = \{\alpha \in \ell^\infty : p(\alpha) < \infty\}$$

with the norm defined as $\|\beta\| = |\beta|_{\mathbb{N}} + p(\beta)$ for $\beta \in C$.

THEOREM 4.2.13. *Consider A_0 the Banach sequence algebra defined before. Then $C = \mathcal{M}(A_0) = C_{BSE}(A_0) = Q(A_0)$.*

PROOF. By Corollary 2.3.27, since A_0 is a Tauberian Banach sequence algebra with a bounded approximate identity, we have that A_0 is a BSE algebra with a BSE norm. Thus $\mathcal{M}(A_0) = C_{BSE}(A_0)$.

Take $\beta \in C$, $\alpha \in A_0$. Let $\varepsilon > 0$. Then there exists $n_1 \in \mathbb{N}$ such that $|\alpha_i| < \varepsilon$ for all $i \geq n_1$. There exists $n_2 \geq n_1$ such that $p_n(\alpha) < \varepsilon$ and such that, for all $n \geq n_2$, $n_2\varepsilon > \sum_{k=1}^{n_1} k|\alpha_{k+1}\beta_{k+1} - \alpha_k\beta_k|$. Then, for $n \geq n_2$, we have

$$\begin{aligned} p_n(\alpha\beta) &\leq \sum_{k=1}^{n_1} \frac{k}{n}|\beta_{k+1}\alpha_{k+1} - \beta_k\alpha_k| \\ &\quad + \sum_{k=n_1+1}^{n} \frac{k}{n}|\beta_k\alpha_{k+1} - \beta_k\alpha_k| + \sum_{k=n_1+1}^{n} \frac{k}{n}|\beta_{k+1}\alpha_{k+1} - \beta_k\alpha_{k+1}| \\ &\leq \frac{n_2\varepsilon}{n} + |\beta|_{\mathbb{N}}p_n(\alpha) + \sup\{|\alpha_k| : k > n_1\}p_n(\beta) \leq \varepsilon(1 + \|\beta\|). \end{aligned}$$

Thus $p_n(\alpha\beta) \to 0$ when n tends to infinity, and so $\alpha\beta \in A_0$. Hence $\beta \in \mathcal{M}(A_0)$. Thus $C \subset \mathcal{M}(A_0) = C_{BSE}(A_0)$.

Now consider $\beta \in C_{BSE}(A_0)$. Then there exists a net $(\beta^{(\nu)})$ in A_0 that is bounded by K and such that it converges pointwise to β. Hence, for each $\varepsilon > 0$ and each $n \in \mathbb{N}$ it is possible to find ν_n such that $\sum_{i=1}^{n} |\beta_i - \beta_i^{(\nu_n)}| < \varepsilon$ and $\sum_{i=1}^{n} |\beta_{i+1} - \beta_{i+1}^{(\nu_n)}| < \varepsilon$. Consider $\varepsilon < \frac{1}{2}$. Then

$$\begin{aligned} p_n(\beta) &\leq \sum_{i=1}^{n} \frac{i}{n}|\beta_{i+1} - \beta_{i+1}^{(\nu_n)}| + \sum_{i=1}^{n} \frac{i}{n}|\beta_i - \beta_i^{(\nu_n)}| + p_n(\beta^{(\nu_n)}) \\ &\leq 2\varepsilon + \|\beta^{(\nu_n)}\| \leq K + 1, \end{aligned}$$

for every $n \in \mathbb{N}$ and so $\beta \in C$.

Finally, by Proposition 2.3.28, $Q(A_0) = C_{BSE}(A_0)$. □

An almost identical proof shows that $C_{BSE(A)} \subset C \subset \mathcal{M}(A)$, since $C_{BSE}(A)$ has an identity, then $\mathcal{M}(A) \subset C_{BSE}(A)$, and so $C_{BSE(A)} = C = \mathcal{M}(A)$. Thus A is a BSE algebra too. In addition, since A is closed in C, Proposition 2.3.29 applies and so A has a BSE norm.

COROLLARY 4.2.14. *The bidual of the Banach algebra A_0 is $A_0'' = C$.*

PROOF. Since A_0 is Tauberian and Arens regular we can apply Corollary 4.1.7 and so A_0'' has a unit. Thus we can apply Proposition 2.3.30 and the result follows. $\square$

4.3. James p^{th} algebra

The James space J_2 was introduced in [55] and [56]. In these, James constructed a separable Banach space such that there is an isomorphism onto its bidual J_2'' but such that J_2 is not reflexive. A study of this space as a Banach algebra was made in [2]. Properties like amenability and weak amenability of the p^{th} James space as a Banach algebra were studied in [78].

For $1 < p < \infty$, we define the p^{th} James space in the following way. Let $\mathcal{F}$ be the set of all finite, strictly increasing subsets of $\mathbb{N}$ containing at least two points. Then, for each $\alpha = (\alpha_n) \in \mathbb{C}^{\mathbb{N}}$ and $F = \{n_1, ..., n_k\} \in \mathcal{F}$, we define

$$N_p(\alpha, F) = 2^{-1/p} \left[\sum_{i=1}^{k-1} |\alpha_{n_{i+1}} - \alpha_{n_i}|^p + |\alpha_{n_k} - \alpha_{n_1}|^p \right]^{1/p}.$$

Then the p^{th} James space is

$$(J_p, N_p) = \{\alpha \in c_0 : N_p(\alpha) = \sup_{F \in \mathcal{F}} N_p(\alpha, F) < \infty\}.$$

As it is shown in [78], J_p with pointwise product is a Banach algebra under the equivalent norm $\|\cdot\|_p$ defined by

$$\|\alpha\|_p := \sup\{N_p(\alpha\beta) : \beta \in J_p \text{ with } N_p(\beta) = 1\} \quad (\alpha \in J_p).$$

Recall that, for $n \in \mathbb{N}$, $\Delta_n \in c_{00}$ is the element such that $\Delta_n(k) = 1$ $(k \leq n)$ and $\Delta_n(k) = 0$ $(k > n)$. It was seen in [78] that

$$N_p(\alpha) = \lim_{n \to \infty} N_p(\Delta_n \alpha) \leq \|\alpha\|_p \leq 2N_p(\alpha), \quad (\alpha \in J_p).$$

Hence (Δ_n) is an approximate identity for J_p with $\|\Delta_n\|_p = 1$, for every $n \in \mathbb{N}$. Thus J_p is a Tauberian Banach sequence algebra on $\mathbb{N}$, and (δ_n) is a Schauder basis for J_p.

Regarding Arens regularity, it follows from [2] that J_2 is Arens regular. For this, they proved that J_2'' is isometrically isomorphic to $J_2^{\#}$, which is commutative. The same argument can be followed to prove that, for $1 < p < \infty$, J_p is Arens regular. We shall see below that the unitization of J_p also plays a role when we study if J_p is a BSE algebra.

For $1 \le p < \infty$, let us consider the linear space $B_p = J_p \oplus \mathbb{C}1 \subset \ell^\infty$, so that

$$B_p = \{\alpha \in \ell^\infty : N_p(\alpha) < \infty\},$$

with the norm defined as

$$\|\alpha\|_p := \sup\{N_p(\alpha\beta) : \beta \in J_p \text{ with } N_p(\beta) = 1\} \quad (\alpha \in B_p).$$

Sometimes it can be useful to calculate the norm of elements of B_p using just elements of J_p with finite support. We shall see below that this is possible.

Lemma 4.3.1. *Let $1 \le p < \infty$, and let $\alpha \in B_p$. Then*

$$\|\alpha\|_p = \sup\{N_p(\alpha\beta) : \beta \in J_p \cap c_{00} \textit{ with } N_p(\beta) = 1\} \quad (\alpha \in B_p).$$

Proof. Let $\alpha \in B_p$. It follows from the definition of $\|\alpha\|_p$ that

$$\sup\{N_p(\alpha\beta) : \beta \in J_p \cap c_{00} \text{ with } N_p(\beta) = 1\} \le \|\alpha\|_p.$$

Let $\varepsilon > 0$. Then there exists $\beta \in J_p$ with $N_p(\beta) = 1$ such that

$$\|\alpha\|_p \le N_p(\alpha\beta) + \varepsilon/2.$$

For $\beta \in J_p$, we know that $N_p(\Delta_n\beta) \longrightarrow N_p(\beta)$ and $N_p(\Delta_n\beta)$ increases with n. Since J_p is an ideal in B_p, $\alpha\beta \in J_p$. Thus there exists $n_0 \in \mathbb{N}$ such that, for $n \ge n_0$,

$$N_p(\alpha\beta) - N_p(\Delta_n\alpha\beta) < \varepsilon/2.$$

So, we have that

$$\|\alpha\|_p \le N_p(\alpha\beta) + \varepsilon/2 \le N_p(\alpha\Delta_n\beta) + \varepsilon.$$

Thus $\|\alpha\|_p \le \sup\{N_p(\alpha\beta) : \beta \in J_p \cap c_{00} \text{ with } N_p(\beta) = 1\}$, as desired. □

Lemma 4.3.2. *Let $1 \le p < \infty$ and $\alpha \in \mathbb{C}^{\mathbb{N}}$. Then*

(a) the sequence $(\|\Delta_n\alpha\|_p)$ is increasing;

(b) for $\alpha \in B_p$, $(\|\Delta_n\alpha\|_p)$ converges to $\|\alpha\|_p$;

(c) $\alpha \in B_p$ if and only if $(\|\Delta_n\alpha\|_p)$ is bounded.

Proof. (a) Let $\alpha \in \mathbb{C}^{\mathbb{N}}$ and $\beta \in c_{00}$ with $N_p(\beta) = 1$. Since $\alpha\beta \in J_p$, for $m \le n$, $N_p(\Delta_m\alpha\beta) \le N_p(\Delta_n\alpha\beta) \le N_p(\alpha\beta)$. Thus, by Lemma 4.3.1,

$$\|\Delta_m\alpha\|_p \le \|\Delta_n\alpha\|_p \le \|\alpha\|_p \quad (m \le n). \tag{4.3.1}$$

(b) For $\alpha \in B_p$ and $\beta \in c_{00}$, there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, $\Delta_n \alpha\beta = \alpha\beta$, and so $\lim_{n\to\infty} \|\Delta_n \alpha\|_p = \|\alpha\|_p$.

(c) If $\alpha \in B_p$, $\|\alpha\|_p < \infty$, and it follows from (4.3.1) that $(\|\Delta_n\alpha\|_p)$ is bounded.

On the other hand, let $\alpha \in \mathbb{C}^{\mathbb{N}}$ such that $(\|\Delta_n\alpha\|_p)$ is bounded. By (b), $(\|\Delta_n\alpha\|_p)$ converges to $\|\alpha\|_p$ and so $\|\alpha\|_p < \infty$. Thus $\alpha \in B_p$. □

PROPOSITION 4.3.3. *Let $1 < p < \infty$. The Banach algebra J_p is a BSE algebra with a BSE norm. What is more, we have that*

$$B_p = C_{BSE}(J_p) = \mathcal{M}(J_p) = Q(J_p).$$

PROOF. By Corollary 2.3.27, since J_p is a Tauberian Banach sequence algebra with a bounded approximate identity, we have that J_p is a BSE algebra with a BSE norm. Thus $\mathcal{M}(J_p) = C_{BSE}(J_p)$. By Proposition 2.3.28, $Q(J_p) = C_{BSE}(J_p)$. Also, since J_p is an ideal in B_p, we have that $B_p \subset \mathcal{M}(J_p)$. Finally, let $\beta \in \mathcal{M}(J_p)$. Then, for $n \in \mathbb{N}$, $\|\Delta_n\beta\|_p \leq \|\beta\|_{op}\|\Delta_n\|_p = \|\beta\|_{op}$, and so by Lemma 4.3.2, $\beta \in B_p$. □

4.4. Tensor products

In this section we shall study the Arens regularity of tensor products of some of the algebras we have studied in previous sections. We shall start with some generic results that will be useful later. We shall proceed later to see some partial results for the tensor products of James p^{th} algebras. Finally, we proceed to study the tensor product of weighted bounded variation algebras, studied in Section 3.3.3.

4.4.1. General results. The following result can be found in [71, Theorem 5.33]. We add it here to facilitate the reading of the document.

THEOREM 4.4.1. *Let X and Y be Banach spaces such that X' has the Radon-Nikodým property and either X' or Y' has the approximation property. Then*

$$\left(X\check{\otimes}Y\right)' = X'\hat{\otimes}Y'.$$

□

PROPOSITION 4.4.2. *Let A and B be dual Banach algebras, with preduals E and F, respectively, such that A has the Radon-Nikodým property and either A or B has the approximation property. Then $A\hat{\otimes}B$ is a dual Banach algebra with Banach-algebra predual $E\check{\otimes}F$.*

PROOF. We can apply Theorem 4.4.1 and we see that

$$(E\check{\otimes}F)' = E'\hat{\otimes}F' = A\hat{\otimes}B.$$

Specifically, the canonical embedding from $A\hat{\otimes}B$ to $(E\check{\otimes}F)'$ is an isometric isomorphism. As $(E\check{\otimes}F)'' = (A\hat{\otimes}B)'$, then $E\check{\otimes}F$ is closed in $(A\hat{\otimes}B)'$.

