Recursive combinatorial constructions and rigidity of frameworks

John Hewetson

This thesis is submitted for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
University of Lancaster
March 2023

Recursive combinatorial constructions and rigidity of frameworks

John Hewetson

Abstract

We present a study of combinatorial constructions that are related to understanding the structure of bar-joint frameworks.

The primary objects of study in Chapters 2 and 3 of this thesis are connected (k, l)-sparsity matroids. Taking inspiration from [22], where connected $(2, 3)$-sparsity matroids are considered, we provide a method of constructing graphs with a connected $(2, 2)$-sparsity matroid. Throughout these chapters we work in as a purely combinatorial a setting as is practical, minimising invocations to theoretic machinery involving frameworks.

In Chapter 4 we show that the aforementioned method of construction is pertinent to characterising globally rigid frameworks in two-dimensional spaces endowed with non-Euclidean norms. This "natural avenue of research" [8, p.181] builds on the characterisation of rigid graphs in such spaces by Dewar. However, when compared to Dewar's characterisation, we make use of an additional constraint in order to link these combinatorial methods to the structure of frameworks in these spaces. Specifically, we demand that the norms we consider are analytic.

We then turn our attention, in Chapter 5, away from (k, l)-sparsity matroids and towards labelled graphs. More precisely, we provide a method of constructing a family of labelled graphs designed to satisfy sparsity conditions relevant to the rigidity of frameworks realised on not necessarily concentric spheres. A precise connection between this family of graphs this notion of rigidity is not provided. Such a description would extend work characterising rigid frameworks on concentric spheres [32], [33].

To conclude there is a short chapter suggesting ways this research may be built upon.

Table of Contents

1 Graphs, Frameworks, and Rigidity **1**

1.1 Graph Theory 1

1.1.1 Structure of Graphs 1

1.1.2 Graph Operations 5

1.2 Matroid Theory 8

1.3 Normed Spaces 12

1.3.1 Topological Vector Spaces 12

1.3.2 Linear Functionals, Smoothness, and Strict Convexity 14

1.4 Rigidity Theory 17

1.4.1 Motivation 17

1.4.2 Frameworks and Rigidity 18

1.4.3 Graphs and Rigidity 20

2 Connected Matroids **26**

2.1 (k, l)-Sparsity of Graphs and Matroids 26

2.2 $(k, 2k-1)$-Connected Graphs 43

2.3 $(2, 2)$-Connected Graphs 49

3 A Construction of $(2, 2)$-Connected Graphs **66**

3.1 Ear Decompositions 66

3.2 Recursive Constructions 78

4 Frameworks in Normed Spaces **97**

4.1 Rigidity of Graphs in Normed Spaces 97

4.1.1 uv-Coincident Rigidity and uv-Sparse Graphs 97

4.1.2 uv-Coincident Graph Operations 100

4.2 Characterising Coincident-Point Independence 105

4.3 Necessary Conditions for Global Rigidity 110

4.4 Global Rigidity in Analytic non-Euclidean Normed Planes 113

5 Vertex-Labelled Graphs **117**

5.1 Motivation 118

5.2 Vertex-Labelling and Vertex-Labelled Graph Operations 119

5.3 Vertex-Labelled Count Sparsity 121

5.3.1 σ_d-Sparse Graphs 121

5.3.2 Vertex-Labelled Graph Operations and σ_d-Sparsity 129

5.4 σ_1-Tight Vertex-Labelled Graphs 133

5.5 σ_2-Tight Vertex-Labelled Graphs 136

5.5.1 σ_d-Critical Sets 136

5.5.2 Vertices of Degree Three 144

5.5.3 σ_2-Cut-Tight Vertex-Labelled Graphs 149

6 Future Research **172**

6.1 Connected Sparsity Matroids 172

6.2 Rigidity in Normed Spaces 174

6.3 Labelled Graphs and Frameworks on Surfaces 175

Bibliography **176**

List of Tables

5.1 Tabulation of the possible labellings of $N_G[v]$ for a vertex, v, of degree three in G where G_χ is a labelled graph. 146

List of Figures

1.1 Illustration of the generalised vertex split and generalised edge-reduction operations.
For $i \in \{1,2\}$, H_i is the generalised vertex split of G at v on (N_1, N_2) adding $\{v_1, v_2, v_1x_i\}$. For $i \in \{1,2\}$, G is the generalised edge-reduction of H_i contracting v_1v_2 to v and deleting v_1x_i. 7

1.2 Illustration of the K_4^--extension and K_4^--reduction operations. G is the K_4^--reduction of H deleting e and adding $\{w_1, w_2\}$ and H is the K_4^--reduction of G deleting $\{w_1, w_2\}$. 7

2.1 Illustrations of some (k,l)-circuits. G_1 is a $(1,0)$-circuit, G_2 is a $(2,2)$-circuit, and G_3 is a $(3,5)$-circuit. 28

2.2 G_1 is a $(2,1)$-circuit, G_2 is a $(1,-1)$-circuit, G_3 is a $(3,5)$-circuit, and G_4 is a $(1,0)$-circuit.
G_1 demonstrates that the lower bounds of Lemma 2.1.0.9 and Lemma 2.1.0.11 are the best possible, and G_3 demonstrates that the lower bounds of Lemma 2.1.0.13 and Lemma 2.1.0.15 are the best possible. G_2 highlights the importance of the condition that $l \geq 0$ in Lemma 2.1.0.11, G_1 highlights the importance of the condition that $l \geq k$ in Lemma 2.1.0.13, and G_4 highlights the importance of the condition that $l \geq k$ in Lemma 2.1.0.15. 33

2.3 Illustration of the 2-cleave and 2-sum operations. (G_1, G_2) is the 2-cleave of G on $(G[V(G_1)], G[V(G_2)])$ and G is the 2-sum of (G_1, G_2). 45

2.4 Illustration of three $(2,2)$-circuits with fewer than eight vertices. We shall regularly refer to a graph as being isomorphic to K_5^-, B_1, or B_2. 50
2.5 Illustration of the 1-separation and 1-join operations. (G_1, G_2) is the 1-separation of G on $(G[V(G_1)], G[V(G) \cap V(G_2)])$ adding $\{v_3, v_4\}$ whereas (G_2', G_1') is the 1-separation of G on $(G[V(G) \cap V(G_2)], G[V(G_1)])$ adding $\{v_3, v_4\}$. G is the 1-join of (G_1, G_2) deleting $\{v_3, v_4\}$, and G is also the 1-join of (G_2', G_1') deleting $\{v_3, v_4\}$. 57
2.6 Illustration of the 2-separation and 2-join operations. (G_1, G_2) is the 2-separation of G on $(G[V(G) \cap V(G_1)], G[V(G) \cap V(G_2)])$ adding $\{v_3, v_4\}$ and G is the 2-join of (G_1, G_2). 57
2.7 Illustration of the 3-separation and 3-join operations. (G_1, G_2) is the 3-separation of G deleting $F = \{a_1a_2, b_1b_2, c_1c_2\}$ on $(G[V(G_1) \backslash \{v\}], G[V(G_2) \backslash \{v\}])$ adding $\{v\}$ and G is the 3-join of (G_1, G_2) adding F. 58

3.1 Possible structures of H_t discussed in the proof of Lemma 3.1.0.7. 74
3.2 Possible structures of H_t when $|\mathcal{Y}| = 1$ discussed in the proof of Lemma 3.1.0.7. 77
3.3 Structure of H_t considered towards the end of the proof of Theorem 3.2.0.1. 79
3.4 Possible structures of G with respect to a 3-edge-separation (H_1', H_2') of G such that $|V(H_1')|, |V(H_2')| \geq 2$ and H_2' is a subgraph of $H_t[Y]$, discussed in the proof of Theorem 3.2.0.4. 86
3.5 The relevant graphs when $G_2 \cong B_1$ in Case 1 in the proof of Theorem 3.2.0.4. 86
3.6 The relevant graphs when $G_2' \cong K_5^-$ in Subcase 3b in the proof of Theorem 3.2.0.4. 87
3.7 The relevant graphs when $G_2' \cong B_1$ in Subcase 3b in the proof of Theorem 3.2.0.4. 94
3.8 The relevant graphs when the set of nodes of G_2' such that there exist $(2,1)$-reductions of G_2' at these nodes that are $(2,2)$-circuits is $\{x, y\}$ in Subcase 3b in the proof of Theorem 3.2.0.4. 94

3.9 The relevant graphs when $G'_2 \cong B_2$ and there exists a generalised edge-reduction of G'_3, contracting e and deleting xy, that is a $(2,2)$-circuit in Subcase 3b in the proof of Theorem 3.2.0.4. 95

4.1 Illustration of a $(2,2)$-sparse graph, $G \cong K_{2,3}$, that is not uv-sparse. If $\mathcal{X} = \{\{u,v,w_1\},\{u,v,w_2\},\{u,v,w_3\}\}$ then $\mathcal{X}$ is a uv-compatible family of G and $i_G(\mathcal{X}) = 6 > 5 = 9 - 2(2) = \text{val}(\mathcal{X})$. 100
4.2 Illustration of a vertex-to-H operation, where $H \cong K_4$. G' is a vertex-to-H operation of G. 100
4.3 Illustration of a vertex-to-4-cycle operation. G' is a vertex-to-4-cycle operation of G. 101
4.4 B_1 is rigid in any normed plane, and it is globally rigid in all analytic non-Euclidean normed planes [14] (see Lemma 4.4.0.1). However, B_1 is not globally rigid in the Euclidean plane since for almost all realisations, the left two vertices may be reflected across the dashed line to obtain an equivalent but non-congruent framework. $\tilde{H}$ is minimally rigid in any non-Euclidean normed plane, however it is flexible in the Euclidean plane. $\tilde{H}$ is not globally rigid in any normed space as it is not 2-vertex-connected (see Theorem 1.4.3.10 and Theorem 4.3.0.1). It is also uv-tight and and appears in the statement of Proposition 4.2.0.1 107

5.1 Illustrations of some (the, up to vertex-labelled isomorphism) labelled graphs that satisfy $|E| = S(G_\chi, \sigma_2)$ but are not σ_2-tight. 124
5.2 Two labelled graphs with the same underlying graph, G. (G,χ_1) is σ_2-tight while (G,χ_2) is not. 137
5.3 Illustration of the possible labellings of $N_G[v]$ for a vertex, v, of degree three in G where G_χ is a labelled graph. 146
5.4 Two σ_2-tight labelled graphs with the same underlying graph, G. (G,χ_1) is σ_2-cut-tight while (G,χ_2) is not. 150
5.5 Illustration of a σ_2-cut-sparse graph, G_χ, such that the $(2,1)$-VL-reduction of G_χ at v adding xy is σ_2-sparse but is not σ_2-cut-sparse. 156

5.6 Illustration of a σ_2-cut-sparse graph, G_χ, such that the $(2, 1)$-VL-reduction of G_χ at v adding xy is σ_2-sparse but is not σ_2-cut-sparse whereas the $(2, 1)$-VL-reduction of G_χ at v adding xz is σ_2-cut-sparse 156

Acknowledgements

First and foremost I would like to thank my supervisor, Tony Nixon, for his support and guidance. I am incredibly grateful for all the advice and knowledge that he imparted, and for his understanding throughout the entirety of the PhD process. The sheer generosity he showed with his time, and his willingness to provide an answer to any question, regardless of how trivial it seemed to me (or indeed was) is something that I will always appreciate. Finally I can only marvel at, and be thankful for, his seemingly endless well of patience.

I would also like to thank my other collaborator, Sean Dewar. His engagement with our joint work, insightful remarks, and guidance regarding the thesis writing process are all greatly appreciated.

Thanks must go to both of my examiners, Bernd Schulze and Jim Cruickshank. The questions they posed and suggestions they made have unquestionably resulted in an improved thesis. That being said, all remaining errors and questionable decisions are of my own making.

My thanks also go to the Lancaster Geometric Rigidity group. It was invaluable to be able to interact with such a welcoming local research group, and I like to think that my presentation skills have come a long way thanks to the opportunities to present my work to them. My thanks, and the thanks of anyone who I present to in future, should go to them for suffering through some of my early presentations.

More broadly, I would like to thank the Mathematics and Statistic Department at Lancaster. Lancaster has been a wonderful home for me throughout my undergraduate degree and now during my PhD studies. The maths PhD community at Lancaster has been a joy to be a part of, and I will miss our heated debates on the most trivial of topics. In particular I would like to thank the other members of the 2018 maths PhD cohort: Maria, Zac, and Frankie. They have been a great bunch of people to go through this process with and I wish each of them all the best for the future.

One slightly hard to define group, who I wouldn't have predicted having to thank at the beginning of my PhD, is all those who were involved in dealing with the COVID-19 pandemic. It is hard to understand the impact that this pandemic had on me and this thesis, certainly a far less meaningful and distressing impact than for many people, but it still played a significant part in the last few years of my life. For that reason my sincere thanks go to all those who helped to get through the worst of the pandemic and return (my) life to some semblance of normality.

Huge thanks should also go to various friends who have always been supportive and there to offer help when needed. I am indebted to Joe, Sam, Grace, Liam, Steve, and many other fantastic people for making day to day life that much more fun over the years.

My deepest thanks go to Beth. Her support, care, excellent cooking, good humour, and so much more have made the last few years a far more rewarding experience than they might otherwise have been. Most of all, her patience (and impatience when required) has been a big help in getting this thesis over the line. Writing this makes me realise just how many people have been patient with me and makes me think that maybe I should do something about that. I will always be amazed that somehow I have been fortunate enough to go through this experience with such a wonderful person by my side.

Finally, I need to express my heartfelt thanks to my family and in particular to my parents and brother. They have always been supportive and inquisitive in the best possible way. It is very difficult to fully appreciate how fortunate one can be to have particular people as their family, as usually it is the only thing one knows. What I am certain of is that I am incredibly grateful to have such excellent and inspiring people as role models and I can but hope to be as good a person as the examples they have set.

Declaration

I hereby declare that this thesis is my own work and none of its contents have previously been submitted for the award of a degree by any university.

The majority of this thesis is the product of research carried out in collaboration with my supervisor, Tony Nixon. Chapter 3 is based on joint work with Tony Nixon and Sean Dewar [13]. Chapter 4 is also based on joint work with Tony Nixon and Sean Dewar [12, 13], and closely follows these pre-prints which were written collaboratively.

John Hewetson
March 2023

Chapter 1

Graphs, Frameworks, and Rigidity

We begin with a chapter outlining some required background material. More specifically, we briefly cover concepts from graph theory, matroid theory, the theory of normed spaces, and rigidity theory.

1.1 Graph Theory

1.1.1 Structure of Graphs

Definition 1.1.1.1. A **graph** is an ordered pair (V, E) where V is a non-empty finite set and E is a set of unordered pairs of distinct elements of V.

This section proceeds by providing terminology and notation that allows us to more easily discuss graphs in a meaningful way. Much of this is standard, but it is preferable to be precise to avoid ambiguity. Whilst true of mathematics in general, this need is perhaps exacerbated by the various competing notions of what it means to be a graph.

Given a graph G, $G = (V(G), E(G))$. We say $v \in V(G)$ is a **vertex** of G and $\{u, v\} \in E(G)$ is an **edge** of G. We denote the edge $\{u, v\}$ by uv or vu. For $e = uv \in E(G)$ we say vertices u and v are **adjacent** in G, while u and e (and v and e) are **incident** in G. In most instances within this thesis, the graph that these properties are occurring 'in' should be clear and hence we may refrain from stating 'in G' without excessive risk of causing confusion.

Take $k \in \mathbb{Z}$ such that $k \geq 2$ and for all $1 \leq i \leq k$ let $G_i = (V_i, E_i)$ be a graph. We say that the **union** of these graphs is the graph $\bigcup_{i=1}^{k} G_i \coloneqq (\bigcup_{i=1}^{k} V_i, \bigcup_{i=1}^{k} E_i)$.

Definition 1.1.1.2. Graphs G and H are **isomorphic**, which we denote by $G \cong H$, if there exists a function $f\colon V(G) \to V(H)$ such that f is bijective and $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$; we call f a **graph isomorphism** (from G to H).

A graph is **complete** if every pair of vertices is an edge. A trivial observation is that if $G \cong H$ then $|V(G)| = |V(H)|$. If G and H are complete then the converse is also true. So, if G and H are complete then $G \cong H$ if and only if $|V(G)| = |V(H)|$. For this reason, we write $G \cong K_n$ to demonstrate that G is a (up to isomorphism, the) complete graph with $|V(G)| = n$. Similarly, if $|E(G)| = |E(K_{|V(G)|})| - 1$ and $|E(H)| = |E(K_{|V(H)|})| - 1$ then $G \cong H$ if and only if $|V(G)| = |V(H)|$. Therefore, we write $G \cong K_n^-$ to indicate that G is a (up to isomorphism, the) graph with $|V(G)| = n$ and $|E(G)| = |E(K_n)| - 1$.

Definition 1.1.1.3. Let G be a graph. A **subgraph** of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For all $\emptyset \neq U \subseteq V(G)$, the graph $G[U] \coloneqq (U, \{xy \in E(G)\colon \{x, y\} \subseteq U\})$ is the subgraph of G **induced** by U. Similarly, for all $\emptyset \neq F \subseteq E(G)$, the graph $G[F] \coloneqq (\{v \in V(G)\colon \exists e \in F$ such that v and e are incident$\}, F)$ is the subgraph of G **induced** by F.

Given a graph G and $X, Y \subseteq V(G)$ we set $d_G(X, Y) \coloneqq |\{xy \in E(G)\colon x \in X \setminus Y$ and $y \in Y \setminus X\}|$ and we set $i_G(X) \coloneqq |\{x_1x_2 \in E(G)\colon \{x_1, x_2\} \subseteq X\}|$.

Remark 1. In their 2005 paper, Jackson and Jordán use $i_G(X)$ to denote $|E(G[X])|$ [22, p.3]. As we do not allow $\emptyset$ to induce a subgraph of G, our rephrasing of this definition allows us to consider $i_G(\emptyset)$ rather than having to specify that $X \neq \emptyset$. Indeed, if $X \neq \emptyset$ then $|E(G[X])| = i_G(X)$ and if $X = \emptyset$ then $i_G(X) = 0$. The decision to give a definition of $d_G(X, Y)$ which is (superficially) different to that of Jackson and Jordán [22, p.5] is made for a similar reason. Note that if $X \neq \emptyset \neq Y$ then $E(G[X \cup Y]) \setminus (E(G[X]) \cup E(G[Y])) = \{xy \in E(G)\colon x \in X \setminus Y$ and $y \in Y \setminus X\}$.

Lemma 1.1.1.4. *[22, Lemma 2.1] Let G be a graph and take $X, Y \subseteq V(G)$.*

$$i_G(X) + i_G(Y) + d_G(X, Y) = i_G(X \cup Y) + i_G(X \cap Y).$$

Definition 1.1.1.5. Let G be a graph and take $v \in V(G)$. The **degree** of v in G is $d_G(v) \coloneqq |\{e \in E(G) \colon v$ and e are incident$\}|$.

Given a graph G and $v \in V(G)$, the **neighbourhood** of v in G is $N_G(v) \coloneqq \{u \in V(G) \colon uv \in E(G)\}$. We note that $|N_G(v)| = d_G(v)$. The **closed neighbourhood** of v in G is $N_G[v] \coloneqq N_G(v) \cup \{v\}$. Two values associated with G are the minimum degree in G and the maximum degree in G, which we denote by $\delta(G) := \min\{d_G(v) \colon v \in V(G)\}$ and $\Delta(G) := \max\{d_G(v) \colon v \in V(G)\}$ respectively. The concept of degree allows us to state the following result, originally due to Euler [15], (see [16] for an English translation) which is often referred to as The First Theorem of Graph Theory. One can find a proof in various places; we refer the reader to Gould's introductory text.

Theorem 1.1.1.6. *[18, Theorem 1.1.1] Let G be a graph, then $\sum_{v \in V(G)} d_G(v) = 2|E(G)|$.*

The remainder of this subsection is primarily concerned with 'connectivity' properties of graphs, namely connectedness, vertex-connectivity, and edge-connectivity. Before considering these connectivity concepts we require terminology to reduce reliance on repetition.

Let G be a graph. A **walk** in G is a non-empty finite sequence $(a_1, \ldots, a_n)$ such that for all $1 \leq i \leq n$, $a_i \in V(G)$ and for all $1 \leq i \leq n-1$, $a_i a_{i+1} \in E(G)$. A **path** in G is a walk $(a_1, \ldots, a_n)$ such that for all $1 \leq i < j \leq n$, $a_i \neq a_j$. A **cycle** in G is a walk $(a_1, \ldots, a_n)$ such that $n \geq 4$, $a_1 = a_n$, and for all $1 \leq i < j \leq n-1$.

Given a graph G we may consider the relation on $V(G)$ whereby vertices u and v are related if and only if there exists a path $(a_1, \ldots, a_n)$ in G such that $a_1 = u$ and $a_n = v$. We note that this is an equivalence relation on $V(G)$ and hence this relation induces a partition of $V(G)$. A **component** of G is a graph $G[U]$ such that U is an element of this partition.

Definition 1.1.1.7. A graph is **connected** if it has exactly one component, otherwise it is **disconnected**.

Before defining vertex-connectivity and edge-connectivity, notions which offer the ability to consider connectedness of graphs in finer detail, we describe some species of graph. A graph, G, is a **cycle graph** if there exists a cycle $(a_1, \dots, a_n)$ in G such that $G = (\{a_1, \dots, a_{n-1}\}, \{a_i a_{i+1} \colon 1 \leq i \leq n-1\})$. Similarly to complete graphs (see above), if G and H are cycle graphs then $G \cong H$ if and only if $|V(G)| = |V(H)|$. Therefore we write $G \cong C_n$ to indicate that G is a (up to isomorphism, the) cycle graph with $|V(G)| = n$.

Definition 1.1.1.8. A graph is a **forest** if there do not exist any cycles in it, and it is a **tree** if it is a connected forest.

Definition 1.1.1.9. Let G be a graph. The **vertex-connectivity** of G is $\kappa(G) \coloneqq \min\{|U| \colon U \subsetneq V(G),$ and $|V(G) \setminus U| = 1$ or $G[V(G) \setminus U]$ is disconnected$\}$. The **edge-connectivity** of G is $\kappa_1(G) \coloneqq \min\{|F| \colon F \subseteq E(G),$ and $|V(G)| = 1$ or $(V(G), E(G) \setminus F)$ is disconnected$\}$. For all $k \in \mathbb{N}$, we say G is ***k*-vertex-connected** if $\kappa(G) \geq k$, and we say that G is ***k*-edge-connected** if $\kappa_1(G) \geq k$.

In Section 2.2 and Section 2.3 we will study graphs with particular structural properties. These properties relate to vertex-connectivity and edge-connectivity, in as much as their presence provides an upper bound on the vertex-connectivity or edge-connectivity of a graph. We this future relevance in mind, for now we have the following definitions. Note that given some $U \subseteq \mathbb{R}$ we set $U^+ \coloneqq \{u \in U \colon u > 0\}$. A typical example of this is that $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.

Definition 1.1.1.10. Let $G = (V, E)$ be a graph and take $k \in \mathbb{N}^+$. A ***k*-vertex-separation** of G is an ordered pair, (G_1, G_2), of subgraphs of G where for $i \in \{1, 2\}$, $G_i = G[V_i] = (V_i, E_i)$ for some $V_i \subseteq V$ such that $V_1 \setminus V_2 \neq \emptyset \neq V_2 \setminus V_1$, $V = V_1 \cup V_2$, $E = E_1 \cup E_2$, and $|V_1 \cap V_2| = k$.

Definition 1.1.1.11. Let $G = (V, E)$ be a graph and take $k \in \mathbb{N}^+$. A ***k*-edge-separation** of G is an ordered pair, (G_1, G_2), of subgraphs of G where for $i \in \{1, 2\}$,

$G_i = G[V_i] = (V_i, E_i)$ for some $V_i \subseteq V$ such that $V_1 \cap V_2 = \emptyset$, $V = V_1 \cup V_2$, and $|E \setminus (E_1 \cup E_2)| = k$.

We conclude this subsection by commenting on the extent to which fixing any one of the vertex-connectivity, edge-connectivity, or minimum degree, of a graph constrains the possible values of the others.

Theorem 1.1.1.12. *[20, Theorem 5.1], [18, Theorem 2.2.2] Let G be a graph, then $\kappa(G) \leq \kappa_1(G) \leq \delta(G)$.*

Theorem 1.1.1.13. *[3, Theorem 3] For all $b, c, d \in \mathbb{N}^+$, if $b \leq c \leq d$ then there exists a graph G such that $\kappa(G) = b$, $\kappa_1(G) = c$, and $\delta(G) = d$.*

Remark 2. Before the statement of [20, Theorem 5.1], Harary credits the result to Whitney via a reference to [40]. The terminology used in Whitney's paper is quite different from current terminology, hence our reference to the proofs of Harary and of Gould. We note that in his paper, Whitney considers vertex-connectivity and edge-connectivity in a more general context than that of graphs (to be precise, he works with objects referred to by Harary as "pseudographs" [20, p.10]). However, the caveats provided by Whitney ("In this section, we allow the graphs to contain 2-circuits, but no 1-circuits." [40, p.158] and "A necessary and sufficient condition that a graph containing no 2-circuit" [40, p.160, Theorem 7]) mean that [40, Theorem 7] is a statement about graphs (in our sense) from which it follows that if a graph is k-vertex-connected then it is also k-edge-connected.

1.1.2 Graph Operations

The idea of a 'graph operation' will appear repeatedly throughout this thesis, and we provide a formal account of such operations here. Rather than speaking of graph operations in a somewhat nebulous way we choose to say that for a graph G, a [name of operation] of G is a graph G' such that G' is related to G in some way. One benefit of this approach is that the connection between one operation and an 'inverse' of that operation can be stated unambiguously. We proceed to define a number of operation and 'inverse' operation pairs. Two of the more involved such pairs are illustrated in Figure 1.1 and Figure 1.2 respectively.

Definition 1.1.2.1. Let $G = (V, E)$ be a graph such that $|V| \geq 2$ and G is not complete. An **edge-addition** of G is a graph $(V, E \cup \{xy\})$ where $xy \notin E$. We say $(V, E \cup \{xy\})$ is the edge-addition of G adding xy.

Let H be a graph such that $E(H) \neq \emptyset$. An **edge-deletion** of H is a graph H' such that H is an edge-addition of H'. If $E(H) \setminus E(H') = \{e\}$, then we say H' is the edge-deletion of H at e.

Definition 1.1.2.2. Take $k \in \mathbb{N}^+$. Let $G = (V, E)$ be a graph such that $|V| \geq k$. A **$(k, 0)$-extension** of G is a graph $(V \cup \{v\}, E \cup \{vu_1, \ldots, vu_k\})$ where $v \notin V$. We say $(V \cup \{v\}, E \cup \{vu_1, \ldots, vu_k\})$ is the $(k, 0)$-extension of G adding v to $\{u_1, \ldots, u_k\}$.

Let H be a graph such that $\{u \in V(H) \colon d_H(u) = k\} \neq \emptyset$. A **$(k, 0)$-reduction** of H is a graph H' such that H is a $(k, 0)$-extension of H'. If $V(H) \setminus V(H') = \{v\}$ then we say H' is the $(k, 0)$-reduction of H at v.

Definition 1.1.2.3. Take $k \in \mathbb{N}^+$. Let $G = (V, E)$ be a graph such that $|V| \geq k+1$ and $|E| \geq 1$. A **$(k, 1)$-extension** of G is a graph $(V \cup \{v\}, (E \setminus \{u_iu_j\}) \cup \{vu_1 \ldots vu_{k+1}\})$ where $v \notin V$, $1 \leq i < j \leq k+1$, and $u_iu_j \in E$. We say $(V \cup \{v\}, (E \setminus \{u_iu_j\}) \cup \{vu_1 \ldots vu_{k+1}\})$ is a $(k, 1)$-extension of G adding v to $\{u_1, \ldots, u_{k+1}\}$, or the $(k, 1)$-extension of G adding v to $\{u_1, \ldots, u_{k+1}\}$ and deleting u_iu_j.

Let H be a graph such that $\{u \in V(H) \colon d_H(u) = k+1 \text{ and } H[N_H(u)] \ncong K_{k+1}\} \neq \emptyset$. A **$(k, 1)$-reduction** of H is a graph H' such that H is a $(k, 1)$-extension of H'. If $V(H) \setminus V(H') = \{v\}$ and $E(H') \setminus E(H) = \{e\}$ then we say H' is a $(k, 1)$-reduction of H at v, or the $(k, 1)$-reduction of H at v adding e.

Definition 1.1.2.4. Let $G = (V, E)$ be a graph such that $|E| \geq 1$. A **generalised vertex split** of G is a graph $((V \setminus \{v\}) \cup \{v_1, v_2\}, (E \setminus \{vu \colon u \in N_G(v)\}) \cup \{v_1u \colon u \in N_1\} \cup \{v_2u \colon u \in N_2\} \cup \{v_1v_2, v_1x\})$ where $d_G(v) \geq 1$, $N_1 \cup N_2 = N_G(v)$, $N_1 \cap N_2 = \emptyset$, and $x \in V \setminus (N_1 \cup \{v\})$. We say $((V \setminus \{v\}) \cup \{v_1, v_2\}, (E \setminus \{vu \colon u \in N_G(v)\}) \cup \{v_1u \colon u \in N_1\} \cup \{v_2u \colon u \in N_2\} \cup \{v_1v_2, v_1x\})$ is the generalised vertex split of G at v on (N_1, N_2) adding $\{v_1, v_2, v_1x\}$.

Let H be a graph such that there exists $v_1v_2 \in E(H)$ with $d_H(v_1) \geq 2$ or $d_H(v_2) \geq 2$.

A **generalised edge-reduction** of H is a graph H' such that H is a generalised vertex split of H'. If $V(H')\setminus V(H) = \{v\}$, and $V(H)\setminus V(H') = \{v_1, v_2\}$, and $|N_H(x)\cap\{v\}| = |N_{H'}\cap\{v_1, v_2\}|-1$, then we say H' is a generalised edge-reduction of H contracting v_1v_2 and deleting v_ix, for some $i \in \{1, 2\}$, or the generalised edge-reduction of H contracting v_1v_2 to v and deleting v_ix, for some $i \in \{1, 2\}$.

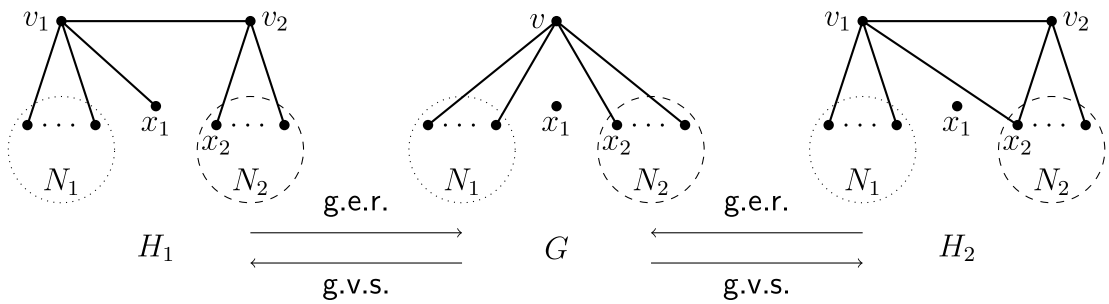


Figure 1.1: Illustration of the generalised vertex split and generalised edge-reduction operations.
For $i \in \{1, 2\}$, H_i is the generalised vertex split of G at v on (N_1, N_2) adding $\{v_1, v_2, v_1x_i\}$. For $i \in \{1, 2\}$, G is the generalised edge-reduction of H_i contracting v_1v_2 to v and deleting v_1x_i.

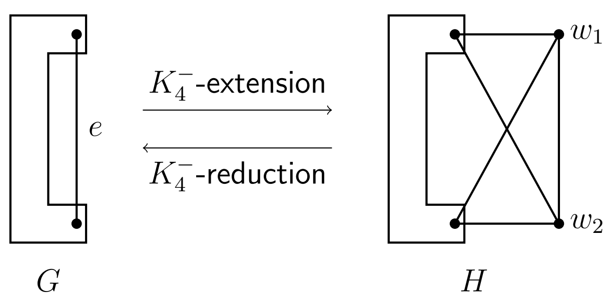


Figure 1.2: Illustration of the K_4^--extension and K_4^--reduction operations. G is the K_4^--reduction of H deleting e and adding $\{w_1, w_2\}$ and H is the K_4^--reduction of G deleting $\{w_1, w_2\}$.

Definition 1.1.2.5. Let $G = (V, E)$ be a graph such that $|E| \geq 1$. A ***K_4^-*-extension** of G is a graph $(V \cup \{w_1, w_2\}, (E\setminus\{xy\})\cup\{xw_1, xw_2, yw_1, yw_2, w_1w_2\})$ where $w_1, w_2 \notin V$ and $xy \in E$. We say $(V \cup \{w_1, w_2\}, (E \setminus \{xy\}) \cup \{xw_1, xw_2, yw_1, yw_2, w_1w_2\})$ is the K_4^--extension of G deleting xy and adding $\{w_1, w_2\}$.

Let H be a graph such that there exists $U \subseteq V(H)$ where $H[U] \cong K_4^-$ and $\{u \in U\colon d_{H[U]}(u) = 3\} \subseteq \{u \in U\colon d_H(u) = 3\}$. A $\boldsymbol{K_4^-}$**-reduction** of H is a graph H' such that H is a K_4^--extension of H'. If $V(H) \setminus V(H') = \{a, b\}$ then we say H' is the K_4^--reduction of H deleting $\{a, b\}$.

1.2 Matroid Theory

In a 1935 paper Whitney showed that various possible definitions of an object, called a matroid, were in fact equivalent [41]. The remainder of Whitney's paper considered how matroids relate to both graphs and matrices. In particular, Whitney introduced a notion of connectivity (which he referred to as non-separability) in the context of matroids.

Whitney's definition of a matroid and of what it means for a matroid to be connected are both based on the 'rank function' of a matroid. We choose to define a matroid via properties satisfied by the 'independent sets' of a matroid, and to define a connected matroid in terms of properties satisfied by the 'circuits' of a matroid.

Note that while one can define a matroid using some concept, e.g. 'independent sets', 'circuits', 'rank function', by demanding that this concept satisfies certain conditions, these conditions are by no means unique. Indeed, it is possible to rephrase the requirements placed on, say, 'circuits' in a way that is ostensibly weaker. An example of this can be seen with the weak circuit elimination axiom and its strong counterpart.

We denote the **power set** of a set E by $\mathcal{P}(E) := \{F \subseteq E\}$.

Definition 1.2.0.1. A **matroid** is an ordered pair $(E, \mathcal{I})$ where E is a finite set, $\mathcal{I} \subseteq \mathcal{P}(E)$, and the following conditions are satisfied:

(I1) $\emptyset \in \mathcal{I}$;

(I2) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$;

(I3) If $X, Y \in \mathcal{I}$ and $|Y| = |X| + 1$ then there exists $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$.

If $X \in \mathcal{I}$ we say that X is an **independent set** of $(E, \mathcal{I})$, otherwise we say that X is a

dependent set of $(E, \mathcal{I})$.

Remark 3. Definition 1.2.0.1 (taken from [39, p.7]) invites a demonstration of the earlier point about how conditions can be replaced. Oxley [34] replaced condition (I3) with a more powerful axiom that only requires $|Y| \geq |X| + 1$ rather than $|Y| = |X| + 1$. Moreover, a further strengthening of condition (I3), where the union of X and some subset (not necessarily a subset with single element) of Y is considered, was given by Welsh [39, Theorem 1.5.1 (The Augmentation Theorem)].

In what follows we do not show the equivalence of various potential definitions but rather consider the behaviour of objects such as independent sets or 'circuits' to be properties of a matroid as defined above.

Given a matroid $\mathcal{M} = (E, \mathcal{I})$ we say that E is the **ground set** of $\mathcal{M}$. Equivalently, we say that $\mathcal{M}$ is a matroid on E. If $C \notin \mathcal{I}$ and for all $b \in C$, $C \setminus \{b\} \in \mathcal{I}$ then we say that C is a **circuit** of $\mathcal{M}$. That is, a circuit of $\mathcal{M}$ is a minimally dependent set of $\mathcal{M}$. The **rank function** of $\mathcal{M}$ is the function $\rho\colon \mathcal{P}(E) \to \mathbb{Z}^+$ such that for $F \in \mathcal{P}(E)$, $\rho(F) = \max\{|X| \colon X \subseteq F \text{ and } X \in \mathcal{I}\}$. The **closure function** of $\mathcal{M}$ is the function $\sigma\colon \mathcal{P}(E) \to \mathcal{P}(E)$ such that for $F \in \mathcal{P}(E)$, $\sigma(F) = \{x \in E \colon \rho(F \cup \{x\}) = \rho(F)\}$.

Definition 1.2.0.2. The matroids $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ are isomorphic, which we denote by $\mathcal{M}_1 \cong \mathcal{M}_2$, if there exists a function $f\colon E_1 \to E_2$ such that f is bijective and $X \in \mathcal{I}_1$ if and only if $\{f(x) \colon x \in X\} \in \mathcal{I}_2$; we call f a **matroid isomorphism** (from $\mathcal{M}_1$ to $\mathcal{M}_2$).

Lemma 1.2.0.3. *[34, Lemma 1.1.3] Let $\mathcal{M}$ be a matroid. If $\mathcal{C}$ is the set of circuits of $\mathcal{M}$ then*

(C1) $\emptyset \notin \mathcal{C}$;

(C2) if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$; and

(C3) if $C_1, C_2 \in \mathcal{C}$ and $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

Lemma 1.2.0.4. *[39, Theorem 1.9.2] Let $\mathcal{M}$ be a matroid. If $\mathcal{C}$ is the set of circuits of*

$\mathcal{M}$ then

(C3)' if $C_1, C_2 \in \mathcal{C}$ and $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ and $f \in C_1 \setminus C_2$ then there exists $C_3 \in \mathcal{C}$ such that $f \in C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

Remark 4. We refer to (C3) as the **weak circuit elimination axiom** and refer to (C3)' as the **strong circuit elimination axiom**. Our reason for referring the reader to Welsh for a proof that condition (C3)' is satisfied is that the proof given by Welsh only involves applying the properties in Lemma 1.2.0.3. This is in contrast to the proof given by Oxley [34, Proposition 1.4.11] which proceeds by considering the closure function of $\mathcal{M}$.

Lemma 1.2.0.5. *[34, Lemma 1.3.1] Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. If ρ is the rank function of $\mathcal{M}$ then*

(R1) if $X \subseteq E$ then $0 \leq \rho(X) \leq |X|$;

(R2) if $X \subseteq Y \subseteq E$ then $\rho(X) \leq \rho(Y)$; and

(R3) if $X, Y \subseteq E$ then $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$.

To conclude this section we echo Subsection 1.1.1 and introduce a notion of connectedness for matroids. The importance of matroid connectedness to this thesis can be inferred from the fact that connected matroids are the eponymous subject of Chapter 2.

Definition 1.2.0.6. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid, and let $\mathcal{C}$ be the set of circuits of $\mathcal{M}$. $\mathcal{M}$ is **connected** if for all $\{e, f\} \subseteq E$ (that is, $e, f \in E$ and $e \neq f$) there exists $C \in \mathcal{C}$ such that $\{e, f\} \subseteq C$, otherwise $\mathcal{M}$ is **disconnected**.

Lemma 1.2.0.7. *[39, Theorem 5.1.2] Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid, let $\mathcal{C}$ be the set of circuits of $\mathcal{M}$, and suppose $\{x, y, z\} \subseteq E$. If there exist $C_1, C_2 \in \mathcal{C}$ such that $\{x, y\} \subseteq C_1$ and $\{y, z\} \subseteq C_2$ then there exists $C_3 \in \mathcal{C}$ such that $\{x, z\} \subseteq C_3$.*

The previous result can be interpreted as stating that for a matroid $\mathcal{M}$ on E, with a set of circuits $\mathcal{C}$, the relation $\sim$ on E, where $e \sim f$ if and only if $e = f$ or there exists $C \in \mathcal{C}$ such that $\{e, f\} \in C$, is transitive and consequently $\sim$ is an equivalence relation (see [34, Proposition 4.1.2]). Rather than dwell on this we state a simple consequence of Lemma 1.2.0.7 that we will make repeated use of in due course.

Lemma 1.2.0.8. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and let $\mathcal{C}$ be the set of circuits of $\mathcal{M}$. If there exists $e \in E$ such that for all $f \in E \setminus \{e\}$ there exists $C \in \mathcal{C}$ such that $\{e, f\} \subseteq C$ then $\mathcal{M}$ is connected.*

Proof. If $|E| = 1$ then $\mathcal{M}$ (vacuously) satisfies the condition of being connected. So we may suppose instead that $|E| \geq 2$ and take $\{f_1, f_2\} \subseteq E$. If $e \in \{f_1, f_2\}$ then by assumption there exists $C \in \mathcal{C}$ such that $\{f_1, f_2\} \subseteq C$. On the other hand, if $e \notin \{f_1, f_2\}$ then by assumption there exist $C_1, C_2 \in \mathcal{C}$ such that $\{e, f_1\} \subseteq C_1$ and $\{e, f_2\} \subseteq C_2$. Then Lemma 1.2.0.7 implies there exists $C_3 \in \mathcal{C}$ such that $\{f_1, f_2\} \subseteq C_3$. Therefore for all $\{f_1, f_2\} \subseteq E$ there exists a circuit containing f_1 and f_2, so $\mathcal{M}$ is connected. □

Remark 5. The proof of the previous result highlights the fact that matroids can be connected by vacuously satisfying the relevant condition. In particular, any matroid on a ground set containing at most one element will always be connected. There is a unique matroid on the empty set and there are, up to isomorphism, two matroids on a set containing one element. Therefore this vacuous satisfaction is only really relevant for three matroids. We mention this now as it impacts on a later definition (see Definition 2.1.0.6) relating graphs and matroids on the edge sets of these graphs.

Definition 1.2.0.9. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and let $\mathcal{C}$ be the set of circuits of $\mathcal{M}$. A **partial ear decomposition** of $\mathcal{M}$ is a non-empty sequence $C_1, \ldots, C_k$, where $C_i \in \mathcal{C}$ for all $1 \leq i \leq k$, such that for all $2 \leq i \leq k$ the following conditions hold:

(ED1) $C_i \cap (\bigcup_{j=1}^{i-1} C_j) \neq \emptyset$;

(ED2) $C_i \setminus (\bigcup_{j=1}^{i-1} C_j) \neq \emptyset$; and

(ED3) for all $C' \in \mathcal{C}$ such that $C' \cap (\bigcup_{j=1}^{i-1} C_j) \neq \emptyset \neq C' \setminus (\bigcup_{j=1}^{i-1} C_j)$, $C' \setminus (\bigcup_{j=1}^{i-1} C_j)$ is not a proper subset of $C_i \setminus (\bigcup_{j=1}^{i-1} C_j)$.

An **ear decomposition** of $\mathcal{M}$ is a partial ear decomposition of $\mathcal{M}$ such that $\bigcup_{i=1}^{k} C_i = E$.

The concept of ear decomposition generalises the graph-theoretic notion of the same name, and the following result describes how the presence of an ear decomposition relates to the connectedness of certain matroids.

Theorem 1.2.0.10. *[7] Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and let r be the rank function of $\mathcal{M}$. If $|E| \geq 2$ then:*

(i) $\mathcal{M}$ is connected if and only if there exists an ear decomposition of $\mathcal{M}$;

(ii) if $\mathcal{M}$ is connected then any partial ear decomposition of $\mathcal{M}$ can be extended to an ear decomposition of $\mathcal{M}$[1]; and

(iii) if $C_1, \ldots, C_t$ is an ear decomposition of $\mathcal{M}$ then for all $2 \leq i \leq t$,

$$r\left(\bigcup_{j=1}^{i} C_i\right) - r\left(\bigcup_{j=1}^{i-1} C_i\right) = \left|C_i \setminus \left(\bigcup_{j=1}^{i-1} C_j\right)\right| - 1.$$

1.3 Normed Spaces

1.3.1 Topological Vector Spaces

We begin this section with some basic definitions, and we assume the reader is familiar with 'vector spaces'[2] as well as 'topological spaces'. The structures gained by combining vector spaces and topological spaces play a key role in what follows. This section is based on work by Dewar [8, 9, 10] which extends initial research by Kitson and Power [27].

For the sake of completeness and clarity we provide various additional information to supplement Dewar's comprehensive translation of various concepts from the setting of 'Euclidean' spaces to that of general 'normed' spaces. Much of this process is detailed and technical, so we aim to strike a balance between providing enough information to allow the reader to readily access Dewar's work and avoiding this section becoming bloated.

Definition 1.3.1.1. A **topological vector space** is an ordered pair (X, τ), where X is a real, finite-dimensional, vector space and τ is a topology on X such that the vector

[1] That is, for any partial ear decomposition, $C_1, \ldots, C_s$, of $\mathcal{M}$ there exists $t \geq s$ such that $C_1, \ldots, C_s, \ldots, C_t$ is an ear decomposition of $\mathcal{M}$.

[2] Vector spaces are sometimes referred to in the literature as 'linear spaces'

space operations are continuous (with respect to τ).

Given a topological vector space, X, and some $x \in X$, a set U is a **neighbourhood** of x if there exists an open set U' such that $x \in U' \subseteq U \subseteq X$. We note that some authors (e.g. [36, p.7] and [37, p.96]) instead demand that a 'neighbourhood' of x is itself an open set (i.e. in our definition U' would be a 'neighbourhood' but U need not be). Our approach is consistent with that in [38, p.2] and, importantly for us, that of Dewar [10].

Definition 1.3.1.2. Let X be a real, finite-dimensional, vector space. A **metric** on X is a function $d\colon X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

(i) $d(x, y) = d(y, x)$;

(ii) $d(x, z) \leq d(x, y) + d(y, z)$; and

(iii) if $d(x, y) = 0$ then $x = y$.

The ordered pair (X, d) is a **metric space**.

Given a metric space (X, d), $x \in X$, and $r \in \mathbb{R}^+$ we employ the following notation. $B_r(x) = \{y \in X : d(x, y) < r\}$ denotes the **open ball** with centre x and radius r, while $S_r[x] = \{y \in X : d(x, y) = r\}$ denotes the **sphere** with centre x and radius r.

Definition 1.3.1.3. Let X be a real, finite-dimensional, vector space. A **seminorm** on X is a function $\|\cdot\|\colon X \to \mathbb{R}$ such that for all $x, y \in X$ and for all $a \in \mathbb{R}$,

(i) $\|ax\| = |a|\|x\|$, and

(ii) $\|x + y\| \leq \|x\| + \|y\|$

The ordered pair $(X, \|\cdot\|)$ is a **seminormed space**.

A **norm** on X is a seminorm on X such that if $\|x\| = 0$ then $x = 0$. If $\|\cdot\|$ is a norm on X then the ordered pair $(X, \|\cdot\|)$ is a **normed space**.

Note that in much of what follows we will perform a standard abuse of notation and refer to a normed space X rather than $(X, \|\cdot\|)$.

Remark 6. The three objects we have just defined form a hierarchy. To be precise, given a normed space $(X, \|\cdot\|)$ there exists a corresponding metric space, namely (X, d) where $d\colon X \times X \to \mathbb{R}$ is defined by $d(x_1, x_2) = \|x_1 + (-x_2)\|$. Similarly, given a metric space (X, d) there exists a corresponding topological vector space, namely (X, τ) where $\tau = \{U \subseteq \mathcal{P}(X)\colon U \text{ is a union of open balls}\}$. In particular, we note that the vector space operations are continuous with respect to τ [38, p.52-53].

A consequence of the previous remark is that any normed space induces a corresponding topological vector space (via the norm-induced metric space). This allows us to discuss notions such as 'continuity' and 'open' in the context of arbitrary normed spaces. We also note that, for a given vector space X, while various norms on X may induce different metrics on X these all result in the same topology, the **norm topology**, on X [8, p.11].

Definition 1.3.1.4. Let X be a finite-dimensional real vector space. An **inner product** on X is a function $\langle\cdot,\cdot\rangle\colon X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ and for all $a, b \in \mathbb{R}$,

(i) $\langle x, y\rangle = \langle y, x\rangle$;

(ii) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$; and

(iii) if $x \neq 0$ then $\langle x, x\rangle > 0$.

The ordered pair $(X, \langle\cdot,\cdot\rangle)$ is a **inner product space**.

Inner product spaces join the hierarchy outlined in Remark 6. To be precise, given an inner product space $(X, \langle\cdot\rangle)$ there exists a corresponding normed space, namely $(X, \|\cdot\|)$ where $\|\cdot\|\colon X \to \mathbb{R}$ is defined by $\|x\| = \sqrt{\langle x, x\rangle}$. For a proof that $\|\cdot\|$ is a norm see [38, Theorem II.6.2]. The dichotomy between those normed spaces that can be associated with an inner product in this way and those that can not is crucial to much of what follows. For that reason we say that a normed space is **Euclidean** if the norm is induced by an inner product as above and we say that it is **non-Euclidean** otherwise.

1.3.2 Linear Functionals, Smoothness, and Strict Convexity

Given real, finite-dimensional, vector spaces X and Y, the set of all linear functions from X to Y underlies a (real, finite-dimensional) vector space where addition and scalar multiplication are defined pointwise [38, p.32]. We denote this vector space by $\mathcal{L}(X,Y)$. Of more relevance to us is the situation where X and Y are not just vector spaces but are in fact normed spaces. For clarity, let us denote these $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$. Then consider the space $(\mathcal{L}(X,Y), \|\cdot\|_3)$, where $\|\cdot\|_3 \colon \mathcal{L}(X,Y) \to \mathbb{R}$ is defined by $\|f\|_3 = \sup\{\|f(z)\|_2 \colon \|z\|_1 \leq 1\}$. We refer to $\|\cdot\|_3$ as the **operator norm** on $\mathcal{L}(X,Y)$, and we shall denote this by $\|\cdot\|_{op}$. Given a normed space X we have a particular interest in working with $\mathcal{L}(X,\mathbb{R})$, where the norm on $\mathbb{R}$ is the standard Euclidean norm (see [8, Remark 1.1.4]). With this in mind we introduce some additional terminology.

Definition 1.3.2.1. Let X be a real, finite-dimensional, vector space. A **linear functional** of X is a linear map from X to $\mathbb{R}$. That is, a map $f \colon X \to \mathbb{R}$ such that for all $x_1, x_2 \in X$ and for all $a_1, a_2 \in \mathbb{R}$, $f(a_1x_1 + a_2x_2) = a_1f(x_1) + a_2f(x_2)$.

For a real, finite-dimensional, normed space X we denote the corresponding normed space $(\mathcal{L}(X,\mathbb{R}), \|\cdot\|_{op})$ by X^* and refer to this as the **dual** (space) of X.

The following result is often referred to as the Hahn-Banach theorem and while we rarely, if ever, make explicit use of it we note that it is fundamental statement concerning linear functionals. Various authors (e.g. [36, p.56] and [29, Chapter 7]) have commented on the fact that multiple statements are referred to as the Hahn-Banach Theorem, so in pursuit of clarity we provide one such statement here. Firstly we need an additional definition.

Definition 1.3.2.2. Let X be a real, finite-dimensional, vector space. A **sublinear functional** of X is a map $f \colon X \to \mathbb{R}^+ \cup \{0\}$ such that for all $x, y \in X$ and for all $a \in \mathbb{R}^+ \cup \{0\}$,

(i) $f(ax) = af(x)$, and

(ii) $f(x+y) \leq f(x) + f(y)$.

Theorem 1.3.2.3. *[29, Theorem 7.3.2] Let X be a real vector space, M a subspace of X, p a sublinear functional on X, and f a linear functional on M. If $f(x) \leq p(x)$ for all $x \in M$, then f extends to a linear functional F on X such that for all $x \in X$, $F(x) \leq p(x)$. Moreover, if p is a seminorm on X then $|F(x)| \leq p(x)$.*

Definition 1.3.2.4. Let X be a normed space and take $x \in X$. A **support functional**

of x is a linear functional f such that $\|f\|_{op} = \|x\|$ and $f(x) = \|x\|^2$.

Lemma 1.3.2.5. *[8, Proposition 1.1.8] Every point in a normed space has a support functional.*[3]

Given a normed space X and $x \in X$, the set of all support functionals of x is

$$\Phi[x] := \{f \in X^* \colon \|f\|_{op} = \|x\| \text{ and } f(x) = \|x\|^2\}.$$

Lemma 1.3.2.5 informs us that for all $x \in X$, $\Phi[x] \neq \emptyset$. Two properties of normed spaces, which one can relate to support functionals, are of particular interest to us.

Definition 1.3.2.6. Let X be a normed space. For all $x \in X \setminus \{0\}$, x is **smooth** if $|\Phi[x]| = 1$. X is **smooth** if $\{x \in X \setminus \{0\} \colon x \text{ is smooth}\} = X \setminus \{0\}$. X is **strictly convex** if for all $x_1, x_2 \in S_1[0]$ and all $t \in (0,1)$, $\|tx + (1-t)y\| < 1$.

Let X be a normed space and take $x \in X \setminus \{0\}$. If x is smooth then we denote the unique support functional of x by φ_x. For examples of normed spaces, including normed spaces that are smooth and/or strictly convex, we direct the reader to [8, Example 1.1.1.12, Example 1.1.1.13, Example 1.1.1.14, Example 1.1.1.15].

Lemma 1.3.2.7. *[8, Proposition 1.1.20] Let X be a normed space. X is strictly convex if and only if for all $x, y \in X$ such that $x \neq y$, $\Phi[x] \cap \Phi[y] = \emptyset$.*

Lemma 1.3.2.8. *[9, Lemma 2.3] Let X be a normed space. If X is strictly convex then for all $x, y \in X$ such that x and y are smooth and linearly independent, φ_x and φ_y are linearly independent.*

The relevance of these concepts to what follows can be seen as a consequence of their use in recent work by Dewar [9, p.1214], who separated their analysis of two-dimensional normed spaces using the following trichotomy. A normed space is either

(i) strictly convex and smooth; or

(ii) strictly convex and not smooth; or

[3] This is an example of a result with a proof that invokes the Hahn-Banach Theorem.

(iii) not strictly convex.

Using this trichotomy, Dewar was able to systematically characterise which graphs are '(infinitesimally) rigid' in any two-dimensional normed space. We aim to build on Dewar's work by understanding which graphs are 'globally rigid' in two-dimensional normed spaces. However for technical reasons we restrict our attention to 'analytic' normed spaces. The final result of this section shows that 'analytic' normed spaces are a subset of one of the three types of normed space outlined above.

Definition 1.3.2.9. Let $(X, \|\cdot\|)$ be a normed space. X is **analytic** if $\|\cdot\||_{X\setminus\{0\}}$ is analytic.

Lemma 1.3.2.10. *[14, Lemma 3.1] Let X be a normed space. If X is analytic then X is smooth and strictly convex.*

1.4 Rigidity Theory

1.4.1 Motivation

It is worth mentioning here that 'rigidity' is an intuitive concept which lends itself to being thought of in various ways. There are collections of structures which one may think of as existing in, and potentially moving around within, some ambient space and at that point it is reasonable to ask whether these structures can move, or be moved, in some meaningful way. That is, there are three aspects to bear in mind when considering the question of whether something is rigid.

Firstly, what is the thing we are considering? The objects that will be the focus of this thesis are so-called 'bar-joint frameworks', specifically finite ones, but various other possibilities such as 'body-bar frameworks' and infinite versions have also been considered. Secondly, what ambient space does the object live within and what, if any, additional constraints are placed on how it lives there. A simple illustration of this point is the idea of taking the same object but realising it in a space of different dimension, or a space with a different notion of distance. Another possibility would be to constrain the object to live on some surface. It seems natural to suppose that how an object can move should be impacted by the space in which it lives. Finally, what does it mean to be 'rigid'? The word rigid is suggestive of restricting the motion of an object in some way, but there are various reasonable restrictions that one could impose. The equivalence, or lack of equivalence, of different versions of rigidity gives rise to a number of interesting questions and highlights some of the nuance hiding behind intuition.

Having commented on the fact that various different objects have been granted the moniker of 'framework', we now clarify which frameworks we shall work with. The objects that we are interested in are often referred to in the literature as 'bar-joint' frameworks. As these are the only types of frameworks that we shall consider, we drop the bar-joint qualifier. Note that given sets A and B, $B^A := \{f\colon A \to B\}$.

1.4.2 Frameworks and Rigidity

Definition 1.4.2.1. Let V be a set and let X be a normed space. A **realisation** of V in X is a map $p \in X^V$. If V is the vertex set of some graph, G, then we also say p is a **realisation** of G. The ordered pair (G, p) is a **framework** in X.

We are now in a position to decide what it means for a framework to be 'rigid'. Firstly we introduce a couple of terms that describe possible relationships between different frameworks (with the same underlying graph). These terms will simplify the process of defining what it means for a framework to be rigid.

Given a graph $G = (V, E)$ and a normed space $(X, \|\cdot\|)$ we may define a rigidity map for G in X as

$$f_G\colon X^V \to \mathbb{R}^E, \quad f_G(p) = \left(\frac{\|p(u) - p(v)\|^2}{2}\right)_{uv \in E}$$

Note that taking the square of the norm allows us to differentiate this function at 0. Moreover, having taken the square, we note that halving $\|p(u) - p(v)\|^2$ means that the derivative of f_G at p is the support functional of p.

Definition 1.4.2.2. Let X be a normed space, let G be a graph, and let p and q be realisations of G in X. We say that (G, p) and (G, q) are **equivalent** if $f_G(p) = f_G(q)$. We say that (G, p) and (G, q) are **congruent** if there exists an isometry, g, of X such that $q = g \circ p$.

Remark 7. We say that two frameworks in X, (G, p) and (G, q) are **quasi-congruent** if for all $\{v_1, v_2\} \subseteq V(G)$, $\|p(v_1) - p(v_2)\| = \|q(v_1) - q(v_2)\|$. Clearly, if (G, p) and (G, q) are congruent then they are quasi-congruent, but the converse need not be true.

The distinction between congruence and quasi-congruence is a subtle one. Asimow and Roth noted that for frameworks in Euclidean spaces, congruence and quasi-congruence are equivalent notions [1, p.280-281]. The fact that there exist contexts where congruence and quasi-congruence are not equivalent notions [8, Proposition 1.1.34]) necessitates a careful approach to considering issues of congruence and equivalence.

Let us now provide two notions of rigidity that bear a clear resemblance to one another.

Definition 1.4.2.3. Let X be a normed space and let (G, p) be a framework in X. (G, p) is **locally rigid** (in X) if there exists $\epsilon > 0$ such that for all $q \in X^{V(G)}$ such that (G, p) and (G, q) are equivalent and $\|p(v) - q(v)\| < \epsilon$ for all $v \in V(G)$, (G, p) and (G, q) are congruent.

Definition 1.4.2.4. Let X be a normed space and let (G, p) be a framework in X. (G, p) is **globally rigid** (in X) if for all $q \in X^{V(G)}$ such that (G, p) and (G, q) are equivalent, (G, p) and (G, q) are congruent.

The similarities and differences between these notions of rigidity are fairly clear, and we note that globally rigidity is a stronger property than local rigidity (i.e. if (G, p) is globally rigid then (G, p) is locally rigid). Note that determining whether or not two frameworks are equivalent boils down to a solving a collection of simultaneous quadratic equations. Because of the inherent difficulty involved in such a process, a common technique in rigidity theory is to consider an alternative conception of rigidity that involves linear equations instead. In order to access this technique we need some additional ideas.

1.4.3 Graphs and Rigidity

Having settled on what it means for a framework to be rigid in some context, a natural question is whether those frameworks that are rigid can be characterised in some interesting way. A common technique when attempting to answer this sort of question is to translate the problem from being about frameworks to being about the underlying graph. There are two key issues which must be taken into consideration when trying to transition from a problem about frameworks to a problem concerning graphs:

(i) Given a graph G and a normed space X, we wish to find some property such that for all realisations of G that satisfy this property the corresponding frameworks are either all rigid or all not rigid.

(ii) Given a graph G and a normed space X, we wish to find some property such that the realisations satisfying that property are, in some sense, typical. That is, we want 'almost all' realisations of G to satisfy this property.

In the context of frameworks in d-dimensional Euclidean spaces, a significant amount of research has been dedicated to frameworks where the realisation is 'generic'. That is, where the multiset of coordinates given by p is 'algebraically independent over $\mathbb{Q}$'. In the more general context of (not necessarily Euclidean) normed spaces, various properties of realisations have been considered. As a starting point let (G, p) be a framework in a normed space X, then p is **well-positioned** if for all $uv \in E(G)$, $p(u) - p(v)$ is smooth. Recall that $x \in X \setminus \{0\}$ is smooth if it has a unique support functional, and we denote this support functional by φ_x.

Let X be a normed space and let G be a graph. If p is a well-positioned realisation of G in X then we can define the rigidity operator of (G, p); the rigidity operator of (G, p) is derivative of f_G at p. That is,

$$df_G|_p \colon X^V \to \mathbb{R}^E, \quad df_G|_p(q) = \left(\varphi_{p(u)-p(v)}(q(u) - q(v))\right)_{uv \in E}.$$

Now, if p is well-positioned then p is:

(i) **regular** if for all well-positioned realisations q of G in X, rank $df_G|_p \geq$ rank $df_G|_q$;

(ii) **strongly regular** if for all q such that (G, q) is equivalent to (G, p), q is regular;

(iii) **completely regular** if for all H such that $V(H) = V(G)$, p is regular; and

(iv) **completely strongly regular** if for all H such that $V(H) = V(G)$, p is strongly regular.

These properties of a realisation will aid us in translating problems from being about frameworks to being about graphs. More details on the different flavours of regularity can be found in [14, Fig. 1., Remark 2.9.] For now we note two useful technical results.

Lemma 1.4.3.1. *[10, Lemma 4.1, Lemma 4.4] Let X be a normed space and let G be a graph. The set of well-positioned realisations of G in X is a conull subset (i.e. the complement of a set with Lebesgue measure zero) of $X^{V(G)}$, and the set of regular realisations of G in X is a non-empty open subset of the set of well-positioned realisations.*

Proposition 1.4.3.2. *[14, Proposition 3.2, Proposition 3.6] Let X be a normed space and let G be a graph. If X is analytic then the sets of regular, strongly regular, completely regular, and completely strongly regular realisations of G in X are all open conull subsets of $X^{V(G)}$.*

Now that we have access to well-positioned realisations we are able to discuss the alternative version of rigidity mentioned at the end of the previous subsection. Recall that we consider normed spaces to have finite dimension.

Definition 1.4.3.3. Let X be a normed space, let G be a graph, and let p be a well-positioned realisation of G in X. An **infinitesimal flex** of (G, p) is an element of the kernel of $df_G|_p$. An infinitesimal flex u is **trivial** if there exists a linear map $T\colon X \to X$ and $x \in X$ such that $u(v) = T(p(v)) + x$ for all $v \in V$, and for all $y \in X$, with support functional f_y, $f_y(T(y)) = 0$[4]. (G, p) is **infinitesimally rigid** if every infinitesimal flex of (G, p) is trivial.

The following result shows that, for certain frameworks, there is a relationship between infinitesimal rigidity and the more natural notion of local rigidity.

[4]For additional details regarding this definition see [11, Subsection 1.2] and [10, Subsection 2.3].

Theorem 1.4.3.4. *Let X be a normed space and let (G,p) be a framework in X.*

(i) [11, Observation 3.4, Theorem 3.7] If p is well-positioned and (G,p) is infinitesimally rigid then (G,p) is locally rigid.

(ii) [10, Theorem 1.1, Lemma 4.4] If p is regular and $\{x \in X : x \text{ is smooth}\}$ is open and (G,p) is locally rigid then (G,p) is infinitesimally rigid.

We are now in a position to introduce the machinery needed to discuss rigidity as a property of the graph underlying a framework.

Definition 1.4.3.5. Let X be a normed space and let (G,p) be a framework in X. (G,p) is **independent** if p is well-positioned and $|E(G)| = \text{rank } df_G|_p$. (G,p) is **minimally (infinitesimally) rigid** if (G,p) is independent and infinitesimally rigid.

For a d-dimensional normed space X, a well-positioned framework (G,p) in X, and a fixed basis $b_1,\dots,b_d$ of X, we can define the **rigidity matrix** to be the $|E| \times d|V|$ matrix $R(G,p)$, where for every $e \in E$, $x \in V$, and $i \in \{1,\dots,d\}$ we have

$$R(G,p)_{e,(x,i)} = \begin{cases} \varphi_{p(x)-p(y)}(b_i) & \text{if } e = xy; \\ 0 & \text{otherwise.} \end{cases}$$

As p is well-positioned and $xy \in E$, $p(x)-p(y)$ is smooth. Hence $p(x)-p(y)$ has a unique support functional which we denote by $\varphi_{p(x)-p(y)}$. Therefore $\varphi_{p(x)-p(y)}(b_i)$ maps b_i to some real number. The choice of basis used to define $R(G,p)$ can be arbitrary as we are only interested in the sets of linearly independent rows of the matrix.

Let X be a normed space and let G be a graph. We say that G is **infinitesimally rigid** in X if there exists a realisation, p, of G in X such that (G,p) is infinitesimally rigid. Similarly, we say that G is **independent** in X if there exists a realisation, q, of G in X such that (G,q) is independent. Slightly differently, we say that G is **globally rigid** in X if the set $\text{GRig}(G;X) := \{p \in X^{V(G)} : (G,p) \text{ is globally rigid}\}$ has a non-empty interior. The following results, the first of which is folklore, show that for a space with sufficiently low dimension, whether a graph is infinitesimally rigid or independent in that space can be determined by simple counting conditions.

Theorem 1.4.3.6. *Let X be a one-dimensional normed space*[5] *and let G be a graph. The following are equivalent:*

(i) G is minimally rigid in X;

(ii) $|E(G)| = |V(G)| - 1$ and for all $\emptyset \neq U \subseteq V(G)$, $i_G(U) \leq |U| - 1$;

(iii) There exists $t \in \mathbb{N}^+$ and a sequence $a_1, \ldots, a_t$, with $a_1 \cong K_1$ and $a_t = G$, such that for all $2 \leq j \leq t$, a_j is a $(1,0)$-extension of a_{j-1}; and

(iv) G is a tree.

Theorem 1.4.3.7. *[35, 28]*[6] *Let X be a two-dimensional Euclidean space and let G be a graph. The following are equivalent:*

(i) G is minimally rigid in X;

(ii) $G \cong K_1$, or $|E(G)| = 2|V(G)| - 3$ and for all $U \subseteq V(G)$ such that $|U| \geq 2$, $i_G(U) \leq 2|U| - 3$; and

(iii) $G \cong K_1$ or there exists $t \in \mathbb{N}^+$ and a sequence $a_1, \ldots, a_t$, with $a_1 \cong K_2$ and $a_t = G$, such that for all $2 \leq j \leq t$, a_j is a $(2,0)$-extension or a $(2,1)$-extension of a_{j-1}.

Theorem 1.4.3.8. *[9] Let X be a two-dimensional non-Euclidean space and let G be a graph. The following are equivalent:*

(i) G is minimally rigid in X; and

(ii) $|E(G)| = 2|V(G)| - 2$ and for all $\emptyset \neq U \subseteq V(G)$, $i_G(U) \leq 2|U| - 2$.

The previous results are examples of the type of goal that one may have in mind when embarking on an attempt to solve a local rigidity problem. We conclude this section, and chapter, with similarly comprehensive results about global rigidity in Euclidean spaces. To do that we first introduce the concept of redundant rigidity.

[5] Every one-dimensional normed space is Euclidean.

[6] This result is often referred to in the literature as 'Laman's Theorem' due to the proof of Laman [28]. However, the rigidity community has recently discovered that an earlier proof was given by Pollaczek-Geiringer [35].

