
A NEW FAMILY OF ISOLATED SYMPLECTIC SINGULARITIES WITH TRIVIAL LOCAL FUNDAMENTAL GROUP

by

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Abstract. — We construct a new infinite family of 4-dimensional isolated symplectic singularities with trivial local fundamental group, answering a question of Beauville raised in 2000. Three constructions are presented for this family: (1) as singularities in blowups of the quotient of \mathbb{C}^4 by the dihedral group of order $2d$, (2) as singular points of Calogero-Moser spaces associated with dihedral groups of order $2d$ at equal parameters, (3) as singularities of a certain Slodowy slice in the d -fold cover of the nilpotent cone in \mathfrak{sl}_d .

1. Introduction

Symplectic singularities were introduced by Beauville [Bea]. Their introduction has led to numerous important developments in both algebraic geometry and geometric representation theory. Basic examples of symplectic singularities include symplectic quotient singularities and singularities in normalizations of nilpotent orbit closures in semisimple Lie algebras [Bea, Section 2]. Note that any two dimensional symplectic singularity is just a rational double point.

An isolated symplectic singularity of dimension ≥ 4 is a normal isolated singularity whose smooth locus admits a holomorphic symplectic 2-form [Bea, (1.2)]. It follows that such a singularity is canonical Gorenstein, hence its local fundamental group is finite [Bra]. Symplectic quotient singularities give many examples of isolated symplectic singularities, but they all have non-trivial local fundamental groups.

Minimal nilpotent orbit closures $\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}$ in simple Lie algebras \mathfrak{g} give examples of isolated symplectic singularities with trivial local fundamental group. In this case, the projective tangent cone $\mathrm{PT}_0(\overline{\mathcal{O}}_{\min}^{\mathfrak{g}})$ of $\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}$ at 0 is isomorphic to $\mathbb{P}\mathcal{O}_{\min}^{\mathfrak{g}} = \mathcal{O}_{\min}^{\mathfrak{g}}/\mathbb{C}^\times$, which is smooth and G -homogeneous (where G is the adjoint Lie group of \mathfrak{g}). It turns out that the smoothness of projective tangent cones characterizes $\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}$, by the following result.

Theorem 1.1 (Beauville). — *Let (\mathcal{X}, x) be an isolated symplectic singularity whose projective tangent cone at x is smooth. Then there exists a simple Lie algebra \mathfrak{g} such that (\mathcal{X}, x) is locally analytically isomorphic to $(\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}, 0)$. Moreover, the singularity $(\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}, 0)$ has trivial local fundamental group if and only if $\mathfrak{g} \neq \mathfrak{sp}_{2n}(\mathbb{C})$ for any $n \geq 1$.*

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If (\mathcal{X}, x) is an isolated symplectic singularity whose projective tangent cone at x is smooth, then the simple Lie algebra \mathfrak{g} such that (\mathcal{X}, x) is locally analytically isomorphic to $(\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}, 0)$ is uniquely determined. It can be recovered as the Lie algebra structure on the cotangent space $T_x^*(\mathcal{X})$ of \mathcal{X} at x induced by the Poisson bracket. In other words,

$$(1.2) \quad T_0^*(\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}) \simeq \mathfrak{g}$$

as Lie algebras.

When $\mathfrak{g} \simeq \mathfrak{sp}_{2n}(\mathbb{C})$, the singularity $(\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}, 0)$ is locally analytically isomorphic to $(\mathbb{C}^{2n}/\mu_2, 0)$, where $\mu_2 = \{1, -1\}$ acts by multiplication on \mathbb{C}^{2n} , so the local fundamental group is isomorphic to μ_2 .

In the same paper, Beauville asked [Bea, (4.3)] whether there exist other isolated symplectic singularities with trivial local fundamental group. In the intervening two decades, no other examples have come to light.

Constructing examples of new isolated symplectic singularities is also motivated by the long-standing conjecture (attributed to LeBrun and Salamon) that any Fano contact manifold is isomorphic to $P\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}$ for some simple Lie algebra \mathfrak{g} . In fact, if Z is a Fano contact manifold and L is the contact line bundle, then there exists a map $L^* \rightarrow X$ which contracts the zero section to one point o . The contact structure on Z induces a symplectic structure on the complement of the zero section in L^* , which gives a symplectic form on $X \setminus \{o\}$. This implies that (X, o) is an isolated symplectic singularity. By the same argument as in [Bea, Proposition 4.2], X has trivial local fundamental group except in the case where $Z \simeq \mathbb{P}^{2n-1}$.

The aim of this paper is to describe a new infinite family of 4-dimensional isolated symplectic singularities with trivial local fundamental group. We will give three different constructions of this family: one from blowups of symplectic quotient singularities, one from Calogero-Moser spaces associated with dihedral groups, and the last one from slices in the d -fold cover of the nilpotent cone of \mathfrak{sl}_d .

For this, let V be a complex vector space of dimension 2, let $d \geq 4$, let $W_d \subset \mathbf{GL}_{\mathbb{C}}(V)$ denote the dihedral group of order $2d$ and let $\mathcal{Q}(d)$ denote the symplectic quotient singularity $(V \times V^*)/W_d$. We define two varieties:

- Let $\tilde{\mathcal{Q}}(d)$ denote the blowup of $\mathcal{Q}(d)$ at its singular locus.
- Let $\mathcal{Z}(d)$ denote the Calogero-Moser space associated with W_d at non-zero equal parameters [Bon1]: it is a Poisson deformation of $\mathcal{Q}(d)$.

Then both $\tilde{\mathcal{Q}}(d)$ and $\mathcal{Z}(d)$ are normal varieties of dimension 4, which have a unique singular point (denoted by 0 for in both cases). Now $(\mathcal{Z}(d), 0)$ is a symplectic singularity by general results about Calogero-Moser spaces [Gor]. It is of course an isolated singularity. As will be shown in the paper, $(\tilde{\mathcal{Q}}(d), 0)$ is also a symplectic singularity (Corollary 3.8), which is locally analytically isomorphic to $(\mathcal{Z}(d), 0)$ (Corollary 2.15). In the case $\tilde{\mathcal{Q}}(5)$, this was observed in [FJLS, Remark 12.8]. Our first main result is the following:

Theorem 1.3. — *The following statements hold:*

- (a) *The symplectic singularity $(\mathcal{Z}(d), 0)$ has trivial local fundamental group.*
- (b) *The symplectic singularity $(\mathcal{Z}(4), 0)$ is locally analytically isomorphic to $(\overline{\mathcal{O}}_{\min}^{\mathfrak{sl}_3}, 0)$.*
- (c) *If $d \geq 5$, then the singularity $(\mathcal{Z}(d), 0)$ is not locally analytically isomorphic to $(\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}, 0)$ for any simple Lie algebra \mathfrak{g} .*
- (d) *If $d' > d \geq 4$, then the singularities $(\mathcal{Z}(d), 0)$ and $(\mathcal{Z}(d'), 0)$ are not locally analytically isomorphic.*

Of course, the analogous statements also hold for the singularity $(\tilde{\mathcal{Q}}(d), 0)$. The proof relies on the explicit computation of the equations for $\mathcal{X}(d)$ (see [Bon2]) and $\tilde{\mathcal{Q}}(d)$. A key role is played by an action of \mathbf{SL}_2 on both $\mathcal{X}(d)$ and $\tilde{\mathcal{Q}}(d)$.

Symplectic singularities of Calogero-Moser spaces associated with the complex reflection groups denoted by $G(d, 1, n)$ in Shephard-Todd classification [ShTo] are relatively well-understood, as they are quiver varieties [EtGi]. However, we think it might be interesting to study the symplectic singularities of Calogero-Moser spaces associated with other complex reflection groups. For the infinite family $G(de, e, n)$ (note that $W_d = G(d, d, 2)$), and the 34 exceptional groups, it is appealing to expect to find more interesting examples of symplectic singularities.

The third construction is a certain cover of a type A Slodowy slice [Slo]. If μ is a partition of d , let \mathcal{O}_μ denote the associated nilpotent orbit in $\mathfrak{sl}_d(\mathbb{C})$. Let \mathcal{N}_d denote the nilpotent cone of $\mathfrak{sl}_d(\mathbb{C})$. We denote by $\mathcal{S}_{d-2,2}$ a Slodowy slice associated with $\mathcal{O}_{d-2,2}$ and set $\mathcal{X}(d) = \mathcal{S}_{d-2,2} \cap \mathcal{N}_d$. The nilpotent cone \mathcal{N}_d admits a μ_d -covering $\pi_d : \tilde{\mathcal{N}}_d \rightarrow \mathcal{N}_d$ which is unramified above the regular nilpotent orbit \mathcal{O}_d and bijective above the branch locus. Define $\tilde{\mathcal{X}}(d) = \pi_d^{-1}(\mathcal{X}(d))$, denote by x_d the unique element of $\mathcal{O}_{d-2,2} \cap \mathcal{S}_{d-2,2}$ and let \tilde{x}_d denote its unique preimage in $\tilde{\mathcal{X}}(d)$. Then \tilde{x}_d is the unique singular point of $\tilde{\mathcal{X}}(d)$ and:

Theorem 1.4. — *If $d \geq 4$, the symplectic singularities $(\mathcal{X}(d), 0)$ and $(\tilde{\mathcal{X}}(d), \tilde{x}_d)$ are locally analytically isomorphic.*

The proof of this theorem is rather indirect. The cyclic group μ_d acts on the singularity $(\mathcal{X}(d), 0)$ and the quotient can be identified with the Calogero-Moser space (at a non-generic parameter) $(\mathcal{X}(d, 1, 2), x_0)$ associated to $G(d, 1, 2)$. By [EtGi, Mar], the latter space is isomorphic to a certain quiver variety $(\mathfrak{M}_\lambda(\mathbf{v}), x_0)$ associated to the framed affine quiver of type A , which can be shown to be isomorphic to $\mathcal{X}(d)$ by using results of Nakajima [Nak] (see also [Ma]). This implies that both $\mathcal{X}(d)$ and $\tilde{\mathcal{X}}(d)$ are d -fold coverings of $\mathcal{X}(d)$, which are shown to be locally holomorphically isomorphic by using the triviality of the local fundamental group of $\mathcal{X}(d)$.

It is interesting to remark that the singularity $(\tilde{\mathcal{X}}(5), \tilde{x}_5)$ is exactly the generic singularity of the nilpotent orbit closure of $A_4 + A_3$ along the codimension 4 boundary $A_4 + A_2 + A_1$ in E_8 . More surprisingly, the whole of $\tilde{\mathcal{N}}_5$ is in fact isomorphic to the Slodowy slice of $(A_4 + A_3, A_4)$ in E_8 (see [FJLS, Section 12.3]).

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Convention, notation. We work over the field \mathbb{C} of complex numbers (polynomials, Lie algebras, algebraic varieties, schemes are supposed to be complex). If \mathcal{X} is an affine scheme, we denote by $\mathbb{C}[\mathcal{X}]$ its ring of regular functions.

We fix a natural number $d \geq 4$, we denote by μ_d the cyclic group of order d and we fix a primitive d -th root of unity ζ . Let $V = \mathbb{C}^2$ and denote by (x, y) its canonical basis and by (X, Y) its dual basis. Thus, $X, Y \in \mathbb{C}[V]$ and we may view x and y as elements of $\mathbb{C}[V^*]$. If $j \in \mathbb{Z}$ or $\mathbb{Z}/d\mathbb{Z}$, we set

$$s_j = \begin{pmatrix} 0 & \zeta^j \\ \zeta^{-j} & 0 \end{pmatrix} \in \mathbf{GL}_2(\mathbb{C}) = \mathbf{GL}_{\mathbb{C}}(V),$$

$s = s_0$, $t = s_1$ and $W_d = \langle s, t \rangle = \langle s_0, s_1, \dots, s_{d-1} \rangle$: it is the dihedral group of order $2d$.

Let U_2 denote the standard representation of \mathfrak{sl}_2 so that $\text{Sym}^d(U_2)$ is the unique simple $\mathfrak{sl}_2(\mathbb{C})$ -module of dimension $d + 1$. We denote by \mathfrak{g}_d the Lie algebra $\mathfrak{sl}_2 \oplus \text{Sym}^d(U_2)$, where the Lie structure is determined by the fact that $\text{Sym}^d(U_2)$ is a commutative ideal of \mathfrak{g}_d on which \mathfrak{sl}_2 acts through its natural action.