Let us see now that $E\check{\otimes}F$ is a left-submodule of $(A\hat{\otimes}B)'$. Let us consider $a, c \in A$, $b, d \in B$ and $x \in E$, $y \in F$. Then, as $x \otimes y \in A' \otimes B' \subset (A\hat{\otimes}B)'$ we have that $\langle a \otimes b, x \otimes y\rangle = \langle a, x\rangle\langle b, y\rangle$. Hence

$$\langle a \otimes b, (c \otimes d) \cdot (x \otimes y)\rangle = \langle (a \otimes b)(c \otimes d), x \otimes y\rangle$$

$$= \langle ac, x\rangle\langle bd, y\rangle = \langle a, c \cdot x\rangle\langle b, d \cdot y\rangle = \langle a \otimes b, (c \cdot x) \otimes (d \cdot y)\rangle.$$

In particular, as E is a submodule of A' and F is a submodule of B', we have that $(c \cdot x) \otimes (d \cdot y) \in E\check{\otimes}F$.

Take now $M_0 = \sum_{i=1}^{m} c_i \otimes d_i \in E \otimes F$ and $N_0 = \sum_{j=1}^{n} x_j \otimes y_j \in E \otimes F$, then

$$M_0 \cdot N_0 = \sum_{i=1}^{m}\sum_{j=1}^{n}(c_i \cdot x_j) \otimes (d_i \cdot y_j) \in E \otimes F.$$

Now let $M \in A\hat{\otimes}B$, $N \in (A\hat{\otimes}B)'$. Let (M_n) be a sequence in $A \otimes B$ such that $M_n \to M$ in $A\check{\otimes}B$, and let (N_n) be a sequence in $E \otimes F$ with $N_n \to N$ in $E\check{\otimes}F \subset (A\hat{\otimes}B)'$. Since the action of any Banach algebra on its dual is (jointly) continuous, and $E\check{\otimes}F$ is closed in $(A\hat{\otimes}B)'$, we have

$$M \cdot N = \lim_n M_n \cdot N_n \in E\check{\otimes}F.$$

The proof that $E\check{\otimes}F$ is a right submodule is symmetrical. □

If A and B are Tauberian Banach sequence algebras, this property is transmitted to the tensor product under certain circumstances. The following result can be found in [26, Proposition 3.3.4]:

PROPOSITION 4.4.3. *Let A and B be natural Banach sequence algebras on S and T, respectively, and suppose that either A or B has the approximation property. Then $A\hat{\otimes}B$ is a natural Banach sequence algebra on $S \times T$, and $A\hat{\otimes}B$ is Tauberian whenever both A and B are Tauberian.* □

The following result is [71, Corollary 5.42, Corollary 5.45]:

COROLLARY 4.4.4. *Every separable dual space and every reflexive Banach space have the Radon-Nikodým property.* □

4.4.2. James p^{th} algebra tensor products. We have not been able to discern if given $1 < p, q < \infty$, the Banach algebra $J_p \hat{\otimes} J_q$ is Arens regular. However, we have some results that might lead to a better understanding of $J_p \hat{\otimes} J_q$ in the future.

PROPOSITION 4.4.5. *Let $1 < p, q < \infty$. Then $B_p \hat{\otimes} B_q$ is a dual Banach algebra.*

PROOF. The fact that it is a dual Banach algebra follows from Proposition 4.4.2 and the following: B_p and B_q are dual Banach algebras, they are separable dual spaces and so they have the Radon-Nikodým property, and they have the approximation property. □

4.4.3. Weighted bounded variation algebras. In this section we shall study the Arens regularity of the tensor products of the algebras studied in Section 3.3.3.

Note that the set $\{\delta_n : n \in \mathbb{N}\}$ is a Schauder basis for M_ω, and so M_ω has the approximation property.

PROPOSITION 4.4.6. *Let $\omega_1, \omega_2 : \mathbb{N} \longrightarrow [1, \infty)$ be sequences. Consider the algebras M_{ω_1} and M_{ω_2}. Then the tensor product $M_{\omega_1} \hat{\otimes} M_{\omega_2}$ is a Tauberian Banach sequence algebra on $\mathbb{N} \times \mathbb{N}$.*

PROOF. Since M_ω is Tauberian for any ω and it has the approximation property, the conclusion follows from Proposition 4.4.3. □

PROPOSITION 4.4.7. *Let $\omega_1, \omega_2 : \mathbb{N} \longrightarrow [1, \infty)$ be sequences such that $\liminf \omega_i(n) < \infty$, $(i = 1, 2)$. Then the Banach algebra $M_{\omega_1} \hat{\otimes} M_{\omega_2}$ is strongly Arens irregular.*

PROOF. By Proposition 3.3.5, $M_{\omega_i} \sim \ell^1$ for $i = 1, 2$, and so, $M_{\omega_1} \hat{\otimes} M_{\omega_2} \sim \ell^1(\mathbb{N} \times \mathbb{N})$. Hence $M_{\omega_1} \hat{\otimes} M_{\omega_2}$ has the Schur property (and so it is weakly sequentially complete). $M_{\omega_1} \hat{\otimes} M_{\omega_2}$ has a bounded approximation identity, since we saw in Section 3.3.3 that for $\liminf \omega(n) < \infty$, M_ω has a bounded approximate identity. Finally, by Proposition 4.4.6, $M_{\omega_1} \hat{\otimes} M_{\omega_2}$ is an ideal in its bidual. Thus, by Theorem 2.3.10, it is strongly Arens irregular as desired. □

PROPOSITION 4.4.8. *Let $\omega_1, \omega_2 : \mathbb{N} \longrightarrow [1, \infty)$ be sequences such that, for $i = 1, 2$, $\liminf \omega_i(n) = \infty$. Then the Banach algebra $M_{\omega_1} \hat{\otimes} M_{\omega_2}$ is Arens regular.*

PROOF. Since for ω such that $\liminf \omega(n) = \infty$ we know that M_ω is also a dual Banach algebra, by Corollary 2.3.23 we know that M_ω'' is compact. In this case, by Corollary 2.3.13, $M_{\omega_1} \hat{\otimes} M_{\omega_2}$ is Arens regular. □

PROPOSITION 4.4.9. *Let* $\omega_1, \omega_2 : \mathbb{N} \longrightarrow [1, \infty)$ *such that* $\liminf \omega_1(n) = \infty$ *and* $\liminf \omega_2(n) < \infty$. *Then* $M_{\omega_1} \hat{\otimes} M_{\omega_2}$ *is neither Arens regular nor strongly Arens irregular.*

PROOF. By Proposition 4.4.6, $M_{\omega_1} \hat{\otimes} M_{\omega_2}$ is again a Tauberian natural Banach sequence algebra on $\mathbb{N} \times \mathbb{N}$. It is isometrically isomorphic to $\ell^1(\mathbb{N} \times \mathbb{N})$ as a Banach space. Thus it also has the Schur property. In this case $(\Delta_n \otimes \Delta_n)$ is an unbounded approximate identity, as otherwise we would have that (Δ_n) is bounded in M_{ω_1}. Also, $(\Delta_n \otimes \Delta_n)$ is a multiplier-bounded approximate identity, so $\|\cdot\|_\pi$ and $\|\cdot\|_{op}$ are not equivalent on $M_{\omega_1} \hat{\otimes} M_{\omega_2}$. By Corollary 2.3.12, the algebra is not strongly Arens irregular. Now, as M_{ω_2} is strongly Arens irregular, by [75, Corollary 3.5], $M_{\omega_1} \hat{\otimes} M_{\omega_2}$ is not Arens regular. $\square$

CHAPTER 5

Decomposable Blaschke products of degree 2^n

5.1. An overview of the new results

For a brief introduction about Blaschke products and their relevance, see Section 1.2.

Let $\mathbb{D}$ denote the open unit disk and $\mathbb{T}$ denote the unit circle. A finite *Blaschke product* B of degree n is a function of the form

$$B(z) = \gamma \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z},$$

where $a_j \in \mathbb{D}$ for $j = 1, \ldots, n$ and $\gamma \in \mathbb{T}$. Note that Blaschke products of degree 1 are the disk automorphisms. Finite Blaschke products are n to 1 maps of the open unit disk $\mathbb{D}$ into itself and the unit circle $\mathbb{T}$ to itself. They are holomorphic on an open set containing the closed unit disk and have finitely many zeros in $\mathbb{D}$. We will consider the set of points in $\mathbb{T}$ that the Blaschke product identifies; in other words, we will be interested in the solutions of $B(z) = \lambda$ for $\lambda \in \mathbb{T}$. Since the constant γ will not play a role in the solution, we will take $\gamma = 1$ in the above description of Blaschke product.

Definition 5.1.1. We say a Blaschke product B is *decomposable* if there exist Blaschke products C and D both of degree $n > 1$ such that

$$B(z) = C(D(z)) = (C \circ D)(z)$$

and B is *indecomposable* otherwise.

If the degree of C and D equals k and m respectively, then the degree of B equals km. Observe that if B is of prime degree, then B is indecomposable.

It is worth noting that in this chapter *langle*$\cdot, \cdot\rangle$ is the (sesquilinear) inner product in ℓ_2^n and $\|\cdot\|$ is the associated norm. With this in mind, given an $n \times n$ matrix A, the *numerical range* $W(A)$ is defined by

$$W(A) := \{\langle Ax, x\rangle : \ x \in \mathbb{C}^n, \ ||x|| = 1\}.$$

The set $W(A)$ contains the spectrum of A, it is a convex set, and its outer boundary is a convex curve. In particular if A is a 2×2 matrix, then its numerical range is either a point, a line segment, or an elliptical disk – all of these can be thought of as elliptical disks. This is called the *elliptical range theorem* [17]. Although this is a theorem about 2×2 matrices, it also sheds light on the numerical range of $n \times n$ matrices.

Let H^2 denote the classical Hardy space, i.e. the set of functions of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. An important operator in the Hardy space is the shift operator, which is defined by $[S(f)](z) = zf(z)$ and its adjoint is S^* defined by $[S^*(f)](z) = (f(z) - f(0))/z$. The following can be defined in a more generic set-up, but we are going to restrict ourselves to the case of a Blaschke product.

Definition 5.1.2. Let B be a Blaschke product, then the *model space* is

$$K_B = H^2 \ominus BH^2,$$

where $\ominus$ is the orthogonal complement of BH^2 in H^2, and BH^2 is the image of H^2 by the Blaschke product B.

The *compression of the shift* is an operator defined on the model space K_B as follows:

$$S_B := P_B S|_{K_B},$$

where P_B is the orthogonal projection from H^2 onto K_B.

In this chapter, we consider a Blaschke product B of degree $n = 2^k - 1$ and $\widehat{B} := zB(z)$ throughout the chapter. For each $\lambda \in \mathbb{T}$ we let $z_{1,\lambda}, \dots, z_{n+1,\lambda}$ denote the points on $\mathbb{T}$ that $\hat{B}$ maps to $B(\lambda)$ ordered by increasing argument. By [43, 63] each convex polygon connecting these points circumscribes a convex smooth curve that we call the *Blaschke curve*, see Definition 5.3.1. In Section 5.3 we show that if the numerical range, $W(S_B)$, is an elliptical disk, then every lower degree curve in the Poncelet package (see that section for a definition) for $\widehat{B}(z) := zB(z)$ is an ellipse and $\widehat{B}$ is a composition of k degree-2 Blaschke products. Here, the boundary of the numerical range is the Blaschke curve associated with $\widehat{B}$. This statement about composition can be generalized to Blaschke products of degree n; see Theorem 5.3.5 below. We also give an example of a Blaschke product of degree-8 for which $W(S_B)$ is elliptical and a "non-example" of a Blaschke product of degree-8 that factors into three degree-2 Blaschke products, but such that $W(S_B)$ is not an elliptical disk.

In Section 5.4, we turn our attention to a deep theorem of Ritt [68] that classifies decomposability of B in terms of the monodromy group associated with B. See also [12, 77] for more recent developments. For this purpose, we start by examining critical values of Blaschke products with elliptical Blaschke curve. As usual, by the set of critical values of B we mean $\{w \in \mathbb{D} : \ w = B(z) \text{ and } B'(z) = 0\}$. We prove that if one writes a Blaschke product B of degree 2^n as composition of n degree-2 Blaschke products, then B has at most n distinct critical values. After obtaining a description of a Blaschke product of degree-n with one critical value, we deduce that if B is such a Blaschke product of degree $n = p_1 p_2 \dots p_m$, then B can be factored in any order as a composition of m Blaschke products of degree $p_1, p_2, \dots, p_m$.

In Section 5.5, we describe the monodromy group of B assuming that B is a normalized Blaschke product (see Section 5.5 for this definition) that is the composition of n degree-2 Blaschke products with n critical values. We prove that in this case the monodromy group associated with B is the wreath product of n cyclic groups of order 2.