Definition 1.4.3.9. Let X be a normed space and let (G,p) be a framework in X. (G,p) is **redundantly (infinitesimally) rigid** if for all $e \in E(G)$, $((V(G), E(G) \setminus \{e\}), p)$ is infinitesimally rigid.

As with independence and infinitesimal rigidity we say that a graph, G, is **redundantly (infinitesimally) rigid** if there exists a realisation, p, of G in X such that (G,p) is redundantly rigid.

Theorem 1.4.3.10. *[21, Theorem 3.1, Theorem 5.9] Let X be a d-dimensional Euclidean space and let G be a graph. If G is globally rigid in X then*

(i) there exists $n \leq d+1$ such that $G \cong K_n$; or

(ii) G is $(d+1)$-vertex-connected and G is redundantly rigid in $\mathbb{E}^d$.

Theorem 1.4.3.11. *[22, Theorem 7.1] Let X be a two-dimensional Euclidean space and let G be a graph. The following are equivalent:*

(i) G is globally rigid in X; and

(ii) there exists $n \leq 3$ such that $G \cong K_n$, or G is 3-vertex-connected and G is redundantly rigid in X.

Jackson and Jordán showed that in the two-dimensional setting Hendrickson's necessary conditions were also sufficient. Their proof involved finding a method of constructing all 3-vertex-connected and redundantly rigid graphs and applying a result of Connelly [5] to guarantee that the graph operations that they made use of would preserve the property of being globally rigid.

A graph (V,E) is **bipartite** if there exists a partition $\{V_1, V_2\}$ of V such that for all $\{u,v\} \in E$ and for all $i \in \{1,2\}$, $\{u,v\} \nsubseteq V_i$. A graph is **complete bipartite** if it is bipartite, with partition $\{V_1, V_2\}$ of V, and every pair of vertices that is not contained within an element of the partition is an edge. Recall that if graphs G and H are complete then $G \cong H$ if and only if $|V(G)| = |V(H)|$. Similarly, if G and H are complete bipartite, with partitions $\{V_1, V_2\}$ and $\{W_1, W_2\}$ respectively, then $G \cong H$ if and only if $\{|V_1|, |V_2|\} = \{|W_1|, |W_2|\}$. For this reason we write $G \cong K_{a,b}$ to demonstrate that G

is a (up to isomorphism, the) complete bipartite graph with partition $\{A, B\}$ of $V(G)$ such that $\{|A|, |B|\} = \{a, b\}$.

The equivalence exhibited in Theorem 1.4.3.11 does not generalise to d-dimensional Euclidean spaces where $d \geq 3$. Connelly [4] showed that for all $d \geq 3$ there exist complete bipartite graphs, with $\binom{d+2}{2}$ vertices, that are $(d+1)$-vertex-connected and redundantly rigid in $(d+1)$-dimensional Euclidean space but are not globally rigid in $(d+1)$-dimensional Euclidean space (e.g. $K_{5,5}$ when $d = 3$). In fact, given a complete bipartite graph G, the following result gives a method of determining whether or not G is globally rigid in d-dimensional Euclidean space.

Theorem 1.4.3.12. *[25, Theorem 63.2.2], [6, Theorem 1.1, Section 5] Take $d, m, n \in \mathbb{N}^+$. $K_{m,n}$ is globally rigid in d-dimensional Euclidean space if and only if $m, n \geq d + 1$ and $m + n \geq \binom{d+2}{2} + 1$.*

The next three chapters of the thesis shall culminate with similar results through an exploration of global rigidity in non-Euclidean spaces. To begin this process we turn our attention to the method of constructing all 3-vertex-connected and redundantly rigid graphs used by Jackson and Jordán.

Chapter 2

Connected Matroids

The focus of this chapter is the study of a particular genre of matroid. These matroids are on the edge sets of some graph, and independence within them is determined by a simple counting condition. We begin by considering these matroids in a relatively general setting, and we gradually narrow our focus until we settle on working with those graphs for which the corresponding '$(2, 2)$-sparsity' matroid is connected.

2.1 (k, l)-Sparsity of Graphs and Matroids

We begin with some results that allow us to introduce the matroids that we shall be working with in this chapter and in Chapter 3.

Lemma 2.1.0.1. *Let $G = (V, E)$ be a graph, take $k, l \in \mathbb{Z}$, and let $\mathcal{I} = \{F \subseteq E\colon$ for all $\emptyset \neq F' \subseteq F, |F'| \leq k|V(G[F'])| - l\}$. $\mathcal{I} \neq \{\emptyset\}$ if and only if $E \neq \emptyset$ and $l \leq 2k - 1$.*

Proof. If $\mathcal{I} \neq \{\emptyset\}$ then clearly $E \neq \emptyset$, and there exists $F \in \mathcal{I}$ such that $|F| = 1$. Hence $1 = |F| \leq k|V(G[F])| - l = 2k - l$ and so $l \leq 2k - 1$. On the other hand, suppose that $E \neq \emptyset$ and $l \leq 2k - 1$. As $E \neq \emptyset$ there exists $F \subseteq E$ such that $|F| = 1$ and hence

$|V(G[F])| = 2$. As $l \leq 2k-1$, $1 = |F| \leq 2k - l = k|V(G[F])| - l$ and hence $F \in \mathcal{I}$. Therefore $\mathcal{I} \neq \emptyset$. □

The notation of the previous result, namely the use of $\mathcal{I}$, is suggestive of a connection to (the independent sets of) matroids. In order to show that such a connection exists we require some additional theory.

Definition 2.1.0.2. Let E be a finite set and let $f\colon \mathcal{P}(E) \to \mathbb{R}$. f is **submodular** if for all $X, Y \subseteq E$, $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$. f is **increasing** if for all $X \subseteq Y \subseteq E$, $f(X) \leq f(Y)$.

Lemma 2.1.0.3. *[34, Proposition 12.1.1, Corollary 12.1.2] Let E be a finite set, let $f\colon \mathcal{P}(E) \to \mathbb{Z}$, and let $\mathcal{I} = \{F \subseteq E\colon \text{for all } \emptyset \neq F' \subseteq F, |F'| \leq f(F')\}$. If f is an increasing submodular function then $(E, \mathcal{I})$ is a matroid.*

Lemma 2.1.0.4. *Let $G = (V, E)$ be a graph, take $k, l \in \mathbb{Z}$, and let $\mathcal{I} = \{F \subseteq E\colon \text{for all } \emptyset \neq F' \subseteq F, |F'| \leq k|V(G[F'])| - l\}$. If $k \geq 0$ then $(E, \mathcal{I})$ is a matroid.*

Proof. We consider the following function.

$$f\colon \mathcal{P}(E) \to \mathbb{Z}, \quad f(F) = \begin{cases} -l & \text{if } F = \emptyset; \\ k|V(G[F])| - l & \text{if } F \neq \emptyset. \end{cases}$$

Then $\mathcal{I} = \{F \subseteq E\colon \text{for all } \emptyset \neq F' \subseteq F, |F'| \leq f(F')\}$. Take $F_1, F_2 \subseteq E$. If $F_1 = \emptyset$ or $F_2 = \emptyset$ then $f(F_1) + f(F_2) = f(F_1 \cup F_2) + f(F_1 \cap F_2)$. If $F_1 \neq \emptyset \neq F_2$ then $F_1 \cup F_2 \neq \emptyset$ and $f(F_1 \cup F_2) = k|V(G[F_1 \cup F_2])| - l$. So,

$$\begin{aligned} f(F_1) + f(F_2) &= (k|V(G[F_1])| - l) + (k|V(G[F_2])| - l) \\ &= k(|V(G[F_1]) \cup V(G[F_2])| + |V(G[F_1]) \cap V(G[F_2])|) - 2l \\ &= k(|V(G[F_1 \cup F_2])| + |V(G[F_1]) \cap V(G[F_2])|) - 2l \\ &= f(F_1 \cup F_2) + k|V(G[F_1]) \cap V(G[F_2])| - l. \end{aligned}$$

If $F_1 \cap F_2 = \emptyset$ then $f(F_1 \cap F_2) = -l$. If $F_1 \cap F_2 \neq \emptyset$ then $f(F_1) \cap F_2) = k|V(G[F_1 \cap$

$F_2])| - l$. Either way, as $k \geq 0$ we see that $f(F_1 \cap F_2) \leq k|V(G[F_1]) \cap V(G[F_2])| - l$. Therefore

$$f(F_1) + f(F_2) = f(F_1 \cup F_2) + k|V(G[F_1]) \cap V(G[F_2])| - l \geq f(F_1 \cup F_2) + f(F_1 \cap F_2),$$

and so f is a submodular function.

Now, take $F_1 \subseteq F_2 \subseteq E$. Since $k \geq 0$, $f(\emptyset) = -l \leq k|V(G[F])| - l = f(F)$ for all $\emptyset \neq F \subseteq E$. So, if $F_1 = \emptyset$ then $f(F_1) \leq f(F_2)$. If $F_1 \neq \emptyset$ then $V(G[F_1]) \leq V(G[F_2])$ and so, as $k \geq 0$, $f(F_1) = k|V(G[F_1])| - l \leq k|V(G[F_2])| - l = f(F_2)$. Therefore f is an increasing function. Lemma 2.1.0.3 implies that $(E, \mathcal{I})$ is a matroid. $\square$

In light of the previous result the following definition is a sensible one.

Definition 2.1.0.5. Let $G = (V, E)$ be a graph, take $k, l \in \mathbb{Z}$ such that $k \geq 0$, and let $\mathcal{I} = \{F \subseteq E\colon \text{for all } \emptyset \neq F' \subseteq F, |F'| \leq k|V(G[F'])| - l\}$. The **$(k,l)$-sparsity matroid** of G, denoted $\mathcal{M}_{(k,l)}(G)$, is the ordered pair $(E, \mathcal{I})$. We say that G is **(k,l)-sparse** if E is independent in $\mathcal{M}_{(k,l)}(G)$.

Definition 2.1.0.6. Let $G = (V, E)$ be a graph and take $k, l \in \mathbb{Z}$ such that $k \geq 0$. G is a **(k,l)-circuit** if E is a circuit of $\mathcal{M}_{(k,l)}(G)$ and $G = G[E]$. G is **(k,l)-connected** if $\mathcal{M}_{(k,l)}(G)$ is connected and $G = G[E]$ and $|E| \geq 2$.

Remark 8. The definition of what it means for a graph to be (k,l)-connected is what Remark 5 was alluding to. Recall that if $|E| \leq 1$ then $\mathcal{M}_{(k,l)}(G)$ is connected. We also infer from Lemma 2.1.0.1, in an informal sense, that for a graph G, the values of k and l such that $\mathcal{M}_{(k,l)}(G)$ is 'interesting' are those where $l \leq 2k - 1$.

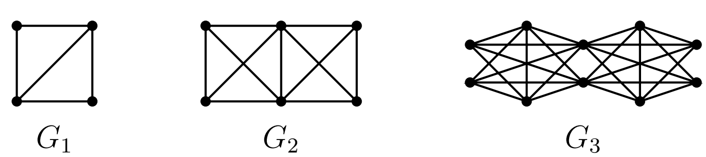


Figure 2.1: Illustrations of some (k,l)-circuits. G_1 is a $(1,0)$-circuit, G_2 is a $(2,2)$-circuit, and G_3 is a $(3,5)$-circuit.

Our next result characterises (k,l)-circuits in a purely graph-theoretic manner. Some examples of (k,l)-circuits are illustrated in Figure 2.1.

Lemma 2.1.0.7. *Let $G=(V,E)$ be a graph, take $k \in \mathbb{N}^+$, and take $l \in \mathbb{Z}$ such that $l \leq 2k-1$. The following are equivalent:*

(i) G is a (k,l)-circuit;

(ii) $3 \leq |E| = k|V|-(l-1)$ and for all $\emptyset \neq F \subsetneq E$, $|F| \leq k|V(G[F])|-l$; and

(iii) $3 \leq |E| = k|V|-(l-1)$ and for all $U \subsetneq V$ such that $|U| \geq 2$, $i_G(U) \leq k|U|-l$.

Proof. Firstly suppose that (ii) holds. By definition, E is a circuit of $\mathcal{M}_{(k,l)}(G)$. As $E \neq \emptyset$ we may take $e \in E$. As $|E| \geq 3$, $E \setminus \{e\} \neq \emptyset$ and hence

$$k|V|-l = (k|V|-(l-1))-1 = |E\setminus\{e\}| \leq k|V(G[E\setminus\{e\}])|-l \leq k|V(G[E])|-l.$$

As $k \in \mathbb{N}^+$ and $V(G[E]) \subseteq V$ it follows that $|V| \leq |V(G[E])| \leq |V|$, and hence $V = V(G[E])$. Therefore $G = G[E]$ and G is a (k,l)-circuit. Moreover, take $U \subsetneq V$ such that $|U| \geq 2$ and let $F = E(G[U])$. As $U \subsetneq V = V(G[E])$ we have that $F \subsetneq E$. If $F = \emptyset$ then $i_G(U) = 0 \leq 2k-(2k-1)$. As $|U| \geq 2$ and $l \leq 2k-1$, this implies that $i_G(U) \leq k|U|-l$. If $F \neq \emptyset$ then $i_G(U) = |F| \leq k|V(G[F])|-l = k|U|-l$. Therefore (ii) implies (i) and (iii).

On the other hand, suppose that (i) holds. Then E is a circuit of $\mathcal{M}_{(k,l)}(G)$ and $G = G[E]$. Then $E \neq \emptyset$, $|E| \geq k|V(G[E])|-(l-1)$, and for all $\emptyset \neq F \subsetneq E$, $|F| \leq k|V(G[F])|-l$. As $l \leq 2k-1$ and $|E| \geq k|V(G[E])|-(l-1)$, $|E| \geq 2$. However, since $|E| \geq 2$ we now have that $|V(G[E])| \geq 3$ and so

$$|E| \geq k|V(G[E])|-(l-1) \geq 3k-(2k-2) = k+2 \geq 3.$$

As $|E| \geq 3$, we may take $f \in E$ such that $\emptyset \neq E\setminus\{f\} \subsetneq E$. So,

$$k|V|-l = k|V(G[E])|-l \leq |E|-1 = |E\setminus\{f\}| \leq k|V(G[E\setminus\{f\}])|-l.$$

As $k \in \mathbb{N}^+$ and $V(G[E \setminus \{f\}]) \subseteq V$ it follows that $|V| \leq |V(G[E \setminus \{f\}])| \leq |V|$ and hence $V = V(G[E \setminus \{f\}])$. Therefore (i) implies (ii).

Finally, suppose that (iii) holds. Take $\emptyset \neq F \subsetneq E$ and let $U = V(G[F])$. As $F \neq \emptyset$, $|U| \geq 2$. If $U = V$ then, as $F \subsetneq E$, it follows that $|F| \leq i_G(U) - 1 = |E| - 1 = k|U| - l$. If $U \subsetneq V$ then $|F| \leq i_G(U) \leq k|U| - l$. Therefore (iii) implies (ii). □

Lemma 2.1.0.8. *Let G be a graph, take $k \in \mathbb{N}^+$, and take $l \in \mathbb{Z}$ such that $l \leq 2k - 1$. If G is a (k, l)-circuit then G is (k, l)-connected.*

Proof. As G is a (k, l)-circuit, $E(G)$ is a circuit of $\mathcal{M}_{(k,l)}(G)$ and hence $\mathcal{M}_{(k,l)}(G)$ is connected. Moreover, $G = G[E(G)]$ and, by Lemma 2.1.0.7, $|E(G)| \geq 3$. Therefore G is (k, l)-connected. □

The next few results in this section consider how placing restrictions on k and l allow us to guarantee that (k, l)-connected graphs satisfy certain graph-theoretic properties. To be more specific we show that if the values of k and l that we allow are sufficiently restricted, then for a (k, l)-circuit G the values $\delta(G)$, $\kappa(G)$, and $\kappa_1(G)$ have sharp lower bounds. Having provided these bounds for (k, l)-circuits we then use the relationship between (k, l)-circuits and (k, l)-connected graphs to show that the same values act as lower bounds in the context of (k, l)-connected graphs. Figure 2.2 illustrates that these lower bounds are the best possible. The same figure also highlights the importance of the restrictions placed on k and l in order to give these lower bounds by showcasing (k, l)-circuits such that k and l are not allowed by the relevant restrictions and the minimum degree, vertex-connectivity, or edge-connectivity is less than the lower bound.

Lemma 2.1.0.9. *Let G be a graph, take $k \in \mathbb{N}^+$, and take $l \in \mathbb{Z}$ such that $l \leq 2k - 1$. If G is a (k, l)-circuit then $\delta(G) \geq k + 1$.*

Proof. Let $G = (V, E)$, take $v \in V$ such that $d_G(v) = \delta(G)$, and let $F = \{vw\colon w \in N_G(v)\}$. Note that $|F| = d_G(v)$. Lemma 2.1.0.7 implies that $G = G[E]$ and $3 \leq |E| = k|V| - (l - 1)$. As $G = G[E]$, $\delta(G) \geq 1$.

If there exists $u \in N_G(v)$ such that $d_G(u) = 1$ then, as $d_G(v) = \delta(G)$, it follows that $N_G(v) = \{u\}$. As $|E| \geq 3$ we have that $\{u, v\} \subsetneq V$ and $F = \{uv\} \subsetneq E$. Let $G[E \setminus F] = (V', E') = G[V \setminus \{u, v\}]$. As G is a (k, l)-circuit and $\emptyset \neq E' \subsetneq E$, it follows that

$$k|V'| - l \geq |E'| = |E| - |F| = (k|V| - (l-1)) - d_G(v) = (k|V'| - l) + 2k.$$

However, this implies that $0 \geq 2k$ which contradicts the fact that $k \in \mathbb{N}^+$.

Therefore $d_G(u) \geq 2$ for all $u \in N_G(v)$. Hence $V(G[E \setminus F]) = V \setminus \{v\}$ and $\emptyset \neq E \setminus F \subsetneq E$. Let $G[E \setminus F] = (V', E') = G[V \setminus \{v\}]$. As G is a (k, l)-circuit and $\emptyset \neq E' \subsetneq E$, it follows that

$$k|V'| - l \geq |E'| = |E| - |F| = (k|V| - (l-1)) - d_G(v) = (k|V'| - l) + k + 1 - d_G(v).$$

Therefore $0 \geq k + 1 - d_G(v)$, and so $\delta(G) = d_G(v) \geq k + 1$. □

Lemma 2.1.0.10. *Let G be a graph, take $k \in \mathbb{N}^+$, and take $l \in \mathbb{Z}$ such that $l \leq 2k-1$. If G is (k, l)-connected then $\delta(G) \geq k + 1$.*

Proof. Let $G = (V, E)$ and take $v \in V$ such that $d_G(v) = \delta(G)$. As G is (k, l)-connected, $\mathcal{M}_{(k,l)}(G)$ is connected, $G = G[E]$, and $|E| \geq 2$. Therefore there exists $e \in E$ such that e is incident to v, there exists $f \in E \setminus \{e\}$, and there exists C, a circuit in $\mathcal{M}_{(k,l)}(G)$, such that $\{e, f\} \subseteq C$. Then $G[C]$ is a (k, l)-circuit. Lemma 2.1.0.9 implies that $\delta(G) = d_G(v) \geq d_{G[C]}(v) \geq \delta(G[C]) \geq k + 1$. □

Lemma 2.1.0.11. *Let G be a graph, take $k \in \mathbb{N}^+$, and take $l \in \mathbb{Z}$ such that $l \leq 2k-1$. If $l \geq 0$ and G is a (k, l)-circuit then $\kappa(G) \geq 1$.*

Proof. Let $G = (V, E)$. Lemma 2.1.0.7 implies that $3 \leq |E| = k|V| - (l-1)$ and for all $\emptyset \neq F \subsetneq E$, $|F| \leq k|V(G[F])| - l$. As $|E| \geq 3$, $|V| \geq 3$. Let $G_1, \ldots, G_n$ be the components of G and for all $1 \leq i \leq n$ let $G_i = (V_i, E_i)$. Then $V = \bigcup_{i=1}^n V_i$ and $E = \bigcup_{i=1}^n E_i$. As each G_i is a component of G we have that for all $1 \leq i < j \leq n$,

$V_i \cap V_j = \emptyset = E_i \cap E_j$. Therefore $|V| = \sum_{i=1}^{n} |V_i|$ and $|E| = \sum_{i=1}^{n} |E_i|$. As G is a (k, l)-circuit, $1 \leq |E_i| \leq k|V(G[E_i])| - (l-1) = k|V_i| - (l-1)$ for all $1 \leq i \leq n$. So,

$$k|V| - (l-1) = |E| = \sum_{i=1}^{n} |E_i| \leq \sum_{i=1}^{n} (k|V_i| - (l-1)) = k|V| - n(l-1).$$

Therefore $l - 1 \geq n(l-1)$.

If $l \geq 2$ then $1 \geq n \geq 1$, so $n = 1$ and hence G is connected. Otherwise $l \in \{0, 1\}$. If $l = 1$ then we have that $k|V| = \sum_{i=1}^{n} |E_i| \leq \sum_{i=1}^{n} (k|V_i|) = k|V|$. Hence for all $1 \leq i \leq n$, $|E_i| = k|V_i| = k|V_i| - (l-1)$. As G is a (k, l)-circuit this implies that $1 = i = n$ and G is connected. If $l = 0$ then we have that $k|V| + 1 = \sum_{i=1}^{n} |E_i| \leq \sum_{i=1}^{n} (k|V_i| + 1) = k|V| + n$ and hence there exists $1 \leq i \leq n$ such that $|E_i| = k|V_i| + 1$. As G is a (k, l)-circuit this implies that $1 = i = n$ and G is connected. So for all $U \subsetneq V$ such that $|U| \leq 0$, $|V \setminus U| = |V| \geq 3$ and $G[V \setminus U] = G$ is connected. Therefore $\kappa(G) \geq 1$. □

Lemma 2.1.0.12. *Let G be a graph, take $k \in \mathbb{N}^+$, and take $l \in \mathbb{Z}$ such that $l \leq 2k - 1$. If $l \geq 0$ and G is (k, l)-connected then $\kappa(G) \geq 1$.*

Proof. Let $G = (V, E)$. As G is (k, l)-connected, $G = G[E]$ and $|E| \geq 2$. As $|E| \geq 2$, $|V| \geq 3$. Let $G_1, \ldots, G_n$ be the components of G and for all $1 \leq i \leq n$ let $G_i = (V_i, E_i)$. Then $V = \bigcup_{i=1}^{n} V_i$ and $E = \bigcup_{i=1}^{n} E_i$. As each G_i is a component of G we have that for all $1 \leq i < j \leq n$, $V_i \cap V_j = \emptyset = E_i \cap E_j$. Moreover, as $G = G[E]$ we have that for all $1 \leq i \leq n$, $E_i \neq \emptyset$.

Take $\{e, f\} \subseteq E$ and let $G[C]$ be a (k, l)-circuit such that $\{e, f\} \subseteq C$, which exists since $\mathcal{M}_{(k,l)}(G)$ is connected. As $k \in \mathbb{N}^+$ and $0 \leq l \leq 2k - 1$, Lemma 2.1.0.11 implies that $\kappa(G[C]) \geq 1$ and so $G[C]$ is connected. Hence there exists $1 \leq i \leq n$ such that $\{e, f\} \subseteq E_i$. Note that e and f were chosen arbitrarily, so for all $\{e, f\} \subseteq E$ there exists $1 \leq i \leq n$ such that $\{e, f\} \subseteq E_i$. As, for all $1 \leq i < j \leq n$, $E_i \neq \emptyset = E_i \cap E_j$ this implies that $n = 1$ and so G is connected. So for all $U \subsetneq V$ such that $|U| \leq 0$, $|V \setminus U| = |V| \geq 3$ and $G[V \setminus U] = G$ is connected. Therefore $\kappa(G) \geq 1$. □

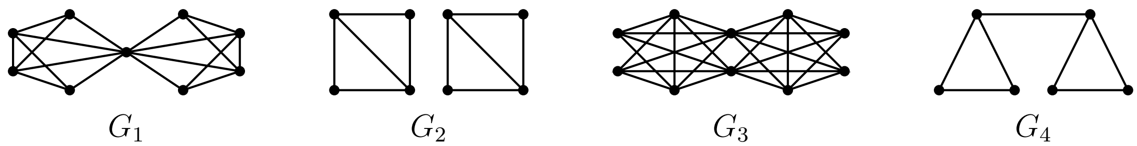


Figure 2.2: G_1 is a $(2, 1)$-circuit, G_2 is a $(1, -1)$-circuit, G_3 is a $(3, 5)$-circuit, and G_4 is a $(1, 0)$-circuit.
G_1 demonstrates that the lower bounds of Lemma 2.1.0.9 and Lemma 2.1.0.11 are the best possible, and G_3 demonstrates that the lower bounds of Lemma 2.1.0.13 and Lemma 2.1.0.15 are the best possible. G_2 highlights the importance of the condition that $l \geq 0$ in Lemma 2.1.0.11, G_1 highlights the importance of the condition that $l \geq k$ in Lemma 2.1.0.13, and G_4 highlights the importance of the condition that $l \geq k$ in Lemma 2.1.0.15.

Lemma 2.1.0.13. *Let G be a graph, take $k \in \mathbb{N}^+$, and take $l \in \mathbb{Z}$ such that $l \leq 2k-1$. If $l \geq k$ and G is a (k, l)-circuit then $\kappa(G) \geq 2$.*

Proof. Let $G = (V, E)$. Lemma 2.1.0.7 implies that $3 \leq |E| = k|V| - (l - 1)$ and for all $\emptyset \neq F \subsetneq E$, $|F| \leq k|V(G[F])| - l$. As $|E| \geq 3$, $|V| \geq 3$. Take $u \in V$ and let $H_1, \ldots, H_n$ be the components of $G[V \setminus \{u\}] = H = (W, F)$. As $|V| \geq 3$, $|W| \geq 2$. For all $1 \leq i \leq n$ let $H_i = (W_i, F_i)$. Then $W = \bigcup_{i=1}^n W_i$ and $F = \bigcup_{i=1}^n F_i$. As each H_i is a component of H we have that for all $1 \leq i < j \leq n$, $W_i \cap W_j = \emptyset = F_i \cap F_j$. Therefore $|W| = \sum_{i=1}^n |W_i|$ and $|F| = \sum_{i=1}^n |F_i|$.

As G is a (k, l)-circuit, Lemma 2.1.0.9 implies that $\delta(G) \geq k + 1 \geq 2$. So for all $1 \leq i \leq n$, $\emptyset \neq F_i \subsetneq E$. Therefore, for all $1 \leq i \leq n$ there exists $a_i \geq l$ such that $|F_i| = k|W_i| - a_i$. So for all $1 \leq i \leq n$,

$$k|W_i \cup \{u\}| - (l - 1) \geq |F_i| + |N_G(u) \cap W_i| = k|W_i \cup \{u\}| + |N_G(u) \cap W_i| - (a_i + k).$$

Therefore for all $1 \leq i \leq n$, $-(l - 1) \geq |N_G(u) \cap W_i| - (a_i + k)$. Let $l = k + x$. Then for all $1 \leq i \leq n$, $|N_G(u) \cap W_i| \leq 1 + a_i - x$. Moreover,

$$\begin{aligned} k|W| + k - ((l-1) + d_G(u)) &= k|V| - ((l-1) + d_G(u)) \\ &= |E| - d_G(u) \\ &= |F| \\ &= \sum_{i=1}^{n} (k|W_i| - a_i) \\ &= k|W| - \sum_{i=1}^{n} a_i. \end{aligned}$$

It follows that $1 + (k - l) - d_G(u) = -\sum_{i=1}^{n} a_i$, and consequently $d_G(u) = (1 - x) + \sum_{i=1}^{n} a_i$. Combining the information we have so far we see that

$$\sum_{i=1}^{n} (1 + a_i - x) \geq \sum_{i=1}^{n} |N_G(u) \cap W_i| = d_G(u) = (1 - x) + \sum_{i=1}^{n} a_i.$$

So $n(1 - x) \geq 1 - x$. As $l \geq k$, $x \geq 0$ and hence $n(1 - x) \leq 1 - x$. Therefore $n(1 - x) = (1 - x)$ which implies that $n = 1$ or $x = 1$.

Recall that for all $1 \leq i \leq n$, $|N_G(u) \cap W_i| \leq 1 + a_i - x$. If $x = 1$ then $l = k + 1$ and $\sum_{i=1}^{n} a_i = d_G(u) = \sum_{i=1}^{n} |N_G(u) \cap W_i| \leq \sum_{i=1}^{n} a_i$. So $|N_G(u) \cap W_i| = a_i$ for all $1 \leq i \leq n$, and hence

$$\begin{aligned} k|W_i \cup \{u\}| - k &= k|W_i \cup \{u\}| - (l - 1) \\ &\geq |F_i| + |N_G(u) \cap W_i| \\ &= (k|W_i| - a_i) + a_i \\ &= k|W_i \cup \{u\}| - k. \end{aligned}$$

Therefore for all $1 \leq i \leq n$, $k|W_i \cup \{u\}| - (l-1) = |F_i| + |N_G(u) \cap W_i| = |F_i \cup \{uv \colon v \in N_G(u) \cap W_i\}|$. As G is a (k, l)-circuit this implies $F_i \cup \{uv \colon v \in N_G(u) \cap W_i\} = E$, so $1 = i = n$. Therefore $n = 1$ and $H = G[V \setminus U]$ is connected. Lemma 2.1.0.11 implies that $\kappa(G) \geq 1$ and so for all $U \subseteq V$ such that $|U| \leq 1$, we have $|V \setminus U| \geq 2$ and

$G[V \setminus U]$ is connected. Therefore $\kappa(G) \geq 2$. □

Lemma 2.1.0.14. *Let G be a graph, take $k \in \mathbb{N}^+$, and take $l \in \mathbb{Z}$ such that $l \leq 2k-1$. If $l \geq k$ and G is (k,l)-connected then $\kappa(G) \geq 2$.*

Proof. Let $G = (V, E)$. As G is (k,l)-connected, $G = G[E]$ and $|E| \geq 2$. As $|E| \geq 2$, $|V| \geq 3$. Take $u \in V$ and let $H_1, \ldots, H_n$ be the components of $G[V \setminus \{u\}] = H = (W, F)$. As $|V| \geq 3$, $|W| \geq 2$. For all $1 \leq i \leq n$ let $H_i = (W_i, F_i)$. Then $W = \bigcup_{i=1}^n W_i$ and $F = \bigcup_{i=1}^n F_i$. As each H_i is a component of H we have that for all $1 \leq i < j \leq n$, $W_i \cap W_j = \emptyset = F_i \cap F_j$. As G is (k,l)-connected, Lemma 2.1.0.10 implies that $\delta(G) \geq k+1 \geq 2$. So for all $1 \leq i \leq n$, $\emptyset \neq F_i \subsetneq E$. If $|F| = 1$ then it follows that $n = 1$ and so H is connected.

Suppose instead that $|F| \geq 2$. Take $\{e, f\} \subseteq F$ and let $G[C]$ be a (k,l)-circuit such that $\{e, f\} \subseteq C$, which exists since $\mathcal{M}_{(k,l)}(G)$ is connected. As $l \geq k$, Lemma 2.1.0.13 implies that $\kappa(G[C]) \geq 2$. So, as $H = G[V \setminus \{u\}]$ and $C \cap F \neq \emptyset$, $\kappa(H[C \cap F]) \geq 1$ and so $H[C \cap F]$ is connected. Therefore there exists $1 \leq i \leq n$ such that $\{e, f\} \subseteq F_i$. Note that e and f were chosen arbitrarily, so for all $\{e, f\} \subseteq F$ there exists $1 \leq i \leq n$ such that $\{e, f\} \subseteq F_i$. As, for all $1 \leq i < j \leq n$, $F_i \neq \emptyset = F_i \cap F_j$ this implies that $n = 1$ and so H is connected. Lemma 2.1.0.12 implies that $\kappa(G) \geq 1$ and so for all $U \subsetneq V$ such that $|U| \leq 1$, $|V \setminus U| \geq 2$ and $G[V \setminus U]$ is connected. Therefore $\kappa(G) \geq 2$. □

Lemma 2.1.0.15. *Let G be a graph, take $k \in \mathbb{N}^+$, and take $l \in \mathbb{Z}$ such that $l \leq 2k-1$. If $l \geq k$ and G is a (k,l)-circuit then $\kappa_1(G) \geq k+1$.*

Proof. Let $G = (V, E)$. Lemma 2.1.0.7 implies that $3 \leq |E| = k|V| - (l-1)$ and for all $\emptyset \neq F \subsetneq E$, $|F| \leq k|V(G[F])| - l$. As $|E| \geq 3$, $|V| \geq 3$. Lemma 2.1.0.9 implies that $\delta(G) \geq k+1$ and hence $|E| \geq k+1$. As $k \in \mathbb{N}^+$ and $|E| \geq k+1$, we can take $F \subsetneq E$ such that $1 \leq |F| \leq k$. Let $G_1, \ldots, G_n$ be the components of $G[E \setminus F]$ and for all $1 \leq i \leq n$ let $G_i = (V_i, E_i)$. As $|F| \leq k$ and $\delta(G) \geq k+1$, $V(G[E \setminus F]) = V = \bigcup_{i=1}^n V_i$. As each G_i is a component of $(V, E \setminus F)$ we have that for all $1 \leq i < j \leq n$, $V_i \cap V_j = \emptyset = E_i \cap E_j$. Therefore $|V| = \sum_{i=1}^n |V_i|$ and

$|E \setminus F| = \sum_{i=1}^{n} |E_i|$. As $\delta(G) \geq k+1$ and $1 \leq |F| \leq k$, for all $1 \leq i \leq n$ we have $\emptyset \neq E_i \subsetneq E$. As G is a (k, l)-circuit, $|E_i| \leq k|V(G[E_i])| - l = k|V_i| - l$ for all $1 \leq i \leq n$. So,

$$\begin{aligned} k|V| - (l-1) &= |F| + \sum_{i=1}^{n} |E_i| \\ &\leq |F| + \sum_{i=1}^{n} (k|V_i| - l) \\ &\leq k|V| - (nl - k). \end{aligned}$$

Therefore $-(l-1) \leq -(nl-k)$, so $k - 1 \geq l(n-1)$. As $l \geq k \in \mathbb{N}^+$, this implies that $n = 1$ and so $(V, E \setminus F)$ is connected. Lemma 2.1.0.11 implies that $\kappa(G) \geq 1$, so Theorem 1.1.1.12 implies that $\kappa_1(G) \geq 1$. So for all $F \subseteq E$ such that $|F| \leq k$, $|V| \geq 3$ and $(V, E \setminus F)$ is connected. Therefore $\kappa_1(G) \geq k+1$. □

Lemma 2.1.0.16. *Let G be a graph, take $k \in \mathbb{N}^+$, and take $l \in \mathbb{Z}$ such that $l \leq 2k-1$. If $l \geq k$ and G is (k, l)-connected then $\kappa_1(G) \geq k+1$.*

Proof. Let $G = (V, E)$. As G is (k, l)-connected, $G = G[E]$ and $|E| \geq 2$. As $|E| \geq 2$, $|V| \geq 3$. Lemma 2.1.0.10 implies that $\delta(G) \geq k+1$ and hence $|E| \geq k+1$. As $k \in \mathbb{N}^+$ and $|E| \geq k+1$, we can take $F \subsetneq E$ such that $1 \leq |F| \leq k$. Let $G_1, \ldots, G_n$ be the components of $G[E \setminus F]$ and for all $1 \leq i \leq n$ let $G_i = (V_i, E_i)$. As $|F| \leq k$ and $\delta(G) \geq k+1$, $V(G[E \setminus F]) = V = \bigcup_{i=1}^{n} V_i$. As each G_i is a component of $(V, E \setminus F)$ we have that for all $1 \leq i < j \leq n$, $V_i \cap V_j = \emptyset = E_i \cap E_j$. As $\delta(G) \geq k+1$ and $1 \leq |F| \leq k$, for all $1 \leq i \leq n$ we have $\emptyset \neq E_i \subsetneq E$. If $|E \setminus F| = 1$ then it follows that $n = 1$ and so $(V, E \setminus F)$ is connected.

Suppose instead that $|E \setminus F| \geq 2$. Take $\{e, f\} \subseteq E \setminus F$ and let $G[C]$ be a (k, l)-circuit such that $\{e, f\} \subseteq C$, which exists since $\mathcal{M}_{(k,l)}(G)$ is connected. As $l \geq k$, Lemma 2.1.0.15 implies that $\kappa_1(G[C]) \geq k+1$. So, as $C \setminus F \neq \emptyset$, $\kappa_1(G[C \setminus F]) \geq 1$ and so $G[C \setminus F]$ is connected. Therefore there exists $1 \leq i \leq n$ such that $\{e, f\} \subseteq E_i$. Note that e and f were chosen arbitrarily, so for all $\{e, f\} \subseteq E \setminus F$ there exists $1 \leq i \leq n$ such that $\{e, f\} \subseteq E_i$. As, for all $1 \leq i < j \leq n$, $E_i \neq \emptyset = E_i \cap E_j$ this implies that $n = 1$ and so $(V, E \setminus F)$ is connected. Lemma 2.1.0.12 implies that $\kappa(G) \geq 1$, so Theorem 1.1.1.12 implies that $\kappa_1(G) \geq 1$. So for all $F \subseteq E$ such that $|F| \leq k$, $|V| \geq 3$ and $(V, E \setminus F)$ is connected. Therefore $\kappa_1(G) \geq k + 1$. □

The remainder of this section considers how (k, l)-connected graphs interact with some of the graph operations introduced in Subsection 1.1.2. In particular, we consider the edge-addition and $(k, 1)$-extension operations. The final result of this section, Lemma 2.1.0.20, is of a different flavour. We use a special case of this result (Lemma 2.3.0.3) in the context of $(2, 2)$-connected graphs in order to better understand how these graphs interact with certain graph operations that we introduce in Section 2.3.

There are a couple of compelling reasons to prove Lemma 2.1.0.20, rather than a more specific result that specifically deals with $(2, 2)$-connected graphs. Firstly, it is interesting to note that this proof requires that $l \neq 2k - 1$. Secondly, as Lemma 2.1.0.14 implies that $\kappa(G) \geq 2$, and the $(3, 5)$-circuit G_3 in Figure 2.2 has $\kappa(G_3) = 2$, it is plausible that this result could prove to be useful in the study of (k, l)-connected graphs where $k \geq 3$.

Lemma 2.1.0.17. *Let G be a graph, take $k \in \mathbb{N}^+$, take $l \in \mathbb{Z}$ such that $l \leq 2k-1$, and suppose G' is an edge-addition of G. If G is (k, l)-connected then G' is (k, l)-connected.*

Proof. Let $G = (V, E)$ and let $G' = (V', E')$. As G' is an edge-addition of G there exists $e' \notin E$ such that $G' = (V, E \cup \{e'\})$. As G is (k, l)-connected, $G = G[E]$ and $|E| \geq 2$. Therefore $V' = V$, $G' = G'[E']$, and $|E'| = |E| + 1 \geq 3$. All that remains is to show that $\mathcal{M}_{(k,l)}(G')$ is connected.

Let $e' = v_1v_2$. As $G = G[E]$ we may take $\{f_1, f_2\} \subseteq E$ such that, for $i \in \{1, 2\}$, f_i is incident to v_i. As G is (k, l)-connected there exists C, a circuit of $\mathcal{M}_{(k,l)}(G)$, such that $\{f_1, f_2\} \subseteq C$. Then $G[C]$ is a (k, l)-circuit, so $|C| = k|V(G[C])| - (l - 1)$ by Lemma 2.1.0.7. Let $C' = (C \setminus \{f_2\}) \cup \{e'\}$. As $G[C]$ is a (k, l)-circuit, Lemma 2.1.0.9 implies that $\delta(G[C]) \geq k + 1 \geq 2$ and hence $V(G'[C']) = V(G[C])$ and $|C'| = |C|$. Therefore $|C'| = |C| = k|V(G[C])| - (l - 1) = k|V(G'[C'])| - (l - 1)$, so C' is a dependent set of $\mathcal{M}_{(k,l)}(G')$ and hence there exists $C'' \subseteq C'$ such that C'' is a circuit of $\mathcal{M}_{(k,l)}(G')$.

If $e' \notin C''$ then $C'' \subseteq C' \setminus \{e'\} \subsetneq C$, which contradicts Lemma 1.2.0.3. Hence $e' \in C''$ and $G'[C'']$ is a (k, l)-circuit. Lemma 2.1.0.7 implies that $|C''| \geq 3$, so we may take $f \in C'' \setminus \{e'\}$. Then $f \in E$ and so, as G is (k, l)-connected, for all $e \in E \setminus \{f\}$ there exists $\tilde{C}$, a circuit of $\mathcal{M}_{(k,l)}(G)$, such that $\{e, f\} \subseteq \tilde{C}$. As G is a subgraph of G', $\tilde{C}$ is also a circuit of $\mathcal{M}_{(k,l)}(G')$. Therefore, for all $e \in E' \setminus \{f\}$ there exists a circuit of $\mathcal{M}_{(k,l)}(G')$ containing e and f. Lemma 1.2.0.8 implies G' is (k, l)-connected. □

Lemma 2.1.0.18. *Let G be a graph, take $k \in \mathbb{N}^+$, take $l \in \mathbb{Z}$ such that $l \leq 2k - 1$, and suppose G' is a $(k, 1)$-extension of G. If G is a (k, l)-circuit then G' is a (k, l)-circuit.*

Proof. Let $G = (V, E)$ and let $G' = (V', E')$. Set $V' \setminus V = \{v\}$, $E \setminus E' = \{u_1 u_{k+1}\}$, and $E' \setminus E = \{vu_1, \ldots, vu_{k+1}\}$. Then $N_{G'}(v) = \{u_1, \ldots, u_{k+1}\}$. Lemma 2.1.0.7 implies that $3 \leq |E| = k|V| - (l - 1)$ and for all $\emptyset \neq F \subsetneq E$, $|F| \leq k|V(G[F])| - l$. We note that $|E'| = |E| + k \geq 3 + k \geq 4$.

Take $\emptyset \neq F' \subseteq E'$. If $F' \subseteq E$ then $F' \subsetneq E$ and $V(G[F]) = V(G'[F])$. Hence $|F'| \leq k|V(G[F])| - l = k|V(G'[F])| - l$. If $F' \subseteq \{vu_1, \ldots, vu_{k+1}\}$ then $|F'| = |V(G'[F'])| - 1$. So, as $k \in \mathbb{N}^+$ and $|V(G'[F'])| \geq 2$,

$$l \leq 2k - 1 = 2(k - 1) + 1 \leq (k - 1)|V(G'[F'])| + 1 = k|V(G'[F'])| - |F'|.$$

Therefore $|F'| \leq k|V(G'[F'])| - l$.

So we may suppose that $F' \cap E \neq \emptyset \neq F' \cap \{vu_1, \ldots, vu_{k+1}\}$. Note that we took $\emptyset \neq F' \subseteq E'$, so we are including the case $F' = E'$. Let $F' \cap E = F$ and let $F' \cap \{vu_1 \ldots, vu_{k+1}\} = F''$, so $F' = F \cup F''$ and $F \cap F'' = \emptyset$. Then

$$|F'| = |F| + |F''| \text{ and } V(G'[F']) = V(G'[F \cup F'']) = V(G[F]) \cup V(G'[F'']),$$

so $|V(G'[F'])| = |V(G[F])| + |V(G'[F''])| - |V(G[F]) \cap V(G'[F''])|$.

As $\emptyset \neq F'' = F' \cap \{vu_1, \ldots, vu_{k+1}\}$, we have that $|V(G[F]) \cap V(G'[F''])| + 1 =$

$|V(G'[F''])| = |F''| + 1 \leq k + 2$. Therefore, as $k \in \mathbb{N}^+$,

$$\begin{aligned}1 &\geq |V(G'[F''])| - (k+1)\\ &= (|V(G[F]) \cap V(G'[F''])| - 1) - (k-1)\\ &= |V(G[F]) \cap V(G'[F''])| - k\\ &= (|V(G[F]) \cap V(G'[F''])| - |V(G'[F''])|) + (|V(G'[F''])| - k)\\ &= k(|V(G[F]) \cap V(G'[F''])| - |V(G'[F''])|) + |V(G'[F''])| - 1.\end{aligned}$$

As $u_1u_{k+1} \notin E'$ it follows that $F \subsetneq E$ and hence there exists $a \geq l$ such that $|F| = k|V(G[F])| - a$. Combining all of this information we see that

$$\begin{aligned}|F'| = |F| + |F''| &= (k|V(G[F])| - a) + (|V(G'[F''])| - 1)\\ &= k(|V(G[F])| + |V(G'[F''])|) - (a + 1 + (k-1)|V(G'[F''])|)\\ &= k(|V(G'[F'])| + |V(G[F]) \cap V(G'[F''])|) - (k-1)|V(G'[F''])|\\ &\quad - (a+1)\\ &\leq (k|V(G'[F'])| - a) + 1\\ &\leq k|V(G'[F'])| - (l-1).\end{aligned}$$

Therefore, $|F'| = k|V(G'[F'])| - (l-1)$ if and only if $|V(G'[F''])| = k+2$ and $|F| = k|V(G[F])| - l$. As G is a (k,l)-circuit it follows that $|F'| = k|V(G'[F'])| - (l-1)$ if and only if $F' = E'$. So $4 \leq |E'| = k|V'| - (l-1)$ and for all $\emptyset \neq F' \subsetneq E'$, $|F'| \leq k|V(G'[F'])| - l$. Lemma 2.1.0.7 implies that G' is a (k,l)-circuit. □

Lemma 2.1.0.19. *Let G be a graph, take $k \in \mathbb{N}^+$, take $l \in \mathbb{Z}$ such that $l \leq 2k-1$, and suppose G' is a $(k,1)$-extension of G. If $k \in \{1,2\}$ and G is (k,l)-connected then G' is (k,l)-connected.*

Proof. Let $G = (V,E)$ and let $G' = (V',E')$. As G' is a $(k,1)$-extension of a (k,l)-connected graph, $G' = G'[E']$ and $|E'| = |E| + k \geq 3$. All that remains is to show that $\mathcal{M}_{(k,l)}(G')$ is connected. Set $V' \setminus V = \{v\}$, $E \setminus E' = \{u_1u_{k+1}\}$, and $E' \setminus E =$

$\{vu_1, \ldots, vu_{k+1}\}$. Then $N_{G'}(v) = \{u_1, \ldots, u_{k+1}\}$. We consider the cases $k = 1$ and $k = 2$ separately.

Case 1: Suppose $k = 1$, so $N_{G'}(v) = \{u_1, u_2\}$ and $E \setminus E' = \{u_1u_2\}$. As $|E| \geq 2$ we may take $e \in E \cap E'$. As G is $(1, l)$-connected there exists C_1, a circuit of $\mathcal{M}_{(1,l)}(G)$, such that $\{e, u_1u_2\} \subseteq C_1$. $G[C_1]$ is a $(1, l)$-circuit. Lemma 2.1.0.18 implies that the $(1, 1)$-extension of $G[C_1]$ adding v and deleting u_1u_2 is a $(1, l)$-circuit. This $(1, 1)$-extension of $G[C_1]$ is a subgraph of G' so there exists a circuit of $\mathcal{M}_{(1,l)}(G')$ containing e, vu_1, and vu_2.

Note that as $G[C_1]$ is a $(1, l)$-circuit, $|E| \geq |C_1| \geq 3$ and hence $|E \cap E'| \geq 2$. Take $f \in (E \cap E') \setminus \{e\}$. As G is $(1, l)$-connected there exists C_2, a circuit of $\mathcal{M}_{(1,l)}(G)$, such that $\{e, f\} \subseteq C_2$. $G[C_2]$ is a $(1, l)$-circuit. If $u_1u_2 \notin C_2$ then C_2 is a circuit of $\mathcal{M}_{(1,l)}(G')$ containing e and f. If $u_1u_2 \in C_2$ then Lemma 2.1.0.18 implies that the $(1, 1)$-extension of $G[C_2]$ adding v and deleting u_1u_2 is a $(1, l)$-circuit. This $(1, 1)$-extension of $G[C_2]$ is a subgraph of G' so there exists a circuit of $\mathcal{M}_{(1,l)}(G')$ containing e and f. Therefore, for all $f' \in E' \setminus \{e\}$ there exists C', a circuit of $\mathcal{M}_{(1,l)}(G')$, such that $\{e, f'\} \subseteq C'$. Lemma 1.2.0.8 implies that G' is $\mathcal{M}_{(1,l)}$-connected.

Case 2: Suppose $k = 2$, so $N_{G'}(v) = \{u_1, u_2, u_3\}$ and $E \setminus E' = \{u_1u_3\}$. As $G = G[E]$ we may take $e \in E \cap E'$ such that e is incident to u_2. As G is $(2, l)$-connected there exists C_1, a circuit of $\mathcal{M}_{(2,l)}(G)$, such that $\{e, u_1u_3\} \subseteq C_1$. $G[C_1]$ is a $(2, l)$-circuit. Lemma 2.1.0.18 implies that the $(2, 1)$-extension of $G[C_1]$ adding v and deleting u_1u_3 is a $(2, l)$-circuit. This $(2, 1)$-extension of $G[C_1]$ is a subgraph of G' so there exists a circuit of $\mathcal{M}_{(2,l)}(G')$ containing e, vu_1, vu_2, and vu_3.

Note that as $G[C_1]$ is a $(2, l)$-circuit, $|E| \geq |C_1| \geq 3$ and hence $|E \cap E'| \geq 2$. Take $f \in (E \cap E') \setminus \{e\}$. As G is $(2, l)$-connected there exists C_2, a circuit of $\mathcal{M}_{(2,l)}(G)$, such that $\{e, f\} \subseteq C_2$. $G[C_2]$ is a $(2, l)$-circuit. If $u_1u_3 \notin C_2$ then C_2 is a circuit of $\mathcal{M}_{(2,l)}(G')$ containing e and f. If $u_1u_3 \in C_2$ then Lemma 2.1.0.18 implies that the $(2, 1)$-extension of $G[C_2]$ adding v and deleting u_1u_3 is a $(2, l)$-circuit. This $(2, 1)$-extension of $G[C_2]$ is a subgraph of G' so there exists a circuit of $\mathcal{M}_{(2,l)}(G')$ containing e and f. Therefore, for all $f' \in E' \setminus \{e\}$ there exists C', a circuit of $\mathcal{M}_{(2,l)}(G')$, such that $\{e, f'\} \subseteq C'$. Lemma

1.2.0.8 implies that G' is $\mathcal{M}_{(2,l)}$-connected. □

Remark 9. In the previous result we had to considerably restrict which values of k we consider. The reason for doing this is suggested in the second sentence of case 2. When $k \leq 1$ we are able to choose $e \in E \cap E'$ in such a way that the existence of C_1, a circuit of $\mathcal{M}_{(k,l)}(G)$, such that $\{e, u_1u_3\} \subseteq C_1$ implies that $G[C_1]$ is a (k,l)-circuit and $N_G(v) \subseteq V(G[C_1])$. However, when $k \geq 3$ it is no longer possible, a priori, to guarantee that there exists a (k,l)-circuit (V', E') such that $N_G(v) \subseteq V'$. If no such (k,l)-circuit exists then we are unable to consider the relevant $(k,1)$-extension of a circuit and so the rest of the proof technique breaks down. In other words, if it can be shown that such a circuit must exist for values of $k \geq 3$ then the previous result could be extended to include these values of k.

Lemma 2.1.0.20. *Let $G = (V,E)$ be a graph, take $k \in \mathbb{N}^+$, and take $l \in \mathbb{Z}$. Suppose there exists a 2-vertex-separation (G_1, G_2) of G, where $G_i = (V_i, E_i)$ for $i \in \{1,2\}$, and $E_1 \cap E_2 \neq \emptyset$. If $k \leq l \leq 2k-2$ and G is (k,l)-connected then for all $e_1 \in E_1 \setminus E_2$ and all $e_2 \in E_2 \setminus E_1$ there exists C', a circuit of $\mathcal{M}_{(k,l)}(G)$, such that $(E_1 \cap E_2) \cup \{e_1, e_2\} \subseteq C'$.*

Proof. As $|V_1 \cap V_2| = 2$ and $E_1 \cap E_2 \neq \emptyset$, $|E_1 \cap E_2| = 1$. Let $E_1 \cap E_2 = \{f\}$, take $e_1 \in E_1 \setminus E_2$, and take $e_2 \in E_2 \setminus E_1$. As G is (k,l)-connected there exists C, a circuit of $\mathcal{M}_{(k,l)}(G)$, such that $\{e_1, e_2\} \subseteq C$. $G[C]$ is a (k,l)-circuit. If $f \in C$ then we are done, so suppose instead that $f \notin C$.

As $k \leq l \leq 2k-1$, Lemma 2.1.0.13 implies that $\kappa(G[C]) \geq 2$ and hence $V_1 \cap V_2 \subseteq V(G[C])$. Let $G' = G[C \cup \{f\}]$. As $V_1 \cap V_2 \subseteq V(G[C])$ and $f \notin C$, G' is the edge-addition of $G[C]$ adding f. We note that, as $G[C]$ is a subgraph of G', C is also a circuit of $\mathcal{M}_{(k,l)}(G')$. Lemma 2.1.0.8 and Lemma 2.1.0.17 together imply that G' is (k,l)-connected. Therefore there exist C_1, C_2, circuits of $\mathcal{M}_{(k,l)}(G')$, such that $\{e_i, f\} \subseteq C_i$ for $i \in \{1,2\}$. If $e_2 \in C_1$ or $e_1 \in C_2$ then we are done, so suppose instead that $e_2 \notin C_1$ and $e_1 \notin C_2$.

As G' is (k,l)-connected there exists C_1', a circuit of $\mathcal{M}_{(k,l)}(G')$, such that $f \in C_1'$. If $C_1' \neq C_1$ then Lemma 1.2.0.3 (C3) (the weak circuit exchange axiom) implies there exists D, a circuit of $\mathcal{M}_{(k,l)}(G')$, such that $D \subseteq (C_1 \cup C_1') \setminus \{f\} \subseteq C$. Lemma 1.2.0.3 (C2) implies that $D = (C_1 \cup C_1') \setminus \{f\} = C$, so $e_2 \in D$ and hence $e_2 \in C_1'$. That is, $C_1' = C_1$ or $e_2 \in C_1'$. Therefore C_1 is the unique circuit of $\mathcal{M}_{(k,l)}(G')$ containing f and not containing e_2. Similarly, C_2 is the unique circuit of $\mathcal{M}_{(k,l)}(G')$ containing f and not containing e_1. Therefore $C_1 \neq C_2$ and $f \in C_1 \cap C_2$. Lemma 1.2.0.3 (C3) implies there exists $\tilde{C}$, a circuit of $\mathcal{M}_{(k,l)}(G')$, such that $\tilde{C} \subseteq (C_1 \cup C_2) \setminus \{f\} \subseteq C$. Then $\tilde{C} = (C_1 \cup C_2) \setminus \{f\} = C$ by Lemma 1.2.0.3 (C2). Consequently,

$$|C_1 \cup C_2| = |C| + 1 = (k|V(G[C])| - (l-1)) + 1 = k|V(G[C])| - (l-2).$$

If $C_1 \cap C_2 = \{f\}$ then

$$\begin{aligned} k|V(G[C])| - (l-2) &= |C_1 \cup C_2| \\ &= |C_1| + |C_2| - |C_1 \cap C_2| \\ &= (k|V(G'[C_1])| - (l-1)) + (k|V(G'[C_2])| - (l-1)) - 1 \\ &= k|V(G[C])| + A - (2l-1), \end{aligned}$$

where $A = k(|V(G'[C_1])| + |V(G'[C_2])| - |V(G[C])|)$. Therefore

$$k(|V(G'[C_1])| + |V(G'[C_2])| - |V(G[C])|) = (l+1).$$

However, as $k \leq l \leq 2k-2$ we observe that $l+1$ is not an integer multiple of k and so we have a contradiction. Hence $(C_1 \cap C_2) \setminus \{f\} \neq \emptyset$.

Take $e \in (C_1 \cap C_2) \setminus \{f\} \subseteq C$. Lemma 1.2.0.3 (C3) implies there exists C', a circuit of $\mathcal{M}_{(k,l)}(G')$, such that $C' \subseteq (C \cup C_1) \setminus \{e\}$. Lemma 1.2.0.3 (C2) implies that $C' \nsubseteq C$. As $e \notin C'$, $C' \neq C, C_1, C_2$. Hence, as $C' \subseteq C \cup \{f\}$, $f \in C'$. Now the uniqueness of C_1 and C_2 implies that $\{e_1, e_2, f\} \subseteq C'$. As $G' = G[C \cup \{f\}]$ is a subgraph of G, C' is a circuit of $\mathcal{M}_{(k,l)}(G)$ such that $(E_1 \cap E_2) \cup \{e_1, e_2\} \subseteq C'$. □

2.2 $(k, 2k-1)$-Connected Graphs

This section acts as a brief aside before proceeding with the bulk of the chapter where our attention turns to $(2, 2)$-connected graphs. The overall aim of this chapter is to extend earlier work on $(2, 2)$-circuits (see [30, 24]) to the setting of $(2, 2)$-connected graphs. This is broadly the same process as employed by Jackson and Jordán [22] who extended Berg and Jordán's [2] study of (3-vertex-connected) $(2, 3)$-circuits in order to characterise (3-vertex-connected) $(2, 3)$-connected graphs. The content of this section is based on the observation that various results from these two papers can be shown to hold true for $(k, 2k-1)$-circuits for arbitrary $k \in \mathbb{N}^+$.

In earlier work on $(2, 3)$-circuits and $(2, 3)$-connected graphs, particular attention was paid to those graphs that were 3-vertex-connected. It should be noted that alongside considering those graphs with sufficient vertex-connectivity, characterisations of $(2, 3)$-circuits and $(2, 3)$-connected graphs, with no additional vertex-connectivity requirements, were provided in [2] and [22] respectively. These characterisations without the requirement of 3-vertex-connectivity laid the foundation for the characterisations with the requirement of 3-vertex-connectivity.[1] Note that even when $k \geq 3$ there may exist a $(k, 2k-1)$-connected graph, G, such that $\kappa(G) = 2$ (see Figure 2.1). Therefore, it is reasonable to suppose that extending [2, Theorem 4.4] and [22, Corollary 5.9] to consider $(k, 2k-1)$-circuits or $(k, 2k-1)$-connected graphs, for arbitrary $k \in \mathbb{N}^+$, could be an important first step in characterising $(k, 2k-1)$-connected graphs.

Lemma 2.2.0.1. *Let $G = (V, E)$ be a graph and take $k \in \mathbb{N}^+$. Suppose there exists a 2-vertex-separation (G_1, G_2) of G, where $G_i = (V_i, E_i)$ for $i \in \{1, 2\}$. If G is a $(k, 2k-1)$-circuit then $E_1 \cap E_2 = \emptyset$ and $|E_i| = i_G(V_i) = k|V_i| - (2k-1)$ for $i \in \{1, 2\}$.*

[1] Berg and Jordán comment on the fact that their characterisation without a vertex-connectivity requirement is not used in their characterisation of 3-vertex-connectivity $(2, 3)$-circuits [2, p.88]. However, graph operations specifically related to $(2, 3)$-circuits that are not 3-vertex-connected (see Figure 2.3) are crucial to their characterisation of 3-vertex-connected $(2, 3)$-circuits.

Proof. As G is a $(k, 2k-1)$-circuit, $|E| = k|V| - (2k-2)$. Therefore,

$$\begin{aligned} k|V| - (2k-2) &= |E| \\ &= |E_1| + |E_2| - |E_1 \cap E_2| \\ &\leq (k|V_1| - (2k-1)) + (k|V_2| - (2k-1)) \\ &= k|V| + k|V_1 \cap V_2| + 2 - 4k \\ &= k|V| - (2k-2). \end{aligned}$$

Hence $|E_1| + |E_2| - |E_1 \cap E_2| = (k|V_1| - (2k-1)) + (k|V_2| - (2k-1))$, which implies that $E_1 \cap E_2 = \emptyset$ and $|E_i| = i_G(V_i) = k|V_i| - (2k-1)$ for $i \in \{1, 2\}$. □

Lemma 2.2.0.2. *Let $G = (V, E)$ be a graph and take $k \in \mathbb{N}^+$. Suppose there exists a 2-vertex-separation (G_1, G_2) of G, where $G_i = (V_i, E_i)$ for $i \in \{1, 2\}$, and $E_1 \cap E_2 \neq \emptyset$. G is $(k, 2k-1)$-connected if and only if $G[E \setminus (E_1 \cap E_2)]$ is $(k, 2k-1)$-connected.*

Proof. As $|V_1 \cap V_2| = 2$ and $E_1 \cap E_2 \neq \emptyset$, $|E_1 \cap E_2| = 1$. Let $E_1 \cap E_2 = \{f\}$ and let $G' = G[E \setminus \{f\}]$. If G' is $(k, 2k-1)$-connected then G is the edge-addition of G' adding f and so Lemma 2.1.0.17 implies G is $(k, 2k-1)$-connected. On the other hand, suppose that G is $(k, 2k-1)$-connected and take $e \in E_1 \setminus \{f\}$. Take $f' \in E \setminus \{e, f\} = E(G') \setminus \{e\}$. As G is $(k, 2k-1)$-connected there exists C, a circuit of $\mathcal{M}_{(k,2k-1)}(G)$, such that $\{e, f'\} \in C$. $G[C]$ is a $(k, 2k-1)$-circuit. Now, $f' \in E_1 \setminus E_2$ or $f' \in E_2 \setminus E_1$.

If $f' \in E_2 \setminus E_1$ then Lemma 2.1.0.13 implies that $V_1 \cap V_2 \subseteq V(G[C])$ and Lemma 2.2.0.1 implies that $f \notin C$. Therefore C is a circuit of $\mathcal{M}_{(k,2k-1)}(G')$. If $f' \in E_1 \setminus E_2$ then take $f'' \in E_2 \setminus E_1$. As G is $(k, 2k-1)$-connected there exist C_1, C_2, circuits of $\mathcal{M}_{(k,2k-1)}(G)$, such that $\{e, f''\} \subseteq C_1$ and $\{f', f''\} \subseteq C_2$. As above, $f \notin C_1, C_2$ and so C_1 and C_2 are circuits of $\mathcal{M}_{(k,2k-1)}(G')$. Lemma 1.2.0.7 implies that there exists C', a circuit of $\mathcal{M}_{(k,2k-1)}(G')$, such that $\{e, f'\} \subseteq C'$.

Therefore, for all $f' \in E(G') \setminus \{e\}$ there exists a $(k, 2k-1)$-circuit of $\mathcal{M}_{(k,2k-1)}(G')$ containing e and f'. Lemma 1.2.0.8 implies that $G' = G[E \setminus (E_1 \cap E_2)]$ is $(k, 2k-1)$-

connected. □

The following graph operations (see Figure 2.3) were introduced by Berg and Jordán in [2], where it was shown that they preserve the property of being a $(2,3)$-circuit. This fact was extended by Jackson and Jordán in [22], who show that these operations also preserve the property of being $(2,3)$-connected. In Section 2.3 we introduce similar operations and consider how they interact with $(2,2)$-circuits and $(2,2)$-connected graphs. Beforehand, we generalise the aforementioned results in [2, 22] and show that the operations of Berg and Jordán interact nicely with $(k,2k-1)$-circuits and $(k,2k-1)$-connected graphs for arbitrary $k \in \mathbb{N}^+$. Note that we refer to one of these operations as a '2-cleave' rather than a '2-separation' in order to avoid confusing this operation with one of the operations introduced in Section 2.3.

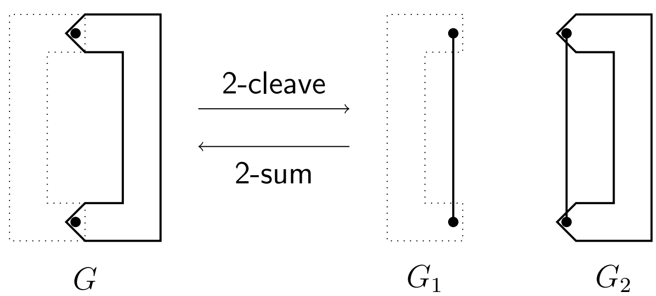


Figure 2.3: Illustration of the 2-cleave and 2-sum operations. (G_1, G_2) is the 2-cleave of G on $(G[V(G_1)], G[V(G_2)])$ and G is the 2-sum of (G_1, G_2).

Definition 2.2.0.3. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that there exists a proper subgraph $(\{u,v\}, \{uv\})$ of both G_1 and G_2, and $V_1 \cap V_2 = \{u,v\}$. The **2-sum** of the ordered pair (G_1, G_2) is the graph $G = (V_1 \cup V_2, (E_1 \cup E_2) \setminus \{uv\})$.

Let G be a graph such that there exists a 2-vertex-separation (H_1, H_2) of G, where $H_i = (U_i, F_i)$ for $i \in \{1,2\}$, and $F_1 \cap F_2 = \emptyset$. A **2-cleave** of G is an ordered pair (G_1, G_2) such that G is the 2-sum of (G_1, G_2). If (G_1, G_2) is a 2-cleave of G, then $(G[V(G_1)], G[V(G_2)])$ is a 2-vertex-separation of G and (G_1, G_2) is the 2-cleave of G on $(G[V(G_1)], G[V(G_2)])$.

Lemma 2.2.0.4. *Let G_1 and G_2 be graphs, take $k \in \mathbb{N}^+$, and suppose G is the 2-sum of (G_1, G_2). If G_1 and G_2 are $(k, 2k-1)$-circuits then G is a $(k, 2k-1)$-circuit.*

Proof. Let $G = (V, E)$, $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and let $E_1 \cap E_2 = \{uv\}$. As G is the 2-sum of (G_1, G_2), $E = (E_1 \cup E_2) \setminus \{uv\}$. As G_1 and G_2 are (k, l)-circuits, $|E_1|, |E_2| \geq 3$ and so $|E| = (|E_1| - 1) + (|E_2| - 1) \geq 4$. Take $\emptyset \neq F \subseteq E$. We can set $F = F_1 \cup F_2$ where $F_i = F \cap E_i$. Since $uv \notin E$, $F_i \subsetneq E_i$ for $i \in \{1, 2\}$. If there exists $i \in \{1, 2\}$ such that $F_i = \emptyset$ then for $j \in \{1, 2\}$ such that $j \neq i$ we have $|F| = |F_j| \leq k|V(G_j[F_j])| - (2k-1) = k|V(G[F_j])| - (2k-1)$.

On the other hand, suppose that $F_1 \neq \emptyset \neq F_2$. Then, as $F_1 \cap F_2 = \emptyset$,

$$\begin{aligned}
|F| &= |F_1| + |F_2| \\
&\leq (k|V(G_1[F_1])| - (2k-1)) + (k|V(G_2[F_2])| - (2k-1)) \\
&= k(|V(G[F_1])| + |V(G[F_2])|) - 2(2k-1) \\
&= k|V(G[F])| + k|V(G[F_1]) \cap V(G[F_2])| - (4k-2) \\
&\leq k|V(G[F])| - (2k-2).
\end{aligned}$$

So, for all $\emptyset \neq F \subseteq E$ we have $|F| \leq k|V(G[F])| - (2k-2)$. Moreover, $|F| = k|V(G[F])| - (2k-2)$ if and only if $|F_1| = k|V(G_1[F_1])| - (2k-1)$ and $|F_2| = k|V(G_2[F_2])| - (2k-1)$ and $|V(G[F_1]) \cap V(G[F_2])| = 2$. As G_1 and G_2 are $(k, 2k-1)$-circuits, it follows that $|F| = k|V(G[F])| - (2k-2)$ if and only if $F_i = E_i \setminus \{uv\}$ for $i \in \{1, 2\}$ if and only if $F = E$. Therefore, $3 \leq |E| = k|V| - (2k-2)$ and for all $\emptyset \neq F \subsetneq E$, $|F| \leq k|V(G[F])| - (2k-1)$. Lemma 2.1.0.7 implies that G is a $(k, 2k-1)$-circuit. □

Lemma 2.2.0.5. *Let G be a graph, take $k \in \mathbb{N}^+$, and suppose that (G_1, G_2) is a 2-cleave of G. If G is a $(k, 2k-1)$-circuit then G_1 and G_2 are $(k, 2k-1)$-circuits.*

Proof. Let $G = (V, E)$, $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$. Set $E_1 \cap E_2 = \{uv\}$. As G is a $(k, 2k-1)$-circuit, Lemma 2.1.0.9 implies that $\delta(G) \geq k+1$ and hence, for $i \in \{1, 2\}$, $|E_i| \geq (k+1) + 1 \geq 3$. Now, take $i \in \{1, 2\}$ and take $\emptyset \neq F \subseteq E_i$. If

$uv \notin F$ then $F \subsetneq E$ and hence $|F| \leq k|V(G[F])| - (2k-1) = k|V(G_i[F])| - (2k-1)$. If $F = \{uv\}$ then $1 = |F| = 2k - (2k-1) = k|V(G_i[F])| - (2k-1)$.

