SOLVING EQUATIONS IN DENSE SIDON SETS

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ABSTRACT. We offer an alternative proof of a result of Conlon, Fox, Sudakov and Zhao [CFSZ20] on solving translation-invariant linear equations in dense Sidon sets. Our proof generalises to equations in more than five variables and yields effective bounds.

CONTENTS

1. INTRODUCTION

A set S of integers is a *Sidon* set if the only solutions to the equation

$$x - x' = y - y' \quad (x, x', y, y' \in S) \tag{1.1}$$

are trivial, in the sense that $x = y$ or $x = x'$. Writing

$$E(S) := \#\left\{(x, x', y, y') \in S^4 : x - x' = y - y'\right\}, \tag{1.2}$$

a finite set S is Sidon if and only if $E(S) \leqslant 2|S|^2 - |S|$.

One can show that if $S \subset [N]$ is Sidon then $|S| \leqslant (1 + o(1))N^{1/2}$, and there are constructions with $|S| \geqslant (1 - o(1))N^{1/2}$, see [O'B04]. Conlon, Fox, Sudakov and Zhao [CFSZ20] have shown that Sidon sets whose cardinality is within a constant of this range possess arithmetic structure, in that they contain a solution to any translation-invariant linear equation in five variables, with all variables distinct. Furthermore they are able to demonstrate that this structure is also possessed by *almost Sidon* sets, that is sets for which

$$E(S) \leqslant (2 + o(1))|S|^2.$$

Their results are deduced using a regularity lemma for graphs with few 4-cycles. We use a Fourier-analytic transference principle developed by Helfgott and de Roton [HdR11] to give an alternative proof of this result, generalising to translation-invariant equations in more variables and extracting bounds. Before stating our main result, we require a function whose existence is guaranteed by the following.

Theorem 1.1 (Counting Roth)**.** *Let $a_1, \dots, a_s \in \mathbb{Z}\setminus\{0\}$ with $a_1+\dots+a_s = 0$. For any $\delta > 0$ there exists $c_{a_i}(\delta) > 0$ such that for all $A \subset [N]$ with $|A| \geqslant \delta N$ we have*

$$\sum_{a_1x_1+\dots+a_sx_s=0} \prod_i 1_A(x_i) \geqslant c_{a_i}(\delta) \sum_{a_1x_1+\dots+a_sx_s=0} \prod_i 1_{[N]}(x_i). \tag{1.3}$$

Proof. This follows from [FGR88, Theorem 2], where the condition that $N \geqslant N_0(a_i, \delta)$ can be removed on adding 'improper'[1] solutions to the count (for instance by including solutions along the diagonal $x_1 = \dots = x_s$). □

We prove an analogous counting result for dense Sidon sets.

Theorem 1.2. *Let $a_1, \dots, a_s \in \mathbb{Z} \setminus \{0\}$ with $a_1 + \dots + a_s = 0$ and $s \geqslant 5$. Given $0 < \delta \leqslant 1/2$, let $c_{a_i}(\delta)$ be a function satisfying (1.3). Recalling the notation (1.2), suppose that $S \subset [N]$ satisfies*

$$|S| \geqslant \delta N^{1/2} \quad \textit{and} \quad E(S) \leqslant (2+\eta)\,|S|^2.$$

Then either

$$N \leqslant \exp\left(O_s\left(c_{a_i}\left(\delta^2/4\right)^{-O(1)}\right)\right), \quad \textit{or} \quad \eta \geqslant \exp\left(-O_s\left(c_{a_i}\left(\delta^2/4\right)^{-O(1)}\right)\right)$$

or

$$\sum_{a_1x_1+\dots+a_sx_s=0} \prod_i 1_S(x_i) \geqslant \frac{c_{a_i}\left(\delta^2/4\right)^2}{2N^{s/2}} \sum_{a_1x_1+\dots+a_sx_s=0} \prod_i 1_{[N]}(x_i). \tag{1.4}$$

Such a counting result yields a bound on the density of Sidon sets lacking solutions to translation-invariant equations in five or more variables.