In Section 5.6, letting $C(\mathbb{T})$ denote the space of continuous functions from the unit circle to itself, we study the *group of invariants* of a finite Blaschke products B; that is, the group

$$\mathcal{G}_B = \{u \in C(\mathbb{T}) : B \circ u = B|\mathbb{T}\}.$$

In [8] Cassier and Chalendar showed that the group of invariants of a Blaschke product of degree n is a cyclic group of order n. The group of invariants for infinite Blaschke products with finitely many singularities was considered in [10]. Here we show that if B is a composition of n degree-2 Blaschke products, say $B = C_n \circ C_{n-1} \circ \cdots \circ C_1$, then the group of invariants of $C_j \circ \cdots \circ C_1$ is of index 2 (and hence normal) in the group of invariants of $C_{j+1} \circ \cdots \circ C_1$. From this, we are able to obtain a connection between elements of the group and a particular automorphism of the unit disk of the form

$$\varphi_a(z) = (z - a)/(1 - \overline{a}z) \tag{5.1.1}$$

for $a \in \mathbb{D}$. From now on, this will be the notation that we shall use for an automorphism of the unit disk unless specified otherwise.

5.2. Closure results

We begin by discussing the background required in projective geometry. Given a field $\mathbb{K}$, for $x, y, z \in \mathbb{K}$, we have an equivalence relation defined on $\mathbb{K}^3 \setminus \{(0,0,0)\}$ by $(x, y, z) \simeq (x', y', z')$ if there exists a scalar $\lambda \neq 0$ such that $x' = \lambda x, y' = \lambda y$, and

$z' = \lambda z$. The projective plane $\mathbb{P}^2(\mathbb{K})$ is defined as the set of equivalence classes for the relation $\simeq$. In this chapter, our field will be $\mathbb{R}$ or $\mathbb{C}$. Points in the real (complex) projective plane $\mathbb{P}^2(\mathbb{R})$ are equivalence classes of triples of real (respectively complex) numbers for the relation above. The real projective plane $\mathbb{P}^2(\mathbb{R})$ is embedded in the complex projective plane $\mathbb{P}^2(\mathbb{C})$.

An algebraic curve in $\mathbb{P}^2(\mathbb{C})$ is the set of zeros of a homogeneous polynomial f with complex coefficients. A *real* algebraic curve is an algebraic curve in the complex projective plane given by an equation $f(x, y, z) = 0$, where f is a homogeneous polynomial with real coefficients. The set of real points of a curve Γ will be denoted by $\Gamma_{\mathbb{R}}$ and is defined by $\Gamma_{\mathbb{R}} = \Gamma \cap \mathbb{P}^2(\mathbb{R})$.

Since polynomial rings over a field are unique factorization domains, every algebraic curve C is the union of finitely many irreducible curves, called its irreducible components. If $C_1, \dots, C_k$ are the irreducible components of C with irreducible defining polynomials $f_1, \dots, f_k$, then $f = f_1 \cdots f_k$ is the minimal polynomial defining C.

5.2.1. Duality and reciprocation about $\mathbb{T}$. If Γ is a real algebraic curve in $\mathbb{P}^2(\mathbb{C})$, then the dual of Γ is denoted by Γ^*, where the points correspond to the tangent lines to Γ.

Note that the dual curve of a general plane algebraic curve Γ is the union of the dual curves of its irreducible components. In particular, Γ and Γ^* have the same number of irreducible components. Moreover, note that the dual to a conic (i.e. a plane algebraic curve of degree 2) is a conic and the dual to a line is a point. By [36, Theorem 5.1], duality is involutive, that is $(\Gamma^*)^* = \Gamma$. Note that the result is stated under the assumption that the curve has no lines as components, because a point is not an algebraic curve.

5.2.2. Poncelet's porism. As we mentioned in Section 1.2, Poncelet discovered that if there exists a polygon of n-sides that is inscribed in a given conic and circumscribed about another conic, then infinitely many such polygons exist. The two conics are said to be *n-Poncelet related*.

Suppose that an ellipse E_1 is inscribed in a convex n-gon that is itself inscribed in the unit circle; that is, E_1 and the unit circle are Poncelet related. Consider the diagonals that connect vertex $k-1$ to vertex $k+m$, for $k = 1, \dots, n$. Then [6, p. 208] the envelope of these diagonals is again an ellipse, E_{m+1}, for $m = 1, \dots, [n/2] - 1$, where $[x]$ denotes the greatest integer less than or equal to x. Below are different versions of the theorems that we use later. These are often referred to as Darboux's theorem. The one we will use can be found in [64] or [6, p. 208].

THEOREM 5.2.1. *Let E_1 be an ellipse inscribed in a convex n-gon that is, in turn, inscribed in $\mathbb{T}$. Consider the diagonals of the Poncelet polygons that leap over m vertices (i.e., all such diagonals that connect vertex $k-1$ with vertex $k+m$ where $k = 1, 2, \ldots, n$ and the vertex numbers are taken modulo n). The envelope of these diagonals is also an ellipse E_{m+1} for $m = 1, \ldots, [n/2] - 1$.*

These curves were also studied in [31], where the author looks at Poncelet–Darboux theorems. There, the relevant result relies on the notion of a conic and a curve being Poncelet–Darboux related, an extension of the notion for two conics being Poncelet related.

The result from Darboux's theorem, and the notion of being Poncelet related, can be extended to the setting of a conic C and a curve S of degree $n-1$ (cf. [31, Subsection 2.1]).

DEFINITION 5.2.2. Let S a curve of degree $n-1$ and C be a conic together with a set of n lines tangent to C. We say that S and C are Poncelet–Darboux n-related, if S contains all the intersection points of the n tangents to C.

Note that the above definition implicitly includes the main result of a theorem by Darboux ([27], [31, Theorem 1]): If S contains all the intersection points of n given tangents, then it will contain the intersection points of any other set of n tangents.

THEOREM 5.2.3 (Darboux's Theorem, cf. [31, Theorem 5]). *If a curve S of degree $n-1$ is n-Poncelet–Darboux related to a conic K and if there is a conic C that is a component of S which is n-Poncelet related to the conic K, then for n odd, the curve S can be completely decomposed into $(n-1)/2$ conics, and for n even, it can be decomposed into $(n-2)/2$ conics and a line.*

In other words, the minimal homogeneous polynomial defining the curve S factors as a product of $(n-1)/2$ degree-2 irreducible polynomials if n is odd, and $(n-2)/2$ degree-2 irreducible polynomials and a degree one polynomial if n is even.

In the case where K is an ellipse, and hence also for $K = \mathbb{T}$, the tangency condition in the theorem above implies that S lies outside K. As we are interested in studying curves inside the unit circle, we need a dual formulation to Theorem 5.2.3. This can be obtained by considering $C = \mathbb{T}$ and S as the "dual curve" to a closed convex curve in $\mathbb{D}$. By reciprocation, S will lie outside of $\mathbb{T}$.

THEOREM 5.2.4 (cf. [54, Theorem B]). *Let C be a closed convex curve in $\mathbb{D}$ and suppose that there is an n-sided polygon inscribed in $\mathbb{T}$ and circumscribed about C. Assume further that C is a connected component of a real algebraic curve Γ in $\mathbb{D}$ of class $n-1$ such that each diagonal of the polygon is tangent to Γ. Then for every point λ of $\mathbb{T}$ there is an n-sided convex polygon that is inscribed in $\mathbb{T}$, circumscribed about Γ, and has λ as a vertex. In the special case when C is an ellipse, the curve Γ decomposes into $(n-1)/2$ ellipses if n is odd, and $(n-2)/2$ ellipses and an isolated point if n is even.*

Motivated by the above formulation, and in accordance with Mirman [64], we have the following definition.

DEFINITION 5.2.5. A smooth closed curve Γ contained in $\mathbb{D}$ is an *n-Poncelet curve* if for every point λ of $\mathbb{T}$ there is an n-sided convex polygon that is inscribed in $\mathbb{T}$, circumscribed about Γ, and has λ as a vertex.

If we begin with a Poncelet curve that is inscribed in a convex n-gon, Mirman considers the diagonals of these polygons and denotes the envelope of the diagonals that skip m vertices, with $m \leq [n/2]-1$, by K_{m+1}. The set $K_1, \ldots, K_{[n/2]}$ is called a package of Poncelet curves. Two recent papers [54], [53] provide many details and examples relevant to Mirman's work as well as this chapter. In our setting, the K_j will be ellipses and for $j > 1$ we sometimes refer to these as lower-degree curves. Thus, $\bigcup_{j=1}^{[n/2]} K_j$ is the package of Poncelet curves generated by the convex n-gons. We will speak of a *package of Poncelet curves associated to a Blaschke product B* whenever the vertices of the polygon come from the function $\widehat{B}(z) := zB(z)$.

5.2.3. Poncelet, Darboux and the numerical range. Given an $n \times n$ matrix A we let the real part of A and the imaginary part be the self-adjoint matrices defined by

$$\mathrm{Re}(A) = \frac{A + A^*}{2} \qquad \text{and} \qquad \mathrm{Im}(A) = \frac{A - A^*}{2i}.$$

Of course, $A = \mathrm{Re}(A) + i\mathrm{Im}(A)$. By [58], we may associate a curve Γ of class n in homogeneous line coordinates via the function $f_A(u, v, w) = \det(u\,\mathrm{Re}(A) + v\,\mathrm{Im}(A) +$

$w\,I)$. Consider the algebraic curve $C(A)$ determined by $f_A = 0$ in $\mathbb{P}^2(\mathbb{C})$; that is

$$C(A) = \{(u, v, w) \in \mathbb{P}^2(\mathbb{C}) : u\,x + v\,y + w\,z = 0 \text{ is a tangent line to } f_A(x, y, z) = 0\}.$$

Kippenhahn's theorem says that $W(A)$ is the convex hull of the real points of $C(A)$. This curve, $C_{\mathbb{R}}(A)$, is called the Kippenhahn curve of A.

We will be interested in the numerical range of compressions of the shift operator associated with finite Blaschke products, see Definition 5.1.2. If we consider S_B, the vertices of each polygon are determined by the function $\widehat{B}(z) := zB(z)$ as follows: Given $\lambda \in \mathbb{T}$, the vertices of the corresponding polygon are the solutions of $\widehat{B}(z) = \widehat{B}(\lambda)$. Such operators have no unitary summand, so by [58, Theorem 2] their eigenvalues are interior to $W(A)$ and the boundary of $W(A)$ is smooth (see, for example, [58, Theorem 12]). Our curves have the property that for $\lambda \in \mathbb{T}$ the two sides of the polygons with vertex at λ are tangent to Γ, and every point of Γ is such a point of tangency, [43]. We see from the expression for f_A that f_A is a homogeneous polynomial of degree n with real coefficients, which tells us that the dual of Γ, denoted by Γ^*, is a real algebraic curve in $\mathbb{P}^2(\mathbb{C})$. It is known [54, Lemma 3.10] that if Γ is a real algebraic curve of class $n-1$, then

$$\bigcup_{j=1}^{[n/2]} K_j = \Gamma.$$

5.3. Ellipses, Numerical Range, and the Blaschke curve

5.3.1. Blaschke products and composition. We begin by considering the Blaschke curve, which is defined as follows.

Definition 5.3.1. Let B be a Blaschke product of degree $n+1$. Then the Blaschke curve $\mathcal{C}$ associated with this Blaschke product is a curve inscribed in the convex polygons with vertices at the solutions $z_j \in \mathbb{T}$ of $B(z_j) = \lambda$ for each $\lambda \in \mathbb{T}$. Each point on the curve is the point of tangency of such a circumscribing convex polygon.

Note that a Blaschke curve is assumed to be contained in $\mathbb{D}$. It is known that the line joining the z_j to z_{j+1} is tangent to $\mathcal{C}$ at a single point, the curve contains no line segments, is a differentiable algebraic curve, and every point on the curve $\mathcal{C}$ can be obtained using the Blaschke product as described in the definition. See [43, 45, 18].

Note that a Blaschke curve is determined by the values the Blaschke product B identifies. The actual value λ associated with a polygon is irrelevant. Therefore, if

φ is an automorphism of the disk and B is a Blaschke product, since $\varphi \circ B$ and B identify the same points on the circle, the two Blaschke curves will be the same, but in general different Blaschke products produce different Blaschke curves. Furthermore, even though a Blaschke curve is a Poncelet curve, not all Poncelet curves are Blaschke curves, [62]. The relationship between the geometry of the numerical range (elliptical numerical range) and composition of degree two Blaschke products is given in [47]. When we assume that $B(0) = 0$, writing $B(z) = zB_1(z)$, it can be shown that the Blaschke curve is the smooth convex curve in $\mathbb{D}$ that is the boundary of the numerical range of an operator that is unitarily equivalent to the compressed shift operator S_{B_1} as defined in Definition 5.1.2 (see [17, 43, 45].) This class consists of contractions T with eigenvalues inside the open unit disk $\mathbb{D}$, that satisfy rank $(I - T^{\star}T) = 1$.