2. Deformation and blowup of a symplectic quotient singularity

2.A. Symplectic quotient $(V \times V^*)/W_d$. — The induced action of W_d on $V \times V^*$ is symplectic, making the quotient $\mathcal{Q}(d) = (V \times V^*)/W_d$ an affine variety with only symplectic singularities. The singular locus $S \subset (V \times V^*)/W_d$ consists of W_d -orbits with non-trivial stabilizer, namely the image of the points of the form $(\lambda(x - \zeta^j y), \mu(Y - \zeta^j X))$, with $\lambda, \mu \in \mathbb{C}$ and $j \in \mathbb{Z}/d\mathbb{Z}$. It is easy to check that S is of codimension 2 and the singularity of $(V \times V^*)/W_d$ along a nonzero point of S is an A_1 -singularity.

We denote by $(\Psi_k)_{k \geq 0}$ the sequence of polynomials in three indeterminates q, Q, e , defined by

$$\begin{cases} \Psi_0 = 1, \\ \Psi_1 = e, \\ \Psi_k = e\Psi_{k-1} - qQ\Psi_{k-2}, \quad \text{if } k \geq 2. \end{cases}$$

(see [Bon2, (1.4)]). Note that Ψ_k is homogeneous of degree k . Alternatively, Ψ_k can be defined by the generating series

$$\sum_{k \geq 0} \Psi_k \mathbf{t}^k = \frac{1}{1 - e\mathbf{t} + qQ\mathbf{t}^2}$$

in $\mathbb{C}[q, Q, e][[\mathbf{t}]]$. In particular,

$$(2.1) \quad \Psi_k(q, 0, e) = \Psi_k(0, Q, e) = e^k \quad \text{and} \quad \Psi_k(q, Q, 0) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ (-qQ)^{k/2} & \text{if } k \text{ is even.} \end{cases}$$

Consider the following W_d -invariant polynomials in $\mathbb{C}[V \times V^*]$.

$$q = xy, Q = XY, e = xX + yY, a_i = x^{d-i}Y^i + y^{d-i}X^i, 0 \leq i \leq d.$$

By [Bon2, Theorem 1.6], we have

$$(2.2) \quad \mathcal{Q}(d) = \{(q, Q, e, a_0, a_1, \dots, a_d) \in \mathbb{C}^{d+4} \mid \\ \forall 1 \leq j \leq k \leq d-1, \begin{cases} e a_j = q a_{j+1} + Q a_{j-1}, \\ a_{j-1} a_{k+1} - a_j a_k = (e^2 - 4qQ) q^{d-k-1} Q^{j-1} \Psi_{k-j}(q, Q, e) \end{cases} \}.$$

2.B. The variety $\mathcal{X}(d)$. — The Calogero-Moser spaces associated with complex reflection groups are extensively studied in [EtGi]. Using the notation of [Bon2], we denote by $\mathcal{X}(d)$ the Calogero-Moser space associated with the dihedral group W_d of order $2d$ at equal parameter $a = 1$. By [EtGi], the variety $\mathcal{X}(d)$ is endowed with a Poisson structure. The case $a \neq 0$ is isomorphic to the case $a = 1$ and the only change is that the Poisson structure is multiplied by a scalar: this is irrelevant for our purpose. It turns out this Calogero-Moser space is a Poisson deformation of the quotient $\mathcal{Q}(d)$, with equations given as follows [Bon2, Theorem 2.9]:

$$\mathcal{X}(d) = \{(q, Q, e, a_0, a_1, \dots, a_d) \in \mathbb{C}^{d+4} \mid \\ \forall 1 \leq j \leq k \leq d-1, \begin{cases} e a_j = q a_{j+1} + Q a_{j-1}, \\ a_{j-1} a_{k+1} - a_j a_k = (e^2 - 4qQ - d^2) q^{d-k-1} Q^{j-1} \Psi_{k-j}(q, Q, e) \end{cases} \}.$$

Note that the point $0 = (0, 0, \dots, 0) \in \mathbb{C}^{d+4}$ belongs to $\mathcal{Z}(d)$ and we denote by $T_0^*(\mathcal{Z}(d))$ the cotangent space of $\mathcal{Z}(d)$ at 0.

Let us first recall some facts about the variety $\mathcal{Z}(d)$ with $d \geq 4$ (recall that the Lie algebra $\mathfrak{g}_d = \mathfrak{sl}_2 \oplus \text{Sym}^d(U_2)$ has been defined at the end of the introduction).

Theorem 2.3. — *The affine variety $\mathcal{Z}(d)$ is irreducible, normal, of dimension 4. Moreover:*

- (a) *The point 0 is an isolated singularity of $\mathcal{Z}(d)$ and its associated maximal ideal of $\mathbb{C}[\mathcal{Z}(d)]$ is a Poisson ideal. This endows $T_0^*(\mathcal{Z}(d))$ with a Lie algebra structure.*
- (b) *We have an isomorphism of Lie algebras*

$$T_0^*(\mathcal{Z}(d)) \simeq \begin{cases} \mathfrak{sl}_3(\mathbb{C}) & \text{if } d = 4, \\ \mathfrak{g}_d & \text{if } d \geq 5, \end{cases}$$

- (c) *If $d \geq 4$, then $\mathcal{Z}(d)$ is a symplectic singularity in the sense of Beauville.*
- (d) *The symplectic singularity $(\mathcal{Z}(4), 0)$ is locally analytically isomorphic to $(\overline{\mathcal{O}}_{\min}^{\mathfrak{sl}_3(\mathbb{C})}, 0)$.*
- (e) *If $d \geq 5$, then the symplectic singularity $(\mathcal{Z}(d), 0)$ is not locally analytically isomorphic to any $(\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}, 0)$ for a simple Lie algebra \mathfrak{g} .*
- (f) *If $d' > d \geq 4$, then the singularities $(\mathcal{Z}(d), 0)$ and $(\mathcal{Z}(d'), 0)$ are not locally analytically isomorphic.*

Proof. — The fact that $\mathcal{Z}(d)$ is irreducible, normal, of dimension 4 is a general fact about Calogero-Moser spaces [EtGi].

(a) was proved in [Bel] (see also [Bon1]).

(b) is proved in [Bon1, Proposition 8.4] and [Bon2, Proposition 2.12].

(c) The fact that $\mathcal{Z}(d)$ is a symplectic singularity is again a general fact about Calogero-Moser spaces [Gor, Proposition 4.5].

(d) A computation with MAGMA [Mag] (and the CHAMP package built by Thiel [Thi]) based on the equations of $\mathcal{Z}(4)$ given in [Bon1, Proposition 8.3] shows that the projective tangent cone $\text{PT}_0(\mathcal{Z}(4))$ of $\mathcal{Z}(4)$ at 0 is smooth. So the result follows from (b) and Beauville's Theorem 1.1 [Bea, Introduction]. We will provide in Remark 3.4 a proof which does not rely on MAGMA computations.

(e) Assume that $d \geq 5$. By (b), $T_0^*(\mathcal{Z}(d)) \simeq \mathfrak{g}_d$ is not a simple Lie algebra. So, by 1.2, the singularity $(\mathcal{Z}(d), 0)$ is not locally analytically isomorphic to $(\overline{\mathcal{O}}_{\min}^{\mathfrak{g}}, 0)$ for any simple Lie algebra \mathfrak{g} .

(f) follows from (b) and the fact that $\dim(\mathfrak{g}_d) = d + 4 < d' + 4 = \dim(\mathfrak{g}_{d'})$. \square

2.C. Blowup. — A polynomial $f \in \mathbb{C}[V \times V^*]$ is called W_d -semi-invariant if $w(f) = \det(w)f$ for all $w \in W_d$. We denote by $\mathbb{C}[V \times V^*]^{W_d\text{-sem}}$ the set of W_d -semi-invariant polynomials: it is a $\mathbb{C}[V \times V^*]^{W_d}$ -module. Let $\delta = xX - yY$ and $\beta_j = x^{d-j}Y^j - y^{d-j}X^j$ for $0 \leq j \leq d$. Then $\delta, \beta_0, \beta_1, \dots, \beta_d$ are W_d -semi-invariants. Let Γ_d denote the (normal) cyclic subgroup of W_d generated by $ts = \text{diag}(\zeta, \zeta^{-1})$. It is easily seen that

$$\mathbb{C}[V \times V^*]^{\Gamma_d} = \mathbb{C}[xy, XY, xX, yY, (x^{d-j}Y^j)_{0 \leq j \leq d}, (y^{d-j}X^j)_{0 \leq j \leq d}].$$

Using the action of $W_d/\Gamma_d \simeq \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}[V \times V^*]^{\Gamma_d}$, this can be rewritten

$$(2.4) \quad \mathbb{C}[V \times V^*]^{\Gamma_d} = \mathbb{C}[V \times V^*]^{W_d}[\delta, \beta_0, \beta_1, \dots, \beta_d].$$

As the product of two elements of the list $\delta, \beta_0, \beta_1, \dots, \beta_d$ is a W_d -invariant, we get that

$$(2.5) \quad \text{The } \mathbb{C}[V \times V^*]^{W_d}\text{-module } \mathbb{C}[V \times V^*]^{\Gamma_d} \text{ is generated by } 1, \delta, \beta_0, \beta_1, \dots, \beta_d.$$

Since $\mathbb{C}[V \times V^*]^{\Gamma_d} = \mathbb{C}[V \times V^*]^{W_d} \oplus \mathbb{C}[V \times V^*]^{W_d\text{-sem}}$, we get that

$$(2.6) \quad \text{The } \mathbb{C}[V \times V^*]^{W_d}\text{-module } \mathbb{C}[V \times V^*]^{W_d\text{-sem}} \text{ is generated by } \delta, \beta_0, \beta_1, \dots, \beta_d.$$

Therefore, the ideal $I = (\delta\mathbb{C}[V \times V^*])^{W_d} = \delta\mathbb{C}[V \times V^*]^{W_d\text{-sem}}$ of $\mathbb{C}[V \times V^*]^{W_d}$ is generated by $\delta^2, \delta\beta_0, \delta\beta_1, \dots, \delta\beta_d$.

Write $D = e^2 - 4qQ = \delta^2$ and consider the following W_d -invariant rational functions:

$$b_j = \frac{\beta_j}{\delta} = \frac{x^{d-j}Y^j - y^{d-j}X^j}{xX - yY}, \quad 0 \leq j \leq d.$$

Then $Db_j = \delta\beta_j$ is W_d -invariant and I is the ideal of $\mathbb{C}[V \times V^*]^{W_d}$ generated by $D, Db_0, Db_1, \dots, Db_d$. Let $\tilde{\mathcal{Q}}(d)$ be the blowup of $\mathcal{Q}(d)$ along this ideal. The (contracting) \mathbb{C}^\times -action on $V \times V^*$ by homothety induces a contracting action on the quotient $\mathcal{Q}(d)$. For this action, D and the b_j 's are homogeneous (of degree 2 and $d-2$ respectively) so the blowup $\tilde{\mathcal{Q}}(d)$ inherits a \mathbb{C}^\times -action.

Recall that the blowup of an affine variety $\text{Spec}(A)$ along an ideal \mathfrak{a} generated by elements f_1, \dots, f_r admits an affine open covering indexed by the f_j 's and such that the affine open subset associated with f_j is isomorphic to $\text{Spec}(A[\mathfrak{a}/f_j])$. In our case, the variety $\tilde{\mathcal{Q}}(d)$ is covered by affine open subsets $\mathcal{Y}(d), \mathcal{Y}_0, \mathcal{Y}_1, \dots, \mathcal{Y}_d$ defined by $\mathcal{Y}(d) = \text{Spec}(\mathbb{C}[V \times V^*]^{W_d}[I/D])$ and $\mathcal{Y}_j = \text{Spec}(\mathbb{C}[V \times V^*]^{W_d}[I/(Db_j)])$, for $j = 0, 1, \dots, d$.

They are all \mathbb{C}^\times -stable. The following relations follow from straightforward computations:

$$(2.7) \quad \begin{cases} a_j = e b_j - 2q b_{j+1} & \text{if } 0 \leq j \leq d-1, \\ a_j = 2Q b_{j-1} - e b_j & \text{if } 1 \leq j \leq d, \end{cases}$$

$$(2.8) \quad \begin{cases} Db_0 = e a_0 - 2q a_1, \\ Db_j = Q a_{j-1} - q a_{j+1} & \text{if } 1 \leq j \leq d-1, \\ Db_d = 2Q a_{d-1} - e a_d. \end{cases}$$

Let η be a primitive $4d$ -th root of unity. The action of $\text{diag}(\eta, \eta^{-1})$ sends a_j to $\sqrt{-1}\beta_j$, while e, q and Q remain invariant. Applying this transformation to the equations (2.2) describing $\mathcal{Q}(d)$, and dividing the first line by δ and the second line by δ^2 , we obtain the following relations (note the sign change in the second line!):

$$(2.9) \quad \begin{cases} e b_j = q b_{j+1} + Q b_{j-1} \\ b_j b_k - b_{j-1} b_{k+1} = q^{d-k-1} Q^{j-1} \Psi_{k-j}(q, Q, e) \end{cases} \quad \text{if } 1 \leq j \leq k \leq d-1.$$

Lemma 2.10. — *With the above notation, we have:*

- (a) $\mathcal{Y}_0 \simeq \mathcal{Y}_d \simeq \mathbb{C}^4$.
- (b) If $1 \leq r \leq d-1$, then \mathcal{Y}_r is smooth.