Corollary 1.3. *Let $a_1, \dots, a_s \in \mathbb{Z} \setminus \{0\}$ with $a_1 + \dots + a_s = 0$ and $s \geqslant 5$. Given $0 < \delta \leqslant 1/2$, let $c_{a_i}(\delta)$ be a function satisfying (1.3). Suppose that $S \subset [N]$ is a Sidon set of size $|S| \geqslant \delta N^{1/2}$ and which lacks solutions to the equation*

$$a_1x_1 + \dots + a_sx_s = 0 \tag{1.5}$$

with $x_1, \dots, x_s \in S$ all distinct. Then

$$N \leqslant \exp\left(O_{a_i}\left(c_{a_i}\left(\delta^2/4\right)^{-O(1)}\right)\right). \tag{1.6}$$

That such bounds are obtainable is noted in [CFSZ20], along with a path to proving them. We depart from the use of weak arithmetic regularity suggested therein. Instead our argument takes advantage of the fact that a Sidon set behaves very nicely with respect to convolution, so that convolving its indicator function with a suitably chosen Bohr set yields a function whose L^1 and L^2 norms are both comparable to that of a dense set of integers (after appropriate renormalisation). Functions whose L^p-norms

[1]See [FGR88, p.253].

behave in this manner are similar enough to dense sets of integers for us to import results from the dense setting to sparse Sidon sets. This observation originates with Helfgott and de Roton [HdR11].

The function $c_{a_i}(\delta)$ satisfying (1.3) does not seem to be estimated in the literature. However, practitioners will be able to derive such estimates on re-working proofs of Roth's theorem. For instance, by modifying Bourgain's argument [Bou99] (accessibly exposited in [TV06, §10.4]) one can prove that

$$c_{a_i}(\delta) = \Omega_{a_i}\left(\exp\left(-\delta^{-O(1)}\right)\right).$$

Corollary 1.3 then implies that a Sidon set $S \subset [N]$ lacking non-trivial solutions to (1.5) satisfies the density bound

$$|S| = O_{a_i}\left(\frac{N^{1/2}}{(\log\log N)^{\Omega(1)}}\right). \tag{1.7}$$

For translation invariant equations in four or more variables, there is a more effective density bound due to Schoen and Sisask [SS16]. One may adapt this argument[2] to more effectively bound $c_{a_i}(\delta)$ and thereby improve (1.7) to

$$|S| = O_{a_i}\left(\frac{N^{1/2}}{\exp\left((\log\log N)^{\Omega(1)}\right)}\right). \tag{1.8}$$

The author would be very interested in any proof which yields a polylogarithmic bound in (1.8). For dense sets of integers, all polylogarithmic bounds require some form of localisation from the interval $[N]$ to a sparser substructure, such as a subprogression or Bohr set. When dealing with sparse sets of integers like Sidon sets, such localisation is lossy, because the sparse set can be even sparser on the substructure. An example to bear in mind is that a subset of $[N]$ of cardinality $\sqrt{N}$ may intersect each subinterval of length $\sqrt{N}$ in at most one point. The author believes that obtaining a polylogarithmic bound in Corollary 1.3 may be a model problem for improving bounds in Roth's theorem in the primes [Gre05, HdR11, Nas15].

Paper Organisation. Theorem 1.2 is proved in §2, assuming three key lemmas. Proving these lemmas occupies §§3–5. Corollary 1.3 is deduced in §6

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Notation.

[2]Replace the indicator function of the sum set $A + A$ in [SS16] with a suitable set of popular sums. Thanks to Thomas Bloom and Olof Sisask for pointing this out.

Standard conventions. We use $[N]$ to denote the interval of integers $\{1, 2, \ldots, N\}$. We use counting measure on $\mathbb{Z}$, so that for $f, g : \mathbb{Z} \to \mathbb{C}$, we have

$$\|f\|_p := \left(\sum_x |f(x)|^p \right)^{\frac{1}{p}} \text{ and } (f * g)(x) := \sum_y f(y)g(x-y).$$

Any sum of the form $\sum_x$ is to be interpreted as a sum over $\mathbb{Z}$. The *support* of f is the set $\mathrm{supp}(f) := \{x \in \mathbb{Z} : f(x) \neq 0\}$.

