ON THE COHOMOLOGY OF PRO-FUSION SYSTEMS

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Abstract. We prove the Cartan-Eilenberg stable elements theorem and construct a Lyndon-Hochschild-Serre type spectral sequence for pro-fusion systems. As an application, we determine the continuous mod-p cohomology ring of $\mathrm{GL}_2(\mathbb{Z}_p)$ for any odd prime p.

1. Introduction

Throughout, let p denote a prime number. Fusion systems for finite groups and compact Lie groups have been successfully defined as algebraic models for their p-completed classifying spaces, see [5]. For profinite groups, fusion was first studied in [12]. More recently, fusions systems have been defined over pro-p groups and are termed *pro-fusion systems* [24].

To compute the mod-p cohomology rings of finite groups and compact Lie groups two tools stand out, namely, the well-known Cartan-Eilenberg stable elements theorem, and the Lyndon-Hochschild-Serre spectral sequence. In the present work, we study the corresponding tools for the continuous mod-p cohomology ring $H^*_c(\cdot;\mathbb{F}_p)$ of pro-fusion systems, where the coefficients are the trivial module $\mathbb{F}_p$. If there is no confusion, we write $H^*_c(\cdot) = H^*_c(\cdot;\mathbb{F}_p)$ for brevity. Our first main result deals with pro-saturated pro-fusion systems (see Definition 2.6).

Theorem 1.1 (Stable Elements Theorem for Pro-Fusion Systems)**.** *Let $\mathcal{F}$ be a pro-saturated pro-fusion system on a pro-p group S, where $\mathcal{F} = \varprojlim_{i\in I} \mathcal{F}_i$ and $S = \varprojlim_{i\in I} S_i$. Then there is a ring isomorphism*

$$H^*_c(S)^{\mathcal{F}} \cong \varinjlim_{i\in I} H^*(S_i)^{\mathcal{F}_i}.$$

Here, $H^*_c(S)^{\mathcal{F}}$ and $H^*(S_i)^{\mathcal{F}_i}$ are the subrings of stable elements for S and its finite quotients S_i, respectively, see Definition 4.3. If $\mathcal{F}$ is finitely generated in the sense of Definition 4.1, the stable elements are determined via a finite number of conditions. As usual, we define the cohomology ring of the pro-fusion system $\mathcal{F}$ (resp. the finite fusion system $\mathcal{F}_i$) to be the subring $H^*_c(\mathcal{F}) := H^*_c(S)^{\mathcal{F}}$ (resp. $H^*(\mathcal{F}_i) := H^*(S_i)^{\mathcal{F}_i}$) of $H^*_c(S)$ (resp. $H^*(S_i)$).

For a profinite group G, we consider the profinite p-completion of the classifying space of G, BG_p, following Morel [19], and we show - analogously to the finite case - that there is a ring isomorphism

$$H^*_c(BG_p) \cong H^*_c(G).$$

Date: January 20, 2021.

2010 *Mathematics Subject Classification.* Primary: 20E18; Secondary: 22E41, 18G40, 55R35.

Key words and phrases. profinite groups, pro-fusion systems, cohomology, p-adic analytic groups.

In addition, we write $\mathcal{F}_S(G)$ for the pro-fusion system defined by the conjugation action of G on a Sylow p-subgroup S. It turns out that $\mathcal{F}_S(G)$ is pro-saturated and saturated and that it is finitely generated if S is open in G. This is the case if G is a compact p-adic analytic group; for example if $G = \mathrm{GL}_n(\mathbb{Z}_p)$.

Theorem 1.2 (Stable Elements Theorem for Profinite Groups)**.** *Let G be a profinite group. Then, there is a ring isomorphism*

$$H^*_c(G) \cong H^*_c(S)^{\mathcal{F}_S(G)}.$$

For both finite and compact Lie groups, a spectral sequence can be built under weaker hypotheses than for the Lyndon-Hochschild-Serre spectral sequence, [9, 14]. We prove a version of this result for the continuous mod-p cohomology of pro-fusion systems.

Theorem 1.3. *Let $\mathcal{F}$ be a pro-saturated pro-fusion system on a pro-p group S and let $T \leq S$ be a strongly $\mathcal{F}$-closed subgroup. Then there is a first quadrant cohomological spectral sequence with second page*

$$E_2^{n,m} = H^n_c(S/T; H^m_c(T))^{\mathcal{F}},$$

and which converges to $H^*_c(S)^{\mathcal{F}}$.