Proof. — (a) We claim that $\mathcal{Y}_0 = \text{Spec}(\mathbb{C}[q, a_0, \frac{1}{b_0}, \frac{b_1}{b_0}])$, which will imply that $\mathcal{Y}_0 \simeq \mathbb{C}^4$ as \mathcal{Y}_0 is of dimension 4. For this, we will express other variables in terms of $q, a_0, \frac{1}{b_0}, \frac{b_1}{b_0}$.

By (2.7), $a_0 = e b_0 - 2q b_1$, whence $e = \frac{1}{b_0} a_0 + 2q \frac{b_1}{b_0}$, which gives the expression for e .

Take $j = k = 1$ in (2.9), we have $b_1^2 - b_0 b_2 = q^{d-2}$ as $\Psi_0 = 1$, hence $\frac{b_2}{b_0} = (\frac{b_1}{b_0})^2 - q^{d-2} (\frac{1}{b_0})^2$, which gives the expression for $\frac{b_2}{b_0}$.

By (2.9), we have $e b_1 = q b_2 + Q b_0$, which gives $Q = e \frac{b_1}{b_0} - q \frac{b_2}{b_0}$, the expression for Q , as $\frac{b_2}{b_0}$ is a polynomial in $q, a_0, \frac{1}{b_0}, \frac{b_1}{b_0}$.

By (2.9), we have $b_1 b_k - b_0 b_{k+1} = q^{d-k-1} \Psi_{k-1}(q, Q, e)$, which gives inductively the expressions for $\frac{b_k}{b_0}$ for all $k \geq 3$.

For $1 \leq j \leq d-1$, we have $a_j = e b_j - 2q b_{j+1}$. Replacing $e = \frac{1}{b_0} a_0 + 2q \frac{b_1}{b_0}$ and using (2.9), we have

$$a_j = a_0 \frac{b_j}{b_0} + 2q \frac{b_1 b_j - b_0 b_{j+1}}{b_0} = a_0 \frac{b_j}{b_0} + 2q^{d-j} \Psi_{j-1}(q, Q, e) \frac{1}{b_0}.$$

It remains to express a_d . Recall that $a_d = 2Q b_{d-1} - e b_d$ and $a_1 = 2Q b_0 - e b_1$ by (2.7). This gives

$$a_d = 2Q b_0 \frac{b_{d-1}}{b_0} - e b_d = a_1 \frac{b_{d-1}}{b_0} + e \frac{b_1 b_{d-1} - b_0 b_d}{b_0} = a_1 \frac{b_{d-1}}{b_0} + e \Psi_{d-2}(q, Q, e) \frac{1}{b_0},$$

as desired. This completes the proof for \mathscr{Y}_0 (and the proof for \mathscr{Y}_d is similar).

(b) For ease of notation, let $B_r = \frac{1}{b_r}$, $B_j = \frac{b_j}{b_r}$ for $0 \leq j \leq d$, $j \neq r$. By (2.7) we have $a_r = e b_r - 2q b_{r+1}$, hence $e = a_r B_r + 2q B_{r+1}$. (We also have $e = 2Q B_{r-1} - a_r B_r$.) For $j > r+1$, the equation $b_{r+1} b_{j-1} - b_r b_j = q^{d-j} Q^r \Psi_{j-r-2}(q, Q, e)$ allows us to iteratively express B_j in terms of q, Q , and e ; we can deal with B_0, \dots, B_{r-2} similarly.

For $0 \leq j < r$ we substitute $e = a_r \cdot \frac{1}{b_r} + 2q \frac{b_{r+1}}{b_r}$ into (2.7) to obtain:

$$a_j = \left(a_r \frac{1}{b_r} + 2q \frac{b_{r+1}}{b_r} \right) b_j - 2q b_{j+1} = a_r B_j + 2q^{d-j+2} Q^r \Psi_{j-r-3}(q, Q, e) B_r.$$

Similarly, we substitute $e = 2Q B_{r-1} - a_r B_r$ into (2.7) to express a_{r+1}, \dots, a_d in terms of a_r, q, Q and e . Thus we have obtained $\mathbb{C}[q, Q, e, a_0, \dots, a_d, B_0, \dots, B_d] = \mathbb{C}[q, Q, a_r, B_r, B_{r-1}, B_{r+1}]$. We have two relations:

$$(2.11) \quad q B_{r+1} - Q B_{r-1} + a_r B_r = 0 \quad \text{and} \quad B_{r-1} B_{r+1} + q^{d-r-1} Q^{r-1} B_r^2 - 1 = 0.$$

The first of these follows from the equality of the two expressions for e ; the second from (2.9). It is straightforward to check that the Jacobian has rank 2 (under the same conditions on $q, Q, a_r, B_r, B_{r-1}, B_{r+1}$), which implies that \mathscr{Y}_r is smooth (of dimension 4). More specifically, it implies that the subvariety of \mathbb{A}^6 with equations (2.11) is smooth of dimension 4; by dimensions we know that \mathscr{Y}_r is isomorphic to one of its irreducible components. \square

2.D. Singularity of the blowup. — By Lemma 2.10, the singularities of the blowup $\widehat{\mathscr{Q}}(d)$ are contained in the affine open subset $\mathscr{Y}(d)$. First, by definition, $\mathbb{C}[\mathscr{Y}(d)] = \mathbb{C}[V \times V^*]^{W_d}[b_0, b_1, \dots, b_d]$. Using (2.7), we see that

$$(2.12) \quad \mathbb{C}[\mathscr{Y}(d)] = \mathbb{C}[q, Q, e, b_0, b_1, \dots, b_d].$$

This shows that $\mathscr{Y}(d)$ is a closed subvariety of \mathbb{C}^{d+4} and the \mathbb{C}^\times -action on $\mathscr{Y}(d)$ is contracting (to 0). Here is a presentation of $\mathscr{Y}(d)$ as well as a first result on its singularities:

Proposition 2.13. — *Equations for $\mathscr{Y}(d) \subset \mathbb{C}^{d+4}$ are given by (2.9). Moreover, $\mathscr{Y}(d)$ is normal and 0 is its only singular point.*

Proof. — Let \mathscr{Y} be the affine scheme defined by the equations (2.9). These equations being homogeneous (recall that e, q and Q have degree 2 and b_0, b_1, \dots, b_d have degree $d-2$), $\mathbb{C}[\mathscr{Y}]$ is \mathbb{N} -graded, with 0-component equal to \mathbb{C} . Equations (2.9) show that we have a surjective morphism $\mathbb{C}[\mathscr{Y}] \rightarrow \mathbb{C}[\mathscr{Y}(d)]$, hence a closed embedding $\mathscr{Y}(d) \hookrightarrow \mathscr{Y}$.

Let $\widehat{\mathbb{C}[\mathscr{Y}]_0}$ denote the completion of $\mathbb{C}[\mathscr{Y}]$ at the point 0 and, similarly, let $\widehat{\mathbb{C}[\mathscr{Y}(d)]_0}$ denote the completion of $\mathbb{C}[\mathscr{Y}(d)]$ at the point 0. The comparison of the definition of $\mathscr{Y}(d)$ with the equations (2.9) shows that we have an isomorphism

$$(\#) \quad \widehat{\mathbb{C}[\mathscr{Y}]_0} \simeq \widehat{\mathbb{C}[\mathscr{Y}(d)]_0}$$

given by $e \mapsto e, q \mapsto q, Q \mapsto Q$ and $b_j \mapsto a_j / \sqrt{d^2 + 4qQ - e^2}$ for $0 \leq j \leq d$.

By Theorem 2.3, the variety $\mathcal{X}(d)$ is normal so $\widehat{\mathbb{C}[\mathcal{X}(d)]}_0$ is a normal domain by Zariski's main Theorem. This shows that $\widehat{\mathbb{C}[\mathcal{Y}]}_0$ is a normal domain. As the natural map $\mathbb{C}[\mathcal{Y}] \rightarrow \widehat{\mathbb{C}[\mathcal{Y}]}_0$ is injective (because $\mathbb{C}[\mathcal{Y}]$ is \mathbb{N} -graded and $\widehat{\mathbb{C}[\mathcal{Y}]}_0$ is the completion with respect to the unique maximal homogeneous ideal), this shows that $\mathbb{C}[\mathcal{Y}]$ is a domain. Therefore,

$$\text{Kdim } \widehat{\mathbb{C}[\mathcal{Y}]}_0 = \text{Kdim } \widehat{\mathbb{C}[\mathcal{X}(d)]}_0 = \text{Kdim } \mathbb{C}[\mathcal{X}(d)] = 4,$$

where Kdim denotes the Krull dimension. Hence \mathcal{Y} is irreducible reduced of dimension 4. As $\mathcal{Y}(d)$ is irreducible of dimension 4, this shows that $\mathcal{Y} = \mathcal{Y}(d)$. In other words, equations for $\mathcal{Y}(d) \subset \mathbb{C}^{d+4}$ are given by (2.9). Moreover, (#) becomes

$$(2.14) \quad \widehat{\mathbb{C}[\mathcal{Y}(d)]}_0 \simeq \widehat{\mathbb{C}[\mathcal{X}(d)]}_0$$

In particular, the singularities $(\mathcal{X}(d), 0)$ and $(\mathcal{Y}(d), 0)$ are locally analytically isomorphic, so 0 is an isolated singularity of $\mathcal{Y}(d)$. Since the \mathbb{C}^\times -action on $\mathcal{Y}(d)$ is contracting, any irreducible component of the singular locus of $\mathcal{Y}(d)$ must contain 0. Thus, 0 is the only singularity of $\mathcal{Y}(d)$.

Finally, $\mathcal{Y}(d)$ is normal at 0 because its completion at 0 is normal, so $\mathcal{Y}(d)$ is normal because all other points are smooth. The proof of the proposition is complete. \square

When viewed inside $\tilde{\mathcal{Q}}(d)$, the point 0 of $\mathcal{Y}(d)$ will still be denoted by 0, as in the Introduction.

Corollary 2.15. — *The blowup $\tilde{\mathcal{Q}}(d)$ is normal and admits a unique singular point, namely the point $0 \in \mathcal{Y}(d)$. The singularities $(\tilde{\mathcal{Q}}(d), 0)$ and $(\mathcal{X}(d), 0)$ are locally analytically isomorphic.*

Remark 2.16. — (1) If V is replaced by \mathbb{C}^n and W_d is replaced by the symmetric group \mathfrak{S}_n , then the analogue of $\tilde{\mathcal{Q}}(d)$ is the Hilbert scheme of n points in the plane by results of Haiman [Hai, Proposition 2.6]. This gives the symplectic resolution of $(\mathbb{C}^n \times \mathbb{C}^n)/\mathfrak{S}_n$.

(2) The fact that $\tilde{\mathcal{Q}}(5)$ has an isolated singularity has been observed in [FJLS, §12.3.1]. \blacksquare

3. Other descriptions of the blowup

3.A. Orbit closures. — We denote by $\mathbf{GSp}(V \times V^*)$ the general symplectic group of $V \times V^*$, which is the group of linear automorphisms of $V \times V^*$ which preserve the symplectic form up to a scalar. Since W_d is a Coxeter group, the finite group W_d commutes with a subgroup of $\mathbf{GSp}(V \times V^*)$ isomorphic to $\mathbf{GL}_2 = \mathbf{GL}_2(\mathbb{C})$ (see for instance [BoRo, §3.6]): to see this, note that $V \simeq V^*$ as a W_d -module, so $V \times V^* \simeq \mathbb{C}^2 \otimes V$ with W_d acting trivially on \mathbb{C}^2 and the action of \mathbf{GL}_2 is on the left hand side of this tensor product decomposition. Note that the restriction of this action to \mathbb{C}^\times (identified with the center of \mathbf{GL}_2) is by homothety, so is the action considered in §2.C.