We use Haar probability measure on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, so that for integrable $F : \mathbb{T} \to \mathbb{C}$, we have

$$\|F\|_p := \left(\int_{\mathbb{T}} |F(\alpha)|^p \mathrm{d}\alpha \right)^{\frac{1}{p}} = \left(\int_0^1 |F(\alpha)|^p \mathrm{d}\alpha \right)^{\frac{1}{p}}$$

and

$$\|F\|_\infty := \sup_{\alpha \in \mathbb{T}} |F(\alpha)|^{\cdot}$$

Write $\|\alpha\|_{\mathbb{T}}$ for the distance from $\alpha \in \mathbb{R}$ to the nearest integer $\min_{n \in \mathbb{Z}} |\alpha - n|$. This remains well-defined on $\mathbb{T}$.

Definition 1.4 (Fourier transform)**.** For $f : \mathbb{Z} \to \mathbb{C}$ with finite support define $\hat{f} : \mathbb{T} \to \mathbb{C}$ by

$$\hat{f}(\alpha) := \sum_{n \in \mathbb{Z}} f(n) e(\alpha n).$$

Here $e(\beta)$ stands for $e^{2\pi i \beta}$.

Asymptotic notation. For a complex-valued function f and positive-valued function g, write $f \lesssim g$ or $f = O(g)$ if there exists a constant C such that $|f(x)| \leq Cg(x)$ for all x. We write $f = \Omega(g)$ if $f \gtrsim g$. The notation $f \asymp g$ means that $f \lesssim g$ and $f \gtrsim g$. We subscript these symbols if the implicit constant depends on additional parameters.

We write $f = o(g)$ if for any $\varepsilon > 0$ there exists $X \in \mathbb{R}$ such that for all $x \geqslant X$ we have $|f(x)| \leqslant \varepsilon g(x)$.

Local conventions. As indicated in the introduction, we define the additive energy of a finitely supported function $f : \mathbb{Z} \to \mathbb{R}$ to be the quantity

$$E(f) := \sum_{x - x' = y - y'} f(x) f(x') f(y) f(y').$$

When $f = 1_S$ is the characteristic function of a finite set $S \subset \mathbb{Z}$ we write $E(S)$. Notice that

$$E(S) = \sum_n r_S(n)^2$$

where

$$r_S(n) := \sum_{n_1 - n_2 = n} 1_S(n_1) 1_S(n_2)$$

is the number of representation of n as a difference of elements of S. In the literature this notation is sometimes used for the number of representations as a *sum* of two elements of S.

2. The transference argument

In this section we prove Theorem 1.2 assuming the following three lemmas.

Lemma 2.1 (L^2 Roth)**.** *Let $a_1, \dots, a_s \in \mathbb{Z} \setminus \{0\}$ with $s \geqslant 5$ and $a_1 + \cdots + a_s = 0$. Given $0 < \delta \leqslant 1/2$, let $c_{a_i}(\delta)$ be a function satisfying (1.3). Let $f : I \to [0, \infty)$ be a function defined on an interval $I \subset \mathbb{Z}$ of length N. If $\sum_n f(n) \geqslant \delta N$ and $\sum_n f(n)^2 \leqslant N$ then we have the lower bound*

$$\sum_{a_1x_1+\cdots+a_sx_s=0} f(x_1) \cdots f(x_s) \geqslant c_{a_i}(\delta^2/4)^2 N^{s-1}.$$

Lemma 2.2 (Counting lemma for bounded energy functions)**.** *Let $s \geqslant 5$ and $a_1, \dots, a_s \in \mathbb{Z} \setminus \{0\}$. Let $\nu : I \to [0, \infty)$ be a function defined on an interval $I \subset \mathbb{Z}$ of length N. Suppose that*

$$\sum_n \nu(n) \leqslant N \quad \textit{and} \quad E(\nu) \leqslant N^3.$$

Then for any $|f_i| \leqslant \nu$ we have

$$\left| \sum_{a_1x_1+\cdots+a_sx_s=0} f_1(x_1) \cdots f_s(x_s) \right| \leqslant N^{s-1} \frac{\min_i \left\| \hat{f}_i \right\|_\infty}{\left\| \hat{1}_{[N]} \right\|_\infty}.$$

Lemma 2.3 (Dense model for almost-Sidon sets)**.** *Let $0 \leqslant \eta \leqslant 1$ and suppose that $S \subset [N]$ satisfies*

$$|S| \geqslant \delta N^{1/2} \quad \textit{and} \quad E(S) \leqslant (2 + \eta)\,|S|^2.$$

Then for any $0 < \varepsilon \leqslant \min\left\{\frac{1}{2}, \delta\right\}$ there exists $f : (-\varepsilon N, (1+\varepsilon)N] \to [0, \infty)$ such that all of the following hold

- $\sum_n f(n) = N^{1/2}|S|$;
- $\left\| \hat{f} - N^{1/2}\hat{1}_S \right\|_\infty \leqslant \varepsilon N$;
- $\sum_n f(n)^2 \leqslant N \left[1 + \left(\eta + N^{-1/2}\right)(1-\eta)^{-1} \exp\left(\varepsilon^{-O(1)}\right)\right]$.

