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2-convexity and 2-concavity in Schatten ideals

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Introduction

The properties p-convexity and q-concavity are fundamental in the study of Banach sequence spaces (see [**L-TzII**]), and in recent years have been shown to be of great significance in the theory of the corresponding Schatten ideals ([**G-TJ**], [**LP-P**] and many other papers). In particular, the notions 2-convex and 2-concave are meaningful in Schatten ideals. It seems to have been noted only recently [**LP-P**] that a Schatten ideal has either of these properties if the underlying sequence space has. One way of establishing this is to use the fact that if $(E, \| \|_E)$ is 2-convex, then there is another Banach sequence space $(F, \| \|_F)$ such that $\|x\|_E^2 = \|x^2\|_F$ for all $x \in E$. The 2-concave case can then be deduced using duality, though this raises some difficulties, for example when E is inseparable.

In this note, we present an alternative approach which proceeds directly from the Markus–Mityagin lemma in the spirit of [**GK**] and [**Si**], by way of a quadratic variant of the well-known Ky Fan Lemma. As well as being (arguably) a natural route to the result just stated, this approach also delivers a theorem characterizing the norm of an operator A as the supremum (in the 2-convex case) or the infimum (in the 2-concave case) of the norms (in E) of the sequences $(\|Ae_j\|)$ for orthonormal bases (e_j).

Notation and definitions

We denote by $x(j)$ the jth term of a numerical sequence x, and by e_j the jth unit vector. For sequences x, y, the product xy and the modulus $|x|$ are defined pointwise in the obvious way. We write P_n for the operator (on any sequence space) that replaces all terms after the first n by 0, so that $P_n(x) = \sum_{j=1}^n x(j)\, e_j$. By a *symmetric Banach sequence space* we mean a Banach lattice $(E, \| \|_E)$ of real null sequences with a symmetric norm satisfying further:

(i) $e_j \in E$ and $\|e_j\|_E = 1$ for all j,
(ii) $\|x\|_E = \lim_{n\to\infty} \|P_n(x)\|_E$ for all $x \in E$.

(We do not exclude the case where E is finite-dimensional.)

Let $(H, \| \|)$ be a separable Hilbert space (of finite or infinite dimension). For a compact operator A on H, let $s_j(A)\,(j = 1, 2, \dots)$ be the singular numbers of A. We denote by $S_E(H)$ the Schatten ideal corresponding to the Banach sequence space E, with norm σ_E defined by $\sigma_E(A) = \|(s_j(A))\|_E$.

Let $A_1, \dots, A_n$ be self-adjoint elements of $S_E(H)$, and let $A_0 = (\sum_{j=1}^n A_j^2)^{1/2}$. Then $A_0 \in S_E(H)$ (this is most easily seen by considering the operator on H^n with first column

$A_1, \dots, A_n$, cf. [**LP-P**]). Hence one can define (as for sequences spaces) $S_E(H)$ to be 2-*convex* if for all such $A_1, \dots, A_n$, we have for some M

$$\sigma_E(A_0) \leqslant M\left(\sum_{j=1}^{n} \sigma_E(A_j)^2\right)^{1/2}$$

and 2-*concave* if we have

$$\left(\sum_{j=1}^{n} \sigma_E(A_j)^2\right)^{1/2} \leqslant M\sigma_E(A_0).$$

The least such constant M is, respectively, the 2-convexity or 2-concavity constant of $S_E(H)$. We say that E, or $S_E(H)$, is *strictly* 2-convex or 2-concave if the constant is 1. Note that in the above definition it is clearly sufficient to consider positive operators A_j.

The results

We will use the two following well-known theorems.

PROPOSITION 1. *Let A be compact, and let (e_i), (f_i) be any two orthonormal sets. Then for each n,*

(i) $\sum_{j=1}^{n} |\langle Ae_j, f_j\rangle| \leqslant \sum_{j=1}^{n} s_j(A)$,

(ii) $\sum_{j=1}^{n} \|Ae_j\|^2 \leqslant \sum_{j=1}^{n} s_j(A)^2$.

Statement (i) is essentially [**GK**, II·4·1]; it also follows in elegant style from [**Si**, propositions 1·11 and 1·12], although it is not stated explicitly there. Statement (ii) follows by applying (i) to A^*A.

For the next result, we denote by D_n the dyadic group $\{-1, 1\}^n$. Elements of D_n belong to $\mathbb{R}^n$, so act on $\mathbb{R}^n$ by multiplication. Also, if $\pi \in S_n$, the group of permutations of $\{1, 2, \dots, n\}$ and $x \in \mathbb{R}^n$, then x_π is the element of $\mathbb{R}^n$ defined by $x_\pi(j) = x[\pi(j)]$.