Remark 3.1. — Note that the above action of \mathbf{GL}_2 on $V \times V^*$ does not coincide with the action of $\mathbf{GL}_{\mathbb{C}}(V) = \mathbf{GL}_2(\mathbb{C})$ given by the natural action on V and the contragredient action on V^* . To distinguish both actions, the natural module for \mathbf{GL}_2 will be denoted by U_2 (as in the subsection about conventions/nomenclature in the Introduction) and we use the notation \mathbf{GL}_2 (instead of $\mathbf{GL}_2(\mathbb{C})$) when we talk about the action respecting the symplectic form on $V \times V^*$ up to scalar. The canonical basis of U_2 will be denoted by $(\varepsilon_1, \varepsilon_2)$.

Here, an explicit isomorphism of W_d -modules $V \xrightarrow{\sim} V^*$ is given for instance by $x \mapsto Y$, $y \mapsto X$. This leads to the following formulas:

$$\text{Action of } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2 \text{ on } V \times V^* \rightsquigarrow \begin{cases} x \mapsto ax + cY, \\ y \mapsto ay + cX, \\ X \mapsto by + dX, \\ Y \mapsto bx + dY. \end{cases}$$

Note that the action of \mathbf{SL}_2 is symplectic while the action of \mathbb{C}^\times scales the symplectic form (and so scales the Poisson bracket). ■

Hence \mathbf{GL}_2 acts on $\mathbb{C}[q, Q, e, a_0, \dots, a_d]$; by a straightforward calculation, q, Q, e span a copy of the adjoint representation \mathfrak{sl}_2 tensored with the determinant, and a_0, a_1, \dots, a_d span a copy of $\text{Sym}^d(U_2)$ where U_2 is the natural representation for \mathbf{GL}_2 . We denote by $\mathfrak{sl}_2^{(1)}$ the above representation of \mathbf{GL}_2 : its underlying space is \mathfrak{sl}_2 (and we will forget the exponent ⁽¹⁾ whenever we forget the \mathbf{GL}_2 -action) but the usual adjoint action of \mathbf{GL}_2 on \mathfrak{sl}_2 is multiplied with the determinant. This discussion shows that $\mathcal{Q}(d)$ may be viewed as a \mathbf{GL}_2 -stable closed subvariety of $\mathfrak{sl}_2^{(1)} \oplus \text{Sym}^d(U_2)$.

This action is very useful for understanding the geometry of $\mathcal{Q}(d)$. While there are infinitely many orbits for the action of \mathbf{SL}_2 on its adjoint representation, there are only two non-zero orbits for the above action of \mathbf{GL}_2 on $\mathfrak{sl}_2^{(1)}$, with representatives

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In particular, the \mathbf{GL}_2 -orbit of any non-zero semisimple element is dense in $\mathfrak{sl}_2^{(1)}$. By the same token:

Lemma 3.2. — *Viewing $\mathcal{Q}(d)$ as a \mathbf{GL}_2 -stable closed subvariety of $\mathfrak{sl}_2^{(1)} \oplus \text{Sym}^d(U_2)$ as above, we have:*

- (a) *The variety $\mathcal{Q}(d)$ is the closure of the \mathbf{GL}_2 -orbit of $(H, \varepsilon_1^d + \varepsilon_2^d)$ in $\mathfrak{sl}_2^{(1)} \oplus \text{Sym}^d(U_2)$.*
- (b) *If d is odd (resp. even), then the singular locus of $\mathcal{Q}(d)$ is the closure of the \mathbf{GL}_2 -orbit of (E, ε_1^d) (resp. of the union of the \mathbf{GL}_2 -orbits of (E, ε_1^d) and $(E, -\varepsilon_1^d)$).*

Proof. — It follows immediately from the generating set for the ring of invariants and the above remarks that $(V \times V^*)/W_d$ is isomorphic (via a closed immersion) to a Zariski closed \mathbf{GL}_2 -stable subset of $\mathfrak{sl}_2^{(1)} \oplus \text{Sym}^d(U_2)$. Inspecting the relations for the generators, we see that if $q = Q = 0$ and $e = 1$ (corresponding to a non-zero multiple of H in \mathfrak{sl}_2) then $a_j = 0$ for $1 \leq j \leq d-1$ and $a_0 a_d = 1$. It is easy to see that the stabilizer of the point $(H, \varepsilon_1^d + \varepsilon_2^d) = (0, 0, 1, 1, 0, \dots, 0, 1)$ in \mathbf{GL}_2 is finite, which (after suitably scaling a basis of $\text{Sym}^d(U_2)$ if necessary) proves (a).

To establish (b), note that the singular locus has dimension 2, and all non-zero \mathbf{GL}_2 -orbits have dimension 2 or 4; the orbit is dense if and only if the projection to $\mathfrak{sl}_2^{(1)}$ is semisimple. Thus, the singular locus is a union of orbits of elements of the form (E, u) or $(0, u)$. Now we recall that the singular locus is the image in $(V \times V^*)/W_d$ of the points with $x = \zeta^k y$, $Y = \zeta^k X$, in which case $q = \zeta^k y^2$, $e = 2\zeta^k yX$, $Q = \zeta^k X^2$ and $a_j = 2y^{d-j} X^j$.

If $q = e = Q = 0$ then clearly $a_j = 0$ for all j , so there are no non-zero orbits in the singular locus of the form $(0, u)$; for the orbits of the form (E, u) , we assume $q = e = 0$ so that $y = 0$. Then $Q = \zeta^k X^2$, $a_0 = \dots = a_{d-1} = 0$ and $a_d = 2X^d$. In particular, $a_d^2 = 4Q^d$, so there are at most two such orbits for each fixed non-zero value of Q ; if d is odd then the two different possibilities for a_d are obviously conjugate via the μ_2 -action; if d is even then $2Q^{d/2} = \pm a_d$ accordingly as k is even or odd, so we do obtain two disjoint orbits. □

An immediate computation shows that $g(\delta) = \det(g)\delta$ for $g \in \mathbf{GL}_2$ (this will be better explained in Remark 3.4) and that the subspace $\mathbb{C}\beta_0 \oplus \mathbb{C}\beta_1 \oplus \cdots \oplus \mathbb{C}\beta_d$ is \mathbf{GL}_2 -stable and is isomorphic to the representation $\mathrm{Sym}^d(U_2)$. In particular, δ is \mathbf{SL}_2 -invariant (note that $D = \delta^2$ corresponds to the determinant in \mathfrak{sl}_2). It follows that \mathbf{GL}_2 acts on the blowup $\hat{\mathcal{Q}}(d)$ and stabilizes the affine open subset $\mathcal{Y}(d) = \mathrm{Spec} \mathbb{C}[q, Q, e, b_0, \dots, b_d]$. Now, the previous remark shows that $\mathbb{C}b_0 \oplus \mathbb{C}b_1 \oplus \cdots \oplus \mathbb{C}b_d$ is \mathbf{GL}_2 stable and is isomorphic to the representation $\mathrm{Sym}^d(U_2)^{(-1)}$, where the exponent (-1) means that the action on $\mathrm{Sym}^d(U_2)^{(-1)}$ is the natural action on $\mathrm{Sym}^d(U_2)$ tensored with the inverse of the determinant. In particular, for the \mathbb{C}^\times -action, b_0, b_1, \dots, b_d have weight $d-2$.

This discussion shows that $\mathcal{Y}(d)$ may be viewed as a \mathbf{GL}_2 -stable closed subvariety of $\mathfrak{sl}_2^{(1)} \oplus \mathrm{Sym}^d(U_2)^{(-1)}$.

Lemma 3.3. — *Through this embedding, $\mathcal{Y}(d)$ is the closure of the \mathbf{GL}_2 -orbit of $(H, \varepsilon_1^d - \varepsilon_2^d)$.*

Proof. — Inspecting the relations (2.9), we see that if $e = 1$ and $q = Q = 0$ then $b_j = 0$ for $1 \leq j \leq d-1$ and $b_0 b_d = -1$. We can now complete the proof with the same argument as in Lemma 3.2(a). \square

Remark 3.4. — Via the inclusion $\mathfrak{sl}_2 \simeq \mathfrak{so}_3 \subset \mathfrak{sl}_3$, we have a \mathbf{GL}_2 -equivariant isomorphism

$$\begin{array}{ccc} \mathfrak{sl}_2^{(1)} \oplus \mathrm{Sym}^4(U_2)^{(-1)} & \longrightarrow & \mathfrak{sl}_3 \\ E & \longmapsto & \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \\ H & \longmapsto & \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \\ F & \longmapsto & \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix} \\ \varepsilon_1^4 & \longmapsto & \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \varepsilon_1^3 \varepsilon_2 & \longmapsto & \begin{pmatrix} 0 & \sqrt{2}/2 & 0 \\ 0 & 0 & \sqrt{2}/2 \\ 0 & 0 & 0 \end{pmatrix} \\ \varepsilon_1^2 \varepsilon_2^2 & \longmapsto & \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} \\ \varepsilon_1 \varepsilon_2^3 & \longmapsto & \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2}/2 & 0 & 0 \\ 0 & -\sqrt{2}/2 & 0 \end{pmatrix} \\ \varepsilon_2^4 & \longmapsto & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \end{array}$$

In particular, $H + \varepsilon_1^4 - \varepsilon_2^4 \longmapsto \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & -2 \end{pmatrix}$, which is of rank 1 and trace 0, i.e. an element of the minimal nilpotent orbit.

Now, the \mathbf{GL}_2 -orbit of $H + \varepsilon_1^4 - \varepsilon_2^4$ has dimension 4 by Lemma 3.3 and is contained in the minimal nilpotent orbit which has also dimension 4. This proves that $\mathcal{Y}(4)$ is isomorphic to the closure of the minimal nilpotent orbit of \mathfrak{sl}_3 . This provides a MAGMA free proof of Theorem 2.3(b) and (d). \blacksquare

3.B. Singular locus of $\mathcal{Q}(d)$. — In the previous subsection, we have identified $\mathcal{Q}(d)$ with a closed subvariety of $\mathfrak{sl}_2^{(1)} \oplus \mathrm{Sym}^d(U_2)$. It is easily seen that $I = (\delta \mathbb{C}[V \times V^*])^{W_d}$ is the ideal vanishing on elements of $\mathcal{Q}(d)$ whose projection onto \mathfrak{sl}_2 is nilpotent. Now, let

I_{sing} denote the ideal of $\mathbb{C}[V \times V^*]^{W_d}$ vanishing on the reduced singular locus $\mathcal{Q}_{\text{sing}}(d)$ of $\mathcal{Q}(d)$.

Lemma 3.5. — *The ideal I_{sing} is generated by I and $(\beta_j \beta_k)_{0 \leq j, k \leq d}$.*

Proof. — Note that $\beta_j \beta_k = D b_j b_k$ obviously vanishes at any singular point. We will use the action of \mathbf{SL}_2 on the coordinate ring; clearly I_{sing} and the singular locus are stable under this action.

Since $\beta_j \beta_k \equiv a_j a_k$ modulo $\mathbb{C}[q, Q, e]$, it follows that $\mathbb{C}[V \times V^*]^{W_d} / I_{\text{sing}}$ is generated as a $\mathbb{C}[q, Q, e] / (e^2 - 4qQ)$ -module by $1, a_0, \dots, a_d$. Now $\mathbb{C}[q, Q, e] / (e^2 - 4qQ)$ is the coordinate ring of the nullcone of \mathfrak{sl}_2 ; by a well-known theorem of Kostant [Kos, Theorem 0.9], it contains one copy of (the \mathbf{SL}_2 -module) $\text{Sym}^{2l}(U_2)$ in each even degree $2l$. Inspecting the relations (2.7) and using the fact that $e^2 - 4qQ \in I_{\text{sing}}$, we find that the following monomials form a basis of the submodule of $\mathbb{C}[V \times V^*]^{W_d} / I_{\text{sing}}$ spanned by a_0, \dots, a_d :

$$a_0, \dots, a_d; qa_0, ea_0, ea_1, \dots, ea_d, Qa_d; q^2 a_0, qea_0, qQa_0, qQa_1, \dots, qQa_d, Qea_d, Q^2 a_d; \dots$$

and in particular there is one copy of $\text{Sym}^{d+2l} U$ in each degree $d+2l$. For d odd, this is especially useful: the quotient $\mathbb{C}[V \times V^*]^{W_d} / I_{\text{sing}}$ decomposes over \mathbf{SL}_2 as a sum of non-isomorphic irreducible submodules, each one appearing in its own degree; for d even, the submodules $\text{Sym}^{2l}(U_2)$ with $2l \leq d$ are doubled. By the \mathbf{SL}_2 -action, it only remains to determine which highest weight vectors vanish on the singular locus. These highest weight vectors are: Q^j and $Q^j a_d$ for d odd; Q^j and any linear span of $Q^j a_d, Q^{j+d/2}$ if d is even. Our result will be proved if we can establish that none of these highest weight vectors vanishes on $\mathcal{Q}_{\text{sing}}(d)$.