The proof that an elliptical numerical range implies that Blaschke products are compositions is an extension of theorems for degree-4 Blaschke products [40, 47], and degree-6 Blaschke products in [53]. We will use the following theorem:

THEOREM 5.3.2. [15, Theorem 2.3] *Given two sets of points $z_1, \ldots, z_n$ and $z'_1, \ldots, z'_n$ interlaced on the unit circle, there is a Blaschke product B of degree n such that $B(0) = 0$, $B(z_j) = B(z_k)$ and $B(z'_j) = B(z'_k)$ for all j and k. This Blaschke product B is unique up to a rotation factor λ with $|\lambda| = 1$.*

THEOREM 5.3.3. *Let B be a Blaschke product of degree $2^n - 1$. If $W(S_B)$ is an elliptical disk then every lower degree curve in the Poncelet package for $\widehat{B}(z) := zB(z)$ is an ellipse or a point and $\widehat{B}$ is a composition of n degree-2 Blaschke products.*

PROOF. By [43, Theorem 2.1], the boundary of $W(S_B)$ is circumscribed by a 2^n-sided convex polygon. Applying Darboux's theorem as given in Theorem 5.2.1 implies, among other things, that all curves inscribed in the appropriate convex polygons with 2^m sides, $m = 2, 3, \ldots, n$ are elliptical or a point. In addition, by Theorem 5.2.4, since the degree of the dual curve is odd, the algebraic curve in question, which is the dual of the dual, will decompose into conics and a point. The set of diagonals joining vertex k with vertex $k + 2^{n-1} \pmod{2^n}$ will yield the point. It is shown in [47] that if $\widehat{B}$ has degree 4 and elliptical numerical range then $\widehat{B}$ is the composition of two degree-2 Blaschke products. We prove the rest by induction.

So suppose that if $\widehat{B_1}$, defined by $\widehat{B_1}(z) = zB_1(z)$, is a Blaschke product with corresponding Blaschke curve (that is, the boundary of $W(S_{B_1})$) elliptical and the degree of $\widehat{B}_1$ is equal to 2^{n_1} with $2 \leq n_1 < n$, then $\widehat{B_1}$ is the composition of n_1 degree-2 Blaschke products. Now consider $\widehat{B}$ of degree 2^n. According to Theorem 5.2.1, if we have an ellipse that is inscribed in a convex n-gon that is itself inscribed in $\mathbb{T}$, then the diagonals of the circumscribing polygons (that connect vertex $k-1$ with vertex $k+m$ for $k = 1, 2, \ldots, 2^n$ and indices chosen modulo 2^n), have as envelope an ellipse E_m. Take the vertices of two polygons, P_z and P_w, and denote them by $\{z_0, \ldots, z_{2^n-1}\}$ and $\{w_0, \ldots, w_{2^n-1}\}$. If we skip a point when connecting points identified by $\widehat{B}$ on $\mathbb{T}$, Theorem 5.2.1 tells us that we will see a Poncelet ellipse and since these polygons have 2^{n-1} interlaced vertices, these line segments will produce a closed convex polygon. So, skipping a point in each set $\{z_j\}$ and $\{w_j\}$, we obtain four convex polygons with 2^{n-1} vertices $\{z_{2j}\}$, $\{z_{2j+1}\}$, $\{w_{2j}\}$, and $\{w_{2j+1}\}$. By Theorem 5.2.1, there is one ellipse inscribed in these convex polygons. That ellipse is a Poncelet ellipse and therefore, by [44, p. 219] it is also a Blaschke curve. Thus, there is a Blaschke product D of degree 2^{n-1} with $D(0) = 0$ that identifies each point in a set with every other point in the same set: For $j \neq l$,

$$D(z_{2j}) = D(z_{2l}) = \lambda_1, D(z_{2j+1}) = D(z_{2l+1}) = \lambda_2,$$

and

$$D(w_{2j}) = D(w_{2l}) = \gamma_1, \text{ and } D(w_{2j+1}) = D(w_{2l+1}) = \gamma_2.$$

Now because the points are interlaced and a Blaschke product has increasing argument on the unit circle, we may assume without loss of generality that

$$\arg(\lambda_1) < \arg(\gamma_1) < \arg(\lambda_2) < \arg(\gamma_2).$$

Therefore, by [46, Theorem 9] there is a degree-2 Blaschke product C mapping 0 to 0 such that

$$C(\lambda_1) = C(\lambda_2) \text{ and } C(\gamma_1) = C(\gamma_2).$$

Thus $C \circ D$ identifies the vertices of P_z and $C \circ D$ identifies the vertices of P_w. Further $C \circ D(0) = 0$.

By the uniqueness guaranteed by Theorem 5.3.2, there exists $\lambda \in \mathbb{T}$ such that $\widehat{B} = \lambda(C \circ D)$. Since D is degree 2^{n-1} with $D(0) = 0$ that identifies every other point in our sets, Theorem 5.2.1 applies to D and therefore the induction hypothesis applies to D. Thus D factors into a composition of $n-1$ Blaschke products of degree 2 and therefore the result holds. $\square$

REMARK 5.3.4. If all the lower-degree curves (that is, when we skip at least one vertex) are ellipses, one can use a counting argument to show that the curve that we obtain by connecting successive points is also a conic: Consider the boundary of the numerical range. By Kippenhahn's theorem [58, Theorem 10] this is an algebraic curve and the dual curve Γ has degree $2^n - 1$. Since the degree of the dual curve is odd, and we assume that each lower-degree component is an ellipse, no component will be a point. Given our assumptions, the numerical range is contained in the open unit disk and the lower-degree curves are all ellipses. Therefore, since the ellipses skipping more than 2^{n-1} points can be matched with one of those skipping fewer than 2^{n-1} points and the one skipping exactly 2^{n-1} points yields a single point, we get $2^{n-1} - 2$ ellipses for the lower-degree cases. (For example, in case we have $8 = 2^3$, we have ellipses when we join every 2nd or 3rd point, and we get a point when we join every 4th point. The case when we join every 5th point is the same as joining every 3rd point and joining every 6th point is the same as joining every second point. The case when we join subsequent points is the one we are trying to determine.) There are $2^{n-1} - 2$ ellipses, a line, and the curve we are trying to identify. Now the dual of an algebraic curve of degree 2 maintains the same degree and the dual of a line is a point, so the degrees of the dual curves corresponding to components that we obtain by skipping at least one point therefore total $2(2^{n-1} - 2) = 2^n - 4$ for those corresponding to ellipses and 1 for the line, or $2^n - 3$. But we should have degree $2^n - 1$, so the component of Γ that we have not yet counted, namely the one corresponding to the curve in which we do not skip any points, must have degree 2. Therefore, it must be a conic and the dual of the dual (the original curve) is contained in the boundary of the numerical range and is a compact convex subset of $\mathbb{D}$. Since the dual of a degree-2 curve is a conic, it must be an ellipse. In Example 5.3.7 we present a Blaschke product of degree 2^n such that $W(S_B)$ is not elliptical, but all lower-degree curves corresponding to polygons that are Poncelet curves inscribed in convex polygons with 2^m sides with $m < n$ are elliptical. Of course, Theorem 5.2.3 tells us that none of the curves that are 2^n-Poncelet can be elliptical. This is also illustrated in this example.

The proof of Theorem 5.3.3 works in greater generality, as indicated below. We have stated it in this way because of our focus on Blaschke products that have degree a power of 2.

THEOREM 5.3.5. *Let B be a degree n Blaschke product with an elliptical Blaschke curve. Then for each factor $k > 1$ of n, there are Blaschke products C of degree k and D of degree n/k such that $B = C \circ D$.*

The proof is essentially the same as that of Theorem 5.3.3 above. We provide a brief outline of the proof, indicating places where the proof will be slightly different.

PROOF. Suppose $n = km$, with $k, m \in \mathbb{N} \setminus \{1\}$. Let P_1 and P_2 denote two Poncelet polygons with n vertices, $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$. Using every k-th point as a vertex, we get k convex m-gons and, applying Theorem 5.2.1, we may conclude that they circumscribe the same ellipse. Since this ellipse is a Poncelet curve contained in $\mathbb{D}$ inscribed in a convex polygon that has all of its vertices on $\mathbb{T}$, as above there is a Blaschke product D of degree m that maps 0 to 0 and identifies each set of m vertices of each of the respective polygons. That is, these m-gons are the convex polygons circumscribing the Blaschke curve of D. Now there are k polygons with m vertices and D is exactly m-to-1, so D takes k values on the k sets, $\{z_j\}$, and k other values on the $\{w_j\}$. This gives us two sets of k values that can be ordered to be interspersed on the unit circle. Therefore, we may choose a Blaschke product C of degree k such that $C(0) = 0$ and C identifies these two sets of k values. Thus $C \circ D$ maps 0 to 0 and identifies the same two sets of points as B. As in the previous theorem, Theorem 5.3.2 implies that $B = C \circ D$. $\square$

5.3.2. Examples of elliptical and non-elliptical curves. In this section, we provide an example of a Blaschke product of degree 8 with an elliptical Blaschke curve as well as a Blaschke product of degree 8 with non-elliptical Blaschke curve.

EXAMPLE 5.3.6. We begin with an example of a Blaschke product C of degree 8 with an elliptical Blaschke curve. To connect this to the numerical range of a compression of the shift, we need $C(0) = 0$. Then $W(S_{C(z)/z})$ will have elliptical numerical range.

Let $a \in \mathbb{D}$. Let $\varphi_a(z)$ as in (5.1.1) and consider the automorphism $\varphi(z) = \varphi_a(z) \circ e^{i\pi/4} z \circ \varphi_a(z)$, which has the property that the composition $\varphi^{[8]}(z) = z$ for all $z \in \mathbb{C}$ and no lower-degree composition satisfies $\varphi^{[j]}(z) = z$ for all $z \in \mathbb{C}$. The Blaschke product $B(z) = (\varphi(z))^8$ is degree 8 and [16, Corollary 11] shows that for each $\lambda \in \mathbb{D}$, the line segments joining the points for which $B(z) = \lambda$ circumscribe an ellipse. By Theorem 5.2.1 we expect that the polygons produced by connecting vertices, as described in that theorem, will also have an ellipse (or point) as their envelope.

To obtain a Blaschke product that maps the origin to itself, we consider the automorphism $\varphi_\alpha(z) := \frac{\alpha - z}{1-\overline{\alpha}z}$ where $\alpha = B(0)$, and note that since the Blaschke product $C = \varphi_\alpha \circ B$ identifies the same points on the unit circle as B, the Blaschke product C has an ellipse as Blaschke curve. It is easy to see that C factors into a composition of degree-2 Blaschke products; for example, we may take $(\varphi_\alpha \circ z^2) \circ z^2 \circ (\varphi_a(z))^2$. (It is shown in [16] that the curves corresponding to skipping 2 and 4 points also product Poncelet ellipses, but this follows directly from Darboux's theorem as well.) When we take $a = .5$ we obtain the pictures below. Note that each of the ellipses in the family has two of the zeros of the Blaschke product as foci; see [64] for more information on the location of foci. The zeros of the Blaschke product C are obtained using Mathematica:

$$-0.158011 + 0.369131i, 0.0141808 + 0.629309i, 0.241238 + 0.685693i,$$

$$0.401172-0.0169046i, 0.42801+0.619984i, 0.555657+0.468632i, 0.58342+0.236332i$$

plus one zero at zero.

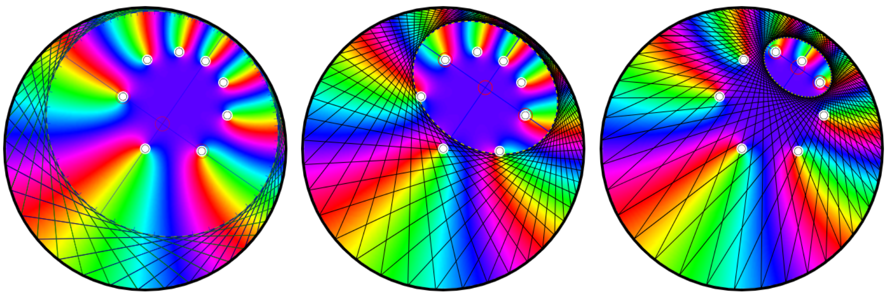

FIGURE 1. Degree-8 Blaschke product example

We say more about this in Proposition 5.4.8.

EXAMPLE 5.3.7. We now turn to an example of a "non-example"; that is, we give an example of a Blaschke product of degree 8 that factors into three degree-2 Blaschke products but such that the Blaschke curve is *not* elliptical. See Figure 2.