Alternatively, suppose that $\{uv\} \subsetneq F$. Then,

$$\begin{aligned} |F| &= |F \setminus \{uv\}| + 1 \\ &\leq (k|V(G[F \setminus \{uv\}])| - (2k-1)) + 1 \\ &= (k|V(G_i[F])| - (2k-2)) - k|\{u,v\} \setminus V(G[F \setminus \{uv\}])| \\ &\leq k|V(G_i[F])| - (2k-2). \end{aligned}$$

So, for all $\emptyset \neq F \subseteq F_i$ we have $|F| \leq k|V(G_i[F])| - (2k-2)$. Moreover, $|F| = k|V(G_i[F])| - (2k-2)$ if and only if $\{uv\} \subsetneq F$ and $|F \setminus \{uv\}| = k|V(G[F \setminus \{uv\}])| - (2k-1)$ and $|\{u,v\} \setminus V(G[F \setminus \{uv\}])| = 0$. As G is a $(k, 2k-1)$-circuit, it follows that $|F| = k|V(G_i[F])| - (2k-2)$ if and only if $F \setminus \{uv\} = E \cap E_i$ if and only if $F = E_i$. Therefore, for $i \in \{1,2\}$, $3 \leq |E_i| = k|V_i| - (2k-2)$ and for all $\emptyset \neq F \subsetneq E_i$, $|F| \leq k|V(G_i[F])| - (2k-1)$. Lemma 2.1.0.7 implies that G_1 and G_2 are $(k, 2k-1)$-circuits. □

Before considering how these operations interact with $(k, 2k-1)$-connected graphs we first note that Lemma 2.1.0.13 implies that if G is a $(k, 2k-1)$-circuit then $\kappa(G) \geq 2$. Figure 2.1 (specifically G_3) illustrates that if $k = 3$ then this lower bound is sharp. We now show that there is nothing special, in this regard, that follows from demanding $k = 3$.

Proposition 2.2.0.6. *For all $k \in \mathbb{N}^+$ there exists a $(k, 2k-1)$-circuit G such that $\kappa(G) = 2$.*

Proof. Let $G' = (V', E')$ be a complete graph such that $|V'| \geq \max\{4, 2k\}$. We observe that

$$\begin{aligned} k|V'| - (2k-2) \leq \binom{|V'|}{2} &\iff |V'|^2 - (2k+1)|V'| + (4k-4) \geq 0 \\ &\iff (|V'| - 2)(|V'| - (2k-1)) \geq 2. \end{aligned}$$

Hence $|E'| \geq k|V'| - (2k-2)$. Therefore E' is a dependent set in $\mathcal{M}_{(k,2k-1)}(G')$, and hence there exists $F' \subseteq E'$ such that F' is a circuit of $\mathcal{M}_{(k,2k-1)}(G')$. Let $H' = G'[F']$ and let H'' be a graph isomorphic to H' such that $|V(H') \cap V(H'')| = 2$ and $|E(H') \cap E(H'')| = 1$. Note that H' and H'' are $(k, 2k-1)$-circuits. Let H be the 2-sum of (H', H''), so $\kappa(H) = 2$. Lemma 2.2.0.4 implies that H is a $(k, 2k-1)$-circuit. □

Lemma 2.2.0.7. *Let G_1 and G_2 be graphs, take $k \in \mathbb{N}^+$, and suppose G is the 2-sum of (G_1, G_2). If G_1 and G_2 are $(k, 2k-1)$-connected then G is $(k, 2k-1)$-connected.*

Proof. Let $G = (V, E)$, $G_1 = (V_1, E_1)$, and let $G_2 = (V_2, E_2)$. As G_1 and G_2 are $(k, 2k-1)$-connected, $G_1 = G_1[E_1]$ and $G_2 = G_2[E_2]$ and $|E_1|, |E_2| \geq 2$. By the definition of 2-sum it follows that $G = G[E]$ and $|E| \geq 2$. All that remains is to show that $\mathcal{M}_{(k,2k-1)}(G)$ is connected. Let $E_1 \cap E_2 = \{e'\}$ and take $e \in E \cap E_1 = E_1 \setminus \{e'\}$. Take $f \in E \setminus \{e\}$.

If $f \in E_1$ then as G_1 is $(k, 2k-1)$-connected there exists C_1, a circuit of $\mathcal{M}_{(k,2k-1)}(G_1)$, such that $\{e, f\} \subseteq C_1$. If $e' \notin C_1$ then C_1 is a circuit of $\mathcal{M}_{(k,2k-1)}(G)$. Alternatively, if $e' \in C_1$ then since G_2 is $(k, 2k-1)$-connected there exists C_2, a circuit of $\mathcal{M}_{(k,2k-1)}(G_2)$, such that $\{e'\} \subsetneq C_2$. Then Lemma 2.2.0.4 implies the 2-sum of $(G_1[C_1], G_2[C_2])$, say G', is a $(k, 2k-1)$-circuit. Let $E(G') = C'$, then C' is a circuit of $\mathcal{M}_{(k,2k-1)}(G)$ such that $\{e, f\} \subseteq C'$.

If $f \notin E_1$ then $f \in E_2 \setminus \{e'\}$. As G_1 and G_2 are (k, l)-connected there exist C_1', a circuit of $\mathcal{M}_{(k,2k-1)}(G_1)$, and C_2', a circuit of $\mathcal{M}_{(k,2k-1)}(G_2)$, such that $\{e, e'\} \subseteq C_1'$ and $\{f, e'\} \subseteq C_2'$. Then Lemma 2.2.0.4 implies the 2-sum of $(G_1[C_1'], G_2[C_2'])$, say G'', is a $(k, 2k-1)$-circuit. Let $E(G'') = C''$, then C'' is a circuit of $\mathcal{M}_{(k,2k-1)}(G)$ such that $\{e, f\} \subseteq C''$. Therefore, for all $f \in E \setminus \{e\}$ there exists C, a circuit of $\mathcal{M}_{(k,2k-1)}(G)$, such that $\{e, f\} \subseteq C$ and so Lemma 1.2.0.8 implies G is $(k, 2k-1)$-connected. □

Lemma 2.2.0.8. *Let G be a graph, take $k \in \mathbb{N}^+$, and suppose (G_1, G_2) is a 2-cleave of G. If G is $(k, 2k-1)$-connected then G_1 and G_2 are $(k, 2k-1)$-connected.*

Proof. Let $G = (V, E)$, $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$. As G is $(k, 2k-1)$-connected, $G = G[E]$ and $|E| \geq 2$. By the definition of 2-cleave it follows that for $i \in \{1, 2\}$, $G_i = G_i[E_i]$ and $|E_i| \geq 2$. All that remains is to show that for $i \in \{1, 2\}$, $\mathcal{M}_{(k,2k-1)}(G_i)$ is connected. Let $E_1 \cap E_2 = \{uv\}$ and for $i \in \{1, 2\}$ take $f_i \in E_i \setminus \{uv\} = E_i \cap E$. As G is $(k, 2k-1)$-connected there exists C, a circuit of $\mathcal{M}_{(k,l)}(G)$, such that $\{f_1, f_2\} \subseteq C$. Lemma 2.1.0.13 implies that $\kappa(G[C]) \geq 2$, so $\{u, v\} \subseteq V(G[C])$, and Lemma 2.2.0.1 implies that $uv \notin C$. Hence $G[C]$ is a $(k, 2k-1)$-circuit and there exists a 2-cleave of $G[C]$.

Let (G'_1, G'_2) be the 2-cleave of $G[C]$ on $(G[C \cap E_1], G[C \cap E_2])$. Lemma 2.2.0.5 implies that G'_1 and G'_2 are both $(k, 2k-1)$-circuits. Let $E(G'_i) = C'_i$, then C'_i is a circuit of $\mathcal{M}_{(k,2k-1)}(G_i)$ such that $\{uv, f_i\} \subseteq C'_i$. Therefore, for $i \in \{1, 2\}$, for all $f'_i \in E_i \setminus \{uv\}$ there exists C_i, a circuit of $\mathcal{M}_{(k,2k-1)}(G_i)$, such that $\{uv, f'_i\} \subseteq C_i$. Lemma 1.2.0.8 implies G_1 and G_2 are $(k, 2k-1)$-connected. □

2.3 $(2, 2)$-Connected Graphs

For the remainder of this chapter we turn our attention from the general setting of (k, l)-connected graphs to the specific case of $(2, 2)$-connected graphs. We begin by examining the structure of $(2, 2)$-circuits, where many of the results are taken or extended from [30]. In particular, we consider three graph operations that were introduced by Nixon in [30] and that were shown to preserve, in some sense, the property of being a $(2, 2)$-circuit. We build on this to conclude the section, and chapter, by showing that they preserve being $(2, 2)$-connected in the same sense. Let us begin by stating some basic consequences of our earlier results on (k, l)-circuits.

Lemma 2.3.0.1. *Let G be a graph. If G is $(2, 2)$-connected then $|V(G)| \geq 5$ and $|E(G)| \geq 9$.*

Proof. As G is $(2, 2)$-connected, $|E(G)| \geq 2$ and $\mathcal{M}_{(2,2)}(G)$ is connected. Therefore, there exists $C \subseteq E(G)$ such that C is a circuit of $\mathcal{M}_{(2,2)}(G)$. That is, $G[C]$ is a $(2, 2)$-circuit. Let $V(G[C]) = V'$. Lemma 2.1.0.7 implies that $|C| = 2|V'| - 1$. Therefore

$2|V'|-1 \leq \binom{|V'|}{2}$. We observe that

$$2|V'|-1 \leq \binom{|V'|}{2} \iff |V'|^2 - 5|V'| + 2 \geq 0 \iff (|V'|-2)(|V'|-3) \geq 4.$$

Hence $|V(G)| \geq |V'| \geq 5$ and $|E(G)| \geq |C| = 2|V'|-1 \geq 9$. □

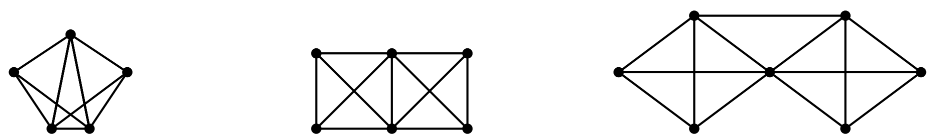


Figure 2.4: Illustration of three $(2,2)$-circuits with fewer than eight vertices. We shall regularly refer to a graph as being isomorphic to K_5^-, B_1, or B_2.

Lemma 2.3.0.2. *Let G be a graph. If G is a $(2,2)$-circuit then $\delta(G) = 3$.*

Proof. Let $G = (V,E)$. As G is a $(2,2)$-circuit, Lemma 2.1.0.7 implies that $2 \leq |E| = 2|V|-1$. Now, by Theorem 1.1.1.6 we have that $4|V|-2 \geq \delta(G)|V|$ and hence $\delta(G) \leq 3$. Lemma 2.1.0.9 implies that $\delta(G) \geq 3$ and therefore $\delta(G) = 3$. □

The following result, that we shall make repeated use of, was alluded to in the paragraphs just before Lemma 2.1.0.17. The proof is a simple application of Lemma 2.3.0.2 to Lemma 2.1.0.20 in the context of $(2,2)$-connected graphs.

Lemma 2.3.0.3. *Let $G = (V,E)$ be a graph and suppose there exists a 2-vertex-separation (G_1, G_2) of G, where $G_i = (V_i, E_i)$ for $i \in \{1,2\}$, and $E_1 \cap E_2 \neq \emptyset$. If $G_2 \cong K_4$ and G is $(2,2)$-connected then for all $e \in E_1 \setminus E_2$ there exists C', a circuit of $\mathcal{M}_{(2,2)}(G)$, such that $E_2 \cup \{e\} \subseteq C'$.*

Proof. Lemma 2.1.0.20 implies that for all $e_1 \in E_1 \setminus E_2$ and for all $e_2 \in E_2 \setminus E_1$ there exists C, a circuit of $\mathcal{M}_{(2,2)}(G)$ such that $(E_1 \cap E_2) \cup \{e_1, e_2\} \subseteq C$. Then $G[C]$ is a $(2,2)$-circuit, so Lemma 2.3.0.2 implies that $\delta(G[C]) = 3$. As $G_2 \cong K_4$, it follows that $E_2 \subseteq C$. So, for all $e \in E_1 \setminus E_2$ there exists C', a circuit of $\mathcal{M}_{(2,2)}(G)$, such that $E_2 \cup \{e\} \subseteq C'$. □

Let G be a $(2,2)$-connected graph. A set X is **critical** in G if $\emptyset \neq X \subseteq V(G)$ and $i_G(X) = 2|X| - 2$. Note that this is equivalent to saying that X is critical in G if $|X| = 1$ or $X = V(G[F])$ for some some $\emptyset \neq F \subseteq E$ such that $|F| = 2|V(G[F])| - 2$.

Lemma 2.3.0.4. *[30, Lemma 2.4] Let $G = (V, E)$ be a $(2,2)$-circuit and take $X \subseteq V$. If X is critical in G then*

(i) $G[X]$ is connected;

(ii) $\delta(G[X]) \geq 2$ if and only if $|X| > 1$; and

(iii) there exists $v \in V \setminus X$ such that $d_G(v) = 3$.

Proof. As X is a critical set $X \neq \emptyset$, so $G[X]$ is defined. We now prove each statement in turn.

(i) Firstly, let us consider $G[X]$. As G is a $(2,2)$-circuit, Lemma 2.1.0.7 implies that $2 \leq |E| = 2|V| - 1$ and for all $\emptyset \neq F \subsetneq E$, $|F| \leq 2|V(G[F])| - 2$. Let $H_1, \ldots, H_n$ be the components of $G[X] = (X, F)$ and for all $1 \leq i \leq n$ let $H_i = (X_i, F_i)$. Then $X = \bigcup_{i=1}^{n} X_i$ and $F = \bigcup_{i=1}^{n} F_i$. As each H_i is a component of $G[X]$ we have that for all $1 \leq i < j \leq n$, $X_i \cap X_j = \emptyset = F_i \cap F_j$. Therefore $|X| = \sum_{i=1}^{n} |X_i|$ and $|F| = \sum_{i=1}^{n} |F_i|$. As X is critical in G, $|F| = 2|X| - 2$ and either $F = \emptyset$ or $\emptyset \neq F \subsetneq E$. If $F = \emptyset$ then $|X| = 1$ and $G[X]$ is connected. If $\emptyset \neq F \subsetneq E$ then

$$2|X| - 2 = |F| = \sum_{i=1}^{n} |F_i| \leq \sum_{i=1}^{n} (2|X_i| - 2) = 2|X| - 2n.$$

As $n \geq 1$ this implies that $n = 1$ and hence $G[X]$ is connected.

(ii) If $\delta(G[X]) \geq 2$ then $|X| > 1$. On the other hand, suppose that $|X| > 1$. As X is critical in G we have that $4|X| - 4 = 2i_G(X) = 2|E(G[X])| = \sum_{v \in X} d_{G[X]}(v)$, where the final equality follows from Theorem 1.1.1.6. Take $v \in X$ such that

$d_{G[X]}(v) = \delta(G[X])$. As $|X| > 1$, $G[X \setminus \{v\}]$ is defined and $i_G(X) = 2|X| - 2 > 2$. So, as G is a $(2,2)$-circuit, $\emptyset \neq E(G[X \setminus \{v\}]) \subsetneq E(G)$ and hence

$$\begin{aligned} 2|X \setminus \{v\}| - 2 \geq E(G[X \setminus \{v\}])| \\ &= i_G(X) - \delta(G[X]) \\ &= 2|X| - (2 + \delta(G[X])) \\ &= 2|X \setminus \{v\}| - \delta(G[X]). \end{aligned}$$

Therefore $\delta(G[X]) \geq 2$.

(iii) Finally let us consider $V \setminus X$, and set $Y = V \setminus X$.[2] As G is a $(2,2)$-circuit and X is critical in G, X and Y are non-empty. We note that $\sum_{v \in Y} d_G(v) = \sum_{v \in Y} d_{G[Y]}(v) + d_G(Y,X) = 2i_G(Y) + d_G(Y,X)$. As G is a $(2,2)$-circuit, Lemma 2.1.0.16 implies that $\kappa_1(G) \geq 3$ and hence $d_G(Y,X) \geq 3$. Hence,

$$\begin{aligned} \sum_{v \in Y} d_G(v) &= 2i_G(Y) + d_G(Y,X) \\ &= 2(|E| - i_G(X) - d_G(Y,X)) + d_G(Y,X) \\ &= 2|E| - (2i_G(X) + d_G(Y,X)) \\ &= 4(|V| - |X|) + 2 - d_G(Y,X) \\ &= 4|Y| - (d_G(Y,X) - 2) \\ &\leq 4|Y| - 1. \end{aligned}$$

So there exists $v \in Y$ such that $d_G(v) < 4$. Lemma 2.1.0.10 implies that $\delta(G) \geq 3$, so there exists $v \in Y$ such that $d_G(v) = 3$.

□

Lemma 2.3.0.5. *[30, Lemma 2.2] Let $G = (V,E)$ be a $(2,2)$-circuit and take $X, Y \subseteq V$. If X and Y are critical in G and $\emptyset \neq X \cap Y \subseteq X \cup Y \subsetneq V$ then $X \cap Y$ and $X \cup Y$ are critical in G and $d_G(X,Y) = 0$.*

[2] This part of the proof is analogous to the proof of [2, Lemma 2.5].

Proof. [3] As X and Y are critical in G, $i_G(X) = 2|X| - 2$ and $i_G(Y) = 2|Y| - 2$. As G is a $(2,2)$-circuit, $G = G[E]$. So, as $\emptyset \neq X \cap Y \subseteq X \cup Y \subsetneq V$ it follows that $\emptyset \neq E(G[X \cap Y]) \subseteq E(G[X \cup Y]) \subsetneq E$. Therefore, as G is a $(2,2)$-circuit, $i_G(X \cap Y) \leq 2|X \cap Y| - 2$ and $i_G(X \cup Y) \leq 2|X \cup Y| - 2$.Then Lemma 1.1.1.4 implies that

$$\begin{aligned}(2|X| - 2) + (2|Y| - 2) + d(X,Y) &= i_G(X \cup Y) + i_G(X \cap Y) \\ &\leq (2|X \cap Y| - 2) + (2|X \cup Y| - 2) \\ &= (2|X| - 2) + (2|Y| - 2).\end{aligned}$$

Therefore $d(X,Y) = 0$ and $i_G(X \cap Y) + i_G(X \cap Y) = (2|X \cap Y| - 2) + (2|X \cup Y| - 2)$. Hence $i_G(X \cup Y) = 2|X \cup Y| - 2$ and $i_G(X \cap Y) = 2|X \cap Y| - 2$, so $X \cap Y$ and $X \cup Y$ are critical in G. □

Lemma 2.1.0.18 implies that a $(2,1)$-extension of a $(2,2)$-circuit is another $(2,2)$-circuit. The following result shows that $(2,1)$-reductions of $(2,2)$-circuits are less well-behaved. This is a consequence of the fact that if G' is a $(2,1)$-extension of G adding v then there may be as many as three graphs that are $(2,1)$-reductions of G' at v.

Lemma 2.3.0.6. *[30, Lemma 2.5] Let $G = (V,E)$ be a $(2,2)$-circuit and take $v \in V$ such that $d_G(v) = 3$, say $N_G(v) = \{x,y,z\}$. There does not exist a $(2,1)$-reduction of G at v adding xy that is a $(2,2)$-circuit if and only if $xy \in E$ or there exists $Z \subseteq V$ such that Z is critical in G and $Z \cap N_G[v] = \{x,y\}$.*

An important technique that we use in characterising $(2,2)$-connected graphs is to consider a subgraph of a $(2,2)$-connected graph that is a $(2,2)$-circuit, and then to investigate the local structure at the vertices of degree three in the $(2,2)$-circuit. With this in mind we introduce some additional notation and terminology.

Let $G = (V,E)$ be a graph. We denote the set of vertices of degree three in G by $V_3(G)$, and we refer to these vertices as **nodes** of G. That is, $V_3(G) := \{v \in V : d_G(v) = 3\} = \{\text{nodes of } G\}$. If G is a $(2,2)$-circuit then Lemma 2.3.0.2 tells us that $V_3(G) \neq \emptyset$. It

[3] This proof is analogous to the proof of [2, Lemma 2.3].

follows from our definition of a $(2,1)$-reduction that if there exists $v \in V_3(G)$ such that $G[N_G(v)] \cong K_3$ then there does not exist a $(2,1)$-reduction of G at v. For this reason we are particularly interested in those nodes of G where there exists a pair of neighbours of the node that are not adjacent. We set $V_3^*(G) \coloneqq \{v \in V_3(G) \colon G[N_G(v)] \ncong K_3\}$.

Given a $(2,2)$-connected graph G and $v \in V_3(G)$ we say that v is a **plausible** node if $v \in V_3^*(G)$ and we say that v is an **implausible** node if $v \notin V_3^*(G)$. The graph G_2 in Figure 2.1 demonstrates that a $(2,2)$-circuit need not contain any plausible nodes. Note that if v is a plausible node of G and $u \in N_G(v)$ then u is a node of G if and only if u is a plausible node of G. That is, for all $v \in V_3^*(G)$, $N_G(v) \cap V_3(G) = N_G(v) \cap V_3^*(G)$. Our next three results provide some details about the structure of $V_3^*(G)$.

Lemma 2.3.0.7. [4] *Let G be a graph. If G is a $(2,2)$-circuit then $G[V_3(G)]$ is a forest.*

Proof. Let $G = (V,E)$. As G is a $(2,2)$-circuit, $V_3(G) \neq \emptyset$ by Lemma 2.3.0.2. If $G[V_3(G)]$ is not a forest then there exists a subgraph $C = (W,F)$ of $G[V_3(G)]$ such that C is a cycle graph and no proper subgraph of C is a cycle graph. By the definition of $V_3(G)$, $V \setminus W \neq \emptyset$ and $d(V \setminus W, W) = |W|$. So,

$$i_G(V \setminus W) = |E| - (i_G(W) + d(V \setminus W, W)) = 2|V| - 1 - (|W| + |W|) = 2|V \setminus W| - 1.$$

However, then $\emptyset \neq E(G[V \setminus W]) \subsetneq E$ which contradicts the fact that G is a $(2,2)$-circuit. Therefore $G[V_3(G)]$ is a forest. $\square$

For our next result we introduce some additional terminology related to critical sets and nodes. Let $G = (V,E)$ be a $(2,2)$-connected graph and let U be a set. We say that U is ***v*-critical** (in G, on $\{x,y\}$) if U is critical in G and there exists $v \in V_3(G)$ with $\{x,y\} \subseteq N_G(v)$ such that $U \cap N_G[v] = \{x,y\}$. We say that U is **node-critical** (in G) if there exists $v \in V_3(G)$ with $N_G(v) = \{x,y,z\}$ such that U is v-critical in G on $\{x,y\}$ and $d_G(z) \geq 4$. We say that v is a **leaf node** (in G) if $v \in V_3^*(G)$ and $|N_G(v) \cap V_3^*(G)| \leq 1$, and we say that v is a **series node** (in G) if $v \in V_3^*(G)$ and $|N_G(v) \cap V_3^*(G)| = 2$. That is, a plausible node v is a leaf node if at most one neighbour

[4] This result and proof are similar to the statement and proof of [30, Lemma 2.7].

of v is a (plausible) node whereas v is a series node if exactly two neighbours of v are (plausible) nodes. Recall that a neighbour of a plausible node is a node if and only if it is a plausible node.

Lemma 2.3.0.8. [5] *Let $G = (V, E)$ be a $(2,2)$-circuit and let v be a node of G with $N_G(v) = \{x, y, z\}$. If there exists a node-critical set, Z, that is v-critical on $\{x, y\}$ in G and there exists a plausible node $u \in V \setminus (Z \cup \{v\})$ such that there does not exist a $(2,1)$-reduction of G at u that is a $(2,2)$-circuit and either*

(i) u is a series node and $Z \cap N_G(u) = \{w\}$ and $w \in V_3(G)$; or

(ii) u is a leaf node and $E(G[N_G(u)]) = \emptyset$,

then there exists a node-critical set Z' in G such that $Z \subsetneq Z'$.

Proof. Firstly, suppose that u is a series node and $Z \cap N_G(u) = \{w\}$ and $w \in V_3(G)$. Let $N_G(u) = \{a, b, w\}$. As u is a series node we may suppose without loss of generality that $d_G(a) = 3$ and $d_G(b) \geq 4$. By Lemma 2.3.0.7 $G[V_3]$ is a forest, so $wa \notin E$. As there does not exist a $(2,1)$-reduction of G at u that is a $(2,2)$-circuit and $wa \notin E$, Lemma 2.3.0.6 implies that there exists a u-critical set, B, in G on $\{w, a\}$. As $w \in Z \cap B$ and $u, b \notin Z \cup B$ and $d_G(b) \geq 4$, Lemma 2.3.0.5 implies that $Z \cup B$ is a node-critical set in G. As $a \in B \setminus Z$, $Z \subsetneq Z \cup B$.

Alternatively, suppose that u is a leaf node and $E(G[N_G(u)]) = \emptyset$. Let $N_G(u) = \{a, b, c\}$. As $v \notin Z \cup \{u\}$ and G is a $(2,2)$-circuit,

$$\begin{aligned} 2|Z \cup \{u\}| - 2 &\geq i_G(Z \cup \{u\}) \\ &= i_G(Z) + |N_G(u) \cap Z| \\ &= 2|Z| - 2 + |N_G(u) \cap Z| \\ &= (2|Z \cup \{u\}| - 2) + (|N_G(u) \cap Z| - 2). \end{aligned}$$

Hence $|N_G(u) \cap Z| \leq 2$. If $|N_G(u) \cap Z| = 2$ then it follows that $2|Z \cup \{u\}| - 2 =$

[5] This result, as stated, is an extension of [30, Lemma 2.10]. However, by replacing [30, Lemma 2.7] with 2.3.0.7 we are able to follow Nixon's proof to obtain this 'extension'.

$i_G(Z \cup \{u\})$ and hence $Z \cup \{u\}$ is a critical set in G. As $u \in V_3(G)$ and $u \notin Z \cup \{v\}$, $u \notin Z \cup \{v, z\}$ and hence $Z \cup \{u\}$ is a node-critical v-critical set in G such that $Z \subsetneq Z \cup \{u\}$ and we are done. So we may suppose instead that $|N_G(u) \cap Z| \leq 1$.

As u is a leaf node we may suppose without loss of generality that $d_G(b), d_G(c) \geq 4$. As $E(G[N_G(u)]) = \emptyset$ and there does not exist a $(2,1)$-reduction of G at u that is a $(2,2)$-circuit, Lemma 2.3.0.6 implies there exists a u-critical set in G on $\{a, c\}$, say B, and there exists a u-critical set in G on $\{a, b\}$, say C. As $a \in B \cap C$ and $u \notin B \cup C$, Lemma 2.3.0.5 implies that $B \cup C$ is a critical set in G. As $N_G(u) \subseteq B \cup C$, similar reasoning as to why $|N_G(u) \cap Z| \leq 2$ gives that $V \setminus \{u\} = B \cup C$. Therefore $Z \cap B \neq \emptyset$ or $Z \cap C \neq \emptyset$.

We may suppose without loss of generality that $Z \cap B \neq \emptyset$. So, as $\{a, c\} \subseteq Z \cup B$ and $|N_G(u) \cap Z| \leq 1$, $Z \subsetneq Z \cup B$. Moreover, as $u \notin Z \cup B$, Lemma 2.3.0.5 implies that $Z \cup B$ is critical in G. If $b \notin Z$ then $Z \cup B$ is a u-critical set in G on $\{a, c\}$. Hence, as $d_G(b) \geq 4$, $Z \cup B$ is a node-critical set in G such that $Z \subsetneq Z \cup B$ and we are done. If $b \in Z$ then $Z \cap C \neq \emptyset$ and $c \notin Z$. Therefore, by similar reasoning to above, we have that $Z \cup C$ is a node-critical set in G such that $Z \subsetneq Z \cup C$. □

The following operations are similar to the 2-sum and 2-cleave operations discussed in Section 2.2. Figures 2.5, 2.6, and 2.7 demonstrate how these operations behave. These operations will consider graphs which have a 2-vertex-separation or 3-edge-separation (see Definition 1.1.1.10 and Definition 1.1.1.11).

Definition 2.3.0.9. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that there exists a proper subgraph $H_2 \cong K_4$ of G_2, $H_1 = G_1[V_1 \cap V_2] \cong K_2$ is a proper subgraph of G_1 and H_2, and $d_{G_2}(u) = 3$ for all $u \in V(H_2) \setminus V(H_1)$. A **1-join** of the ordered pair (G_1, G_2) is a graph $G = (V_1 \cup (V_2 \setminus U_2), (E_1 \cup E_2) \setminus F_2)$, where $(U_2, F_2) \cong K_4$ is a proper subgraph of G_2, $V_1 \cap V_2 \subsetneq U_2$, and $d_{G_2}(u) = 3$ for all $u \in U_2 \setminus (V_1 \cap V_2)$. If G is a 1-join of (G_1, G_2) and $V_2 \setminus V(G) = \{u_1, u_2\}$ then G is the 1-join of (G_1, G_2) deleting $\{u_1, u_2\}$.

Let $G = (V, E)$ be a graph such that there exists a 2-vertex-separation (H_1, H_2) of G, where $H_i = (U_i, F_i)$ for $i \in \{1,2\}$, and $F_1 \cap F_2 = \emptyset$. A **1-separation** of G is an ordered pair (G_1, G_2) such that G is the 1-join of (G_1, G_2). If (G_1, G_2) is a 1-separation of G then $(G[V(G_1)], G[V \cap V(G_2)])$ is a 2-vertex-separation of G and (G_1, G_2) is the 1-separation of G on $(G[V(G_1)], G[V \cap V(G_2)])$ adding $V(G_2) \setminus V$.

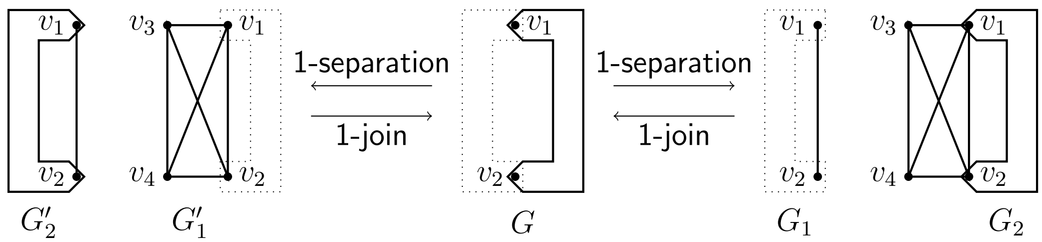


Figure 2.5: Illustration of the 1-separation and 1-join operations.
(G_1, G_2) is the 1-separation of G on $(G[V(G_1)], G[V(G) \cap V(G_2)])$ adding $\{v_3, v_4\}$ whereas (G'_2, G'_1) is the 1-separation of G on $(G[V(G) \cap V(G_2)], G[V(G_1)])$ adding $\{v_3, v_4\}$. G is the 1-join of (G_1, G_2) deleting $\{v_3, v_4\}$, and G is also the 1-join of (G'_2, G'_1) deleting $\{v_3, v_4\}$.

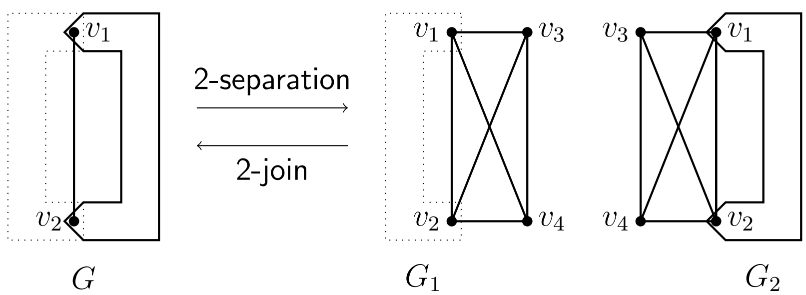


Figure 2.6: Illustration of the 2-separation and 2-join operations.
(G_1, G_2) is the 2-separation of G on $(G[V(G) \cap V(G_1)], G[V(G) \cap V(G_2)])$ adding $\{v_3, v_4\}$ and G is the 2-join of (G_1, G_2).

Definition 2.3.0.10. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that there exists a proper subgraph (U, F), equal to the complete graph on $\{v_1, v_2, v_3, v_4\}$, of both G_1 and G_2, $d_{G_i}(v_3) = 3 = d_{G_i}(v_4)$ for $i \in \{1,2\}$, and $V_1 \cap V_2 = U$. The **2-join** of the ordered pair (G_1, G_2) is the graph $G = ((V_1 \cup V_2) \setminus \{v_3, v_4\}, ((E_1 \cup E_2) \setminus F) \cup \{v_1 v_2\})$.

Let $G = (V, E)$ be a graph such that there exists a 2-vertex-separation (H_1, H_2) of G, where $H_i = (U_i, F_i)$ for $i \in \{1,2\}$, and $F_1 \cap F_2 \neq \emptyset$. A **2-separation** of G is an ordered pair (G_1, G_2) such that G is the 2-join of (G_1, G_2). If (G_1, G_2) is a 2-separation of G then $(G[V \cap V(G_1)], G[V \cap V(G_2)])$ is a 2-vertex-separation of G and (G_1, G_2) is the 2-separation of G on $(G[V \cap V(G_1)], G[V \cap V(G_2)])$ adding $(V(G_1) \cap V(G_2)) \setminus V$.

Definition 2.3.0.11. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $V_1 \cap V_2 = \{v\} = V_3(G_1) \cap V_3(G_2)$ and, for $i \in \{1,2\}$, $d_{G_i}(v) = 3$. A **3-join** of the ordered pair (G_1, G_2) is a graph $G = ((V_1 \cup V_2) \setminus \{v\}, E(G_1[V_1 \setminus \{v\}]) \cup E(G_2[V_2 \setminus \{v\}]) \cup F)$, where $F \subseteq \{u_1u_2 \colon u_i \in N_{G_i}(v)\}$ and $|F| = 3 = \frac{|V(G[F])|}{2}$. If G is a 3-join of (G_1, G_2) and $F = E(G) \setminus (E_1 \cup E_2)$ then G is the 3-join of (G_1, G_2) adding F.

Let $G = (V, E)$ be a graph such that there exists a 3-edge-separation (H_1, H_2) of G, where $H_i = (U_i, F_i)$ for $i \in \{1,2\}$, $F = E \setminus (F_1 \cup F_2)$, and $|F| = 3 = \frac{|V(G[F])|}{2}$. A **3-separation** of G deleting F is an ordered pair (G_1, G_2) such that G is the 3-join of (G_1, G_2) adding F. If (G_1, G_2) is a 3-separation of G deleting F then $(G[V \cap V(G_1)], G[V \cap V(G_2)])$ is a 3-edge-separation of G and (G_1, G_2) is the 3-separation of G deleting F on $(G[V \cap V(G_1)], G[V \cap V(G_2)])$ adding $V(G_1) \cap V(G_2)$.

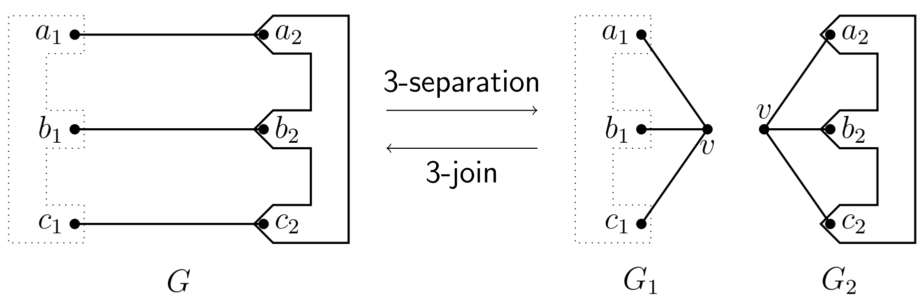


Figure 2.7: Illustration of the 3-separation and 3-join operations. (G_1, G_2) is the 3-separation of G deleting $F = \{a_1a_2, b_1b_2, c_1c_2\}$ on $(G[V(G_1) \setminus \{v\}], G[V(G_2) \setminus \{v\}])$ adding $\{v\}$ and G is the 3-join of (G_1, G_2) adding F.

Remark 10. Observe that given an appropriate ordered pair of graphs, (G_1, G_2), there is a unique 2-join of (G_1, G_2). This is due to the constraints placed on the relationship between the vertex sets of G_1 and G_2. A similar uniqueness is present with the 2-sum of Berg and Jordán, however there may be multiple 1-joins of 3-joins of (G_1, G_2).

Now that we have defined this collection of operations we proceed to state a number of results, originally proved in [30], analogous to Lemma 2.2.0.4 and Lemma 2.2.0.5. In Section 2.2 these results explaining how $(k, 2k-1)$-circuit behave with respect to 2-sum and 2-cleave operations were applied to prove corresponding results (Lemma 2.2.0.7 and Lemma 2.2.0.8) about $(k, 2k-1)$-connected graphs. We take the same approach by extending Nixon's results to show how $(2,2)$-connected graphs behave with respect to i-join and i-separation operations.

Lemma 2.3.0.12. *[30, Lemma 3.1] Let G_1 and G_2 be graphs and suppose G is a 1-join of (G_1, G_2). If G_1 and G_2 are $(2,2)$-circuits then G is a $(2,2)$-circuit.*

Lemma 2.3.0.13. *[30, Lemma 3.1] Let G be a graph and suppose (G_1, G_2) is the 1-separation of G on $(G[V(G_1)], G[V(G) \cap V(G_2)])$ adding $V(G_2) \setminus V(G)$ and (G_2', G_1') is the 1-separation of G on $(G[V(G_2) \cap V(G)], G[V(G_1)])$ adding $V(G_2) \setminus V(G)$. If G is a $(2,2)$-circuit then either:*

(i) G_1 and G_2 are $(2,2)$-circuits, and G_1' and G_2' are not $(2,2)$-circuits; or

(ii) G_1' and G_2' are $(2,2)$-circuits, and G_1 and G_2 are not $(2,2)$-circuits.

Lemma 2.3.0.14. *[30, Lemma 3.2] Let G_1 and G_2 be graphs and suppose G is the 2-join of (G_1, G_2). If G_1 and G_2 are $(2,2)$-circuits then G is a $(2,2)$-circuit.*

Lemma 2.3.0.15. *[30, Lemma 3.2] Let G be a graph and suppose (G_1, G_2) is a 2-separation of G. If G is a $(2,2)$-circuit then G_1 and G_2 are $(2,2)$-circuits.*

Lemma 2.3.0.16. *[30, Lemma 3.3] Let G_1 and G_2 be graphs and suppose G is a 3-join of (G_1, G_2). If G_1 and G_2 are $(2,2)$-circuits then G is a $(2,2)$-circuit.*

Lemma 2.3.0.17. *[30, Lemma 3.3] Let G be a graph and suppose (G_1, G_2) is a 3-separation of G. If G is a $(2,2)$-circuit then G_1 and G_2 are $(2,2)$-circuits.*

Note that, in what follows, we deal with the 2-join/separation and 3-join/separation operations before the 1-join/separation operations. We order the results in this way because our proof of the extension to Lemma 2.3.0.13 will make use of the extension to Lemma 2.3.0.15.

Lemma 2.3.0.18. *Let G_1 and G_2 be graphs and suppose G is a 2-join of (G_1, G_2). If*

G_1 and G_2 are $(2,2)$-connected then G is $(2,2)$-connected.

Proof. Let $G = (V, E)$, $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$. Let H denote the subgraph of both G_1 and G_2 that is isomorphic to K_4. As G_1 and G_2 are $(2,2)$-connected $G_1 = G_1[E_1]$ and $G_2 = G_2[E_2]$ and, by Lemma 2.3.0.1, $|E_1|, |E_2| \geq 9$. By the definition of 2-join it follows that $G = G[E]$ and $|E| \geq 2$. All that remains is to show that $\mathcal{M}_{(2,2)}(G)$ is connected.

Let $E \cap E(H) = \{e\}$ and take $f \in E \setminus \{e\}$. We may suppose without loss of generality that $f \in E_1$. As G_1 and G_2 are $(2,2)$-connected, Lemma 2.3.0.3 implies that for $i \in \{1,2\}$ there exists C_i, a circuit of $\mathcal{M}_{(2,2)}(G_i)$, such that $E(H) \cup \{f\} \subseteq C_1$ and $E(H) \subsetneq C_2$. Lemma 2.3.0.14 implies the 2-join of $(G_1[C_1], G_2[C_2])$, say G', is a $(2,2)$-circuit. Let $E(G') = C'$, then C' is a circuit of $\mathcal{M}_{(2,2)}(G)$ such that $\{e, f\} \subseteq C'$. Therefore, for all $f \in E \setminus \{e\}$ there exists C, a circuit of $\mathcal{M}_{(2,2)}(G)$, such that $\{e, f\} \subseteq C$ and so Lemma 1.2.0.8 implies that G is $(2,2)$-connected. □

Lemma 2.3.0.19. *Let G be a graph and suppose (G_1, G_2) is a 2-separation of G. If G is $(2,2)$-connected then G_1 and G_2 are $(2,2)$-connected.*

Proof. Let $G = (V, E)$, and for $i \in \{1,2\}$ let $G_i = (V_i, E_i)$ and $H_i = (U_i, F_i) = G[V \cap V_i]$. Then (G_1, G_2) is the 2-separation of G on (H_1, H_2) adding $(V_1 \cap V_2) \setminus V$. As G is $(2,2)$-connected $G = G[E]$ and, by Lemma 2.3.0.1, $|E| \geq 9$. By the definition of 2-separation it follows that, for $i \in \{1,2\}$, $G_i = G_i[E_i]$ and $|E_i| \geq 2$. All that remains is to show that for $i \in \{1,2\}$, $\mathcal{M}_{(2,2)}(G_i)$ is connected.

Let $F_1 \cap F_2 = \{e\}$ and for $i \in \{1,2\}$ take $f_i \in F_i \setminus \{e\}$. As G is $(2,2)$-connected, Lemma 2.1.0.20 implies there exists C, a circuit of $\mathcal{M}_{(2,2)}(G)$, such that $\{e, f_1, f_2\} \subseteq C$. Let (G_1', G_2') be the 2-separation of $G[C]$ on $(H_1[C \cap F_1], H_2[C \cap F_2])$ adding $(V_1 \cap V_2) \setminus V$. Lemma 2.3.0.15 implies that G_1' and G_2' are both $(2,2)$-circuits. For $i \in \{1,2\}$ let $E(G_i') = C_i'$, then C_i' is a circuit of $\mathcal{M}_{(2,2)}(G_i)$ such that $\{f_i\} \cup (E_1 \cap E_2) \subseteq C_i'$. Therefore, for $i \in \{1,2\}$, for all $f_i \in E_i \setminus \{e\}$ there exists C_i, a circuit of $\mathcal{M}_{(2,2)}(G_i)$, such that $\{e, f_i\} \subseteq C_i$ and so Lemma 1.2.0.8 implies that G_1 and G_2 are $(2,2)$-connected. □

Lemma 2.3.0.20. *Let G_1 and G_2 be graphs and suppose G is a 3-join of (G_1, G_2). If G_1 and G_2 are $(2,2)$-connected then G is $(2,2)$-connected.*

Proof. Let $G = (V, E)$, $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$. Let $V_1 \cap V_2 = \{v\}$ and let G be the 3-join of (G_1, G_2) adding F. Let $F = \{a_1a_2, b_1b_2, c_1c_2\}$ where, for $i \in \{1, 2\}$, $\{a_i, b_i, c_i\} \subseteq V_i$. As G_1 and G_2 are $(2,2)$-connected $G_1 = G_1[E_1]$ and $G_2 = G_2[E_2]$ and, by Lemma 2.3.0.1, $|E_1|, |E_2| \geq 9$. By the definition of 3-join it follows that $G = G[E]$ and $|E| \geq 2$. All that remains is to show that $\mathcal{M}_{(2,2)}(G)$ is connected.

Take $e \in F$, take $f_1 \in E \cap E_1$, and take $f_2 \in E \cap E_2$. As G_1 and G_2 are $(2,2)$-connected, for $i \in \{1, 2\}$ there exists C_i, a circuit of $\mathcal{M}_{(2,2)}(G_i)$ such that $\{va_i, f_i\} \subseteq C_i$. $G_i[C_i]$ is a $(2,2)$-circuit for $i \in \{1, 2\}$. Lemma 2.3.0.2 implies that for $i \in \{1, 2\}$, $\delta(G_i[C_i]) = 3$ and hence $\{va_i, vb_i, vc_i, f_i\} \subseteq C_i$. Then Lemma 2.3.0.16 implies the 3-join of $(G_1[C_1], G_2[C_2])$ adding F, say G', is a $(2,2)$-circuit. Let $E(G') = C'$, then C' is a circuit of $\mathcal{M}_{(2,2)}(G)$ such that $F \cup \{f_1, f_2\} \subseteq C'$. Therefore, for all $f \in E \setminus \{e\}$ there exists C, a circuit of $\mathcal{M}_{(2,2)}(G)$, such that $\{e, f\} \subseteq C$ and so Lemma 1.2.0.8 implies that G is $(2,2)$-connected. □

Lemma 2.3.0.21. *Let G be a graph and suppose (G_1, G_2) is a 3-separation of G. If G is $(2,2)$-connected then G_1 and G_2 are $(2,2)$-connected.*

Proof. Let $G = (V, E)$, let (G_1, G_2) be a 3-separation of G deleting F, and for $i \in \{1, 2\}$ let $G_i = (V_i, E_i)$ and $H_i = (U_i, F_i) = G[V \cap V_i]$. Then (G_1, G_2) is the 3-separation of G deleting F on (H_1, H_2) adding $V_1 \cap V_2$. As G is $(2,2)$-connected $G = G[E]$ and, by Lemma 2.3.0.1, $|E| \geq 9$. By the definition of 3-separation it follows that, for $i \in \{1, 2\}$, $G_i = G_i[E_i]$ and $|E_i| \geq 2$. All that remains is to show that for $i \in \{1, 2\}$, $\mathcal{M}_{(2,2)}(G_i)$ is connected.

Let $F = \{a_1a_2, b_1b_2, c_1c_2\}$ and let $V_1 \cap V_2 = \{v\}$. For $i \in \{1, 2\}$, let $e_i = va_i$ and take $f_i \in F_i$. As G is $(2,2)$-connected there exists C, a circuit of $\mathcal{M}_{(2,2)}(G)$, such that $\{f_1, f_2\} \subseteq C$. $G[C]$ is a $(2,2)$-circuit. Lemma 2.1.0.16 implies that $\kappa_1(G[C]) \geq 3$ and hence $F \cup \{f_1, f_2\} \subseteq C$. Let (G_1', G_2') be the 3-separation of $G[C]$ deleting F on $(H_1[C \cap F_1], H_2[C \cap F_2])$ adding $\{v\}$. Lemma 2.3.0.17 implies that G_1' and G_2' are both

$(2,2)$-circuits. For $i \in \{1,2\}$ let $E(G_i') = C_i'$, then C_i' is a circuit of $\mathcal{M}_{(2,2)}(G_i)$ such that $\{f_i, va_i, vb_i, vc_i\} \subseteq C_i$. Therefore, for $i \in \{1,2\}$, for all $f_i \in E_i \setminus e_i$ there exists C_i, a circuit of $\mathcal{M}_{(2,2)}(G_i)$, such that $\{e_i, f_i\} \subseteq C_i$ and so Lemma 1.2.0.8 implies that G_1 and G_2 are $(2,2)$-connected. □

Lemma 2.3.0.22. *Let G_1 and G_2 be graphs and suppose G is a 1-join of (G_1, G_2). If G_1 and G_2 are $(2,2)$-connected then G is $(2,2)$-connected.*

Proof. Let $G = (V, E)$, $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$. Let $V_1 \cap V_2 = \{v_1, v_2\}$ and let $V_2 \setminus V = \{v_3, v_4\}$. As G_1 and G_2 are $(2,2)$-connected $G_1 = G_1[E_1]$ and $G_2 = G_2[E_2]$ and, by Lemma 2.3.0.1, $|E_1|, |E_2| \geq 9$. By the definition of 1-join it follows that $G = G[E]$ and $|E| \geq 2$. All that remains is to show that $\mathcal{M}_{(2,2)}(G)$ is connected. Take $e \in E_1 \cap E$, and take $f \in E \setminus \{e\}$.

Suppose $f \in E_1$. As G_1 is $(2,2)$-connected there exists C_1, a circuit of $\mathcal{M}_{(2,2)}(G_1)$, such that $\{e, f\} \subseteq C_1$. If $v_1v_2 \notin C_1$ then C_1 is a circuit of $\mathcal{M}_{(2,2)}(G)$. Alternatively, suppose $v_1v_2 \in C_1$. Lemma 2.3.0.3 implies there exists C_2, a circuit of $\mathcal{M}_{(2,2)}(G_2)$, such that $(E_2 \setminus E) \subsetneq C_2$. Lemma 2.3.0.12 implies the 1-join of $(G_1[C_1], G_2[C_2])$ deleting $\{v_3, v_4\}$, say G', is a $(2,2)$-circuit. Let $E(G') = C'$, then C' is a circuit of $\mathcal{M}_{(2,2)}(G)$ such that $\{e, f\} \subseteq C'$. So, for all $f \in (E \cap E_1) \setminus \{e\}$ there exists C', a circuit of $\mathcal{M}_{(2,2)}(G)$, such that $\{e, f\} \subseteq C'$.

On the other hand, suppose $f \in E_2$. As G_1 is $(2,2)$-connected there exists C_1', a circuit of $\mathcal{M}_{(2,2)}(G_1)$, such that $\{e, v_1v_2\} \subseteq C_1'$. Moreover, Lemma 2.3.0.3 implies there exists C_2', a circuit of $\mathcal{M}_{(2,2)}(G_2)$, such that $(E_2 \setminus E) \cup \{f\} \subseteq C_2'$. Lemma 2.3.0.12 implies the 1-join of $(G_1[C_1'], G_2[C_2'])$ deleting $\{v_3, v_4\}$, say G'', is a $(2,2)$-circuit. Let $E(G'') = C''$, then C'' is a circuit of $\mathcal{M}_{(2,2)}(G)$ such that $\{e, f\} \subseteq C'$. So, for all $f \in E \cap E_2$ there exists C', a circuit of $\mathcal{M}_{(2,2)}(G)$, such that $\{e, f\} \subseteq C'$. Therefore, for all $f \in E \setminus \{e\}$ there exists C, a circuit of $\mathcal{M}_{(2,2)}(G)$, such that $\{e, f\} \subseteq C$ and so Lemma 1.2.0.8 implies that G is $(2,2)$-connected. □

The proof of the following result boils down to the observation that a K_4^--extension of a graph can be thought of as arising from a particular 1-join involving the same graph.

Lemma 2.3.0.23. *Let G be a graph and let G' be a K_4^--extension of G. If G is $(2,2)$-connected then G' is $(2,2)$-connected.*

Proof. Take $e \in E(G)$, and let G' be the K_4^--extension of G deleting e and adding $\{v_1, v_2\}$. Let $e = xy$ and set

$$\tilde{G} = (\{v_1, v_2, v_3, v_4, x, y\}, \{v_1x, v_2x, v_3x, v_4x, v_1y, v_2y, v_3y, v_4y, v_1v_2, v_3v_4, xy\}),$$

where $V(\tilde{G}) \cap V(G) = \{x, y\}$. Observe that $\tilde{G} \cong B_1$ (see Figure 2.4). Then G' is the 1-join of $(G, \tilde{G})$ deleting $\{v_3, v_4\}$). As G is $(2,2)$-connected and B_1 is $(2,2)$-connected, Lemma 2.3.0.22 implies G' is $(2,2)$-connected. □

Lemma 2.3.0.24. *Let G be a graph and suppose (H_1, H_2) is a 2-vertex-separation of G, where $H_i = (U_i, F_i)$ for $i \in \{1, 2\}$, and $F_1 \cap F_2 = \emptyset$. Let (G_1, G_2) be the 1-separation of G on (H_1, H_2) adding U, and let (G_2', G_1') be the 1-separation of G on (H_2, H_1) adding U. If G is $(2,2)$-connected then:*

(i) G_2 and G_1' are $(2,2)$-connected; and

(ii) G_1 is $(2,2)$-connected or G_2' is $(2,2)$-connected.

Proof. Let $G = (V, E)$, and for $i \in \{1, 2\}$ let $G_i = (V_i, E_i)$ and $G_i' = (V_i', E_i')$. Then $V_1 = U_1$, $V_2 = U_2 \cup U$, $V_1' = U_1 \cup U$, and $V_2' = U_2$. Let $U_1 \cap U_2 = \{v_1, v_2\}$ and let $U = \{v_3, v_4\}$. We prove parts (i) and (ii) in turn.

(i) As $v_1v_2 \notin E$, let G' denote the edge-addition of G adding v_1v_2. Lemma 2.1.0.17 implies that G' is $(2,2)$-connected. Observe that (G_1', G_2) is the 2-separation of G on $(G'[U_1], G'[U_2])$ adding $\{v_3, v_4\}$. Lemma 2.3.0.19 implies that G_1' and G_2 are $(2,2)$-connected.

(ii) As G is $(2,2)$-connected $G = G[E]$ and, by Lemma 2.3.0.1, $|E| \geq 9$. By the definition of 1-separation it follows that $G_1 = G_1[E_1]$, $G_2' = G_2'[E_2']$, and $|E_1|, |E_2'| \geq 2$. All that remains is to show that $\mathcal{M}_{(2,2)}(G_1)$ is $(2,2)$-connected or $\mathcal{M}_{(2,2)}(G_2')$ is $(2,2)$-connected.

Let us suppose, in pursuit of a contradiction, that both $\mathcal{M}_{(2,2)}(G_1)$ and $\mathcal{M}_{(2,2)}(G'_2)$ are not connected. As $\mathcal{M}_{(2,2)}(G_1)$ is not connected Lemma 1.2.0.8 implies that for all $e \in E_1$ there exists $f \in E_1 \setminus \{e\}$ such that no circuit in $\mathcal{M}_{(2,2)}(G_1)$ contains e and f. In particular, we can take $f \in E_1 \setminus \{v_1v_2\}$ such that no circuit in $\mathcal{M}_{(2,2)}(G'_2)$ contains f and v_1v_2. Similarly, as $\mathcal{M}_{(2,2)}(G'_2)$ is not connected Lemma 1.2.0.8 implies that for all $e' \in E'_2$ there exists $f' \in E'_2 \setminus \{e'\}$ such that no circuit in $\mathcal{M}_{(2,2)}(G'_2)$ contains e' and f'. In particular, we can take $f' \in E'_2 \setminus \{v_1v_2\}$ such that no circuit in $\mathcal{M}_{(2,2)}(G'_2)$ contains f' and v_1v_2. Now, $\{f, f'\} \subseteq E$. Since G is $(2, 2)$-connected there exists $\tilde{C}$, a circuit of $\mathcal{M}_{(2,2)}(G)$, such that $\{f, f'\} \subseteq \tilde{C}$. Lemma 2.1.0.14 implies that $\kappa(G[\tilde{C}]) \geq 2$ and so $\{v_1, v_2\} \subseteq V(G[\tilde{C}])$.

Let $(\tilde{G}_1, \tilde{G}_2)$ be the 1-separation of $G[\tilde{C}]$ on $(G[\tilde{C} \cap E_1], G[\tilde{C} \cap E'_2])$ adding $\{v_3, v_4\}$ and let (G''_2, G''_1) be the 1-separation of $G[\tilde{C}]$ on $(G[\tilde{C} \cap E'_2], G[\tilde{C} \cap E_1])$ adding $\{v_3, v_4\}$. Lemma 2.3.0.13 implies that $\tilde{G}_1$ is a $(2, 2)$-circuit or G''_2 is a $(2, 2)$-circuit. It follows that $E(\tilde{G}_1)$ is a circuit of $\mathcal{M}_{(2,2)}(G_1)$ or $E(G''_2)$ is a circuit of $\mathcal{M}_{(2,2)}(G'_2)$. However, $\{f, v_1v_2\} \subseteq E(\tilde{G}_1)$ and $\{f', v_1v_2\} \subseteq E(G''_2)$ which provides a contradiction. Therefore our supposition that both $\mathcal{M}_{(2,2)}(G_1)$ and $\mathcal{M}_{(2,2)}(G'_2)$ are not connected must be false, so G_1 is $(2, 2)$-connected or G'_2 is $(2, 2)$-connected.

□

We conclude this chapter with a result which shows that if there exists a plausible node of a $(2, 2)$-circuit such that some neighbours of the node are adjacent then there is some reduction of our $(2, 2)$-circuit that gives another $(2, 2)$-circuit.

Lemma 2.3.0.25. [6] *Let $G = (V, E)$ be a $(2, 2)$-circuit and let v be a node of G with $N_G(v) = \{x, y, z\}$. If $xz \in E$ and $yz \notin E$ then we have the following trichotomy:*

(i) $xy \in E$, $d_G(x) \geq 4$, *and the $(2, 1)$-reduction of G at v adding yz is a $(2, 2)$-circuit; or*

(ii) $xy \in E$, $d_G(x) = 3$, *and the K_4^--reduction of G deleting $\{v, x\}$ is a $(2, 2)$-circuit; or*

[6] This result is a minor extension of [30, Lemma 2.8].

(iii) $xy \notin E$ *and there exists a* $(2,1)$*-reduction,* G'*, of* G *at* v *such that* G' *is a* $(2,2)$*-circuit.*

Proof. Firstly, suppose that $xy \in E$ and $d_G(x) \geq 4$. Then we may take $u \in N_G(x) \setminus \{v, y, z\}$. Take $X \subseteq V$ such that $X \cap N_G[v] = \{y, z\}$, then there exists $a \in \mathbb{Z}$ such that $i_G(X) = 2|X| - a$. If $u \in X$ then $X \cup \{v, x\} \subseteq V$ and so

$$2|X \cup \{v,x\}| - 1 \geq i_G(X \cup \{v,x\}) \geq i_G(X) + 6 = 2|X| - a + 6 = 2|X \cup \{v,x\}| - (a-2).$$

This implies that $a \geq 3$, so X is not critical in G. If $u \notin X$ then $X \cup \{v, x\} \subsetneq V$ and so

$$2|X \cup \{v,x\}| - 2 \geq i_G(X \cup \{v,x\}) \geq i_G(X) + 5 = 2|X| - a + 5 = 2|X \cup \{v,x\}| - (a-1).$$

This implies that $a \geq 3$, so X is not critical in G. So for all $X \subseteq V$ such that $X \cap N_G[v] = \{y, z\}$, X is not critical in G. Lemma 2.3.0.6 implies that the $(2,1)$-reduction of G at v adding yz is a $(2,2)$-circuit.

Next, suppose that $xy \in E$ and $d_G(x) = 3$. Let $H_1 = (U_1, F_1) = G[V \setminus \{v, x\}]$ and $H_2 = (U_2, F_2) = (\{v, x, y, z\}, \{vx, vy, vz, xy, xz\})$. Then (H_1, H_2) and (H_2, H_1) are both 2-vertex-separations of G such that $E(H_1) \cap E(H_2) = \emptyset$. Let (G_1, G_2) be a 1-separation of G on (H_1, H_2) and let (G'_2, G'_1) be a 1-separation of G on (H_2, H_1). We observe that G_1 is the K_4^--reduction of G deleting $\{v, x\}$. As $G'_2 \cong K_4$ which is not a $(2,2)$-circuit, Lemma 2.3.0.13 implies that G_1 is a $(2,2)$-circuit.

Finally, suppose that $xy \notin E$. That there exists a $(2,1)$-reduction of G at v that is a $(2,2)$-circuit follows from [30, Lemma 2.8]. $\square$

Chapter 3

A Construction of $(2,2)$-Connected Graphs

Our third chapter picks up directly from where the previous chapter left off and is the most combinatorially technical part of the thesis. The results in this chapter can often be seen as analogous to results in [22]. In that paper, Jackson and Jordán built on earlier work in [2] studying $(2,3)$-circuits. This chapter proceeds similarly by building on an understanding of $(2,2)$-circuits, gained from Section 2.3, [30], and [24], and culminating with a method of constructing all $(2,2)$-connected graphs.

3.1 Ear Decompositions

The first section of this chapter sees us repeatedly invoke the relationship (recall Theorem 1.2.0.10) between ear decompositions of matroids and connected matroids in order to understand when there exists some graph operation of a $(2,2)$-connected graph that is a $(2,2)$-connected graph with fewer edges.

Lemma 3.1.0.1. *Let G be a $(2,2)$-connected graph and let $C_1,\dots,C_t$ be an ear decomposition of $\mathcal{M}_{(2,2)}(G)$. For $1 \le i \le t$ let $G[C_i] = H_i = (V_i, C_i)$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and let $X = V_t \setminus Y$. If $t \ge 2$ then:*

(i) either $Y = \emptyset$ *and* $|C_t \setminus (\bigcup_{j=1}^{t-1} C_j)| = 1$*, or* $Y \neq \emptyset$ *and* $C_t \setminus (\bigcup_{j=1}^{t-1} C_j) = \{e \in E(G)\colon$ *there exists* $y \in Y$ *such that* e *is incident to* $y\}$*;*

(ii) $|C_t \setminus (\bigcup_{j=1}^{t-1} C_j)| = 2|Y| + 1$*;*

(iii) if $Y \neq \emptyset$ *then* X *is critical in* H_t*;*

(iv) if $Y \neq \emptyset$ *then* $H_t[Y] = G[Y]$*,* $d_{H_t}(y) = d_G(y)$ *for all* $y \in Y$*, and* $Y \cap V_3(G) \neq \emptyset$*;*

(v) if $Y \neq \emptyset$ *then* $G[Y]$ *is connected;*

(vi) $|X| \geq 4$*.*

Proof. We prove each statement in turn.

(i) If $Y = \emptyset$ then Definition 1.2.0.9 (ED3) implies that $|C_t \setminus (\bigcup_{j=1}^{t-1} C_j)| = 1$. If $Y \neq \emptyset$ then Definition 1.2.0.9 (ED3) implies $C_t \setminus (\bigcup_{j=1}^{t-1} C_j) \subseteq \{e \in E(G)\colon$ there exists $y \in Y$ that is incident to $e\}$. As $C_1, \dots, C_t$ is an ear decomposition of $\mathcal{M}_{(2,2)}(G)$, $\{e \in E(G)\colon$ there exists $y \in Y$ that is incident to $e\} \subseteq C_t \setminus (\bigcup_{j=1}^{t-1} C_j)$. So, if $Y \neq \emptyset$ then $C_t \setminus (\bigcup_{j=1}^{t-1} C_j) = \{e \in E(G)\colon$ there exists $y \in Y$ such that e is incident to $y\}$.

(ii) Theorem 1.2.0.10 (i) gives us that $G[\bigcup_{j=1}^{t-1} C_j]$ is a $(2,2)$-connected graph, and so Theorem 1.2.0.10 (iii) implies that $|C_t \setminus (\bigcup_{j=1}^{t-1} C_j)| - 1 = r(E) - r(\bigcup_{j=1}^{t-1} C_j)$, where r is the rank function of $\mathcal{M}_{(2,2)}(G)$. Now, by Theorem 1.4.3.8 we note that $(2,2)$-connected graphs are rigid in any two-dimensional non-Euclidean normed space and hence $r(E) = 2|V| - 2$ and $r(\bigcup_{j=1}^{t-1} C_j) = 2|V \setminus Y| - 2$. Therefore

$$|C_t \setminus (\bigcup_{j=1}^{t-1} C_j)| = 1 + (2|V| - 2) - (2|V \setminus Y| - 2) = 2|Y| + 1.$$

(iii) Note that as $t \geq 2$, $X \neq \emptyset$. As $Y \neq \emptyset$ it follows from (i) that $i_{H_t}(Y) + d_{H_t}(X, Y) = |C_t \setminus (\bigcup_{j=1}^{t-1} C_j)|$. Then (ii) implies that $i_{H_t}(Y) + d_{H_t}(X, Y) = 2|Y| + 1$, and so

$$i_{H_t}(X) = |C_t| - (i_{H_t}(Y) + d_{H_t}(X, Y)) = (2|V_t| - 1) - (2|Y| + 1) = 2|X| - 2.$$

Hence X is critical in H_t.

(iv) As $Y \neq \emptyset$, we can consider $H_t[Y]$ and $G[Y]$. That $H_t[Y] = G[Y]$ follows from the definition of Y. That $d_{H_t}(y) = d_G(y)$ for all $y \in Y$ follow from (i). Lastly, as X is critical in H_t by (iii), Lemma 2.3.0.4 (iii) implies that $Y \cap V_3(H_t) \neq \emptyset$ and hence by the previous sentence we have that $Y \cap V_3(G) \neq \emptyset$.

(v) As $Y \neq \emptyset$ we can consider $G[Y]$. Let $G_1, \ldots, G_n$ be the components of $G[Y]$ and for all $1 \leq i \leq n$ let $G_i = (Y_i, F_i)$. Then $Y = \bigcup_{i=1}^n Y_i$ and $F = \bigcup_{i=1}^n F_i$. As each G_i is a component of $G[Y]$ we have that for all $1 \leq i < j \leq n$, $Y_i \cap Y_j = \emptyset = F_i \cap F_j$. Therefore $|Y| = \sum_{i=1}^n |Y_i|$ and $|F| = \sum_{i=1}^n |F_i|$. Moreover, for all $1 \leq i \leq n$ we have that $H_t[Y_i] = G[Y_i]$ and so $i_{H_t}(Y_i) = |F_i|$.

As H_t is a $(2,2)$-circuit and X is critical in H_t by (iii), it follows that for all $1 \leq i \leq n$ there exists $a_i \geq 1$ such that,

$$\begin{aligned}
|F_i| + d_{H_t}(X, Y_i) &= i_{H_t}(X \cup Y_i) - i_{H_t}(X) \\
&= (2|X \cup Y_i| - a_i) - (2|X| - 2) \\
&= 2(1 + |X| + |Y_i| - |X \cap Y_i|) - (2|X| + a_i) \\
&= 2|Y_i| + (2 - a_i).
\end{aligned}$$

Now, by (ii) and (i) we have

$$\begin{aligned}
2|Y| + 1 = \left| C_t \setminus \left(\bigcup_{j=1}^{t-1} C_j \right) \right| &= i_{H_t}(Y) + d_{H_t}(X, Y) \\
&= \sum_{i=1}^{n} (|F_i| + d_{H_t}(X, Y_i)) \\
&= \sum_{i=1}^{n} (2|Y_i| + 2 - a_i) \\
&= 2|Y| + 2n - \sum_{i=1}^{n} a_i.
\end{aligned}$$

Therefore $\sum_{i=1}^{n} a_i = 2n-1$ and so, as $a_i \geq 1$ for all $1 \leq i \leq n$, there exists $1 \leq i \leq n$ such that $a_i = 1$. As H_t is a $(2,2)$-circuit it follows that $X \cup Y_i = V_t$, so $i = 1 = n$ and $H_t[Y] = G[Y]$ is connected.