Recall that the singular locus is the set of orbits of points satisfying $x = \zeta^k y, Y = \zeta^k X$ for some k . Then $Q = \zeta^k X^2$ and $a_d = 2X^d$. None of these vanish, so for d odd, we obtain our result immediately. If $d = 2m$ is even then $Q^m = \pm X^d$ accordingly as k is even or odd; these two possibilities correspond to the two irreducible components of $\mathcal{Q}_{\text{sing}}(d)$. It follows that there do exist highest weight vectors in degree $d+2l$ vanishing on each of these irreducible components (specifically, $Q^l(a_d \mp 2Q^m)$). However, no non-zero linear span of $Q^l a_d$ and Q^{m+l} can vanish on the *whole* singular locus, so our proof is complete. \square

Corollary 3.6. — *For $d = 2m$ even, the ideals of the two irreducible components of the singular locus are:*

$$J_1 = (D, a_0 - 2q^m, a_1 - q^{m-1}e, a_2 - 2q^{m-1}Q, \dots, a_d - 2Q^m), \quad \text{and} \\ J_2 = (D, a_0 + 2q^m, a_1 + q^{m-1}e, a_2 + 2q^{m-1}Q, \dots, a_d + 2Q^m).$$

For ease of notation, let $a_{2j}^\pm := a_{2j} \pm 2q^{m-j}Q^j$ and $a_{2j+1}^\pm := a_{2j+1} \pm q^{m-j-1}Q^{j+1}e$. Note that:

$$a_{2j}^\pm = (x^{m-j}Y^j \pm y^{m-j}X^j)^2 \quad \text{and} \\ a_{2j+1}^\pm = (x^{m-j}Y^j \pm y^{m-j}X^j)(x^{m-j-1}Y^{j+1} \pm y^{m-j-1}X^{j+1}),$$

so that the generators of J_1 (resp. J_2) can be expressed naturally in terms of the generating invariants (resp. semi-invariants) for the dihedral group of order $2m$.

Proof. — By the proof of Lemma 3.5, it will suffice to show that $I_{\text{sing}} \subset J_1$ and $I_{\text{sing}} \subset J_2$. It follows easily from (2.7) that $D b_j \in J_1$ and $D b_j \in J_2$. Since $D b_j b_k - D b_{j-1} b_{k+1} \in (D)$, it will therefore suffice to show that each of $D b_0^2, D b_0 b_1, D b_1^2, \dots, D b_d^2$ belongs to $J_1 \cap J_2$. We recall that

$$D b_{2j}^2 = (x^{2m-2j}Y^{2j} - y^{2m-2j}X^{2j})^2 = a_{2j}^- a_{2j}^+ \in J_1 \cap J_2.$$

Similarly,

$$Db_{2j+1}^2 = \frac{1}{4} \left(a_{2j}^+ a_{2j+2}^- + 2a_{2j+1}^+ a_{2j+1}^- + a_{2j}^- a_{2j+2}^+ \right) \text{ and } Db_j b_{j+1} = \frac{1}{2} \left(a_j^+ a_{j+1}^- + a_j^- a_{j+1}^+ \right).$$

□

Proposition 3.7. — *The blowup $\tilde{\mathcal{Q}}(d)$ of $\mathcal{Q}(d)$ at I is isomorphic to the blowup of $\mathcal{Q}(d)$ at its reduced singular locus. Moreover, the symplectic form on $V \times V^*$ induces a symplectic structure on $\tilde{\mathcal{Q}}(d) \setminus \{0\}$.*

Proof. — The first assertion follows from the description of the blowup at I_{sing} as the closure of the set of elements of $\mathbb{A}^{d+4} \times \mathbb{P}^{(d+1)(d+2)/2}$ of the form

$$((q, Q, e, a_0, \dots, a_d), [D : Db_0 : \dots : Db_d : Db_0^2 : Db_0 b_1 : \dots : Db_d^2]),$$

where at least one of D, Db_0, \dots, Db_d^2 is non-zero; the projection to $\mathbb{P}^{(d+1)(d+2)/2}$ is just the Veronese embedding applied to $[D : Db_0 : \dots : Db_d] \in \mathbb{P}^{d+1}$.

Let us now prove the second statement. We can consider $\mathcal{Q}(d)$ as the μ_2 -quotient of $(V \times V^*)/\Gamma_d$. Since there are no non-zero fixed points for the action of Γ_d , the quotient $(V \times V^*)/\Gamma_d$ has an isolated singularity. For $p \in \mathcal{Q}_{\text{sing}}(d) \setminus \{0\}$, $\mathcal{Q}(d)$ has A_1 -singularity at s . As $\pi : \tilde{\mathcal{Q}}(d) \rightarrow \mathcal{Q}(d)$ is the blowup of the singular locus, it restricts to the minimal resolution to $\mathcal{Q}(d) \setminus \{0\}$. This implies that $\tilde{\mathcal{Q}}(d) \setminus \pi^{-1}(0)$ admits a symplectic structure (say $\tilde{\omega}$) coming from that on $V \times V^*$. As $\pi^{-1}(0)$ has dimension 2, $\tilde{\omega}$ extends to a symplectic structure on the smooth locus of $\tilde{\mathcal{Q}}(d)$, which is just $\tilde{\mathcal{Q}}(d) \setminus \{0\}$ by Corollary 2.15. □

By Corollary 2.15, the singularities $(\tilde{\mathcal{Q}}(d), 0)$ and $(\mathcal{X}(d), 0)$ are analytically isomorphic but this does not ensure that the variety $\tilde{\mathcal{Q}}(d)$ inherits a symplectic form on its smooth locus. Thanks to the previous proposition, we can now deduce:

Corollary 3.8. — *The variety $\tilde{\mathcal{Q}}(d)$ has an isolated symplectic singularity.*

The singularity $\tilde{\mathcal{Q}}(d)$ does not admit a projective symplectic resolution. Since the blowup morphism is crepant (being Poisson) it would imply by composition that $(V \times V^*)/W_d$ admits a projective symplectic resolution, which is not the case. In fact, this also follows from the general fact that the only 4-dimensional isolated symplectic singularity admitting a symplectic resolution is analytically isomorphic to the minimal nilpotent orbit closure in \mathfrak{sl}_3 (see [WiWi, Theorem 1.1]).

Note that the symplectic singularity $(\mathcal{X}(d), 0)$ does not admit a contracting \mathbb{C}^\times -action. Therefore, the fact that symplectic singularity $(\mathcal{Y}(d), 0)$ is locally analytically isomorphic to $(\mathcal{X}(d), 0)$ and admits a contracting \mathbb{C}^\times -action can be viewed as confirmation of a general conjecture of Kaledin [Kal, Conjecture 1.8] in this particular case.

4. Local fundamental group

In order to compute the local fundamental group of $\mathcal{Y}(d)$ around 0, we will consider the following smooth irreducible surface

$$\mathcal{S} = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 - y^2 z = 1\}.$$

We define the morphism $\varphi : \mathcal{S} \rightarrow \mathcal{Y}(d) \setminus \{0\}$ by

$$\varphi(x, y, z) = (1, -z, 0, y, x, yz, xz, \dots, \underbrace{yz^k}_{b_{2k}}, \underbrace{xz^k}_{b_{2k+1}}, \dots).$$

It's easy to see that φ is a closed immersion, whose image is equal to

$$\{(q, Q, e, b_0, b_1, \dots, b_d) \in \mathcal{Y}(d) \mid q = 1 \text{ and } e = 0\}.$$

Lemma 4.1. — *The smooth surface \mathcal{S} is simply-connected.*

Proof. — Let $p_0 = (1, 1, 0) \in \mathcal{S}$. Let \mathcal{S}° denote the open subset $\{(x, y, z) \in \mathcal{S} \mid y \neq 0\}$. Then $p_0 \in \mathcal{S}^\circ$ and the map $\pi_1(\mathcal{S}^\circ, p_0) \rightarrow \pi_1(\mathcal{S}, p_0)$ is surjective [God, Theorem 2.3]. Now, $\mathbb{C} \times \mathbb{C}^\times \simeq \mathcal{S}^\circ$ through the variables (x, y) and the map $(x, y) \mapsto (x, y, y^{-2}(x^2 - 1))$, so $\pi_1(\mathcal{S}^\circ, p_0)$ is generated by the loop γ defined by $\gamma(t) = (1, e^{2\sqrt{-1}\pi t}, 0)$. Then it remains to show that γ is homotopic, in \mathcal{S} , to the trivial loop. But γ is contained in $\{1\} \times \mathbb{C} \times \{0\} \subset \mathcal{S}$, so the result follows. \square

Coming back to our aim of proving Theorem 1.3, it remains to prove that the local fundamental group of the singularity $(\mathcal{Y}(d), 0)$ is trivial. Moreover, the existence of a contracting \mathbb{C}^\times -action implies that the local fundamental group of $(\mathcal{Y}(d), 0)$ is just the fundamental group of $\mathcal{Y}(d) \setminus \{0\}$. The above discussion shows that it only remains to check the following proposition, whose proof uses in an essential way the \mathbf{SL}_2 -action:

Proposition 4.2. — *The variety $\mathcal{Y}(d) \setminus \{0\}$ is simply-connected.*

Proof. — We fix a base point $p = (0, 0, 1, 1, 0, \dots, 0, 1) \in \mathcal{Y}(d) \setminus \{0\}$ and we want to show that $\pi_1(\mathcal{Y}(d) \setminus \{0\}, p) = 1$. We divide the proof into several steps.

• *First step: projection to \mathfrak{sl}_2 .* First, if $(q, Q, e) \in \mathbb{C}^3$, we denote by $M(q, Q, e)$ the matrix

$$M(q, Q, e) = \begin{pmatrix} e & 2Q \\ -2q & -e \end{pmatrix} \in \mathfrak{sl}_2.$$

Viewing $\mathcal{Y}(d)$ as a closed subvariety of $\mathfrak{sl}_2 \oplus \text{Sym}^d(U_2)$ as in Lemma 3.3 (we have forgotten the exponents (1) and (-1) in $\mathfrak{sl}_2 \oplus \text{Sym}^d(U_2)$ because we are only considering the \mathbf{SL}_2 -action, and not the \mathbf{GL}_2 -action), the map $\text{pr} : \mathcal{Y}(d) \rightarrow \mathfrak{sl}_2$ obtained by projecting to the first component is given by

$$\text{pr}(q, Q, e, b_0, b_1, \dots, b_d) = M(q, Q, e).$$

It is \mathbf{SL}_2 -equivariant. Also, the matrices $\text{pr}(p) = M(0, 0, 1) = \text{diag}(1, -1) = H$ and $M(1, -1/4, 0)$ are in the same $\mathbf{SL}_2(\mathbb{C})$ -orbit because they have the same non-zero determinant. We denote by g_0 an element of \mathbf{SL}_2 such that $g_0 M(0, 0, 1) g_0^{-1} = M(1, -1/4, 0)$.

Another easy property of this action is the description of its restriction to the diagonal torus of \mathbf{SL}_2 : let $\Delta^{\text{op}} : \mathbb{C}^\times \rightarrow \mathbf{SL}_2$, $\xi \mapsto \text{diag}(\xi, \xi^{-1})$. Then

$$(4.3) \quad \Delta^{\text{op}}(\xi) \cdot (q, Q, e, b_0, b_1, \dots, b_d) = (\xi^{-2}q, \xi^2Q, e, \xi^{-d}b_0, \xi^{2-d}b_1, \dots, \xi^d b_d).$$

• *Second step: fibration.* Now, let \mathcal{U} denote the open subset $(\det \circ \text{pr})^{-1}(\mathbb{C}^\times)$ of $\mathcal{Y}(d)$ and let \mathcal{V} denote the open subset $\det^{-1}(\mathbb{C}^\times)$ of $\mathfrak{sl}_2(\mathbb{C})$, so that $\mathcal{U} = \text{pr}^{-1}(\mathcal{V})$. Then \mathcal{V} is the open subset of regular semisimple elements. Moreover:

Fact 1. *The map $\det \circ \text{pr} : \mathcal{U} \rightarrow \mathbb{C}^\times$ is a principal fibration with fiber isomorphic to $\mathbf{SL}_2(\mathbb{C})/\Delta^{\text{op}}(\mu_d)$.*

Proof. — Let $\zeta \in \mathbb{C}^\times$. Let ξ be a square root of $-\zeta$ and set $\tau = \text{diag}(\xi, -\xi)$, so that $\tau \in \det^{-1}(\zeta)$. More precisely, $\det^{-1}(\zeta)$ is the \mathbf{SL}_2 -orbit of τ , and the stabilizer of τ in \mathbf{SL}_2 is the diagonal torus $\Delta^{\text{op}}(\mathbb{C}^\times)$. So it is sufficient to prove that $\Delta^{\text{op}}(\mathbb{C}^\times)$ acts transitively on $\text{pr}^{-1}(\tau)$ and that the stabilizer of some (any) point in $\text{pr}^{-1}(\tau)$ is equal to $\Delta^{\text{op}}(\mu_d)$.

