# Some Inequalities for Power Means; a Problem from "The Logarithmic Mean Revisited"

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**Abstract.** We establish some inequalities comparing power means of two numbers with combinations of the arithmetic and geometric means. A conjecture from [**1**] is confirmed.

Given positive numbers $a$, $b$, the arithmetic, geometric and $p$th power means are

$$A(a,b) = \tfrac{1}{2}(a+b), \quad G(a,b) = (ab)^{1/2}, \quad M_p(a,b) = \left(\tfrac{1}{2}(a^p + b^p)\right)^{1/p}$$

for $p \neq 0$. With $a$, $b$ fixed, we denote these just by $A$, $G$ and $M_p$.

Of course, these definitions extend in a natural way to more than two numbers. For any finite set of positive numbers, it is clear that $M_1 = A$, and well known that $G \le M_p \le A$ for $0 < p \le 1$, $M_p \ge A$ for $p \ge 1$ and $M_p \le G$ for $p < 0$. (Also, one defines $M_0$ to be $G$).

For two numbers, it is easily seen that $M_{1/2} = \frac{1}{2}G + \frac{1}{2}A$. This equality does not extend to more than two numbers: for the triple $(4,1,1)$ we have $M_{1/2} < \frac{1}{2}G + \frac{1}{2}A$, while the opposite inequality holds for $(4,4,1)$. From now on, we restrict considerations to two numbers $a$, $b$. It was shown in the note [**1**] that $M_{1/3} \le \frac{2}{3}G + \frac{1}{3}A$, and conjectured that $M_p \le (1-p)G + pA$ for $0 < p \le \frac{1}{2}$, together with the opposite inequality for $\frac{1}{2} \le p \le 1$. As reported in [**1**], the conjecture was confirmed by Gord Sinnamon; his proof (communicated privately) is ingenious, but it involves some fairly heavy manipulation.

A more complete picture is obtained if at the same time we compare $M_p$ with $G^{1-p}A^p$. Equality holds for $p = 1$, and it is easily verified that $M_{-1} = G^2/A$ (this is the harmonic mean), so equality also holds for $p = -1$. The results in [**1**] imply that $M_{1/3} \ge G^{2/3}A^{1/3}$ (with the logarithmic mean coming between these two quantities), suggesting that a similar inequality holds for $0 < p \le 1$, though this was not explicitly stated as a conjecture.

Here we offer a simple unified treatment of both comparisons, based on the substitution that was used in [**1**]. The results are as follows.

**Theorem 1.** *The inequality* $M_p \le (1-p)G + pA$ *holds for* $0 < p \le \frac{1}{2}$ *and for* $p \ge 1$. *The opposite inequality holds for* $\frac{1}{2} \le p \le 1$ *and for* $p < 0$.

**Theorem 2.** *The inequality* $G^{1-p}A^p \le M_p$ *holds for* $0 < p \le 1$ *and for* $p \le -1$. *The opposite inequality holds for* $p \ge 1$ *and for* $-1 \le p < 0$.

Note first that if $x = a/b$, then $A(a,b) = bA(x,1)$ and similarly for $G$ and $M_p$, so it is sufficient to consider the pair $(x,1)$: henceforth the notation $A$, $G$, $M_p$ applies to this pair. Now substitute $x = e^{2t}$. Then $G = e^t$ and

$$A = \tfrac{1}{2}(e^{2t}+1) = e^t \cosh t, \qquad M_p = \left(\tfrac{1}{2}(e^{2pt}+1)\right)^{1/p} = e^t (\cosh pt)^{1/p}.$$

So, for example, the inequality $M_p \geq G$ stated above translates to $(\cosh pt)^{1/p} \geq 1$, which is obvious for $p > 0$. The inequality in Theorem 1 translates to

$$(\cosh pt)^{1/p} \leq (1-p) + p\cosh t. \tag{1}$$

For both theorems, we will use the following Lemma, essentially an adaption of the "integrating factor" method to inequalities.

**Lemma 3.** *Let $f$ be a function satisfying $f(0) = f'(0) = 0$ and $f''(t) \geq f(t)$ for $t \geq 0$. Then $f(t) \geq 0$ for $t > 0$. The reverse applies if $f''(t) \leq f(t)$ for $t > 0$.*

*Proof.* Let $g(t) = f'(t) + f(t)$ and $h(t) = f'(t) - f(t)$. Then $g(0) = h(0) = 0$ and

$$g'(t) - g(t) = h'(t) + h(t) = f''(t) - f(t) \geq 0,$$

hence

$$\frac{d}{dt}\Big(e^{-t}g(t)\Big) = e^{-t}\Big(g'(t) - g(t)\Big) \geq 0,$$

$$\frac{d}{dt}\Big(e^{t}h(t)\Big) = e^{t}\Big(h'(t) + h(t)\Big) \geq 0.$$

Consequently $e^{-t}g(t)$ and $e^t h(t)$ are increasing. So for $t > 0$, we have $g(t) \geq 0$ and $h(t) \geq 0$, hence $f'(t) \geq 0$, so also $f(t) \geq 0$. The inequalities reverse if $f''(t) \leq f(t)$. ■