(vi) If $Y = \emptyset$ then $|X| = |V_t|$ so, as H_t is $(2,2)$-circuit, $|X| = |V_t| \geq 5$. If $Y \neq \emptyset$ then X is critical in H_t by (iii), so $|X| = 1$ or $|X| \geq 4$. As $C_1, \ldots, C_t$ is an ear decomposition of $\mathcal{M}_{(2,2)}(G)$, $C_t \cap (\bigcup_{i=1}^{t-1} C_i) \neq \emptyset$ and hence $E(H_t[X]) \neq \emptyset$. Therefore $|X| \neq 1$ and so $|X| \geq 4$.

□

Remark 11. We note that in the proof of parts (i) and (v) of Lemma 3.1.0.1 no use is made of the fact that $t \geq 2$. Indeed, if $t = 1$ then we see that $Y = V_t \neq \emptyset$ and $G[Y] = G = H_t$ is connected by Lemma 2.1.0.12. Also, although the proof of part (iv) invokes part (iii), and so uses the fact that $t \geq 2$, Lemma 2.3.0.2 implies that part (iv) also holds in the case that $t = 1$. However, the condition that $t \geq 2$ is necessary for parts (ii), (iii), and (vi) of Lemma 3.1.0.1.

Lemma 3.1.0.2. *Let G be a $(2,2)$-circuit, take $v \in V_3(G)$, and let $N_G(v) = \{x, y, z\}$. If $xy \notin E(G)$ then the $(2,1)$-reduction of G at v adding xy, G', has exactly one subgraph, J, that is a $(2,2)$-circuit. Moreover, $V(J)$ is the unique minimal v-critical set in G on $\{x, y\}$ (i.e. no proper subset of $V(J)$ is v-critical set in G on $\{x, y\}$) if and only if $J \neq G'$.*

Proof. Lemma 2.3.0.2 gives us that $V_3(G) \neq \emptyset$, so we may take $v \in V_3(G)$. Let $\mathcal{Z} = \{Z \colon Z$ is v-critical in G on $\{x, y\}\}$. If $\mathcal{Z} = \emptyset$ then Lemma 2.3.0.6 implies G' is a $(2,2)$-circuit, so $J = G'$. Then $V(J) = V(G) \setminus \{v\}$ so $V(J)$ is not v-critical in G on $\{x, y\}$.

If $\mathcal{Z} \neq \emptyset$ then let $Z' = \bigcap_{Z \in \mathcal{Z}} Z$. As $x \in Z'$ and $v \notin Z'$, Lemma 2.3.0.5 implies that $Z' \in \mathcal{Z}$. As $z \notin Z'$, $G'[Z'] \neq G'$. By definition, Z' is the unique minimal v-critical set in G on $\{x, y\}$. Let $J' = G'[Z']$. As G' is a $(2,1)$-reduction of G, J' must be a subgraph of any subgraph of G' that is a $(2,2)$-circuit. Moreover, by the definition of Z', J' is a $(2,2)$-circuit and so J' is the unique subgraph of G' that is a $(2,2)$-circuit. □

Lemma 3.1.0.3. *Let G be a $(2,2)$-connected graph and let $C_1,\dots,C_t$ be an ear decomposition of $\mathcal{M}_{(2,2)}(G)$. For $1 \le i \le t$ let $G[C_i] = H_i = (V_i, C_i)$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and let $X = V_t \setminus Y$. Suppose that $t \ge 2$ and $Y \ne \emptyset$. Take $v \in Y \cap V_3(G)$ and suppose there exists $\{x,y\} \subseteq N_G(v)$ such that $xy \notin E(G)$. Let H_t' be the $(2,1)$-reduction of H_t at v adding xy, and let J be the unique subgraph of H_t' that is a $(2,2)$-circuit. If $E(H_t') \setminus E(H_t[X]) \subsetneq E(J)$ then the $(2,1)$-reduction of G at v adding xy is $(2,2)$-connected.*

Proof. Let $G = (V,E)$. As $t \ge 2$ and $Y \ne \emptyset$, Lemma 3.1.0.1 (v) implies that $Y \cap V_3(G) \ne \emptyset$. Hence we can take $v \in Y \cap V_3(G) = Y \cap V_3(H_t)$. Let $N_G(v) = \{x,y,z\}$ and suppose $xy \notin E(G)$. Then $xy \notin C_t$ and there exists a $(2,1)$-reduction of H_t at v adding xy, which we call H_t'. Lemma 3.1.0.2 implies there exists a unique subgraph, J, of H_t' such that J is a $(2,2)$-circuit. As $xy \notin E(G)$, there exists a $(2,1)$-reduction of G at v adding xy, which we call G'. Let $G' = (V',E')$. As G' is a $(2,1)$-reduction of G, $G' = G'[E']$ and $|E'| = |E| - 2 \ge 7$. All that remains is to show that $\mathcal{M}_{(2,2)}(G')$ is connected.

Now,

$$
\begin{aligned}
E' &= (E \setminus \{vx, vy, vz\}) \cup \{xy\} \\
&= \left(\left(\bigcup_{i=1}^{t} C_i \right) \setminus \{vx, vy, vz\} \right) \cup \{xy\} \\
&= \bigcup_{i=1}^{t-1} C_i \cup (C_t \setminus \{vx, vy, vz\}) \cup \{xy\} \\
&= \left(\bigcup_{i=1}^{t-1} C_i \right) \cup E(H_t').
\end{aligned}
$$

Lemma 3.1.0.1 (i) implies that $E(H_t[X]) \subseteq \bigcup_{i=1}^{t-1} C_i$. So, as $E(H_t') \setminus E(H_t[X]) \subsetneq E(J)$,

$$
E' = \left(\bigcup_{i=1}^{t-1} C_i \right) \cup (E(H_t') \setminus E(H_t[X])) \subseteq \left(\bigcup_{i=1}^{t-1} C_i \right) \cup E(J) \subseteq E'.
$$

Therefore $(\bigcup_{i=1}^{t-1} C_i) \cup E(J) = E'$. As $|\bigcup_{i=1}^{t-1} C_i| \geq 2$ and $\bigcup_{i=1}^{t-1} C_i$ is an ear decomposition of $G[\bigcup_{i=1}^{t-1} C_i] = G'[\bigcup_{i=1}^{t-1} C_i]$, Theorem 1.2.0.10 implies that $\mathcal{M}_{(2,2)}(G'[\bigcup_{i=1}^{t-1} C_i])$ is connected and hence $G'[\bigcup_{i=1}^{t-1} C_i]$ is $(2,2)$-connected. As J is a $(2,2)$-circuit, Lemma 2.1.0.8 implies that J is $(2,2)$-connected.

As $E(H_t') \setminus E(H_t[X]) \subsetneq E(J) \subseteq E(H_t')$, $E(J) \cap E(H_t[X]) \neq \emptyset$. Recall that $E(J) \cap E(H_t[X]) \subseteq E' = E(J) \cup (\bigcup_{i=1}^{t-1} C_i)$ and take $e \in E(J) \cap (\bigcup_{i=1}^{t-1} C_i)$. Take $f \in E' \setminus \{e\}$. As both $G'[\bigcup_{i=1}^{t-1} C_i]$ and J are $(2,2)$-connected there exists C, a circuit of $\mathcal{M}_{(2,2)}(G')$, such that $\{e, f\} \subseteq C$. So for all $f \in E' \setminus \{e\}$ there exists C, a circuit of $\mathcal{M}_{(2,2)}(G')$, such that $\{e, f\} \subseteq C$. Therefore Lemma 1.2.0.8 implies that is $\mathcal{M}_{(2,2)}(G')$-connected. $\square$

Our next result is a consequence of Lemma 3.1.0.3 that we will make repeated use of.

Lemma 3.1.0.4. *Let G be a $(2,2)$-connected graph and let $C_1, \ldots, C_t$ be an ear decomposition of $\mathcal{M}_{(2,2)}(G)$. For $1 \leq i \leq t$ let $G[C_i] = H_i = (V_i, C_i)$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and let $X = V_t \setminus Y$. Suppose that $t \geq 2$ and $Y \neq \emptyset$. Take $v \in Y \cap V_3(G)$ and suppose there exists $\{x, y\} \subseteq N_G(v)$ such that $xy \notin C_t$. If the $(2,1)$-reduction of H_t at v adding xy is a $(2,2)$-circuit and $xy \notin E(G)$ then the $(2,1)$-reduction of G at v adding xy is $(2,2)$-connected.*

Proof. Let $H_t' = (V_t', C_t')$ denote the $(2,1)$-reduction of H_t at v adding xy. By Lemma 3.1.0.1 (iii) and (vi), $E(H_t[X]) \neq \emptyset$. Hence $C_t' \setminus E(H_t[X]) \subsetneq C_t'$ and so, as $xy \notin E(G)$, Lemma 3.1.0.3 implies that the $(2,1)$-reduction of G at v adding xy is $(2,2)$-connected. $\square$

Lemma 3.1.0.5. *Let G be a $(2,2)$-connected graph and let $C_1, \ldots, C_t$ be an ear decomposition of $\mathcal{M}_{(2,2)}(G)$. For $1 \leq i \leq t$ let $G[C_i] = H_i = (V_i, C_i)$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and let $X = V_t \setminus Y$. Suppose that $t \geq 2$ and $Y \neq \emptyset$. Take $v \in Y \cap V_3(G)$, let $N_G(v) = \{x, y, z\}$, and suppose that $d_{H_t}(x) = 3$ and $\{xy, xz, yz\} \cap C_t = \{xy, xz\}$. If $|N_{H_t}(v) \cap X| \leq 2$ then*

(i) there exists a K_4^--reduction of G deleting $\{v, x\}$ if and only if $yz \notin E(G)$; and

(ii) if there exists a K_4^--reduction of G deleting $\{v, x\}$, G', then G' is $(2,2)$-connected.

Proof. As $t \geq 2$ and $Y \neq \emptyset$, Lemma 3.1.0.1 (iv) implies that $Y \cap V_3(G) \neq \emptyset$. Hence we can take $v \in Y \cap V_3(G) = Y \cap V_3(H_t)$ and let $N_G(v) = \{x, y, z\}$. If there exists a K_4^--reduction of G deleting $\{v, x\}$ then $yz \notin E(G)$. On the other hand, suppose that $yz \notin E(G)$. Then $\{xy, xz, yz\} \cap E(G) = \{xy, xz\}$. Note that Lemma 3.1.0.1 (iii) and (vi) together give us that X is critical in H_t and $|X| \geq 4$. Therefore, if $x \in X$ then $d_{H_t[X]}(x) \geq 2$ by Lemma 2.3.0.4 (ii), and so $y, z \in X$. However this implies that $|N_{H_t}(v) \cap X| \geq 3$, a contradiction. Therefore $x \in Y$ and so $d_G(x) = 3$ by Lemma 3.1.0.1 (iv). Therefore there exists a K_4^--reduction of G deleting $\{v, x\}$, and we denote this graph by G'.

Take $v_1, v_2 \notin V$, let $H'_1 = G[V \setminus \{v, x\}]$, and let $H'_2 = G[N_G[v]]$. Observe that $G = (V(H'_1) \cup V(H'_2), E(H'_1) \cup E(H'_2))$, $|V(H'_1) \cap V(H'_2)| = 2$, $V(H'_1) \setminus V(H'_2) \neq \emptyset \neq V(H'_2) \setminus V(H'_1)$, and $E(G[V(H'_1) \cap V(H'_2)]) = \emptyset$. Let (G_1, G_2) be the 1-separation of G on (H'_1, H'_2) adding $\{v_1, v_2\}$, and let (G'_2, G'_1) be the 1-separation of G on (H_2, H_1) adding $\{v_1, v_2\}$. Then $G' = G_1$. We see that $G'_2 \cong K_4$, so Lemma 2.3.0.24 implies that $G_1 = G'$ is $(2, 2)$-connected. $\square$

The inspiration for our next result is [30, Lemma 2.6], although our result is not quite a direct analogue. Nixon proves that if a $(2, 2)$-circuit has sufficiently large vertex-connectivity and edge-connectivity then this circuit contains nodes which do not lie in any critical set in the circuit. Instead of size, we consider structure. That is, we consider those $(2, 2)$-connected graphs that have, or rather do not have, a particular structure which is related to vertex-connectivity and edge-connectivity. Nixon's result also considers a condition relating to K_4 subgraphs, and while this idea is absent here it does appear in the proof of Lemma 3.2.0.1.

Lemma 3.1.0.6. *Let G be a $(2, 2)$-connected graph and let $C_1, \ldots, C_t$ be an ear decomposition of $\mathcal{M}_{(2,2)}(G)$. For $1 \leq i \leq t$ let $G[C_i] = H_i = (V_i, C_i)$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and let $X = V_t \setminus Y$. Suppose $Y \neq \emptyset$ and there does not exist a 3-edge-separation (H'_1, H'_2) of H_t such that $|V(H'_1)|, |V(H'_2)| \geq 2$ and H'_i is a subgraph of $H_t[Y]$ for some $i \in \{1, 2\}$. Let $\mathcal{U} = \{U \subseteq V_t \colon |U| \geq 2$ and U is critical in $H_t\}$. Take $\emptyset \neq \mathcal{X} \subseteq \mathcal{U}$ such that there exists $X_0 \in \mathcal{X}$ such that $X \subseteq X_0$. Let $\mathcal{Y} = V_t \setminus (\bigcup_{U \in \mathcal{X}} U)$. If $|\mathcal{Y}| \geq 2$ or*

$\bigcup_{U\in\mathcal{X}} H_t[U]$ *is disconnected then* $|\mathcal{Y}\cap V_3(G)|\geq 2$.

Proof. We begin by showing that $\mathcal{U}\neq\emptyset$ and that there exists $X_0\in\mathcal{U}$ such that $X\subseteq X_0$. If $t=1$ then $G=(V_t,C_t)=(Y,C_t)$ is a $(2,2)$-circuit and $X=\emptyset$. It follows from Lemma 2.3.0.2 that $\delta(G)=3$, so we may take $v\in V_3(G)$. Lemma 2.3.0.1 implies that $|V_t|\geq 5$ and so $V_t\setminus\{v\}\neq\emptyset$ and $i_G(V_t\setminus\{v\})=|C_t|-3=(2|V_t|-1)-3=2|V_t\setminus\{v\}|-2$. Therefore $V_t\setminus\{v\}\in\mathcal{U}$ and for all $U\in\mathcal{U}$, $X\subseteq U$. Alternatively, if $t\geq 2$ then as $Y\neq\emptyset$ Lemma 3.1.0.1 (iii) and (vi) together imply that X is critical in H_t and $|X|\geq 4$. Hence $X\in\mathcal{U}$. Therefore $\mathcal{U}\neq\emptyset$ and there exists $U\in\mathcal{U}$ such that $X\subseteq U$. Therefore we may take $\emptyset\neq\mathcal{X}\subseteq\mathcal{U}$ and $X_0\in\mathcal{X}$ such that $X\subseteq X_0$.

Let $\mathcal{X}=\{X_0,\ldots,X_k\}$ for some $k\geq 0$. Then $\mathcal{Y}=V_t\setminus(\bigcup_{j=0}^{k}X_j)$. Let $G_1,\ldots,G_n$ be the components of $\bigcup_{j=0}^{k}H_t[X_j]$, and for all $1\leq i\leq n$ let $G_i=(Z_i,F_i)$. As $|\mathcal{Y}|\geq 2$ or $\bigcup_{j=0}^{k}H_t[X_j]$ is disconnected, we have that for all $1\leq i\leq n$, $Z_i\neq V_t$. Hence Lemma 2.3.0.5 implies that for all $1\leq i\leq n$, Z_i is critical in H_t. That is, $i_{H_t}(Z_i)=2|Z_i|-2$. Now, Theorem 1.1.1.6 implies that for all $1\leq i\leq n$,

$$\sum_{u\in Z_i} d_{H_t[Z_i]}(u)=2i_{H_t}(Z_i)=4|Z_i|-4.$$

So for all $1\leq i\leq n$, $\sum_{u\in Z_i}(4-d_{H_t[Z_i]}(u))=4$. As $|Z_i|\geq 2$ for all $1\leq i\leq n$, and $|\mathcal{Y}|\geq 2$ or $\bigcup_{j=0}^{k}H_t[X_j]$ is disconnected, it follows that for all $1\leq i\leq n$ we have $|V_t\setminus Z_i|\geq 2$.

We supposed that there there does not exist a 3-edge-separation (H_1',H_2') of H_t such that $|V(H_1')|,|V(H_2')|\geq 2$ and H_i' is a subgraph of $H_t[Y]$ for some $i\in\{1,2\}$. So since there exists $1\leq i\leq n$ such that $X\subseteq X_0\subseteq Z_i$ and since for all $1\leq i\leq n$, $Z_i\neq V_t$, it follows that $d_{H_t}(Z_i, V_t \setminus Z_i) \geq 4$ for all $1 \leq i \leq n$. This implies that for all $1 \leq i \leq n$,

$$\begin{aligned}\sum_{u \in Z_i}(4 - d_{H_t}(u)) &= \sum_{u \in Z_i}(4 - (d_{H_t[Z_i]}(u) + d_{H_t}(\{u\}, V_t \setminus Z_i))) \\ &= \sum_{u \in Z_i}(4 - d_{H_t[Z_i]}(u)) - \sum_{u \in Z_i} d_{H_t}(\{u\}, V_t \setminus Z_i) \\ &= 4 - d_{H_t}(Z_i, V_t \setminus Z_i) \\ &\leq 0.\end{aligned}$$

Therefore,

$$\sum_{u \in V_t \setminus \mathcal{Y}} (4 - d_{H_t}(u)) = \sum_{i=1}^{n} \sum_{u \in Z_i}(4 - d_{H_t}(u)) \leq 0.$$

Now, as H_t is a $(2,2)$-circuit we have that $|C_t| = 2|V_t| - 1$. So Theorem 1.1.1.6 implies that $\sum_{v \in V_t} d_{H_t}(v) = 2|C_t| = 4|V_t| - 2$. That is,

$$2 = \sum_{v \in V_t}(4 - d_{H_t}(v)) = \sum_{v \in \mathcal{Y}}(4 - d_{H_t}(v)) + \sum_{v \in V_t \setminus \mathcal{Y}}(4 - d_{H_t}(v)) \leq \sum_{v \in \mathcal{Y}}(4 - d_{H_t}(v)).$$

Therefore, $|\{v \in \mathcal{Y} \colon d_{H_t}(v) \leq 3\}| \geq 2$. If $t = 1$ then $G = H_t$ and if $t \geq 2$ then Lemma 3.1.0.1 (iv) implies $d_{H_t}(y) = d_G(y)$ for all $y \in Y$. Either way, $\{v \in \mathcal{Y} \colon d_{H_t}(v) \leq 3\} = \{v \in \mathcal{Y} \colon d_G(v) \leq 3\}$. Lemma 2.1.0.10 implies that $\delta(G) \geq 3$ and hence we conclude that $|\mathcal{Y} \cap V_3(G)| \geq 3$. □

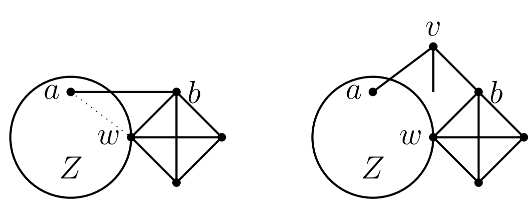


Figure 3.1: Possible structures of H_t discussed in the proof of Lemma 3.1.0.7.

Lemma 3.1.0.7. *Let G be a $(2,2)$-connected graph and let $C_1, \ldots, C_t$ be an ear decomposition of $\mathcal{M}_{(2,2)}(G)$. For $1 \leq i \leq t$ let $G[C_i] = H_i = (V_i, C_i)$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and let $X = V_t \setminus Y$. Suppose $t \geq 2$ and $Y \neq \emptyset$. Let $\mathcal{X} = \{X\} \cup \{U \subseteq V_t \colon H_t[U] \cong K_4\}$*

and let $\mathcal{Y} = V_t \setminus (\bigcup_{U\in\mathcal{X}} U)$. Suppose that $|\mathcal{Y}| \leq 1$ and $\bigcup_{U\in\mathcal{X}} H_t[U]$ is connected. If $\bigcup_{U\in\mathcal{X}} U \neq X$ and $|N_{H_t}(v) \cap X| \leq 1$ for all $v \in Y \cap V_3(H_t)$, then there exists a generalised edge-reduction, an edge-deletion, or a K_4^--reduction of G that is $(2,2)$-connected.

Proof. Let $X = X_0$ and set $\mathcal{X} = \{X_0, \ldots, X_k\}$ for some $k \geq 0$, so $\bigcup_{U\in\mathcal{X}} U = \bigcup_{j=0}^{k} X_j$. As $\bigcup_{U\in\mathcal{X}} U \neq X$, $k \geq 1$. As $\bigcup_{j=0}^{k} H_t[X_j]$ is connected and $k \geq 1$ we may, by reordering $X_1, \ldots, X_k$ if necessary, suppose without loss of generality that $X_j \cap (\bigcup_{i=0}^{j-1} X_i) \neq \emptyset$ for all $1 \leq j \leq k$. Let $s = \min\{j\colon \bigcup_{i=0}^{j} X_i = \bigcup_{i=0}^{k} X_i\}$. As $\bigcup_{U\in\mathcal{X}} U \neq X$, $s \geq 1$. Therefore, for all $0 \leq j \leq s-1$, $\emptyset \neq \bigcup_{i=0}^{j} X_j \subsetneq V_t$. As we have that $X_j \cap (\bigcup_{i=0}^{j-1} X_i) \neq \emptyset$ for all $1 \leq j \leq k$, Lemma 2.3.0.5 now implies that for all $1 \leq j \leq s-1$, $\bigcup_{i=0}^{j} X_i$ is critical in H_t and $d_{H_t}\left(\left(\bigcup_{i=0}^{j-1} X_i\right) \setminus X_j, X_j \setminus \bigcup_{i=0}^{j-1} X_i\right) = 0$.

If $|(\bigcup_{i=0}^{s-1} X_i) \cap X_s| \geq 2$ then, as $H_t[X_s] \cong K_4$, $|E(H_t[(\bigcup_{i=0}^{s-1} X_i) \cap X_s])| \geq 1$. As we have that $d_{H_t}\left(\left(\bigcup_{i=0}^{j-1} X_i\right) \setminus X_j, X_j \setminus \bigcup_{i=0}^{j-1} X_i\right) = 0$ for all $1 \leq j \leq s-1$, it follows that there exists $0 \leq j \leq s-1$ such that $|X_j \cap X_s| \geq 2$. By the definition of s we then have that $|X_j \cap X_s| \in \{2,3\}$. Hence $X_j \cap X_s$ is not critical in H_t and so Lemma 2.3.0.5 implies that $X_j \cup X_s = V_t$. Hence $X_j = X_0 = X$ and $Y = X_s \setminus X$. However, then there exists $y \in Y \cap V_3(H_t)$ such that $|N_{H_t}(y) \cap X| \geq 2$ which gives a contradiction. Therefore $|(\bigcup_{i=0}^{s-1} X_i) \cap X_s| = 1$.

Let $Z = \bigcup_{i=0}^{s-1} X_i$, let $Z \cap X_s = \{w\}$, and recall that Z is critical in H_t. We consider the cases $\mathcal{Y} = \emptyset$ and $|\mathcal{Y}| = 1$ separately. We observe that if $\mathcal{Y} = \emptyset$ then H_t has a particular structure (see the left of Figure 3.1), whereas if $|\mathcal{Y}| = 1$ then H_t has one of three possible structures (see the right of Figure 3.1).

Firstly, suppose that $\mathcal{Y} = \emptyset$. Then $Z \cup X_s = V_t$. As H_t is a $(2,2)$-circuit it follows that

$$\begin{aligned}
i_{H_t}(Z) + i_{H_t}(X_s) + d_{H_t}(Z, X_s) - i_{H_t}(Z \cap X_s) &= |C_t| \\
&= 2|V_t| - 1 \\
&= 2|Z| + 2|X_s| - (2+1) \\
&= i_{H_t}(Z) + i_{H_t}(X_s) + 1.
\end{aligned}$$

As $|Z \cap X_s| = 1$, $i_{H_t}(Z \cap X_s) = 0$ and hence $d_{H_t}(Z, X_s) = 1$. Let ab denote the unique edge of H_t such that $a \in Z \setminus X_s$ and $b \in X_s \setminus Z = X_s \setminus \{w\}$. Let $H_1' = G[V \setminus (X_s \setminus \{w\})]$ and let $H_2' = G[X_s \cup \{a\}]$. As $X_s \setminus \{w\} \subseteq Y$ we have that (H_1', H_2') is a 2-vertex-separation of G and $E(G[V(H_1') \cap V(H_2')]) = E(G[\{a, w\}])$. Hence we have two cases to consider depending on whether $aw \in E(G)$ or $aw \notin E(G)$. Take $v_1, v_2 \notin V(G)$ and let G' be a generalised edge-reduction of G that contracts the edge ab and deletes the edge bw.

If $aw \in E(G)$ then let (G_1, G_2) be the 2-separation of G on (H_1', H_2') adding $\{v_1, v_2\}$. Lemma 2.3.0.19 implies that G_1 is $(2,2)$-connected. Then $G' \cong G_1$ and so G' is $(2,2)$-connected. If $aw \notin E(G)$ then $aw \notin E(H_i')$ for $i \in \{1, 2\}$. Let (G_1, G_2) be the 1-separation of G on (H_1', H_2') adding $\{v_1, v_2\}$ and let (G_2', G_1') be the 1-separation of G on (H_2', H_1') adding $\{v_1, v_2\}$. As $\delta(G_2') = 2$, Lemma 2.1.0.10 implies G_2' is not $(2,2)$-connected. Hence Lemma 2.3.0.24 implies that G_1 is $(2,2)$-connected. Then G' is a K_4^--extension of G_1 and so Lemma 2.3.0.23 implies G' is $(2,2)$-connected.

On the other hand suppose that $|\mathcal{Y}| = 1$ and let $\mathcal{Y} = \{v\}$. Then $\emptyset \neq Z \cup X_s = V_t \setminus \{v\}$ and so, by Lemma 2.3.0.5, $Z \cup X_s$ is critical in H_t and $d_{H_t}(Z, X_s) = 0$. Lemma 2.3.0.4 (iii) then implies that $d_{H_t}(v) = 3$. Let $N_{H_t}(v) = \{a, b, c\}$. As $\kappa(H_t) \geq 2$ by Lemma 2.1.0.13, we may suppose without loss of generality that $a \in Z \setminus \{w\}$ and $b \in X_s \setminus \{w\}$. Then either $c \in Z \setminus \{w\}$, $c \in X_s \setminus \{w\}$, or $c = w$. Let $H_1' = G[V \setminus (X_s \setminus \{w\})]$ and let $H_2' = G[X_s \cup \{v\}]$. As $(X_s \setminus \{w\}) \cup \{v\} \subseteq Y$ we have that (H_1', H_2') is a 2-vertex-separation of G and $E(G[V(H_1') \cap V(H_2')]) = E(G[\{v, w\}])$. Hence we have two cases to consider depending on whether $vw \in E(G)$ or $vw \notin E(G)$. That is, depending on whether $c = w$ or $c \neq w$. Take $v_1, v_2 \notin V(G)$.

If $c = w$ (see the middle of Figure 3.2) then let G' be a generalised edge-reduction of G that contracts the edge vb and deletes the edge bw. Let (G_1, G_2) be the 2-separation of G on (H_1', H_2') adding $\{v_1, v_2\}$ and note that $G' \cong G_1$. Lemma 2.3.0.19 implies that G_1 is $(2,2)$-connected, so G' is $(2,2)$-connected.

If $c \in Z \setminus \{w\}$ (see the left of Figure 3.2) then let G' be a generalised edge-reduction of G that contracts the edge vb and deletes the edge vc. Let (G_1, G_2) be the 1-separation of G on (H_1', H_2') adding $\{v_1, v_2\}$, let (G_2', G_1') be the 1-separation of G on (H_2', H_1') adding

$\{v_1, v_2\}$, and note that $G' \cong G'_1$. Lemma 2.3.0.24 implies that G'_1 is $(2,2)$-connected, so G' is $(2,2)$-connected.

If $c \in X_s \setminus \{w\}$ (see the right of Figure 3.2) then let G' be a generalised edge-reduction of G that contracts the edge vb and deletes the edge bw. Let (G_1, G_2) be the 1-separation of G on (H'_1, H'_2) adding $\{v_1, v_2\}$ and let (G'_2, G'_1) be the 1-separation of G on (H'_2, H'_1) adding $\{v_1, v_2\}$. As $\delta(G'_2) = 2$, Lemma 2.1.0.10 implies that G'_2 is not $(2,2)$-connected. Hence Lemma 2.3.0.24 implies that G_1 is $(2,2)$-connected. Then G' is a K_4^--extension of G_1 and so Lemma 2.3.0.23 implies G' is $(2,2)$-connected. □

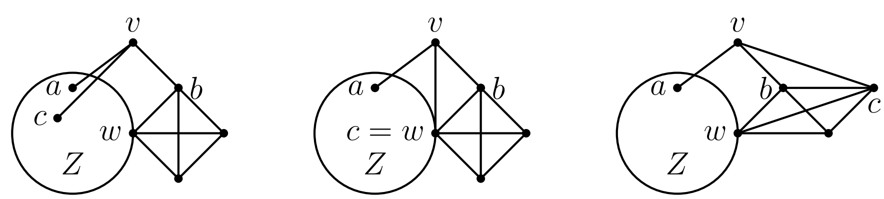


Figure 3.2: Possible structures of H_t when $|\mathcal{Y}| = 1$ discussed in the proof of Lemma 3.1.0.7.

Lemma 3.1.0.8. *Let G be a $(2,2)$-connected graph and let $C_1, \dots, C_t$ be an ear decomposition of $\mathcal{M}_{(2,2)}(G)$. For $1 \le i \le t$ let $G[C_i] = H_i = (V_i, C_i)$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and let $X = V_t \setminus Y$. Suppose that $t \ge 2$ and that for all $v \in Y \cap V_3^*(H_t)$ the following hold:*

(a) $|N_{H_t}(v) \cap X| \le 1$;

(b) for all $w \in N_{H_t}(v)$, there does not exist a K_4^--reduction of G deleting $\{v, w\}$; and

(c) there does not exist a $(2,1)$-reduction of G at v that is $(2,2)$-connected.

If $Y \cap V_3^*(H_t) \neq \emptyset$ then $H_t[Y \cap V_3^*(H_t)]$ is a forest, and for all $u \in Y \cap V_3^*(H_t)$ such that u is a leaf of $H_t[Y \cap V_3^*(H_t)]$ the following hold:

(i) $E(H_t[N_{H_t}(u)]) = \emptyset = E(G[N_{H_t}(u)])$; *and*

(ii) there exists a u-critical set, X^*, in H_t such that X^* is node-critical in H_t and $X \subsetneq X^*$.

Proof. As $Y \cap V_3^*(H_t) \neq \emptyset$, $H_t[Y \cap V_3^*(H_t)]$ is a subgraph of $H_t[V_3(H_t)]$ and hence is a forest by Lemma 2.3.0.7. So we may take $u \in Y \cap V_3^*(H_t)$ such that u is a leaf of $H_t[Y \cap V_3^*(H_t)]$. Let $N_{H_t}(u) = \{x, y, z\}$. By condition (a) we have that $E(H_t[\{x, y, z\}]) = E(G[\{x, y, z\}])$, and we may suppose without loss of generality that $y, z \notin X$ so $y, z \in Y$. Recall that as u is a plausible node of H_t, $N_{H_t}(u) \cap V_3^*(H_t) = N_{H_t}(u) \cap V_3(H_t)$. So, as u is a leaf of $H_t[Y \cap V_3^*(H_t)]$, $d_{H_t}(y) \geq 4$ or $d_{H_t}(z) \geq 4$.

By conditions (a) and (b), and the fact that $E(H_t[\{x, y, z\}]) = E(G[\{x, y, z\}])$, Lemma 3.1.0.5 implies that, for all $w \in N_{H_t}(u)$, there does not exist a K_4^--reduction of H_t deleting $\{u, w\}$. By condition (c) and the fact that $E(H_t[\{x, y, z\}]) = E(G[\{x, y, z\}])$, Lemma 3.1.0.4 implies there does not exist a $(2, 1)$-reduction of H_t at v that is a $(2, 2)$-circuit. Consequently, Lemma 2.3.0.25 implies that $E(H_t[N_{H_t}(u)]) = \emptyset$. As $E(H_t[N_{H_t}(u)]) = \emptyset$ and there does not exist a $(2, 1)$-reduction of H_t at u that is a $(2, 2)$-circuit, Lemma 2.3.0.6 implies there exist minimal u-critical sets (i.e. no proper subset is a u-critical set) X_1, X_2, and X_3 in H_t on $\{x, y\}$, $\{x, z\}$, and $\{y, z\}$ respectively. As $Y \neq \emptyset$, Lemma 3.1.0.1 (iii) implies that X is critical in H_t.

If $x \in X$ then, as $u \notin X \cup X_1 \cup X_2$, Lemma 2.3.0.5 implies that $X \cup X_1$ and $X \cup X_2$ are critical in H_t. As $y, z \notin X$, $X \subsetneq X \cup X_1$ and $X \subsetneq X \cup X_2$. As $d_{H_t}(y) \geq 4$ or $d_{H_t}(z) \geq 4$ it follows that $X \cup X_1$ is a node-critical in H_t or $X \cup X_2$ is node-critical in H_t. If $x \notin X$ then $\{x, y, z\} \subseteq Y$ and so, as u is a leaf of $H_t[Y \cap V_3^*(H_t)]$, we may suppose without loss of generality that $d_{H_t}(y), d_{H_t}(z) \geq 4$. As $u \notin X_1 \cup X_2$, Lemma 2.3.0.5 implies that $X_1 \cup X_2$ is critical in H_t. So as H_t is a $(2, 2)$-circuit and $N_{H_t}(u) \subseteq X_1 \cup X_2$ it follows that $X_1 \cup X_2 = V_t \setminus \{u\}$. Hence there exists $i \in \{1, 2\}$ such that $X \cap X_i \neq \emptyset$ and so , by Lemma 2.3.0.5, $X \cup X_i$ is a node-critical u-critical set in H_t. Moreover, as $z \in X_1 \setminus X$ and $y \in X_2 \setminus X$ we have that $X \subsetneq X \cup X_i$. As u was chosen to be an arbitrary leaf of $H_t[Y \cap V_3^*(H_t)]$ the result follows. □

3.2 Recursive Constructions

In this section we provide a characterisation of $(2, 2)$-connected graphs (see Theorem 3.2.0.6). This characterisation may be considered the culmination of all our work on

$(2, 2)$-connected graphs thus far. Before stating a full characterisation, Theorem 3.2.0.1 gives a useful partial characterisation of $(2, 2)$-connected graphs. In particular, we show that any $(2, 2)$-connected graph that is not a $(2, 2)$-circuit and does not contain a very specific structure may be reduced to a smaller $(2, 2)$-connected graph by one of three graph operations. The proof we present is quite involved, and makes use of many of the technical lemmas above, so we give an outline here.

We begin by setting up a $(2, 2)$-connected graph G, an ear decomposition of $\mathcal{M}_{(2,2)}(G)$, and sets X and Y as in, for example, Lemma 3.1.0.6. We then show that if $Y = \emptyset$ then G can be reduced. Then we suppose that $Y \neq \emptyset$ and proceed to show that there exists a node of G in Y and that any such node must have at most one neighbour in X. We then combine Lemma 3.1.0.7 and Lemma 3.1.0.6 to show that there exists a node of G in Y that is not contained in a subgraph of G isomorphic to K_4. Finally, we apply Lemma 3.1.0.8 to guarantee the existence of a particular node-critical set which we use to show that G must have a certain structure. This allows us to show that G can be reduced.

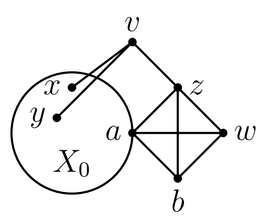


Figure 3.3: Structure of H_t considered towards the end of the proof of Theorem 3.2.0.1.

Theorem 3.2.0.1. *Let G be a $(2, 2)$-connected graph and let $C_1, \ldots, C_t$ be an ear decomposition of $\mathcal{M}_{(2,2)}(G)$. For $1 \leq i \leq t$ let $G[C_i] = H_i = (V_i, C_i)$. Let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and let $X = V_t \setminus Y$. Suppose $t \geq 2$. If there does not exist a 3-edge-separation (H_1', H_2') of H_t such that $|V(H_1')|, |V(H_2')| \geq 2$ and H_i' is a subgraph of $H_t[Y]$ for some $i \in \{1, 2\}$ then there exists a generalised edge-reduction, an edge-deletion, or a K_4^--reduction of G that is $(2, 2)$-connected.*

Proof. Let $G = (V, E)$. First suppose that $Y = \emptyset$. Lemma 3.1.0.1 (i) implies that $|C_t \setminus (\bigcup_{j=1}^{t-1} C_j)| = 1$, say $C_t \setminus (\bigcup_{j=1}^{t-1} C_j) = \{e\}$. Then $C_1, \ldots, C_{t-1}$ is an ear decomposition of

$\mathcal{M}_{(2,2)}(G[E \setminus \{e\}])$. Lemma 2.3.0.1 implies that $|E \setminus \{e\}| \geq 2$ and so Theorem 1.2.0.10 implies that $\mathcal{M}_{(2,2)}(G[E \setminus \{e\}])$ is connected. Therefore $G[E \setminus \{e\}]$ is an edge-deletion of G that is $(2,2)$-connected. Hence we may suppose instead that $Y \neq \emptyset$.

As $Y \neq \emptyset$, Lemma 3.1.0.1 (iii), (iv), (v), and (vi) together give us that X is critical in H_t, $Y \cap V_3(G) = Y \cap V_3(H_t) \neq \emptyset$, $G[Y]$ is connected, and $|X| \geq 4$. We now proceed by considering the various possible values of $\max\{|N_{H_t}(u) \cap X| \colon u \in Y \cap V_3(H_t)\}$. Note that $0 \leq \max\{|N_{H_t}(u) \cap X| \colon u \in Y \cap V_3(H_t)\} \leq 3$.

Suppose that $\max\{|N_{H_t}(u) \cap X| \colon u \in Y \cap V_3(H_t)\} = 3$. That is, suppose there exists $v \in Y \cap V_3(H_t)$ such that $|N_{H_t}(v) \cap X| = 3$. Then $N_{H_t}(v) \subseteq X$, let $N_{H_t}(v) = \{x, y, z\}$. Since $G[Y]$ is connected, $Y = \{v\}$. Moreover, Lemma 3.1.0.1 (i) implies that $\bigcup_{i=1}^{t-1} C_i = E \setminus \{vx, vy, vz\}$ and hence $C_1, \ldots, C_{t-1}$ is an ear decomposition of $\mathcal{M}_{(2,2)}(G[V \setminus \{v\}])$. Lemma 2.3.0.1 implies that $|E(G[V \setminus \{v\}])| \geq 2$ and so Theorem 1.2.0.10 implies that $\mathcal{M}_{(2,2)}(G[V \setminus \{v\}])$ is connected. Therefore $G[V \setminus \{v\}]$ is $(2,2)$-connected.

If $xy \notin E$ then Lemma 2.1.0.17 implies that the edge-addition of $G[V \setminus \{v\}]$ adding xy, which is equal to the $(2,1)$-reduction of G at v adding xy, is $(2,2)$-connected. Alternatively, if $xy \in E$ then $G[E \setminus \{xy\}]$, the edge-deletion of G at xy, is the $(2,1)$-extension of $G[V \setminus \{v\}]$ adding v and deleting xy. Lemma 2.1.0.19 implies that $G[E \setminus \{xy\}]$ is $(2,2)$-connected. Therefore we have that for all $u \in Y \cap V_3(H_t)$, if $|N_{H_t}(u) \cap X| = 3$ then there exists a generalised edge-reduction or an edge-deletion of G that is $(2,2)$-connected. So we may suppose that $0 \leq \max\{|N_{H_t}(u) \cap X| \colon u \in Y \cap V_3(H_t)\} \leq 2$.

Next suppose that $\max\{|N_{H_t}(u) \cap X| \colon u \in Y \cap V_3(H_t)\} = 2$. So there exists $v \in Y \cap V_3(H_t)$ such that $|N_{H_t}(v) \cap X| = 2$. Let $N_{H_t}(v) = \{x, y, z\}$. We may suppose without loss of generality that $\{y, z\} \subseteq X$ and $x \in Y$. We now consider which edges are present between the neighbours of v. If $\{xy, xz\} \subseteq E$ then $\{xy, xz\} \subseteq C_t$ and so, as X is critical in H_t,

$$i_{H_t}(X \cup \{v, x\}) = (2|X| - 2) + 1 + d_{H_t}(X, \{v, x\}) \geq 2|X| + 3 = 2|X \cup \{v, x\}| - 1.$$

As H_t is a $(2,2)$-circuit we have that $V_t = X \cup \{v, x\}$ and $d_{H_t}(X, \{v, x\}) = 4$. Hence, as

$\{v,x\} \subseteq Y$, $Y = \{v,x\}$ and $d_{H_t}(x) = 3 = d_G(x)$. So $(G[V \setminus \{v,x\}], G[\{v,x,y,z\}])$ is a 2-vertex-separation of G. Lemma 2.3.0.1 implies that $|E(G[V \setminus \{v,x\}])| \geq 2$. Lemma 3.1.0.1 (i) implies that $C_1 \ldots, C_{t-1}$ is an ear decomposition of $\mathcal{M}_{(2,2)}(G[V \setminus \{v,x\}])$, so Theorem 1.2.0.10 implies that $\mathcal{M}_{(2,2)}(G[V \setminus \{v,x\}])$ is connected and hence $G[V \setminus \{v,x\}]$ is $(2,2)$-connected. If $yz \notin E$ then the K_4^--reduction of G deleting $\{v,x\}$ equals the edge-addition of $G[V \setminus \{v,x\}]$ adding yz, and so this graph is $(2,2)$-connected by Lemma 2.1.0.17. If $yz \in E$ then the edge-deletion of G at yz is a K_4^--extension of $G[V \setminus \{v,x\}]$ and so is $(2,2)$-connected by Lemma 2.3.0.23. So if $\{xy, xz\} \subseteq E$ then there exists a K_4^--reduction or edge-deletion of G that is $(2,2)$-connected.

Suppose instead that $|\{xy, xz\} \cap E| \leq 1$. We may suppose without loss of generality that $xz \notin E$. As $xz \notin E$, $xz \notin C_t$ and so we may consider the $(2,1)$-reduction of H_t at v adding the edge xz. Denote this graph by H_t' and let J_1 be the unique subgraph of H_t' that is a $(2,2)$-circuit, which exists by Lemma 3.1.0.2. If $J_1 = H_t'$ then Lemma 3.1.0.4 implies that the $(2,1)$-reduction of G at v adding xz is $(2,2)$-connected. Alternatively, if $J_1 \neq H_t'$ then Lemma 3.1.0.2 implies that $V(J_1)$ is the minimal v-critical set in H_t on $\{x,z\}$. As $z \in X \cap V(J_1)$ and $v \notin X \cup V(J_1)$, Lemma 2.3.0.5 implies that both $V(J_1) \cap X$ and $V(J_1) \cup X$ are critical in H_t, and that $d_{H_t}(X, V(J_1)) = 0$. Therefore $xy \notin C_t$ and so, as $x \in Y$, $xy \notin E$. Let H_t'' denote the $(2,1)$-reduction of H_t at v adding xy and let J_2 be the unique subgraph of H_t'' that is a $(2,2)$-circuit, which exists by Lemma 3.1.0.2. By a similar argument as with J_1, we may suppose that $J_2 \neq H_t'$ and so $V(J_2)$ is the minimal v-critical set in H_t on $\{x,y\}$. As $N_{H_t}(v) \subseteq V(J_1) \cup X$,

$$i_{H_t}(V(J_1) \cup X \cup \{v\}) = (2|V(J_1) \cup X| - 2) + 3 = 2|V(J_1) \cup X \cup \{v\}| - 1.$$

As H_t is a $(2,2)$-circuit it follows that $V(J_1) \cup X = V_t \setminus \{v\}$, and hence

$$V(J_1) \cap Y = V(J_1) \setminus X = V_t \setminus (X \cup \{v\}) = Y \setminus \{v\}.$$

Let $F_1 = \{ab \in C_t \colon a \in V(J_1) \cap X, b \in V(J_1) \cap Y\}$, so $|F_1| = d_{H_t}(V(J_1) \cap X, V(J_1) \cap Y)$.

Then, as $V(J_1)$ is the minimal v-critical set in H_t on $\{x, z\}$,

$$\begin{aligned} E(J_1) &= E(H_t[V(J_1)]) \cup \{xz\} \\ &= E(H_t[V(J_1) \cap X]) \cup E(H_t[V(J_1) \cap Y]) \cup F_1 \cup \{xz\} \\ &= E(H_t[V(J_1) \cap X]) \cup E(H_t[Y \setminus \{v\}]) \cup F_1 \cup \{xz\}. \end{aligned}$$

Moreover,

$$0 = d_{H_t}(X, V(J_1)) = d_{H_t}(X \setminus V(J_1), V(J_1) \setminus X) = d_{H_t}(X \setminus V(J_1), V(J_1) \cap Y).$$

So,

$$d_{H_t}(X, Y \setminus \{v\}) = d_{H_t}(X \setminus V(J_1), V(J_1) \cap Y) + d_{H_t}(V(J_1) \cap X, V(J_1) \cap Y) = |F_1|.$$

Consequently, $F_1 = \{ab \in C_t \colon a \in X, b \in Y \setminus \{v\}\}$. Now,

$$\begin{aligned} E(H_t') &= E(H_t[V \setminus \{v\}]) \cup \{xz\} \\ &= E(H_t[X]) \cup E(H_t[Y \setminus \{v\}]) \cup \{ab \in C_t \colon a \in X, b \in Y \setminus \{v\}\} \cup \{xz\} \\ &= E(H_t[X]) \cup E(H_t[Y \setminus \{v\}]) \cup F_1 \cup \{x\}. \end{aligned}$$

Therefore,

$$E(H_t') \setminus E(H_t[X]) = E(H_t[Y \setminus \{v\}]) \cup F_1 \cup \{xz\} = E(J_1) \setminus E(H_t[V(J_1) \cap X]).$$

By a similar argument, considering J_2 rather than J_1, we see that

$$E(H_t') \setminus E(H_t[X]) = E(H_t[Y \setminus \{v\}]) \cup F_2 \cup \{xy\} = E(J_2) \setminus E(H_t[V(J_2) \cap X]).$$

Now as $V(J_1)$ and $V(J_2)$ are both critical in H_t and $x \in V(J_1) \cap V(J_2)$ and $v \notin V(J_1) \cup V(J_2)$, Lemma 2.3.0.5 implies that $V(J_1) \cup V(J_2)$ is critical in H_t. As $N_{H_t}(v) \subseteq$

$V(J_1) \cup V(J_2)$,

$$\begin{aligned} i_{H_t}(V(J_1) \cup V(J_2) \cup \{v\}) &= i_{H_t}(V(J_1) \cup V(J_2)) + 3 \\ &= (2|V(J_1) \cup V(J_2)| - 2) + 3 \\ &= 2|V(J_1) \cup V(J_2) \cup \{v\}| - 1. \end{aligned}$$

As H_t is a $(2,2)$-circuit it follows that $V(J_1) \cup V(J_2) = V_t \backslash \{v\}$. So, as $|X| \geq 4$ by Lemma 3.1.0.1 (vi), we observe that there exists $i \in \{1,2\}$ such that $V(J_i) \cap X$ is critical in H_t and $|V(J_i) \cap X| \geq 2$. Therefore $E(H_t[V(J_i) \cap X]) \neq \emptyset$ and so $E(H_t') \setminus E(H_t[X]) \subsetneq E(J_i)$. Then Lemma 3.1.0.3 implies that the $(2,1)$-reduction of G at v adding xz is $(2,2)$-connected (if $i=1$) or the $(2,1)$-reduction of G at v adding xy is $(2,2)$-connected (if $i=2$). Therefore we have that for all $u \in Y \cap V_3(H_t)$, if $|N_{H_t}(u) \cap X| \leq 2$ then there exists a generalised edge-reduction or an edge-deletion of G that is $(2,2)$-connected. So we may suppose that $0 \leq \max\{|N_{H_t}(u) \cap X| \colon u \in Y \cap V_3(H_t)\} \leq 1$.

So we have now shown that $Y \cap V_3(H_t) \neq \emptyset$ and that for all $u \in Y \cap V_3(H_t)$, $|N_{H_t}(u) \cap X| \leq 1$. Let $\mathcal{X} = \{X\} \cup \{U \subseteq V_t \colon H_t[U] \cong K_4\}$ and let $\mathcal{Y} = V_t \setminus (\bigcup_{U \in \mathcal{X}} U)$. Lemma 3.1.0.7 implies that if $|\mathcal{Y}| \leq 1$ and $\bigcup_{U \in \mathcal{X}} H_t[U]$ is connected then either $\bigcup_{U \in \mathcal{X}} U = X$ or there exists a generalised edge-reduction, an edge-deletion, or a K_4^--reduction of G that is $(2,2)$-connected. If $\bigcup_{U \in \mathcal{X}} U = X$ then $Y \cap V_3(H_t) = Y \cap V_3^*(H_t)$, so $Y \cap V_3^*(H_t) \neq \emptyset$. Alternatively, if $|\mathcal{Y}| \geq 2$ or $\bigcup_{U \in \mathcal{X}} H_t[U]$ is disconnected then, since there does not exist a 3-edge-separation (H_1', H_2') of H_t such that $|V(H_1')|, |V(H_2')| \geq 2$ and H_i' is a subgraph of $H_t[Y]$ for some $i \in \{1,2\}$, Lemma 3.1.0.6 implies that $|\mathcal{Y} \cap V_3(G)| \geq 2$ and hence $|Y \cap V_3^*(H_t)| \geq 2$. Moreover, as $|N_{H_t}(u) \cap X| \leq 1$ for all $u \in Y \cap V_3(H_t)$ we have that $\emptyset \neq Y \cap V_3^*(H_t) = Y \cap V_3^*(G)$. That is, the set of plausible nodes of G is non-empty.

Take $v \in Y \cap V_3^*(H_t)$. As $|N_{H_t}(v) \cap X| \leq 1$, $E(H_t[N_{H_t}(v)]) = E(G[N_{H_t}(v)])$. If there exists a $(2,1)$-reduction of H_t at v that is a $(2,2)$-circuit then, as $E(H_t[N_{H_t}(v)]) = E(G[N_{H_t}(v)])$, Lemma 3.1.0.4 implies there exists a $(2,1)$-reduction of G at v that is $(2,2)$-connected. If there exists a K_4^--reduction of H_t deleting $\{v,w\}$, for some $w \in N_{H_t}(v)$, then, as $|N_{H_t}(v) \cap X| \leq 1$ and $E(H_t[N_{H_t}(v)]) = E(G[N_{H_t}(v)])$, Lemma 3.1.0.5 implies there exists a K_4^--reduction of G deleting $\{v,w\}$ that is $(2,2)$-connected.

Therefore we may suppose that no such $(2,1)$-reductions or K_4^--reductions of H_t exist, and so Lemma 2.3.0.25 implies that $E(H_t[N_{H_t}(v)]) = \emptyset$. As v was chosen arbitrarily from $Y \cap V_3^*(H_t)$ it follows that for all $u \in Y \cap V_3^*(H_t)$, $|N_{H_t}(u) \cap X| \leq 1$ and $E(H_t[N_{H_t}(u)]) = \emptyset = E(G[N_{H_t}(u)])$.

As $Y \cap V_3^*(H_t) \neq \emptyset$, Lemma 3.1.0.8 implies that if there does not exist a $(2,1)$-reduction of G that is $(2,2)$-connected, and there does not exist a K_4^--reduction of G (that is $(2,2)$-connected by Lemma 3.1.0.5), then $H_t[Y \cap V_3^*(H_t)]$ is a forest and for all $u \in Y \cap V_3^*(H_t)$ such that u is a leaf of $H_t[Y \cap V_3^*(H_t)]$ we have $E(H_t[N_{H_t}(u)]) = \emptyset = E(G[N_{H_t}(u)])$ and there exists a node-critical u-critical set in H_t that contains X as a proper subset. As $Y \cap V_3^*(H_t)$ is finite, we may choose $v \in Y \cap V_3^*(H_t)$ (not necessarily a leaf of $H_t[Y \cap V_3^*(H_t)]$) and a node-critical v-critical set in H_t, X^*, such that $X \subsetneq X^*$ and if $u \in Y \cap V_3^*(H_t)$ and Y^* is a node-critical u-critical set in H_t such that $X \subseteq Y^*$ then $|X^*| \geq |Y^*|$. Let $N_{H_t}(v) \setminus X^* = \{z\}$, so $d_{H_t}(z) \geq 4$.

Let $Z = V_t \setminus (X^* \cup \{v\})$. As $|N_{H_t}(v) \cap X^*| = 2$, $X^* \cup \{v\}$ is critical in H_t and so Lemma 2.3.0.4 (iii) implies that $Z \cap V_3(H_t) \neq \emptyset$. Then $H_t[Z \cap V_3(H_t)]$ is a subgraph of $H_t[V_3(H_t)]$ and so is a forest by Lemma 2.3.0.7. Therefore we can take $w \in Z \cap V_3(H_t)$ such that w is a leaf of $H_t[Z \cap V_3(H_t)]$. As $d_{H_t}(z) \geq 4$, $w \neq z$. If $|N_{H_t}(w) \cap X^*| = 3$ then $i_{H_t}(X^* \cup \{w\}) = 2|X^* \cup \{w\}| - 1$, but as $v \notin X^* \cup \{w\}$ this contradicts the fact that H_t is a $(2,2)$-circuit. So $|N_{H_t}(w) \cap X^*| \leq 2$. If $|N_{H_t}(w) \cap X^*| = 2$ then $X^* \cup \{w\}$ is a node-critical v-critical set such that $X \subsetneq X^* \cup \{w\}$ and $|X^* \cup \{w\}| > |X^*|$, which contradicts the maximality of $|X^*|$. Therefore $|N_{H_t}(w) \cap X^*| \leq 1$. Hence, either w is a leaf node of H_t or w is a series node of H_t and $|N_{H_t}(w) \cap X^*| = 1$ and the neighbour of w in X^* is a node, or $H_t[N_{H_t}[w]] \cong K_4$.

If $H_t[N_{H_t}[w]] \not\cong K_4$ then recall that $E(H_t[N_{H_t}(w)]) = \emptyset$, and hence Lemma 2.3.0.8 implies that there exists Z', a w-critical node-critical set in H_t, such that $X \subsetneq X^* \subsetneq Z \subsetneq Z'$. This contradicts the maximality of $|X^*|$, so instead we must have that $H_t[N_{H_t}[w]] \cong K_4$. Now suppose that Z contains a node of H_t that is not a leaf of $H_t[Z \cap V_3(H_t)]$. Then there exists $w' \in Z \cap V_3^*(H_t)$ such that w' is not a leaf of $H_t[Z \cap V_3(H_t)]$ with $a, b \in N_{H_t}(w) \cap Z \cap V_3(H_t)$ where a is a leaf of $H_t[Z \cap V_3(H_t)]$. Then $H_t[N_{H_t}[a]] \cong K_4$, and so as $a, w' \in V_3(H_t)$ it follows that $ab \in C_t$. However, as $b \in V_3(H_t)$ this contradicts the fact that $H_t[Z \cap V_3]$ is a forest. Therefore, for all $w \in Z \cap V_3(H_t)$ we have that w is a leaf of $H_t[Z \cap V_3(H_t)]$ and $H_t[N_{H_t}(w)] \cong K_4$.

Let $X_0 = X^*$ and let $\mathcal{X} = \{X_0\} \cup \{U \subseteq V_t \colon U \nsubseteq X_0 \text{ and } H_t[U] \cong K_4\}$. As $Z \cap V_3(H_t) \neq \emptyset$ we have that $\mathcal{X} \neq X_0$, so let $\mathcal{X} = \{X_0, \dots, X_k\}$ for some $k \geq 1$. If $|V_t \setminus (\bigcup_{i=0}^{k} X_i)| \geq 2$ or $\bigcup_{i=0}^{k} H_t[X_i]$ is disconnected then Lemma 3.1.0.6 implies that $|(V_t \setminus (\bigcup_{i=0}^{k} X_i)) \cap V_3(G)| \geq 2$. As $V_t \setminus (\bigcup_{i=0}^{k} X_i) \subseteq V_t \setminus X \subseteq Y$, it follows that there exists $v' \in Y \setminus \{v\} \subseteq Z$ such that $d_{H_t}(v') = 3$. However as $v' \notin \bigcup_{i=1}^{k} X_i$ we see that $H_t[N_{H_t}(v')] \ncong K_4$, which is a contradiction. Therefore we have $|V_t \setminus (\bigcup_{i=0}^{k} X_i)| \leq 2$ and $\bigcup_{i=0}^{k} H_t[X_i]$ is connected. As $v \in V_t \setminus (\bigcup_{i=0}^{k} X_i)$, $V_t \setminus (\bigcup_{i=0}^{k} X_i) = \{v\}$.

Take $w \in Z \cap V_3(H_t)$. As $\bigcup_{i=0}^{k} H_t[X_i]$ is connected and $k \geq 1$ we may, by reordering $X_1, \dots, X_k$ if necessary, suppose without loss of generality that $X_j \cap (\bigcup_{i=0}^{j-1} X_i) \neq \emptyset$ for all $1 \leq j \leq k$. As $v \notin \bigcup_{i=0}^{k} X_i$ and $X_0 \cap X_1 \neq \emptyset$, Lemma 2.3.0.5 implies that $X_0 \cup X_1$ and $X_0 \cap X_1$ are critical in H_t, and $d_{H_t}(X_0, X_1) = 0$. As $X_1 \nsubseteq X_0$, the maximality of $X^* = X_0$ implies that $z \in X_1$. As H_t is a $(2,2)$-circuit and $N_{H_t}(v) \subseteq X_0 \cup X_1$, it follows that $k = 1$ and so $X_0 \cup X_1 = V_t \setminus \{v\}$. Hence $w \in X_1 \setminus X_0$ and $z \in N_{H_t}(w) \cap (X_1 \setminus X_0)$. Let $N_{H_t}(w) = \{a, b, z\}$. As $X_0 \cap X_1$ is critical in H_t and $X_1 \nsubseteq X_0$, $|X_0 \cap X_1| = 1$. So we may suppose without loss of generality that $N_{H_t}(w) \cap X_0 = \{a\}$. As $d_{H_t}(X_0, X_1) = 0$, $N_{H_t}(z) = \{v, a, b, w\}$ and $d_{H_t}(b) = 3$ (see Figure 3.3).

Let $H_1' = G[V \setminus \{b, w, z\}]$ and let $H_2' = G[\{v, a, b, w, z\}]$. As $\{v, b, w, z\} \subseteq Y$, $va \notin E$ and (H_1', H_2') is a 2-vertex-separation of G. Take $v_1, v_2 \notin V$. Let (G_1, G_2) be the 1-separation of G on (H_1', H_2') adding $\{v_1, v_2\}$, and let (G_2', G_1') be the 1-separation of G on (H_2', H_1') adding $\{v_1, v_2\}$. As $\delta(G_2') = 2$, Lemma 2.1.0.10 implies that G_2' is not $(2,2)$-connected. Then Lemma 2.3.0.24 implies that G_1 is $(2,2)$-connected. Let G' be a generalised edge-reduction of G that contracts vz and deletes za. Then G' is isomorphic to a K_4^--extension of G_1 and so Lemma 2.3.0.23 implies G' is $(2,2)$-connected. Therefore we have shown that there exists a generalised edge-reduction, an edge-deletion, or a K_4^--reduction of G that is $(2,2)$-connected. $\square$

In order to extend the previous result to apply to $(2,2)$-connected graphs without the condition on 3-edge-separations we introduce some additional terminology and results

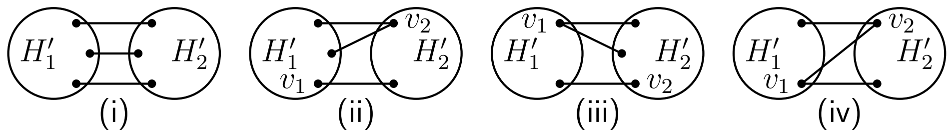


Figure 3.4: Possible structures of G with respect to a 3-edge-separation (H_1', H_2') of G such that $|V(H_1')|, |V(H_2')| \geq 2$ and H_2' is a subgraph of $H_t[Y]$, discussed in the proof of Theorem 3.2.0.4.

from [24]. In particular, their concept of an 'atom' of a graph G will be central to our proof of Theorem 3.2.0.4. Let (H_1, H_2) and (H_3, H_4) be a 2-vertex-separation and a 3-edge-separation respectively of some graph G. We say (H_1, H_2) is a **trivial** 2-vertex-separation of G if $H_i \cong K_4$ for some $i \in \{1, 2\}$, and (H_1, H_2) is a **non-trivial** 2-vertex-separation of G otherwise. Similarly, we say (H_3, H_4) is a **non-trivial** 3-edge-separation of G if $|V(G[E(G) \setminus (E(H_3) \cup E(H_4))])| = 6$, and (H_3, H_4) is a **trivial** 3-edge-separation of G otherwise. An **atom** of G is a subgraph, H, of G such that H is part of a non-trivial 2-vertex-separation or non-trivial 3-edge-separation of G and no proper subgraph of H is. Note that there exists an atom of G if and only if there exists a non-trivial 2-vertex-separation or non-trivial 3-edge-separation of G.

Theorem 3.2.0.2. *[24, Theorem 2.2] Let G be a $(2,2)$-circuit and suppose that $G \not\cong K_5^-, B_1, B_2$. If G has no non-trivial 2-vertex-separations and no non-trivial 3-edge-separations then there exists $\{u, v\} \subseteq V_3(G)$ such that there exist $(2,1)$-reductions of G at u and at v that are $(2,2)$-circuits.*

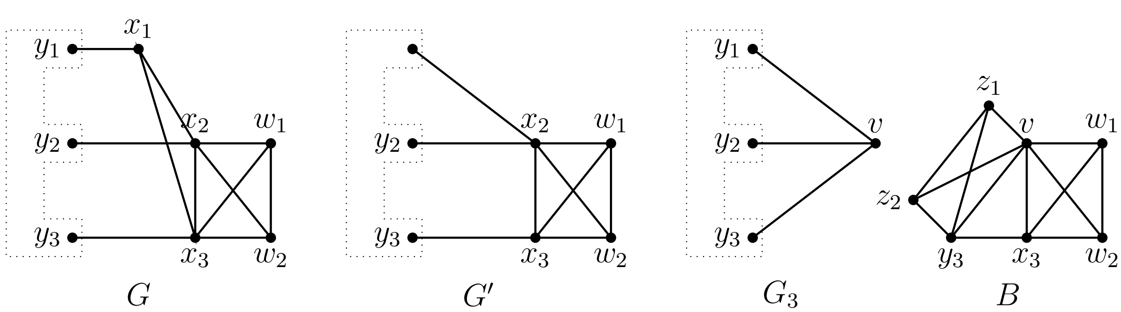


Figure 3.5: The relevant graphs when $G_2 \cong B_1$ in Case 1 in the proof of Theorem 3.2.0.4.

Theorem 3.2.0.3. *[24, Theorem 2.3] Let G be a $(2,2)$-circuit. If $G \ncong K_5^-, B_1$, then there exists G', a K_4^--reduction or a generalised edge-reduction of G, such that G' is a $(2,2)$-circuit.*

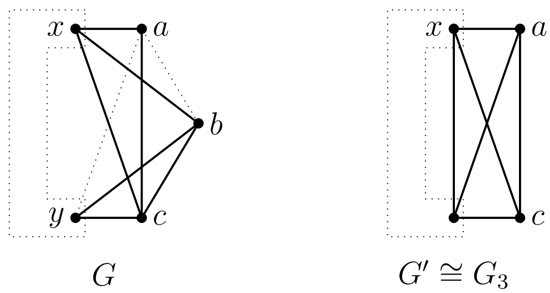


Figure 3.6: The relevant graphs when $G_2' \cong K_5^-$ in Subcase 3b in the proof of Theorem 3.2.0.4.

Theorem 3.2.0.4. *Let G be a graph. If G is $(2,2)$-connected then there exists G', an edge-deletion or a K_4^--reduction or a generalised edge-reduction of G, such that G' is $(2,2)$-connected if and only if $G \ncong K_5^-, B_1$.*

Proof. Let $G = (V, E)$ and let $C_1, \dots, C_t$ be an ear decomposition of $\mathcal{M}_{(2,2)}(G)$. For $1 \leq i \leq t$ let $G[C_i] = H_i = (V_i, C_i)$, let $Y = V_t \setminus \bigcup_{i=1}^{t-1} V_i$ and let $X = V_t \setminus Y$. If $t = 1$ then G is a $(2,2)$-circuit and Theorem 3.2.0.3 implies we are done. If $t \geq 2$ and there does not exist a 3-edge-separation (A_1, A_2) of H_t such that $|V(A_1)|, |V(A_2)| \geq 2$ and A_i is a subgraph of $H_t[Y]$ for some $i \in \{1, 2\}$ then Theorem 3.2.0.1 implies we are done. Therefore we may suppose that $t \geq 2$ and there exists a 3-edge-separation (A_1, A_2) of H_t such that $|V(A_1)|, |V(A_2)| \geq 2$ and A_2 is a subgraph of $H_t[Y]$. Let $F = E \setminus (E(A_1) \cup E(A_2))$.

As G is $(2,2)$-connected, Lemma 2.1.0.14 implies that $\kappa(G) \geq 2$ and so G has one of four possible structures. To be more precise, either (A_1, A_2) is a non-trivial 3-edge-separation of G or (A_1, A_2) is a 2-vertex-separation of G and we have three possibilities to consider regarding the three edges between $V(A_1)$ and $V(A_2)$. See Figure 3.4 for an illustration of this. We proceed to show that there exists a subgraph of A_2 that is an atom of H_t.

If (A_1, A_2) is a non-trivial 3-edge-separation of H_t (see Figure 3.4 (i)) then some subgraph of A_2 is an atom of H_t. If (A_1, A_2) is a trivial 3-edge-separation of G (see Figure 3.4 (ii), (iii), or (iv)) then in case (ii) let v_2 be the vertex of A_2 with two neighbours in $V(A_1)$ and let v_1 be the vertex of A_1 that is not adjacent to v_2 and has a neighbour in $V(A_2)$, in case (iii) let v_1 be the vertex of A_1 with two neighbours in $V(A_2)$ and let v_2 be the vertex of A_2 that is not adjacent to v_1 and has a neighbour in $V(A_2)$, in case (iv) let v_i be the vertex of A_i with exactly two neighbours in $V(A_j)$ for $i, j \in \{1, 2\}$ and $i \neq j$. Then in each of cases (ii), (iii), and (iv) we have that $(H_t[V(A_1) \cup \{v_2\}], H_t[V(A_2) \cup \{v_1\}])$ is a non-trivial 2-vertex-separation of H_t and so some subgraph of $H_t[V(A_2) \cup \{v_1\}]$ is an atom of H_t. Let A_2' denote a subgraph of A_2 (in case (i)) or $H_t[V(A_2) \cup \{v_1\}]$ (in cases (ii), (iii), and (iv)) that is an atom of H_t, and let (A_1', A_2') denote the corresponding 2-vertex-separation or 3-edge-separation of H_t. We split the remainder of the proof into three cases based on whether (A_1', A_2') is a 3-edge-separation or a 2-vertex-separation of H_t, and if it is a 2-vertex-separation then on whether $E(A_1') \cap E(A_2') = \emptyset$ or $E(A_1') \cap E(A_2') \neq \emptyset$.