Using (2.1) and the equations for $\mathcal{Y}(d)$, it follows that

$$\begin{aligned} \text{pr}^{-1}(\tau) = \{ & (q, Q, e, b_0, b_1, \dots, b_d) \in \mathbb{C}^{d+4} \mid q = Q = b_1 = \dots = b_{d-1} = 0, \\ & e = \xi \text{ and } b_0 b_d = -\xi^{d-2}\}. \end{aligned}$$

Now, the result follows from (4.3). \square

• *Third step: long exact sequence in homotopy.* Applying the long exact sequence in homotopy to Fact 1 yields a short exact sequence of groups

$$(4.4) \quad 1 \longrightarrow \pi_1(\mathbf{SL}_2(\mathbb{C})/\Delta^{\text{op}}(\boldsymbol{\mu}_d), 1) \longrightarrow \pi_1(\mathcal{U}, p) \longrightarrow \pi_1(\mathbb{C}^\times, -1) \longrightarrow 1.$$

But $\pi_1(\mathbf{SL}_2(\mathbb{C})/\Delta^{\text{op}}(\boldsymbol{\mu}_d), 1) \simeq \boldsymbol{\mu}_d$, generated by the loop $\alpha_0 : t \mapsto \Delta^{\text{op}}(e^{2\sqrt{-1}\pi t/d})\Delta^{\text{op}}(\boldsymbol{\mu}_d)$ while $\pi_1(\mathbb{C}^\times, -1) \simeq \mathbb{Z}$, generated by the loop $\beta_0 : t \mapsto e^{2\sqrt{-1}\pi t}$. Now, let

$$\begin{aligned} \alpha : [0, 1] &\longrightarrow \mathcal{U} \\ t &\longmapsto (0, 0, 1, e^{-2\sqrt{-1}\pi t}, 0, \dots, 0, -e^{2\sqrt{-1}\pi t}) \end{aligned}$$

and

$$\begin{aligned} \beta : [0, 1] &\longrightarrow \mathcal{U} \\ t &\longmapsto g_0^{-1}\varphi(1, 0, -e^{2\sqrt{-1}\pi t}/4)g_0. \end{aligned}$$

Then α is the image of α_0 in $\pi_1(\mathcal{U}, p)$ while β is a lift of β_0 in $\pi_1(\mathcal{U}, p)$. The short exact sequence of groups (4.4) implies that

$$(4.5) \quad \pi_1(\mathcal{U}, p) = \langle \alpha, \beta \rangle.$$

• *Last step: conclusion.* Recall the following classical fact:

Fact 2. *The map $\pi_1(\mathcal{U}, p) \longrightarrow \pi_1(\mathcal{Y}(d) \setminus \{0\}, p)$ is surjective.*

Proof. — By Proposition 2.13, the variety $\mathcal{Y}(d) \setminus \{0\}$ is smooth, and \mathcal{U} is a Zariski open subset of $\mathcal{Y}(d) \setminus \{0\}$, so the result follows from [God, Theorem 2.3]. \square

Fact 2 shows that it suffices to check that α and β are homotopy equivalent, in $\mathcal{Y}(d) \setminus \{0\}$, to the trivial loop. But the loops α and β are both contained in $g_0^{-1}\varphi(\mathcal{S})g_0$, which is simply-connected by Lemma 4.1. So the proof of the proposition is complete. \square

Together with Theorem 2.3, this concludes the proof of Theorem 1.3.

5. Quiver varieties and Slodowy slices

In this section we show that the isolated singularity $(\mathcal{X}(d), 0)$ is locally analytically isomorphic to a certain singularity $(\tilde{\mathcal{X}}(d), \tilde{x}_d)$ constructed as a cover of a type A Slodowy slice. The isomorphism passes through a Nakajima quiver variety.

We begin with a quiver of type \tilde{A} with d vertices, labelled $\rho_0, \dots, \rho_{d-1}$ and d arrows $\rho_i \rightarrow \rho_{i+1}$, where indices are taken modulo d . We extend this by adding a framing $\rho_\infty \rightarrow \rho_0$ and denote by Q the resulting quiver. Let \bar{Q} be the doubled quiver, which is independent of the choice of orientation of Q .

We let δ be the minimal imaginary root for the \tilde{A} root system, thought of as a dimension vector for \bar{Q} , and set $\mathbf{v} = \rho_\infty + 2\delta$. Let $Q = \bigoplus_{i=1}^{d-1} \mathbb{Z}\rho_i$ be the root lattice of the root system Φ of type A_{d-1} , with set of simple roots $\{\rho_1, \dots, \rho_{d-1}\}$. Then $Q^+ := \bigoplus_{i=1}^{d-1} \mathbb{Z}_{\geq 0}\rho_i$ and the set of positive roots is $\Phi^+ = Q^+ \cap \Phi$. Let $\alpha_h = \rho_1 + \dots + \rho_{d-1}$ be the highest positive root. For $1 \leq i < j \leq d-1$, we define $\alpha_{i,j} = \rho_i + \dots + \rho_j$, an element of Φ^+ . The symbol $<$ will refer to the dominance ordering on Q . We have

$$(5.1) \quad \{\alpha \in Q^+ \mid (\alpha, \alpha) = 2\} = \Phi^+.$$

Lemma 5.2. — *The following holds:*

$$(5.3) \quad \{\alpha \in Q^+ \mid (\alpha, \alpha) = 4, \text{ and } \alpha \leq 2\alpha_h\} = \{\alpha_{i,j} + \alpha_{k,l} \mid 1 \leq i < k < l < j \leq d-1\}$$

$$(5.4) \quad \cup \{\alpha_{i,j} + \alpha_{k,l} \mid 1 \leq i < j < k < l \leq d-1\}.$$

Proof. — When $1 \leq i < k < l < j \leq d-1$ (where the "shorter" root $\alpha_{k,l}$ sits on top of the "higher" root $\alpha_{i,j}$) or when $1 \leq i < j < k < l \leq d-1$ (where we have a disjoint union of two roots) we have $(\alpha_{i,j}, \alpha_{k,l}) = 0$ and hence

$$(\alpha_{i,j} + \alpha_{k,l}, \alpha_{i,j} + \alpha_{k,l}) = (\alpha_{i,j}, \alpha_{i,j}) + (\alpha_{k,l}, \alpha_{k,l}) = 2 + 2 = 4.$$

Therefore, the right hand sides of (5.3) and (5.4) are contained in the left hand side.

Assume now we have $\alpha \in Q^+$ with $(\alpha, \alpha) = 4$. If the support of α is disconnected, we can write $\alpha = \alpha^{(1)} + \alpha^{(2)}$ with the $\alpha^{(i)} \neq 0$ having orthogonal supports. Then $4 = (\alpha, \alpha) = (\alpha^{(1)}, \alpha^{(1)}) + (\alpha^{(2)}, \alpha^{(2)})$. Since $(\beta, \beta) \geq 2$ for all non-zero $\beta \in Q^+$, we deduce from (5.1) that $\alpha^{(i)} \in \Phi^+$ and hence α belongs to the set in (5.4).

Thus, we assume that the support of α is connected. As a (connected) subgraph of a type A Dynkin diagram is again a type A Dynkin diagram, we may assume without loss of generality that the support of α is the whole Dynkin diagram. Thus, $\alpha_h \leq \alpha \leq 2\alpha_h$ and we can write

$$\alpha = 2\alpha_h - (\alpha_{i_1, j_1} + \cdots + \alpha_{i_k, j_k})$$

where $1 \leq i_1 \leq j_1 < i_2 \leq \cdots < i_k \leq j_k \leq d-1$ and moreover $j_m + 1 < i_{m+1}$. We compute

$$\begin{aligned} (\alpha, \alpha) &= 4(\alpha_h, \alpha_h) + \sum_{m=1}^k (\alpha_{i_m, j_m}, \alpha_{i_m, j_m}) - 4 \sum_{m=1}^k (\alpha_h, \alpha_{i_m, j_m}) \\ &= 8 + 2k - 4(\delta_{i_1, 1} + \delta_{j_k, d-1}). \end{aligned}$$

If this equals 4 then we must have $k = 2$ and $i_1 = 1, j_2 = d-1$. Hence $\alpha = \alpha_{1, d-1} + \alpha_{j_1+1, i_2-1}$ belongs to the right hand side of (5.3). \square

Now we consider the parameter λ , where $\lambda_\infty = -2, \lambda_0 = 1$ and $\lambda_i = 0$ otherwise, for the quiver variety associated to Q . Notice that $\lambda \cdot \mathbf{v} = \lambda \cdot (\rho_\infty + 2\delta) = 0$. Therefore we can associate to it the affine quiver variety $\mathfrak{M}_\lambda(\mathbf{v})$ as defined in [BeSc, Section 1.1]. This space parameterises semi-simple representations of the deformed preprojective algebra $\Pi^\lambda(Q)$ of dimension \mathbf{v} . We define the set $\Sigma_\lambda(\mathbf{v})$ to be all positive roots α for Q such that

- (A) $\alpha \leq \mathbf{v}$ and $\lambda \cdot \alpha = 0$,
- (B) $p(\alpha) > p(\beta^{(1)}) + \cdots + p(\beta^{(k)})$ for all proper decompositions $\alpha = \beta^{(1)} + \cdots + \beta^{(k)}$ with $\beta^{(i)}$ a positive root, $\lambda \cdot \beta^{(i)} = 0$.

Here $p(\alpha) := 1 - (1/2)(\alpha, \alpha)$. A vector α belongs to $\Sigma_\lambda(\mathbf{v})$ if and only if there is a simple representation of $\Pi^\lambda(Q)$ of dimension $\alpha \leq \mathbf{v}$.

Lemma 5.5. — $\Sigma_\lambda(\mathbf{v}) = \{\mathbf{v}, \rho_\infty + 2\rho_0 + \alpha_h, \rho_\infty + 2\rho_0 + \rho_1 + \rho_{d-1}, \rho_1, \dots, \rho_{d-1}\}$.

Proof. — If $\alpha \in \Sigma_\lambda(\mathbf{v})$ then either $\alpha_\infty = 1$ or $\alpha_\infty = 0$. Consider first the latter. Since $\lambda \cdot \alpha = 0$, this implies that $\alpha_0 = 0$ too. That is, $\alpha \in \Phi^+$. But $\Sigma_\lambda(\mathbf{v}) \cap \Phi^+ = \{\rho_1, \dots, \rho_{d-1}\}$.

Therefore, we assume that $\alpha_\infty = 1$. As noted in [BeCr, Lemma 4.3], this implies that $\alpha = \rho_\infty + m\delta + (1/2)(v, v)\delta - v$, for some $v \in Q$ and $m \in \mathbb{Z}_{\geq 0}$. Since $\lambda \cdot \alpha = 0$, we must have $m + (1/2)(v, v) = 2$. Moreover, for $i = 1, \dots, d-1$, $m + (1/2)(v, v) - v_i \leq 2$ because $\delta_i = 1$. We deduce that $v \in Q^+$.