As first example application, we compute the cohomology ring $H^*_c(\mathbb{Z}3^{1+2}_+; \mathbb{F}_3)$, where $\mathbb{Z}3^{1+2}_+$ is the 3-adic version of the finite extraspecial group 3^{1+2}_+ (see Example 3.1). Then we determine the cohomology rings of the general linear groups of dimension 2 over the p-adic integers.

Theorem 1.4. *We have:*

(a) *For $p = 3$, $H^*_c(\mathrm{GL}_2(\mathbb{Z}_3); \mathbb{F}_3) \cong \mathbb{F}_3[X] \otimes \Lambda(Z_1, Z_2, Z_3)$, with degrees $|Z_1| = 1$, $|Z_2| = |Z_3| = 3$ and $|X| = 4$.*
(b) *For $p > 3$, $H^*_c(\mathrm{GL}_2(\mathbb{Z}_p); \mathbb{F}_p) \cong \Lambda(Z_1, Z_2)$ with degrees $|Z_1| = 1$, $|Z_2| = 3$.*

The result for $p = 3$ was already obtained in [15] employing different tools. The classes of degrees $2p - 3$ and $2p - 2$ found by Aguadé [2, Corollary 1.2] in the ring $H^*(\mathrm{GL}_2(\mathbb{Z}/p); \mathbb{F}_p)$ survive to $H^*_c(\mathrm{GL}_2(\mathbb{Z}_p); \mathbb{F}_p)$ only for $p = 3$. Note that both rings in the statement of Theorem 1.4 are Cohen-Macaulay and hence, by Benson-Carlson duality for p-adic analytic groups [4, §12.3], they satisfy the Poincaré duality after quotienting out the polynomial parts. The dualizing degrees are 7 and 4, respectively, and they are obtained after a degree shift of 4, which is the p-adic dimension of $\mathrm{GL}_2(\mathbb{Z}_p)$.

Remarks and notation: In Theorems 1.1, 1.2 and 1.3, taking stable elements with respect to the category $\mathcal{F}^\circ$ instead of $\mathcal{F}$ gives isomorphic rings. Here, $\mathcal{F}^\circ$ is the full subcategory of $\mathcal{F}$ with objects the open subgroups of S. We write $H^*_c(\cdot)$ instead of $H^*_c(\cdot; \mathbb{F}_p)$ for the continuous mod-p cohomology ring and $H^*(\cdot)$ instead of $H^*(\cdot; \mathbb{F}_p)$ for the singular and for the discrete mod-p cohomology rings. For a group G and an element $g \in G$, we let c_g denote the conjugation morphism which maps x to $c_g(x) = {}^g x = gxg^{-1}$. If there is no confusion, we denote the elements in a quotient by the same symbols as the elements that they represent. As usual in the context of profinite groups, subgroups are assumed to be closed, generation is considered topologically, and homomorphisms are continuous.

Outline of the paper: In Section 2 we present the necessary background on pro-fusion systems, in Section 3, we briefly discuss cohomology, classifying spaces and p-completion for profinite groups, in Section 4, we prove Theorems 1.1, 1.2 and 1.3, and in Section 5, we determine the cohomology ring of $\mathrm{GL}_2(\mathbb{Z}_p)$, for $p = 3$ and $p > 3$ separately.

Acknowledgements: We are grateful to Peter Symonds for helpful conversations, in particular, those regarding stable elements, see Remark 4.6. We are also grateful to Jon González-Sánchez for useful advice concerning the computations in Section 5.

2. Pro-fusion systems

In this section, we briefly introduce pro-fusion systems on pro-p groups, and we review the known results that we will need in the sequel. Loosely, pro-fusion systems on pro-p groups generalize the fusion systems on finite p-groups. We refer the reader to [24] and to [5] for more background and more details on the topic.

First, recall that a fusion system on a finite p-group S is a category whose objects are the subgroups of S and the morphisms are injective group homomorphisms subject to certain axioms.

Definition 2.1 (Morphisms of fusion systems)**.** Let $\mathcal{F}$ and $\mathcal{G}$ be fusion systems on finite p-groups S and T, respectively. A *morphism of fusion systems* $(\alpha, A)\colon \mathcal{F} \to \mathcal{G}$ consists of a group homomorphism $\alpha\colon S \to T$ and a functor $A\colon \mathcal{F} \to \mathcal{G}$ satisfying the following properties.