Let $a = .84^4$ and consider the Blaschke product

$$B(z) = z^4 \left(\frac{z^4 - a}{1 - az^4} \right).$$

This Blaschke product has four zeros at $.84, -.84, .84i, -.84i$ and a zero of order 4 at zero. It is also clear that $B = B_2 \circ B_1 \circ B_1$ where $B_1(z) = z^2$ and $B_2(z) = \frac{z(z-a)}{1-az}$. Due to the symmetry of the problem, we obtain vertical and horizontal tangent lines to the Blaschke curve when $B(z) = -1$. The solutions to this equation are denoted by $z_1, z_2, \ldots, z_8$. Using Mathematica we obtain the eight solutions and we are able to compute the semi-major and semi-minor axis and we find that they are both equal to .965767. Therefore, if this were an ellipse, the equation would be

$$x^2 + y^2 = .965767^2.$$

By construction this circle will be tangent to the horizontal and vertical segments in the circumscribing polygon, but it must be tangent to all other sides as well. Since it must be tangent to the line segment joining z_1 and z_2, we compute the distance from the origin to this line segment. The line segment has equation:

$$x + y = 1.22518.$$

The distance from the origin to this line is

$$\frac{1.22518}{\sqrt{2}} = .866333 \neq .965767.$$

Therefore, our assumption that this is a Poncelet ellipse must be incorrect. When we connect every third point, things look quite different, see Figure 3.

Recall [54, Theorem 3.8], which says that if d is a divisor of n and $d \geq 3$, then the number of curves C_k, $1 \leq k \leq [n/2]$, that have the d-Poncelet property is $\Phi(d)/2$, where Φ is Euler's totient function, counting the positive integers up to d that are relatively prime to d. This implies that for $n = 8$, if Γ is a complete Poncelet curve, then C_1, C_3 are 8-Poncelet curves, C_2 is a 4-Poncelet curve and C_4 a 2-Poncelet curve, possibly consisting of a single point (cf. [54, Example 3.12]). Thus, Theorem 5.2.3 does not apply in this setting.

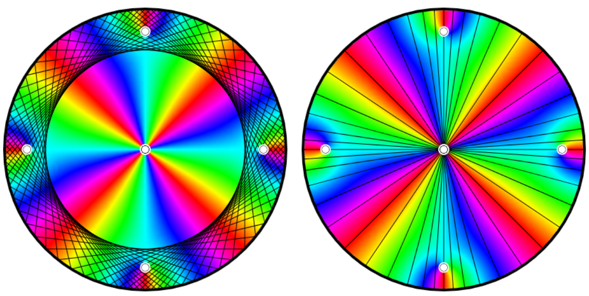

FIGURE 2. Degree-8 Blaschke product Poncelet curves (or point)

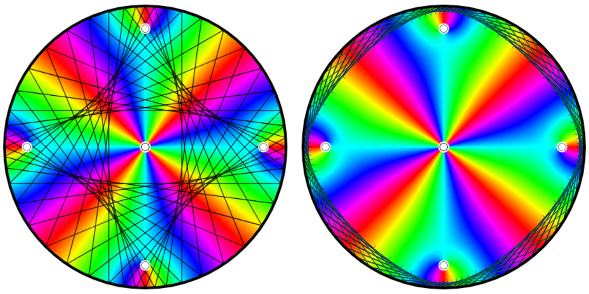

FIGURE 3. Degree-8 Blaschke product non-conics

Now suppose that we connect every other point, which yields a convex quadrilateral. Solving

$$z^4(z^4 - a)/(1 - az^4) = 1,$$

we see that four vertices are $1, i, -1, -i$. The other four are $\pm 1/\sqrt{2} \pm 1/\sqrt{2}i$. Both of these quadrilaterals circumscribe the circle $|z| = 1/\sqrt{2}$. In this case, $D(z) = z^4$ identifies two sets of vertices of circumscribing quadrilaterals. Since two such sets of points determine such a Poncelet curve [45], D identifies all points in a set of vertices and we see that the circle is the Blaschke curve associated with D. As a consistency check, the matrix associated with $S_{D(z)/z}$ is the 3×3 Jordan block and its numerical range is the closed disk of radius $\cos(\pi/4)$.

5.4. Critical Values of Blaschke Products with Elliptical Blaschke Curve

In this section, we study results that follow from understanding the critical values of a Blaschke product. That this is connected to composition of Blaschke products follows from results of Ritt [68] and Cowen [12], and this will be discussed in Section 5.5. A discussion of this also appears in the book, [42]. Note that by Theorem 5.3.3, if B is of degree 2^n and has an elliptical Blaschke curve, then B is a composition of n degree-2 Blaschke products.

The next theorem is useful when we count the number of critical points of a Blaschke product in $\mathbb{D}$.

THEOREM 5.4.1 (Walsh's Blaschke product theorem [73, p. 377]). *Let B be a Blaschke product of degree m with zeros $a_1, \ldots, a_m \in \mathbb{D}$. Then B has exactly $m-1$ critical points in $\mathbb{D}$ and they lie in the convex hull of the set $\{0, a_1, \ldots, a_m\}$. The critical points of B outside $\mathbb{D}$ are the conjugates, relative to $\mathbb{T}$, of those in $\mathbb{D}$.*

5.4.1. Blaschke product with few critical values. The following shows that when we have a Blaschke product of degree 2^n that has a Blaschke curve that is an ellipse, then we have far fewer critical values than $2^n - 1$. Recall that the set of critical values is the set

$$\{w \in \mathbb{D} : w = B(z) \text{ and } B'(z) = 0\},$$

and the set of critical values is

$$\{z \in \mathbb{D} : B'(z) = 0\}.$$

EXAMPLE 5.4.2. Suppose $B = C \circ D$ with C and D degree 2. Then, because B is degree 4, we know that there are 3 critical points in $\mathbb{D}$ (for example, see Theorem 5.4.1) and, therefore, at most 3 critical values. Let z_0 be the point with $D'(z_0) = 0$ and w_0 the point with $C'(w_0) = 0$. Let $D(z_1) = D(z_2) = w_0$. Computing $B'(z) = C'(D(z))D'(z)$, we get critical points of B at z_0, where $D'(z_0) = 0$, and at z_1, z_2 where $C'(D(z_1)) = C'(D(z_2)) = C'(w_0) = 0$. But $B(z_0)$ is one critical value and $B(z_1) = C(D(z_1)) = C(D(z_2)) = B(z_2)$ is the only possibility for the second. So B has at most two critical values. We generalize this argument below in Proposition 5.4.3.

PROPOSITION 5.4.3. *Let $B = B_1 \circ B_2 ... \circ B_n$ be a composition of n Blaschke products with $\deg B_j = k_j$ for $j = 1, \ldots, n$. Then B has at most $\sum_{i=1}^{n}(k_i - 1)$ (distinct) critical values.*

PROOF. We shall prove this by induction. For $n = 1$, the proposition is true. Suppose that the statement is true for $B = B_1 \circ ... \circ B_{n-1}$, a composition of $n - 1$ Blaschke products with $\deg B_j = k_j$ where $1 \leq j \leq n - 1$; that is, B has a maximum of $\sum_{j=1}^{n-1}(k_j - 1)$ critical values. Let $p = \Pi_{j=1}^{n-1} k_j$ denote the degree of B and consider $C = B_n \circ B$, with $\deg B_n = k_n$. Since $C'(z) = B_n'(B(z))B'(z)$, all the critical points of B are critical points of C that generate at most $\sum_{j=1}^{n-1}(k_j - 1)$ critical values of C. We know that B_n has $k_n - 1$ critical points and we denote them by $w_1, ..., w_{k_n - 1}$. Since B is a p to 1 map, there are p values $z_1^{(w_j)}, ..., z_p^{(w_j)}$ such that $B(z_j^{(w_j)}) = w_j$ for every j. Hence, $z_1^{(w_j)}, ..., z_{p-1}^{(w_j)}$ are critical points of C for every w_j, but $C(z_1^{(w_j)}) = ... = C(z_{k_1 - 1}^{(w_j)}) = B_n(w_j)$, and so these critical points generate at most $k_n - 1$ critical values. Then C has a maximum of $(k_n - 1) + \sum_{j=1}^{n-1}(k_j - 1)$ critical values, as desired. ∎

COROLLARY 5.4.4. *Let B be a composition of n-degree 2 Blaschke products. Then B has at most n (distinct) critical values.*

The following results will be useful in this section (see also [72, 80]):

THEOREM 5.4.5. [50] *Let $z_1, \ldots, z_d$ denote (not necessarily distinct) points in $\mathbb{D}$. Then there exists a unique finite Blaschke product of degree $d+1$ with $B(0) = 0$, $B(1) = 1$ and critical points at z_j.*

COROLLARY 5.4.6. *Two proper holomorphic maps $f, g : \mathbb{D} \to \mathbb{D}$ have the same critical points, counted with multiplicity, if and only if $f = \tau \circ g$ for some conformal automorphism τ of $\mathbb{D}$.*

The following theorem will be useful in Section 5.5 when we discuss the monodromy group associated with Blaschke products with one critical value. We have been unable to locate a reference for this exact result in the literature, so we present the proof below. (A related result can be found in [38].)

THEOREM 5.4.7. *Let B be a Blaschke product of degree n with one critical value, w. Then all critical points of B are equal to a point $a \in \mathbb{D}$, and there exists an automorphism τ such that*

$$B(z) = \tau\left(\frac{z-a}{1-\overline{a}z}\right)^n.$$

PROOF. There exist critical points $z_1, \ldots, z_{n-1}$ such that $B(z_j) = w$. There are n points (possibly the same) satisfying $B(z) = w$, so we let these be denoted z_0, and z_j for $j = 1, \ldots, n-1$. Let

$$C(z) = \frac{B(z) - w}{1 - \overline{w}B(z)} = \left(\frac{z - z_0}{1 - \overline{z_0}z}\right)\left(\frac{z - z_1}{1 - \overline{z_1}z}\right)\cdots\left(\frac{z - z_{n-1}}{1 - \overline{z_{n-1}}z}\right).$$

Now,

$$C'(z) = \frac{(1 - |w|^2)B'(z)}{(1 - \overline{w}B(z))^2},$$

so $C'(z_j) = 0$ for $j = 1, \ldots, n-1$. So, C has the same critical points as B and therefore each zero, $z_1, \ldots, z_{n-1}$, is a zero of order greater than 1, and we may write C as

$$C(z) = \left(\frac{z - z_{m_1}}{1 - \overline{z_{m_1}}z}\right)^{j_1}\cdots\left(\frac{z - z_{m_l}}{1 - \overline{z_{m_l}}z}\right)^{j_l},$$

where $l < n-1$ and $z_{m_1}, \ldots, z_{m_l}$ distinct. (If no $z_j = z_0$ for $j \geq 1$, then one $j_k = 1$ and $z_{m_k} = z_0$.)

Now let us write $C(z) = \left(\frac{z-z_{m_1}}{1-\overline{z_{m_1}}z}\right)^{j_1} \times D(z)$, where $D(z_{m_1}) \neq 0$. Then $C'(z)$ has a zero at z_{m_1} of at least order $j_1 - 1$. Suppose it has a zero of order strictly greater than $j_1 - 1$. Then

$$C'(z) = j_1 \left(\frac{z - z_{m_1}}{1 - \overline{z_{m_1}}z}\right)^{j_1 - 1} \times D(z) + \left(\frac{z - z_{m_1}}{1 - \overline{z_{m_1}}z}\right)^{j_1} \times D'(z).$$

Since C' is assumed divisible by $\left(\frac{z-z_{m_1}}{1-\overline{z_{m_1}}z}\right)^{j_1}$, we would have D divisible by $\left(\frac{z-z_{m_1}}{1-\overline{z_{m_1}}z}\right)$. But this is impossible because $D(z_{m_1}) \neq 0$.

Applying this argument to each factor involving z_{m_k} for $k = 1, \ldots, l$, we see that each such z_{m_k} can contribute at most $j_k - 1$ critical points, so the total number of critical points that we get from the z_{m_k} is $\sum_{k=1}^{l}(j_k - 1) = n - l$. But there are $n - 1$ critical points, so we must have $l = 1$; in other words, all z_j must be equal. Thus, C has the same critical points as $\left(\frac{z-z_1}{1-\overline{z_1}z}\right)^n$ and by Corollary 5.4.6, there is an automorphism τ_1 such that

$$C(z) = \tau_1 \circ \left(\frac{z - z_1}{1 - \overline{z_1}z}\right)^n.$$

Letting $\tau_w(z) = \frac{z-w}{1-\overline{w}z}$, we have $C = \tau_w \circ B$. Thus, $B = \tau_w^{-1} \circ C$ and

$$B(z) = \tau_w^{-1} \circ C(z) = \tau \circ \left(\frac{z - z_1}{1 - \overline{z_1}z}\right)^n,$$

with $\tau = \tau_w^{-1} \circ \tau_1$.

□

PROPOSITION 5.4.8. *Let B be a Blaschke product of degree n with one critical value. Then B has an elliptic Blaschke curve.*

PROOF. By Theorem 4.7, B can be written as

$$B(z) = \tau \circ \left(\frac{z - a}{1 - \bar{a}z}\right)^n,$$

where τ is an automorphism. But by [16, Corollary 10] a Blaschke product of the form $\left(\frac{z-a}{1-\bar{a}z}\right)^n$ has an elliptic Blaschke curve, and this does not change by composing with automorphisms. □

COROLLARY 5.4.9. *Suppose B is a Blaschke product of degree $n = p_1 p_2 \ldots p_m$ with one critical value. Then B can be factored in any order as a composition of m Blaschke products of degree $p_1, \ldots, p_m$.*

PROOF. This follows from the form of B in Theorem 5.4.7. □

5.5. The monodromy group and compositions of Blaschke products

We begin by considering the following from Cowen's paper [12], [42, Chapter 9], or [77]. This is closely related to a theorem of Ritt.