*Proof of Theorem 1.* As we have seen, the substitution $x = e^{2t}$ translates $M_p \leq (1-p)G + pA$ to $f(t) \geq 0$ (for all $t$), where

$$f(t) = p\cosh t + (1-p) - (\cosh pt)^{1/p}.$$

Since $f$ is even, it is enough to consider $t > 0$. Then $f(0) = 0$ and

$$f'(t) = p\sinh t - (\cosh pt)^{1/p-1}\sinh pt.$$

So $f'(0) = 0$ and

$$\begin{aligned} f''(t) &= p\cosh t - p(\cosh pt)^{1/p} - (1-p)(\cosh pt)^{1/p-2}(\sinh pt)^2 \\ &= p\cosh t - (\cosh pt)^{1/p} + (1-p)(\cosh pt)^{1/p-2} \\ &= f(t) - (1-p) + (1-p)(\cosh pt)^{1/p-2}. \end{aligned}$$

If $0 < p \leq \frac{1}{2}$, then $\frac{1}{p} - 2 \geq 0$, so $(\cosh pt)^{1/p-2} \geq 1$ and $f''(t) \geq f(t)$ for all $t$. If $p \geq \frac{1}{2}$ or $p < 0$, then $(\cosh pt)^{1/p-2} \leq 1$. So if $\frac{1}{2} \leq p \leq 1$ or $p < 0$, then $f''(t) \leq f(t)$, and if $p \geq 1$, then $f''(t) \geq f(t)$. The statements follow, by the Lemma. ■

*Proof of Theorem 2.* The inequality $G^{1-p}A^p \leq M_p$ translates to

$$(\cosh pt)^{1/p} \geq (\cosh t)^p \tag{2}$$

(this inequality is perhaps of some interest in its own right). Let

$$f(t) = (\cosh pt)^{1/p^2} - \cosh t.$$

Then $f(0) = 0$ and

$$f'(t) = \frac{1}{p}(\cosh pt)^{1/p^2-1} \sinh pt - \sinh t,$$

$$f''(t) = (\cosh pt)^{1/p^2} + r(t) - \cosh t = f(t) + r(t),$$

where

$$r(t) = \left(\frac{1}{p^2} - 1\right)(\cosh pt)^{1/p^2-2}(\sinh pt)^2.$$

If $|p| \leq 1$, then $1/p^2 - 1 \geq 0$, so $r(t) \geq 0$, hence $f''(t) \geq f(t)$, so $f(t) \geq 0$ for $t \geq 0$. This implies (2) if $0 < p \leq 1$ and the reverse of (2) if $-1 \leq p < 0$. If $|p| \geq 1$, then $f''(t) \leq f(t)$, so $f(t) \leq 0$ for $t \geq 0$. This implies the reverse of (2) for $p \geq 1$ and (2) for $p \leq -1$. ■

It remains to compare and combine the inequalities in Theorems 1 and 2. There are five intervals to consider. For $0 < p \leq \frac{1}{2}$, we have

$$G^{1-p}A^p \leq M_p \leq (1-p)G + pA.$$

For $-1 \leq p < 0$, we have

$$(1-p)G + pA \leq M_p \leq G^{1-p}A^p.$$

In the other cases, we have either two upper bounds or two lower ones. We compare them. For this purpose, write $(1-p)G + pA = C$. For $\frac{1}{2} \leq p \leq 1$, we have $G^{1-p}A^p \leq C$, so the better estimate is $(1-p)G + pA \leq M_p$, given by Theorem 1. (Recall that in this case we have the upper bound $M_p \leq A$).

For $p \geq 1$, we have $C \leq G^{1-p}A^p$, by the weighted AM-GM inequality applied to $A = \frac{1}{p}C + (1 - \frac{1}{p})G$. So the better estimate is $M_p \leq pA - (p-1)G$, again from Theorem 1. (Also $M_p \geq A$).

For $p \leq -1$, we have again $C \leq G^{1-p}A^p$, seen by writing $G = [1/(1+q)]C + [q/(1+q)]A$, where $q = -p$. So the better estimate is $G^{1-p}A^p \leq M_p$, given by Theorem 2. (Also $M_p \leq G$).

REFERENCES

1. Jameson, G.J.O., Mercer, P.R. (2019). The logarithmic mean revisited. *Amer. Math. Monthly* 126 (7): 641–645.

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