Lemma 2.1 is deduced from Theorem 1.1 in §3. Lemma 2.2 is proved in §4. Lemma 2.3 constitutes the main idea in our approach and is proved in §5.

Proof of Theorem 1.2. We may assume that $\eta \leqslant 1/2$, for the second possible conclusion of the theorem is that η is large. Let us apply Lemma 2.3, with ε to be chosen. This gives $f : (-\varepsilon N, (1+\varepsilon)N] \to [0,\infty)$ satisfying $\sum_n f(n) \geqslant \delta N$, $\left\|\hat{f} - N^{1/2}\hat{1}_S\right\|_\infty \leqslant \varepsilon N$ and

$$\sum_n f(n)^2 \leqslant N\left[1 + \left(\eta + N^{-1/2}\right)\exp\left(\varepsilon^{-O(1)}\right)\right]. \tag{2.1}$$

By (2.1), either $\sum_n f(n)^2 \leqslant 2N$ or one of the following two possibilities holds

$$\eta \geqslant \exp\left(-\varepsilon^{-O(1)}\right) \quad \text{or} \quad N \leqslant \exp\left(\varepsilon^{-O(1)}\right). \tag{2.2}$$

Notice that $(-\varepsilon N, (1+\varepsilon)N]$ is an interval of length at most $2N$. Hence, assuming that neither option in (2.2) holds, Lemma 2.1 gives that

$$\sum_{a_1x_1+\dots+a_sx_s=0} f(x_1)\cdots f(x_s) \geqslant c_{a_i}(\delta^2/4)^2 N^{s-1}.$$

Define $\nu := f + N^{1/2}1_S$. We claim that, provided we divide through by a suitable absolute constant, the function ν satisfies the hypotheses of Lemma 2.2 on the interval $I = (-\varepsilon N, (1+\varepsilon)N]$. By the triangle inequality in L^4, and the Fourier-analytic interpretation of energy, we have

$$\begin{aligned} E(\nu)^{1/4} = \|\hat{\nu}\|_4 \leqslant \left\|\hat{f}\right\|_4 + N^{1/2}\left\|\hat{1}_S\right\|_4 &\leqslant \|f\|_1^{1/2}\left\|\hat{f}\right\|_2^{1/2} + N^{1/2}E(S)^{1/4} \\ &\lesssim N^{1/2}\|f\|_2 + N^{1/2}N^{1/4} \lesssim N^{3/4}. \end{aligned}$$

Assuming that neither option in (2.2) holds, we compare Fourier coefficients at zero to deduce that

$$\begin{aligned} \sum_n \nu(n) = \hat{f}(0) + N^{1/2}\hat{1}_S(0) &\leqslant 2\hat{f}(0) + \varepsilon N \\ &\leqslant 2(2N)^{1/2}\Big(\sum_n f(n)^2\Big)^{1/2} + \varepsilon N \lesssim N. \end{aligned}$$

We may therefore apply Lemma 2.2 together with a telescoping identity to deduce that

$$\left|\sum_{a_1x_1+\dots+a_sx_s=0}\left(\prod_i f(x_i) - \prod_i N^{1/2}1_S(x_i)\right)\right| \lesssim s\varepsilon N^{s-1}.$$

Hence either we deduce (1.4), or one of the following holds

- $\varepsilon \gtrsim s^{-1}c_{a_i}(\delta^2/4)^2$;
- $\eta \geqslant \exp\left(-\varepsilon^{-O(1)}\right)$;
- $N \leqslant \exp\left(\varepsilon^{-O(1)}\right)$.

We obtain Theorem 1.2 on taking ε sufficiently small to preclude the first possibility. □

3. Results on dense sets of integers

The purpose of this section is to deduce Lemma 2.1 from Theorem 1.1.