Case 1. (A_1', A_2') is a non-trivial 3-edge-separation of H_t.

Let $A_3' = G[V \setminus V(A_2')]$. As (A_1', A_2') is a non-trivial 3-edge-separation of H_t, $V(A_2') \subseteq V(A_2) \subseteq Y$. Therefore, (A_3', A_2') is a non-trivial 3-edge-separation of G. Let $F = E \setminus (E(A_2') \cup E(A_3')) = \{x_1y_1, x_2y_2, x_3y_3\}$, where $x_i \in V(A_2')$ for $1 \leq i \leq 3$. Take $v \notin V$. Let (G_1, G_2) be the 3-separation of H_t on (A_1', A_2') adding $\{v\}$ and let (G_3, G_2) be the 3-separation of G adding on (A_3', A_2') adding $\{v\}$. Lemma 2.3.0.17 implies that G_2 is a $(2, 2)$-circuit and Lemma 2.3.0.21 implies that G_3 is $(2, 2)$-connected. As A_2' is an atom of H_t, there do not exist any non-trivial 2-vertex-separations or non-trivial 3-edge-separations of G_2 and $G_2 \not\cong B_2$. Hence Theorem 3.2.0.2 implies that $G_2 \cong K_5^-$ or $G_2 \cong B_1$ or there exist two nodes of G_2 such that there exist $(2, 1)$-reductions of G_2 at these nodes that are $(2, 2)$-circuits.

If $G_2 \cong K_5^-$ then $A_2' \cong K_4$. Let G' be a generalised edge-reduction of G that contracts x_1y_1 and deletes x_1x_2. Then there exists a $(2, 1)$-extension of a $(2, 1)$-extension of G_2 that is isomorphic to G', and so G' is $(2, 2)$-connected by Lemma 2.1.0.19. If $G_2 \cong B_1$ then we may suppose without loss of generality that $d_{G_2}(x_1) = 3$ and $d_{G_2}(x_2), d_{G_2}(x_3) \geq 5$ (see Figure 3.5). Therefore $d_{A_2'}(x_1) = 2$. Let G' be a generalised edge-reduction of G that contracts x_1y_1 and deletes x_1x_3. Then let B be a graph such that $B \cong B_2$, with $V(B) = \{v, x_3, y_3, w_1, w_2, z_1, z_2\}$ where $\{w_1, w_2\} = V(A'_2) \setminus \{x_1, x_2, x_3\}$, and $z_1, z_2 \notin V$, where $B[\{v, y_3, z_1, z_2\}] \cong K_4 \cong B[\{v, x_3, w_1, w_2\}]$. Then G' is isomorphic to the 1-join of (G_3, B) and hence G' is $(2,2)$-connected by Lemma 2.3.0.22.

Alternatively, suppose there exist two nodes of G_2 such that there exist $(2,1)$-reductions of G_2 at these nodes that are $(2,2)$-circuits. Then there exists $u \in V_3(G_2) \setminus \{v\} = V_3(G_2) \cap V(A'_2)$ such that there exists a $(2,1)$-reduction of G_2 at u that is a $(2,2)$-circuit. Let $N_{G_2}(u) = \{r, s, t\}$ and suppose without loss of generality that the $(2,1)$-reduction of G_2 at u adding rs is a $(2,2)$-circuit. Denote this graph by G'_2. Note that G'_2 is isomorphic to a generalised edge-reduction of G_2 that contracts ur (or us) and deletes ut. As G'_2 is a $(2,2)$-circuit, $t \neq v$. Without loss of generality we may also suppose that $s \neq v$. Let $N_G(u) = \{r', s, t\}$, where $r' = r$ if $r \neq v$ and $r' \in \{y_1, y_2, y_3\}$ if $r = v$ (i.e., if $u \in \{x_1, x_2, x_3\}$). Let G' be generalised edge-reduction of G that contracts ur' and deletes ut. Then G' is isomorphic to a 3-join of (G_3, G'_2), so Lemma 2.3.0.20 implies G' is $(2,2)$-connected.

Case 2. (A'_1, A'_2) is a non-trivial 2-vertex-separation of H_t and $E(A'_1) \cap E(A'_2) \neq \emptyset$.

Let $V(A'_1) \cap V(A'_2) = \{x, y\}$ and let $A'_3 = G[V \setminus V(A'_2)]$. As (A'_1, A'_2) is a non-trivial 2-vertex-separation of H_t and $V(A'_2) \subseteq V(A_2) \cup \{v_1\}$, $|V(A'_2) \cap X| \leq 1$. Therefore (A'_3, A'_2) is a non-trivial 2-vertex-separation of G and $E(A'_2) \cap E(A'_3) = \{xy\}$. Take $w_1, w_2 \notin V$. Let (G_1, G_2) be the 2-separation of H_t on (A'_1, A'_2) adding $\{w_1, w_2\}$ and let (G_3, G_2) be the 2-separation of G on (A'_3, A'_2) adding $\{v_1, v_2\}$. Lemma 2.3.0.15 implies that G_2 is a $(2,2)$-circuit and Lemma 2.3.0.19 implies that G_3 is $(2,2)$-connected. As A'_2 is an atom of H_t, there do not exist any non-trivial 3-edge-separations of G_2. If there exists a non-trivial 2-vertex-separation, (Z_1, Z_2), of G_2 such that $V(Z_1) \cap V(Z_2) = \{z_1, z_2\}$ then as A'_2 is an atom of H_t it follows that $\{z_1, z_2\} = \{x, y\}$ and there exists $i \in \{1, 2\}$ such that $Z_i \cong K_4$. However, this contradicts the fact that (Z_1, Z_2) is a non-trivial 2-vertex-separation. So, there do not exist any non-trivial 2-vertex-separations or non-trivial 3-edge-separations of G_2. As A'_2 is an atom of H_t we see that $G_2 \ncong B_1$. Also, as $\kappa(G_2) \leq 2$, $G_2 \ncong K_5^-$. Consequently, Theorem 3.2.0.2 implies that $G_2 \cong B_2$ or there exist two nodes of G_2 such that there exist $(2,1)$-reductions of G_2 at these nodes that are $(2,2)$-circuits.

If $G_2 \cong B_2$ then we may suppose without loss of generality that $d_{A'_2}(x) = 2$ and set $N_{A'_2}(x) = \{y, a\}$. Let G' be a generalised edge-reduction of G that contracts xa and deletes xy. Then $G' \cong G_3$, so G' is $(2,2)$-connected. Alternatively, suppose there exist two nodes of G_2 such that there exist $(2,1)$-reductions of G_2 at these nodes that are $(2,2)$-circuits. Then there exists $u \in V_3(G_2) \cap V(A'_2)$ such that $u \notin \{x, y\}$ and there exists a $(2,1)$-reduction of G_2 at u that is a $(2,2)$-circuit. Take $e \in E(K[N_{G_2}(u)])$ and suppose without loss of generality that the $(2,1)$-reduction of G_2 at u adding e is a $(2,2)$-circuit. Denote this graph by G'_2. Note that $u \notin \{w_1, w_2, x, y\}$, and so $u \in V(A'_2) \cap Y$ and hence $u \in V_3(G)$ and $e \notin E$. Let G' be the $(2,1)$-reduction of G at u adding e. Then G' is the 2-join of (G_3, G'_2) and so G' is $(2,2)$-connected by Lemma 2.3.0.18.

Case 3. (A'_1, A'_2) is a non-trivial 2-vertex-separation of H_t and $E(A'_1) \cap E(A'_2) = \emptyset$.

Let $V(A'_1) \cap V(A'_2) = \{x, y\}$ and let $A'_3 = G[V \setminus V(A'_2)]$. As (A'_1, A'_2) is a non-trivial 2-vertex-separation of H_t and $V(A'_2) \subseteq V(A_2) \cup \{v_1\}$, $|V(A'_2) \cap X| \leq 1$. Therefore (A'_3, A'_2) is a non-trivial 2-vertex-separation of G and $E(A'_2) \cap E(A'_3) = \emptyset$. Take $w_1, w_2 \notin V$. Let (G_1, G_2) be the 1-separation of H_t on (A'_1, A'_2) adding $\{w_1, w_2\}$, and let (G_3, G_2) be the 1-separation of G on (A'_3, A'_2) adding $\{w_1, w_2\}$. Let (G'_2, G'_1) be the 1-separation of H_t on (A'_2, A'_1) adding $\{w_1, w_2\}$ and let (G'_2, G'_3) be the 1-separation of G on (A'_2, A'_3) adding $\{w_1, w_2\}$. Lemma 2.3.0.13 implies that G_2 is a $(2,2)$-circuit and G'_2 is not a $(2,2)$-circuit, or vice versa. We consider these possibilities as separate subcases.

Subcase 3a. G_2 is a $(2,2)$-circuit.

As G'_2 is a proper subgraph of G_2, G'_2 is not $(2,2)$-connected. Therefore Lemma 2.3.0.24 implies that G_3 is $(2,2)$-connected. As A'_2 is an atom of H_t, there do not exist any non-trivial 3-edge-separations of G_2. If there exists a non-trivial 2-vertex-separation, (Z_1, Z_2), of G_2 such that $V(Z_1) \cap V(Z_2) = \{z_1, z_2\}$ then as A'_2 is an atom of H_t it follows that $\{z_1, z_2\} = \{x, y\}$ and there exists $i \in \{1, 2\}$ such that $Z_i \cong K_4$. However, this contradicts the fact that (Z_1, Z_2) is a non-trivial 2-vertex-separation. So, there do not exist any non-trivial 2-vertex-separations or non-trivial 3-edge-separations of G_2. Also, as $\kappa(G_2) \leq 2$, $G_2 \ncong K_5^-$. Consequently, Theorem 3.2.0.2 implies that $G_2 \cong B_1$ or $G_2 \cong B_2$ or there exist two nodes of G_2 such that there exist $(2,1)$-reductions of G_2 at these nodes that are $(2,2)$-circuits.

If $G_2 \cong B_1$ then $A'_2 \cong K_4^-$. Let $V(A'_2) \setminus \{x,y\} = \{r,s\}$ and let G' denote the K_4^--reduction of G deleting $\{r,s\}$. Then $G' = G_3$ and so G' is $(2,2)$-connected. If $G_2 \cong B_2$ then let $V(A'_2) \setminus \{x,y\} = \{a,b,c\}$. We may suppose without loss of generality that $N_{G_2}(x) \cap \{a,b,c\} = \{a\}$. Let G' be a generalised edge-reduction of G that contracts ax and deletes ay. Then G' is isomorphic to a K_4^--reduction of G_3 and so Lemma 2.3.0.23 implies G' is $(2,2)$-connected.

Alternatively, suppose there exist two nodes of G_2 such that there exist $(2,1)$-reductions of G_2 at these nodes that are $(2,2)$-circuits. Then there exists $u \in V_3(G_2) \cap V(A'_2)$ such that $u \notin \{x,y\}$ and there exists a $(2,1)$-reduction of G_2 at u that is a $(2,2)$-circuit. Take $e \in E(K[N_{G_2}(u)])$ and suppose without loss of generality that the $(2,1)$-reduction of G_2 at u adding e is a $(2,2)$-circuit. Denote this graph by $\tilde{G}_2$. Note that $u \notin \{w_1, w_2, x, y\}$, and so $u \in V(A'_2) \cap Y$ and hence $u \in V_3(G)$ and $e \notin E$. Let G' be the $(2,1)$-reduction of G at u adding e. Then G' is the 1-join of $(G_3, \tilde{G}_2)$ and so G' is $(2,2)$-connected by Lemma 2.3.0.22.

Subcase 3b. G'_2 is a $(2,2)$-circuit.

Lemma 2.3.0.24 implies that G'_3 is $(2,2)$-connected. As A'_2 is an atom of H_t, there do not exist any non-trivial 2-vertex-separations or non-trivial 3-edge-separations of G'_2. Hence Theorem 3.2.0.2 implies that $G'_2 \cong K_5^-$ or $G'_2 \cong B_1$ or $G'_2 \cong B_2$ or there exist two nodes of G'_2 such that there exist $(2,1)$-reductions of G'_2 at these nodes that are $(2,2)$-circuits.

If $G'_2 \cong K_5^-$ then let $\{a,b,c\} = V(A'_2) \setminus \{x,y\}$. We may suppose without loss of generality that $d_{G'_2}(c) = 4$ and that $E(K[V(A'_2)]) \setminus E(G'_2) \in \{ab, ay\}$ (see Figure 3.6). Let G' be a generalised edge-reduction of G that contracts by and deletes bc. Then $G' \cong G'_3$ and so G' is $(2,2)$-connected.

If $G'_2 \cong B_1$ then let $\{a,b,c,d\} = V(A'_2) \setminus \{x,y\}$. As A'_2 is an atom of H_t we may suppose without loss of generality that $3 = d_{G'_2}(x) \leq d_{G'_2}(y)$, $N_{G'_2}(x) = \{a,b,y\}$, and $d_{G'_2}(a) \leq d_{G'_2}(b)$ (see Figure 3.7). Let G' be the generalised edge-reduction of G that contracts ay (to u) and deletes ab. Then let B be a graph such that $B \cong B_2$, with $V(B) = \{y, b, c, d, x, w_1, w_2\}$ where $\{w_1, w_2\} = V(G'_3) \setminus V(G)$, where $B[\{y, b, c, d\}] \cong K_4 \cong B[\{y, x, w_1, w_2\}]$. Then G' is isomorphic to the 2-join of (G'_3, B) and hence G' is $(2, 2)$-connected by Lemma 2.3.0.18.

Alternatively, suppose there exist two nodes of G'_2 such that there exist $(2, 1)$-reductions of G'_2 at these nodes that are $(2, 2)$-circuits. Then either there exists $u \in V(A'_2) \setminus \{x, y\}$ such that there exists a $(2, 1)$-reduction of G'_2 at u that is a $(2, 2)$-circuit or the set of nodes of G'_2 such that there exist $(2, 1)$-reductions of G'_2 at these nodes that are $(2, 2)$-circuits is $\{x, y\}$. Firstly, suppose there exists $u \in V(A'_2) \setminus \{x, y\}$ and $e \in E(K[N_{G'_2}(u)])$ such that the $(2, 1)$-reduction of G'_2 at u adding e is a $(2, 2)$-circuit. Denote this graph $\tilde{G}'_2$. As $e \notin E(G'_2)$, $e \neq xy$. As $u \notin \{x, y\}$, $u \in V(A'_2) \cap Y$ and hence $u \in V_3(G)$ and $e \notin E$. Let G' be the $(2, 1)$-reduction of G at u adding e. Then G' is the 1-join of $(G'_3, \tilde{G}'_2)$ and hence G' is $(2, 2)$-connected by Lemma 2.3.0.22. On the other hand, suppose that the set of nodes of G'_2 such that there exist $(2, 1)$-reductions of G'_2 at these nodes that are $(2, 2)$-circuits is $\{x, y\}$. Let $N_{G'_2}(x) = \{a, b, y\}$ and take $e \in E(K[\{a, b, y\}])$ such that the $(2, 1)$-reduction of G'_2 at x adding e is a $(2, 2)$-circuit (see Figure 3.8). Denote this graph by $\tilde{G}'_2$. As $\tilde{G}'_2$ is a $(2, 2)$-circuit and $d_{G'_2}(y) = 3$, Lemma 2.3.0.2 implies $e \in \{ay, by\}$. We may suppose without loss of generality that $e = ay$. So, as $a \in V(A'_2) \cap Y$, $ay \notin E$. Let G' be the generalised edge-reduction of G that contracts xa (to u) and deletes xb, and let $\tilde{G}'_3$ be the graph given by the isomorphism of G'_3 that maps every vertex except x to itself, and maps x to a. Then G' is isomorphic to the 1-join of $(\tilde{G}'_3, \tilde{G}'_2)$ and hence G' is $(2, 2)$-connected by Lemma 2.3.0.22.

Finally, suppose that $G'_2 \cong B_2$. Then, as A'_2 is an atom of H_t, we must have that $d_{G'_2}(x) = 4 = d_{G'_2}(y)$. Let $\{z\} = N_{G'_2}(x) \cap N_{G'_2}(y)$. We now turn out attention to G'_3. As $d_{G'_3}(x) = 3 = d_{G'_3}(y)$, $G'_3 \ncong K_5^-$. At this point we invoke an induction argument. We may suppose that our result holds for all $(2, 2)$-connected graphs with sufficiently few vertices, say at most n, and that $|V(G)| = n + 1$. Note also that the result holds (trivially) when $n = 5$. Now, as $|V(G'_3)| < |V(G)|$ and $G'_3 \ncong K_5^-$, it follows from our induction hypothesis that that $G'_3 \cong B_1$, or there exists a generalised edge-reduction, an edge-deletion, or a K_4^--reduction of G_3' that is $(2,2)$-connected. If $G_3' \cong B_1$, then G is the 1-join of (G_2', G_3'), and hence G_2' is a K_4^--reduction of G and so we are done. So we may instead suppose that there exists a generalised edge-reduction, an edge-deletion, or a K_4^--reduction of G_3' that is $(2,2)$-connected.

If there exists a K_4^--reduction of G_3', say deleting $\{u_1, u_2\}$, that is $(2,2)$-connected then denote this graph by $\tilde{G}_3'$. As $xy \in E(G_3')$, $u_1, u_2 \notin \{x, y, w_1, w_2\}$. Hence we can consider the K_4^--reduction of G deleting $\{u_1, u_2\}$. Denote this graph by G', then G' is the 1-join of $(G_2', \tilde{G}_3')$ and hence G' is $(2,2)$-connected by Lemma 2.3.0.22.

If there exists a generalised edge-reduction of G_3', say contracting e to u and deleting f, that is $(2,2)$-connected then denote this graph by $\tilde{G}_3'$. As G_3' is $(2,2)$-connected $e \notin \{xy, xw_1, xw_2, yw_1, yw_2, w_1w_2\}$, so $e \in E$, and $f \notin \{xw_1, xw_2, yw_1, yw_2, w_1w_2\}$. If $f \neq xy$ then $f \in E$. Let G' denote a generalised edge-reduction of G contracting e and deleting f. If e is incident to x, then say that e contracts to u and let $\tilde{G}_2'$ denote the graph given by the isomorphism that maps every vertex of G_2' except x to itself, and maps x to u. If e is not incident to x then let $\tilde{G}_2' = G_2'$. Then, G' is the 1-join of $(\tilde{G}_2', \tilde{G}_3')$ and hence G' is $(2,2)$-connected by Lemma 2.3.0.22. If $f = xy$ then $f \notin E$. We may suppose without loss of generality that e is incident to x, say $e = xa$ for some $a \in V(A_3') \setminus \{y\}$. Let G' denote the generalised edge-reduction of G contracting e to u and deleting xz (see Figure 3.9). Let $\tilde{G}_2$ denote the isomorphism of the edge-deletion of G_2 at xz that maps every vertex of G_2 except x to itself and maps x to u. We observe that $\tilde{G}_2$ is isomorphic to a K_4^--extension of the subgraph $G_2[N_{G_2}(y)] \cong B_2$, and hence $\tilde{G}_2$ is $(2,2)$-connected by Lemma 2.3.0.23. If $ay \in E$ then $uy \in E(G')$, so G' is the 2-join of $(\tilde{G}_3', \tilde{G}_2)$ and hence G' is $(2,2)$-connected by Lemma 2.3.0.18. If $ay \notin E$ then $uy \notin E(G')$. Let (G_4, G_5) denote a 1-separation of $\tilde{G}_3'$ on $(\tilde{G}_3'[V(A_3')], \tilde{G}_3'[\{u, y, w_1, w_2\}])$, and let (G_5', G_4') denote a 1-separation of $\tilde{G}_3'$ on $(\tilde{G}'3[\{u, y, w_1, w_2\}], \tilde{G}_3'[V(A_3')])$. Then G' is the 1-join of $(G_4, \tilde{G}_2)$. As $G_5' \cong K_4$, Lemma 2.3.0.24 implies that G_4 is $(2,2)$-connected and hence G' is $(2,2)$-connected by Lemma 2.3.0.22.

Finally, if there exists an edge-deletion of G_3', say at e, that is $(2,2)$-connected then denote this graph by $\tilde{G}_3'$. Lemma 2.1.0.10 implies $e \notin \{xw_1, xw_2, yw_1, yw_2, w_1w_2\}$. If

$e \neq xy$ then $e \in E$ and we can denote the edge-deletion of G at e by G'. Then G' is the 1-join of $(G_2', \tilde{G}_3')$ and hence G' is $(2,2)$-connected by Lemma 2.3.0.22.If $e = xy$ then let G' be the edge-deletion of G at xu. Let (G_4, G_5) denote a 1-separation of $\tilde{G}_3'$ on $(\tilde{G}_3'[V(A_3')], \tilde{G}_3'[\{x, y, w_1, w_2\}])$ and let (G_5', G_4') denote a 1-separation of $\tilde{G}_3'$ on $(\tilde{G}_3'[\{x, y, w_1, w_2\}], \tilde{G}_3'[V(A_3')])$. Let $\tilde{G}_2$ denote the edge-deletion of G_2 at xu. Then G' is the 1-join of $(G_4, \tilde{G}_2)$. As $G_5' \cong K_4$, Lemma 2.3.0.24 implies that G_4 is $(2,2)$-connected and hence G' is $(2,2)$-connected by Lemma 2.3.0.22. □

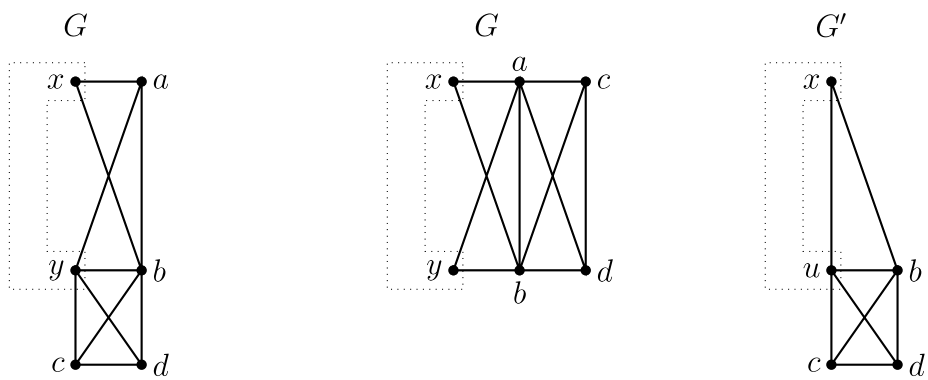


Figure 3.7: The relevant graphs when $G_2' \cong B_1$ in Subcase 3b in the proof of Theorem 3.2.0.4.

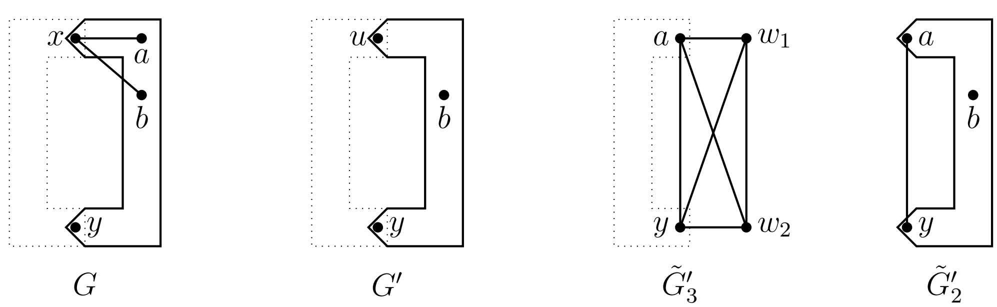


Figure 3.8: The relevant graphs when the set of nodes of G_2' such that there exist $(2,1)$-reductions of G_2' at these nodes that are $(2,2)$-circuits is $\{x, y\}$ in Subcase 3b in the proof of Theorem 3.2.0.4.

Theorem 3.2.0.5. *[24, Theorem 2.1] Let G be a graph. The following are equivalent:*

(i) G *is a* $(2,2)$*-circuit; and*

(ii) there exists $t \in \mathbb{N}^+$ *and a sequence* $a_1, \ldots, a_t$*, with* $a_1 \cong K_5^-$ *or* $a_1 \cong B_1$*, and* $a_t = G$*, such that for all* $2 \leq j \leq t$*,*

(a) a_j *is a* K_4^-*-extension or a generalised vertex split of* a_{j-1}*; and*

(b) if a_j *is a generalised vertex split of* a_{j-1} *then* a_j *is a* $(2,2)$*-circuit.*

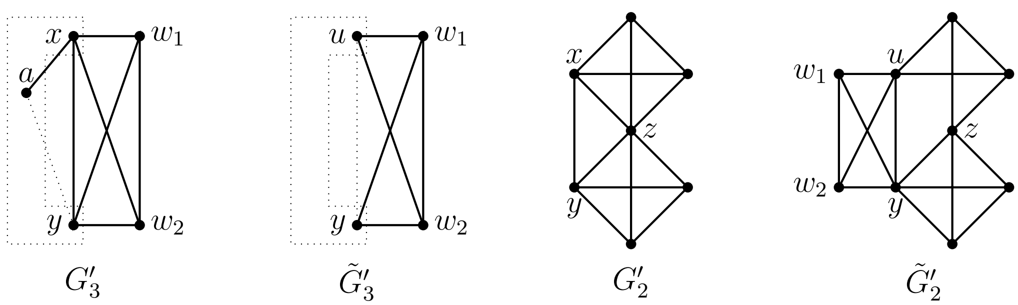


Figure 3.9: The relevant graphs when $G_2' \cong B_2$ and there exists a generalised edge-reduction of G_3', contracting e and deleting xy, that is a $(2,2)$-circuit in Subcase 3b in the proof of Theorem 3.2.0.4.

Theorem 3.2.0.6. *Let* G *be a graph. The following are equivalent:*

(i) G *is* $(2,2)$*-connected; and*

(ii) there exists $t \in \mathbb{N}^+$ *and a sequence* $a_1, \ldots, a_t$*, with* $a_1 \cong K_5^-$ *or* $a_1 \cong B_1$*,* $a_t = G$*, such that for all* $2 \leq j \leq t$*,*

(a) a_j *is a* K_4^-*-extension, or an edge-addition, or a generalised vertex split of* a_{j-1}*; and*

(b) if a_j *is a generalised vertex split of* a_{j-1} *then* a_j *is* $(2,2)$*-connected.*

Proof. Suppose (i) holds. We proceed by induction on $|V|$, and note that clearly if $G \cong K_5^-$ or $G \cong B_1$ then there exists a sequence of the form claimed. By Lemma 2.1.0.19, for all $s \in \mathbb{N}$ such that $s \geq 5$, there exists a $(2,2)$-connected graph with s vertices. Take $n \in \mathbb{N}$ such that $n \geq 6$ and suppose that (ii) holds for all $(2,2)$-connected graphs with at most n vertices. Now suppose that $|V| = n + 1 \geq 7$. Theorem 3.2.0.4 implies there exists a K_4^--reduction or an edge-deletion or a generalised edge-reduction of G that is $(2,2)$-connected. Let this graph be G'. As $|V(G')| \leq n$, it follows from our induction hypothesis that there exists $t \in \mathbb{N}^+$ and a sequence $a_1, \ldots, a_t$, with $a_1 \cong K_5^-$ or $a_1 \cong B_1$, $a_t = G'$, such that for all $2 \leq j \leq t$,

(a) a_j is a K_4^--extension, or an edge-addition, or a generalised vertex split of a_{j-1}; and

(b) if a_j is a generalised vertex split of a_{j-1} then a_j is $(2,2)$-connected.

Therefore, $a_1, \ldots, a_t, G$ is a sequence of the form claimed.

On the other hand, if (ii) holds then, as K_5^- and B_1 are $(2,2)$-circuits and hence $(2,2)$-connected by Lemma 2.1.0.8, repeated applications of Lemma 2.1.0.17, Lemma 2.3.0.23, and condition (b) together imply that G is $(2,2)$-connected. □

Chapter 4

Frameworks in Normed Spaces

In this chapter we provide a geometric companion to the combinatorics of the previous two chapters. We begin by providing some background on the study of frameworks and rigidity in normed spaces and proceed to consider global rigidity of frameworks in normed planes. This chapter concludes by combining this study with the results of Chapter 3 in order to give an appropriately 'generic' characterisation of globally rigid graphs in analytic normed planes.

4.1 Rigidity of Graphs in Normed Spaces

4.1.1 uv-Coincident Rigidity and uv-Sparse Graphs

In 2005, a characterisation of global rigidity in two-dimensional Euclidean space arose from a combination of Jackson and Jordán's combinatorial result [22, Theorem 6.15] with Connelly's geometric result [5, Theorem 1.5]. So, in order to characterise global rigidity in (a family of) non-Euclidean normed spaces we look to combine Theorem 3.2.0.6 with geometric results outlined in this chapter. To that end we spend the first two sections of this chapter considering a particular type of framework. These frameworks are certainly not well-positioned, but they fail to be well-positioned in a very specific way and are in some sense 'close enough' to being well-positioned to them to be useful. These frameworks have previously been used to study rigidity [23] and global rigidity [24] of frameworks realised on cylinders.

Let $G = (V, E)$ be a graph with distinct vertices $u, v \in V$, and let X be a normed space. A realisation, p, of V in X is $\boldsymbol{uv}$**-coincident** if $p(u) = p(v)$; then the framework (G, p) is also $\boldsymbol{uv}$**-coincident**[1]. We say that a uv-coincident realisation, p, of a graph (V, E) is **well-positioned** if p is a well-positioned realisation of $(V, E \setminus \{uv\})$.

Let (V, E) be a graph and let X be a normed space. Then we denote the set of uv-coincident realisations of (V, E) in X by $X^V/uv := \{p \in X^V : p(u) = p(v)\}$. Now, if p is a well-positioned uv-coincident realisation of (V, E) in X then:

(i) p is **regular** if for all $q \in X^V/uv$,

$$\text{rank } R((V, E \setminus \{uv\}), p) \geq \text{rank } R((V, E \setminus \{uv\}), q); \text{ and}$$

(ii) (G, p) is **independent** if $uv \notin E$ and (G, p) is independent in X.

Definition 4.1.1.1. Let X be a normed space, let (V, E) be a graph, and let p be a well-positioned uv-coincident realisation of (V, E) in X. The uv-coincident framework $((V, E), p)$ is **infinitesimally rigid** in X if $((V, E \setminus \{uv\}), p)$ is infinitesimally rigid in X.

Let p be a well-positioned uv-coincident realisation of a graph G in a normed space X. (G, p) is **minimally (infinitesimally) rigid** if (G, p) is independent and infinitesimally rigid. We say that G is **(minimally)** $\boldsymbol{uv}$**-rigid** in X if there exists a uv-coincident realisation, p, of G in X such that (G, p) is (minimally) infinitesimally rigid. Similarly, we say that G is $\boldsymbol{uv}$**-independent** in X if there exists a uv-coincident framework (G, q) that is independent.

Lemma 4.1.1.2. *[12, Lemma 2.5][2] Let X be a normed space and let G be a graph. The set of well-positioned uv-coincident realisations of G in X is a conull subset (i.e.*

[1] For the remainder of this chapter, while discussing uv-coincidence we nay not repeat that the graphs in question have distinct vertices u and v.

[2] The proof is analogous to that of Lemma 1.4.3.1.

the complement of a set with Lebesgue measure zero) of $X^V(G)/uv$, and the set of regular uv-coincident realisations of G in X is a non-empty open subset of the set of well-positioned uv-coincident realisations.

If a normed space X is two-dimensional then we say X is a **normed plane**. As we shall see, uv-rigidity in normed planes is closely related to the following sparsity property of graphs. Let $G = (V, E)$ be a graph and let u, v be two distinct vertices of G. Let $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ be a family of sets such that for all $1 \leq i \leq k$, $X_i \subseteq V$. We say that $\mathcal{X}$ is a ***uv*-compatible family** of G if, for all $1 \leq i \leq k$, $\{u, v\} \subsetneq X_i$.

Let $G = (V, E)$ be a graph with distinct vertices $u, v \in V$. Then we introduce a function, t_G, that assigns a value to non-empty subsets of V. That is, $t_G \colon V \to \mathbb{Z}$ is the function defined by

$$t_G(U) = \begin{cases} 4 & \text{if } U = \{u, v\}; \\ 3 & \text{if } U \neq \{u, v\} \text{ and } |U| \in \{2, 3\}; \\ 2 & \text{otherwise.} \end{cases}$$

With t_G in hand we can assign value to both non-empty subsets of G and also to uv-compatible families of G. Given $\emptyset \neq U \subseteq V$, (respectively, a uv-compatible family $\mathcal{X} = \{X_1, \ldots, X_k\}$) the **value** of U (respectively, $\mathcal{X}$) is

$$\text{val}(U) := 2|U| - t_G(u) \quad (\text{respectively, } \text{val}(\mathcal{X}) := \left(\sum_{i=1}^{k} \text{val}(X_i)\right) - 2(k-1)).$$

Note that $\text{val}(\mathcal{X}) = 2 + \sum_{i=1}^{k}(2|X_i| - (t_G(X_i) + 2))$. Also, if $|\mathcal{X}| = 1$, say $\mathcal{X} = X$, then $\text{val}(\mathcal{X}) = \text{val}(X)$.

Let $G = (V, E)$ be a graph. G is ***uv*-sparse** if $i_G(U) \leq \text{val}(U)$ for all $U \subseteq V$ such that $|U| \geq 2$, and $i_G(\mathcal{X}) := \left|\bigcup_{i=1}^{k} E(G[X_i])\right| \leq \text{val}(\mathcal{X})$ for all uv-compatible families $\mathcal{X}$. G is ***uv*-tight** if G is uv-sparse and $|E| = 2|V| - 2$. Figure 4.1 illustrates a graph that is $(2, 2)$-sparse and not uv-sparse. Note that if G is uv-sparse then $uv \notin E$. Jackson, Kaszanitzky, and Nixon, showed that the edge sets of the uv-sparse subgraphs of G form the independent sets of a matroid [23, Lemma 7], and when $|V| \geq 5$ the maximally independent sets (i.e. independent sets such that no superset of them is also independent) of this matroid matroid has rank $2|V|-2$.

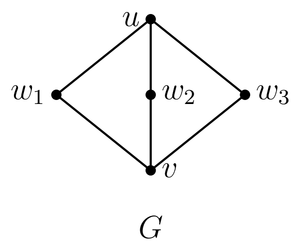


Figure 4.1: Illustration of a $(2,2)$-sparse graph, $G \cong K_{2,3}$, that is not uv-sparse. If $\mathcal{X} = \{\{u,v,w_1\},\{u,v,w_2\},\{u,v,w_3\}\}$ then $\mathcal{X}$ is a uv-compatible family of G and $i_G(\mathcal{X}) = 6 > 5 = 9-2(2) = \mathsf{val}(\mathcal{X})$.

4.1.2 uv-Coincident Graph Operations

Definition 4.1.2.1. Let $G=(V,E)$ and $H=(W,F)$ be graphs such that $V \cap W = \{u\}$. A **vertex-to-$\boldsymbol{H}$** operation of G is a graph $G' = (V \cup W, (E \setminus \{uv\colon v \in N_G(u)\}) \cup F \cup F')$, where $F' \subseteq \{vw\colon v \in N_G(u) \text{ and } w \in W\}$ such that for all $v \in N_G(u)$, $d_{G'}(\{v\},W) = 1$.

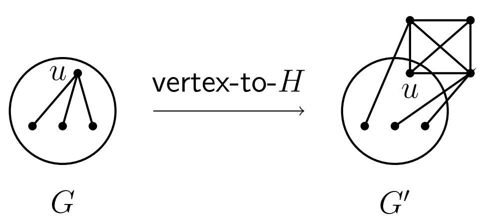


Figure 4.2: Illustration of a vertex-to-H operation, where $H \cong K_4$. G' is a vertex-to-H operation of G.

Definition 4.1.2.2. Let $G=(V,E)$ such that $\Delta(G) \geq 2$. A **vertex-to-4-cycle** operation of G is a graph $(V \cup \{w'\}, (E \setminus \{wv_i\colon 1 \leq i \leq k\}) \cup \{wv_1, wv_2, w'v_1, w'v_2\} \cup F')$, where there exists $k \geq 2$ such that $N_G(w) = \{v_1,\ldots,v_k\}$, and $F' \subseteq \{wv_i\colon 3 \leq i \leq k\} \cup \{w'v_i\colon 3 \leq i \leq k\}$ such that for all $3 \leq i \leq k$, $|F' \cap \{wv_i, w'v_i\}| = 1$. We say

$(V \cup \{w'\}, (E \setminus \{wv_i \colon 1 \leq i \leq k\}) \cup \{wv_1, wv_2, w'v_1, w'v_2\} \cup F')$ is a vertex-to-4-cycle operation of G at w.

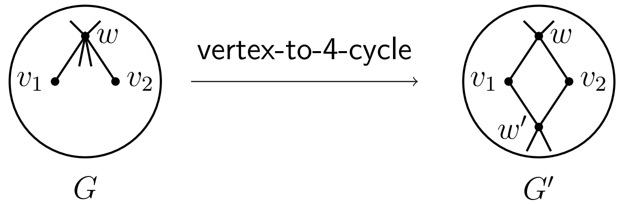


Figure 4.3: Illustration of a vertex-to-4-cycle operation. G' is a vertex-to-4-cycle operation of G.

A graph $G = (V, E)$ is **$(2,2)$-sparse** if $i_G(U) \leq 2|U| - 2$ for all $\emptyset \neq U \subseteq V$; and G is **$(2,2)$-tight** if G is $(2,2)$-sparse and $|E| = 2|V| - 2$. All $(2,2)$-tight graphs can be constructed from a single vertex by a sequence of $(2,0)$- and $(2,1)$-extensions, vertex-to-4-cycle operations, and vertex-to-H operations where $H \cong K_4$ (see [33, Theorem 3.1] for details). Figure 4.2 illustrates vertex-to-H operations, while Figure 4.3 illustrates vertex-to-4-cycle operations. We shall make use of various specialised versions of $(2,0)$-extensions, $(2,1)$-extensions, vertex-to-4-cycle operations, and vertex-to-H operations.

Let $G = (V, E)$ be a graph and suppose that $V \cap \{u, v\} = \{u\}$. A **$(2,0)$-extension that adds v**, of G, is a $(2,0)$-extension of G, say G', such that $V(G') \setminus V = \{v\}$ and $u \notin N_{G'}(v)$. A **vertex-to-4-cycle operation that adds v**, of G, is a vertex-to-4-cycle operation of G at u, say G', such that $V(G') \setminus V = \{v\}$ A **vertex-to-H operation that adds v**, of G, is a vertex-to-H operation of G where $V \cap V(H) = \{u\}$ and $v \in V(H) \setminus V$.

Let $G = (V, E)$ be a graph and suppose that $\{u, v\} \subseteq V$. A **$uv - (2,0)$-extension** of G is a $(2,0)$-extension of G, say G', such that $\{w \in V \colon d_G(w) \neq d_{G'}(w)\} \neq \{u, v\}$. A **$uv - (2,1)$-extension** of G is a $(2,1)$-extension of G, say G' where $V(G') \setminus V = \{w\}$ and $E \setminus E(G') = \{a, b\}$, such that $\{u, v\} \nsubseteq \{a, b, w\}$.

As at most one of u or v is 'involved' in a $uv - (2,0)$-extension or a $uv - (2,1)$-extension of a graph, the techniques used to prove [9, Lemma 5.1, Lemma 5.2] can be applied to prove the following result.

Lemma 4.1.2.3. *Let X be a normed plane, let G be a graph such that $\{u, v\} \subseteq V(G)$, and let G' be either a $uv - (2,0)$-extension or a $uv - (2,1)$-extension of G. If G is uv-independent in X then G' is uv-independent in X.*

Lemma 4.1.2.4. *Let X be a normed plane, let G be a graph such that $\{u, v\} \cap V(G) = \{u\}$, and let G' be a $(2,0)$-extension that adds v, of G. If X is strictly convex then G' is uv-independent in X if and only if G is independent in X.*

Proof. We note that as G' contains G as a subgraph, if G' is uv-independent then G will be independent. Suppose there exists an independent realisation, p, of G in X. By applying translations, we may suppose that $p(u) = 0$. Let $N_{G'}(v) = \{v_1, v_2\}$. We may also assume that $p(v_1)$ and $p(v_2)$ are linearly independent and smooth, as if not then we could apply Lemma 1.4.3.1 to find a realisation, q, of G in X such that $q(v_1)$ and $q(v_2)$ are linearly independent. Define p' to be the well-positioned realisation of G' in X such that $p'(x) = p(x)$ for all $x \in V(G)$, and $p'(v) = p(u)$. We see that there exist $1 \times 2|V(G)|$ matrices A and B such that

$$R(G', p') = \left[\begin{array}{c|c} R(G,p) & \mathbf{0}_{|E(G)| \times 2} \\ \hline A & -\varphi_{p(v_1)} \\ B & -\varphi_{p(v_2)} \end{array}\right].$$

Since $p(v_1)$ and $p(v_2)$ are linearly independent and X is strictly convex, Lemma 1.3.2.8 implies $\varphi_{p(v_1)}$ and $\varphi_{p(v_2)}$ are linearly independent. Therefore (G', p') is independent and so G' is uv-independent. □

Observe that Lemma 4.1.2.4 specified that our non-Euclidean normed plane was strictly convex, and that this condition was used in the penultimate sentence of the proof. For vertex-to-4-cycle operations we use a technique from [23, Lemma 11] to show that a vertex-to-4-cycle operation that has two coincident vertices preserves independence. As with the previous result our proof requires that the normed plane in question is strictly convex.

Lemma 4.1.2.5. *Let X be a strictly convex normed plane and let G and G' be graphs.*

(i) If $V(G) \cap \{u,v\} = \{u\}$, G is independent in X, and G' a vertex-to-4-cycle operation that adds v, of G, then G' is uv-independent in X.

(ii) If $\{u,v\} \subseteq V(G)$, G is uv-independent in X, and G' is a vertex-to-4-cycle operation of G, then G' is uv-independent in X.

Proof. Let w be the vertex of G such that G' is a vertex-to-4-cycle operation of G at w, let $N_G(w) = \{v_1, \ldots, v_k\}$ for some $k \geq 2$, and let $V(G') \backslash V(G) = \{w'\}$. Suppose that G is uv-independent (respectively, independent). By Lemma 4.1.1.2 (respectively, Lemma 1.4.3.1) we may choose a uv-independent (respectively, independent) realisation, p, of G in X such that $p(w)$, $p(v_1)$, and $p(v_2)$ are not collinear. By applying translations to p, we may assume that $p(w) = 0$. Define p' to be the realisation of G' such that $p'(x) = p(x)$ for all $x \in V(G)$, and $p(w') = p(w)$. Then p' is a well-positioned uv-coincident realisation of G'. Let $G'' = (V(G'), (E(G') \setminus \{w'v_i \colon 3 \leq i \leq k\}) \cup \{wv_i \colon 3 \leq i \leq k\})$. We see that there exist $1 \times 2|V(G)|$ matrices A and B such that

$$R(G'', p') = \left[\begin{array}{c|c} R(G,p) & \mathbf{0}_{|E(G)|\times 2} \\ \hline A & \varphi_{p'(w')-p'(v_1)} \\ B & \varphi_{p'(w')-p'(v_2)} \end{array}\right] = \left[\begin{array}{c|c} R(G,p) & \mathbf{0}_{|E(G)|\times 2} \\ \hline A & -\varphi_{p(v_1)} \\ B & -\varphi_{p(v_2)} \end{array}\right].$$

Since $p(v_1)$ and $p(v_2)$ are linearly independent and X is strictly convex, Lemma 1.3.2.8 implies $\varphi_{p(v_1)}, \varphi_{p(v_2)}$ are linearly independent. Hence $R(G'', p')$ has linearly independent rows.

We proceed to describe a sequence of rank-preserving row operations that will take $R(G'', p')$ to $R(G', p')$. As $\varphi_{p(v_1)}$ and $\varphi_{p(v_2)}$ are linearly independent, for all $3 \leq i \leq k$ there exist a unique α_i and β_i such that

$$\alpha_i \varphi_{p(v_1)} + \beta_i \varphi_{p(v_2)} = \varphi_{p(v_i)} = \varphi_{p'(v_i)-p'(z)},$$

where $z \in \{w, w'\}$ is chosen such that $v_i z \in E(G')$. For all $1 \leq i \leq k$, let (wv_i) denote the row of $R(G'', p')$ corresponding to the edge wv_i, and similarly let $(w'v_1)$ and $(w'v_2)$ denote the rows of $R(G'', p')$ corresponding to edges $w'v_1$ and $w'v_2$ respectively. For

$v_i \in N_{G'}(w')$, let $[w'v_i]$ denote the row of $R(G', p')$ corresponding to the edge $w'v_i$. Now, for all $v_i \in N_{G'}(w')\backslash\{v_1, v_2\}$, we have

$$[w'v_i] = (wv_i) - \alpha_i(wv_1) - \beta_i(wv_2) + \alpha_i(w'v_1) + \beta_i(w'v_2).$$

Applying these row operations to $R(G'', p')$, preserves linear independence and results in the matrix $R(G', p')$. Therefore the rows of $R(G', p')$ are linearly independent, and so G' is uv-independent in X. □

Our next result shows that vertex-to-H operations, where H is some $(2, 2)$-tight graph, that has two coincident vertices preserves independence.

Lemma 4.1.2.6. *Let X be a non-Euclidean normed plane, and let G, G', and H be graphs.*

(i) Suppose $V(G) \cap \{u, v\} = \{v\}$, G is independent in X, H is uv-tight, and G' is a vertex-to-H operation that adds v, of G. If H is minimally uv-rigid in X then G' is uv-independent in X.

(ii) Suppose $\{u, v\} \subseteq V(G)$, G is uv-independent in X, H is $(2, 2)$-tight, and G' is a vertex-to-H operation of G. If H is minimally rigid in X, then G' is uv-independent in X.

Proof. Let w be the vertex of G such that $V(G) \cap V(H) = \{w\}$. If (i) holds then let (G, p) be an independent framework in X and let (H, q) be a minimally rigid uv-coincident framework in X. If (ii) holds then let (G, p) be an independent uv-coincident framework in X and let (H, q) be a minimally rigid framework in X. By applying translations to p and q we may assume $p(w) = q(w) = 0$. For a matrix A with columns corresponding to a subset of $V(G) \cup V(H)$, define A_w to be the submatrix of A created by deleting all columns of A corresponding to w. Given a fixed basis b_1, b_2 of X used to define rigidity matrices in X, we define the matrix

$$M := \left[\begin{array}{c|c} R(H,q)_w & \mathbf{0}_{|E(H)|\times(2|V(G)|-2)} \\ \hline A & R(G,p)_w \end{array}\right]$$

where A is the $|E(G)| \times (2|V(H)|-2)$ matrix with entries

$$A_{e,(x,i)} = \begin{cases} \varphi_{p(x)-p(y)}(b_i) & \text{if } e = xy; \\ 0 & \text{otherwise.} \end{cases}$$

By our choice of p and q, the rows of M are linearly independent.

By Lemma 4.1.1.2 we may choose, for all $n \in \mathbb{N}$, well-positioned uv-coincident realisations, p^n, of G' such that $p^n(x) = q(x)$ for all $x \in V(H)$, and $\|p^n(x) - p(x)\| < \frac{1}{n}$ for all $x \in V(G)$. For all $n \in \mathbb{N}$ define M_n to be the matrix created by multiplying each row of $R(G', p^n)_w$ that corresponds to an edge of H by n. As the map $x \to \varphi_x$ is continuous on the set of smooth points of X it follows that for all $xy \in E$, the limit as n tends to infinity of $(\varphi_{p^n(x)-p^n(y)})_{n\in\mathbb{N}}$ is $\varphi_{p(x)-p(y)}$, and so the limit as n tends to infinity of $(M_n)_{n\in\mathbb{N}}$ is M. Hence, for sufficiently large $N \in \mathbb{N}$, the matrix M_N (and hence $R(G', p^N)_w$) will have linearly independent rows. Then $p' = p^N$ is a an independent uv-coincident realisation of G' in X, and so G' is uv-independent. □

4.2 Characterising Coincident-Point Independence

With the geometric results of the previous section in hand, we consider some combinatorial ideas from [23]. In particular, combining these combinatorial ideas with the results from the previous section allows us to state a sufficient condition for a graph G to be uv-rigid in a strictly convex normed plane. We begin with the following result which can be extracted from the proof of [23, Theorem 4].

Proposition 4.2.0.1. *[23] If G is a uv-tight graph and $|V(G)| \geq 5$ then there exists $t \in \mathbb{N}^+$ and a sequence $a_1 \ldots a_t$, with a_1 a $(2,2)$-tight graph such that $|V(a_1)| \geq 4$ and $V(a_1) \cap \{u,v\} = \{u\}$, or $a_1 \cong \tilde{H}$ (see the right of Figure 4.4) and $d_{a_1}(u) = 3 = d_{a_2}(v)$*

and $uv \notin E(a_1)$, and $a_t \cong G$, such that for all $2 \leq i \leq t$, a_j is a $(2,0)$-extension that adds v or, a vertex-to-4-cycle operation that adds v, or a vertex-to-H operation that adds v where H is a uv-tight graph, or a $uv-(2,0)$-extension, or a $uv-(2,1)$-extension, or a vertex-to-4-cycle operation or a vertex-to-H operation where H is a $(2,2)$-tight graph.

We will also require the following lemmas.

Lemma 4.2.0.2. *Let X be a non-Euclidean normed plane and let G be a graph such that $\{u,v\} \subseteq V(G)$ and $|V(G)| \leq 4$. If X is strictly convex then G is uv-independent in X if and only if G is uv-sparse.*

Proof. As $|V(G)| \leq 4$, if G is not uv-sparse then $uv \in E(G)$ and so G is not uv-independent. Alternatively, suppose that G is uv-sparse so $uv \notin E$. Note that G is a subgraph of some graph isomorphic to K_4^- and so it is sufficient to consider the case where $|V(G)| = 4$ and $G = (V(G), E(K[V(G)] \setminus \{uv\}))$. As G is a $(2,0)$-extension that adds v, of $G[V(G) \setminus \{v\}]$, G is uv-independent by Theorem 1.4.3.8 and Lemma 4.1.2.4. □

Lemma 4.2.0.3. *Let X be a non-Euclidean normed plane. If G is a graph such that $G \cong \tilde{H}$ (see the right of Figure 4.4), and $d_G(u) = 3 = d_G(v)$, and $uv \notin E(G)$, then G is minimally uv-rigid in X.*

Proof. Let $G = (V,E)$, let $N_G(u) = \{x, a_1, a_2\}$, and let $N_G(v) = \{x, b_1, b_2\}$. By Theorem 1.4.3.8 there exists a realisation, p_1, of $N_G[u]$ in X such that the framework $(K[N_G[u]], p_1)$ is minimally rigid in X. Let $p\colon V \to X$ be the realisation of V in X such that $p(b_1) = p_1(a_1)$, $p(b_2) = p_1(a_2)$, $p(v) = p_1(u)$, and $p(w) = p_1(w)$ for all $w \in N_G[u]$. Note that (G,p) is a minimally rigid uv-coincident framework; this follows from the fact that joining two minimally rigid frameworks in a non-Euclidean normed plane produces a minimally rigid framework, since the trivial infinitesimal flexes in a non-Euclidean normed plane correspond to translations. Therefore G is minimally uv-rigid. □

Let $G = (V,E)$ be a graph and take $F \subseteq E$. For a family $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$, where $S_i \subseteq V$ for all $1 \leq i \leq k$, we say that $\mathcal{S}$ is a **cover** of F if $F \subseteq \{xy \in$

E: there exists $1 \leq i \leq k$ such that $\{x,y\} \subseteq S_i\}$. By combining Theorem 1.4.3.8 with [23, Subsection 3.1] we obtain the following result.

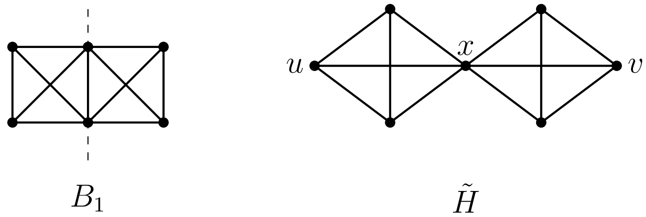


Figure 4.4: B_1 is rigid in any normed plane, and it is globally rigid in all analytic non-Euclidean normed planes [14] (see Lemma 4.4.0.1). However, B_1 is not globally rigid in the Euclidean plane since for almost all realisations, the left two vertices may be reflected across the dashed line to obtain an equivalent but non-congruent framework. $\tilde{H}$ is minimally rigid in any non-Euclidean normed plane, however it is flexible in the Euclidean plane. $\tilde{H}$ is not globally rigid in any normed space as it is not 2-vertex-connected (see Theorem 1.4.3.10 and Theorem 4.3.0.1). It is also uv-tight and and appears in the statement of Proposition 4.2.0.1

.

Lemma 4.2.0.4. *Let X be a non-Euclidean normed plane and let G be a graph. Let p be a well-positioned realisation of G in X. Let $\mathcal{S}$ be the set of all covers $\mathcal{X} \coloneqq \{X_1, \dots, X_k\}$ of $E(G)$. Let $s\colon \mathbb{N} \to \{0,1\}$ be the function such that $s(x) = 1$ if $x = 2$ and $s(x) = 0$ otherwise. Then,*

$$\text{rank } R(G,p) \leq \min_{\mathcal{X} \in \mathcal{S}} \sum_{i=1}^{k} \left(2|X_i| - (2 + s(|X_i|))\right),$$

with equality if and only if p is regular. Moreover, it suffices to minimise over all covers $\mathcal{Y} \coloneqq \{Y_1, \dots, Y_{k'}\}$ of $E(G)$ such that $|Y_i| \geq 2$ for all $1 \leq i \leq k'$ and $|Y_i \cap Y_j| \leq 1$ for all $1 \leq i < j \leq k'$, with equality only if $\min\{|Y_i|, |Y_j|\} = 2$.

We are now able to remove the condition on $|V(G)|$ from Lemma 4.2.0.2. Given a graph $G = (V,E)$ such that $\{u,v\} \subseteq V$, set $G/uv \coloneqq ((V \setminus \{u,v\}) \cup \{z\}, (E(G[V \setminus \{u,v\}]) \cup \{zw\colon w \in (N_G(u) \cup N_G(v)) \setminus \{u,v\}\})$, where $z \notin V$. We say that G/uv is a **contraction** of G at $\{u, v\}$, or the contraction of G at $\{u, v\}$ adding z[3].

Theorem 4.2.0.5. *Let X be a non-Euclidean normed plane and let G be a graph such that $\{u, v\} \subseteq V(G)$. If X is strictly convex then G is uv-independent in X if and only if G is uv-sparse.*

Proof. Let $G = (V, E)$. Firstly, suppose G is uv-independent in X. Let p be a regular (and hence independent) uv-coincident realisation of G in X and let G/uv be the contraction of G at $\{u, v\}$ adding z. Let p' be the realisation of G/uv in X such that $p'(z) = p(u) = p(v)$ and $p'(x) = p(x)$ for all $x \in V \setminus \{u, v\}$. For all $\emptyset \neq U \subseteq V$, the (possibly uv-coincident) framework $(G[U], p|_U)$ is independent. Hence, if $\{u, v\} \nsubseteq U$ then $i_G(U) \leq \text{val}(U)$ by Theorem 1.4.3.8. Since the case when $U = \{u, v\}$ is trivial, it remains to show that $i_G(\mathcal{X}) \leq \text{val}(\mathcal{X})$ for all uv-compatible families $\mathcal{X}$ of G. (Note that the case when $U \subseteq V$ and $\{u, v\} \subseteq U$ will be included by taking $\mathcal{X} = \{U\}$).

Let $\mathcal{X} = \{X_1, \ldots, X_k\}$ be a uv-compatible family of G and consider the subgraph $H = (U, F)$ be the subgraph of G such that $U = \bigcup_{i=1}^k X_i$ and $F = \bigcup_{i=1}^k E(G[X_i])$. Let H/uv be the contraction of H at $\{u, v\}$ adding z. Let $q = p|_U$ and let $q' = p'|_{(U \setminus \{u,v\}) \cup \{z\}}$. Note that if, for all $1 \leq i \leq k$, $X_i/uv = (X_i \setminus \{u, v\}) \cup \{z\}$ then $\mathcal{X}' = \{X_1/uv, \ldots, X_k/uv\}$ is a cover of $E(H/uv)$ By Lemma 4.2.0.4 we have

$$\begin{aligned}
\text{rank } R(H/uv, q') &\leq \sum_{i=1}^{k} (2|X_i/uv| - (2 + s(|X_i/uv|))) \\
&= \sum_{i=1}^{k} (2|X_i| - 2 - t(X_i) \\
&= \text{val}(\mathcal{X}) - 2.
\end{aligned}$$

Every μ' in the kernel of $R(H/uv, q')$ determines a unique vector μ in the kernel of $R(H, q)$ via setting $\mu(u) = \mu(v) = \mu'(z)$ and $\mu(x) = \mu'(x)$ for all for all $x \in U \setminus \{u, v\}$. Theorem $\dim \ker R(H, q) \geq \dim \ker R(H/uv, q')$. The rigidity matrix $R(H, q)$

[3]For us, a contraction will always be the more general vertex-contraction (which does not require u and v be adjacent) not the stricter edge-contraction (which does require u and v be adjacent).

has linearly independent rows, since $R(G,p)$ has linearly independent rows, and so

$$i_G(\mathcal{X}) = \text{rank } R(H,q) \leq \text{rank } R(H/uv,q') + 2 \leq \text{val}(\mathcal{X}).$$

Therefore G is uv-sparse.

On the other hand, suppose that G is uv-sparse. If $|V| \leq 4$ then G is uv-independent in X by Lemma 4.2.0.2. Therefore we may suppose that $|V| \geq 5$. By adding additional edges, if necessary, we may assume that G is uv-tight[4]. Proposition 4.2.0.1 now gives a method of constructing G. Furthermore, as X is strictly convex the geometric processes corresponding to this method of construction geometric operations preserve minimal rigidity in X (see Subsection 4.1.2). So, as any graph that begins the process of constructing G is uv-independent in X by Theorem 1.4.3.8 (i.e., every $(2,2)$-tight graph is independent in X) and Lemma 4.2.0.3, it follows that G is uv-independent in X. □

To conclude this section we use Theorem 4.2.0.5 result to provide a following delete-contract characterisation of uv-rigidity in (strictly convex) non-Euclidean normed planes. This bears comparison to a delete-characterisation of uv-rigidity in Euclidean normed planes [17, Theorem 15].

Theorem 4.2.0.6. *Let X be a non-Euclidean normed plane and let G be a graph such that $\{u,v\} \subseteq V(G)$. If X is strictly convex, then G is uv-rigid in X if and only if $(V(G), E(G) \setminus \{uv\})$ and G/uv are both rigid in X.*

Proof. Let $G = (V,E)$ and let G/uv be the contraction of G at $\{u,v\}$ adding z. Suppose that G is uv-rigid. It is immediate from the definition that $(V(G), E(G) \setminus \{uv\})$ is rigid. Let p be a regular uv-coincident realisation of G in X, and define p' to be the realisation of G/uv in X such that $p'(x) = p(x)$ for all $x \in V \setminus \{u,v\}$ and $p'(z) = p(u) = p(v)$. Given an infinitesimal flex μ' of $(G/uv, p')$ we can form an infinitesimal flex μ of (G,p) via setting $\mu(x) = \mu'(x)$ for all $x \in V \setminus \{u,v\}$ and $\mu(u) = \mu(v) = \mu'(z)$. Since (G,p) is an infinitesimally rigid uv-coincident framework, there exists $\lambda \in X$ such that μ must

[4]Recall that the edge sets of uv-sparse graphs are the independent sets of a matroid, and when $|V| \geq 5$ the maximally independent sets of this matroid have rank $2|V| - 2$.

have that $\mu = \lambda|_{x \in V}$ (and hence $\mu' = \lambda|_{x \in V(G/uv)}$) for some $\lambda \in X$ (i.e. mu is a restriction of a translation to the vertices of G). Hence $(G/uv, p')$ is infinitesimally rigid and G/uv is rigid. In a similar manner to the proof of [23, Theorem 1], the converse follows from Theorem 4.2.0.5. □

4.3 Necessary Conditions for Global Rigidity

In this section we develop Hendrickson-type conditions [21] necessary for graphs to be globally rigid; we work in the generality of normed spaces, although we will occasionally require the additional assumption that the normed space contains only finitely many linear isometries. After this section we will focus on non-Euclidean normed planes, which do have finitely many linear isometries. Before that though, we prove that all globally rigid graphs are 2-vertex-connected by extending Theorem 1.4.3.10 to the context of non-Euclidean normed spaces.

Theorem 4.3.0.1. *Take $d \in \mathbb{N}^+$, let X be a d-dimensional non-Euclidean normed space, and let G be a graph. If $|V(G)| \geq 2$ and G is globally rigid in X then G is 2-vertex-connected.*

Proof. Let $G = (V, E)$ and suppose for in pursuit of a contradiction that G is not 2-vertex-connected and is globally rigid in X. Firstly, suppose that $G = K[\{v, w\}]$ and let p be a realisation of G in X such that (G, p) is globally rigid. By applying translations to p, if necessary, we may suppose that $p(w) = 0$ and hence there exists $r \in \mathbb{R}^+$ such that $p(v) \in \{x \in X : \|x\| = r\}$. For all $y \in \{x \in X : \|x\| = r\}$, note that the framework (G, q), where $q(w) = 0$ and $q(v) = y$, is equivalent to (G, p). As (G, p) is globally rigid there exists a linear isometry of X that maps $p(v)$ to y. Hence linear isometries of X act transitively on $\{x \in X : \|x\| = r\}$. However this implies X is Euclidean, a contradiction.

So we may suppose that $G \not\cong K_2$, so there exists $u \in V$ and a partition $\{V_1, V_2\}$ of $V \setminus \{u\}$ such that $\{xy \in E : x \in V_1, y \in V_2\} = \emptyset$. As G is globally rigid in X there exists a well-positioned realisation, p, of G in X such that p is in the interior of GRig$(G; X)$ and $p(u) = 0$. By perturbing, if necessary, we may also assume that $p(v) \neq p(w)$ for all $\{v, w\} \subseteq V$, and, for $i \in \{1,2\}$, there exists $v_i \in V_i$ such that $\|p(v_1) - p(v_2)\| \neq \|p(v_1) + p(v_2)\|$[5]. Let p' be the realisation of G in X such that $p'(v) = p(v)$ for all $v \in V_1 \setminus \{u\}$ and $p'(v) = -p(v)$ for all $v \in V_2$. Then (G,p) and (G,p') are equivalent. However (G,p) and (G,p') are not congruent, since $\|p'(v_1)-p'(v_2)\| = \|p(v_1)+p(v_2)\| \neq \|p(v_1)-p(v_2)\|$. Therefore (G,p) is not globally rigid, a contradiction. □

Let G be a graph and take $d \in \mathbb{N}^+$. Recall from Theorem 1.4.3.10 that if $|V(G)| \geq d+2$ and G is globally rigid in a d-dimensional Euclidean space then G is $(d+1)$-vertex-connected. Therefore the vertex-connectivity requirement that it is necessary for a graph to have in order to be rigid in d-dimensional normed spaces is far more relaxed in the non-Euclidean setting. One may wonder whether Theorem 4.3.0.1 could be strengthened to give a statement more akin to Theorem 1.4.3.10, however our next result shows that if X is a normed space with finitely many linear isometries then this bound on the vertex-connectivity of globally rigid graphs in X can not be improved.

Proposition 4.3.0.2. *Take $d \in \mathbb{N}^+$ and let X be a d-dimensional normed space with finitely many linear isometries. If there exists a graph, G, with $|V(G)| \geq 2$ that is globally rigid in X then there exists a graph, G', with $\kappa(G') = 2$ that is globally rigid in X.*

Proof. Let $G = (V,E)$. By Theorem 4.3.0.1, G is 2-vertex-connected and so $|E(G)| \geq 3$. Take $v_1 v_2 \in E$, and let G_1 and G_2 be graphs such that $G_1 \cong G \cong G_2$ and $V(G_1) \cap V = \{v_1, v_2\} = V(G_2) \cap V$ and $E(G_1) \cap E = \{v_1 v_2\} = E(G_2) \cap E$. Let $G' = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$, so $\kappa(G') = 2$. As X has finitely many linear isometries, there exists an open dense set of points that are not invariant under any non-trivial linear isometry of X. Hence we may choose an open set $U \subseteq \mathrm{GRig}(G;X)$ such that, for each $p \in U$, $p(v_1) - p(v_2)$ is not invariant under any non-trivial linear isometry of X. Let $U' = \{p \in X^{V(G')} \colon \text{for } i \in \{1,2\}, p|_{V_i} \in U\}$. Since U is an open set, it follows that U' is an open set.

[5] To see why we may assume this, note that if $\|p(v_1) - p(v_2)\| = \|p(v_1) + p(v_2)\|$ then for all $\delta > 0$ we may take q in the interior of $\mathrm{GRig}(G;X)$ such that $q(v_1) = p(v_1) + \delta(p(v_1) + p(v_2))$, $q(v_2) = p(v_2) + \delta(p(v_1) + p(v_2))$, and $q(v) = p(v)$ for all $v \in V \setminus \{v_1, v_2\}$. Then $\|q(v_1) - q(v_2)\| = \|p(v_1) - p(v_2)\|$ and $\|q(v_1) + q(v_2)\| = (1+\delta)(\|p(v_1) + p(v_2)\|$. As p is injective this implies that $\|q(v_1) - q(v_2)\| \neq \|q(v_1) + q(v_2)\|$.

Take $p' \in U'$ take $q' \in X^{V(G')}$ such that (G', p') is equivalent to (G', q') and $p'(v_1) = q'(v_1)$. By applying translations, if necessary, we may assume $p(v_1) = 0$. Since both $(G_1, p'|_{V_1})$ and $(G_2, p'|_{V_2})$ are globally rigid, there exist linear isometries T_1 and T_2 of X such that, for $i \in \{1, 2\}$, $T_i(p'(v)) = q'(v)$ for all $v \in V_i$. Note that $T_1(p'(v_2)) = T_2(p'(v_2)) = q'(v_2)$. As $p'(v_2) - p'(v_1) = p'(v_2)$ is invariant under linear isometries, and $p(v_2) = T_2^{-1}(T_1(p(v_2)))$, we have that $T_1 = T_2$. Therefore (G', p') and (G', q') are congruent and so (G, p) is globally rigid. Hence G is globally rigid in X. □

Having extended Theorem 1.4.3.10 to the context of non-Euclidean normed spaces, from the point of view of vertex-connectivity, our next step is to extend Theorem 1.4.3.10 from the point of view of redundant rigidity. A proof of the following result can be found in [13], however due to the technical details related to normed spaces that are used in that proof we omit the details here. We remark that the following result concerns non-Euclidean normed spaces that are smooth and have finitely many linear isometries.

Theorem 4.3.0.3. *[13, Theorem 3.7] Take $d \in \mathbb{N}^+$, let X be a d-dimensional non-Euclidean normed space, let G be a graph, and let p be a realisation of G in X. If X is smooth and has finitely many linear isometries, p is completely strongly regular, and (G, p) is globally rigid in X then (G, p) is redundantly rigid in X[6].*

It remains to show that these necessary conditions for frameworks to be globally rigid correspond to necessary conditions for graphs to be globally rigid. To show this we restrict our attention to analytic normed spaces. While any realisation, p, of a graph, G, such that (G, p) is globally and infinitesimally rigid must be strongly regular, p need not be completely strongly regular. In fact, we do not know if such a realisation would exist in a given normed space (see [13, Remark 2.4, Subsection 3.3] and [26]). Fortunately, Proposition 1.4.3.2 informs us that this is not the case for analytic normed spaces and so we can obtain the main result of this section.