5.5.1. The decompositions of Ritt and Cowen. The decompositions of Ritt and Cowen require consideration of the critical values and a normalization of the Blaschke product. In general, if a Blaschke product has degree n, there are at most $n-1$ critical points in $\mathbb{D}$ (as is the case for polynomials) and at most $n-1$ critical values. However, as we have seen in Section 5.4, when the Blaschke product is a composition, there are fewer critical values. Following Cowen, we say that a finite Blaschke product is normalized if $B(0)=0$, $B'(0)>0$, and $B(a)=0$ implies that $B'(a)\neq 0$. Given a Blaschke product B it is always possible to find $\alpha,\beta\in\mathbb{D}$ and $\lambda\in\mathbb{T}$ so that

$$\lambda\varphi_\alpha\circ B\circ\varphi_\beta$$

is in normalized form; see [42, Proposition 9.2.6] for details. Let S denote the set of critical points, so that $B(S)$ denotes the critical values. The oriented closed loops in $\mathbb{D}\setminus B(S)$ based at the point 0 form a group. Given two loops γ, δ, the product $\gamma\cdot\delta$ is obtained by "gluing" them; that is, since they both start and end at zero, we begin by following δ and then continue with γ. We consider homotopy classes of such loops, recalling that loops are homotopy equivalent if one can be deformed to another in $\mathbb{D}\setminus B(S)$. We denote by γ^* the equivalence class of the curve γ. Since we assume that 0 is not a critical point, B^{-1} has n branches at 0 that will be denoted by $g_1,g_2,\ldots,g_n$, where $g_1(0)=0$. We let G_B be the group associated with B that consists of the set of permutations of $\{g_1,\ldots,g_n\}$ induced by the loops in $\mathbb{D}\setminus B(S)$ based at 0. The connection to composition (or, more precisely, decomposition) is described in Theorem 5.5.1. We say that a group G respects a partition $\mathcal{P}$ if for each $g\in G$ and $P\in\mathcal{P}$, there exists $P'\in\mathcal{P}$ so that $gP\subset P'$. If G respects a partition, then each element of the partition will have the same cardinality and this is called the order of $\mathcal{P}$.

For a given Blaschke product B, we consider the set, $B(S)$, of critical values of B and by L_B we mean the set of continuous curves in $\mathbb{D}\setminus B(S)$ for which $\gamma(0)=\gamma(1)=0$. Cowen showed [12] that the monodromy group

$$G_B:=\{\gamma^*:\ \gamma\in L_B\}$$

can be computed from a given Blaschke product B and its local inverses; the precise statement appears in Theorem 5.5.1. He states that if one knows all of the normal subgroups of the group G_B then one can construct all possible non-trivial compositional factorization of B, but that this association of normal subgroups and compositions is more complicated than "one would hope." Thus, we focus on the generators, rather than the group itself.

THEOREM 5.5.1 (Ritt, Cowen). *Let B be a finite normalized Blaschke product. If $\mathcal{P}$ is a partition of the set of branches of B^{-1} at $0, \{g_1, g_2, \dots, g_n\}$ that G_B respects, then there are finite Blaschke products $J_{\mathcal{P}}$ and $b_{\mathcal{P}}$ with the order of $b_{\mathcal{P}}$ the same as the order of $\mathcal{P}$ so that $B = J_{\mathcal{P}} \circ b_{\mathcal{P}}$. Conversely, if J and b are finite Blaschke products so that $B = J \circ b$ then there is a partition $\mathcal{P}_b$ of the set of branches at 0 that G_B respects such that the order of $\mathcal{P}_b$ is the same as the order of b.*

Cowen makes no claim about the equivalence of factorizations, though that is discussed in [65]. Obviously, we may write $B = (C \circ \varphi_a) \circ (\varphi_a^{-1} \circ D)$ where φ_a is an automorphism. But what is perhaps less obvious is that the degrees of the decompositions may vary. For example, $\left(\frac{z-a}{1-\overline{a}z}\right)^6 = z^2 \circ \left(\frac{z-a}{1-\overline{a}z}\right)^3 = z^3 \circ \left(\frac{z-a}{1-\overline{a}z}\right)^2$. However, there is a notion of length for a Blaschke product (which requires factoring into prime factors) and the length in this case is an invariant under such factorizations, [65, p. 24]. Decompositions were also considered in [77], where a method to visualize the monodromy group is presented. We give a sense of the main ideas here.

5.5.2. Visualizing the monodromy group. Following Wegert, we consider the basins of attraction, Z_k, for the Blaschke product and their images, D_k. Recall that the basins are simply connected and their boundaries are formed by the stable manifolds S_j of the critical values and arcs on the unit circle. Then $B(S_j)$ is a radial segment that has one endpoint at the critical value and B maps each basin Z_k in a one-to-one fashion onto slit disks $D_k = \mathbb{D} \setminus R_k$, where R_k denotes the union of all radial slits $B(S_j)$ of stable manifolds S_j that belong to the boundary of Z_k. The Riemann surface is obtained from the D_k as follows: two slit disks are connected if Z_k and Z_j have a common boundary component; we glue D_k and D_j along the image of the common component; that is, along the slit. Because Wegert's description depends on the phase plot, it is not possible to distinguish points that are sent to values with the same argument. To handle this problem, in what follows we will assume that the Blaschke product B under consideration is regularized; that is, B maps zero to zero, has simple zeros, and has the property that whenever z_1 and z_2 are critical points that satisfy $B(z_1)/B(z_2) > 0$, then $B(z_1) = B(z_2)$. A Blaschke product B can be regularized by composing with a disk automorphism, $\varphi \circ B$.

We have the following theorem from [77].

Theorem 5.5.2. *Let B be a regularized Blaschke product and let S denote the union of all stable manifolds of all saddle points of B. Then the basins $Z_1, \dots, Z_n$ of the zeros $z_1, \dots, z_n$ of B are the simply connected components of $\mathbb{D} \setminus S$. The restriction B_k of B to an arbitrary basin Z_k maps Z_k bijectively and conformally onto $\mathbb{D} \setminus R_k$, the unit disk with some radials slits, denoted here by R_k.*

Example 5.5.3. Consider a degree-8 Blaschke product B. In visualizing what is happening using Wegert's method, due to the coloring, we assume the Blaschke product is regularized. To visualize the monodromy group, we consider loops (one from each homotopy equivalence class) that encircle the critical value exactly once. Wegert's idea is to show how the generators of the monodromy group can be read off the phase plot.

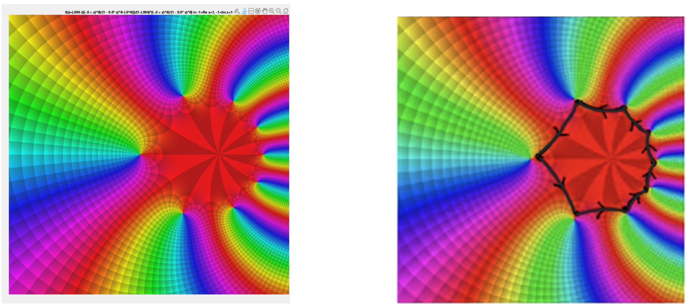

Figure 4. Blaschke product tiling and a possible generator

Consider a Blaschke product with a 0 at 0 and seven other zeros. Figure 4 is obtained via a coloring and tiling of the plane. It shows the pullback of the plane under B or the phase plot of B. There are seven critical points counted according to multiplicity, eight zeros, and one critical value in the red region. On a plot such as the one in this example, one can spot a critical point in the grid as a point where the grid does not form a square; that is, where the function is not conformal. These tiles, with more than four vertices, are referred to as the exceptional tiles. In this case, we see the critical value in red. To compute the monodromy group, one needs to find the eight zeros. They are generally easy to spot because they are the places where all the colors come together. Thus, we see a critical point surrounded by 8 zeros. Now a loop in the plane either circles the critical value or misses it. As in the particular degree-4 case in [12] (see also [42]), not circling the critical value corresponds to the identity map in the monodromy group. If the map circles the critical value, then when we compute the final element of the continuation, each zero will move to the next one and we obtain the generator (12345678) of the cyclic group on 8 elements. Essentially, zeros that are associated to critical points corresponding to the same critical value are moved simultaneously.

5.5.3. Computing monodromy groups. To illustrate this method and for later reference, we provide a detailed proof that generalizes an example of Cowen. We use his construction for Blaschke products here. We note that this also follows immediately from an observation in [77, p. 970] that generators of G_B are in a one-to-one correspondence with the critical values of B.

PROPOSITION 5.5.4. *Let B be a normalized Blaschke product of degree n with one critical value. Then the monodromy group associated with B is a cyclic group of order n.*

PROOF. The proof is illustrated in Figure 5 and Figure 6. For each Blaschke product with a single critical value, we know from Theorem 5.4.7 that there will be one critical point of order $n-1$ in $\mathbb{D}$. Let A and B be two generic points. Draw a path starting at A, through the critical value and ending at B (picture on the right). If the Blaschke product is of degree n, the inverse image will have n curves, each passing through the critical point (picture on the left). Note that because the argument of the Blaschke product is increasing, the inverse images (A) and (B) of the points A and B are interlaced. The inverse image of a loop based at the origin will begin at a zero of the Blaschke product and, if oriented counterclockwise, will always pass through the curve associated with (A), that is, the curve from the critical point to (A) (drawn in purple in Fig. 5.) The inverse image of the loop must then pass between the critical point and B and it must end at a zero. Thus, it has moved from a zero z_1 to a zero z_2. This will be repeated until the curve returns to z_1. Thus, the permutation associated with this is $(123\ldots n)$. If the loop does not contain the critical value, then we obtain the identity. So the monodromy group is the cyclic group on n elements. $\square$

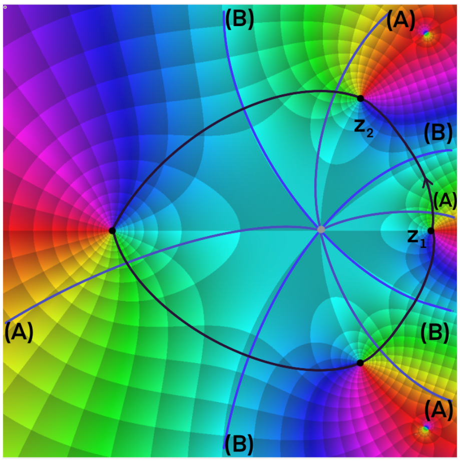

FIGURE 5. Inverse image.

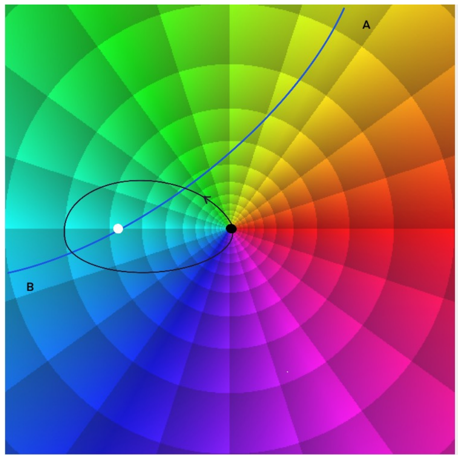

FIGURE 6. Loop.

We begin with more discussion of Blaschke products of degree 4. These will also serve as the first step of the induction in Theorem 5.5.8.

In Figure 7, we illustrate the effect of composing two Blaschke products C and D that vanish at 0. The tiled phase plots of D and C are shown on the left and in the middle, respectively, and on the right, we see their composition, $B = C \circ D$. For further information on phase plots see [72].

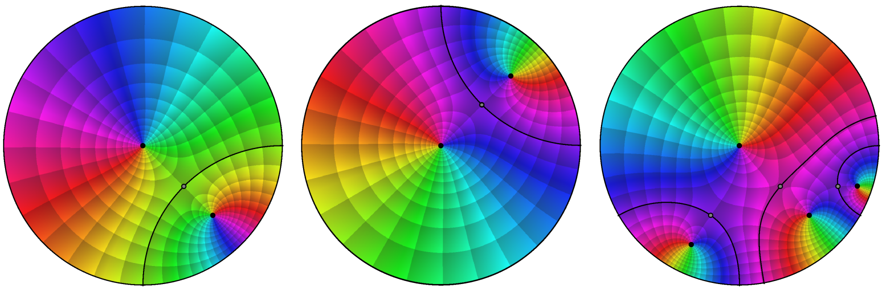

FIGURE 7. Composition of two Blaschke products

The zeros are represented by the black dots and the critical points (which lie on the black lines) are represented by gray dots. In the first picture on the left, the Blaschke product D has two zeros and there are two basins each of which is mapped by D onto the unit disk. When composed with C, we obtain the Blaschke product B, which is illustrated on the right. Note that B has four zeros and because $C(0) = 0$, two of these are the zeros of D. We see that one zero has been added to each of the two basins that appear in the first picture on the left (note that the common boundary of the basins of D is not a boundary of the basins of B; see Figure 8). We are really grateful to Elias Wegert for providing us with Figures 7 and 8.