Proof of Lemma 2.1. Translating, we may assume that $I = [N]$. Define

$$A := \{x \in [N] : f(x) \geqslant \delta/2\}\,.$$

Then, employing the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\delta N \leqslant \sum_x f(x) = \sum_{x \notin A} f(x) &+ \sum_{x \in A} f(x)\\ &\leqslant \tfrac{1}{2}\delta N + |A|^{1/2}\left(\sum_x f(x)^2\right)^{1/2} \\ &\leqslant \tfrac{1}{2}\delta N + (|A|N)^{1/2}\,.\end{aligned} \tag{3.1}$$

Therefore

$$|A| \geqslant \tfrac{\delta^2}{4}N. \tag{3.2}$$

Applying Theorem 1.1 we deduce that

$$\begin{aligned}\sum_{a_1x_1+\cdots+a_sx_s=0} f(x_1)\cdots f(x_s) &\geqslant (\delta/2)^s \sum_{a_1x_1+\cdots+a_sx_s=0} 1_A(x_1)\cdots 1_A(x_s)\\ &\geqslant (\delta/2)^s c_{a_i}(\delta^2/4)N^{s-1}.\end{aligned}$$

For fixed $x_1, \dots, x_{s-1}$ there is at most one x_s solving $a_1x_1+\cdots+a_sx_s = 0$. This leads to the trivial estimate $c_{a_i}(\delta) \leqslant \delta^{s-1}$. In particular $(\delta/2)^s \geqslant c_{a_i}(\delta^2/4)$. $\square$

4. An almost-Sidon counting lemma

Proof of Lemma 2.2. For any finitely supported $f : \mathbb{Z} \to \mathbb{C}$ and $a \in \mathbb{Z}\setminus\{0\}$ we have

$$\int_{\mathbb{T}} \big|\hat{f}(a\alpha)\big|^{s-1} \mathrm{d}\alpha \leqslant \Big(\sum_n |f(n)|\Big)^{s-5} \int_{\mathbb{T}} \big|\hat{f}(a\alpha)\big|^{4} \mathrm{d}\alpha.$$

If $|f| \leqslant \nu$, then

$$\sum_n |f(n)| \leqslant \sum_n \nu(n) \leqslant N.$$

By orthogonality

$$\begin{aligned}\int_{\mathbb{T}} \big|\hat{f}(a\alpha)\big|^{4} \mathrm{d}\alpha = \sum_{x-x'=y-y'} & f(x)\overline{f(x')f(y)}f(y')\\ &\leqslant \sum_{x-x'=y-y'} \nu(x)\nu(x')\nu(y)\nu(y') \leqslant N^3.\end{aligned}$$

Therefore

$$\int_{\mathbb{T}} \big|\hat{f}(a\alpha)\big|^{s-1} \mathrm{d}\alpha \leqslant N^{s-2}.$$

Again by orthogonality, together with Hölder's inequality

$$\left|\sum_{a_1x_1+\cdots+a_sx_s=0} f_1(x_1)\cdots f_s(x_s)\right| = \left|\int_{\mathbb{T}} \hat{f}_1(a_1\alpha)\cdots\hat{f}_s(a_s\alpha)\mathrm{d}\alpha\right|$$
$$\leqslant \|\hat{f}_i\|_\infty \prod_{j\neq i}\left(\int_{\mathbb{T}} \left|\hat{f}(a\alpha)\right|^{s-1}\mathrm{d}\alpha\right)^{\frac{1}{s-1}} \leqslant \|\hat{f}_i\|_\infty N^{s-2}. \quad \square$$

5. A modelling lemma for almost-Sidon sets

We begin our proof of Lemma 2.3 with two subsidiary results on almost Sidon sets.