1. $A(P) = \alpha(P)$ for each subgroup P of S.
2. For each morphism $\varphi\colon P \to Q$ in $\mathcal{F}$ we have a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & \alpha(P) \arrow[d, "A(\varphi)"] \\
Q \arrow[r, "\alpha"] & \alpha(Q).
\end{tikzcd}$$

Note that if $(\alpha, A)\colon \mathcal{F} \to \mathcal{G}$ is a morphism of fusion systems, then the homomorphism α completely determines the functor A. Explicitly, in the above definition, $A(\varphi)$ is defined by the formula $A(\varphi)(\alpha(u)) = \alpha(\varphi(u))$ for $u \in P$. From this formula, we see immediately that if a homomorphism $\alpha\colon S \to T$ defines a morphism of fusion systems from $\mathcal{F}$ to $\mathcal{G}$, then $\mathrm{Ker}(\alpha)$ is a strongly $\mathcal{F}$-closed subgroup of S. Conversely, if $\alpha\colon S \to T$ is a homomorphism such that $\mathrm{Ker}(\alpha)$ is strongly $\mathcal{F}$-closed in S, then for any $\mathcal{F}$-morphism $\varphi\colon P \to Q$, the induced homomorphism $\alpha_*(\varphi)\colon \alpha(P) \to \alpha(Q)$ sending $\alpha(u)$ to $\alpha(\varphi(u))$ for $u \in P$ is well-defined. In this case, α defines a morphism of fusion systems from $\mathcal{F}$ to $\mathcal{G}$ if and only if $\alpha_*(\varphi) \in \mathcal{G}$ whenever $\varphi \in \mathcal{F}$.

Now we give a slightly expanded reformulation of pro-fusion systems on pro-p groups given in [24]. Recall that a poset $(I, \geq)$ is *directed* if for all $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$.

Definition 2.2 (Pro-fusion systems)**.** Suppose that we have an inverse system of fusion systems $\mathcal{F}_i$ on finite p-groups S_i, indexed by a directed poset I. This means that there is a functor from I (as a category) to the category of fusion systems on finite p-groups. Explicitly, we have a fusion system $\mathcal{F}_i$ on a finite p-group S_i for all $i \in I$ and we have morphisms of fusion systems $(f_{ij}, F_{ij})\colon \mathcal{F}_j \to \mathcal{F}_i$ for all $i, j \in I$ with $j \geq i$ such that $f_{ii} = \mathrm{id}_{S_i}$ (and hence $F_{ii} = \mathrm{id}_{\mathcal{F}_i}$) for all $i \in I$, and such that $f_{ij} f_{jk} = f_{ik}$ (and hence $F_{ij} F_{jk} = F_{ik}$) for all $i, j, k \in I$ with

$k \geq j \geq i$. Set $S = \varprojlim_{i\in I} S_i$, and let $f_i \colon S \to S_i$ be the canonical homomorphisms and $N_i = \mathrm{Ker}(f_i)$ for all $i \in I$. Then S is a pro-p group and $\{N_i \mid i \in I\}$ is a basis of open neighborhoods of 1 in S such that $N_j \leq N_i$ if $j \geq i$. We define a category $\mathcal{F}$ as follows. The objects of $\mathcal{F}$ are the closed subgroups of S. If P is a closed subgroup of S, we have $P = \varprojlim_{i\in I} f_i(P)$. If Q is another closed subgroup of S, then the functors $F_{ij} \colon \mathcal{F}_j \to \mathcal{F}_i$ define an inverse system of finite sets $\mathrm{Hom}_{\mathcal{F}_i}(f_i(P), f_i(Q))$ and we have a map

$$\begin{array}{ccc} \varprojlim_{i\in I} \mathrm{Hom}_{\mathcal{F}_i}(f_i(P), f_i(Q)) & \xrightarrow{\theta_{P,Q}} & \mathrm{Hom}_c(P, Q) \\ (\varphi_i)_{i\in I} & \mapsto & (\varphi\colon (x_i) \mapsto (\varphi_i(x_i)))_{i\in I}. \end{array}$$