Next, for all $\alpha' = \rho_\infty + m'\delta + (1/2)(v', v')\delta - v'$ with $\alpha' < \alpha$ and $m' + (1/2)(v', v') = 2$, we require $p(\alpha) > p(\alpha')$ because $\alpha - \alpha'$ is a sum of real roots. If $\alpha > \alpha'$ then $v' > v$. Moreover, $2 - (1/2)(v, v) = p(\alpha) > p(\alpha') = 2 - (1/2)(v', v')$ if and only if $(v', v') > (v, v)$. Thus, we require:

$$(v', v') > (v, v) \text{ for all } v' \in Q^+, v' > v \text{ and } (v', v') \leq 4.$$

First, if $v = 0$ then $(v, v) = 0$ so the condition is vacuous. Next, we assume that $(v, v) = 2$. Then $v \in \Phi^+$ by (5.1) and the condition says v is maximal with respect to \leq on Φ^+ . In other words, $v = \alpha_h$ is the highest root. This corresponds to $\alpha = \rho_\infty + 2\rho_0 + \alpha_h$. Finally, if $(v, v) = 4$ then the condition says that v should be a maximal vector in the union of (5.3)

and (5.4). There is only one maximal vector, which is $\alpha_{1,d-1} + \alpha_{2,d-2}$. This corresponds to $\alpha = \rho_\infty + 2\rho_0 + \rho_1 + \rho_{d-1}$. \square

Lemma 5.6. — *The quiver variety $\mathfrak{M}_\lambda(\mathbf{v})$ has three symplectic leaves, $\mathfrak{M}_\lambda(\mathbf{v})_{\tau_0}$, $\mathfrak{M}_\lambda(\mathbf{v})_{\tau_2}$ and $\mathfrak{M}_\lambda(\mathbf{v})_{\tau_4}$, of dimension 0, 2 and 4 respectively.*

Proof. — As shown in [BeSc], the symplectic leaves of $\mathfrak{M}_\lambda(\mathbf{v})$ are labelled by the representation types $\tau = (\beta^{(1)}, n_1; \dots; \beta^{(k)}, n_k)$ of (\mathbf{v}, λ) . Here $\beta^{(i)} \in \Sigma_\lambda(\mathbf{v})$, $\mathbf{v} = n_1\beta^{(1)} + \dots + n_k\beta^{(k)}$ and the real roots in $\Sigma_\lambda(\mathbf{v})$ occur at most once amongst the $\beta^{(i)}$. The leaf labelled by τ has dimension $\sum_i 2p(\beta^{(i)})$. In our case, Lemma 5.5 implies that the possible representation types are

$$\begin{aligned}\tau_0 &= (\rho_\infty + 2\rho_0 + \rho_1 + \rho_{d-1}, 1; \rho_1, 1; \rho_2, 2; \dots; \rho_{d-1}, 1) \\ \tau_2 &= (\rho_\infty + 2\rho_0 + \alpha_h, 1; \rho_1, 1; \rho_2, 1; \dots; \rho_{d-1}, 1) \\ \tau_4 &= (\mathbf{v}, 1).\end{aligned}$$

\square

Since the symplectic leaf $\mathfrak{M}_\lambda(\mathbf{v})_{\tau_0}$ is zero-dimensional and connected, it equals $\{x_0\}$ for some point $x_0 \in \mathfrak{M}_\lambda(\mathbf{v})$.

Recall from the introduction that $\mathcal{S}_{d-2,2}$ is a Slodowy slice associated with the nilpotent orbit $\mathcal{O}_{d-2,2}$, that $\mathcal{X}(d) = \mathcal{S}_{d-2,2} \cap \mathcal{N}_d$, and that x_d is the unique element of $\mathcal{O}_{d-2,2} \cap \mathcal{S}_{d-2,2}$.

Theorem 5.7. — *The singularities $(\mathfrak{M}_\lambda(\mathbf{v}), x_0)$ and $(\mathcal{X}(d), x_d)$ are locally analytically isomorphic.*

Proof. — This is an application of Crawley-Boevey's étale local picture [C-B], together with the isomorphism of Nakajima [Nak] (see also [Ma]). The point $x_0 \in \mathfrak{M}_\lambda(\mathbf{v})_{\tau_0}$ corresponds to a semi-simple representation of the deformed preprojective algebra $\Pi^\lambda(\mathbb{Q})$ of the form

$$M = M_\infty \oplus M_1 \oplus M_2^{\oplus 2} \oplus \dots \oplus M_{d-2}^{\oplus 2} \oplus M_{d-1},$$

where all summands are simple, $\dim M_\infty = \rho_\infty + 2\rho_0 + \rho_1 + \rho_{d-1}$ and $\dim M_i = \rho_i$. Since the dimension vector of each simple summand is real, there is a unique (up to isomorphism) simple representation of that dimension.

Corollary 4.10 of [C-B] says that $(\mathfrak{M}_\lambda(\mathbf{v}), x_0)$ is (étale locally) isomorphic to 0 in another (framed) quiver variety; as shown in [BeSc, Theorem 3.3] this isomorphism is Poisson. The vertices e_i of the new quiver are in bijection with the M_i , and M_∞ corresponds to the framing data. There are $-(\dim M_i, \dim M_j) = \delta_{i,j-1} + \delta_{i,j+1}$ arrows from e_i to e_j in the doubled quiver. That is, the new (undoubled) quiver has underlying graph of Dynkin type A_{d-1} . Since all vectors are real there are no loops at any vertices. Finally there are

$$-(\dim M_\infty, \dim M_i) = \begin{cases} 1 & i = 2, d-2, \\ 0 & \text{otherwise,} \end{cases}$$

arrows from the framing vertex e_∞ to e_i . As a (framed) quiver variety,

$$\begin{array}{ccccccc} w : & & 1 & & & & 1 \\ & & | & & & & | \\ v : & 1 & \text{---} & 2 & \text{---} & 2 & \dots & 2 & \text{---} & 2 & \text{---} & 1 \end{array}$$

Here w is given by $w_2 = w_{d-2} = 1$ and $w_i = 0$ otherwise and $v_1 = v_{d-1} = 1$ and $v_2 = 2$ otherwise. Following the construction given in section 8 of [Nak], we see that $\mu = (d)$ and $\lambda = (d-2, 2)$. Therefore, the isomorphism of [Nak, Remark 8.5(2)] says that the above framed quiver variety is isomorphic to $\mathcal{X}(d)$. It follows from [Los, Lemma 4.6.4] that this isomorphism is as Poisson varieties. \square

Remark 5.8. — If M is a representation of Π^λ lying on the 2-dimensional symplectic leaf $\mathfrak{M}_\lambda(\mathbf{v})_{\tau_2}$ then

$$M = M_\infty \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_{d-1} \oplus M_{d-1},$$

with $\dim M_i = \rho_i$. The simple summands correspond to vertices $e_\infty, e_1, \dots, e_{d-1}$. The associated doubled quiver is the affine type A quiver, except now $p(\rho_\infty + 2\rho_0 + a_h) = 1$, which implies that there are two loops at e_∞ . The dimension vector is the minimal imaginary root $\delta = (1, \dots, 1)$. This implies that $(\mathfrak{M}_\lambda(\mathbf{v}), [M])$ is isomorphic to $(\mathbb{C}^2 \times (\mathbb{C}^2 / \mu_d), 0)$.

In the notation of [Mar, Section 7.3], we choose c_1 generic, $\underline{c} = 0$, and let $\mathcal{Z}(d, 1, 2)$ denote the spectrum of the centre of the rational Cherednik algebra associated to the wreath product group $G(d, 1, 2) = \mu_d \wr \mathfrak{S}_2$ at $t = 0$ and (c_1, \underline{c}) . By [Mar, Theorem 1.4], we have an isomorphism $\mathcal{Z}(d, 1, 2) \cong \mathfrak{M}_\lambda(\mathbf{v})$, which is Poisson up to a scalar factor. Therefore, Theorem 5.7 can be reinterpreted as saying that there is an isomorphism of symplectic singularities $(\mathcal{Z}(d, 1, 2), x_0) \cong (\mathcal{X}(d), x_d)$. Recall that the nilpotent cone \mathcal{N}_d of $\mathfrak{sl}_d(\mathbb{C})$ admits a μ_d -covering $\pi_d : \tilde{\mathcal{N}}_d \rightarrow \mathcal{N}_d$ which is unramified above the regular nilpotent orbit \mathcal{O}_d and bijective above the branch locus. We have set $\tilde{\mathcal{X}}(d) = \pi_d^{-1}(\mathcal{X}(d))$ and we denote by \tilde{x}_d the unique element in $\pi_d^{-1}(x_d)$. We are now ready to prove our second main result:

Theorem 5.9. — *If $d \geq 4$, the symplectic singularities $(\mathcal{Z}(d), 0)$ and $(\tilde{\mathcal{X}}(d), x_d)$ are locally analytically isomorphic.*

Proof. — The group W_d is a normal subgroup of $G(d, 1, 2)$ with quotient μ_d . By [BeTh, Proposition 4.17], the group μ_d acts on $\mathcal{Z}(d)$ such that $\mathcal{Z}(d)/\mu_d \cong \mathcal{Z}(d, 1, 2)$. From now on, we consider all spaces as complex analytic spaces.

Theorem 5.7 says that, in the analytic topology, there exist (analytic) open balls B and D around x_0 and x_d in $\mathcal{Z}(d, 1, 2)$ and $\mathcal{X}(d)$ respectively such that $\varphi : D \xrightarrow{\sim} B$, sending x_d to x_0 . Let \tilde{B} and \tilde{D} denote the preimages of B and D in $\mathcal{Z}(d)$ and $\tilde{\mathcal{X}}(d)$ respectively. By [Kuh, Theorem 4], \tilde{B} and \tilde{D} are normal complex spaces.

The preimage of the smooth locus of B and D in \tilde{B} and \tilde{D} are denoted by \tilde{B}° and \tilde{D}° respectively. The complement to these open sets has (complex) codimension two. Thus Proposition 4.2 says that $\pi_1(\tilde{B}^\circ) = 1$. Hence $\pi_1(B_{\text{sm}}) = \mu_d$ since the map $\tilde{B}^\circ \rightarrow B_{\text{sm}}$ is unramified and Galois with group μ_d . Therefore, there exists a (necessarily holomorphic; see [AnSt]) covering map $\psi^\circ : \tilde{B}^\circ \rightarrow \tilde{D}^\circ$ making the diagram

$$\begin{array}{ccc} \tilde{B}^\circ & \xrightarrow{\psi^\circ} & \tilde{D}^\circ \\ \downarrow & & \downarrow \\ B_{\text{sm}} & \xleftarrow{\varphi} & D_{\text{sm}} \end{array}$$

commutative. Since the degree of both vertical maps is d , we deduce that ψ° is an isomorphism. Composing ψ° with the embedding $\tilde{D}^\circ \hookrightarrow \tilde{\mathcal{X}}(d)$ gives a map $\psi : \tilde{B}^\circ \rightarrow \tilde{\mathcal{X}}(d)$. By [Har, VI, Proposition 3.1], the complex space $\tilde{\mathcal{X}}(d)$ is Stein. Therefore, [AnSt, Theorem 2] says that ψ extends to a holomorphic map $\psi : \tilde{B} \rightarrow \tilde{\mathcal{X}}(d)$.

Consider next the diagram

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\psi} & \tilde{\mathcal{X}}(d) \\ \downarrow & & \downarrow \pi_d \\ B & \xrightarrow{\varphi^{-1}} D \hookrightarrow & \mathcal{X}(d). \end{array}$$

It is commutative when restricted to the dense open set \tilde{B}° . Therefore, it is everywhere commutative. Since the composition $\tilde{B} \rightarrow B \rightarrow D \hookrightarrow \mathcal{X}(d)$ has image equal to D , we

deduce that the image of $\pi_d \circ \psi$ equals D . The map ψ is μ_d -equivariant (since it is so generically). Therefore, the fact that $\tilde{D} = \pi_d^{-1}(D)$ means that the image of ψ equals \tilde{D} and ψ is bijective. Finally, since both \tilde{B} and \tilde{D} are normal, the analytic version of Zariski's main theorem implies that ψ is biholomorphic. In particular, the singularities $(\mathcal{Z}(d), 0)$ and $(\tilde{\mathcal{X}}(d), \tilde{x}_d)$ are locally analytically isomorphic. \square

Corollary 5.10. — *The isolated symplectic singularity $(\tilde{\mathcal{X}}(d), \tilde{x}_d)$ has trivial local fundamental group.*

6. Complements

6.A. Further deformation of $\mathcal{Z}(d)$. — If $d \geq 4$ is even, then the Poisson variety $\mathcal{Z}(d)$ admits a further deformation [EtGi], which is still a Calogero-Moser space associated with the dihedral group W_d (see [Bon1]). Let $\mathcal{Z}^\#(d)$ be a generic such deformation. The variety $\mathcal{Z}^\#(4)$ is smooth so is uninteresting for our purpose but, if $d \geq 6$ then $\mathcal{Z}^\#(d)$ admits a single singular point that we still denote by 0 (see [Bel] or [Bon1]).