Lemma 5.1. *Let $S\subset[N]$ satisfy*

$$E(S) := \sum_{x-x'=y-y'} 1_S(x)1_S(x')1_S(y)1_S(y') \leqslant (2+\eta)|S|^2.$$

Then, on writing

$$r_S(n) := \sum_{n_1-n_2=n} 1_S(n_1)1_S(n_2),$$

we have

$$\sum_{\substack{r_S(n)>1\\ n\neq 0}} r_S(n) \leqslant \eta|S|^2+|S|.$$

Proof. We observe that

$$\sum_{\substack{r_S(n)>1\\ n\neq 0}} r_S(n) \leqslant \sum_{n\neq 0} r_S(n)(r_S(n)-1) = \sum_{n\neq 0} r_S(n)^2 - \sum_{n\neq 0} r_S(n)$$
$$\leqslant (1+\eta)|S|^2 - \big(|S|^2-|S|\big) = \eta|S|^2+|S|. \quad \square$$

Lemma 5.2. *Let $\eta\in[0,1)$ and suppose that $S\subset[N]$ satisfies*

$$E(S) := \sum_{x-x'=y-y'} 1_S(x)1_S(x')1_S(y)1_S(y') \leqslant (2+\eta)|S|^2.$$

Then

$$|S| \leqslant 2\left(\tfrac{N}{1-\eta}\right)^{1/2}.$$

Proof. Using Lemma 5.1 we have

$$|S|^2 = \sum_{\substack{r_S(n)\leqslant 1\\ n\neq 0}} r_S(n) + \sum_{\substack{r_S(n)>1\\ n\neq 0}} r_S(n) + |S| \leqslant 2N+\eta|S|^2+2|S|. \quad \square$$

We are now in a position to prove Lemma 2.3 in earnest.

Proof of Lemma 2.3. Define the large spectrum of S to be the set

$$\mathrm{Spec}(S,\varepsilon) := \left\{\alpha \in \mathbb{T} : |\hat{1}_S(\alpha)| \geqslant \varepsilon |S|\right\}.$$

Define the Bohr set

$$B := \{n \in [-\varepsilon N, \varepsilon N] : \|n\alpha\|_{\mathbb{T}} \leqslant \varepsilon \quad \forall \alpha \in \mathrm{Spec}(S,\varepsilon)\}. \tag{5.1}$$

Write μ_B for the normalised characteristic function of B, so that

$$\mu_B := |B|^{-1} 1_B.$$

Then we define

$$f := N^{1/2} 1_S * \mu_B, \tag{5.2}$$

where, for finitely supported f_i, we set

$$f_1 * f_2(n) := \sum_{m_1+m_2=n} f_1(m_1) f_2(m_2).$$

It is straightforward to check that f is supported on $(-\varepsilon N, (1+\varepsilon)N]$ and that $\sum_n f(n) = N^{1/2}|S|$. Let us next estimate $|N^{1/2}\hat{1}_S - \hat{f}|$. The key identity is

$$\widehat{f_1 * f_2} = \hat{f}_1 \hat{f}_2,$$

so that $|N^{1/2}\hat{1}_S - \hat{f}| = N^{1/2}|\hat{1}_S||1-\hat{\mu}_B|$.

If $\alpha \notin \mathrm{Spec}(S,\varepsilon)$ then we have

$$|N^{1/2}\hat{1}_S(\alpha) - \hat{f}(\alpha)| = N^{1/2}|\hat{1}_S(\alpha)||1-\hat{\mu}_B(\alpha)| \leqslant 2N^{1/2}\varepsilon|S|.$$

If $\alpha \in \mathrm{Spec}(S,\varepsilon)$, then for each $n \in B$ we have $e(\alpha n) = 1 + O(\varepsilon)$. Hence $\hat{\mu}_B(\alpha) = 1 + O(\varepsilon)$, and consequently

$$|N^{1/2}\hat{1}_S(\alpha) - \hat{f}(\alpha)| = N^{1/2}|\hat{1}_S(\alpha)||1-\hat{\mu}_B(\alpha)| \lesssim N^{1/2}|S|\varepsilon.$$

Combining both cases and Lemma 5.2 gives

$$\left\|N^{1/2}\hat{1}_S - \hat{f}\right\|_\infty \lesssim \varepsilon N. \tag{5.3}$$

We have

$$\sum_n f(n)^2 = N|B|^{-2} \sum_{n_1-n_2=m_1-m_2} 1_S(n_1)1_S(n_2)1_B(m_1)1_B(m_2). \tag{5.4}$$

Write

$$r_S(n) := \sum_{n_1-n_2=n} 1_S(n_1)1_S(n_2).$$

Then by Lemma 5.1, the inner sum in (5.4) is

$$\sum_n r_S(n) r_B(n) \leqslant |B|^2 + |B| \sum_{\substack{r_S(n)>1 \\ n\neq 0}} r_S(n) + |B||S|$$
$$\leqslant |B|^2 + \left(\eta|S|^2 + 2|S|\right)|B|.$$