The relation between the maps φ_i and φ can be summarized by the following commutative diagram

$$\begin{tikzcd}
P \arrow[r, "f_j"] \arrow[rr, bend left, "f_i"] \arrow[d, "\varphi"'] & f_j(P) \arrow[r, "f_{ij}"] \arrow[d, "\varphi_j"] & f_i(P) \arrow[d, "\varphi_i"] \\
Q \arrow[r, "f_j"'] \arrow[rr, bend right, "f_i"'] & f_j(Q) \arrow[r, "f_{ij}"'] & f_i(Q)
\end{tikzcd}$$

where $i, j \in I$ with $j \geq i$: by the universal property of the inverse limit, φ is the unique homomorphism making the left square of the above diagram commutative for all $j \in I$. Here $\mathrm{Hom}_c(P, Q)$ denotes the set of continuous homomorphisms from P to Q with respect to the profinite topology. Indeed $\varphi \colon P \to Q$ is continuous because we have $\varphi(P \cap N_i) \subseteq Q \cap N_i$ for all $i \in I$ by the above diagram. We set $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ to be the image of the above map $\theta_{P,Q}$. Since $\theta_{P,Q}$ is injective, we may identify

$$\mathrm{Hom}_{\mathcal{F}}(P, Q) = \varprojlim_{i\in I} \mathrm{Hom}_{\mathcal{F}_i}(f_i(P), f_i(Q)).$$

It is straightforward to see that $\mathcal{F}$ is indeed a category under the usual composition of maps. We say that $\mathcal{F}$ is a *pro-fusion system* on the pro-p group S.

Pro-fusion systems satisfy all the axioms of fusion systems (cf. [24, Lemmas 2.9, 2.12]). In particular, every fusion system $\mathcal{F}$ on a finite p-group is a pro-fusion system because it is isomorphic to the pro-fusion system defined by the constant inverse system $\{\mathcal{F}\}$. Also note that the inclusions $\varphi(P \cap N_i) \subseteq Q \cap N_i$ show the continuity of the $\mathcal{F}$-morphisms φ, and imply that the N_i are strongly $\mathcal{F}$-closed subgroups of S.

The notion of morphisms of pro-fusion systems on pro-p groups is identical to that of fusion systems on finite p-groups, with the additional requirements that homomorphisms be continuous and that subgroups be closed.

Definition 2.3 (Morphisms of pro-fusion systems)**.** Let $\mathcal{F}$, $\mathcal{G}$ be pro-fusion systems on pro-p groups S, T, respectively. A *morphism of pro-fusion systems* $(\alpha, A) \colon \mathcal{F} \to \mathcal{G}$ consists of a continuous homomorphism $\alpha \colon S \to T$ and a functor $A \colon \mathcal{F} \to \mathcal{G}$ satisfying the following properties.

(1) $A(P) = \alpha(P)$ for each closed subgroup P of S.

(2) For each morphism $\varphi\colon P \to Q$ in $\mathcal{F}$ we have a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & \alpha(P) \arrow[d, "A(\varphi)"] \\
Q \arrow[r, "\alpha"] & \alpha(Q)
\end{tikzcd}$$

Morphisms of pro-fusion systems can be composed in the obvious way, thus forming the category of pro-fusion systems. As for fusion systems, if $(\alpha, A)\colon \mathcal{F} \to \mathcal{G}$ is a morphism of pro-fusion systems, then the continuous homomorphism α completely determines the functor A, and a continuous homomorphism $\alpha\colon S \to T$ defines a morphism of pro-fusion systems from $\mathcal{F}$ to $\mathcal{G}$ if and only if $\mathrm{Ker}(\alpha)$ is strongly $\mathcal{F}$-closed in S and for each morphism $\varphi\colon P \to Q$ in $\mathcal{F}$ the induced homomorphism $\alpha_*(\varphi)\colon \alpha(P) \to \alpha(Q)$ belongs to $\mathcal{G}$.

Example 2.4 (Canonical morphisms). Let $\mathcal{F}$ be the pro-fusion system defined by an inverse system of fusion systems $\mathcal{F}_i$ on finite p-groups S_i. We use the notations in Definition 2.2. Since each $N_i = \mathrm{Ker}(f_i)$ is an open strongly $\mathcal{F}$-closed subgroup of S, the canonical homomorphism $f_i\colon S \to S_i$ is continuous and sends each morphism $\varphi = (\varphi_i)$ in $\mathcal{F}$ to the morphism φ_i in $\mathcal{F}_i$. Thus f_i defines a morphism of pro-fusion systems $(f_i, F_i)\colon \mathcal{F} \to \mathcal{F}_i$. We call the morphisms $(f_i, F_i)\colon \mathcal{F} \to \mathcal{F}_i$ the *canonical morphisms* associated to the pro-fusion system $\mathcal{F}$.