Theorem 4.3.0.4. *Take $d \in \mathbb{N}^+$, let X be a d-dimensional analytic non-Euclidean normed space, and let G be a graph. If X has finitely many linear isometries, $|V(G)| \geq 2$,*

[6]Note that, as stated in [13], the proof of this result also uses the condition that $|V(G)| \geq 2$. However if $|V(G)| < 2$ then $G \cong K_1$ and so (G, p) is, trivially, redundantly rigid in X.

and G is globally rigid in X then G is 2-vertex-connected and redundantly rigid in X.

Proof. By Theorem 4.3.0.1, G is 2-vertex-connected. As G is globally rigid in X there exists an open set $U \subseteq X^V$ such that for all $q \in U$, (G, q) is globally rigid in X. By Proposition 1.4.3.2, we may choose a completely strongly regular realisation $p \in U$. Since analytic normed spaces are smooth, Theorem 4.3.0.3 implies that (G, p) is redundantly rigid in X, and hence G is redundantly rigid in X. $\square$

4.4 Global Rigidity in Analytic non-Euclidean Normed Planes

We now bring our study of globally rigid frameworks in non-Euclidean normed spaces to a close by gathering together various results that allow us, in Theorem 4.4.0.6 to characterise those graphs that are rigid in analytic non-Euclidean normed planes.

Lemma 4.4.0.1. *[14, Theorem 5.3, Theorem 5.4] Let X be a non-Euclidean normed plane and G be a graph. If X is analytic and $G \cong K_5^-$ or $G \cong B_1$, then G is globally rigid in X.*

The previous result informs us that the graphs that begin our construction in Theorem 4.4.0.6 are globally rigid in analytic non-Euclidean normed planes. Our next result informs us that one of the graph operations we will use in our construction behaves well with respect to global rigidity. Similarly to Theorem 4.3.0.3 a proof can be found in [13] but we omit it here due to the technical details involved; we refer the reader to [13, Subsection 7.1] for more information. We remark that while the fact that the normed plane in question is analytic is not explicitly commented on in the proof of [13, Theorem 7.5], it is required in order to apply [13, Lemma 7.4].

Lemma 4.4.0.2. *[13, Theorem 7.5] Let X be a non-Euclidean normed plane, let G be a graph, and let G' be a K_4^--extension of G. If X is analytic and G is globally rigid in X then G' is globally rigid in X.*

The next two results are concerned with showing that another graph operation used in the construction in Theorem 4.4.0.6 behaves well with respect to global rigidity.

Lemma 4.4.0.3. *[12, Lemma 5.3], [14, Theorem 3.10][7] Take $d \in \mathbb{N}^+$, let X be a d-dimensional non-Euclidean normed space, let G be a graph, and let p be a uv-coincident realisation of G in X. If X is smooth, and has finitely many linear isometries, and (G,p) is infinitesimally rigid in X, and (G,p) is globally rigid in X then there exists an open neighbourhood, $U \subseteq X^{V(G)}$, of p such that (G,q) is globally rigid in X for all $q \in U$.*

Lemma 4.4.0.4. *[12, Theorem 5.4] Let X be a non-Euclidean normed plane and let G be a graph such that $|E(G)| \geq 1$. Take $z \in X$ such that $d_G(z) \geq 1$, $N_G(z) = N_1 \cup N_2$, and $N_1 \cap N_2 = 0$. Let G' be the generalised vertex split of G at z on (N_1, N_2) adding $\{u,v\}$. If X is analytic, G is globally rigid in X, and $(V(G'), E(G') \setminus \{uv\})$ is rigid in X then G' is globally rigid in X.*

Proof. Let G'/uv be the contraction of G' at $\{u,v\}$ adding z, so $G'/uv = G$. As G is globally rigid in X, G'/uv is rigid in X by Theorem 1.4.3.4. As $(V(G'), E(G') \setminus \{uv\})$ is rigid in X, Theorem 4.2.0.6 implies G' is uv-rigid in X. Now, [13, Lemma 5.1] implies we may take a realisation, p, of G in X such that (G,p) is infinitesimally rigid in X, (G,p) is globally rigid in X, and the realisation, p', of G' given by setting $p'(x) = p(x)$ for all $x \in V(G)$ and $p'(u) = p(z) = p'(v)$ is a uv-coincident realisation of G' such that (G',p') is infinitesimally rigid in X and (G',p') is globally rigid in X. So, Lemma 4.4.0.3 gives us that there exists a neighbourhood, $U \subseteq X^{V(G')}$, of p such that (G',q) is globally rigid in X for all $q \in U$. Therefore G' is globally rigid in X. □

We give one more result before providing our characterisation of globally rigid graphs in analytic non-Euclidean normed planes. This next result acts as a bridge between this chapter and Chapter 3. In particular, it opens the door for Theorem 3.2.0.6 to be applied to aid understanding of global rigidity in analytic non-Euclidean normed planes.

[7] See [12, Section 5] and [14, Subsection 3.2] for additional details. Note that [12, Lemma 5.3] is analogous to [14, Theorem 3.7] and so we have added the condition that X must have finitely many linear isometries to the result as stated in [12].

Lemma 4.4.0.5. *Let X be a non-Euclidean normed plane, let G be a graph, and let p be a completely regular realisation of G in X. If X is analytic then G is $(2,2)$-connected if and only if G is 2-vertex-connected and (G,p) is redundantly rigid in X.*

Proof. Since p is a completely regular realisation of G in X, (G,p) is redundantly rigid in X if and only if $G-e$ is rigid in X for all $e \in E(G)$. When combined with Theorem 1.4.3.8 this implies that (G,p) is redundantly rigid in X if and only if G contains a spanning $(2,2)$-tight subgraph and every edge of G is contained in a $(2,2)$-circuit.

Now, G contains a spanning $(2,2)$-tight subgraph and every edge of G is contained in a $(2,2)$-circuit if and only if G is redundantly rigid on the cylinder (see [32, Theorem 5.4]). Also, G is redundantly rigid on the cylinder and 2-vertex-connected if and only if G is $(2,2)$-connected (see [30, Theorem 5.4]). Therefore, combining all these equivalences gives us that G is $(2,2)$-connected if and only if G is 2-vertex-connected and (G,p) is redundantly rigid in X. □

We are now able to present the main result of this chapter.

Theorem 4.4.0.6. *Let X be a non-Euclidean normed plane and let G be a graph. If X is analytic then the following are equivalent:*

(i) G is globally rigid in X;

(ii) $G \cong K_1$, or G is 2-vertex-connected and G is redundantly rigid in X;

(iii) $G \cong K_1$, or G is $(2,2)$-connected; and

(iv) $G \cong K_1$, or there exists $t \in \mathbb{N}^+$ and a sequence $a_1, \dots, a_t$, with $a_1 \cong K_5^-$ or $a_1 \cong B_1$, $a_t = G$, such that for all $2 \leq j \leq t$,

(a) a_j is a K_4^--extension, or an edge-addition, or a generalised vertex split of a_{j-1}; and

(b) if a_j is a generalised vertex split of a_{j-1} then a_j is $(2,2)$-connected.

Proof. By Theorem 4.3.0.4, (i) implies (ii). Theorem 4.4.0.5 gives that (ii) holds if and only if (iii) holds. Theorem 3.2.0.6 gives that (iii) holds if and only if (iv) holds. Finally, the combination of Lemma 4.4.0.1, the fact that edge-addition clearly preserves global rigidity of a graph in X, Lemma 4.4.0.2, and Lemma 4.4.0.4 together with fact that (ii) holds if and only if (iii) holds, shows that (iv) implies (i). $\square$

Chapter 5

Vertex-Labelled Graphs

The penultimate chapter of this thesis a combinatorial analysis of structures that one hopes will aid understanding of which graphs are rigid when realised on some collection of non-concentric spheres. The rigidity of frameworks on spheres was studied by Nixon, Owen, and Power [32, 33], and in particular these authors considered frameworks realised on a family of concentric spheres, or a family of concentric cylinders. In these situations, the relevant graphs were shown to be $(2,3)$-tight and $(2,2)$-tight respectively. We look to extend this study by considering the graphs likely to be relevant to the rigidity of frameworks realised on a family of non-concentric spheres. To that end we consider labelled graphs, where the labelling of vertices can be thought of as corresponding to the sphere that a vertex may be realised on. By removing the hypothesis of concentricity, additional geometric complications arise. In particular, the constraints placed on a subgraph may not be as restrictive as those placed on the graph it is contained within. While we refrain from providing an analysis of these geometric issues, this chapter proceeds to introduce and work with a notion of sparsity that appears appropriate to consider this problem.

5.1 Motivation

Let us begin this chapter by discussing the motivation for combinatorial analysis that follows. We do not provide formal terminology and machinery for this discussion, take a heuristic approach.

The frameworks considered earlier in this thesis have been graphs realised in some normed space. One may restrict the possible locations that vertices may be mapped to by a realisation in various ways; this is the idea behind generic frameworks for example. One idea that has been widely researched [32, 33, 23, 24] is to restrict the possible locations that vertices are mapped to such that the framework can be thought of as living on some surface.

If we have a framework on some surface, then it is natural to consider what it means for a framework to be rigid. Three possibilities spring to mind. The first is to simply rigidity with respect to the ambient space that the surface is living in. However, this somehow ignores the fact that the framework is living on a surface and so is not particularly interesting. Secondly, one could require that if two frameworks are equivalent then there is an isometry of the surface that takes one framework to the other (i.e. the frameworks are congruent). This definitely captures the fact that a framework is living on some surface, but we argue that in some settings (e.g. non-concentric spheres).

The third possibility, and the one inspires the upcoming combinatorical analysis, is to require that if two frameworks are equivalent then they are quasi-congruent (recall Remark 7). We argue for this formulation of rigidity of frameworks on surfaces by means of examples. Let G be a graph with at least five vertices, and let p be a realisation of G such that (G, p) lives on a pair of non-concentric spheres, S_1 and S_2. There is one isometry of the surface $S_1 \cup S_2$, namely the rotation about the line through the centres of S_1 and S_2.

If no vertices of G are mapped to S_2 by p then any of the three isometries of S_1 will take (G, p) to an equivalent and quasi-congruent framework that is non-congruent. The previous example may seem contrived, as S_2 is somehow irrelevant, but it is indicative of the situation with more interesting examples. If that exactly one vertex of G is mapped to S_2 then either applying either the isometry of $S_1 \cup S_2$ to (G, p), or rotating about the line through the centre of S_1 and $p(v)$, where v is the unique vertex of G mapped to S_2, will take (G, p) to an equivalent and quasi-congruent framework that is non-congruent. These examples, as well as the fact that there is one isometry of $S_1 \cup S_2$ (and no isometry of three pairwise non-concentric spheres such that centres of these three spheres are not collinear) provide an explanation for the rationale behind Definition 5.3.1.2. Similar reasoning applied to circles motivations Definition 5.3.1.1.

Before turning out attention back to combinatorial matters we pre-empt an additional quirk of frameworks on non-concentric spheres that motivates the change of direction in Subsection 5.5.3. Consider the graph $\tilde{H}$ illustrated in Figure 4.4, and suppose this graph is realised on a pair of non-concentric spheres, S_1 and S_2, such that x is the unique vertex of $\tilde{H}$ realised on S_2. If one fixes either of the subgraphs of $\tilde{H}$ isomorphic to K_4 and rotates the other about the line through the centre of S_1 and $p(x)$ then this results in an equivalent non-quasi-congruent framework. Therefore, despite satisfying the conditions of Definition 5.3.1.2, this graph would not be rigid on $S_1 \cup S_2$. A similar issue can be seen in Figure 5.4. The existence of these graphs prompts us to give Definition 5.5.3.1.

5.2 Vertex-Labelling and Vertex-Labelled Graph Operations

We begin this section by formalising the key objects that we work with in this chapter.

Definition 5.2.0.1. Let $G = (V, E)$ be a graph and let X be a finite non-empty set. An **(X-) vertex-labelling** of G is a map $\chi\colon V \to X$. The ordered pair (G, χ), which we often denote G_χ, is an **(X-) vertex-labelled graph**.

We shall borrow much of the terminology that we use in this chapter from graph theory. For example, concepts such as connectivity are not meaningfully impacted by labelling vertices. However, notions such as subgraphs and isomorphisms require additional detail in the context of vertex-labelled graphs. Throughout this chapter the only labellings of graphs that we consider are vertex-labellings and so for sake of brevity we will often dispense with the qualifier 'vertex'. Similarly, we will only make reference to a graph being X-labelled in those rare cases where the particular set X is relevant.

Definition 5.2.0.2. Let X and Y be finite non-empty sets, let G_χ be an X-labelled graph, and let H_ψ be a Y-labelled graph. G_χ and H_ψ are **vertex-labelled isomorphic**, which we denote by $G_\chi \cong H_\psi$, if $|X| = |Y|$ and there exists a graph isomorphism $f\colon V(G) \to V(H)$ such that $\chi(u) = \chi(v)$ if and only if $\psi(f(u)) = \psi(f(v))$.

A **vertex-labelled subgraph** of G_χ is a vertex-labelled graph $G'_{\chi'}$ such that G' is a subgraph of G and $\chi' = \chi|_{V(G')}$.

Let $G_\chi = (V, E)_\chi$ be an X-vertex-labelled graph. Given $\emptyset \neq U \subseteq V$, $G_\chi[U] \coloneqq (G[U], \chi|_U)$ is the vertex-labelled subgraph of G_χ **induced** by U. Similarly, give $\emptyset \neq F \subseteq E$, $G_\chi[F] \coloneqq (G[F], \chi|_{V(G[F])})$ is the vertex-labelled subgraph of G **induced** by F. Consider the relation on V where two vertices are related if and only if they are mapped to the same element of X by χ. We note that this is an equivalence relation on V and hence this relation induces a partition of V. We denote this partition by $\mathcal{V}(G_\chi)$.

We complete this preliminary section by translating some additional concepts from the world of graphs to that of labelled graphs. Specifically we consider some operations, originally introduced in Subsection 1.1.2, in a labelled-graph context.

Definition 5.2.0.3. Take $d \in \mathbb{N}^+$ and let $G_\chi = (V, E)_\chi$ be a labelled graph such that $|V| \geq d$. A **$(d, 0)$-VL-extension**[1] of G_χ is a labelled graph $G'_{\chi'}$ where G' is a $(d, 0)$-extension of G and $\chi'|_V = \chi$.

Let H_ψ be a labelled graph such that $\{u \in V(H)\colon d_H(u) = d\} \neq \emptyset$. A **$(d, 0)$-VL-reduction** of H_ψ is a labelled graph $H'_{\psi'}$ such that H_ψ is a $(d, 0)$-VL-extension of H'.

Definition 5.2.0.4. Take $d \in \mathbb{N}^+$ and let $G_\chi = (V, E)_\chi$ be a labelled graph such that $|V| \geq d + 1$ and $|E| \geq 1$. A **$(d, 1)$-VL-extension** of G_χ is a labelled graph $G'_{\chi'}$ where

[1] Despite choosing to refer to vertex-labelled graphs as labelled graphs, we continue to use VL, rather than L, to discuss these graph operations. This is less of an encumbrance than repeatedly stating "vertex" and acts as a reminder of what is being labelled.

G' is a $(d,1)$-extension of G and $\chi'|_V = \chi$.

Let H_ψ be a labelled graph such that there exists $u \in V(H)$ with $d_H(u) = d+1$ and $I_H(N_H(v)) < \binom{d+1}{2}$. A **$(d,1)$-VL-reduction** of H is a labelled graph $H'_{\psi'}$ such that H_ψ is a $(d,1)$-VL-extension of $H'_{\psi'}$.

5.3 Vertex-Labelled Count Sparsity

The purpose of this section is to introduce a vertex-labelling of graphs pertinent to studying the rigidity of graphs realised on collections of spheres living in d-dimensional Euclidean space. To that end we consider graphs which satisfy a condition similar to the one given in Definition 2.1.0.5. Our definition of what it means for a labelled graph to be 'sparse' is subtly different in that no reference is made to an intermediate matroid. However, the labelled graphs that we call sparse satisfy similar properties to sparse graphs in the literature and, indeed, those seen in Definition 2.1.0.5.

5.3.1 σ_d-Sparse Graphs

Definition 5.3.1.1. Let G_χ be a labelled graph. The **$\boldsymbol{\sigma_1}$-count** of G_χ is the function $\sigma_1\colon \mathcal{P}(V(G_\chi)) \setminus \emptyset \to \mathbb{Z}$ defined by

$$\sigma_1(U) = \begin{cases} 1 & \text{if there exists } W \in \mathcal{V}(G_\chi)\colon U \subseteq W; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 5.3.1.2. Let G_χ be a labelled graph. The **$\boldsymbol{\sigma_2}$-count** of G_χ is the function

$\sigma_2\colon \mathcal{P}(V(G_\chi)) \setminus \emptyset \to \mathbb{Z}$ defined by

$$\sigma_2(U) = \begin{cases} 3 & \text{if there exists } W \in \mathcal{V}(G_\chi)\colon U \subseteq W; \\ 2 & \text{if } |\{W \in \mathcal{V}(G_\chi)\colon U \cap W \neq \emptyset\}| = 2 \text{ and} \\ & \exists W \in \mathcal{V}(G_\chi) \text{ such that } |U \cap W| = 1; \\ 1 & \text{if } |\{W \in \mathcal{V}(G_\chi)\colon U \cap W \neq \emptyset\}| = 2 \text{ and } \forall W \in \mathcal{V}(G_\chi), |U \cap W| \neq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Take $d \in \{1, 2\}$ and let $G_\chi = (V, E)_\chi$ be a labelled graph. Then

$$S(G_\chi, \sigma_d) := \begin{cases} \min\left\{d|V| - \sigma_d(V), (d+1)|V| - \binom{d+2}{2}\right\} & \text{if } |V| \geq d+2; \\ \binom{|V|}{2} & \text{if } |V| \leq d+1. \end{cases}$$

Definition 5.3.1.3. Take $d \in \{1, 2\}$ and let G_χ be a labelled graph. G_χ is **$\boldsymbol{\sigma_d}$-sparse** if for all labelled subgraphs $G'_{\chi'}$ of G_χ, $|E(G')| \leq S(G'_{\chi'}, \sigma_d|_{V(G')})$. G_χ is **$\boldsymbol{\sigma_d}$-tight** if it is σ_d-sparse and $|E(G)| = S(G_\chi, \sigma_d) = d|V| - \sigma_d(V)$.

Lemma 5.3.1.4. *Take $d \in \{1, 2\}$ and let $G_\chi = (V, E)_\chi$ be a labelled graph. Then,*

(i) $\binom{|V|}{2} = (d+1)|V| - \binom{d+2}{2} \iff |V| \in \{d+1, d+2\}$*; and*

(ii) $\binom{|V|}{2} > (d+1)|V| - \binom{d+2}{2} \iff |V| < d+1$ *or* $|V| > d+2$.

Proof. Take $\sim \in \{=, >\}$. Then,

$$\begin{aligned} \binom{|V|}{2} \sim (d+1)|V| - \binom{d+2}{2} &\iff |V|^2 - |V| \sim 2(d+1)|V| - (d+2)(d+1) \\ &\iff |V|^2 + (-3-2d)|V| + (d+2)(d+1) \sim 0 \\ &\iff (|V| - (d+1))(|V| - (d+2)) \sim 0. \end{aligned}$$

□

Lemma 5.3.1.5. *Take $d \in \{1,2\}$ and let $G_\chi = (V,E)_\chi$ be a labelled graph. For all $\sim \in \{<,\leq,=,\geq,>\}$,*

$$\binom{|V|}{2} \sim d|V| - \sigma_d(V) \iff (|V|-d)(|V|-(d+1)) \sim d(d+1) - 2\sigma_d(V).$$

Proof.

$$\begin{aligned}\binom{|V|}{2} \sim d|V| - \sigma_d(V) &\iff |V|^2 - |V| \sim 2d|V| - 2\sigma_d(V) \\ &\iff |V|^2 + (-1-2d)|V| + 2\sigma_d(V) \sim 0 \\ &\iff (|V|-d)(|V|-(d+1)) \sim d(d+1) - 2\sigma_d(V).\end{aligned}$$

□

Our next result is similar to, but arguably more illuminating than, the previous pair of lemmas. In particular, it highlights the nuances that arise when studying the labelled graphs that are relevant to the study of rigidity of frameworks realised on higher dimensional spaces.

Lemma 5.3.1.6. *Take $d \in \{1,2\}$ and let $G_\chi = (V,E)_\chi$ be a labelled graph. Suppose $|V| \geq d+2$.*

(i) If $d = 1$ then $\min\{d|V| - \sigma_d(V), (d+1)|V| - \binom{d+2}{2}\} = d|V| - \sigma_d(V)$.

(ii) If $d = 2$ then $\min\{d|V| - \sigma_d(V), (d+1)|V| - \binom{d+2}{2}\} \neq d|V| - \sigma_d(V)$ if and only if

(a) $|V| = 4$ and $\sigma_2(V) \in \{0,1\}$; or

(b) $|V| = 5$ and $\sigma_2(V) = 0$.

Proof. For all $\sim \in \{<,\leq,=,\geq,>\}$,

$$(d+1)|V| - \binom{d+2}{2} \sim d|V| - \sigma_d(V) \iff |V| \sim \binom{d+2}{2} - \sigma_d(V).$$

If $d = 1$ then $|V| \geq d+2 = 3 \geq \binom{d+2}{2} - \sigma_d(V)$, so $d|V| - \sigma_d(V) \leq (d+1)|V| - \binom{d+2}{2}$. If $d = 2$ then $\min\{d|V| - \sigma_d(V), (d+1)|V| - \binom{d+2}{2}\} \neq d|V| - \sigma_d(V)$ if and only if $d|V| - \sigma_d(V) > (d+1)|V| - \binom{d+2}{2}$ if and only if $4 \leq |V| < 6 - \sigma_d(V)$ if and only if either $|V| = 4$ and $\sigma_d(V) \in \{0, 1\}$, or $|V| = 5$ and $\sigma_d(V) = 0$. □

Suppose that G_χ is a σ_d-sparse labelled graph, with $|V| \geq d+2$, where $d \in \{1, 2\}$. The previous result says that if $d = 1$ then G_χ is σ_1-tight if and only if $|E| = S(G_\chi, \sigma_1)$, whereas if $d = 2$ then it is possible for $|E| = S(G_\chi, \sigma_2)$ to hold but for G_χ to not be σ_2-tight. Examples of this phenomena are illustrated in Figure 5.1.

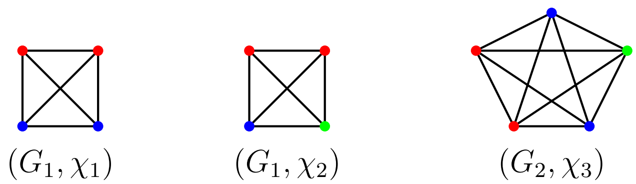


Figure 5.1: Illustrations of some (the, up to vertex-labelled isomorphism) labelled graphs that satisfy $|E| = S(G_\chi, \sigma_2)$ but are not σ_2-tight.

We conclude this subsection by proving some basic properties of σ_d-sparse labelled graphs.

Lemma 5.3.1.7. *Take $d \in \{1, 2\}$ and let $G_\chi = (V, E)_\chi$ be a labelled graph. For all $\emptyset \neq X \subseteq Y \subseteq V$, $\sigma_d(X) \geq \sigma_d(Y)$.*

Proof. Firstly suppose that $d = 1$, so $\sigma_1(X), \sigma_1(Y) \in \{0, 1\}$. If $\sigma_1(Y) = 0$ then trivially $\sigma_1(X) \geq \sigma_1(Y)$. If $\sigma_1(Y) = 1$ then there exists $W \in \mathcal{V}(G_\chi)$ such that $Y \subseteq W$. Hence $X \subseteq Y \subseteq W$ and $\sigma_1(X) = 1 = \sigma_1(Y)$.

Now suppose that $d = 2$, so $\sigma_2(X), \sigma_2(Y) \in \{0, 1, 2, 3\}$. If $\sigma_2(Y) = 0$ then trivially $\sigma_2(X) \geq \sigma_2(Y)$. If $\sigma_2(Y) = 1$ then $|\{W \in \mathcal{V}(G_\chi) \colon Y \cap W \neq \emptyset\}| = 2$. Hence $|\{W \in \mathcal{V}(G_\chi) \colon X \cap W \neq \emptyset\}| \leq 2$ and so $\sigma_2(X) \geq \sigma_2(Y)$. If $\sigma_2(Y) = 2$ then $|\{W \in \mathcal{V}(G_\chi) \colon Y \cap W \neq \emptyset\}| = 2$ and there exists $U \in \mathcal{V}(G_\chi)$ such that $|Y \cap U| = 1$. It follows that $|\{W \in \mathcal{V}(G_\chi) \colon X \cap W \neq \emptyset\}| \leq 2$, and if $|\{W \in \mathcal{V}(G_\chi) \colon X \cap W \neq \emptyset\}| = 2$ then $|X \cap U| = 1$. Therefore $\sigma_2(X) \geq 2$. If $\sigma_d(Y) = 3$ then there exists $W \in \mathcal{V}(G_\chi)$ such that $Y \subseteq W$. Hence $X \subseteq Y \subseteq W$ and $\sigma_2(X) = 3 = \sigma_2(Y)$. □

Lemma 5.3.1.8. *Take $d \in \{1,2\}$ and let $G_\chi = (V,E)_\chi$ be a labelled graph. If G_χ is σ_d-sparse then $\delta(G) \leq 2d - \min\{1, \sigma_2(V)\}$. If G_χ is σ_d-tight then $\delta(G) \geq d$ if and only if $G \ncong K_d$.*

Proof. By Theorem 1.1.1.6, $\delta(G)|V| \leq \sum_{v \in V} d_G(v) = 2|E|$. Suppose that G_χ is σ_d-sparse. If $|V| \leq d+1$ then $|E| \leq \binom{|V|}{2}$ and so $\delta(G) \leq d \leq 2d - \min\{1, \sigma_2(V)\}$. If $|V| \geq d+2$ then $2|E| \leq 2(d|V| - \sigma_d(V))$ and so $\delta(G) \leq 2d - \left(\frac{2}{|V|}\right)\sigma_d(V)$, and as $\delta(G) \in \mathbb{Z}$ it follows that $\delta(G) \leq 2d - \min\{1, \sigma_2(V)\}$.

Suppose G_χ is σ_2-tight, so $|E| = S(G_\chi, \sigma_d) = d|V| - \sigma_d(V)$. If $\delta(G) \geq d$ then clearly $G \ncong K_d$. On the other hand, suppose $G \ncong K_d$. If $|V| \leq d+1$ then $d|V| - \sigma_d(V) = \binom{|V|}{2}$ and so Lemma 5.3.1.5 implies $|V| \in \{d, d+1\}$. Then $G \cong K_{|V|}$ and so, as $G \ncong K_d$, $G \cong K_{d+1}$ and $\delta(G) = d$. If $|V| \geq d+2$ then take $v \in V$ such that $d_G(v) = \delta(G)$, then

$$i_G(V \setminus \{v\}) = |E| - d_G(v) = d|V| - (\sigma_d(V) + \delta(G)) = d|V \setminus \{v\}| + d - (\sigma_d(V) + \delta(G)).$$

Now, let $G'_{\chi'} = (V', E')_{\chi'}$ denote the $G_\chi[V \setminus \{v\}]$. As G_χ is σ_d-sparse, $i_G(V \setminus \{v\}) = |E'| \leq S(G'_{\chi'}, \sigma_d|_{V(G')})$. As $|V| \geq d+2$, $|V'| \geq d+1$ and so Lemma 5.3.1.4 and Lemma 5.3.1.5 together imply that $|E'| \leq |V'| - \sigma_d(V')$. Lemma 5.3.1.7 implies $\sigma_d(V') \geq \sigma_d(V)$. So,

$$d|V'| + d - (\sigma_d(V) + \delta(G)) = |E'| \leq d|V'| - \sigma_d(V') \leq d|V'| - \sigma_d(V).$$

Therefore $\delta(G) \geq d$. □

Lemma 5.3.1.9. *Take $d \in \{1,2\}$ and let $G_\chi = (V,E)_\chi$ be a labelled graph. Suppose G_χ is σ_d-tight and $G \ncong K_1$.*

(i) If $\sigma_d(V) > 0$ then $\kappa(G), \kappa_1(G) \geq 1$.

(ii) If $\sigma_d(V) = 0$ then for all labelled subgraphs H_ψ of G_χ such that H is a component of G, $\sigma_d(V(H)) = 0$ and H_ψ is σ_d-tight.

Proof. As $G \ncong K_1$, $|V| \geq 2$. Let $G_1, \ldots, G_n$ be the components of G and for all $1 \leq i \leq n$ let $G_i = (V_i, E_i)$. Then $V = \bigcup_{i=1}^n V_i$ and $E = \bigcup_{i=1}^n E_i$. As each G_i is a component of G we have that for all $1 \leq i < j \leq n$, $V_i \cap V_j = \emptyset = E_i \cap E_j$. Therefore $|V| = \sum_{i=1}^{n} |V_i|$ and $|E| = \sum_{i=1}^{n} |E_i|$. As G_χ is σ_d-tight and $|V| \geq 2 \geq d$, Lemma 5.3.1.8 implies $G \cong K_2 = K_d$, and so $\kappa(G) = 1$, or $\delta(G) \geq d$. Suppose that $\delta(G) \geq d$ and hence for all $1 \leq i \leq n$, $d \leq |E_i|$ and $|V_i| \geq d+1$. Therefore, as G_χ is σ_d-sparse, Lemma 5.3.1.4 and Lemma 5.3.1.5 together imply that $|E_i| \leq d|V_i| - \sigma_d(V_i)$ for all $1 \leq i \leq n$. So,

$$d|V| - \sigma_d(V) = |E| = \sum_{i=1}^{n} |E_i| \leq \sum_{i=1}^{n} (d|V_i| - \sigma_d(V_i)) = d|V| - \sum_{i=1}^{n} \sigma_d(V_i).$$

Hence $\sigma_d(V) = \sum_{i=1}^{n} \sigma_d(V_i)$. Now, Lemma 5.3.1.7 implies that $\sigma_2(V) \leq \sigma_2(V_i)$ for all $1 \leq i \leq n$. So, as $\sigma_d(V) = \sum_{i=1}^{n} \sigma_d(V_i)$ we have that either $n = 1$ or $\sigma_2(V) = 0 = \sigma_2(V_i)$ for all $1 \leq i \leq n$. If $n = 1$ then $\kappa(G) \geq 1$ and so Theorem 1.1.1.12 implies $\kappa_1(G) \geq 1$. If $\sigma_2(V) = 0 = \sigma_2(V_i)$ for all $1 \leq i \leq n$ then $G_\chi[V_i]$ is σ_d-tight for all $1 \leq i \leq n$. □

Lemma 5.3.1.10. *Let $G_\chi = (V, E)_\chi$ be a σ_2-tight labelled graph. If $\sigma_2(V) = 3$ and $|V| \geq 3$ then $\kappa(G) \geq 2$.*

Proof. As G_χ is σ_2-tight and $\sigma_2(V) = 3$, $|E| = 2|V| - 3$. As $|V| \geq 2$ we can take $U \subsetneq V$ such that $|U| = 1$, say $U = \{u\}$, and let $H_1, \ldots, H_n$ be the components of $G[V \setminus \{u\}] = H = (W, F)$. As $|V| \geq 3$, $|W| \geq 2$. For all $1 \leq i \leq n$ let $H_i = (W_i, F_i)$. Then $W = \bigcup_{i=1}^{n} W_i$ and $F = \bigcup_{i=1}^{n} F_i$. As each H_i is a component of H we have that for all $1 \leq i < j \leq n$, $W_i \cap W_j = \emptyset = F_i \cap F_j$. Therefore $|W| = \sum_{i=1}^{n} |W_i|$ and $|F| = \sum_{i=1}^{n} |F_i|$. As G_χ is σ_2-tight and $|V| \geq 3$, Lemma 5.3.1.8 implies $\delta(G) \geq 2$. So for all $1 \leq i \leq n$, $\emptyset \neq F_i \subsetneq E$. Therefore, for all $1 \leq i \leq n$ there exists $a_i \geq 3$ such that $|F_i| = 2|W_i| - a_i$. So for all $1 \leq i \leq n$,

$$2|W_i \cup \{u\}| - 3 \geq |F_i| + |N_G(u) \cap W_i| = 2|W_i \cup \{u\}| + |N_G(u) \cap W_i| - (a_i + 2).$$

Therefore, for all $1 \leq i \leq n$, $|N_G(u) \cap W_i| \leq a_i - 1$. Moreover,

$$\begin{aligned} 2|W| - (1 + d_G(u)) &= 2|V| - (3 + d_G(u)) \\ &= |E| - d_G(u) \\ &= |F| \\ &= \sum_{i=1}^{n} (2|W_i| - a_i) \\ &= 2|W| - \sum_{i=1}^{n} a_i. \end{aligned}$$

It follows that $1 + d_G(u) = \sum_{i=1}^{n} a_i$ and consequently $d_G(u) = (\sum_{i=1}^{n} a_i) - 1$. Combining the information we have so far we see that

$$\sum_{i=1}^{n}(a_i - 1) \geq \sum_{i=1}^{n} |N_G(u) \cap W_i| = d_G(u) = \left(\sum_{i=1}^{n} a_i\right) - 1.$$

So $-n \geq -1$ and hence $n = 1$. Therefore $H = G[V \setminus U]$ is connected. Lemma 5.3.1.9 implies that $\kappa(G) \geq 1$ and so for all $U \subseteq V$ such that $|U| \leq 1$, we have $|V \setminus U| \geq 2$ and $G[V \setminus U]$ is connected. Therefore $\kappa(G) \geq 2$. □

Lemma 5.3.1.11. *Let $G_\chi = (V, E)_\chi$ be a σ_2-sparse labelled graph and take $U \subseteq V$. Suppose that $|U| \geq 3$ and $\kappa(G[U]) = 1$. Take $u \in U$ and let $G_1, \ldots, G_n$ be the components of $G[U \setminus \{u\}]$. If U is critical in G_χ and $n \geq 2$ then for all $1 \leq i \leq n$, $\sigma_2(V(G_i) \cup \{u\}) \leq 2$.*

Proof. For all $1 \leq i \leq n$, let $G_i = (U_i, F_i)$. Now, for all $1 \leq j \leq n$ let $W_j = \{u\} \cup (\bigcup_{i=1}^{n}(U_i \setminus U_j))$. As G_χ is σ_2-sparse and U is σ_2-critical in G_χ we observe that, for all $1 \leq j \leq n$,

$$\begin{aligned} 2|U| - \sigma_2(U) &= i_G(U_j \cup \{u\}) + i_G(W_j) \\ &\leq (2|U_j \cup \{u\}| - \sigma_2(U_j \cup \{u\})) + 2|W_j| - \sigma_2(W_j) \\ &= 2(|U| + 1) - (\sigma_2(U_j \cup \{u\}) + \sigma_2(W_j). \end{aligned}$$

Therefore, $\sigma_2(U_j \cup \{u\}) \leq 2 + (\sigma_2(U) - \sigma_2(W_j))$. As $W_j \subseteq U$, Lemma 5.3.1.7 implies that $\sigma_2(U_j \cup \{u\}) \leq 2$. □

Lemma 5.3.1.12. *Let $G_\chi = (V, E)_\chi$ be a σ_2-tight labelled graph. If $\sigma_2(V) \in \{2, 3\}$ and $|V| \geq 3$ then $\kappa_1(G) \geq 2$.*

Proof. As $|V| \geq 3$, Lemma 5.3.1.8 implies $\delta(G) \geq 2$. Hence $|E| \geq 2$ and we may take $F \subsetneq E$ such that $|F| = 1$. Let $G_1, \ldots, G_n$ be the components of $G[E \setminus F]$ and for all $1 \leq i \leq n$ let $G_i = (V_i, E_i)$. As $|F| = 1 < \delta(G)$, $V(G[E \setminus F]) = V = \bigcup_{i=1}^n V_i$. As each G_i is a component of $(V, E \setminus F)$ we have that for all $1 \leq i < j \leq n$, $V_i \cap V_j = \emptyset = E_i \cap E_j$. As $\delta(G) \geq 2$ and $|F| = 1$, $|V_i| \geq d$ for all $1 \leq i \leq n$. As G_χ is σ_2-sparse Lemma 5.3.1.5 implies that for all $1 \leq i \leq n$, $|E_i| \leq S(G_\chi[V_i], \sigma_2|_{V_i}) \leq 2|V_i| - \sigma_2(V_i)$. So as G_χ is σ_2-tight,

$$\begin{aligned} 2|V| - \sigma_2(V) &= |F| + \sum_{i=1}^n |E_i| \\ &\leq 1 + \sum_{i=1}^n (2|V_i| - \sigma_2(V_i) \\ &= 2|V| + 1 - (\sum_{i=1}^n \sigma_2(V_i). \end{aligned}$$

Lemma 5.3.1.7 gives us that for all $1 \leq i \leq n$, $\sigma_2(V_i) \geq \sigma_2(V)$. So $1 + \sigma_2(V) \geq \sum_{i=1}^n \sigma_2(V_i) \geq n\sigma_2(V)$. As $\sigma_2(V) \in \{2, 3\}$ it follows that $n = 1$ and so $(V, E \setminus F)$ is connected. Lemma 5.3.1.9 implies that $\kappa_1(G) \geq 1$. So for all $F \subsetneq E$ such that $|F| \leq 1$, $|V| \geq 3$ and $(V, E \setminus F)$ is connected. Therefore $\kappa_1(G) \geq 2$. □

Lemma 5.3.1.13. *Let $G_\chi = (V,E)_\chi$ be a labelled graph. Suppose G_χ is σ_2-tight, $|V| \geq 3$, and $\kappa_1(G) = 1$. Take $e \in E$ and let $G_1, \ldots, G_n$ be the components of $G[E \setminus \{e\}]$. If $n \geq 2$ then for all $1 \leq i \leq n$, $\sigma_2(V(G_i)) \leq 1$.*

Proof. As G_χ is σ_2-tight and $|V| \geq 3$ and $\kappa_1(G) = 1$, Lemma 5.3.1.12 implies $\sigma_2(V) \in \{0,1\}$. For all $1 \leq i \leq n$ let $G_i = (V_i, E_i)$. As $\delta(G) \geq 2$, $V(G[E \setminus \{e\}]) = V = \bigcup_{i=1}^n V_i$. As each G_i is a component of $(V, E \setminus \{e\})$ we have that for all $1 \leq i < j \leq n$, $V_i \cap V_j = \emptyset = E_i \cap E_j$. As G_χ is σ_2-sparse Lemma 5.3.1.5 implies that for all $1 \leq i \leq n$, $|E_i| \leq S(G_\chi[V_i], \sigma_2|_{V_i}) \leq 2|V_i| - \sigma_2(V_i)$. So as G_χ is σ_2-tight,

$$\begin{aligned} 2|V| - \sigma_2(V) &= 1 + \sum_{i=1}^n |E_i| \\ &\leq 1 + \sum_{i=1}^n (2|V_i| - \sigma_2(V_i)) \\ &= 2|V| + 1 - \left(\sum_{i=1}^n \sigma_2(V_i) \right). \end{aligned}$$

Lemma 5.3.1.7 gives us that for all $1 \leq i \leq n$, $\sigma_2(V_i) \geq \sigma_2(V)$. So $\sum_{i=1}^n \sigma_2(V_i) \leq 1 + \sigma_2(V)$. As $n \geq 2$ and $\sigma_2(V) \in \{0,1\}$ it follows that for all $1 \leq i \leq n$, $\sigma_2(V_i) \leq 1$. □

5.3.2 Vertex-Labelled Graph Operations and σ_d-Sparsity

The aim of this section is to understand the conditions under which certain labelled graph operations preserve, or fail to preserve, the property of being σ_d-sparse (or σ_d-tight). These results will, in due course, be used to construct families of labelled graphs.

Lemma 5.3.2.1. *Take $d \in \{1,2\}$ and let $G_\chi = (V,E)_\chi$ be a labelled graph. Suppose $G'_{\chi'}$ is a $(d,0)$-VL-reduction of G_χ. G_χ is σ_d-sparse if and only if $G'_{\chi'}$ is σ_d-sparse.*

Proof. Let $V \setminus V(G') = \{v\}$. If G_χ is σ_d-sparse then, as $G'_{\chi'}$ is a labelled subgraph of G_χ, $G'_{\chi'}$ is σ_d-sparse. On the other hand, suppose that $G'_{\chi'}$ is σ_d-sparse and let H_ψ be a labelled subgraph of G_χ. If $|V(H)| \leq d+1$ then $|E(H)| \leq \binom{|V|}{2} = S(H_\psi, \sigma_d|_{V(H)})$. If

$|V(H)| \geq d+2$ then $|V(H) \setminus \{v\}| \geq d+1$. Let $V(H) \setminus \{v\} = U$. Lemma 5.3.1.4 and Lemma 5.3.1.5 together imply that

$$i_{G'}(U) \leq \min \left\{ d|U| - \sigma_d(U), (d+1)|U| - \binom{d+2}{2} \right\}.$$

So it follows from Lemma 5.3.1.7 that

$$\begin{aligned}
|E(H)| &\leq i_G(U) + (|N_G(v) \cap V(H)|)(|V(H) \cap \{v\}|) \\
&\leq i_{G'}(U) + d|V(H) \cap \{v\}| \\
&\leq \min \left\{ d|U| - \sigma_d(U), (d+1)|U| - \binom{d+2}{2} \right\} + d|V(H) \cap \{v\}| \\
&= \min \left\{ d|V(H)| - \sigma_d(U), d|V(H)| + |U| - \binom{d+2}{2} \right\} \\
&\leq \min \left\{ d|V(H)| - \sigma_d(V(H)), (d+1)|V(H)| - \binom{d+2}{2} \right\} \\
&= S(H_\psi, \sigma_d|_{V(H)}).
\end{aligned}$$

As we chose an arbitrary labelled subgraph of G_χ, for every labelled subgraph, H_ψ, of G_χ we have $|E(H)| \leq S(H_\psi, \sigma_d|_{V(H)})$ and therefore G_χ is σ_d-sparse. □

Lemma 5.3.2.2. *Take $d \in \{1, 2\}$ and let $G_\chi = (V, E)_\chi$ be a labelled graph. Suppose $G'_{\chi'}$ is a $(d, 0)$-VL-reduction of G_χ. If G_χ is σ_d-tight then $G'_{\chi'}$ is σ_d-tight. If $G'_{\chi'}$ is σ_d-tight then G_χ is σ_d-tight if and only if $\sigma_d(V) = \sigma_d(V')$.*

Proof. Let $V \setminus V(G') = \{v\}$. Firstly, suppose that G_χ is σ_d-tight. Then $|E| = S(G_\chi, \sigma_d) = d|V| - \sigma_d(V)$ and Lemma 5.3.2.1 implies that $G'_{\chi'}$ is σ_d-sparse. As $|E| = d|V| - \sigma_d(V)$, Lemma 5.3.1.7 implies that

$$|E(G')| = |E| - d = d|V(G')| - \sigma_d(V) \geq d|V'| - \sigma_d(V(G')).$$

Since $G'_{\chi'}$ is σ_d-sparse this implies $d|V(G')| - \sigma_d(V(G')) \leq |E(G')| \leq S(G'_{\chi'}, \sigma_d|_{V(G')})$. If $|V(G')| \geq d+1$ then $S(G'_{\chi'}, \sigma_d|_{V(G')}) \leq d|V'| - \sigma_d(V(G'))$, by Lemma 5.3.1.4 and

Lemma 5.3.1.5, and so $G'_{\chi'}$ is σ_d-tight. If $|V(G')| \leq d$ then, as G' is a $(d,0)$-reduction of G, $|V(G')| = d$ and so $|V| = d$. As G_χ is σ_d-sparse it follows that $\binom{d+1}{2} \geq S(G_\chi, \sigma_d) = d|V| - \sigma_d(V) = d(d+1) - \sigma_d(V)$ and hence $\sigma_d(V) \geq \binom{d+1}{2}$. Therefore $\sigma_d = \binom{d+1}{2}$, $G \cong K_{d+1}$, $G' \cong K_d$, and by Lemma 5.3.1.7 we have $\sigma_d(V(G')) = \binom{d+1}{2}$. Consequently, $|E(G')| = d|V(G')| - \sigma_d(V(G')) = d^2 - \binom{d+1}{2} = \binom{d}{2} = S(G'_{\chi'}, \sigma_d|_{V(G')})$ and so $G'_{\chi'}$ is σ_d-tight.

On the other hand, suppose that $G'_{\chi'}$ is σ_d-tight. Then $|E(G')| = S(G'_{\chi'}, \sigma_d|_{V(G')}) = d|V(G')| - \sigma_d(V')$ and Lemma 5.3.2.1 implies that G_χ is σ_d-sparse. As $|E(G')| = d|V(G')| - \sigma_d(V(G'))$, Lemma 5.3.1.7 implies that

$$|E| = |E(G')| + d = d|V| - \sigma_d(V(G')) \leq d|V| - \sigma_d(V).$$

If G_χ is σ_d-tight then $|E| = d|V| - \sigma_d(V)$, so $\sigma_d(V) = \sigma_d(V')$. Alternatively, suppose that $\sigma_d(V) = \sigma_d(V')$. As G is a $(d,0)$-extension of G' we have that $|V| \geq d+1$, so Lemma 5.3.1.4 and Lemma 5.3.1.5 together imply that $S(G_\chi, \sigma_d) \leq d|V| - \sigma_d(V)$. So as G_χ is σ_d-sparse we have that

$$d|V| - \sigma_d(V) = |E| \leq S(G_\chi, \sigma_d) \leq d|V| - \sigma_d(V).$$

Hence $|E| = S(G_\chi, \sigma_d) = d|V| - \sigma_d(V)$ and G_χ is σ_d-tight. □

Lemma 5.3.2.3. *Take $d \in \{1,2\}$ and let $G_\chi = (V,E)_\chi$ be a labelled graph. Suppose $G'_{\chi'}$ is a $(d,1)$-VL-reduction of G_χ, with $v \in V \setminus V(G')$ and $e \in E(G') \setminus E$. If G_χ is σ_d-sparse then $G'_{\chi'}$ is σ_d-sparse if and only if for all $U \subseteq V(G')$ such that the endpoints of e are in U, $i_G(U) \leq S(G_\chi[U], \sigma_d|_U) - 1$. If $G'_{\chi'}$ is σ_d-sparse then G_χ is σ_d-sparse.*

Proof. Let $e = xy$. Firstly, suppose that G_χ is σ_d-sparse. If $G'_{\chi'}$ is σ_d-sparse then take $U \subseteq V(G')$ such $\{x,y\} \subseteq U$ and note that

$$S(G_\chi[U], \sigma_d|_U) - 1 = S(G'_{\chi'}[U], \sigma_d|_U) - 1 \geq i_{G'}(U) - 1 = i_G(U).$$

As U was chosen arbitrarily it follows that for all $U \subseteq V(G')$ such that $\{x,y\} \subseteq U$,

$i_G(U) \leq S(G_\chi[U], \sigma_d|_U) - 1$. On the other hand suppose that for all $U \subseteq V(G)$ such that $\{x, y\} \subseteq U$, $i_G(U) \leq S(G_\chi[U], \sigma_d|_U) - 1$. Take a labelled subgraph, H_ψ, of $G'_{\chi'}$. If $e \notin E(H)$ then H_ψ is a labelled subgraph of G_χ and so $|E(H)| \leq S(H_\psi, \sigma_d|_{V(H)})$. If $e \in E(H)$ then $\{x, y\} \subseteq V(H)$ and so $i_{G'}(V(H)) = i_G(V(H))+1 \leq S(G_\chi[V(H)], \sigma_d|_{V(H)}) = S(G'_{\chi'}[V(H)], \sigma_d|_{V(H)})$. As H_ψ was chosen arbitrarily, $G'_{\chi'}$ is σ_d-sparse.

Now suppose that $G'_{\chi'}$ is σ_d-sparse and let H_ψ be a labelled subgraph of G_χ. If $|V(H)| \leq d+1$ then $|E(H)| \leq \binom{|V(H)|}{2}) = S(H_\psi, \sigma_d|_{V(H)})$. If $|V(H)| \geq d+2$ then $|V(H) \setminus \{v\}| \geq d+1$. Let $V(H) \setminus \{v\} = W$. Lemma 5.3.1.4 and Lemma 5.3.1.5 together imply that

$$i_{G'}(W) \leq \min\left\{d|W| - \sigma_d(W), (d+1)|W| - \binom{d+2}{2}\right\}.$$

So it follows from Lemma 5.3.1.7 that

$$\begin{aligned}
|E(H)| &\leq i_G(W) + (|N_G(v) \cap V(H)|)(|V(H) \cap \{v\}|) \\
&= (i_{G'}(W) - |E(G'[W]) \cap \{e\}|) + (|N_G(v) \cap V(H)|)(|V(H) \cap \{v\}|) \\
&\leq i_{G'}(W) + d|V(H) \cap \{v\}| \\
&\leq \min\left\{d|W| - \sigma_d(W), (d+1)|W| - \binom{d+2}{2}\right\} + d|V(H) \cap \{v\}| \\
&= \min\left\{d|V(H)| - \sigma_d(W), d|V(H)| + |V(H) \setminus \{v\}| - \binom{d+2}{2}\right\} \\
&\leq \min\left\{d|V(H)| - \sigma_d(V(H)), (d+1)|V(H)| - \binom{d+2}{2}\right\} \\
&= S(H_\psi, \sigma_d|_{V(H)}).
\end{aligned}$$

As we chose an arbitrary labelled subgraph of G_χ, for every labelled subgraph, H_ψ, of G_χ we have $|E(H)| \leq S(H_\psi, \sigma_d|_{V(H)})$ and therefore G_χ is σ_d-sparse. □

Lemma 5.3.2.4. *Take $d \in \{1, 2\}$ and let $G_\chi = (V, E)_\chi$ be a labelled graph. Suppose $G'_{\chi'}$ is a $(d, 1)$-VL-reduction of G_χ, with $v \in V \setminus V(G')$ and $e \in E(G') \setminus E$. If G_χ is σ_d-tight then $G'_{\chi'}$ is σ_d-tight if and only if $G'_{\chi'}$ is σ_d-sparse. If $G'_{\chi'}$ is σ_d-tight then G_χ is σ_d-tight if and only if $\sigma_d(V) = \sigma_d(V(G'))$.*

Proof. Let $e = xy$. Firstly, suppose that G_χ is σ_d-tight. If $G'_{\chi'}$ is σ_d-tight then $|E| = S(G_\chi, \sigma_d) = d|V| - \sigma_d(V)$. As G' is a $(d,1)$-reduction of G, $|V(G')| \geq d+1$ and so Lemma 5.3.1.4 and Lemma 5.3.1.5 together imply that $S(G'_{\chi'}, \sigma_d|_{V(G')}) \leq d|V'| - \sigma_d(V(G'))$. So, Lemma 5.3.1.7 implies that

$$|E(G')| = |E|+1-(d+1) = d|V(G')|-\sigma_d(V) \geq d|V'|-\sigma_d(V(G')) \geq S(G'_{\chi'}, \sigma_d|_{V(G')}).$$

Therefore, $G'_{\chi'}$ is σ_d-tight if and only if $G'_{\chi'}$ is σ_d-sparse.

On the other hand, suppose that $G'_{\chi'}$ is σ_d-tight. Then $|E(G')| = S(G'_{\chi'}, \sigma_d|_{V(G')}) = d|V(G')| - \sigma_d(V(G'))$. Lemma 5.3.2.3 implies that G_χ is σ_d-sparse. As G is a $(d,1)$-extension of G', $|V| \geq d+2$ and so $S(G_\chi, \sigma_d) \leq d|V| - \sigma_d(V(G))$. So, as $|E(G')| = d|V(G')| - \sigma_d(V(G'))$,

$$d|V| - \sigma_d(V) \geq S(G_\chi, \sigma_d) \geq |E| = (|E(G')| - 1) + (d+1) = d|V| - \sigma_d(V(G')).$$

Therefore, G_χ is σ_d-tight if and only if $d|V| - \sigma_d(V) = S(G_\chi, \sigma_d) = |E|$ if and only if $d|V| - \sigma_d(V) = d|V| - \sigma_d(V(G'))$ if and only if $\sigma_d(V) = \sigma_d(V(G'))$. □

5.4 σ_1-Tight Vertex-Labelled Graphs

In this short section we turn our focus to the σ_1-count function. Our aim is to provide a method of constructing all σ_1-tight labelled graphs. Firstly we introduce some additional notation. Recall that if $G_1 \ldots, G_k$ are graphs then $\bigcup_{i=1}^k G_i = (\bigcup_{i=1}^k V(G_i), \bigcup_{i=1}^k E(G_i))$. If $V(G_i) \cap V(G_j) = \emptyset$ for all $1 \leq i < j \leq k$, then $\oplus_{i=1}^k G_i := \bigcup_{i=1}^k G_i$.

Lemma 5.4.0.1. *Let $G_\chi = (V,E)_\chi$ be a labelled graph. Suppose that G_χ is σ_1-tight. If G is 2-regular then there exists a $(1,1)$-VL-reduction of G_χ that is σ_1-tight if and only if there does not exist $m \in \mathbb{N}^+$ such that $G \cong \oplus_{i=1}^m K_3$.*

Proof. Suppose there exists a $(1,1)$-VL-reduction of G_χ that is σ_1-tight, say at v adding e. Then $e \notin E$, so $G[N_G[v]] \ncong K_3$ and hence there does not exist $m \in \mathbb{N}^+$ such that $G \cong \oplus_{i=1}^m K_3$.

Alternatively, suppose that there does not exist $m \in \mathbb{N}^+$ such that $G \cong \oplus_{i=1}^m K_3$. Then we may consider some component, H, of G such that $H \ncong K_3$. As G is 2-regular and H is a component of G, H is 2-regular. As H is 2-regular and $H \ncong K_3$, there exists $n \in \mathbb{N}$ such that $n \geq 4$ and $H \cong C_n$. As G_χ is σ_1-sparse Lemma 5.3.1.7 implies $\sigma_1(V(H)) = 0 = \sigma_1(V)$. As $|V(H)| \geq 4$ there exists $v \in V(H)$ such that $\sigma_1(V(H)) = \sigma_1(V(H) \setminus \{v\})$.

Let $G'_{\chi'}$ denote the $(1,1)$-VL-reduction of G at v, let $N_G(v) = \{x, y\}$, and take $U \subseteq (V \setminus \{v\})$ such that $\{x, y\} \subseteq U$. If $|U| = 2$ then $i_G(U) = 0 = \binom{|U|}{2} - 1 = S(G_\chi[U], \sigma_1|_U) - 1$. If $|U| \geq 3$ then Lemma 5.3.1.6 implies that for all $U \subseteq W \subseteq V$, $S(G_\chi[W], \sigma_1|_U) = |W| - \sigma_1(W)$. Hence,

$$i_G(U) = i_G(U \cup \{v\}) - 2 \leq S(G_\chi[U \cup \{v\}], \sigma_1|_{U \cup \{v\}}) - 2 = |U| - (\sigma_1(U \cup \{v\}) + 1).$$

If $\sigma_1(U \cup \{v\}) = \sigma_1(U)$ then $i_G(U) \leq |U| - (\sigma_1(U) + 1) = S(G_\chi[U], \sigma_1|_U) - 1$. If $\sigma_1(U \cup \{v\}) \neq \sigma_1(U)$ then Lemma 5.3.1.7 implies $\sigma_1(U \cup \{v\}) = 0$ and $\sigma_1(U) = 1$. As $\sigma_1(V(H) \setminus \{v\}) = 0$, it follows that $G[U]$ is not connected and so $i_G(U) \leq |U| - 2 = S(G_\chi[U], \sigma_1|_U) - 1$. As U was chosen arbitrarily, for all $U \subseteq (V \setminus \{v\})$ such that $\{x, y\} \subseteq U$ we have $i_G(U) \leq S(G_\chi[U], \sigma_1|_U) - 1$. Consequently, Lemma 5.3.2.4 implies that $G'_{\chi'}$ is σ_1-tight. □

Lemma 5.4.0.2. *Let $G_\chi = (V, E)_\chi$ be a labelled graph. Suppose that G_χ is σ_1-tight. There exists a $(1,0)$-VL-reduction or a $(1,1)$-VL-reduction of G_χ, that is σ_1-tight, if and only if $|V| \geq 2$ and there does not exist $m \in \mathbb{N}^+$ such that $G \cong \oplus_{i=1}^m K_3$.*

Proof. Suppose there exists a $(1,0)$-VL-reduction, or a $(1,1)$-VL-reduction, of G_χ that is σ_1-tight. This implies that $\Delta(G) \geq 1$, so $|V| \geq 2$. As G_χ is σ_2-tight, either $\delta(G) = 1$ or G is 2-regular. If $\delta(G) = 1$ then there does not exist $m \in \mathbb{N}^+$ such that $G \cong \oplus_{i=1}^m K_3$. If G is 2-regular then there does not exist a $(1,0)$-VL-reduction of G_χ. Hence there exists a $(1,1)$-VL-reduction of G_χ that is σ_1-tight and so Lemma 5.4.0.1 implies there does not exist $m \in \mathbb{N}^+$ such that $G \cong \oplus_{i=1}^m K_3$.

On the other hand suppose that $|V| \geq 2$ and there does not exist $m \in \mathbb{N}^+$ such that $G \cong \oplus_{i=1}^{m} K_3$. If $\delta(G) = 2$ then, as G_χ is σ_1-sparse, Lemma 5.3.1.8 implies that $\sigma_1(V) = 0$. As G_χ is σ_1-tight, $|E| = S(G_\chi, \sigma_1) = |V|$. Consequently, Theorem 1.1.1.6 implies that G is 2-regular and so Lemma 5.4.0.1 implies there exists a $(1,1)$-VL-reduction of G_χ that is σ_1-tight. If $\delta(G) \neq 2$ then, as $|V| \geq 2$, Lemma 5.3.1.8 implies that $\delta(G) = 1$. Then there exists a $(1,0)$-VL-reduction of G_χ and Lemma 5.3.2.2 implies this is σ_1-tight. □

Proposition 5.4.0.3. *Let $G_\chi = (V,E)_\chi$ be a labelled graph. The following are equivalent:*

(i) G_χ is σ_1-tight; and

(ii) there exists $t \in \mathbb{N}^+$ and a sequence $a_1, \dots, a_t$, with $a_1 \cong H_\psi$, where $H \cong K_1$ or there exists $m \in \mathbb{N}^+$ such that $H \cong \oplus_{i=1}^{m} K_3$, $\sigma_1(V(a_1)) = \max\{0, 2 - |V|\}$, $a_t = G_\chi$, such that, for all $2 \leq j \leq t$, a_j is a $(1,0)$-VL-extension or a $(1,1)$-VL-extension of a_{j-1} and $\sigma_1(V(a_j)) = \sigma_1(V(a_{j-1}))$.

Proof. Suppose (i) holds. We proceed by induction on $|V|$, and note that clearly if $G \cong K_1$ then there exists a sequence of the form claimed. By Lemma 5.3.2.2, for all $s \in \mathbb{N}^+$ there exists a σ_1-tight graph with s vertices. Take $n \in \mathbb{N}^+ \setminus \{1\}$ and suppose that (ii) holds for all σ_1-tight graphs with at most n vertices. Now suppose that $|V| = n+1$. If there exists $m \in \mathbb{N}^+$ such that $G \cong \oplus_{i=1}^{m} K_3$ then there exists a sequence of the form claimed. If no such m exists then Lemma 5.4.0.2 implies there exists a $(1,0)$-VL-reduction or $(1,1)$-VL-reduction of G_χ that is σ_1-tight. Let this labelled graph be $G'_{\chi'}$. As $|V(G')| \leq n$, it follows from our induction hypothesis that there exists $t \in \mathbb{N}^+$ and a sequence $a_1, \dots, a_t$ with $a_1 \cong H_\psi$, where $H \cong K_1$ or there exists $m \in \mathbb{N}^+$ such that $H \cong \oplus_{i=1}^{m} K_3$, $\sigma_1(V(a_1)) = \max\{0, 2 - |V|\}$, $a_t = G'_{\chi'}$, such that, for all $2 \leq j \leq t$, a_j is a $(1,0)$-VL-extension or a $(1,1)$-VL-extension of a_{j-1} and $\sigma_1(V(a_j)) = \sigma_1(V(a_{j-1}))$. As H_ψ is σ_1-tight, repeated applications of Lemma 5.3.2.2 and Lemma 5.3.2.4 imply that $G'_{\chi'}$ is σ_1-tight. As G_χ is σ_1-tight, one more application of Lemma 5.3.2.2 or Lemma 5.3.2.4 implies that $\sigma_1(V(G)) = \sigma_1(V(G'))$. Therefore, $a_1, \dots, a_{t-1}, G_\chi$ is a sequence of the form claimed. On the other hand, if (ii) holds then as $\sigma_1(V(a_1)) = \max\{0, 2 - |V|\}$

we note that H_ψ is σ_1-tight. Hence repeated applications of Lemma 5.3.2.2 or Lemma 5.3.2.4 imply that G_χ is σ_1-tight. □

5.5 σ_2-Tight Vertex-Labelled Graphs

The focus of the remainder of this chapter will be σ_2-tight labelled graphs, although we continue to state and prove results for $d \in \{1,2\}$ in places. In Section 2.3 we introduced the notion of a vertex set being critical, and we make use of a similar idea in the context of labelled graphs. Before getting to that we briefly consider, for $d \in \{1,2\}$, the relationship between σ_d-sparsity of a labelled graph G_χ and related sparsity condition on G. For $d \in \{1,2\}$ and $0 \le k \le \binom{d+1}{2}$ we say that G is **(d,k)-sparse** if $i_G(U) \le d|U| - k$ for all $U \subseteq V(G)$ such that $|U| \ge d+1$. We say that G is **(d,k)-tight** if G is (d,k)-sparse and $|E(G)| = d|V(G)| - k$.

5.5.1 σ_d-Critical Sets

Lemma 5.5.1.1. *Take $d \in \{1,2\}$ and let $G_\chi = (V,E)_\chi$ be a labelled graph. If G_χ is σ_d-sparse then G is $(d,\sigma_d(V))$-sparse. Moreover, if G_χ is σ_d-tight then G is $(d,\sigma_d(V))$-tight.*

Proof. Suppose that G_χ is σ_d-sparse. Then Lemma 5.3.1.4 and Lemma 5.3.1.5 together imply that for all $U \subseteq V$ such that $|U| \ge d+1$, $i_G(U) \le S(G_\chi[U], \sigma_d|_U) \le d|U| - \sigma_d(U)$. Consequently, Lemma 5.3.1.7 implies that for all $U \subseteq V$ such that $|U| \ge d+1$, $i_G(U) \le d|U| - \sigma_d(V)$. Therefore G is $(d,\sigma_d(V))$-sparse. If G_χ is also σ_d-tight then it follows that $|E| = S(G_\chi, \sigma_d) = d|V| - \sigma_d(V)$ and hence G_χ is $(d,\sigma_d(V))$-tight. □

Our next result shows that, unsurprisingly, if we specify that the labelling of the vertex set of a graph $G = (V,E)$ is 'trivial'[2], then the converse of the previous result holds. However, if we allow the vertices to be labelled in a non-trivial way then the converse is no longer true. An example of this can be seen in Figure 5.2.

[2]That is, $\mathcal{V}(G_\chi) = \{V\}$. If one considers the labels to be colours then this is equivalent to the resulting labelled graph being monochrome.

Proposition 5.5.1.2. *Take $d \in \{1,2\}$ and let $G_\chi = (V,E)_\chi$ be a labelled graph. If $\sigma_d(V) = \binom{d+1}{2}$ then the following are equivalent:*

(i) G_χ is σ_d-tight;

(ii) G is $(d, \binom{d+1}{2})$-tight; and

(iii) there exists $t \in \mathbb{N}^+$ and a sequence $a_1, \dots, a_t$, with $a_1 \cong H_\psi$ where $H \cong K_d$, and $a_t = G_\chi$, such that for all $2 \le j \le t$, a_j is a $(d,0)$-VL-extension or a $(d,1)$-VL-extension of a_{j-1} and $\sigma_d(a_j) = \sigma_d(a_{j-1}) = \binom{d+1}{2}$.

Proof. If (i) holds then Lemma 5.5.1.1 implies (ii) holds. Suppose that (ii) holds. Then by Theorem 1.4.3.6 and Theorem 1.4.3.7 we have that there exists $t \in \mathbb{N}^+$ and a sequence $a'_1, \dots, a'_t$, with $a'_1 \cong K_d$, and $a'_t = G$, such that for all $2 \le j \le t$, a'_j is a $(d,0)$-extension or a $(d,1)$-extension of a_{j-1} and a_j is $(d, \binom{d+1}{2})$-tight. For all $1 \le j \le t$, let $a_j = (a'_j, \chi'_j)$, where $\chi'_j = \chi|_{V(a'_j)}$. Then for all $2 \le j \le t$, a_j is a $(d,0)$-VL-extension or a $(d,1)$-VL-extension of a_{j-1}. As $\sigma_2(V) = \binom{d+1}{2}$, Lemma 5.3.1.7 implies $\sigma_2(V(a_j)) = \binom{d+1}{2}$ for all $1 \le j \le t$. Therefore (iii) holds. Finally, if (iii) holds then as $\sigma_2(V(H)) = \binom{d+1}{2}$ we have that H_ψ is σ_d-tight. Repeated applications of Lemma 5.3.2.2 or Lemma 5.3.2.4 imply that G_χ is σ_d-tight, so (i) holds. □

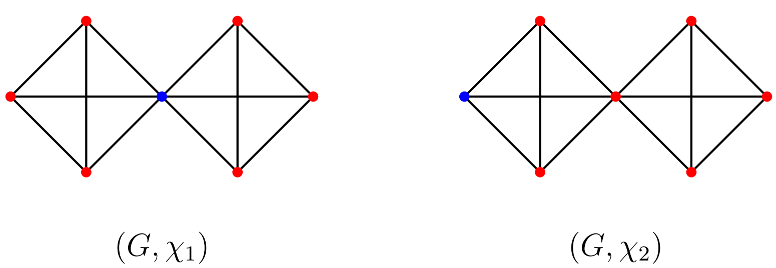


Figure 5.2: Two labelled graphs with the same underlying graph, G. (G, χ_1) is σ_2-tight while (G, χ_2) is not.

In the proof of Proposition 5.5.1.2 we made use of pre-existing methods for constructing $(1,1)$-tight or $(2,3)$-tight graphs. Analogous methods exist for $(2,2)$-tight graphs ([32, Theorem 2.13], [31, Theorem 1.5]), $(2,1)$-tight graphs [31, Theorem 1.4], and $(2,0)$-tight graphs such that no subgraph is isomorphic to K_5 [19, Theorem 1.4.3].

Given that there are pre-existing methods of constructing these different families of graphs, it is reasonable to suppose that we may provide a similar method for σ_2-tight labelled graphs. There are two main reasons for not doing this. Firstly, the translation to the labelled setting creates a lot of additional work. In particular, we are not dealing with a single $(2, k)$-tightness condition but a hierarchy of such conditions. Secondly, our primary interest is in those graphs relevant to the rigidity of frameworks realised on spheres and in this context additional constraints arise which make the resulting subfamily of σ_2-tight labelled graphs more complicated to work with.

Definition 5.5.1.3. Let G_χ be a labelled graph. A set X is **$\boldsymbol{\sigma_d}$-critical** in G_χ if $\emptyset \neq X \subseteq V(G)$ and $G_\chi[X]$ is σ_d-tight.

With this concept in hand we are able to reconsider how, given a σ_2-sparse labelled graph G_χ, the existence of a $(2,1)$-VL-reduction of G_χ that is σ_2-sparse may be prevented. Before doing so we conclude this section with some basic, but useful, results about critical sets. We are particularly interested in σ_2-critical sets.