Using the estimate $|S| \lesssim (1-\eta)^{-1/2}N^{1/2}$ afforded by Lemma 5.2, it remains to establish the lower bound

$$|B| \geqslant \exp\left(-\varepsilon^{O(1)}\right) N. \tag{5.5}$$

Let $\alpha_1, \dots, \alpha_R$ be a maximal $(1/N)$-separated subset of $\mathrm{Spec}(S, \varepsilon)$. Since every element of $\mathrm{Spec}(S, \varepsilon)$ is within $1/N$ of some α_i, one can check that

$$B \supset \{n \in [-\varepsilon N/2, \varepsilon N/2] : \|\alpha_i n\|_{\mathbb{T}} \leqslant \varepsilon/2 \quad \forall i = 1, \dots, R\}. \tag{5.6}$$

Hence the argument proving the standard lower bound for Bohr sets (e.g. [TV06, Lemma 4.2]) gives

$$|B| \geqslant \lceil 4/\varepsilon \rceil^{1+R} N.$$

By the large sieve inequality (e.g. [Vau97, Lemma 5.3]) we have

$$R\varepsilon^4 |S|^4 \leqslant \sum_{i=1}^{R} \left| \hat{1}_S(\alpha_i) \right|^4 \lesssim N \sum_n r_S(n)^2 \lesssim N\,(2+\eta)\,|S|^2.$$

Hence $R \lesssim \delta^{-2}\varepsilon^{-4}$. □

6. Proof of Corollary 1.3

Proof of Corollary 1.3. Let us first obtain an upper bound for the number of solutions in S to the equation

$$a_1 x_1 + \cdots + a_s x_s = 0.$$

By our hypotheses, all such solutions should have $x_i = x_j$ for some $i \neq j$. At the cost of a factor of $\binom{s}{2}$, we may assume that $x_{s-1} = x_s$. Writing $n := a_4 x_4 + \cdots + a_s x_s$, the number of choices for the remaining three variables is at most

$$\begin{aligned} \sum_{a_1x_1 + a_2x_2 + a_3x_3 = -n} 1_S(x_1) 1_S(x_2) 1_S(x_3) &= \int_{\mathbb{T}} e(\alpha n) \prod_{i=1}^{3} \hat{1}_S(a_i x_i) \mathrm{d}\alpha \\ &\leqslant \prod_{i=1}^{3} \left(\int_{\mathbb{T}} \left| \hat{1}_S(a_i \alpha_i) \right|^3 \mathrm{d}\alpha \right)^{\frac{1}{3}} \leqslant \prod_{i=1}^{3} \left(\int_{\mathbb{T}} \left| \hat{1}_S(a_i \alpha_i) \right|^4 \mathrm{d}\alpha \right)^{\frac{1}{4}} = E(S)^{\frac{3}{4}}. \end{aligned}$$

We may assume that $\eta \leqslant 1/2$ (otherwise we are done). Using Lemma 5.2 we deduce that the number of choices for x_1, x_2, x_3 is $O(N^{3/4})$. Since there are $N^{\frac{s-4}{2}}$ choices for the remaining variables, we deduce that

$$\sum_{a_1x_1 + \cdots + a_s x_s = 0} \prod_i 1_S(x_i) \lesssim_s N^{\frac{s}{2} - \frac{5}{4}}.$$

Comparing this with the lower bound given in Theorem 1.2, we deduce that either

$$N \leqslant \exp\left(O_s \left(c_{a_i} \left(\delta^2/4\right)^{-O(1)} \right) \right)$$

or

$$N^{\frac{s}{2} - \frac{5}{4}} \gtrsim_s \frac{c_{a_i} \left(\delta^2/4\right)^2}{2N^{s/2}} \sum_{a_1x_1 + \cdots + a_s x_s = 0} \prod_i 1_{[N]}(x_i) \gtrsim_{a_i} c_{a_i} \left(\delta^2/4\right)^2 N^{\frac{s}{2} - 1}.$$

The latter implies that $N \lesssim_{a_i} c_{a_i} (\delta^2/4)^{O(1)}$, which in turn implies (1.6). □

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