The proof of the next proposition is easy and we omit it.

Proposition 2.5 (Pro-fusion systems as inverse limits). *Let $\mathcal{F}$ be the pro-fusion system defined by an inverse system of fusion systems $\mathcal{F}_i$ on finite p-groups. Then $\mathcal{F} = \varprojlim_i \mathcal{F}_i$ is the inverse limit of the $\mathcal{F}_i$ in the category of pro-fusion systems.*

Saturation for pro-fusion systems may be defined in a similar fashion to that in the finite case, see [24, Definition 2.16]. The next discussion illustrates some of the subtleties underneath the concepts of (pro-)saturation.

Definition 2.6. A *pro-saturated fusion system* is a pro-fusion system $\mathcal{F} = \varprojlim_{i\in I} \mathcal{F}_i$ on a pro-p group $S = \varprojlim_{i\in I} S_i$, where each $\mathcal{F}_i$ is a saturated fusion system on the finite p-group S_i.

A profinite group or pro-fusion system is termed *countably based* if it can be expressed as an inverse limit indexed by the set of natural numbers $\mathbb{N}$.

Lemma 2.7 ([24, Theorems 3.9]). *A pro-fusion system $\mathcal{F}$ on a pro-p group S is countably based if and only if S is countably based (as pro-p group).*

Theorem 2.8 ([24, Theorems 5.1, 5.2]). *If $\mathcal{F}$ is a pro-saturated fusion system and $\mathcal{F}$ is countably based, then $\mathcal{F}$ is saturated.*

Example 2.9. If G is a profinite group with Sylow pro-p subgroup S, then we define the pro-fusion system that G induces on S as the category $\mathcal{F}_S(G)$ with objects the closed subgroups of S and morphisms all homomorphisms induced by conjugation by elements of G [24, Example 2.8]. Then $\mathcal{F}_S(G)$ is a pro-fusion system in the sense of Definition 2.2, it is pro-saturated, and it is saturated by [24, Example 2.18].

3. Cohomology, classifying spaces and p-completion

In this section, we first discuss continuous cohomology of profinite groups and some of its properties. Since we are concerned about p-local phenomena, we always consider coefficients in the trivial module $\mathbb{F}_p$. We refer the reader to [26], [22] for more details and background. Then we discuss classifying spaces and p-completion for simplicial sets [6] and for profinite simplicial sets [19], [13].

Let G be a profinite group and let $H^*_c(G)$ be its continuous mod-p cohomology ring. The group $H^*_c(G)$ can be recovered from the finite images of G, i.e., if $G \cong \varprojlim_{i\in I} G_i$ for some inverse system of finite groups G_i, then for all $* \geq 0$,

$$H^*_c(G) \cong \varinjlim_{i\in I} H^*(G_i). \tag{1}$$

If G is profinite and K is a closed normal subgroup of G, there exists a first quadrant cohomological spectral sequence converging to $H^*_c(G)$ [22, §7.2]. It is the Lyndon-Hochschild-Serre spectral sequence, LHS s.s. for short,

$$E_2^{n,m} = H^n_c(G/K; H^m_c(K)) \Rightarrow H^{n+m}_c(G). \tag{2}$$

It is easy to check that this is a spectral sequence of $\mathbb{F}_p$-algebras. It is the profinite counterpart to the standard Lyndon-Hochschild-Serre spectral sequence for abstract (discrete) groups. Below, in Theorem 3.2, we provide an example of computation using (1) and (2).

Example 3.1. Consider the extraspecial group of order p^3 and exponent p,

$$p^{1+2}_+ = \langle A, B, C \mid A^p = B^p = C^p = [A, C] = [B, C] = 1, [A, B] = C\rangle. \tag{3}$$

The mod-p cohomology ring $H^*(p^{1+2}_+)$ was computed by Leary in [17]. Define $\mathbb{Z}3^{1+2}_+$ to be the 3-adic version of the extraspecial group 3^{1+2}_+, namely $\mathbb{Z}3^{1+2}_+ = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}/3$, with $\mathbb{Z}/3$-action on $\mathbb{Z}_3 \times \mathbb{Z}_3$ given by the integral matrix $\left(\begin{smallmatrix} 1 & -3 \\ 1 & -2 \end{smallmatrix}\right)$.