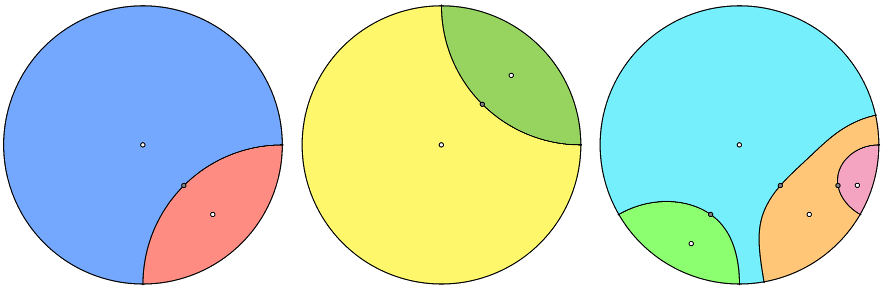

FIGURE 8. Basins of the Blaschke products

Considering this example in more detail, we impose a grid on the phase plot of B obtaining Figure 9.

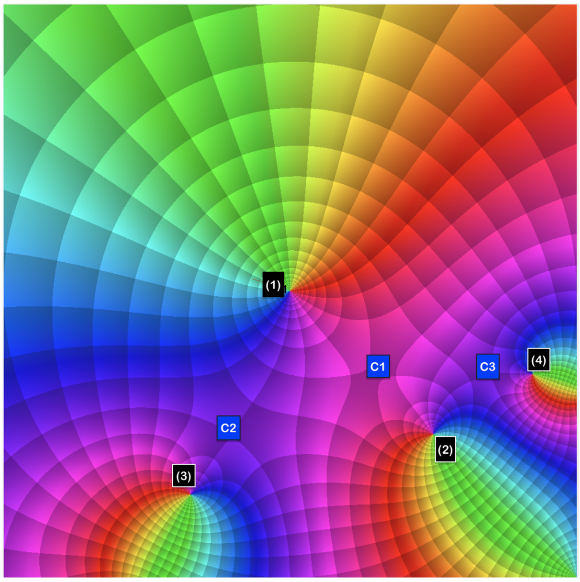

FIGURE 9. Phase plot with grid

Here, we see that when a loop based at 0 circles the critical value associated with the critical points labelled $C2$ and $C3$ in the figure, the inverse image will produce two loops, one moving from one of the zeros of D (labeled (1)) to a zero introduced by composition that is labeled (3) and, simultaneously, the second loop will move from the zero (2) of D to the newly introduced zero (4). This yields the product of the two transpositions, (13)(24).

Note that there is a difference between our work here and that in [77] in that we assume that $B(0) = 0$ and base our loops at 0, while the loops in [77] are arbitrarily small loops about critical points, labeled in blue.

EXAMPLE 5.5.5. Let B be a normalized Blaschke product of degree 4 with $B = C \circ D$ with $B(0) = 0$. As in [15, Proposition 2.1] we may assume that $B(0) = C(0) =$

$D(0) = 0$, C and D degree 2, and B, C, and D regularized. In [42, Section 9.4] two (modified) examples of Cowen's Blaschke products of degree 4 that are compositions are presented. In one case, the group is computed to be the cyclic group of order 4 generated by (1234). This example has one critical value. The second example has two critical values. Here, the monodromy group is shown to be the dihedral group on 4 elements (order 8). We show that this works in general, for regularized Blaschke products of degree 4 that are compositions of two (regularized) degree-2 Blaschke products and have two distinct critical values. We consider two cases.

(1) If B has one critical value, then by Proposition 5.5.4, the monodromy group is cyclic and generated by the permutation (1234).

(2) Suppose that B has two critical values. Note that since we assume $D(0) = 0$ we must have $D(z) = z(a - z)/(1 - \bar{a}z) = z\varphi_a(z)$ for some $a \in \mathbb{D}$. Since we assume that B has simple zeros, we may assume that $a \neq 0$. So, $D(\varphi_a(z)) = \varphi_a(z)\,(\varphi_a(\varphi_a(z))) = z\varphi_a(z) = D(z)$. Therefore,

$$B(\varphi_a(z)) = C \circ D(\varphi_a(z)) = C \circ D(z) = B(z).$$

So,

$$B'(\varphi_a(z))\varphi_a'(z) = B'(z).$$

But φ_a has no critical point in $\mathbb{D}$ so we have $B'(z) = 0$ if and only if $B'(\varphi_a(z)) = 0$.

Since the set of critical points $\{z_1, z_2, z_3\}$ must be invariant under φ_a and φ_a is self-inversive, either $\varphi_a(z_3) = z_j$ for $j \in \{1, 2\}$ or $\varphi_a(z_3) = z_3$. If the critical points are distinct, then we may assume that $z_2 = \varphi_a(z_1)$. Because we assume the points are distinct, we can only have $\varphi_a(z_3) = z_3$. A computation shows that, since $a \neq 0$, we have $z_3 = \frac{1-\sqrt{1-|a|^2}}{\bar{a}} = a_\star$.

If the critical points are not distinct, then two must be equal, say $z_1 = z_2$. If $\varphi_a(z_1) \neq z_1$, then the third must be $\varphi_a(z_1) = z_3$. But then $B(z_1) = B(z_2) = B(\varphi_a(z_1)) = B(z_3)$ and there is only one critical value, which is the case that we have already handled. Therefore $z_1 = z_2 = a_\star$ and either $\varphi_a(z_3) = z_3$ or $\varphi_a(z_3) = z_1$. Either way, all three points must be equal and B would have only one critical value.

We now turn to the monodromy group in this case. We know that B has three critical points $z_1, \varphi_a(z_1)$, and $a_\star$ and four zeros. In this case, a loop circling a critical value corresponding to one critical point (the critical point of D) will yield a transposition. (See the detailed discussion above.) We assume, upon re-numbering, that it is (12). We know that there are just two critical values and therefore two generators. Each zero of D will remain when we compose with C and we add two more zeros, one to each basin, and two more critical points. Thus, a loop circling one critical value corresponding to two critical points will move points simultaneously about the other critical points and therefore will yield a generator that is a product of transpositions, say (13)(24), while a generator circling both will yield (1324), which is the product of these two. We may choose either as our second generator, since $(12)(13)(24) = (1423) = (1324)^{-1}$ and $(12)(1324)^{-1} = (12)(1423) = (13)(24)$, so both groups will be the same. Thus we can replace these three generators with just two, namely (13) and (12)(34), which is the number of critical values.

This second set of generators yields a group that is isomorphic to the wreath product of two cyclic groups of order two; that is, the group, $\mathbb{Z}_2 \wr \mathbb{Z}_2$. We say more about this below.

5.5.4. Wreath products and trees. The theory of wreath products is often explained by thinking of them as groups acting on a finite rooted tree. We are grateful to Peter Brooksbank for providing this background on wreath products.

Let $\mathcal{T} = \mathcal{T}_k$ denote a binary tree of height k, where $k \geq 1$ and let $n = 2^k$. Setting $\Omega = \Omega_k = \{1, \ldots, 2^k\}$, one can label nodes and leaves of the tree as follows: start labelling the root node by Ω and rest of the nodes of $\mathcal{T}$ with subsets of Ω. If a node at level $0 \leq j < k$ is labelled with $\{m+1, m+2, \ldots, m+2^{k-j}\}$ for some integer m, one can label its left child at the level $j+1$ with $\{m+1, \ldots, m+2^{k-(j+1)}\}$ and the right child with $\{m+2^{k-(j+1)}+1, \ldots, m+2^{k-j}\}$. In this manner the leaves of the tree are labelled from left to right with $\{1\}, \ldots, \{n\}$ or, equivalently, $1, \ldots, n$. Let $\Gamma = \Gamma_k$ denote the group of automorphisms of the binary tree $\mathcal{T}$ and let $\mathrm{Sym}(\Omega)$ be the symmetric group on Ω. One can identify Γ with its image in $\mathrm{Sym}(\Omega)$ by noticing that the action of Γ on Ω gives a faithful representation $\Gamma \to \mathrm{Sym}(\Omega)$ (See [30, Pp. 45-50] for the definition of wreath product). Additionally, since Γ permutes the nodes at each level, the block labels are permuted in the action at each level making it possible to view Γ and its iterated wreath product structure.

Consider the following elements of $\mathrm{Sym}(\Omega)$

$$\sigma_k := (12),\ \sigma_{k-1} = (13)(24), \ldots, \sigma_1 := \prod_{l=1}^{2^{k-1}} (l(l+2^{k-1})).$$

Then each σ_j is an automorphism of the labelled tree $\mathcal{T}$; in fact σ_j is an automorphism of the leftmost subtree rooted on level $j-1$.

For example, the tree for $k=3$ is presented in Figure 10. The group is a semi-direct product generated by $\sigma_3 = (12), \sigma_2 = (13)(24), \sigma_1 = (15)(26)(37)(48)$ and has $128 = 2^{1+2+4}$ elements.

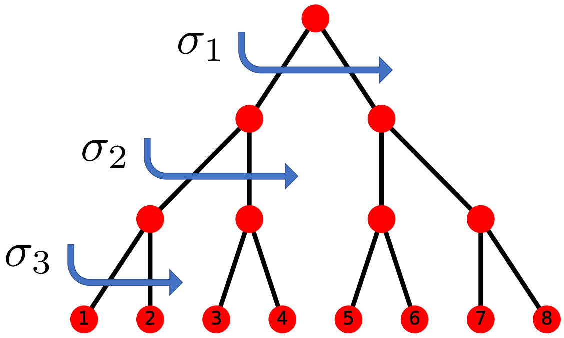

FIGURE 10. Tree for $k = 3$

Next we present a well-known result on three basic properties of the group of automorphisms of $\mathcal{T}$ (see, for example, [70, p. 140] for ((2)) and ((3))). We include a short proof.

PROPOSITION 5.5.6. *The following hold true for the group of automorphisms of* $\mathcal{T}$*:*

(1) $\Gamma = \langle \sigma_1, \ldots \sigma_k \rangle$;
(2) Γ *is an iterated wreath product of* k *cyclic groups of order* 2;
(3) Γ *is a Sylow* 2*-subgroup of* $\mathrm{Sym}(\Omega)$

PROOF. The proof of ((1)) is done by induction on k, which is the height of the tree. Since the result is clear for $k=1$, let us assume $k>1$ and assume the result holds true for trees of height less than k. Notice that each automorphism $\alpha \in \Gamma$ permutes the two nodes on level 1; these two nodes are the children of the root node. Observe that α either fixes both children or interchanges them and we want to prove that $\alpha \in \langle \sigma_1, \ldots \sigma_k \rangle$. If α interchanges the children, replace α with $\alpha\sigma_1$ so that α now fixes both children. We may now write

$$\alpha = \beta\gamma = \gamma\beta,$$

where β is an automorphism of the left subtree that is the identity on $\{\frac{n}{2}+1,\ldots,n\}$ and γ is an automorphism of the right subtree that is the identity on $\{1,\ldots,\frac{n}{2}\}$.

By induction, $\beta \in \langle \sigma_2, \ldots, \sigma_k \rangle$. Additionally, $\sigma_1\gamma\sigma_1$ is an automorphism of the left subtree and it is the identity on the right subtree. Therefore $\sigma_1\gamma\sigma_1 \in \langle \sigma_2, \ldots, \sigma_k \rangle$. Hence,

$$\alpha = \beta\gamma \in \langle \sigma_1, \ldots \sigma_k \rangle.$$

That Γ is an iterated wreath product of k cyclic groups of order 2 follows from the construction. To check that Γ is a Sylow 2-subgroup of $\mathrm{Sym}(\Omega)$, we have only to compute the order of Γ, which is

$$|\Gamma| = 2^{1+2+\cdots+2^{k-1}} = 2^{2^k-1}.$$

□

REMARK 5.5.7. Consider a Blaschke product of the form $B(z) = z^2\left(\frac{a-z}{1-az}\right)^2$ with $a \in (0,1]$. After normalizing to obtain a Blaschke product with two critical values, it follows from Example 5.5.5 that the monodromy group is the dihedral group D_8. Thus, its order is 8 and this group is non-abelian. It has two subgroups, $H = \{1, s\}$ of order 2 and $K = \{1, r, r^2, r^3\}$ of order 4. Since all groups of order at most 5 are abelian, we know both H and K are abelian. Since the direct product of two abelian groups is abelian, D_8 cannot be a direct product of its subgroups (this shows that [77, Theorem 4] is not correct as stated (see [76])). However, D_8 is a semidirect product of the same subgroups H (reflection over the diagonal) and K (rotation by an angle $\pi/2$). Furthermore, since every subgroup of index 2 is a normal subgroup, K is a normal subgroup. Since the wreath product is a special combination of two groups based on the semidirect product, it is also possible to express D_8 as the wreath product of $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$. By the definition of the wreath product this means:

$$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

If we think of D_8 as the group of automorphisms of the square, the term $(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$ corresponds to the swapping the sides of the square (rotations) and the term at the end $\mathbb{Z}/2\mathbb{Z}$ corresponds to swapping through the diagonals (reflections).