Lemma 5.5.1.4. *Take $d \in \{1, 2\}$ and let $G_\chi = (V, E)_\chi$ be a labelled graph and suppose there exists $v \in V$ such that $d_G(v) \geq d$. If G_χ is σ_d-sparse then for all $U \subseteq V \setminus \{v\}$ such that U is σ_d-critical in G_χ and $|U \cap N_G(v)| \geq d$, $\sigma_d(U \cup \{v\}) \leq \sigma_d(U) - (|U \cap N_G(v)| - d)$.*

Proof. Suppose such a U exists, then

$$i_G(U \cup \{v\}) = i_G(U) + |N_G(v) \cap U| = (d|U \cup \{v\}| - \sigma_d(U)) + (|U \cap N_G(v)| - d).$$

As $|U \cap N_G(v)| \geq d$ and $U \subseteq V \setminus \{v\}$, $|U \cup \{v\}| \geq d + 1$. As G_χ is σ_2-sparse, it follows from Lemma 5.3.1.4 and Lemma 5.3.1.5 that

$$i_G(U \cup \{v\}) \leq S(G[U \cup \{v\}], \sigma_d|_{U \cup \{v\}}) \leq d|U \cup \{v\}| - \sigma_d(U \cup \{v\}).$$

Therefore, $\sigma_d(U \cup \{v\}) \leq \sigma_d(U) - (|U \cap N_G(v)| - d)$. □

Lemma 5.5.1.5. *Let $G_\chi = (V, E)_\chi$ be a labelled graph. Suppose that X, Y, and Z are σ_2-critical in G_χ, then $i_G(X \cup Y) + i_G(X \cap Y) = (2|X \cup Y| - \min\{\sigma_2(X), \sigma_2(Y)\}) + (2|X \cap$*

$Y| - \max\{\sigma_2(X), \sigma_2(Y)\}) + d_G(X, Y)$. *Moreover, if* $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$ *then* $i_G(X \cup Y \cup Z) = 2|X \cup Y \cup Z| + 6 + d_G(X, Y) + d_G(X \cup Y, Z) - (2|X \cap Y \cap Z| + \sigma_2(X) + \sigma_2(Y) + \sigma_2(Z) + d_G(X \cap Z, Y \cap Z))$.

Proof. As X, Y, and Z are σ_2-critical in G_χ, $i_G(X) = 2|X| - \sigma_2(X)$, $i_G(Y) = 2|Y| - \sigma_2(Y)$, and $i_G(Z) = 2|Z| - \sigma_2(Z)$. Lemma 1.1.1.4 implies

$$\begin{aligned} i_G(X \cup Y) + i_G(X \cap Y) &= (2|X| - \sigma_2(X)) + (2|Y| - \sigma_2(Y)) + d_G(X, Y) \\ &= (2|X \cup Y| - \min\{\sigma_2(X), \sigma_2(Y)\}) \\ &\quad + (2|X \cap Y| - \max\{\sigma_2(X), \sigma_2(Y)\}) + d_G(X, Y). \end{aligned}$$

Moreover, if $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$ then $i_G(X \cup Y) = (2|X \cup Y| + 2) - (\sigma_2(X) + \sigma_2(Y)) + d_G(X, Y)$ and

$$\begin{aligned} |X \cup Y \cup Z| &= |X \cup Y| + |Z| - |(X \cup Y) \cap Z| \\ &= |X \cup Y| + |Z| - |(X \cap Z) \cup (Y \cap Z)| \\ &= |X \cup Y| + |Z| - (|X \cap Z| + |Y \cap Z| - |X \cap Y \cap Z|) \\ &= |X \cup Y| + |Z| + |X \cap Y \cap Z| - 2. \end{aligned}$$

So,

$$\begin{aligned} i_G(X \cup Y \cup Z) &= i_G(X \cup Y) + i_G(Z) + d_G(X \cup Y, Z) - i_G((X \cup Y) \cap Z) \\ &= (2|X \cup Y| + 2) - (\sigma_2(X) + \sigma_2(Y)) + d_G(X, Y) + 2|Z| - \sigma_2(Z) \\ &\quad + d_G(X \cup Y, Z) - i_G((X \cap Z) \cup (Y \cap Z)) \\ &= 2(|X \cup Y \cup Z| + 3 - |X \cap Y \cap Z|) - (\sigma_2(X) + \sigma_2(Y) + \sigma_2(Z)) \\ &\quad + d_G(X, Y) + d_G(X \cup Y, Z) - d_G(X \cap Z, Y \cap Z). \end{aligned}$$

□

Lemma 5.5.1.6. *Let* $G_\chi = (V, E)_\chi$ *be a labelled graph. Suppose that* X *and* Y *are* σ_2*-critical in* G_χ. *If* $\sigma_2(V) \geq 1$, G_χ *is* σ_2*-sparse, and* $|X \cap Y| \geq 2$ *then* $d_G(X, Y) = 0$,

$X \cup Y$ is σ_2-critical in G_χ, and we have the following trichotomy. Either

(i) $\sigma_2(X \cup Y) = 1$, $\sigma_2(X) = 2 = \sigma_2(Y)$, $\sigma_2(X \cap Y) = 3$, and $X \cap Y$ is σ_2-critical in G_χ; or

(ii) $\sigma_2(X \cup Y) = 1$, $\sigma_2(X) = 2 = \sigma_2(Y) = \sigma_2(X \cap Y)$, and $G[X \cap Y] \cong K_2$; or

(iii) $\sigma_2(X \cup Y) = \min\{\sigma_2(X), \sigma_2(Y)\}$, $\sigma_2(X \cap Y) = \max\{\sigma_2(X), \sigma_2(Y)\}$, and $X \cap Y$ is σ_2-critical in G_χ.

Proof. Note that for all $U \subseteq V$, if $|U| \geq 2$ then $S(G_\chi[U], \sigma_2|_U) = 1 \leq 2|U| - \sigma_2(U)$. As G_χ is σ_2-sparse, $i_G(X \cup Y) \leq S(G_\chi[X \cup Y], \sigma_2|_{X \cup Y}) \leq 2|X \cup Y| - \sigma_2(X \cup Y)$ and $i_G(X \cap Y) \leq S(G_\chi[X \cap Y], \sigma_2|_{X \cap Y}) \leq 2|X \cap Y| - \sigma_2(X \cap Y)$. So Lemma 5.5.1.5 implies

$$\sigma_2(X) + \sigma_2(Y) \geq \sigma_2(X \cup Y) + \sigma_2(X \cap Y) + d_G(X, Y). \tag{5.1}$$

We proceed by considering the difference between $\min\{\sigma_2(X), \sigma_2(Y)\}$ and $\sigma_2(X \cup Y)$.

Let $T = \min\{\sigma_2(X), \sigma_2(Y)\} - \sigma_2(X \cup Y)$ and observe that, as $\sigma_2(V) \geq 1$, Lemma 5.3.1.7 implies $T \in \{0, 1, 2\}$. If $T = 2$ then $\sigma_2(X \cup Y) = 1$ and $\sigma_2(X) = 3 = \sigma_2(Y)$. As $\sigma_2(X) = 3 = \sigma_2(Y)$, it follows that there exist $W_1, W_2 \in \mathcal{V}(G_\chi)$ such that $X \subseteq W_1$ and $Y \subseteq W_2$. As $X \cap Y \neq \emptyset$, $W_1 = W_2$. However this implies that $\sigma_2(X \cup Y) = 3$, a contradiction. Therefore $T \in \{0, 1\}$.

Suppose next that $T = 1$. If $\sigma_2(X \cup Y) = 2$ then $\sigma_2(X) = 3 = \sigma_2(Y)$ and, similarly to the case $T = 2$, it follows that $\sigma_2(X \cup Y) = 3$, a contradiction. So $\sigma_2(X \cup Y) = 1$, $\min\{\sigma_2(X), \sigma_2(Y)\} = 2$, and $\max\{\sigma_2(X), \sigma_2(Y)\} \in \{2, 3\}$. If $\max\{\sigma_2(X), \sigma_2(Y)\} = 3$ then we may suppose without loss of generality that $\sigma_2(X) = 2$ and $\sigma_2(Y) = 3$. As $X \cap Y \neq \emptyset$, Lemma 5.3.1.7 implies $\sigma_2(X \cap Y) = 3$. Consequently there exists $W \subseteq \mathcal{V}(G_\chi)$ such that $X \cap Y \subseteq Y \subseteq W$. As $|X \cap Y| \geq 2$ it follows that $|\{x \in X \setminus W\}| = 1 = |\{x \in (X \cup Y) \setminus W\}|$ and so $\sigma_2(X \cup Y) = 2$, a contradiction. Hence $\sigma_2(X) = 2 = \sigma_2(Y)$. As X and Y are both σ_2-critical in G_χ, $|X|, |Y| \geq 4$ and so there exist $W_1, W_2 \in \mathcal{V}(G_\chi)$ such that $X \cup Y \subseteq W_1 \cup W_2$, $|X \cap W_1| \geq 3$, and $|X \cap W_2| = 1$. Now, either $|Y \cap W_1| \geq 3$ and $|Y \cap W_2| = 1$, or $|Y \cap W_1| = 1$ and $|Y \cap W_2| \geq 3$.

If $|Y \cap W_1| \geq 3$ and $|Y \cap W_2| = 1$ then $|(X \cup Y) \cap W_2| \in \{1, 2\}$. So, as $\sigma_2(X \cup Y) = 1$, $|(X \cup Y) \cap W_2| = 2$ and $X \cap Y \subseteq W_1$. So $\sigma_2(X \cap Y) = 3$ and 5.1 implies $d_G(X, Y) = 0$. Then it follows from Lemma 5.5.1.5 that

$$(2|X \cup Y| - 1) + (2|X \cap Y| - 3) \geq i_G(X \cup Y) + i_G(X \cap Y) = (2|X \cup Y| - 2) + (2|X \cap Y| - 2).$$

Therefore $i_G(X \cup Y) = 2|X \cup Y| - 1$ and $i_G(X \cap Y) = 2|X \cap Y| - 3$, so $X \cup Y$ and $X \cap Y$ are σ_2-critical in G_χ.

On the other hand, if $|Y \cap W_1| = 1$ and $|Y \cap W_2| \geq 3$ then, as $|X \cap Y| \geq 2$, $X \cap Y = (X \cap W_2) \cup (Y \cap W_1)$. So $|X \cap Y| = 2$, $\sigma_2(X \cap Y) = 2$, and $i_G(X \cap Y) \leq 2|X \cap Y| - 3$. It follows from Lemma 5.5.1.5 that

$$(2|X \cup Y| - 1) + (2|X \cap Y| - 3) \geq i_G(X \cup Y) + i_G(X \cap Y) = (2|X \cup Y| - 2) + (2|X \cap Y| - 2).$$

Therefore $i_G(X \cup Y) = 2|X \cup Y| - 1$ and $i_G(X \cap Y) = 2|X \cap Y| - 3$, so $X \cup Y$ is σ_2-critical in G_χ and $G[X \cap Y] \cong K_2$.

Finally suppose that $T = 0$, so $\sigma_2(X \cup Y) = \min\{\sigma_2(X), \sigma_2(Y)\}$. Then Lemma 5.3.1.7 and 5.1 together imply that $d_G(X, Y) = 0$ and $\sigma_2(X \cap Y) = \max\{\sigma_2(X), \sigma_2(Y)\}$. Consequently, Lemma 5.5.1.5 gives us that $X \cup Y$ and $X \cap Y$ are σ_2-critical in G_χ. $\square$

Lemma 5.5.1.7. *Let $G_\chi = (V, E)_\chi$ be a labelled graph. Suppose that X and Y are σ_2-critical in G_χ. If $\sigma_2(V) \geq 1$, G_χ is σ_2-sparse, $|X \cap Y| = 1$, and $d_G(X, Y) \geq 1$ then one of the following holds:*

(i) $\sigma_2(X \cup Y) = 1$, $\min\{\sigma_2(X), \sigma_2(Y)\} = 2$, $\max\{\sigma_2(X), \sigma_2(Y)\} = 3$, $\sigma_2(X \setminus Y) = 3 = \sigma_2(Y \setminus X)$, $d_G(X, Y) = 1$ *and $X \cup Y$ is not σ_2-critical in G_χ; or*

(ii) $\sigma_2(X \cup Y) = 1$, $\min\{\sigma_2(X), \sigma_2(Y)\} = 2$, $\max\{\sigma_2(X), \sigma_2(Y)\} = 3$, $\sigma_2(X \setminus Y) = 3 = \sigma_2(Y \setminus X)$, $d_G(X, Y) = 2$ *and $X \cup Y$ is σ_2-critical in G_χ; or*

(iii) $\sigma_2(X \cup Y) = 1$, $\sigma_2(X) = 2 = \sigma_2(Y)$, $d_G(X, Y) = 1$, $X \cup Y$ *is σ_2-critical in G_χ,* $\min\{\sigma_2(X \setminus Y), \sigma_2(Y \setminus X)\} = 2$, *and* $\max\{\sigma_2(X \setminus Y), \sigma_2(Y \setminus X)\} = 3$*; or*

(iv) $\sigma_2(X \cup Y) = 1$, $\sigma_2(X) = 2 = \sigma_2(Y)$, $d_G(X,Y) = 1$, $X \cup Y$ *is* σ_2*-critical in* G_χ, *and* $\sigma_2(X \setminus Y) = 2 = \sigma_2(Y \setminus X)$*; or*

(v) $\sigma_2(X \cup Y) = \min\{\sigma_2(X), \sigma_2(Y)\}$, $d_G(X,Y) = 1$, $X \cup Y$ *is* σ_2*-critical in* G_χ*, and* $\max\{\sigma_2(X), \sigma_2(Y)\} = 3$.

Proof. As $|X \cap Y| = 1$, $i_G(X \cap Y) = 0 = 2|X \cap Y| - 2$. As G_χ is σ_2-sparse, $i_G(X \cup Y) \leq S(G_\chi[X \cup Y], \sigma_2|_{X \cup Y}) \leq 2|X \cup Y| - \sigma_2(X \cup Y)$. So Lemma 5.5.1.5 implies

$$\sigma_2(X) + \sigma_2(Y) \geq \sigma_2(X \cup Y) + 2 + d_G(X,Y) \geq \sigma_2(X \cup Y) + 3. \tag{5.2}$$

Let $T = \min\{\sigma_2(X), \sigma_2(Y)\} - \sigma_2(X \cup Y)$ and observe that, as $\sigma_2(V) \geq 1$, Lemma 5.3.1.7 implies $T \in \{0, 1, 2\}$. If $T = 2$ then $\sigma_2(X \cup Y) = 1$ and $\sigma_2(X) = 3 = \sigma_2(Y)$. As $\sigma_2(X) = 3 = \sigma_2(Y)$, it follows that there exist $W_1, W_2 \in \mathcal{V}(G_\chi)$ such that $X \subseteq W_1$ and $Y \subseteq W_2$. As $X \cap Y \neq \emptyset$, $W_1 = W_2$. However this implies that $\sigma_2(X \cup Y) = 3$, a contradiction. Therefore $T \in \{0, 1\}$.

Suppose next that $T = 1$. If $\sigma_2(X \cup Y) = 2$ then $\sigma_2(X) = 3 = \sigma_2(Y)$ and, similarly to the case $T = 2$, it follows that $\sigma_2(X \cup Y) = 3$, a contradiction. So $\sigma_2(X \cup Y) = 1$, $\min\{\sigma_2(X), \sigma_2(Y)\} = 2$, and $\max\{\sigma_2(X), \sigma_2(Y)\} \in \{2, 3\}$. If $\max\{\sigma_2(X), \sigma_2(Y)\} = 3$ then we may suppose without loss of generality that $\sigma_2(X) = 2$ and $\sigma_2(Y) = 3$. Now, as $\sigma_2(X \cup Y) = 1$ and $\sigma_2(X) = 2$ and $\sigma_2(Y) = 3$ and $|X \cap Y| = 1$ there exist $W_1, W_2 \subseteq \mathcal{V}(G_\chi)$ such that $Y \subseteq W_2$ and $X \setminus Y = X \cap W_1$. Hence $\sigma_2(X \setminus Y) = 3 = \sigma_2(Y \setminus X)$. As G_χ is σ_2-sparse Lemma 5.5.1.5 implies that $d_G(U_1, U_2) \in \{1, 2\}$. If $d_G(U_1, U_2) = 1$ then $U_1 \cup U_2$ is not σ_2-critical in G_χ and (i) holds, whereas if $d_G(U_1, U_2) = 2$ then $U_1 \cup U_2$ is σ_2-critical in G_χ and (ii) holds.

If $\max\{\sigma_2(X), \sigma_2(Y)\} = 2$ then, as X and Y are both σ_2-critical in G_χ, $|X|, |Y| \geq 4$ and so there exist $W_1, W_2 \in \mathcal{V}(G_\chi)$ such that $X \cup Y \subseteq W_1 \cup W_2$, $|X \cap W_1| \geq 3$, and $|X \cap W_2| = 1$. Moreover, as $d_G(X,Y) \geq 1$, 5.2 implies $d_G(X,Y) = 1$. Consequently, Lemma 5.5.1.5 implies that $i_G(X \cup Y) + i_G(X \cap Y) = (2|X \cup Y| - 2) + (2|X \cap Y| - 2) + 1$. Therefore $X \cup Y$ is σ_2-critical in G_χ. Now, either $|Y \cap W_1| = 1$ and $|Y \cap W_2| \geq 3$, or $|Y \cap W_1| \geq 3$ and $|Y \cap W_2| = 1$.

If $|Y \cap W_1| = 1$ and $|Y \cap W_2| \geq 3$ then $X \cap Y \in \{X \cap W_2, Y \cap W_1\}$. If $X \cap Y = X \cap W_2$ then $\sigma_2(X \setminus Y) = 3$ and $\sigma_2(Y \setminus X) = 2$. If $X \cap Y = Y \cap W_1$ then $\sigma_2(Y \setminus X) = 3$ and $\sigma_2(X \setminus Y) = 2$. Therefore, $\min\{\sigma_2(X \setminus Y), \sigma_2(Y \setminus X)\} = 2$ and $\max\{\sigma_2(X \setminus Y), \sigma_2(Y \setminus X)\} = 3$. That is, (ii) holds. On the other hand, if $|Y \cap W_1| \geq 3$ and $|Y \cap W_2| = 1$ then $|(X \cup Y) \cap W_2| \in \{1, 2\}$. So, as $\sigma_2(X \cup Y) = 1$, $|(X \cup Y) \cap W_2| = 2$ and hence $X \cap Y \subseteq W_1$. Therefore $\sigma_2(X \setminus Y) = 2 = \sigma_2(Y \setminus X)$. That is, (iii) holds.

Finally, suppose that $T = 0$. Then $\sigma_2(X \cup Y) = \min\{\sigma_2(X), \sigma_2(Y)\}$ and, since $d_G(X, Y) \geq 1$, Lemma 5.3.1.7 and 5.2 together imply that $d_G(X, Y) = 1$, $X \cup Y$ is σ_2-critical in G_χ, and $\max\{\sigma_2(X), \sigma_2(Y)\} = 3$. That is, (iv) holds. □

Lemma 5.5.1.8. *Let $G_\chi = (V, E)_\chi$ be a labelled graph. Suppose that X, Y, and Z are σ_2-critical in G_χ. If $\sigma_2(V) \geq 1$, G_χ is σ_2-sparse, $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$, and $X \cap Y \cap Z = \emptyset$ then $d_G(X \cup Y, Z) = 0$, $d_G(X \cap Z, Y \cap Z) = d_G(X, Y)$, $X \cup Y \cup Z$ is σ_2-critical in G_χ, and we have the following dichotomy. Either*

(i) $\sigma_2(X \cup Y \cup Z) = 1$, $\min\{\sigma_2(X), \sigma_2(Y), \sigma_2(Z)\} = 2 = |\{A \in \{X, Y, Z\} \colon \sigma_2(A) = 2\}|$, *and* $\max\{\sigma_2(X), \sigma_2(Y), \sigma_2(Z)\} = 3$*; or*

(ii) $\sigma_2(X \cup Y \cup Z) = \min\{\sigma_2(X), \sigma_2(Y), \sigma_2(Z)\}$, *and* $|\{A \in \{X, Y, Z\} \colon \sigma_2(A) = 3\}| \geq 2$.

Proof. As G_χ is σ_2-sparse and $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$ and $X \cap Y \cap Z = \emptyset$, Lemma 5.5.1.5 implies that

$$\sigma_2(X) + \sigma_2(Y) + \sigma_2(Z) + d_G(X \cap Z, Y \cap Z) \geq \sigma_2(X \cup Y \cup Z) + 6 + d_G(X, Y) + d_G(X \cup Y, Z). \tag{5.3}$$

Hence, as $d_G(X \cap Z, Y \cap Z) \leq d_G(X, Y)$,

$$\sigma_2(X) + \sigma_2(Y) + \sigma_2(Z) \geq \sigma_2(X \cup Y \cup Z) + 6 + d_G(X \cup Y, Z). \tag{5.4}$$

Let $T = \min\{\sigma_2(X), \sigma_2(Y), \sigma_2(Z)\} - \sigma_2(X \cup Y \cup Z)$ and observe that, as $\sigma_2(V) \geq 1$, Lemma 5.3.1.7 implies $T \in \{0, 1, 2\}$. If $T = 2$ then it follows that $\sigma_2(X \cup Y \cup Z) = 1$ and $\sigma_2(X) = \sigma_2(Y) = \sigma_2(Z) = 3$. As $\sigma_2(X) = \sigma_2(Y) = \sigma_2(Z) = 3$, it follows that there exist $W_1, W_2, W_3 \in \mathcal{V}(G_\chi)$ such that $X \subseteq W_1$, $Y \subseteq W_2$, and $Z \subseteq W_3$. As $X \cap Y \neq \emptyset \neq Y \cap Z$, $W_1 = W_2 = W_3$. However this implies that $\sigma_2(X \cup Y \cup Z) = 3$, a contradiction. Therefore $T \in \{0, 1\}$.

Suppose next that $T = 1$. If $\sigma_2(X \cup Y \cup Z) = 2$ then $\sigma_2(X) = \sigma_2(Y) = \sigma_2(Z) = 3$ and, similarly to the case $T = 2$, it follows that $\sigma_2(X \cup Y \cup Z) = 3$, a contradiction. So $\sigma_2(X \cup Y \cup Z) = 1$ and $\min\{\sigma_2(X), \sigma_2(Y), \sigma_2(Z)\} = 2$. So $\max\{\sigma_2(X), \sigma_2(Y), \sigma_2(Z)\} = 3$ by 5.4. We may suppose without loss of generality that $\sigma_2(X) = 2$ and $\sigma_2(Z) = 3$. If $\sigma_2(Y) = 3$ then there exist $W_2, W_3 \in \mathcal{V}(G_\chi)$ such that $Y \subseteq W_2$ and $Z \subseteq W_3$. As $Y \cap Z \neq \emptyset$, $W_2 = W_3$. As $X \cap Y \cap Z = \emptyset$, $|X \cap (Y \cup Z)| = 2$ and so $|\{x \in X \colon x \notin W_2\}| = 1$. However this implies that $\sigma_2(X \cup Y \cup Z) = 2$, a contradiction. Therefore $\sigma_2(Y) = 2 = |\{A \in \{X, Y, Z\} \colon \sigma_2(A) = 2\}|$. It follows from 5.4 that $d_G(X \cup Y, Z) = 0$ and so, as $d_G(X \cap Z, Y \cap Z) \leq d_G(X, Y)$, it follows from 5.3 that $d_G(X \cap Z, Y \cap Z) = d_G(X, Y)$. Then Lemma 5.5.1.5 implies $X \cup Y \cup Z$ is σ_2-critical in G_χ. Finally, suppose that $T = 0$. Then $\sigma_2(X \cup Y \cup Z) = \min\{\sigma_2(X), \sigma_2(Y), \sigma_2(Z)\}$, and Lemma 5.3.1.7 and 5.4 together imply that $|\{A \in \{X, Y, Z\} \colon \sigma_2(A) = 3\}| \geq 2$ and $d_G(X \cup Y, Z) = 0$. Moreover, as $d_G(X \cap Z, Y \cap Z) \leq d_G(X, Y)$ 5.3 implies $d_G(X \cap Z, Y \cap Z) = d_G(X, Y)$. Then Lemma 5.5.1.5 gives us that $X \cup Y \cup Z$ is σ_2-critical in G_χ. □

5.5.2 Vertices of Degree Three

Now that we have access to the idea of a σ_2-critical set the remainder of this chapter will see us use these sets to derive a method of constructing certain σ_2-tight graphs. In this short section we see how σ_2-critical sets are, when we restrict the possible labellings of the vertex set, precisely the correct concept to use in order to understand what may prevent the existence of a σ_2-tight $(2, 1)$-VL-reduction of a σ_2-tight labelled graph. The different ways that the local structure of a labelled graph, at a vertex of degree three, can be impacted by the choice of labelling are illustrated in Figure 5.3 and Table 5.1.

Lemma 5.5.2.1. *Let $G_\chi = (V, E)_\chi$ be a labelled graph and suppose there exists $v \in V$ such that $d_G(v) = 3$. If G_χ is σ_2-sparse then there does not exist a $(2, 1)$-VL-reduction* *of G_χ at v adding e that is σ_2-sparse if and only if either $e \in E$, or $e \notin E$ and there exists $U \subseteq V \setminus \{v\}$ such that the endpoints of e are in U and*

(i) U is σ_2-critical in G_χ; or

(ii) $G[U] \cong K_5^-$ and $\sigma_2(U) = 0 = \sigma_2(V)$.

Moreover, if G_χ is σ_2-tight then there does not exist a $(2,1)$-VL-reduction of G_χ at v adding e that is σ_2-tight if and only if there does not exist a $(2,1)$-VL-reduction of G_χ at v adding e that is σ_2-sparse.

Proof. Let $e = xy$. We note that, by definition, there does not exist a $(2,1)$-VL-reduction of G_χ at v adding xy if and only if $xy \in E$. So we may suppose instead that there does exist such a $(2,1)$-VL-reduction of G_χ and denote this $G'_{\chi'} = (V', E')_{\chi'}$. As G_χ is σ_2-sparse, Lemma 5.3.2.3 implies that $G'_{\chi'}$ is not σ_2-sparse if and only if there exists $U \subseteq V'$ such that $\{x,y\} \subseteq U$ and $i_G(U) = S(G_\chi[U], \sigma_2|_U)$. As $xy \notin E$ this implies that $G'_{\chi'}$ is not σ_2-sparse if and only if there exists $n \geq 4$ such that $G[U] \ncong K_n$, and

(a) $i_G(U) = S(G_\chi[U], \sigma_2|_U) = 2|U| - \sigma_2(U)$; or

(b) $i_G(U) = S(G_\chi[U], \sigma_2|_U) = 3|U| - 6 < 2|U| - \sigma_2(U)$.

Consequently, Lemma 5.3.1.6 implies that $G'_{\chi'}$ is not σ_2-sparse if and only if U is σ_2-critical in G_χ, or $G[U] \cong K_5^-$ and $\sigma_2(W) = 0 = \sigma_2(V)$.

Suppose G_χ is σ_2-tight. As G' is a $(2,1)$-reduction of G, $|V'| \geq 3$ and so Lemma 5.3.1.4 and Lemma 5.3.1.5 together imply that $S(G'_{\chi'}, \sigma_2|_{V'}) \leq 2|V'| - \sigma_2(V')$. Moreover, Lemma 5.3.1.7 implies that $\sigma_d(V) \leq \sigma_d(V')$. So, as G_χ is σ_d-tight,

$$|E'| = (|E| - 3) + 1 = 2|V'| - \sigma_2(V) \geq 2|V'| - \sigma_2(V') \geq S(G'_{\chi'}, \sigma_2|_{V'}).$$

Therefore, $|E'| \geq S(G'_{\chi'}, \sigma_2|_{V'})$ and $|E'| = 2|V'| - \sigma_2(V') = S(G'_{\chi'}, \sigma_2|_{V'})$ if and only if $|E'| = S(G'_{\chi'}, \sigma_2|_{V'})$. Consequently, there does not exist a $(2,1)$-VL-reduction of G_χ at v adding xy that is σ_2-tight if and only if there does not exist a $(2,1)$-VL-reduction of G_χ at v adding xy that is σ_2-sparse. □

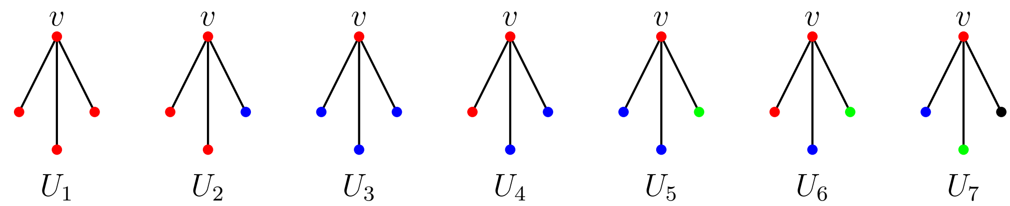


Figure 5.3: Illustration of the possible labellings of $N_G[v]$ for a vertex, v, of degree three in G where G_χ is a labelled graph.

For the remainder of this chapter we constrain ourselves to working with labelled graphs G_χ such that $\sigma_2(V(G)) \geq 1$. Lemma 5.3.1.6 (ii) and, in particular, Lemma 5.5.2.1 show that restricting our focus in this way will make it easier to discuss the structure around those vertices of degree three at which there does not exist a $(2,1)$-VL-reduction of G_χ that is σ_2-sparse.

Case	U_1	U_2	U_3	U_4	U_5	U_6	U_7
$\sigma_2(N_G(v))$	3	2	3	2	2	0	0
$\sigma_2(N_G[v])$	3	2	2	1	0	0	0
$\lvert\mathcal{V}(G_\chi)\rvert$	≥ 1	≥ 2	≥ 2	≥ 2	≥ 3	≥ 3	≥ 4

Table 5.1: Tabulation of the possible labellings of $N_G[v]$ for a vertex, v, of degree three in G where G_χ is a labelled graph.

So far we have considered when a specific $(2,1)$-VL-reduction of a σ_2-sparse labelled graph will be $(2,1)$-sparse. Note that, given a labelled graph G_χ and $v \in V(G)$ such that $d_G(v) = 3$, the number of $(2,1)$-VL-reductions of G_χ at v is $3 - E(G[N_G(v)])$. We wish to understand when there exists any σ_2-sparse $(2,1)$-VL-reduction of G_χ at v.

Lemma 5.5.2.2. *Let $G_\chi = (V,E)_\chi$ be a σ_2-sparse labelled graph, and suppose that $\sigma_2(V) \geq 1$. If there exists $v \in V$ such that $d_G(v) = 3$ and $\sigma_2(N_G[v]) = 3$ then there exists a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse.*

Proof. Let us define H to be the complete graph with vertex set $N_G(v)$ and let $F = E(H) \setminus E$. We proceed by considering $i_G(N_G(v))$. As G_χ is σ_2-sparse and $|N_G[v]| = 4$ and $\sigma_2(N_G[v]) = 3$, $i_G(N_G[v]) \leq 2|N_G[v]| - 3 = 5$ and so $i_G(N_G(v)) \leq 2$.

If $i_G(N_G(v)) = 2$ then let $F = \{e\}$. Suppose there exists $U \subseteq V \setminus \{v\}$ such that the endpoints of e are in U and U is σ_2-critical in G_χ. As $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$, Lemma 5.5.1.4 implies $|N_G(v) \cap U| = 2$. Moreover, as $\sigma_2(N_G([v]) = 3$ we note that $\sigma_2(U) = \sigma_2(U \cup N_G[v])$ and consequently

$$i_G(U \cup N_G[v]) \geq i_G(U) + 5 = (2|U| - \sigma_2(U)) + 5 = 2|U \cup N_G[v]| - (\sigma_2(U \cup N_G[v]) - 1).$$

However this contradicts the fact that G_χ is σ_2-sparse, so no such set U can exist and hence there exists a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse by Lemma 5.5.2.1.

If $i_G(N_G(v)) = 1$ then let $F = \{e_1, e_2\}$, so $V(G[F]) = N_G(v)$. Suppose there exist $U_1, U_2 \subseteq V \setminus \{v\}$ such that, for $i \in \{1,2\}$, the endpoints of e_i are in U_i and U_i is σ_2-critical in G_χ. We observe that, since $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$ and $N_G(v) \subseteq U_1 \cup U_2$, $\sigma_2(U_1 \cup U_2 \cup \{v\}) = \sigma_2(U_1 \cup U_2)$. So Lemma 5.5.1.4 implies that $U_1 \cup U_2$ is not σ_2-critical in G_χ and hence, as $U_1 \cap U_2 \neq \emptyset$, Lemma 5.5.1.6 implies that $|U_1 \cap U_2| = 1$. Therefore, for $i, j \in \{1,2\}$ where $i \neq j$, the endpoint of e_i that is not an endpoint of e_j is in $U_i \setminus U_j$ and hence $d_G(U_1, U_2) \geq 1$. So, as $U_1 \cup U_2$ is not σ_2-critical in G_χ, Lemma 5.5.1.7 implies that $\sigma_2(U_1 \cup U_2) = 1$, $\min\{\sigma_2(U_1), \sigma_2(U_2)\} = 2$, and $\max\{\sigma_2(U_1), \sigma_2(U_2)\} = 3$. However, as $N_G(v) \subseteq U_1 \cup U_2$ and $\sigma_2(N_G(v)) = 3$ it follows that that $\sigma_2(U_1 \cup U_2) = \min\{\sigma_2(U_1, U_2)\}$, a contradiction. So such sets U_1 and U_2 can not both exist and hence there exists a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse by Lemma 5.5.2.1.

If $i_G(N_G(v)) = 0$ then let $F = \{e_1, e_2, e_3\}$, so $V(G[F]) = N_G(v)$. Suppose there exist $U_1, U_2, U_3 \subseteq V \setminus \{v\}$ such that, for $i \in \{1,2,3\}$, the endpoints of e_i are in U_i and U_i is σ_2-critical in G_χ. We observe that, since $\sigma_2(N_G[v]) = 3$ and $N_G(v) \subseteq U_i \cup U_j$ for all $i, j \in \{1,2,3\}$ such that $i \neq j$, for all $i, j \in \{1,2,3\}$ such that $i \neq j$, $\sigma_2(U_i \cup U_j \cup \{v\}) = \sigma_2(U_i \cup U_j)$. Similarly, $\sigma_2(U_1 \cup U_2 \cup U_3 \cup \{v\}) = \sigma_2(U_1 \cup U_2 \cup U_3)$. Therefore, Lemma 5.5.1.4 implies that for all $i, j \in \{1,2,3\}$ such that $i \neq j$, $U_i \cup U_j$ is not σ_2-critical in G_χ and similarly that $U_1 \cup U_2 \cup U_3$ is not σ_2-critical in G_χ. Then, since for all $i, j \in \{1,2,3\}$ such that $i \neq j$, $U_i \cap U_j \neq \emptyset$, Lemma 5.5.1.6 implies that for all $i, j \in \{1,2,3\}$ such that $i \neq j$, $|U_i \cap U_j| = 1$. However, as $U_1 \cap U_2 \cap U_3 = \emptyset$, Lemma 5.5.1.8 implies that $U_1 \cup U_2 \cup U_3$ is σ_2-critical in G_χ, a contradiction. So such sets U_1, U_2, and U_3 can not exist and hence there exists a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse by Lemma 5.5.2.1. $\square$

Lemma 5.5.2.3. *Let $G_\chi = (V,E)_\chi$ be a σ_2-sparse labelled graph and suppose that $\sigma_2(V) \geq 1$. If there exists $v \in V$ such that $d_G(v) = 3$ and $\sigma_2(N_G[v]) = 2 = \sigma_2(N_G(v))$ then there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse if and only if either*

(i) $i_G(N_G(v)) = 3$; *or*

(ii) $i_G(N_G(v)) = 2$ *and there exists* $U \subseteq V \setminus \{v\}$ *such that* $U \cap N_G(v) = \{u \in N_G(v) \colon d_{G[N_G(v)]}(u) = 1\}$, U *is* σ_2*-critical in* G_χ, $\sigma_2(U) = 2$, *and* $\sigma_2(U \cup \{v\}) = 1$; *or*

(iii) $i_G(N_G(v)) = 2$ *and there exists* $U \subseteq V \setminus \{v\}$ *such that* $U \cap N_G(v) = \{u \in N_G(v) \colon d_{G[N_G(v)]}(u) = 1\}$, U *is* σ_2*-critical in* G_χ, *and* $\sigma_2(U) = 3$; *or*

(iv) $i_G(N_G(v)) = 1$ *and there exist* $U_1, U_2 \subseteq V \setminus \{v\}$ *such that* $U_1 \cap U_2 = \{u \in N_G(v) \colon d_{G[N_G(v)]}(u) = 0\}$, U_1 *and* U_2 *are* σ_2*-critical in* G_χ, $\sigma_2(U_1 \cup U_2) = 1$, $\min\{\sigma_2(U_1), \sigma_2(U_2)\} = 2$, $\max\{\sigma_2(U_1), \sigma_2(U_2)\} = 3$, *and* $d_G(U_1, U_2) = 1$.

Proof. Lemma 5.5.2.1 implies that if any of (i), (ii), or (iii) hold then there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse. On the other hand, let us suppose there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse. Let us define H to be the complete graph with vertex set $N_G(v)$ and let $F = E(H) \setminus E$. We proceed by considering $i_G(N_G(v))$.

If $i_G(N_G(v)) = 2$ then let $F = \{e\}$. As there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse, Lemma 5.5.2.1 implies there exists $U \subseteq V \setminus \{v\}$ such that the endpoints of e are in U and U is σ_2-critical in G_χ. As $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$, Lemma 5.5.1.4 implies $|N_G(v) \cap U| = 2$. That is, $U \cap N_G(v) = \{u \in N_G(v) \colon d_{G[N_G(v)]}(u) = 1\}$. Consequently,

$$i_G(U \cup N_G[v]) \geq i_G(U) + 5 = (2|U| - \sigma_2(U)) + 5 = 2|U \cup N_G[v]| - (\sigma_2(U) - 1).$$

As G_χ is σ_2-sparse it follows that $\sigma_2(U) \geq \sigma_2(U \cup N_G[v]) + 1$. As $\sigma_2(V) \geq 1$, Lemma 5.3.1.7 implies $\sigma_2(U) \in \{2, 3\}$. If $\sigma_2(U) = 2$ then $\sigma_2(U \cup N_G[v]) = 1 = \sigma_2(U \cup \{v\})$. Therefore (ii) holds. If $\sigma_2(U) = 3$ then (iii) holds.

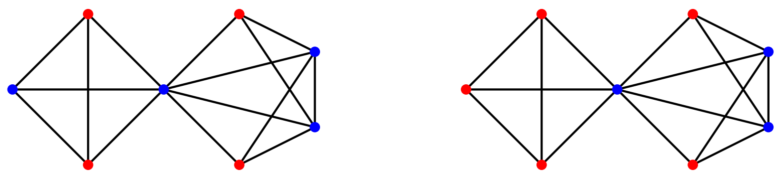
If $i_G(N_G(v)) = 1$ then let $F = \{e_1, e_2\}$, so $V(G[F]) = N_G(v)$. As there does not exist a $(2, 1)$-VL-reduction of G_χ at v that is σ_2-sparse, Lemma 5.5.2.1 implies there exist $U_1, U_2 \subseteq V \setminus \{v\}$ such that, for $i \in \{1, 2\}$, the endpoints of e_i are in U_i and U_i is σ_2-critical in G_χ. We observe that, since $\sigma_2(N_G[v]) = 2 = \sigma_2(N_G(v))$ and $N_G(v) \subseteq U_1 \cup U_2$, $\sigma_2(U_1 \cup U_2 \cup \{v\}) = \sigma_2(U_1 \cup U_2)$. So Lemma 5.5.1.4 implies that $U_1 \cup U_2$ is not σ_2-critical in G_χ and hence, as $U_1 \cap U_2 \neq \emptyset$, Lemma 5.5.1.6 implies that $|U_1 \cap U_2| = 1$. That is, $U_1 \cap U_2 = \{u \in N_G(v) \colon d_{G[N_G(v)]}(u) = 0\}$. Therefore, for $i, j \in \{1, 2\}$ where $i \neq j$, the endpoint of e_i that is not an endpoint of e_j is in $U_i \setminus U_j$ and hence $d_G(U_1, U_2) \geq 1$. So, as $U_1 \cup U_2$ is not σ_2-critical in G_χ, Lemma 5.5.1.7 implies that $\sigma_2(U_1 \cup U_2) = 1$, $\min\{\sigma_2(U_1), \sigma_2(U_2)\} = 2$, $\max\{\sigma_2(U_1), \sigma_2(U_2)\} = 3$, and $d_G(U_1, U_2) = 1$. That is, (iv) holds. If $i_G(N_G(v)) = 0$ then let $F = \{e_1, e_2, e_3\}$, so $V(G[F]) = N_G(v)$. As there does not exist a $(2, 1)$-VL-reduction of G_χ at v that is σ_2-sparse, Lemma 5.5.2.1 implies there exist $U_1, U_2, U_3 \subseteq V \setminus \{v\}$ such that, for $i \in \{1, 2, 3\}$, the endpoints of e_i are in U_i and U_i is σ_2-critical in G_χ. We observe that, since $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$ and $N_G(v) \subseteq U_i \cup U_j$ for all $i, j \in \{1, 2, 3\}$ such that $i \neq j$, for all $i, j \in \{1, 2, 3\}$ such that $i \neq j$, $\sigma_2(U_i \cup U_j \cup \{v\}) = \sigma_2(U_i \cup U_j)$. Similarly, $\sigma_2(U_1 \cup U_2 \cup U_3 \cup \{v\}) = \sigma_2(U_1 \cup U_2 \cup U_3)$. Therefore, Lemma 5.5.1.4 implies that for all $i, j \in \{1, 2, 3\}$ such that $i \neq j$, $U_i \cup U_j$ is not σ_2-critical in G_χ and similarly that $U_1 \cup U_2 \cup U_3$ is not σ_2-critical in G_χ. Then, since for all $i, j \in \{1, 2, 3\}$ such that $i \neq j$, $U_i \cap U_j \neq \emptyset$, Lemma 5.5.1.6 implies that for all $i, j \in \{1, 2, 3\}$ such that $i \neq j$, $|U_i \cap U_j| = 1$. However, as $U_1 \cap U_2 \cap U_3 = \emptyset$, Lemma 5.5.1.8 implies that $U_1 \cup U_2 \cup U_3$ is σ_2-critical in G_χ, a contradiction. So such sets U_1, U_2, and U_3 can not all exist, a contradiction. □

5.5.3 σ_2-Cut-Tight Vertex-Labelled Graphs

While the study of general σ_2-sparse labelled graphs may be of independent interest, our primary goal is to study graphs related to the rigidity of graphs realised on non-concentric spheres. It is for that reason that we introduce the following additional property of

σ_2-sparse labelled graphs and this chapter concludes with a method of constructing a subfamily of those σ_2-tight labelled graphs that have this property.

Definition 5.5.3.1. Let G_χ be a labelled graph. G_χ is **$\boldsymbol{\sigma_2}$-cut-sparse** if G_χ is σ_2-sparse and for all σ_2-tight labelled subgraphs $G'_{\chi'}$ of G_χ such that $\kappa(G') = 1$, for all $v' \in V(G')$ such that $G'[V(G') \setminus \{v'\}]$ is not connected, with components $H'_1, \ldots, H'_n$, for all $1 \leq i \leq n$, $\sigma_2(V(H'_i)) \leq 2$. G_χ is **$\boldsymbol{\sigma_2}$-cut-tight** if G_χ is σ_2-cut-sparse and σ_2-tight.



(G, χ_1) (G, χ_2)

Figure 5.4: Two σ_2-tight labelled graphs with the same underlying graph, G. (G, χ_1) is σ_2-cut-tight while (G, χ_2) is not.

Our first step is to attempt to give the interaction between the operations introduced in Section 5.2 and σ_2-cut-sparsity the same treatment as we gave the interaction between those operations and σ_2-sparsity in Subsection 5.3.2. We shall see that when considering $(2,0)$-VL-extension, $(2,0)$-VL-reduction, and $(2,1)$-VL-extension operations the addition of this 'cut' condition makes no difference. However, additional complications arise when analysing the $(2,1)$-VL-reduction operation which necessitate us taking a more global view of the situation.

Lemma 5.5.3.2. *Let $G_\chi = (V, E)_\chi$ be a labelled graph. Suppose $G'_{\chi'}$ is a $(2,0)$-VL-reduction of G_χ. G_χ is σ_2-cut-sparse if and only if $G'_{\chi'}$ is σ_2-cut-sparse.*

Proof. Let $V \setminus V(G') = \{v\}$. If G_χ is σ_2-cut-sparse then, as $G'_{\chi'}$ is a labelled subgraph of G_χ, $G'_{\chi'}$ is σ_2-cut-sparse. On the other hand, suppose that $G'_{\chi'}$ is σ_2-cut-sparse. Lemma 5.3.2.1 implies G_χ is σ_2-sparse. If there does not exist a σ_2-tight labelled subgraph, $G[W]_\psi$, of G_χ such that $|W| \geq 3$ and $\kappa(G[W]) = 1$ then G_χ is σ_2-cut-tight. Alternatively, suppose that such a subgraph exists and take $w \in W$ such that $G[W \setminus \{w\}]$ is not connected. Let $W' = W \setminus \{v\}$ and let $H_1, \dots, H_n$ be the components of $G[W \setminus \{w\}]$.

If $v \notin W$ then $G_\chi[W] = G'_{\chi'}[W]$. As $G'_{\chi'}$ is σ_2-cut-sparse, $\sigma_2(V(H_i)) \leq 2$ for all $1 \leq i \leq n$. If $v \in W$ then, as $G'_{\chi'}$ is a $(2,0)$-VL-reduction of G_χ, $G_\chi[W'] = G'_{\chi'}[W']$. If $w = v$ then $G[W']$ is not connected. As $G[W]_\psi$ is σ_2-tight, Lemma 5.3.1.9 implies that $\sigma_2(V(H_i)) = 0$ for all $1 \leq i \leq n$.

If $w \neq v$ then $v \in W'$ and, as $G[W]_\psi$ is σ_2-tight and $|W| \geq 3$, Lemma 5.3.1.8 implies $d_{G[W]}(v) \geq 2$. Therefore $G'_{\chi'}[W']$ is a $(2,0)$-VL-reduction of $G_\chi[W]$ and so, by Lemma 5.3.2.2, $G'_{\chi'}[W'] = G_\chi[W']$ is σ_2-tight. As $d_{G[W]}(v) \geq 2$, $G'[W' \setminus \{w\}]$ is not connected. Let $H'_1, \dots, H'_m$ be the components of $G'[W' \setminus \{w\}]$. As $G'_{\chi'}$ is σ_2-cut-sparse it follows that $\sigma_2(V(H'_j)) \leq 2$ for all $1 \leq j \leq m$. Now, for all $1 \leq i \leq n$ there exists $1 \leq j \leq m$ such that $V(H'_j) \subseteq V(H_i)$. Therefore Lemma 5.3.1.7 implies that $\sigma_2(V(H_i)) \leq 2$ for all $1 \leq i \leq n$. □

Lemma 5.5.3.3. *Let $G_\chi = (V,E)_\chi$ be a labelled graph. Suppose $G'_{\chi'}$ is a $(2,0)$-VL-reduction of G_χ. If G_χ is σ_2-cut-tight then $G'_{\chi'}$ is σ_2-cut-tight. If $G'_{\chi'}$ is σ_2-cut-tight then G_χ is σ_2-cut-tight if and only if $\sigma_2(V) = \sigma_2(V(G'))$.*

Proof. Let $V \setminus V(G') = \{v\}$. Firstly, suppose that G_χ is σ_2-cut-tight. Lemma 5.5.3.2 implies that $G'_{\chi'}$ is σ_2-cut-sparse and Lemma 5.3.2.2 implies that $G'_{\chi'}$ is σ_2-tight. On the other hand, suppose that $G'_{\chi'}$ is σ_2-cut-tight. Lemma 5.5.3.2 implies that G_χ is σ_2-cut-sparse. Therefore Lemma 5.3.2.2 implies G_χ is σ_2-cut-tight if and only if G_χ is σ_2-tight if and only if and only if $\sigma_2(V) = \sigma_2(V(G'))$. □

Lemma 5.5.3.4. *Let $G_\chi = (V,E)_\chi$ be a labelled graph. Suppose $G'_{\chi'}$ is a $(2,1)$-VL-reduction of G_χ. If $G'_{\chi'}$ is σ_2-cut-sparse then G_χ is σ_2-cut-sparse. If $G'_{\chi'}$ is σ_2-cut-tight then G_χ is σ_2-cut-tight if and only if $\sigma_2(V) = \sigma_2(V(G'))$.*

Proof. Let $V \setminus V(G') = \{v\}$, let $E(G') \setminus E = \{e\}$, and suppose that $G'_{\chi'}$ is σ_2-cut-sparse. Lemma 5.3.2.3 implies G_χ is σ_2-sparse. If there does not exist a σ_2-tight labelled subgraph, $G[W]_\psi$, of G_χ such that $|W| \geq 3$ and $\kappa(G[W]) = 1$ then G_χ is σ_2-cut-sparse.

Alternatively, suppose that such a subgraph exists. That is, there exists $W \subseteq V$ such that $|W| \geq 3$, W is σ_2-critical in G_χ, and $\kappa(G[W]) = 1$. Let $e = xy$ and take $w \in W$ such that $G[W \setminus \{w\}]$ is not connected. Let $W' = W \setminus \{v\}$, let $G_1, \dots, G_n$ be the components of $G[W \setminus \{w\}]$, and for all $1 \leq i \leq n$ let $V(G_i) = V_i$. Firstly suppose that $|\{x, y\} \cap W| \leq 1$. Then $G_\chi[W'] = G'_{\chi'}[W']$. If $v \notin W$ then $G_\chi[W] = G'_{\chi'}[W]$ and $G_1, \dots G_n$ are the components of $G'[W \setminus \{w\}]$. As $G'_{\chi'}$ is σ_2-cut-sparse, $\sigma_2(V_i) \leq 2$ for all $1 \leq i \leq n$. If $v \in W$ then $d_{G[W]}(v) = 2$ by Lemma 5.3.1.8, and so $G_\chi[W]$ is a $(2, 0)$-VL-extension of $G'_{\chi'}[W']$. As $G'_{\chi'}$ is σ_2-cut-sparse, $G'_{\chi'}[W]$ is σ_2-cut-sparse and so Lemma 5.5.3.2 implies $G_\chi[W]$ is σ_2-cut-sparse.

On the other hand, suppose that $\{x, y\} \subseteq W$. As W is σ_2-critical in G_χ and $G'_{\chi'}$ is σ_2-sparse,

$$\begin{aligned}
2|W| - \sigma_2(W) &= i_G(W) \\
&= (i_{G'}(W') - 1) + (|W \cap \{v\}| \cdot |W \cap N_G(v)|) \\
&\leq 2|W'| - (\sigma_2(W') + 1) + (|W \cap \{v\}| \cdot |W \cap N_G(v)|) \\
&= 2|W| - (\sigma_2(W') + 1 + 2|W \cap \{v\}|) + (|W \cap \{v\}| \cdot |W \cap N_G(v)|) \\
&= 2|W| - (\sigma_2(W') + 1 + |W \cap \{v\}| \cdot (2 - |W \cap N_G(v)|)).
\end{aligned}$$

So $\sigma_2(W) \geq \sigma_2(W') + 1 + |W \cap \{v\}| \cdot (2 - |N_G(v) \cap W|)$. Lemma 5.3.1.7 implies that $\sigma_2(W') \geq \sigma_2(W)$ and hence $N_G[v] \subseteq W$ and $\sigma_2(W) = \sigma_2(W')$. Therefore $G'_{\chi'}[W']$ is a $(2, 1)$-VL-reduction of $G_\chi[W]$. As $G'_{\chi'}$ is σ_2-sparse, Lemma 5.3.2.3 and Lemma 5.3.2.4 together imply that $G'_{\chi'}[W']$ is σ_2-tight. As $N_G[v] \subseteq W$, $|W'| \geq 3$. Let $G'_1, \dots, G'_m$ denote the components of $G'[W' \setminus \{w\}]$ and for all $1 \leq j \leq m$ let $V(G'_j) = V'_j$.

Suppose that $w = v$. Then for all $1 \leq i \leq n$ there exists $1 \leq j \leq m$ such that $V_i \subseteq V'_j$. In particular, there exists $1 \leq i \leq n$ such that $\{x, y\} \subseteq V_i$ if and only if for all $1 \leq i \leq n$ there exists $1 \leq j \leq m$ such that $V_i = V'_j$. Alternatively, there exist $1 \leq a < b \leq n$ such that $x \in V_a$ and $y \in V_b$ if and only if there exists $1 \leq j \leq m$ such that $V_a \cup V_b = V'_j$ and for all $1 \leq i \leq n$ such that $i \notin \{a, b\}$ there exists $1 \leq k \leq m$ such that $V_i = V'_j$. Note that, by construction, $V_i = V_j$ if and only if $i = j$ and similarly $V'_i = V'_j$ if and only if $i = j$.

If there exists $1 \leq i \leq n$ such that $\{x, y\} \subseteq V_i$ then $G'[W']$ is not connected and so, as W' is σ_2-critical in $G'_{\chi'}$, Lemma 5.3.1.9 implies that for all $1 \leq j \leq m$, $\sigma_2(V'_j) = 0$. Therefore $\sigma_2(V_i) = 0$ for all $1 \leq i \leq n$. If there exist $1 \leq a < b \leq n$ such that $x \in V_a$ and $y \in V_b$ then, as W' is σ_2-critical in $G'_{\chi'}$, Lemma 5.3.1.9 implies that for all $1 \leq j \leq m$ such that $V'_j \neq V_a \cup V_b$, $\sigma_2(V'_j) = 0$. We observe that either $V_a \cup V_b = W'$, and so is σ_2-critical in $G'_{\chi'}$, or $G'[V_a \cup V_b]$ is a component of $G'[W']$ and so $V_a \cup V_b$ is σ_2-critical in $G'_{\chi'}$ by Lemma 5.3.1.9. As $G'[W']$ is σ_2-tight and $|W'| \geq 3$, Lemma 5.3.1.8 implies $d_{G'[W']}(x) \geq 2$ and so $|V_a \cup V_b| \geq 3$. Furthermore, $\kappa_1(G'[V_a \cup V_b]) = 1$ and $G'[E(G'[V_a \cup V_b]) \setminus \{xy\}] = G[V_a \cup V_b]$ is not connected. So, by Lemma 5.3.1.13, $\sigma_2(V_a), \sigma_2(V_b) \leq 1$ and hence $\sigma_2(V_i) \leq 1$ for all $1 \leq i \leq n$.

Suppose instead that $w \neq v$, so there exists $1 \leq i \leq n$, say a, such that $N_G[v] \setminus \{w\} \subseteq V_a$ and so $d_{G[V_a \setminus \{w\}]}(v) \geq 2$. Hence, as $G[W \setminus \{w\}]$ is not connected, $G'[W' \setminus \{w\}]$ is not connected. If $G'[W']$ is connected then, as W' is critical in $G'_{\chi'}$ and $|W'| \geq 3$ and $G'_{\chi'}$ is σ_2-cut-sparse, $\sigma_2(V'_j) \leq 2$ for all $1 \leq j \leq m$. Moreover, for all $1 \leq i \leq n$ such that $i \neq a$ there exists $1 \leq j \leq m$ such that $V_i = V'_j$, and there exists $1 \leq k \leq m$ such that $V_a \setminus \{v\} = V'_k$ so $V_a \supset V'_k$. Therefore $\sigma_2(V_i) \leq 2$ for all $1 \leq i \leq n$ by Lemma 5.3.1.7.

On the other hand, if $G'[W']$ is not connected then let $H_1, \ldots, H_t$ denote the components of $G'[W']$ and for all $1 \leq i \leq t$ let $V(H_i) = U_i$. As W' is σ_2-critical in $G'_{\chi'}$, Lemma 5.3.1.9 implies that, for all $1 \leq i \leq t$, U_i is σ_2-critical in $G'_{\chi'}$ and $\sigma_2(U_i) = 0$. There exists $1 \leq i \leq t$, say a, such that $w \in U_a$ and we observe that $(\bigcup_{i=1}^{t} U_i) \setminus (U_a \setminus V_a) = V_a \setminus \{v\}$. Therefore $V_a \supseteq (\bigcup_{i=1}^{t} U_i) \setminus (U_a \setminus V_a)$ and so $\sigma_2(V_a) = 0$ by Lemma 5.3.1.7. As U_a is σ_2-critical in $G'_{\chi'}$, $|U_a| \geq 3$. If $G'[U_a \setminus \{w\}]$ is connected then $\{V_1, V_2\} = \{(V_a \cup \{v\}), (U_a \setminus \{w\})\}$ and $\sigma_2(V_1) = 0 = \sigma_2(V_2)$. If $G'[U_a \setminus \{w\}]$ is not connected then for all $1 \leq i \leq n$ such that $i \neq a$, $V_i \subseteq U_a \setminus \{w\}$. So, as $G'_{\chi'}$ is σ_2-cut-sparse, $\sigma_2(V_i) \leq 2$ for all $1 \leq i \leq n$ such that $i \neq a$.

Finally, if $G'_{\chi'}$ is σ_2-cut-tight then by above and Lemma 5.3.2.4 we have that G_χ is σ_2-cut-tight if and only if G_χ is σ_2-tight if and only if $\sigma_2(V(G')) = \sigma_2(V)$. □

The three previous results provide justification for the earlier comment suggesting that for $(2, 0)$-VL-reductions/extensions and $(2, 1)$-VL-extensions the move from σ_2-sparsity to σ_2-cut-sparsity would not have any impact. However we shall have to give a more piecemeal analysis for $(2,1)$-VL-reductions. Lemma 5.3.2.3 gives a necessary and sufficient condition for a $(2,1)$-VL-reduction of a σ_2-sparse graph to be σ_2-sparse. Our next step is to build on this result and show that there exist σ_2-sparse $(2,1)$-VL-reductions of σ_2-sparse graphs that are not σ_2-cut-sparse.

Building on this, given a vertex v of degree three in a σ_2-sparse labelled graph G_χ, Lemma 5.5.2.2 and Lemma 5.5.2.3 allow us to understand when there exists a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse by considering $\sigma_2(N_G(v))$ and $\sigma_2(N_G[v])$.

There is an important point here about how information about $\sigma_2(V)$ is required in order to apply these two results, which both use the hypothesis that $\sigma_2(V) \geq 1$. This information is used in two ways. First and foremost, Lemma 5.5.2.1 tells us that restriction guarantees that if $(2,1)$-VL-reduction does not preserve σ_2-sparsity then a particular σ_2-critical set exists. Secondly, the fact that $\sigma_2(V) \geq 1$ allows us to make use of the results concerning the intersection of σ_2-critical sets. This second usage is seemingly less important, as it's plausible that the relevant properties of intersecting σ_2-critical sets could be extended to the situation where $\sigma_2(V) = 0$. This shows that, at least if we demand that $\sigma_2(V) \geq 1$, then whether a $(2,1)$-VL-reduction preserves σ_2-sparsity is, in some sense, a purely local question.

On the other hand, our next result, which plays a similar role to that of Lemma 5.5.2.1 but in the context of σ_2-cut-sparsity, has a much more tangible reliance on $\sigma_2(V)$. In this setting we must now demand that $\sigma_2(V) \geq 2$ in order to make the question of whether a $(2,1)$-VL-reduction preserves σ_2-cut-sparsity local in the same way. Figure 5.5 highlights this point. This is clearly unsatisfactory, as by allowing $\sigma_2(V)$ to equal one we have access to a far more varied collection of labelled graphs, but this potentially acts as evidence to support the idea that σ_2-cut-sparsity is a notably more complicated notion than that of σ_2-sparsity.

Lemma 5.5.3.5. *Let $G_\chi = (V,E)_\chi$ be a vertex-labelled graph, and suppose there exists $v \in V$ such that $d_G(v) = 3$. Let $N_G(v) = \{x,y,z\}$ and suppose G_χ is σ_2-cut-sparse. Suppose there exists a $(2,1)$-VL-reduction of G_χ at v adding xy that is*

σ_2-sparse, and denote this by $G'_{\chi'}$. If $\sigma_2(V) \geq 2$ then $G'_{\chi'}$ is not σ_2-cut-sparse if and only if there exist $W_1, W_2 \subseteq V \setminus \{v\}$ such that $|W_1|, |W_2| \geq 3$, $N_G(v) \cap W_1 = \{x, y\}$, $W_1 \cap W_2 = \{v \in V : \sigma_2(V \setminus \{v\}) > \sigma_2(V)\}$, $i_G(W_1) = 2|W_1| - 3$, W_2 is σ_2-critical in G_χ, and $d_G(W_1, W_2) = 0$.

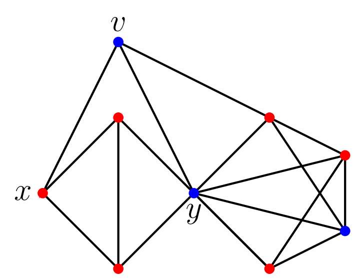
Proof. Firstly suppose that there exist $W_1, W_2 \subseteq V \setminus \{v\}$ such that $|W_1|, |W_2| \geq 3$, $N_G(v) \cap W_1 = \{x, y\}$, $W_1 \cap W_2 = \{v \in V : \sigma_2(V \setminus \{v\}) > \sigma_2(V)\}$, $i_G(W_1) = 2|W_1| - 3$, W_2 is σ_2-critical in G_χ, and $d_G(W_1, W_2) = 0$. Then W_1 and W_2 are σ_2-critical in G_χ, so Lemma 5.5.1.5 implies $i_{G'}(W_1 \cup W_2) = 2|W_1 \cup W_2| - 2$ and hence $W_1 \cup W_2$ is σ_2-critical in G_χ. Moreover, $|W_1 \cup W_2| \geq |W_1| \geq 4$ and $d_{G'}(W_1, W_2) = d_G(W_1, W_2) = 0$. Consequently $G'[(W_1 \cup W_2) \setminus (W_1 \cap W_2)]$ is not connected and $\sigma_2(W_1 \setminus W_2) = 3$. Therefore $G'_{\chi'}$ is not σ_2-cut-sparse.

On the other hand, suppose $G'_{\chi'}$ is not σ_2-cut-sparse. Then there exists $U \subseteq V(G') = V \setminus \{v\}$ such that $|U| \geq 3$, U is σ_2-critical in $G'_{\chi'}$, and $\kappa(G'[U]) = 1$. Moreover, there exists $u \in U$ such that $G'[U \setminus \{u\}]$ is not connected, say the components of $G'[U \setminus \{u\}]$ are $G'_1, \ldots, G'_n$, and there exists $1 \leq i \leq n$ such that $\sigma_2(V(G'_i)) = 3$. For all $1 \leq i \leq n$, let $V(G'_i) = U'_i$ and let $U'_i \cup \{u\} = U_i$. Lemma 5.3.1.11 implies that for all $1 \leq i \leq n$, $\sigma_2(U_i) \leq 2$. So, as $\sigma_2(V) = 2$ Lemma 5.3.1.7 gives us that for all $1 \leq i \leq n$, $\sigma_2(U_i) = 2 = \sigma_2(U)$. It follows that, for all $1 \leq i < j \leq n$, U_i is σ_2-critical in $G'_{\chi'}$ and $d_G(U_i, U_j) = 0$.

As G_χ is σ_2-cut-sparse and $E(G') \setminus E = \{xy\}$, $\{x, y\} \subseteq U$ and there exists $1 \leq i \leq n$ such that $\{x, y\} \subseteq U_i$. We may suppose without loss of generality that U_1. If $z \in U_1$ then $G_\chi[U \cup \{v\}]$ is a $(2,1)$-VL-extension of $G'_{\chi'}[U]$ and so Lemma 5.3.2.2 implies that $U \cup \{v\}$ is σ_2-critical in G_χ. However, as $|U \cup \{v\}| \geq 3$ and $N_G(v) \subseteq U_1$ this contradicts the fact that G_χ is σ_2-cut-sparse. Hence $N_G(v) \cap U_1 = \{x, y\}$. So, $|U_1|, |U_2| \geq 3$, $N_G(v) \cap U_1 = \{x, y\}$, $U_1 \cap U_2 = \{u\} = \{v \in V : \sigma_2(V \setminus \{v\}) > \sigma_2(V)\}$, $i_G(U_1) = i_{G'}(U_1) - 1 = 2|U_1| - 3$, and $i_G(U_2) = i_{G'}(U_2) = 2|U_2| - 2$, so U_2 is σ_2-critical in G, and $d_G(U_1, U_2) = 0$. □

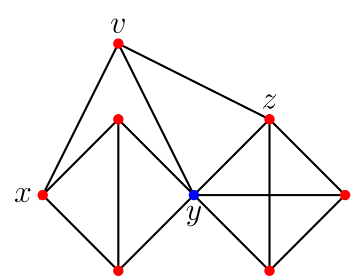
As discussed above, the previous result is a not wholly satisfactory analogue of Lemma

5.5.2.1 and does not give rise to an easy to work with object as was the case with σ_2-critical sets. However, part of the reason for giving Lemma 5.5.3.5 in the form above is that we are interested in σ_2-tight graphs with $\sigma_2(V) = 2$. It is certainly plausible that a full analogue of Lemma 5.5.2.1 (i.e., with no restriction on $\sigma_2(V)$) could be found, although it is unclear how useful such a result would be.



G_χ

Figure 5.5: Illustration of a σ_2-cut-sparse graph, G_χ, such that the $(2,1)$-VL-reduction of G_χ at v adding xy is σ_2-sparse but is not σ_2-cut-sparse.



G_χ

Figure 5.6: Illustration of a σ_2-cut-sparse graph, G_χ, such that the $(2,1)$-VL-reduction of G_χ at v adding xy is σ_2-sparse but is not σ_2-cut-sparse whereas the $(2,1)$-VL-reduction of G_χ at v adding xz is σ_2-cut-sparse

Lemma 5.5.3.6. *Let G_χ be a σ_2-cut-sparse labelled graph. If $\sigma_2(V) \geq 2$ and there exists $v \in V$ such that $d_G(v) = 3$ and $\sigma_2(N_G[v]) = 3$ then there exists a $(2,1)$-VL-reduction of G_χ at v that is σ_2-cut-sparse.*

Proof. If $\sigma_2(V) = 3$ then this follows from Lemma 5.5.2.2 and Lemma 5.5.3.5, so we may suppose that $\sigma_2(V) = 2$. As $d_G(v) = 3$, $|V| \geq 4$ so let $\{w\} = \{u \in V(G) \colon \sigma_2(V \setminus \{u\}) > \sigma_2(V)\}$. As $\sigma_2(N_G(v)) = 3$, $w \notin N_G(v)$. Let $N_G(v) = \{x, y, z\}$, define H to be the complete graph with vertex set $N_G(v)$, and let $F = E(H) \setminus E$. We proceed by considering $i_G(N_G(v))$. As G_χ is σ_2-sparse and $|N_G[v]| = 4$ and $\sigma_2(N_G[v]) = 3$, $i_G(N_G[v]) \leq 2|N_G[v]| - 3 = 5$ and so $i_G(N_G(v)) \leq 2$.