Theorem 3.2. *The continuous mod-*3 *cohomology of the group* $\mathbb{Z}3^{1+2}_+$ *is given by*

$$H^*_c(\mathbb{Z}3^{1+2}_+; \mathbb{F}_3) \cong \mathbb{F}_3[x'] \otimes \Lambda(y, y', Y, Y')/\{yy', yY, y'Y', YY', yY' - y'Y\},$$

with degrees $|y| = |y'| = 1$, $|Y| = |Y'| = |x'| = 2$.

Proof. For $i \geq 1$, let $N_i = 3^i\mathbb{Z}\times 3^i\mathbb{Z}$ and let $G_i = G/N_i = (\mathbb{Z}/3^i\times\mathbb{Z}/3^i)\rtimes\mathbb{Z}/3$ be the semidirect product with $\mathbb{Z}/3$ acting via the integral matrix $\left(\begin{smallmatrix} 1 & -3 \\ 1 & -2 \end{smallmatrix}\right)$. In particular, $G_1 \cong 3^{1+2}_+$ as described in (3) and we have surjective group homomorphisms $G_{i+1} \twoheadrightarrow G_i$. We set G to be the pro-3 group $\varprojlim_{i\in I} G_i \cong \mathbb{Z}3^{1+2}_+$.

The graded $\mathbb{F}_3$-modules $H^*(G_i)$ are known to be isomorphic for all $i \geq 1$ [10, Proposition 5.8], but the rings $\{H^*(G_i)\}_{i>1}$ are still unknown. Nevertheless, here we determine the ring $H^*_c(G)$.

Consider the LHS spectral sequence (2) associated to the normal subgroup $K = \mathbb{Z}_3 \times \mathbb{Z}_3$ of G. Note that $H^*_c(K)$ is an exterior algebra on two generators of degree 1. Moreover, $G/K \cong \mathbb{Z}/3$ acts on $H^1_c(K)$ via $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and trivially on $H^2_c(K)$. Using the results in [23, Corollary 4(ii)], we obtain the $\mathbb{F}_3$-generators for the corner of $E_2^{*,*}$,

$$\begin{array}{c|ccccc}
2 & \overline{Y} & \overline{Y}\overline{y'} & \overline{Y}\overline{x'} & \overline{Y}\overline{y'}\overline{x'} & \overline{Y}\overline{x'}^2 \\
1 & \overline{y} & \overline{Y'} & \overline{y}\overline{x'} & \overline{Y'}\overline{x'} & \overline{y}\overline{x'}^2 \\
0 & 1 & \overline{y'} & \overline{x'} & \overline{y'x'} & \overline{x'}^2 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

Then, $\overline{y}, \overline{y'}, \overline{x'}, \overline{Y}, \overline{Y'}$ generate the bigraded algebra $E_2^{*,*}$. The homomorphism $\pi\colon G \to G_1$ induces a morphism of spectral sequences from the LHS s.s. of G_1 to that of G and all five generators of $E_2^{*,*}$ are in the image of this morphism. By [23, Theorem 5(i)], the LHS spectral sequence of G_1 collapses at the second page, and we have $E_2 = E_\infty$ for the LHS spectral sequence of G. Consider now the induced morphism $\pi^*\colon H^*(G_1) \to H_c^*(G)$ and define y, y', x', Y, Y' to be the image by π^* of the generators of the same name in [17, Theorem 7]. Then they give rise to the aforementioned overlined generators of $E_2^{*,*}$, and they satisfy the relations:

$$yy' = yY = y'Y' = YY' = Y^2 = Y'^2 = 0,\ yY' = y'Y. \tag{4}$$

The corresponding relations between the overlined generators give a bigraded algebra isomorphic to $E_2^{*,*}$. Hence, $H_c^*(G)$ is generated by the elements y, y', x', Y, Y' of degrees $1, 1, 2, 2, 2$ respectively subject to the relations (4). □

The notion of continuous and discrete cohomology groups may be extended to the categories $\widehat{\mathsf{SSet}}$ of simplicial profinite sets and SSet of simplicial (discrete) sets respectively. In fact, if $X = \{X_n\}_{n\geq 0}$ belongs to either of these categories, then $H_c^*(X)$ if $X \in \widehat{\mathsf{SSet}}$ or $H^*(X)$ if $X \in \mathsf{SSet}$. We consider cohomology groups of classifying spaces of either finite or profinite groups.