Before we prove the next theorem, we note that an example from [47] shows that even if a Blaschke product is decomposable, the order of the composition matters and can change the shape of the boundary of $W(S_B)$. In fact, in [53, Theorem 10],

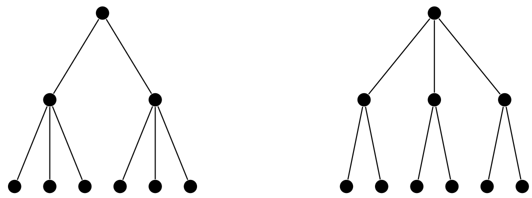

FIGURE 11. The tree for $S_2 \wr S_3$ (left) and $S_3 \wr S_2$ (right).

the authors show that for a degree-6 Blaschke product, the numerical range being elliptical is equivalent to the corresponding Blaschke product $\widehat{B}$ having the property that it factors as $C_1 \circ D_1$ with C_1 degree-2 and D_1 degree 3 and it also factors as $D_2 \circ C_2$ with C_2 degree 2 and D_2 degree 3. For example, if we let $\widehat{B}_1$ be a Blaschke product of degree-6 such that $\widehat{B}_1 = C_1 \circ D_1$ where

$$C_1(z) = z\left(\frac{z-a}{1-\overline{a}z}\right)^2 \quad \text{and} \quad D_1(z) = z^2.$$

then one can show $W(S_{B_1})$ is an elliptic disk, where $\widehat{B}_1(z) = zB_1(z)$. Note that

$$\widehat{B}_1(z) = z^2\left(\frac{z^2-a}{1-\overline{a}z^2}\right)^2 = z^2 \circ z\left(\frac{z^2-a}{1-\overline{a}z^2}\right) = z\left(\frac{z-a}{1-\overline{a}z}\right)^2 \circ z^2.$$

On the other hand, if we consider $\widehat{B}_2 = C_2 \circ D_2$ where

$$C_2(z) = z\left(\frac{z-.5}{1-.5z}\right) \quad \text{and} \quad D_2(z) = z^3,$$

then letting $\widehat{B}_2(z) = zB_2(z)$, it turns out that $W(S_{B_2})$ is not an elliptical disk.

As discussed above, the groups that we consider will have monodromy groups that can be represented by trees. Here we see the same effect on order: Suppose we have a Blaschke product of degree 6 decomposed as C of degree 2 and D of degree 3. The tree representation of the wreath product $S_2 \wr S_3$ of the monodromy group G_B with $B = C \circ D$ is quite different from the tree representation of the wreath product of $S_3 \wr S_2$ of the group G_B when $B = D \circ C$ as shown in the Figure 11.

THEOREM 5.5.8. *Let B be a regularized Blaschke product with distinct critical points and n distinct critical values that is the composition of n degree-2 regularized Blaschke products; that is, $B = B_n \circ \cdots \circ B_1$ with B_j regularized for each j. Further assume that $B_j(0) = 0$ for $j = 1, \ldots, n$. Then the monodromy group associated with B is the wreath product of n cyclic groups of order 2, or $\underbrace{\mathbb{Z}_2 \wr \mathbb{Z}_2 \cdots \wr \mathbb{Z}_2}_{n \text{ times}}$.*

The general form of an element in such a wreath product, depending on the numbering, would be

$$\prod_{1}^{1}(i(i+1)) = (12),\ \prod_{i=1}^{2}(i(i+2)) = (13)(24),\ \ldots,\ \prod_{i=1}^{2^{n-1}}(i(i+2^{n-1})).$$

Proof. Note that the assumption that $B_j(0) = 0$ for each j implies that $B_j \circ \cdots \circ B_1$ is a subfactor of $B_{j+1} \circ \cdots \circ B_1$ for each j. Thus, the zeros of $B_j \circ \cdots \circ B_1$ are a subset of the zeros of $B_{j+1} \circ B_j \circ \cdots \circ B_1$ and all such zeros are simple. We have shown in Example 5.5.5 that the result is true for the composition of two degree-2 Blaschke products. So suppose that it is true whenever we have the composition of $n-1$ degree-2 Blaschke products. Consider $B_n \circ B_{n-1} \circ \cdots \circ B_1$. Now by our assumption, each B_j must give rise to a distinct critical value. Notice that B_1 will correspond to one critical point, B_2 to two additional critical points, and, in general, B_j will add 2^{j-1} critical points to those obtained from $B_{j-1} \circ \cdots \circ B_1$. Considering the loops that do not circle the critical value associated with B_n we obtain, by induction, $n-1$ generators that yield the group $\underbrace{\mathbb{Z}_2 \wr \mathbb{Z}_2 \cdots \wr \mathbb{Z}_2}_{n-1 \text{ times}}$. If a loop circles the new critical value that we obtain from B_n, then with an appropriate numbering, all critical points that are added are associated with the same critical value and so the action on each will be simultaneous. Note that the additional composition with B_n adds a zero to each basin and a critical point, which is the one that will be encircled. Thus each new zero is paired with a zero that was in the previous basin and we obtain a generator $(1(2^{n-1}+1)) \cdots (2^{n-1}2^n)$ as our generator. Since the number of generators is the same as the number of critical values, we now have the complete set of generators. Thinking of our wreath product as a group acting on a finite rooted tree, [9, Theorem 2.1.6] completes the proof of the theorem. $\square$

Remark: As in Example (2), this is not the only generator one can choose. With proper ordering, there is a cycle of length n that serves as generator.

5.6. Groups of invariants

Given a finite Blaschke product B of degree n, denote the set of continuous functions $u : \mathbb{T} \to \mathbb{T}$ by $C(\mathbb{T})$ and consider the group of invariants of u defined by

$$\mathcal{G}_B = \{u \in C(\mathbb{T}) : B \circ u = B|\mathbb{T}\}.$$

This set $\mathcal{G}_B$ is a group under the operation of composition; that is, the composition of two functions in $\mathcal{G}_B$ is again in the set, the identity is in the set and the operation is associative. We also note that in order to be in the group of invariants a continuous function u must be a bijective mapping of the circle (see [10, Lemma 4.1], where these results are extended to infinite Blaschke products) and, for later reference, we note that the argument of B (appropriately chosen) is increasing on the unit circle and therefore the zeros of $B(z) - \lambda$ are simple for every point $\lambda \in \mathbb{T}$. (This is well known; for a reference see for example, [10, Remark 2.1]). Thus, each element in $\mathcal{G}_B$ has an inverse. Cassier and Chalendar showed [8] that $\mathcal{G}_B$ is a cyclic group of order n. In this section, we consider composition and its relation to the group of invariants. One such theorem is given below.

THEOREM 5.6.1. [15, Theorem 5.13] *A Blaschke product B of degree $n = mk$ with $m > 1$ is a composition of two non-trivial Blaschke products if and only if there exists a Blaschke product D of degree $k > 1$ such that $\mathcal{G}_D$ is generated by g^m for some generator g of $\mathcal{G}_B$.*

The authors of [15] note that if the Blaschke product D exists, then there is a finite Blaschke product C such that $B = C \circ D$. If D has degree 2, then $\mathcal{G}_D$ has order 2. A generator of the group can be found using an observation of Frantz [39]: Let $a \in \mathbb{D}$. If $z \in \mathbb{T}$ and consider the line through z and a. Since this is not tangent to $\mathbb{T}$, Frantz shows that $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ is the other point of intersection of the line joining z and a with $\mathbb{T}$. We use this in Theorem 5.6.2.

For compositions of degree-2 Blaschke products, it's possible to say more. For example, if C_j are degree-2 Blaschke products for $j = 1, 2, 3$ and $B = C_3 \circ C_2 \circ C_1$, there is a generator w such that the corresponding group is $\langle w \rangle$ where $w^2 = v$ is the generator of $\mathcal{G}_{C_2 \circ C_1}$, and $w^4 = v^2 = u$ is the generator of $\mathcal{G}_{C_1}$. We extend this observation below.

Note that we may write a factorization of B as $B = C_n \circ C_{n-1} \circ \cdots \circ C_1 = C_n \circ C_{n-1} \circ \cdots \circ \varphi_{C_1} \circ \varphi_{C_1} \circ C_1$, where φ_{C_1} is an automorphism, and $\varphi_{C_1} \circ C_1$ has the same group of invariants as C_1. Thus, we may assume that $C_1(0) = 0$.

THEOREM 5.6.2. *Let B be a Blaschke product of order 2^n and let g denote a generator of $\mathcal{G}_B$. If $B = C_n \circ C_{n-1} \circ \cdots \circ C_1$ is a composition of n Blaschke products of order 2, then the group of invariants of $C_j \circ \cdots \circ C_1$ is a normal subgroup of index 2 of* *the group of invariants of $C_{j+1} \circ C_j \circ \cdots \circ C_1$ for each j. If $C_1(z) = z\left(\frac{a-z}{1-\bar{a}z}\right)$, then $g^{2^{n-1}} = \varphi_a$.*

Proof. The first statement follows by using the fact that the group of invariants of $C_j \circ \cdots \circ C_1$ is a cyclic group of order 2^j: Every subgroup of a cyclic group is normal and using Lagrange's theorem we conclude that the index is 2. Therefore, the group of invariants of $C_j \circ \cdots \circ C_1$ is a normal subgroup of index two of the group of invariants of $C_{j+1} \circ C_j \circ \cdots \circ C_1$. We also know that since $\mathcal{G}_B$ is a cyclic group, if g is a generator of $\mathcal{G}_B$ and H is a non-trivial subgroup, then g^m is a generator of H, where m is the smallest positive integer with g^m in H. In our case, we take the subgroup H_j corresponding to the group of invariants of $C_j \circ \cdots \circ C_1$, which has order 2^j. Thus, if g^m generates $\mathcal{G}_{C_j \circ \cdots \circ C_1}$, then m is the smallest integer such that $(g^m)^{2^j} = e = g^{2^n}$. So, $m = 2^{n-j}$.

To prove the remaining assertion, first suppose that B has degree 2; that is, that $n = 1$. As above, we may assume that $B(0) = 0$ and we may write $B(z) = \lambda z\left(\frac{a-z}{1-\bar{a}z}\right)$. We will suppose that $\lambda = 1$.

We have for every $a \in \mathbb{D}$, $\varphi_a \neq id$ and $B \circ \varphi_a(z) = \varphi_a(z)(\varphi_a \circ \varphi_a)(z) = B(z)$. Thus φ_a is a generator of $\mathcal{G}_B$. Note that φ_a has no fixed point on the unit circle and maps a point of $w_1 \in \mathbb{T}$ to a second (distinct) point w_2 on $\mathbb{T}$. Since $B \circ \varphi_a = B$, we see that $B(w_1) = B(w_2)$.

Now suppose that $u \neq e$ is another generator of $\mathcal{G}_B$, and consider two points z_1, z_2 that B identifies. Either $u(z_1) = z_2 = \varphi_a(z_1)$ and $u(z_2) = z_1 = \varphi_a(z_2)$, or there exist z_1, z_2 with $u(z_1) = z_1$ and $u(z_2) = z_2$. If the latter case occurs for some z_1 and z_2, then this divides the unit circle into two arcs, ℓ_1 and ℓ_2 with endpoints z_1 and z_2, see Figure 12. Now either $u(\ell_1) \subseteq \ell_1$ or $u(\ell_1) \subseteq \ell_2$. In the first case, all points in ℓ_1 would be mapped to themselves under u and the same is true for $u(\ell_2)$. So $u = e$. Thus, u must interchange the two arcs. Choose a sequence (w_n) in ℓ_2 tending to z_1. Then, by the discussion above, (see also [39]) $u(w_n)$ is equal to the endpoint of the line segment on $\mathbb{T}$ that passes through the point a. Therefore $u(w_n) \to z_2$. So $u(z_1) = z_2 = \varphi_a(z_1)$. Since this is true for all points on $\mathbb{T}$, we see that φ_a is the only generator of $\mathcal{G}_B = \mathcal{G}_{C_1}$.

Now suppose that $n > 1$ and $B = C_n \circ \cdots \circ C_1$. Then by our work thus far, $\mathcal{G}_B$ has a generator g of order n and $g^{2^{n-1}}$ is a generator of G_{C_1}. But every generator of $\mathcal{G}_{C_1}$ must identify the same points as φ_a. Thus, $g^{2^{n-1}} = \varphi_a$.

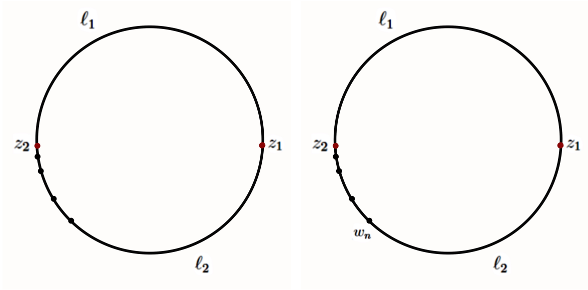

FIGURE 12. Arcs ℓ_1, ℓ_2 and points z_1, z_2

$\square$

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