PROPOSITION 2. *Let x, y be decreasing, non-negative members of $\mathbb{R}^n$. Define $X(k) = \sum_{j=1}^{k} x(j)$, and $Y(k)$ similarly. Suppose that $X(k) \leqslant Y(k)$ for each k. Then*

$$y \in \operatorname{conv}\{\epsilon x_\pi : \epsilon \in D_n, \pi \in S_n\}.$$

If, further, $X(n) = Y(n)$, then

$$y \in \operatorname{conv}\{x_\pi : \pi \in S_n\}.$$

Proof. The first statement is the standard Markus–Mityagin lemma (see, for example, [**GK**, III·3]). The second statement is surely well known: it is stated without proof in [**Sch**, lemma 4·2], where it is observed that something like it already appears in [**HLP**]. For completeness, we mention how the proof of [**GK**] can be adapted for this case. Suppose the statement is false. Then there is a linear functional ϕ such that $\phi(y) > \phi(x_\pi)$ for all $\pi \in S_n$. Let $\phi(u) = \sum_{j=1}^{n} a_j u(j)$. Since $X(n) = Y(n)$, we can add a constant c to each a_j, and hence we may assume that $a_j \geqslant 0$ for each j. The proof now proceeds as before, but without the need for terms $\epsilon_j \in \{-1, 1\}$ to convert negative a_j's to $|a_j|$.

We deduce a quadratic variant of the Ky Fan lemma.

PROPOSITION 3. *Let E be a Banach sequence space. Let x, y be decreasing, non-negative null sequences such that $\sum_{j=1}^{k} y(j)^2 \leqslant \sum_{j=1}^{k} x(j)^2$ for all k. Then:*

(i) *if E is strictly 2-convex and $x \in E$, then $y \in E$ and $\|y\|_E \leqslant \|x\|_E$;*
(ii) *if E is strictly 2-concave, $y \in E$ and also $\sum_{j=1}^{\infty} y(j)^2 = \sum_{j=1}^{\infty} x(j)^2$, then $x \in E$ and $\|y\|_E \geqslant \|x\|_E$.*

Proof. It is clearly enough to prove both statements for finitely non-zero sequences x, y: the statement is then obtained by considering limits (with a small adjustment to the nth term to ensure the required equality in case (ii)).

(i) By the first statement in Proposition 2, there exist rearrangements z_r of x (for $1 \leqslant r \leqslant R$, say), $\lambda_r > 0$ and $\epsilon_r \in D_n$ such that $\sum_{r=1}^{R} \lambda_r = 1$ and $y^2 = \sum_{r=1}^{R} \lambda_r \epsilon_r z_r^2$, hence $y^2 \leqslant \sum_{r=1}^{R} \lambda_r z_r^2$. By 2-convexity and the fact that $\|z_r\|_E = \|x\|_E$ for each r, we have

$$\begin{aligned} \|y\|_E^2 &\leqslant \sum_{r=1}^{R} \lambda_r \|z_r\|_E^2 \\ &= \|x\|_E^2. \end{aligned}$$

(ii) By the second statement in Proposition 2, there exist z_r and $\lambda_r > 0$ such that $\sum_{r=1}^{R} \lambda_r = 1$ and $y^2 = \sum_{r=1}^{R} \lambda_r z_r^2$. 2-concavity gives the stated inequality.

LEMMA 1. *If the Banach sequence space E is strictly 2-concave, then E is contained in l_2 and $\|x\|_2 \leqslant \|x\|_E$ for all $x \in E$.*

Proof. Take $x \in E$. Since $(E, \| \, \|_E)$ is a Banach lattice, $\|P_n x\|_E \leqslant \|x\|_E$ for each n. Write $x_j = x(j)\, e_j$. Then $(P_n x)^2 = \sum_{j=1}^{n} x(j)^2 e_j = \sum_{j=1}^{n} x_j^2$, so by 2-concavity,

$$\sum_{j=1}^{n} x(j)^2 = \sum_{j=1}^{n} \|x_j\|_E^2 \leqslant \|P_n x\|_E^2 \leqslant \|x\|_E^2.$$

The statement follows.

In the same way, if E is 2-convex, then E contains l_2 and $\|x\|_E \leqslant \|x\|_2$.

It is now easy to characterize the Schatten ideal norm of an operator in the way stated in the introduction. The following result is well known for the classical ideals $S_p(H)$ given by $E = l_p$ (see, for example, [**GK**], p. 95).

THEOREM 1. *Let A be an element of $S_E(H)$.*
(i) *If E is strictly 2-convex, then for any orthonormal set (e_j),*

$$\|(\|Ae_j\|)\|_E \leqslant \sigma_E(T).$$

(ii) *If E is strictly 2-concave, then for any orthonormal basis (e_j),*

$$\sigma_E(A) \leqslant \|(\|Ae_j\|)\|_E$$

whenever the right-hand side is finite.