If $i_G(N_G(v)) = 2$ then let $F = \{e\}$. As $\sigma_2(N_G(v)) = 3$, we may suppose without loss of generality that $e = xy$. Suppose there exist $W_1, W_2 \subseteq V \setminus \{v\}$ such that $|W_1|, |W_2| \geq 3$, $N_G(v) \cap W_1 = \{x, y\}$, $W_1 \cap W_2 = \{w\}$, $i_G(W_1) = 2|W_1| - 3$, W_2 is σ_2-critical in G_χ, and $d_G(W_1, W_2) = 0$. As $w \notin N_G(v)$, and $xz, xy \in E$, and $d_G(W_1, W_2) = 0$, $z \notin W_2$. Lemma 1.1.1.4 implies

$$\begin{aligned} i_G(W_1 \cup W_2 \cup \{v, z\}) &= i_G(W_1 \cup W_2) + i_G(\{v, z\}) + d_G(\{v, z\}, W_1 \cup W_2) \\ &= (2|W_1 \cup W_2| - 3) + 5 \\ &= 2|W_1 \cup W_2 \cup \{v, z\}| - 2. \end{aligned}$$

As G_χ is σ_2-sparse it follows that $d_G(\{v, z\}, W_1 \cup W_2) = 4$ and $W_1 \cup W_2 \cup \{v, z\}$ is σ_2-critical in G_χ. Moreover, $\kappa(G[W_1 \cup W_2 \cup \{v, z\}]) = 1$ and $G[(W_1 \cup W_2 \cup \{v, z\}) \setminus \{w\}]$ is not connected. As $G[W_2 \setminus \{w\}]$ is a component of $G[(W_1 \cup W_2 \cup \{v, z\}) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(W_2 \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) \geq 2$. Therefore such sets W_1 and W_2 can not both exist and so Lemma 5.5.2.2 and Lemma 5.5.3.5 together imply there exists a $(2, 1)$-VL-reduction of G_χ at v that is σ_2-cut-sparse.

If $i_G(N_G(v)) = 1$ then let $F = \{e_1, e_2\} = \{xy, xz\}$, so $V(G[F]) = N_G(v)$. By Lemma 5.5.2.2 we may suppose, without loss of generality since $\sigma_2(N_G(v)) = 3$, that there does not exist $W \subseteq V \setminus \{v\}$ such that W is σ_2-critical in G_χ and $W \cap N_G(v) = \{x, y\}$. Let us suppose instead that there exist $W_1, W_2 \subseteq V \setminus \{v\}$ such that $|W_1|, |W_2| \geq 3$, $N_G(v) \cap W_1 = \{x, y\}$, $W_1 \cap W_2 = \{w\}$, $i_G(W_1) = 2|W_1| - 3$, W_2 is σ_2-critical in G_χ, and $d_G(W_1, W_2) = 0$. We now have two cases to consider.

Firstly, suppose there exists $U \subseteq V \setminus \{v\}$ such that $U \cap N_G(v) = \{x, z\}$ and U is

σ_2-critical in G_χ. Then $W_1 \cap U \neq \emptyset$ and Lemma 1.1.1.4 implies

$$\begin{aligned} i_G(W_1 \cup U) &= i_G(W_1) + i_G(U) + d_G(W_1, U) - i_G(W_1 \cap U) \\ &\geq 2(|W_1| + |U|) + 1 - (3 + \sigma_2(U) + i_G(W_1 \cap U) \\ &= 2|W_1 \cup U| + 2|W_1 \cap U| - (2 + \sigma_2(U) + i_G(W_1 \cap U)). \end{aligned}$$

If $|W_1 \cap U| = 1$ then $\sigma_2(U) = 3$ and $i_G(W_1 \cap U) = 2|W_1 \cap U| - 2$, so $i_G(W_1 \cup U) \geq 2|W_1 \cup U| - 3$. If $|W_1 \cap U| \geq 2$ then, as G_χ is σ_2-sparse, $i_G(W_1 \cap U) \leq 2|W_1 \cap U| - \sigma_2(W_1 \cap U)$ and hence Lemma 5.3.1.7 implies $i_G(W_1 \cup U) \geq 2|W_1 \cup U| - 3$. So, regardless of $|W_1 \cap U|$, as $N_G(v) \subseteq W_1 \cup U$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$ and G_χ is σ_2-sparse, Lemma 5.5.1.4 implies $W_1 \cup U$ is not σ_2-critical in G_χ. Therefore $i_G(W_1 \cup U) = 2|W_1 \cup U| - 3$, $d_G(W_1, U) = 1$, and $W_1 \cup U \cup \{v\}$ is σ_2-critical in G_χ.

If $|W_2 \cap (W_1 \cup U \cup \{v\})| \geq 2$ then Lemma 5.5.1.6 implies $W_2 \cap (W_1 \cup U \cup \{v\})$ is σ_2-critical in G_χ and, as $\sigma_2(W_1 \cup U \cup \{v\}) = 2 = \sigma_2(W_2)$, $|W_2 \cap (W_1 \cup U \cup \{v\})| \geq 4$. Hence $|W_2 \cap (U \setminus W_1)| \geq 3$. As $d_G(W_1, U) = 1$, Lemma 5.3.1.8 implies $w \in U$. So $|W_1 \cap U|, |(W_1 \cap U) \cap W_2| \geq 2$. Hence, by previous calculations, $W_1 \cap U$ is σ_2-critical in G_χ and by Lemma 5.5.1.6 $(W_1 \cap U) \cup W_2$ is σ_2-critical in G_χ. Moreover, Lemma 1.1.1.4 gives us that $d_G(W_1 \cap U, W_2) = 0$ and so $\kappa(G[(W_1 \cap U) \cup W_2]) = 1$ and $G[((W_1 \cap U) \cup W_2) \setminus \{w\}]$ is not connected. As $G[W_2 \setminus \{w\}]$ is a component of $G[((W_1 \cap U) \cup W_2) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(W_2 \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) \geq 2$, so we must have $W_2 \cap (W_1 \cup U \cup \{v\}) = \{w\}$.

As $W_2 \cap (W_1 \cup U \cup \{v\}) = \{w\}$, Lemma 1.1.1.4 implies

$$i_G(W_1 \cup U \cup \{v\} \cup W_2) = |W_1 \cup U \cup \{v\}| + d_G(W_1 \cup U \cup \{v\}, W_2) - 2.$$

So, as G_χ is σ_2-sparse, $d_G(W_1 \cup U \cup \{v\}, W_2) = 0$ and $W_1 \cup U \cup \{v\} \cup W_2$ is σ_2-critical in G_χ. Therefore $\kappa(G[W_1 \cup U \cup \{v\} \cup W_2]) = 1$ and $G[(W_1 \cup U \cup \{v\} \cup W_2) \setminus \{w\}]$ is not connected. As $G[W_2 \setminus \{w\}]$ is a component of $G[(W_1 \cup U \cup \{v\} \cup W_2) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(W_2 \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) \geq 2$. Therefore no such set U can exist.

Now let us suppose instead that there exist $A_1, A_2 \subseteq V \setminus \{v\}$ such that $|A_1|, |A_2| \geq 3$, $N_G(v) \cap A_1 = \{x, z\}$, $A_1 \cap A_2 = \{w\}$, $i_G(A_1) = 2|A_1| - 3$, A_2 is σ_2-critical in G_χ, and $d_G(A_1, A_2) = 0$. If $A_2 \cap W_2 = \{w\}$ then Lemma 5.5.1.7 implies $d_G(A_2, W_2) = 0$ and so Lemma 5.5.1.5 implies $A_2 \cup W_2$ is σ_2-critical in G_χ. Therefore $\kappa(G[A_2 \cup W_2]) = 1$, and $G[(A_2 \cup W_2) \setminus \{w\}]$ is not connected. As $G[W_2 \setminus \{w\}]$ is a component of $G[(A_2 \cup W_2) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(W_2 \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) \geq 2$, so we must have $|A_2 \cap W_2| \geq 2$. Therefore, Lemma 5.5.1.6 implies that $A_2 \cap W_2$ is σ_2-critical in G_χ. Let $C = A_2 \cap W_2$ and note that as $\sigma_2(C) = 2$, $|C| \geq 4$.

As $d_G(A_1, W_1) \geq 1$, Lemma 1.1.1.4 implies $i_G(A_1 \cup W_1) \geq 2|A_1 \cup W_1| - 3$. So, as $N_G(v) \subseteq A_1 \cup W_1$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v)$, Lemma 5.5.1.4 implies that $i_G(A_1 \cup W_1) = 2|A_1 \cup W_1| - 3$. Hence $d_G(A_1, W_1) = 1$ and $A_1 \cup W_1 \cup \{v\}$ is σ_2-critical in G_χ. So, as $C \cap A_1 = \{w\} = C \cap W_1$ and $v \notin C$, $\kappa(G[A_1 \cup W_1 \cup \{v\} \cup C]) = 1$ and $G[(A_1 \cup W_1 \cup \{v\} \cup C) \setminus \{w\}]$ is not connected. As $G[C \setminus \{w\}]$ is a component of $G[(A_1 \cup W_1 \cup \{v\} \cup C) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(C \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) \geq 2$. Therefore such sets A_1 and A_2 can not both exist.

So either W_1 and W_2 exist in which case, by the above, Lemma 5.5.2.2 and Lemma 5.5.3.5 together imply the $(2,1)$-VL-reduction of G_χ at v adding xy is σ_2-cut-sparse or W_1 and W_2 do not both exist in which case Lemma 5.5.2.2 and Lemma 5.5.3.5 together imply the $(2,1)$-VL-reduction of G_χ at v adding xz is σ_2-cut-sparse.

If $i_G(N_G(v)) = 0$ then $F = \{xy, xz, yz\}$. By Lemma 5.5.2.2 we may suppose, without loss of generality since $\sigma_2(N_G(v)) = 3$, that there does not exist $W \subseteq V \setminus \{v\}$ such that W is σ_2-critical in G_χ and $W \cap N_G(v) = \{x, y\}$. Let us suppose instead that there exist $W_1, W_2 \subseteq V \setminus \{v\}$ such that $|W_1|, |W_2| \geq 3$, $N_G(v) \cap W_1 = \{x, y\}$, $W_1 \cap W_2 = \{w\}$, $i_G(W_1) = 2|W_1| - 3$, W_2 is σ_2-critical in G_χ, and $d_G(W_1, W_2) = 0$. We now have two cases to consider.

Firstly, suppose there exist $A, B \subseteq V \setminus \{v\}$ such that $A \cap N_G(v) = \{x, z\}$, $B \cap N_G(v) = \{y, z\}$, and A and B are σ_2-critical in G_χ. As $N_G(v) \subseteq A \cup B$ and $\sigma_2(N_G(v)) = \sigma_2(N_G[v])$, Lemma 5.5.1.4 implies $A \cup B$ is not σ_2-critical in G_χ. Consequently, Lemma 5.5.1.6 implies $A \cap B = \{z\}$ and Lemma 5.5.1.7 implies $d_G(A, B) = 0$. As $A \cap B =$

$\{z\}$ and $\sigma_2(V) = 2$, $\max\{\sigma_2(A), \sigma_2(B)\} = 3$ and $\min\{\sigma_2(A), \sigma_2(B)\} = \sigma_2(A \cup B)$. Therefore Lemma 5.5.1.5 implies $i_G(A \cup B) = 2|A \cup B| - (\sigma_2(A \cup B) + 1)$.

As $N_G(v) \cap ((A \cup B) \cap W_1) = \{x, y\}$ and there does not exist $W \subseteq V \setminus \{v\}$ such that W is σ_2-critical in G_χ and $W \cap N_G(v) = \{x, y\}$, $i_G((A \cup B) \cap W_1) \leq 2|(A \cup B) \cap W_1| - (\sigma_2((A \cup B) \cap W_1) + 1)$. Therefore Lemma 5.5.1.5 implies $i_G(A \cup B \cup W_1) \geq 2|A \cup B \cup W_1| - 3$. Moreover, as G_χ is σ_2-sparse and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$ and $N_G(v) \subseteq A \cup B \cup W_1$ it follows that $i_G(A \cup B \cup W_1) = 2|A \cup B \cup W_1| - 3$ and $d_G(A \cup B, W_1) = 0$ and $A \cup B \cup W_1 \cup \{v\}$ is σ_2-critical in G_χ.

If $|W_2 \cap (A \cup B \cup W_1 \cup \{v\})| \geq 2$ then Lemma 5.5.1.6 implies $W_2 \cap (A \cup B \cup W_1 \cup \{v\})$ is σ_2-critical in G_χ. So, as $\sigma_2(A \cup B \cup W_1) = 2 = \sigma_2(W_2)$, $|W_2 \cap (A \cup B \cup W_1 \cup \{v\})| \geq 4$ and hence $|W_2 \cap ((A \cup B) \setminus W_1)| \geq 3$. As $d_G(A \cup B, W_1) = 0$, Lemma 5.3.1.8 implies $w \in A \cup B$. As $A \cap B = \{z\}$ and $d_G(A, B) = 0$, there exists $S \in \{A, B\}$ such that $w \in S$ and $|W_2 \cap S| \geq 2$. We may suppose, without loss of generality since $\sigma_2(N_G(v)) = 3$ that $w \in A$ and $|A \cap W_2| \geq 2$.

As $w \in A$, $|A \cap W_1| \geq 2$ and so Lemma 5.5.1.5 implies $i_G(A \cup W_1) \geq 2|A \cup W_1| - 3$. As G_χ is σ_2-sparse and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$ and $N_G(v) \subseteq A \cup W_1$, Lemma 5.5.1.4 implies $i_G(A \cup W_1) = 2|A \cup W_1| - 3$ and therefore $A \cap W_1$ is σ_2-critical in G_χ. Now, Lemma 5.5.1.5 implies $i_G((W_1 \cap A) \cup W_2) = 2|(W_1 \cap A) \cup W_2| - 2 + d_G(W_1 \cap A, W_2)$. So, as G_χ is σ_2-sparse, $(W_1 \cap A) \cup W_2$ is σ_2-critical in G_χ and $d_G(W_1 \cap A, W_2) = 0$. Moreover, $\kappa(G[(W_1 \cap U) \cup W_2]) = 1$, and $G[((W_1 \cap A) \cup W_2) \setminus \{w\}]$ is not connected. As $G[W_2 \setminus \{w\}]$ is a component of $G[((W_1 \cap A) \cup W_2) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(W_2 \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) \geq 2$, so we must have that $W_2 \cap (W_1 \cup A \cup B \cup \{v\}) = \{w\}$. Now, Lemma 5.5.1.5 implies $i_G(W_1 \cup A \cup B \cup \{v\} \cup W_2) = 2|W_1 \cup A \cup B \cup \{v\}| + d_G(W_1 \cup A \cup B \cup \{v\}, W_2) - 2$. So, as G_χ is σ_2-sparse, $W_1 \cup A \cup B \cup \{v\} \cup W_2$ is σ_2-critical in G_χ and $d_G(W_1 \cup A \cup B \cup \{v\}, W_2) = 0$. Moreover, $\kappa(G[(W_1 \cup A \cup B \cup \{v\} \cup W_2]) = 1$, and $G[(W_1 \cup A \cup B \cup \{v\} \cup W_2) \setminus \{w\}]$ is not connected. As $G[W_2 \setminus \{w\}$ is a component of $G[(W_1 \cup A \cup B \cup \{v\} \cup W_2) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(W_2 \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) \geq 2$. Therefore such sets A and B can not both exist.

Alternatively, suppose there exist $B_1, B_2 \subseteq V \setminus \{v\}$ such that $|B_1|, |B_2| \geq 3$, $N_G(v) \cap B_1 = \{y, z\}$, $B_1 \cap B_2 = \{w\}$, $i_G(B_1) = 2|B_1| - 3$, B_2 is σ_2-critical in G_χ, and $d_G(B_1, B_2) = 0$. If $B_2 \cap W_2 = \{w\}$ then Lemma 5.5.1.7 implies $d_G(B_2, W_2) = 0$ and so Lemma 5.5.1.5 implies $B_2 \cup W_2$ is σ_2-critical in G_χ. Therefore $\kappa(G[B_2 \cup W_2]) = 1$, and $G[(B_2 \cup W_2) \setminus \{w\}]$ is not connected. As $G[W_2 \setminus \{w\}]$ is a component of $G[(B_2 \cup W_2) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(W \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) \geq 2$, so we must have $|B_2 \cap W_2| \geq 2$. Therefore, Lemma 5.5.1.6 implies that $B_2 \cap W_2$ is σ_2-critical in G_χ. Let $C = B_2 \cap W_2$ and note that as $\sigma_2(C) = 2$, $|C| \geq 4$.

If $B_1 \cap W_1$ is σ_2-critical in G_χ then, as $\sigma_2(B_1 \cap W_1) = 2$, $|B_1 \cap W_1| \geq 4$ and, as $(B_1 \cap W_1) \cap C = \{w\}$, Lemma 5.5.1.5 implies $(B_1 \cap W_1) \cup C$ is σ_2-critical in G_χ and $d_G(B_1 \cap W_1, C) = 0$. Therefore $\kappa(G[(B_1 \cap W_1) \cup C]) = 1$ and $G[((B_1 \cap W_1) \cup C) \setminus \{w\}]$ is not connected. As $G[C \setminus \{w\}]$ is a component of $G[((B_1 \cap W_1) \cup C) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(C \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) \geq 2$, so we must have that $B_1 \cap W_1$ is not σ_2-critical in G_χ. Hence, as $|B_1 \cap W_1| \geq 2$, $i_G(B_1 \cap W_1) \leq 2|B_1 \cap W_1| - 3$. As $N_G(v) \subseteq B_1 \cup W_1$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$, Lemma 5.5.1.4 implies $B_1 \cup W_1$ is not σ_2-critical in G_χ and so $i_G(B_1 \cup W_1) \leq 2|B_1 \cup W_1| - 3$. Therefore Lemma 1.1.1.4 implies

$$\begin{aligned} i_G(B_1 \cup W_1) + i_G(B_1 \cap W_1) &\leq (2|B_1 \cup W_1| - 3) + (2|B_1 \cap W_1|) - 3 \\ &= (2|B_1| - 3) + (2|W_1| - 3) \\ &= i_G(B_1) + i_G(W_1) \\ &= i_G(B_1 \cup W_1) + i_G(B_1 \cap W_1) - d_G(B_1, W_1). \end{aligned}$$

Consequently, $d_G(B_1, W_1) = 0$, $i_G(B_1 \cup W_1) = 2|B_1 \cup W_1| - 3$, $i_G(B_1 \cap W_1) = 2|B_1 \cap W_1| - 3$, and $B_1 \cup W_1 \cup \{v\}$ is σ_2-critical in G_χ.

As $(B_1 \cup W_1 \cup \{v\}) \cap C = \{w\}$, Lemma 5.5.1.7 implies $d_G(B_1 \cup W_1 \cup \{v\}, C) = 0$ and so Lemma 5.5.1.5 implies $B_1 \cup W_1 \cup \{v\} \cup C$ is σ_2-critical in G_χ. Therefore $\kappa(G[B_1 \cup W_1 \cup \{v\} \cup C]) = 1$ and $G[(B_1 \cup W_1 \cup \{v\} \cup C) \setminus \{w\}]$ is not connected. As $G[C \setminus \{w\}]$ is a component of $G[(B_1 \cup W_1 \cup \{v\} \cup C) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(C \setminus \{w\}) \leq 2$. However, this contradicts the fact that $\sigma_2(V) \geq 2$. Therefore such sets B_1 and B_2 can not both exist.

So either W_1 and W_2 exist in which case, by the above, Lemma 5.5.2.2 and Lemma 5.5.3.5 together imply the $(2,1)$-VL-reduction of G_χ at v adding xy is σ_2-cut-sparse or W_1 and W_2 do not both exist in which case Lemma 5.5.2.2 and Lemma 5.5.3.5 together imply the $(2,1)$-VL-reduction of G_χ at v adding xz is σ_2-cut-sparse or the $(2,1)$-VL-reduction of G_χ at v adding yz is σ_2-cut-sparse. □

Lemma 5.5.3.7. *Let G_χ be a σ_2-cut-sparse labelled graph. If $\sigma_2(V) = 2$ and there exists $v \in V$ such that $d_G(v) = 3$ and $\sigma_2(N_G[v]) = 2 = \sigma_2(N_G(v))$ then there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-cut-sparse if and only if there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse.*

Proof. If there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse then there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-cut-sparse. On the other hand, let us suppose there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-cut-sparse. As $d_G(v) = 3$, $|V| \geq 4$ so we set $\{w\} = \{u \in V(G) \colon \sigma_2(V \setminus \{u\}) > \sigma_2(V)\}$. As $\sigma_2(N_G(v)) = 2$, $w \in N_G(v)$ so let $N_G(v) = \{w, x, y\}$. Define H to be the complete graph with vertex set $N_G(v)$, and let $F = E(H) \setminus E$. We proceed by considering $i_G(N_G(v))$. If $i_G(N_G(v)) = 3$ then Lemma 5.5.2.3 implies there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse.

If $i_G(N_G(v)) = 2$ then let $F = \{e\}$. Suppose there exist $W_1, W_2 \subseteq V \setminus \{v\}$ such that $|W_1|, |W_2| \geq 3$, $N_G(v) \cap W_1 = \{\text{endpoints of } e\}$, $W_1 \cap W_2 = \{w\}$, $i_G(W_1) = 2|W_1| - 3$, W_2 is σ_2-critical in G_χ, and $d_G(W_1, W_2) = 0$. We may suppose, without loss of generality since $\sigma_2(N_G(v)) = 2 = \sigma_2(V)$, that $e = wx$. As $wy, xy \in E$, and $d_G(W_1, W_2) = 0$, $y \notin W_2$. Lemma 1.1.1.4 implies

$$\begin{aligned} i_G(W_1 \cup W_2 \cup \{v, y\}) &= i_G(W_1 \cup W_2) + i_G(\{v, y\}) + d_G(\{v, y\}, W_1 \cup W_2) \\ &\geq (2|W_1 \cup W_2| - 3) + 5 \\ &= 2|W_1 \cup W_2 \cup \{v, y\}| - 2. \end{aligned}$$

As G_χ is σ_2-sparse it follows that $d_G(\{v,y\}, W_1 \cup W_2) = 4$ and $W_1 \cup W_2 \cup \{v,y\}$ is σ_2-critical in G_χ. Moreover, $\kappa(G[W_1 \cup W_2 \cup \{v,y\}]) = 1$ and $G[(W_1 \cup W_2 \cup \{v,y\}) \setminus \{w\}]$ is not connected. As $G[W_2 \setminus \{w\}]$ is a component of $G[(W_1 \cup W_2 \cup \{v,y\}) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(W_2 \setminus \{w\}) \leq 2$. However, this contradicts the fact that $\sigma_2(V) \geq 2$. Therefore such sets W_1 and W_2 can not both exist. Lemma 5.5.3.5 now implies that there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-sparse.

If $i_G(N_G(v)) = 1$ then let $F = \{e_1, e_2\}$, so $V(G[F]) = N_G(v)$. By Lemma 5.5.2.3 we may suppose, without loss of generality since $\sigma_2(N_G(v)) = 2 = \sigma_2(V)$, that $wx \in F$ and there does not exist $W \subseteq V \setminus \{v\}$ such that W is σ_2-critical in G_χ and $W \cap N_G(v) = \{w,x\}$. Let us suppose instead that there exist $W_1, W_2 \subseteq V \setminus \{v\}$ such that $|W_1|, |W_2| \geq 3$, $N_G(v) \cap W_1 = \{w,x\}$, $W_1 \cap W_2 = \{w\}$, $i_G(W_1) = 2|W_1| - 3$, W_2 is σ_2-critical in G_χ, and $d_G(W_1, W_2) = 0$. We now have three cases to consider.

Firstly, suppose $F = \{wx, xy\}$. Suppose there exists $U \subseteq V \setminus \{v\}$ such that $U \cap N_G(v) = \{x,y\}$ and U is σ_2-critical in G_χ. Then $W_1 \cap U \neq \emptyset$ and Lemma 1.1.1.4 implies

$$\begin{aligned}
i_G(W_1 \cup U) &= i_G(W_1) + i_G(U) + d_G(W_1, U) - i_G(W_1 \cap U) \\
&\geq 2(|W_1| + |U|) + 1 - (3 + \sigma_2(U) + i_G(W_1 \cap U) \\
&= 2|W_1 \cup U| + 2|W_1 \cap U| - (2 + \sigma_2(U) + i_G(W_1 \cap U)).
\end{aligned}$$

If $|W_1 \cap U| = 1$ then $\sigma_2(U) = 3$ and $i_G(W_1 \cap U) = 2|W_1 \cap U| - 2$, so $i_G(W_1 \cup U) \geq 2|W_1 \cup U| - 3$. If $|W_1 \cap U| \geq 2$ then, as G_χ is σ_2-sparse, $i_G(W_1 \cap U) \leq 2|W_1 \cap U| - \sigma_2(W_1 \cap U)$ and hence Lemma 5.3.1.7 implies $i_G(W_1 \cup U) \geq 2|W_1 \cup U| - 3$. So, regardless of $|W_1 \cap U|$, as $N_G(v) \subseteq W_1 \cup U$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$ and G_χ is σ_2-sparse, Lemma 5.5.1.4 implies $W_1 \cup U$ is not σ_2-critical in G_χ. Therefore $i_G(W_1 \cup U) = 2|W_1 \cup U| - 3$, $d_G(W_1, U) = 1$, and $W_1 \cup U \cup \{v\}$ is σ_2-critical in G_χ.

If $|W_2 \cap (W_1 \cup U \cup \{v\})| \geq 2$ then Lemma 5.5.1.6 implies $W_2 \cap (W_1 \cup U \cup \{v\})$ is σ_2-critical in G_χ and, as $\sigma_2(W_1 \cup U \cup \{v\}) = 2 = \sigma_2(W_2)$, $|W_2 \cap (W_1 \cup U \cup \{v\})| \geq 4$. Hence $|W_2 \cap (U \setminus W_1)| \geq 3$. As $d_G(W_1, U) = 1$ and $w \notin U$ we have that $d_{G[W_2 \cap (W_1 \cup U \cup \{v\})]}(w) \leq 1$. However this contradicts Lemma 5.3.1.8, so we must have $W_2 \cap (W_1 \cup U \cup \{v\}) = \{w\}$.

As $W_2 \cap (W_1 \cup U \cup \{v\}) = \{w\}$, Lemma 1.1.1.4 implies

$$i_G(W_1 \cup U \cup \{v\} \cup W_2) = |W_1 \cup U \cup \{v\}| + d_G(W_1 \cup U \cup \{v\}, W_2) - 2.$$

So, as G_χ is σ_2-sparse, $d_G(W_1 \cup U \cup \{v\}, W_2) = 0$ and $W_1 \cup U \cup \{v\} \cup W_2$ is σ_2-critical in G_χ. Therefore $\kappa(G[W_1 \cup U \cup \{v\} \cup W_2]) = 1$ and $G[(W_1 \cup U \cup \{v\} \cup W_2) \setminus \{w\}]$ is not connected. As $G[W_2 \setminus \{w\}]$ is a component of $G[(W_1 \cup U \cup \{v\} \cup W_2) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(W_2 \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) \geq 2$. Therefore no such set U can exist.

So either W_1 and W_2 exist in which case, by above, Lemma 5.5.3.5 and Lemma 5.5.2.1 together imply that the $(2,1)$-VL-reduction of G_χ at v adding xy is σ_2-cut-sparse, or W_1 and W_2 do not exist in which case Lemma 5.5.3.5 and Lemma 5.5.2.1 together imply that the $(2,1)$-VL-reduction of G_χ at v adding wx is σ_2-cut-sparse. Either way we contradict that there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-cut-sparse.

On the other hand, suppose $F = \{wx, wy\}$. Then two cases remain to be considered. Firstly, suppose there exists $U \subseteq V \setminus \{v\}$ such that $U \cap N_G(v) = \{w, y\}$ and U is σ_2-critical in G_χ. Then $W_1 \cap U \neq \emptyset$ and Lemma 1.1.1.4 implies

$$\begin{aligned} i_G(W_1 \cup U) &= i_G(W_1) + i_G(U) + d_G(W_1, U) - i_G(W_1 \cap U) \\ &\geq 2(|W_1| + |U|) + 1 - (3 + 2 + i_G(W_1 \cap U) \\ &= 2|W_1 \cup U| + 2|W_1 \cap U| - (4 + i_G(W_1 \cap U)). \end{aligned}$$

If $|W_1 \cap U| = 1$ then $i_G(W_1 \cup U) \geq 2|W_1 \cup U| - 2$. If $|W_1 \cap U| \geq 2$ then $\sigma_2(W_1 \cap U) = 2$ and $i_G(W_1 \cap U) \leq 2|W_1 \cap U| - 2$, so $i_G(W_1 \cup U) \geq 2|W_1 \cup U| - 2$. So, regardless of $|W_1 \cap U|$, as G_χ is σ_2-sparse we have that $W_1 \cup U$ is σ_2-critical in G_χ. However, as $N_G(v) \subseteq W_1 \cup U$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$ this contradicts Lemma 5.5.1.4. Therefore no such set U can exist.

Alternatively, suppose there exist $A_1, A_2 \subseteq V \setminus \{v\}$ such that $|A_1|, |A_2| \geq 3$, $N_G(v) \cap A_1 = \{w, y\}$, $A_1 \cap A_2 = \{w\}$, $i_G(A_1) = 2|A_1| - 3$, A_2 is σ_2-critical in G_χ, and $d_G(A_1, A_2) = 0$. If $A_2 \cap W_2 = \{w\}$ then Lemma 5.5.1.7 implies $d_G(A_2, W_2) = 0$ and so

Lemma 5.5.1.5 implies $A_2 \cup W_2$ is σ_2-critical in G_χ. Therefore $\kappa(G[A_2 \cup W_2]) = 1$ and $G[(A_2 \cup W_2) \setminus \{w\}]$ is not connected. As $G[W_2 \setminus \{w\}]$ is a component of $G[(A_2 \cup W_2) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(W_2 \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) = 2$, so we must have $|A_2 \cap W_2| \geq 2$. Therefore Lemma 5.5.1.6 implies that $A_2 \cap W_2$ is σ_2-critical in G_χ. Let $C = A_2 \cap W_2$ and note that as $\sigma_2(C) = 2$, $|C| \geq 4$.

As $d_G(A_1, W_1) \geq 1$, Lemma 1.1.1.4 implies $i_G(A_1 \cup W_1) \geq 2|A_1 \cup W_1| - 3$. So, as $N_G(v) \subseteq A_1 \cup W_1$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$, Lemma 5.5.1.4 implies that $i_G(A_1 \cup W_1) = 2|A_1 \cup W_1| - 3$. Hence $d_G(A_1, W_1) = 1$ and $A_1 \cup W_1 \cup \{v\}$ is σ_2-critical in G_χ. So, as $C \cap A_1 = \{w\} = C \cap W_1$ and $v \notin C$, $\kappa(G[A_1 \cup W_1 \cup \{v\} \cup C]) = 1$ and $G[(A_1 \cup W_1 \cup \{v\} \cup C) \setminus \{w\}]$ is not connected. As $G[C \setminus \{w\}]$ is a component of $G[(A_1 \cup W_1 \cup \{v\} \cup C) \setminus \{w\}$ and G_χ is σ_2-cut-sparse, $\sigma_2(C \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) = 2$. Therefore such sets A_1 and A_2 can not both exist.

So either W_1 and W_2 exist in which case, by above, Lemma 5.5.3.5 and Lemma 5.5.2.1 together imply that the $(2,1)$-VL-reduction of G_χ at v adding wy is σ_2-cut-sparse, or W_1 and W_2 do not exist in which case Lemma 5.5.3.5 and Lemma 5.5.2.1 together imply that the $(2,1)$-VL-reduction of G_χ at v adding wx is σ_2-cut-sparse. Either way we contradict that there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-cut-sparse.

If $i_G(N_G(v)) = 0$ then $F = \{wx, wy, xy\}$. By Lemma 5.5.2.3 and Lemma 5.5.3.5 as there does not exist a $(2,1)$-VL-reduction of G_χ that is σ_2-cut-sparse we may suppose, without loss of generality since $\sigma_2(N_G(v)) = 2 = \sigma_2(V)$, that there exist $W_1, W_2 \subseteq V \setminus \{v\}$ such that $|W_1|, |W_2| \geq 3$, $N_G(v) \cap W_1 = \{w, x\}$, $W_1 \cap W_2 = \{w\}$, $i_G(W_1) = 2|W_1| - 3$, W_2 is σ_2-critical in G_χ, and $d_G(W_1, W_2) = 0$. Similarly, we may suppose there exists $A \subseteq V \setminus \{v\}$ such that $A \cap N_G(v) = \{x, y\}$ and A is σ_2-critical in G_χ. Note that $\sigma_2(A) = 3$. We now have two cases to consider.

Firstly, suppose there exists $B \subseteq V \setminus \{v\}$ such that $B \cap N_G(v) = \{w, y\}$ and B is

σ_2-critical in G_χ. Note that $\sigma_2(B) = 2$. Then $B \cap W_1 \neq \emptyset$ and Lemma 1.1.1.4 implies

$$\begin{aligned} i_G(B \cup W_1) &= i_G(B) + i_G(W_1) + d_G(B, W_1) - i_G(B \cap W_1) \\ &\geq (2|B| - 2) + (2|W_1| - 3) - (2|B \cap W_1| - 2) \\ &= 2|B \cup W_1| - 3. \end{aligned}$$

As $N_G(v) \subseteq B \cup W_1$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$, Lemma 5.5.1.4 implies $B \cup W_1$ is not σ_2-critical in G_χ, so $i_G(B \cup W_1) = 2|B \cup W_1| - 3$. Hence $d_G(B, W_1) = 0$ and $i_G(B \cap W_1) = 2|B \cap W_1| - 2$. As $|(B \cup W_1) \cap A| \geq 2$, another application of Lemma 1.1.1.4 give us that

$$\begin{aligned} i_G(B \cup W_1 \cup A) &= i_G(B \cup W_1) + i_G(A) + d_G(B \cup W_1, A) - i_G((B \cup W_1) \cap A) \\ &\geq (2|B \cup W_1| - 3) + 2|A| - 3 - (2|(B \cup W_1) \cap A| - 3) \\ &= 2|B \cup W_1 \cup A| - 3. \end{aligned}$$

Similarly to above, as $N_G(v) \subseteq B \cup W_1 \cup A$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$, Lemma 5.5.1.4 implies $B \cup W_1 \cup A$ is not σ_2-critical in G_χ, so $i_G(B \cup W_1 \cup A) = 2|B \cup W_1 \cup A| - 3$. Hence $d_G(B \cup W_1, A) = 0$ and $(B \cup W_1) \cap A$ is σ_2-critical in G_χ.

As $\sigma_2(N_G(v)) \subseteq A \cup B$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$, Lemma 5.5.1.4 implies $A \cup B$ is not σ_2-critical in G_χ. Consequently, Lemma 5.5.1.6 implies $A \cap B = \{y\}$ and Lemma 5.5.1.7 implies $d_G(A, B) = 0$. Therefore Lemma 5.5.1.5 implies $i_G(A \cup B) = 2|A \cup B| - 3$. So $(B \cup W_1) \cap A$ is σ_2-critical in G_χ and $B \cap A = \{y\}$. Therefore $(B \cup W_1) \cap A = \{y\} \cup (W_1 \cap A)$ and, as $xy \notin E$, $|(B \cup W_1) \cap B| \geq 4$ which implies $|W_1 \cap A| \geq 3$. As $d_G(B, W_1) = 0$, it follows that $d_{G[(B \cup W_1) \cap A]}(y) = 0$. However this contradicts Lemma 5.3.1.8. Therefore no such set B can exist.

Alternatively, suppose there exist $B_1, B_2 \subseteq V \setminus \{v\}$ such that $|B_1|, |B_2| \geq 3$, $N_G(v) \cap B_1 = \{w, y\}$, $B_1 \cap B_2 = \{w\}$, $i_G(B_1) = 2|B_1| - 3$, B_2 is σ_2-critical in G_χ, and $d_G(B_1, B_2) = 0$. If $B_2 \cap W_2 = \{w\}$ then Lemma 5.5.1.7 implies $d_G(B_2, W_2) = 0$ and so Lemma 5.5.1.5 implies $B_2 \cup W_2$ is σ_2-critical in G_χ. Therefore $\kappa(G[B_2 \cup W_2]) = 1$ and $G[(B_2 \cup W_2) \setminus \{w\}]$ is not connected. As $G[W_2 \setminus \{w\}]$ is a component of $G[(B_2 \cup W_2) \setminus \{w\}]$

and G_χ is σ_2-cut-sparse, $\sigma_2(W_2 \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) = 2$, so we must have $|B_2 \cap W_2| \geq 2$. Therefore Lemma 5.5.1.6 implies that $B_2 \cap W_2$ is σ_2-critical in G_χ. Let $C = B_2 \cap W_2$ and note that as $\sigma_2(C) = 2$, $|C| \geq 4$.

If $A \cap (B_1 \cup W_1)$ is σ_2-critical in G_χ then, as $xy \notin E$, $|A \cap (B_1 \cup W_1)| \geq 4$. So we may suppose, without loss of generality since $\sigma_2(N_G(v)) = 2$, that $|A \cap B_1| \geq 2$. As $N_G(v) \subseteq A \cup B_1$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$, Lemma 5.5.1.4 implies $A \cup B_1$ is not σ_2-critical in G_χ. Therefore $i_G(A \cup B_1) \leq 2|A \cup B_1| - 3$. So, Lemma 1.1.1.4 implies

$$\begin{aligned} 2|A| - 3 + 2|B_1| - 3 + d_G(A, B_1) &= i_G(A \cup B_1) + i_G(A \cap B_1) \\ &\leq (2|A \cup B_1| - 3) + (2|A \cap B_1| - 3) \\ &= (2|A| - 3) + (2|B_1| - 3). \end{aligned}$$

Therefore, as G_χ is σ_2-sparse, $d_G(A, B_1) = 0$ and $i_G(A \cup B_1) = 2|A \cup B_1| - 3$ and $A \cup B_1 \cup \{v\}$ is σ_2-critical in G_χ.

As $A \cup B_1 \cup \{v\}$ and C are σ_2-critical in G_χ and $(A \cup B_1 \cup \{v\}) \cap C \neq \emptyset$, and G_χ is σ_2-sparse, Lemma 1.1.1.4 implies

$$\begin{aligned} (2|A \cup B_1 \cup \{v\}| - 2) + (2|C| - 2) &= i_G(A \cup B_1 \cup \{v\} \cup C) \\ &\quad + i_G((A \cup B_1 \cup \{v\}) \cap C) \\ &\leq (2|A \cup B_1 \cup \{v\} \cup C| - 2) \\ &\quad + (2|(A \cup B_1 \cup \{v\}) \cap C| - 2) \\ &= (2|A \cup B_1 \cup \{v\}| - 2) + (2|C| - 2). \end{aligned}$$

Therefore $d_G(A \cup B_1 \cup \{v\}, C) = 0$, $A \cup B_1 \cup \{v\} \cup C$ is σ_2-critical in G_χ and $i_G((A \cup B_1 \cup \{v\}) \cap C) = (2|(A \cup B_1 \cup \{v\}) \cap C| - 2)$.

If $|(A \cup B_1 \cup \{v\}) \cap C| \geq 2$ then $(A \cup B_1 \cup \{v\}) \cap C$ is σ_2-critical in G_χ and $|(A \cup B_1 \cup \{v\}) \cap C| \geq 4$. Now, $(A \cup B_1 \cup \{v\}) \cap C = (A \cap C) \cup \{w\}$, so $|A \cap C| \geq 3$ and hence Lemma 5.5.1.6 implies $d_G(A, C) = 0$. It follows that $d_{G[(A \cup B_1 \cup \{v\}) \cap C]}(w) = 0$. However this contradicts Lemma 5.3.1.8. So $(A \cup B_1 \cup \{v\}) \cap C = \{w\}$. Therefore

$\kappa(G[A \cup B_1 \cup \{v\} \cup C]) = 1$ and $G[(A \cup B_1 \cup \{v\} \cup C) \setminus \{w\}]$ is not connected. As $G[C \setminus \{w\}]$ is a component of $G[(A \cup B_1 \cup \{v\} \cup C) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(C \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) = 2$. Therefore $A \cap (B_1 \cup W_1)$ is not σ_2-critical in G_χ, so $i_G(A \cap (B_1 \cup W_1)) \leq 2|A \cap (B_1 \cup W_1)| - 4$.

As $N_G(v) \subseteq B_1 \cup W_1$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$, Lemma 5.5.1.4 implies $B_1 \cup W_1$ is not σ_2-critical in G_χ. Therefore $i_G(B_1 \cup W_1) \leq 2|B_1 \cup W_1| - 3$. Also, Lemma 1.1.1.4 gives us that

$$\begin{aligned} i_G(B_1 \cup W_1) &= i_G(B_1) + i_G(W_1) + d_G(B_1, W_1) - i_G(B_1 \cap W_1) \\ &\geq 2(|B_1| + |W_1|) - (6 + i_G(B_1 \cap W_1) \\ &\geq 2|B_1 \cup W_1| - 4. \end{aligned}$$

So $2|B_1 \cup W_1| - 4 \leq i_G(B_1 \cup W_1) \leq 2|B_1 \cup W_1| - 3$.

As $N_G(v) \subseteq A \cup B_1 \cup W_1$ and $\sigma_2(N_G[v]) = \sigma_2(N_G(v))$, Lemma 5.5.1.4 implies $A \cup B_1 \cup W_1$ is not σ_2-critical in G_χ. Therefore $i_G(A \cup B_1 \cup W_1) \leq 2|B_1 \cup W_1| - 2$. Combining the information we have so far, Lemma 1.1.1.4 implies

$$\begin{aligned} (2|A| - 3) + (2|B_1 \cup W_1| - 4) &= (2|A \cup B_1 \cup W_1| - 3) + (2|A \cap (B_1 \cup W_1)| - 4) \\ &\geq i_G(A \cup B_1 \cup W_1) + i_G(A \cap (B_1 \cup W_1)) \\ &\geq (2|A| - 3) + (2|B_1 \cup W_1| - 4) + d_G(A, B_1 \cup W_1) \end{aligned}$$

Therefore $d_G(A, B_1 \cup W_1) = 0$ and $i_G(A \cup B_1 \cup W_1) = 2|A \cup B_1 \cup W_1| - 3$ and $i_G(A \cap (B_1 \cup W_1)) = 2|A \cap (B_1 \cup W_1)| - 4$ and $A \cup B_1 \cup W_1 \cup \{v\}$ is σ_2-critical in G_χ.

If $|(A \cup B_1 \cup W_1 \cup \{v\}) \cap C| \geq 2$ then $(A \cup B_1 \cup W_1 \cup \{v\}) \cap C$ is σ_2-critical in G_χ and $|(A \cup B_1 \cup W_1 \cup \{v\}) \cap C| \geq 4$. Now, $(A \cup B_1 \cup W_1 \cup \{v\}) \cap C = (A \cap C) \cup \{w\}$, so $|A \cap C| \geq 3$ and hence Lemma 5.5.1.6 implies $d_G(A, C) = 0$. It follows that $d_{G[(A \cup B_1 \cup W_1 \cup \{v\}) \cap C]}(w) = 0$. However this contradicts Lemma 5.3.1.8. So $(A \cup B_1 \cup W_1 \cup \{v\}) \cap C = \{w\}$. Therefore $\kappa(G[A \cup B_1 \cup W_1 \cup \{v\} \cup C]) = 1$ and $G[(A \cup B_1 \cup W_1 \cup \{v\} \cup C) \setminus \{w\}]$ is not connected. As $G[C \setminus \{w\}]$ is a component of $G[(A \cup B_1 \cup W_1 \cup \{v\} \cup C) \setminus \{w\}]$ and G_χ is σ_2-cut-sparse, $\sigma_2(C \setminus \{w\}) \leq 2$. However this contradicts the fact that $\sigma_2(V) = 2$.

Therefore such set B_1 and B_2 can not both exist.

So either W_1 and W_2 and A exist in which case, by above, Lemma 5.5.3.5 and Lemma 5.5.2.1 together imply that the $(2,1)$-VL-reduction of G_χ at v adding wy is σ_2-cut-sparse, or W_1 and W_2 do not exist in which case Lemma 5.5.3.5 and Lemma 5.5.2.1 together imply that the $(2,1)$-VL-reduction of G_χ at v adding wx is σ_2-cut-sparse or A does not exist in which case Lemma 5.5.3.5 and Lemma 5.5.2.1 together imply that the $(2,1)$-VL-reduction of G_χ at v adding xy is σ_2-cut-sparse. Either way we contradict that there does not exist a $(2,1)$-VL-reduction of G_χ at v that is σ_2-cut-sparse. □

Note that neither of the previous two results say that if a $(2,1)$-VL-reduction of a σ_2-cut-sparse graph is σ_2-sparse then that $(2,1)$-VL-reduction is σ_2-cut-sparse. This distinction is illustrated in Figure 5.6

Lemma 5.5.3.8. *Let G_χ be a labelled graph. Suppose that G_χ is σ_2-cut-tight and $\sigma_2(V) = 2$. There exists a $(2,0)$-VL-reduction or a $(2,1)$-VL-reduction of G_χ that is σ_2-cut-tight if and only if $|V| \geq 5$.*

Proof. Suppose there exists a $(2,0)$-VL-reduction or a $(2,1)$-VL-reduction of G_χ that is σ_2-cut-tight. This implies $\Delta(G) \geq 2$, so $|V| \geq 3$. As G_χ is σ_2-tight and $|V| \geq 3$ and $\sigma_2(V) = 2$, Lemma 5.3.1.5 and Lemma 5.3.1.6 together imply $|E| = S(G_\chi, \sigma_2) = 2|V| - 2$. Theorem 1.1.1.6 now implies $|V| \geq 4$, and as there exists a $(2,0)$-VL-reduction or a $(2,1)$-VL-reduction of G_χ that is σ_2-cut-tight we note that G is not a complete graph so in fact $|V| \geq 5$.

On the other hand suppose that $|V| \geq 5$. As $\sigma_2(V) = 2$, $|\{u \in V(G) : \sigma_2(V \setminus \{u\}) > \sigma_2(V)\}| = 1$, so we set $\{u \in V(G) : \sigma_2(V \setminus \{u\}) > \sigma_2(V)\} = \{w\}$. If $\delta(G) \neq 3$ then Lemma 5.3.1.8 implies that $\delta(G) = 2$. Then there exists a $(2,0)$-VL-reduction of G_χ and Lemma 5.5.3.3 implies this is σ_2-cut-tight. So we may suppose that $\delta(G) = 3$ then take $v \in V$ such that $d_G(v) = 3$ and $\sigma_2(N_G[v]) = \max\{\sigma_2(N_G[u]) : u \in V \text{ and } d_G(u) = 3\}$. As $\sigma_2(V) = 2$, $\sigma_2(N_G[v]) \in \{2,3\}$. If $\sigma_2(N_G[v]) = 3$ then Lemma 5.5.3.6 and Lemma 5.3.2.4 together imply there exists a $(2,1)$-VL-reduction of G_χ that is σ_2-cut-tight.

If $\sigma_2(N_G[v]) \neq 3$ then, as $\sigma_2(N_G[v]) = \max\{\sigma_2(N_G[u]) : u \in V \text{ and } d_G(u) = 3\}$,

$\sigma_2(N_G[u]) = 2$ for all $u \in V$ such that $d_G(u) = 3$. Let $U = \{u \in V : d_G(u) = 3\}$ and let $U' = \{u \in U : \sigma_2(N_G(u)) = 2\} = U \setminus \{w\}$. Note that $w \in N_G(u)$ for all $u \in U'$, so $d_G(w) \geq |U'|$. As $|E| = 2|V| - 2$, Theorem 1.1.1.6 implies $|U| \geq 4$ and

$$4|V|-4 = \sum_{v \in V} d_G(v) \geq d_G(w)+3|U'|+4(|V|-(|U'|+1)) = (4|V|-4)+(d_G(w)-|U'|).$$

Hence $d_G(w) \leq |U'|$ and so $d_G(w) = |U'|$. Take $u \in U'$ and let $N_G(u) = \{w, x, y\}$. If $xw \notin E$ or $yw \notin E$ then Lemma 5.5.3.7, Lemma 5.5.2.3, and Lemma 5.3.2.4 together imply there exists a $(2,1)$-VL-reduction of G_χ that is σ_2-cut-tight. So we may suppose that $\{xw, yw\} \subseteq E$ and hence $d_G(x) = 3 = d_G(y)$. If $xy \notin E$ then Lemma 5.3.1.8 implies there does not exist $U \subseteq V \setminus \{v\}$ such that $U \cap N_G(v) = \{x, y\}$, U is σ_2-critical in G_χ and $\sigma_2(U) = 3$. Then Lemma 5.5.3.7, Lemma 5.5.2.3, and Lemma 5.3.2.4 together imply there exists a $(2,1)$-VL-reduction of G_χ that is σ_2-cut-tight. If $xy \in E$ then $G[N_G[u]] \cong K_4$. As G_χ is σ_2-cut-tight and u was chosen arbitrarily, it follows that $G = G[N_G[u]]$. However, then $|V| = 4$ which is a contradiction. So if $\delta(G) = 3$ then there exists a $(2,1)$-VL-reduction of G_χ that is σ_2-cut-tight. □

We are now able to state and prove the main theorem of this section. This is an analogue of Proposition 5.4.0.3 where we replace σ_1-tight with σ_2-cut-tight, although we also have the additional constraint that $\sigma_2(V) = 2$.

Theorem 5.5.3.9. *Let $G_\chi = (V, E)_\chi$ be a labelled graph. The following are equivalent:*

(i) $\sigma_2(V) = 2$ and G_χ is σ_2-cut-tight; and

(ii) there exists $t \in \mathbb{N}^+$ and a sequence $a_1, \ldots, a_t$, with $a_1 \cong (K_4, \psi)$, $\sigma_1(V(a_1)) = 2$, $a_t = G_\chi$, such that, for all $2 \leq j \leq t$, a_j is a $(2,0)$-VL-extension or a $(2,1)$-VL-extension of a_{j-1} and $\sigma_1(a_j) = \sigma_1(a_{j-1})$.

Proof. Suppose (i) holds, we proceed by induction on $|V|$. By Lemma 5.5.3.3, for all $s \in \mathbb{N}$ such that $s \geq 4$, there exists a σ_2-cut-tight graph with s vertices. Take $n \in \mathbb{N}$ such that $n \geq 4$ and suppose that (ii) holds for all σ_2-cut-tight graphs with at most n vertices. Now suppose that $|V| = n+1$. As $|V| \geq 5$, Lemma 5.5.3.8 implies there exists a $(2,0)$-VL-reduction or $(2,1)$-VL-reduction of G_χ that is σ_2-cut-tight. Let this labelled graph be $G'_{\chi'}$. As $|V(G')| \leq n$, it follows from our induction hypothesis that there exists $t \in \mathbb{N}^+$ and a sequence $a_1, \ldots, a_t$ with $a_1 \cong (K_4, \psi)$, $\sigma_2(V(a_1)) = 2$, $a_t = G'_{\chi'}$, such that, for all $2 \leq j \leq t$, a_j is a $(2,0)$-VL-extension or a $(2,1)$-VL-extension of a_{j-1} and $\sigma_1(a_j) = \sigma_1(a_{j-1})$. As a_1 is σ_2-cut-tight, repeated applications of Lemma 5.5.3.3 and Lemma 5.5.3.4 imply that $G'_{\chi'}$ is σ_2-cut-tight. As G_χ is σ_2-cut-tight, one more application of Lemma 5.5.3.3 or Lemma 5.5.3.4 implies that $\sigma_2(V(G')) = \sigma_2(V)$. Therefore, $a_1, \ldots, a_t, G_\chi$ is a sequence of the form claimed. On the other hand, if (ii) holds then as $\sigma_2(V(a_1)) = 2$ we note that a_1 is σ_2-cut-tight. Hence repeated applications of Lemma 5.5.3.3 or Lemma 5.5.3.4 imply that G_χ is σ_2-cut-tight. $\square$

Chapter 6

Future Research

To finally bring things to a close we conclude with a brief discussion of some of the possible directions that future research could take. This chapter is organised as to echo the order of earlier chapters. Some of the ideas put forward are relatively nebulous and would, one imagines, require a significant investment of time and energy. On the other hand, we also note a few quirks present in earlier results and contemplate the possibility of providing comparatively minor optimisations.

6.1 Connected Sparsity Matroids

During Chapter 2 and Chapter 3 we acknowledged that, despite an interest in rigidity providing some motivation for studying (k,l)-sparse graphs, there were many purely combinatorial questions about these graphs that could be asked and answered. In Section 2.1 we endeavoured to treat these objects in as great a level of generality as possible. Typically we were able to answer questions for all $k \in \mathbb{N}^+$ and a large range of values of l. One notable exception to this was Lemma 2.1.0.19, where we specified that $k \in \{1,2\}$ for reasons discussed in Remark 9. That being said, we did not show that the result could not be generalised to apply for values of k greater than three, and perhaps even to arbitrary $k \in \mathbb{N}^+$. As $(k,1)$-extensions of graphs have been widely studied, particularly for $k \leq 3$ but also for larger values of k, and so it would be nice to know for which values of k the conclusion of Lemma 2.1.0.19 still holds.

Our next question(s) are also related to graph operations, however this time it is the more involved 2-sum, 2-cleave, i-join, and i-separation operations (where $i \in \{1, 2, 3\}$) that could prove fruitful to consider. In Section 2.2, by considering $(k, 2k-1)$-connected graphs, we extended earlier work [2, 22] which focused on the particular case of this where $k = 2$. Of note was the fact that the 2-sum and 2-cleave operations behaved well, i.e. preserved the properties of being a $(k, 2k-1)$-circuit or $(k, 2k-1)$-connected, for all $k \in \mathbb{N}^+$. This good behaviour was fundamental to Berg and Jordán's method of constructing $(2, 3)$-circuits and to Jackson and Jordán's method of constructing $(2, 3)$-connected graphs. A natural question to consider is whether these operations could prove to be similarly useful in allowing us to find a method of constructing $(3, 4)$-connected, $(4, 7)$-connected, or general $(k, 2k-1)$-connected graphs (or indeed $(k, 2k-1)$-circuits). It is plausible that these families of graphs correspond to globally rigid graphs in some unusual settings.

While the 2-sum and 2-cleave operation were able to be applied to $(k, 2k-1)$-connected graphs for arbitrary $k \in \mathbb{N}^+$, the i-join and i-separation operations appear to be more closely wedded to the specific setting of $(2, 2)$-connectivity. However, we note that Lemma 2.1.0.20 highlights an important distinction between $(k, 2k-1)$-connected graphs and (k, l)-connected graphs where $k \leq l \leq 2k-2$. It could be illuminating to investigate whether graph operations analogous to the i-join and i-separation operations may behave similarly well for different (k, l)-connected graphs for specific values of k and l. We note that Lemma 2.1.0.14 only gives a lower bound of two for a $\kappa(G)$, where G is (k, l)-connected, for all $k \leq l \leq 2k-1$ so it is possible that 2-vertex-separations could continue to be present as k gets larger. Also, if G is (k, l)-connected then Lemma 2.1.0.10 gives us that $\delta(G) \geq k+1$ and so perhaps repurposing the 3-join and 3-separation operations as $(k+1)$-join and $(k+1)$-separation operations could shed some light on these graphs for larger values of k. An inviting first step could be to consider how these operations, and other structural results, may be extended to (k, k)-connected graphs or even just to the $(3, 3)$ case. The final comment we make regarding connected sparsity matroids is to observe the only place where the combinatorial purity of the second and third chapters is disturbed. The proof of Lemma 3.1.0.1 (ii) makes use of a Theorem 1.4.3.8 which is a result concerning rigid frameworks in two-dimensional non-Euclidean spaces. The same issue arose in [22, Lemma 5.2]. Rigidity reared its head in other parts of Jackson and Jordán's work, for example [22, Lemma 3.9], but in these other cases we were able to give a purely combinatorial proof of the analogous result (e.g. Lemma 2.1.0.17). If similar methods are to be employed to better understand these matroids for a variety of values of k then this issue would need to be addressed. Comparatively little is known about rigidity of frameworks in higher-dimensional (i.e. larger k value) spaces and so this could be problematic. It would be satisfying to give a combinatorial proof of this result, and it would also allow these matroids to be studied with only an indirect relationship to rigidity which could lead to new insights.

6.2 Rigidity in Normed Spaces

The first, and arguably most obvious, comment to make here is to acknowledge the fact that the characterisation given in Chapter 4 only concerns analytic (non-Euclidean) normed spaces. The corresponding characterisation of rigidity in non-Euclidean normed spaces has no such restriction. A very inviting problem to consider is whether this restriction to the analytic setting can be loosened and if so then how and by how much. This feels like a challenging problem, as the various technical details interact in quite subtle ways as is sometimes evidenced by the wording of results. Note that references to a completely regular realisation, smooth space, or strictly convex space, may appear to pop up quite sporadically. There are various results (e.g. Lemma 4.1.2.4) where the constraints required to allow the proof of that result to work can have a significant impact further down the line when that result is the primary reason that another result (e.g. Theorem 4.2.0.6) involves some additional condition.

Another possible direction to take this research would be to consider higher-dimensional normed spaces. As we have mentioned, relatively little is known concerning higher-dimensional (global) rigidity. There does not appear to be an obvious reason why considering the non-Euclidean setting would prove to be more fruitful, but as this is still a fairly novel area it is certainly something worth exploring.

6.3 Labelled Graphs and Frameworks on Surfaces

Finally we consider how one may build upon the work from Chapter 5. First and foremost, as was alluded to in that chapter, we believe that the graphs studied here should prove to be relevant to studying rigidity of frameworks on non-concentric spheres. This is a natural extension of earlier work [32] that characterised rigid frameworks on concentric spheres. We claim that, given Theorem 5.5.3.9, it should be possible to characterise a particular notion of rigidity in this context. The only labelled graph operations used are based on the well-understood $(2,0)$-extension and $(2,1)$-extension operations and this construction begins from a single labelled graph. Showing that the labelled graph (K_4, χ) is rigid (when χ is such that $\sigma_2(V(K_4)) = 2$) and that the labelled graph operations preserve rigidity would provide a sufficient condition for a graph to be rigid and it seems that necessity of these labelled graphs could be confirmed using similar arguments to those in [32].

Spheres are not the only surface that has been considered from a rigidity perspective, not even in [32]. Cylinders are also well-understood and some more exotic surfaces have also been investigated. One possible application of the ideas in Chapter 5 would be to investigating rigidity of frameworks realised on other formations of surfaces. The notion of a labelled graph may prove to be very useful and by tweaking the σ_d-functions it may be possible to model any (labelled) graphs of interest in a similar manner.

Of course, it may be wise to first confirm that these ideas are as useful in the spherical setting as seems likely. The combinatorial analysis of Chapter 5 is detailed and would greatly benefit from be combined with a formal rigidity theoretic motivation rather than the general description in Section 5.1.

Bibliography

[1] L. Asimow and B. Roth. "The Rigidity of Graphs". In: *Transactions of the American Mathematical Society* 245 (1978), pp. 279–289. ISSN: 1088-6850. DOI: https://doi.org/10.2307/1998867.

[2] A.R. Berg and T. Jordán. "A proof of Connelly's conjecture on 3-connected circuits of the rigidity matroid". In: *Journal of Combinatorial Theory, Series B* 88.1 (2003), pp. 77–97. ISSN: 1096-0902. DOI: https://doi.org/10.1016/S0095-8956(02)00037-0.

[3] G. Chartrand and F. Harary. "Graphs with prescribed connectivities". In: *Theory of graphs. Proceedings of the colloquium held at Tihany, Hungary*. September 1966. Ed. by P. Erdős and G. Katona. Academic Press, 1968, pp. 61–63.

[4] R. Connelly. "On generic global rigidity". In: *Applied Geometry and Discrete Mathematics. The Victor Klee Festschrift*. Ed. by P. Gritzmann and B. Sturmfels. Vol. 4. DIMACS Series in Discrete Mathematics and Theoretical Computer Science. American Mathematical Society and Association of Computing Machinery, 1991, pp. 147–155. ISBN: 978-0-8218-6593-4.

[5] R. Connelly. "Generic Global Rigidity". In: *Discrete & Computational Geometry* 33 (2005), pp. 549–563. ISSN: 1432-0444. DOI: https://doi.org/10.1007/s00454-004-1124-4.

[6] R. Connelly, S.J. Gortler, and L. Theran. "Global rigidity of complete bipartite graphs". 2021. arXiv: 2105.01389v1 [math.MG].

[7] C.R. Coullard and L. Hellerstein. "Independence and port oracles for matroids, with an application to computational learning theory". In: *Combinatorica* 16.2 (1996), pp. 189–208. ISSN: 1439-6912. DOI: https://doi.org/10.1007/BF01844845.

[8] S. Dewar. "The rigidity of countable frameworks in normed spaces". PhD thesis. Lancaster University, 2019. 243 pp. DOI: https://doi.org/10.17635/lancaster/thesis/756.

[9] S. Dewar. "Infinitesimal Rigidity in Normed Planes". In: *SIAM Journal on Discrete Mathematics* 34.2 (2020), pp. 1205–1231. ISSN: 1095-7146. DOI: https://doi.org/10.1137/19M1284051.

[10] S. Dewar. "Equivalence of Continuous, Local and Infinitesimal Rigidity in Normed Spaces". In: *Discrete & Computational Geometry* 65.3 (2021), pp. 655–679. ISSN: 1432-0444. DOI: https://doi.org/10.1007/s00454-019-00135-5.

[11] S. Dewar. "Infinitesimal rigidity and prestress stability for frameworks in normed spaces". In: *Discrete Applied Mathematics* 322 (2022), pp. 425–438. ISSN: 1872-6771. DOI: https://doi.org/10.1016/j.dam.2022.09.001.

[12] S. Dewar, J. Hewetson, and A. Nixon. "Coincident-point rigidity in normed planes". 2021. arXiv: 2112.10480v1 [math.CO].

[13] S. Dewar, J. Hewetson, and A. Nixon. "Uniquely realisable graphs in analytic normed planes". 2022. arXiv: 2206.07426v1 [math.CO].

[14] S. Dewar and A. Nixon. "Generalised rigid body motions in non-Euclidean planes with applications to global rigidity". In: *Journal of Mathematical Analysis and Applications* 514.1 (2022), p. 126259. ISSN: 1096-0813. DOI: https://doi.org/10.1016/j.jmaa.2022.126259.

[15] L. Euler. "Solutio problematis ad geometriam situs pertinentis". In: *Commentarii Academiae scientiarum imperialis Petropolitanae* 8 (1736), pp. 128–140. ISSN: 2686-7079.

[16] L. Euler. "Solutio problematis ad geometriam situs pertinentis". In: N.L. Biggs, E.K. Lloyd, and R.J. Wilson. *Graph Theory 1736-1936*. 1st ed. Clarendon Press, 1976. ISBN: 9780198539018.

[17] Z. Fekete, T. Jordán, and V. E. Kaszanitzky. "Rigid Two-Dimensional Frameworks with Two Coincident Points". In: *Graphs and Combinatorics* 31.3 (2015), pp. 585–599. ISSN: 1435-5914. DOI: https://doi.org/10.1007/s00373-013-1390-0.

[18] R. Gould. *Graph Theory*. 1st ed. Bejamin/Cummings, 1988. ISBN: 9780805360301.

[19] H. Guler. "Rigidity of Frameworks". PhD thesis. Queen Mary University of London, 2018. 149 pp. URL: http://qmro.qmul.ac.uk/xmlui/handle/123456789/36220.

[20] F. Harary. *Graph Theory*. 1st ed. Addison-Wesley, 1972. ISBN: 9780201027877.

[21] B. Hendrickson. "Conditions for unique graph realizations". In: *SIAM Journal on Computing* 21.1 (1992), pp. 65–84. ISSN: 1095-7111. DOI: https://doi.org/10.1137/0221008.

[22] B. Jackson and T. Jordán. "Connected rigidity matroids and unique realizations of graphs". In: *Journal of Combinatorial Theory, Series B* 94.1 (2005), pp. 1–29. ISSN: 1096-0902. DOI: https://doi.org/10.1016/j.jctb.2004.11.002.

[23] B. Jackson, V. E. Kaszanitzky, and A. Nixon. "Rigid Cylindrical Frameworks with Two Coincident Points". In: *Graphs and Combinatorics* 35.1 (2019), pp. 141–168. ISSN: 1435-5914. DOI: https://doi.org/10.1007/s00373-018-1983-8.

[24] B. Jackson and A. Nixon. "Global rigidity of generic frameworks on the cylinder". In: *Journal of Combinatorial Theory, Series B* 139 (2019), pp. 193–229. ISSN: 1096-0902. DOI: https://doi.org/10.1016/j.jctb.2019.03.002.

[25] T. Jordán and W. Whiteley. "Global Rigidity". In: *Handbook of Discrete and Computational Geometry*. Ed. by J.E. Goodman, J. O'Rourke, and C.D. Tóth. 3rd ed. CRC Press, 2018, pp. 1661–1694. ISBN: 9781498711395.

[26] D. Kitson. "Finite and Infinitesimal Rigidity with Polyhedral Norms". In: *Discrete & Computational Geometry* 54.2 (2014), pp. 390–411. ISSN: 1432-0444. DOI: https://doi.org/10.1007/s00454-015-9706-x.

[27] D. Kitson and S. C. Power. "Infinitesimal rigidity for non-Euclidean bar-joint frameworks". In: *Bulletin of the London Mathematical Society* 46.4 (2014), pp. 685–697. ISSN: 1469-2120. DOI: https://doi.org/10.1112/blms/bdu017.

[28] G. Laman. "On graphs and rigidity of plane skeletal structures". In: *Journal of Engineering Mathematics* 4.4 (1970), pp. 331–340. ISSN: 1573-2703. DOI: https://doi.org/10.1007/BF01534980.

[29] L. Narici and E. Beckenstein. *Topological Vector Spaces*. 2nd ed. Taylor and Francis Group, 2010. ISBN: 9780429147890.

[30] A. Nixon. "A Constructive Characterisation of Circuits in the Simple (2,2)-sparsity Matroid". In: *European Journal of Combinatorics* 42 (2014), pp. 92–106. ISSN: 1095-9971. DOI: https://doi.org/10.1016/j.ejc.2014.05.009.

[31] A. Nixon and J.C. Owen. "An inductive construction of (2, 1)-tight graphs". In: *Contributions to Discrete Mathematics* 9.2 (2014), pp. 1–16. ISSN: 1715-0868. DOI: https://doi.org/10.11575/cdm.v9i2.62096.

[32] A. Nixon, J.C. Owen, and S.C. Power. "Rigidity of Frameworks Supported on Surfaces". In: *SIAM Journal on Discrete Mathematics* 26.4 (2012), pp. 1733–1757. ISSN: 1095-7146. DOI: https://doi.org/10.1137/110848852.

[33] A. Nixon, J.C. Owen, and S.C. Power. "A Characterisation of Generically Rigid Frameworks on Surfaces of Revolution". In: *SIAM Journal on Discrete Mathematics* 28.4 (2014), pp. 2008–2028. ISSN: 1095-7146. DOI: https://doi.org/10.1137/130913195.

[34] J.G. Oxley. *Matroid Theory*. Oxford University Press, 1992. ISBN: 9780198535638.

[35] H. Pollaczek-Geiringer. "Über die Gliederung ebener Fachwerke". In: *Zeitschrift für angewandte Mathematik und Mechanik* 7.1 (1927), pp. 58–72. ISSN: 1521-4001. DOI: https://doi.org/10.1002/zamm.19270070107.

[36] W. Rudin. *Functional Analysis*. 2nd ed. McGraw-Hill, 1991. ISBN: 9780070542365.

[37] G. F. Simmons. *Introduction to Topology and Modern Analysis*. 1st ed. McGraw-Hill, 1963. ISBN: 9780070573895.

[38] A.E. Taylor and D.C. Lay. *Introduction to Functional Analysis*. 2nd ed. Wiley, 1980. ISBN: 9780471846468.

[39] D.J.A. Welsh. *Matroid Theory*. Academic Press, 1976. ISBN: 9780127440507.

[40] H. Whitney. "Congruent Graphs and the Connectivity of Graphs". In: *American Journal of Mathematics* 54.1 (1932), pp. 150–168. ISSN: 1080-6377. DOI: https://doi.org/10.2307/2371086.

[41] H. Whitney. "On the Abstract Properties of Linear Dependence". In: *American Journal of Mathematics* 57.3 (1935), pp. 509–533. ISSN: 1080-6377. DOI: https://doi.org/10.2307/2371182.