So assume from now on that $d \geq 6$ is even. Again, general facts about Calogero-Moser spaces say that $(\mathcal{Z}^\#(d), 0)$ is a symplectic singularity [Gor, Proposition 4.5] and we denote by $T_0^*(\mathcal{Z}^\#(d))$ the cotangent space of $\mathcal{Z}^\#(d)$ at 0, endowed with its Lie algebra structure induced by the Poisson bracket on $\mathcal{Z}^\#(d)$. The smallest singular case $d = 6$ is somewhat particular:

Proposition 6.1. — *The singularities $(\mathcal{Z}^\#(6), 0)$ and $(\overline{\mathcal{O}}_{\min}^{\text{sp}_4}, 0)$ are locally analytically isomorphic.*

Proof. — By [Bon1, Proposition 8.8], we have an isomorphism of Lie algebras $T_0^*(\mathcal{Z}^\#(6)) \simeq \text{sp}_4$ and computations with MAGMA [Mag], [Thi], based on the equations of $\mathcal{Z}^\#(6)$ given in [Bon1, Table 5] show that the projective tangent cone of $\mathcal{Z}^\#(6)$ at 0 is smooth. So the result follows from Beauville's Theorem 1.1. \square

In other words, the singularity $(\mathcal{Z}^\#(6), 0)$ is locally analytically isomorphic to the singularity $(\mathbb{C}^4/\mu_2, 0)$. This shows in particular that the local fundamental group of $(\mathcal{Z}^\#(6), 0)$ is isomorphic to μ_2 .

In the general even case, the action of μ_2 on $\mathcal{Z}(d)$ given by

$$(q, Q, e, a_0, \dots, a_m) \mapsto (q, Q, e, -a_0, \dots, -a_d)$$

is free on the smooth part of an open neighbourhood of 0, hence $\mathcal{Z}(d)/\mu_2$ has an isolated singularity at 0. By the argument in the proof of Prop. 2.13, this is locally analytically isomorphic to $(\mathcal{Y}(d)/\mu_2, 0)$ (with the same action on \mathbb{C}^{d+4}). It is not difficult to show that the Calogero-Moser space associated with W_{2d} and (non-generic) parameters $(c_1, c_2) = (0, 1)$ is isomorphic to $\mathcal{Z}(d)/\mu_2$.

Question: *Assume $d \geq 4$, is the singularity $(\mathcal{Z}^\#(2d), 0)$ locally analytically isomorphic to $(\mathcal{Z}(d)/\mu_2, 0)$?*

We note here a further description of $\mathcal{Y}(d)/\mu_2$. Recall by Corollary 3.6 that the singular locus of $\mathcal{Q}(2d)$ has two irreducible components C_1 and C_2 , with ideals J_1 and J_2 . We can cover the blowup of $\mathcal{Q}(2d)$ at C_1 by affine open subsets as we did in §2.C. It is easy to see that the first affine open subset $\text{Spec}(\mathbb{C}[V \times V^*]^{W_{2d}}[J_1/D])$ is isomorphic to $\mathcal{Y}(d)/\mu_2$. In fact, the singular locus of this blowup is a disjoint union of a 2-dimensional subset (the pre-image of C_2 , which can be resolved by blowing up once more) and the singular point $0 \in \mathcal{Y}(d)/\mu_2$. Similarly, $\text{Spec}(\mathbb{C}[V \times V^*]^{W_{2d}}[J_2/D])$ is isomorphic to $\mathcal{Y}(d)/\mu_2$.

6.B. Hilbert series. — We compute here the Hilbert series of the graded algebra $\mathbb{C}[\mathcal{Y}(d)]$. For this, recall that any symplectic singularity is Cohen-Macaulay (and even Gorenstein [Bea, Proposition 1.3]). Therefore, $\mathbb{C}[\mathcal{Y}(d)]$ is a graded Cohen-Macaulay ring. Let $R = \mathbb{C}[q, Q, b_0, b_d]$ and let \mathfrak{m} denote the unique graded maximal ideal of R . If $f \in \mathbb{C}[\mathcal{Y}(d)]$, we denote by \bar{f} its image in $\mathbb{C}[\mathcal{Y}(d)]/\langle \mathfrak{m} \rangle$. Then

$$(6.2) \quad \mathbb{C}[\mathcal{Y}(d)]/\langle \mathfrak{m} \rangle = \mathbb{C} \oplus \mathbb{C}\bar{e} \oplus \cdots \oplus \mathbb{C}\bar{e}^{d-2} \oplus \mathbb{C}\bar{b}_1 \oplus \mathbb{C}\bar{b}_2 \oplus \cdots \oplus \mathbb{C}\bar{b}_{d-1}$$

Proof of (6.2). — By definition, $\mathbb{C}[\mathcal{Y}(d)]/\langle \mathfrak{m} \rangle$ is the commutative \mathbb{C} -algebra whose presentation is given by:

$$\left\{ \begin{array}{l} \text{Generators: } \bar{e}, \bar{b}_1, \dots, \bar{b}_{d-1} \\ \text{Relations: } (\#) \quad \left\{ \begin{array}{l} \forall 1 \leq j \leq d-1, \bar{e}\bar{b}_j = 0 \\ \forall 1 \leq j \leq k \leq d-1, \bar{b}_j\bar{b}_k = \begin{cases} 0 & \text{if } j+k \neq d, \\ -\bar{e}^{d-2} & \text{if } j+k = d. \end{cases} \end{array} \right. \end{array} \right.$$

Then $\mathbb{C} + \mathbb{C}\bar{e} + \cdots + \mathbb{C}\bar{e}^{d-2} + \mathbb{C}\bar{b}_1 + \mathbb{C}\bar{b}_2 + \cdots + \mathbb{C}\bar{b}_{d-1}$ is a subalgebra of $\mathbb{C}[\mathcal{Y}(d)]/\langle \mathfrak{m} \rangle$ (because $\bar{e}^{d-1} = \bar{e}\bar{b}_1\bar{b}_{d-1} = 0$), which contains all the generators. So it is equal to $\mathbb{C}[\mathcal{Y}(d)]/\langle \mathfrak{m} \rangle$. It only remains to show that $\dim \mathbb{C}[\mathcal{Y}(d)]/\langle \mathfrak{m} \rangle \geq 2d-2$.

For this, let $E_{k,l}$ denote the elementary $(2d-2) \times (2d-2)$ -matrix whose only non-zero entry is the (k, l) -entry, which is equal to 1. We set

$$\bar{E} = E_{2,1} + E_{3,2} + \cdots + E_{d-1,d-2} \quad \text{and} \quad \bar{B}_j = E_{d-1+j,1} - E_{d-1,2d-j-1}$$

for $1 \leq j \leq d-1$. Then the relations (#) show that there is a unique morphism of algebras $\theta : \mathbb{C}[\mathcal{Y}(d)]/\langle \mathfrak{m} \rangle \rightarrow \text{Mat}_{2d-2}(\mathbb{C})$ sending \bar{e} to \bar{E} and \bar{b}_j to \bar{B}_j . As $\theta(1), \theta(\bar{e}), \dots, \theta(\bar{e}^{d-2}), \theta(\bar{b}_1), \dots, \theta(\bar{b}_{d-1})$ are linearly independent, this proves the result. \square

By (6.2), $\mathbb{C}[\mathcal{Y}(d)]/\langle \mathfrak{m} \rangle$ is finite-dimensional, so the graded Nakayama Lemma implies that $\mathbb{C}[\mathcal{Y}(d)]$ is a finitely generated R -module. This shows that (q, Q, b_0, b_d) is a system of parameters: since $\mathbb{C}[\mathcal{Y}(d)]$ is Cohen-Macaulay, this shows that

$$(6.3) \quad \mathbb{C}[\mathcal{Y}(d)] = R \oplus Re \oplus \cdots \oplus Re^{d-2} \oplus Rb_1 \oplus Rb_2 \oplus \cdots \oplus Rb_{d-1}.$$

The Hilbert series $H_{\mathcal{Y}(d)}(\mathbf{t})$ of $\mathbb{C}[\mathcal{Y}(d)]$ is then easily computed:

$$(6.4) \quad H_{\mathcal{Y}(d)}(\mathbf{t}) = \frac{1 + \mathbf{t}^2 + \cdots + \mathbf{t}^{2d-4} + (d-1)\mathbf{t}^{d-2}}{(1-\mathbf{t}^2)^2(1-\mathbf{t}^{d-2})^2}.$$

6.C. Higher dimension. — We explain here why the \mathbf{SL}_2 -equivariant morphism $\mathcal{Q}(d) \rightarrow \mathfrak{sl}_2$ is a particular case of a more general situation. For this, let $(O, (\cdot, \cdot))$ be a finite dimensional orthogonal vector space and let (S, ω) be a finite dimensional symplectic vector space. For $f \in \text{Hom}(S, O)$ (resp. $g \in \text{Hom}(O, S)$), we denote by f^* (resp. g^*) the unique element of $\text{Hom}(O, S)$ (resp. $\text{Hom}(S, O)$) such that $(f(s), o) = \omega(s, f^*(o))$ (resp. $\omega(g(o), s) = (o, g^*(s))$) for all $o \in O$ and $s \in S$. Note that $f^{**} = -f$ and $g^{**} = -g$. Moreover, the morphisms

$$\begin{array}{ccc} & \text{Hom}(S, O) & \\ \pi_{\text{sp}} \swarrow & & \searrow \pi_{\text{so}} \\ \mathfrak{sp}(S) & & \mathfrak{so}(O) \end{array}$$

where $\pi_{\text{sp}}(f) = f^*f$ and $\pi_{\text{so}}(f) = ff^*$ are the quotient maps by $\mathbf{O}(O)$ and $\mathbf{Sp}(S)$ respectively, which are equivariant with respect to $\mathbf{Sp}(S)$ and $\mathbf{O}(O)$ respectively [KP82].

Example 6.5. — Assume here that $S = U_2$, the natural module of \mathbf{GL}_2 endowed with the symplectic form given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and that $O = V$ endowed with the symmetric bilinear form given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that the quadratic form associated with (\cdot, \cdot) is $Q \in \mathbb{C}[V]$. Then

$$V \times V^* \simeq V \times V \simeq U_2 \otimes V \simeq U_2^* \otimes V \simeq \text{Hom}(U_2, V) \simeq \text{Mat}_2(\mathbb{C})$$

(see Remark 3.1 for the first two isomorphisms). Through this sequence of isomorphisms, the action of $\mathbf{O}(V)$ (resp. \mathbf{GL}_2) is by left (resp. right) multiplication. Note that

$$\mathfrak{sp}(S) = \mathfrak{sl}_2 \quad \text{and} \quad \mathfrak{so}(V) = \{\text{diag}(z, -z) \mid z \in \mathbb{C}\}.$$

To respect the notation of the paper, if $f \in \text{Mat}_2(\mathbb{C})$ has matrix $\begin{pmatrix} x & Y \\ y & X \end{pmatrix}$, then f^* has matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & Y \\ y & X \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} X & Y \\ -y & -x \end{pmatrix}$. Computing $\pi_{\text{sp}}(f) = f^* f$ and $\pi_{\text{so}}(f) = f f^*$, we get the following commutative diagrams

$$\begin{array}{ccc} V \times V^* \simeq \text{Mat}_2(\mathbb{C}) & & \begin{pmatrix} x & Y \\ y & X \end{pmatrix} \\ \pi_{\text{sp}} \swarrow & & \swarrow \pi_{\text{sp}} \\ \mathfrak{sp}(U_2) \simeq \mathfrak{sl}_2 & & \begin{pmatrix} xX+yY & 2XY \\ -2xy & -xX-yY \end{pmatrix} \\ \det \searrow & & \searrow \det \\ \mathbb{C} & & \mathbb{C} \end{array} \quad \begin{array}{ccc} & & \begin{pmatrix} xX-yY & 0 \\ 0 & -xX+yY \end{pmatrix} \\ & & \swarrow \pi_{\text{so}} \\ & & \begin{pmatrix} xX-yY & 0 \\ 0 & -xX+yY \end{pmatrix} \\ & & \searrow \det \\ & & -(xX-yY)^2 \end{array}$$

The map π_{sp} factorizes through the quotient by $\mathbf{O}(V)$ but, since $W_d \subset \mathbf{O}(V)$, it factorizes through a map $\mathcal{Q}(d) \rightarrow \mathfrak{sl}_2$ which is nothing but the composition of the first projection with the closed immersion $\mathcal{Q}(d) \hookrightarrow \mathfrak{sl}_2 \oplus \text{Sym}^d(U_2)$. On the other hand, π_{so} identifies with the map δ (once we identify $\mathfrak{so}(V)$ with \mathbb{C}). The relation $\det f^* f = \det f f^*$ gives $\delta^2 = e^2 - 4qQ$ (which can also be checked directly, of course). ■

It is a natural question whether the construction of Example 6.5 can be extended to higher dimensional settings to produce other interesting symplectic singularities.

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