In both cases, equality occurs for the (e_j) appearing in the spectral representation of A.

Proof. We may assume that (e_j) is ordered so that $(\|Ae_j\|)$ is decreasing. Statement (i) follows at once from Proposition 1 (ii) and Proposition 3 (i). If E is 2-concave, then by Lemma 1, $E \subseteq l_2$, so $A \in S_2(H)$ (the Hilbert–Schmidt operators) and for any orthonormal basis (e_j),

$$\sum_{j=1}^{\infty} \|Ae_j\|^2 = \sum_{j=1}^{\infty} s_j(A)^2.$$

Proposition 3(ii) now gives statement (ii).

We remark that elementary examples (e.g. with $E = l_1$) show that the right-hand side in statement (ii) is not always finite.

A further application of 2-convexity or 2-concavity now yields the result stated at the beginning.

THEOREM 2. *If E is strictly 2-convex or 2-concave, then so is $S_E(H)$.*

Proof. Let $A_1, \dots, A_n$ be positive elements of $S_E(H)$, and let $A = (\sum_{i=1}^n A_i^2)^{1/2}$. Let the spectral representation of A be $\sum_{j=1}^\infty \mu_j e_j \otimes e_j$, so that $\mu_j = s_j(A)$ and $Ae_j = \mu_j e_j$, hence

$$\mu_j^2 = \langle A^2 e_j, e_j \rangle = \sum_{i=1}^n \langle A_i^2 e_j, e_j \rangle = \sum_{i=1}^n \|A_i e_j\|^2.$$

Define scalar sequences a, a_i by:

$$a(j) = \mu_j,$$
$$a_i(j) = \|A_i e_j\|.$$

Then $a^2 = \sum_{i=1}^n a_i^2$ and $\|a\|_E = \sigma_E(A)$.

If E is 2-convex, then $\|a\|_E^2 \leqslant \sum_{i=1}^n \|a_i\|_E^2$ and Theorem 1(i) gives $\|a_i\|_E \leqslant \sigma_E(A_i)$, hence

$$\sigma_E(A)^2 \leqslant \sum_{i=1}^n \sigma_E(A_i)^2$$

as required. If E is 2-concave, the same applies with both inequalities reversed.

2-convexity and 2-concavity constants. There are plenty of examples of Banach sequence spaces that are 2-convex or 2-concave, but not with constant 1, for example: (i) finite-dimensional spaces in general, (ii) certain Lorentz sequence spaces (see [**R**], [J]). If E has 2-convexity or 2-concavity constant $M (\neq 1)$, then clearly Proposition 3 and Theorem 1 hold with the constant M inserted. Owing to the second use of 2-convexity or 2-concavity in Theorem 2, the above method requires the insertion of M^2 in this Theorem. Actually, $S_E(H)$ has the same 2-convexity or 2-concavity constant as E. To show this, we amend the method as follows. With $a(j) = s_j(A)$ as above, Proposition 1 gives

$$\sum_{j=1}^k a(j)^2 = \sum_{j=1}^k \sum_{i=1}^n \|A_i e_j\|^2 \leqslant \sum_{i=1}^n \sum_{j=1}^k s_j(A_i)^2.$$

Write $b_i(j) = s_j(A_i)$. The stated result is now given by the following variant of Proposition 3:

PROPOSITION 4. *Let E be a Banach sequence space. Let $a, b_1, \dots, b_n$ be decreasing non-negative sequences belonging to E, and let $b^2 = \sum_{i=1}^n b_i^2$. Suppose that $\sum_{j=1}^k a(j)^2 \leqslant \sum_{j=1}^k b(j)^2$ for all k. Then:*

(i) *if E has 2-convexity constant M, then $\|a\|_E^2 \leqslant M^2 \sum_{i=1}^n \|b_i\|_E^2$;*

(ii) *if* E *has 2-concavity constant* M *and also* $\sum_{j=1}^{\infty} a(j)^2 = \sum_{j=1}^{\infty} b(j)^2$, *then* $\sum_{i=1}^{n} \|b_i\|_E^2 \leqslant M^2 \|a\|_E^2$.

Proof. Again it is enough to consider the finite-dimensional case. As in Proposition 3, there exist $\lambda_r > 0$ (for $r = 1, \ldots, R$) and $\pi_r \in S_n$ such that $\sum_{r=1}^{R} \lambda_r = 1$ and

$$a^2 \leqslant \sum_{r=1}^{R} \lambda_r b_{\pi_r}^2 = \sum_{r=1}^{R} \sum_{i=1}^{n} \lambda_r b_{i,\pi_r}^2,$$

with the $\leqslant$ replaced by equality in case (ii). Both statements now follow.

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