Definition 3.3. Let G be a finite group or a profinite group. Then its *classifying space* $BG \in \mathsf{SSet}$ or $BG \in \widehat{\mathsf{SSet}}$, respectively, is the simplicial set or simplicial profinite set, respectively, with n-simplices $(BG)_n = G \times \ldots \times G$ (n-copies).

Here, the face and degeneracy maps are the usual ones, and we have ring isomorphisms

$$H^*(G) \cong H^*(BG) \text{ for } G \text{ finite and } H_c^*(G) \cong H_c^*(BG) \text{ for } G \text{ profinite.} \tag{5}$$

Morel builds in [19, Proposition 2] a functor $\widehat{\mathsf{SSet}} \to \widehat{\mathsf{SSet}}$, that we denote as $X \mapsto X_p$, together with a natural transformation $X \to X_p$. The object X_p is the fibrant replacement of X in a certain model category structure on $\widehat{\mathsf{SSet}}$ for which the weak equivalences are the $H_c^*(\cdot)$-isomorphisms. In particular, the induced map,

$$H_c^*(X_p) \to H_c^*(X) \text{ is an isomorphism.} \tag{6}$$

Sullivan's profinite completion can be obtained from Morel's construction after forgetting the profinite topology, see [19, p. 368].

Definition 3.4. Let $X \in \widehat{\mathsf{SSet}}$ be a simplicial profinite set. Then its *p-completion* is the simplicial profinite set X_p.

If G is a profinite group, Equations (5) and (6) give a ring isomorphism

$$H_c^*(G) \cong H_c^*(BG_p), \tag{7}$$

i.e., we recover the continuous cohomology of G via the p-completion BG_p of its classifying space. Finally, let G be a finite group and let $(\mathbb{F}_p)_\infty BG$ denote the Bousfield-Kan p-completion [6] of its classifying space. Note that G may be considered as a profinite group and BG as a simplicial profinite set. By [13, Corollary 3.16], there is a weak equivalence of simplicial sets,

$$(\mathbb{F}_p)_\infty BG \to |BG_p|,$$

where $|\cdot|\colon \widehat{\mathsf{SSet}} \to \mathsf{SSet}$ is the forgetful functor. So Definition 3.4 extends the notion of the Bousfield-Kan p-completion for classifying spaces of finite groups to the p-completion for classifying spaces of profinite groups.

4. Stable elements and a spectral sequence

We start this section with the definition of a subset of morphisms that generates a pro-fusion system. Then we introduce the notion of stable elements in the present context, before proving Theorems 1.1, 1.2 and 1.3.

Definition 4.1. Let $\mathcal{F}$ be a pro-fusion system on a pro-p group S and let $\mathcal{M}$ be a set of morphisms of $\mathcal{F}$. We say that $\mathcal{F}$ is *generated by* $\mathcal{M}$ if every morphism in $\mathcal{F}$ is equal to the composition of a finite sequence of restrictions of morphisms in $\mathcal{M} \cup \mathrm{Inn}(S)$. In this case, we call $\mathcal{M}$ a set of *generators* of $\mathcal{F}$. If $\mathcal{M}$ is finite, we say that $\mathcal{F}$ is *finitely generated*.

As mentioned in the introduction, $\mathcal{F}^\circ$ denotes the full subcategory of $\mathcal{F}$ with objects the open subgroups of S. The notions of generation and finite generation may analogously be defined for $\mathcal{F}^\circ$.

Lemma 4.2. *Let $\mathcal{F}$ be a pro-fusion system on a pro-p group S. If $\mathcal{F}$ is finitely generated, then $\mathcal{F}^\circ$ is also finitely generated.*

Proof. Let $\mathcal{M}$ be a finite set of generators of $\mathcal{F}$. We show that $\mathcal{M}^\circ = \{\varphi \in \mathcal{M} \mid \varphi \text{ belongs to } \mathcal{F}^\circ\}$ is a finite set of generators of $\mathcal{F}^\circ$. Let $\varphi\colon P \to Q$ be an $\mathcal{F}^\circ$-isomorphism. Then there exists a finite sequence of $\mathcal{F}$-isomorphisms $\varphi_i\colon P_{i-1} \to P_i$ $(1 \le i \le n)$ such that $P_0 = P$, $P_n = Q$, $\varphi = \varphi_n \circ \cdots \circ \varphi_1$, where each φ_i is a restriction of some $\mathcal{F}$-morphism in $\mathcal{M} \cup \mathrm{Inn}(S)$. Now the $\mathcal{F}$-isomorphisms preserve the open subgroups and also their indices in S by [24, Lemma 2.11]. Thus all the subgroups P_i are open, and hence all the morphisms φ_i are restrictions of some $\mathcal{F}^\circ$-morphisms in $\mathcal{M} \cup \mathrm{Inn}(S)$. This makes $\mathcal{M}^\circ$ a finite set of generators of $\mathcal{F}^\circ$. ☐

Next we define the $\mathcal{F}$-stable elements for general functors with domain $\mathcal{F}$.

Definition 4.3. Let $\mathcal{F}$ be a pro-fusion system on a pro-p group S and consider a functor $H\colon \mathcal{F}^{op} \to \mathsf{Ab}$ from the category $\mathcal{F}$ to the category of abelian groups. We say that $x \in H(S)$ is *$\mathcal{F}$-stable* (resp. *$\mathcal{F}^\circ$-stable*) if $H(\varphi)(x) = H(\iota_P^S)(x)$ for all closed (resp. open) subgroups $P \le S$ and all $\mathcal{F}$-morphisms $\varphi\colon P \to S$. We write $H(S)^{\mathcal{F}}$ (resp. $H(S)^{\mathcal{F}^\circ}$) for the abelian subgroup of $\mathcal{F}$-stable (resp. $\mathcal{F}^\circ$-stable) elements of $H(S)$.

Here and throughout, ι_P^S denotes the inclusion homomorphism from P to S, and we write res_P^S instead of $H(\iota_P^S)$ whenever the functor H is clear from the context. Note that the abelian group $H(S)^{\mathcal{F}}$ (resp. $H(S)^{\mathcal{F}^\circ}$) is exactly the inverse limit $\varprojlim_{P\in\mathcal{F}} H(\cdot)$ (resp. $\varprojlim_{P\in\mathcal{F}^\circ} H(\cdot)$). The functors $H\colon \mathcal{F}^{op} \to \mathsf{Ab}$ that we consider satisfy the following condition: given any closed subgroup P of S and any $x \in P$, then $H(c_x) = 1_{H(P)} \colon H(P) \to H(P)$ is the identity. This is the case for the cohomology functors used in Theorems 4.5 and 4.9 below. The next lemma is easy to prove. It shows, in particular, that if $\mathcal{F}$ is finitely generated, it is enough to consider a finite number of conditions to determine the $\mathcal{F}$-stable elements.

Lemma 4.4. *Let $\mathcal{F}$ be a pro-fusion system on a pro-p group S. Assume that $\mathcal{F}$ is generated by $\mathcal{M}$. Let $H \colon \mathcal{F}^{op} \to \mathsf{Ab}$ be a functor such that $H(c_x) = 1_{H(P)}$ for $c_x \in \mathrm{Inn}(P)$, $P \leq S$. Then,*

$$H(S)^{\mathcal{F}} = \{x \in H(S) \mid H(\iota_Q^S \circ \varphi)(x) = H(\iota_P^S)(x) \text{ for all } \varphi \colon P \to Q, \varphi \in \mathcal{M}\}.$$

A similar statement holds for $\mathcal{F}^\circ$-stable elements if $\mathcal{F}^\circ$ is generated by $\mathcal{M}^\circ$.

For the continuous cohomology functor $H = H_c^*(\cdot) \colon \mathcal{F} \to \mathbb{F}_p\text{-}\mathsf{Alg} \subseteq \mathsf{Ab}$, note that $H_c^*(S)^{\mathcal{F}}$ and $H_c^*(S)^{\mathcal{F}^\circ}$ are $\mathbb{F}_p$-subalgebras of $H_c^*(S)$. The classical stable elements theorem of Cartan and Eilenberg [8, XII.10.1] states that, for a finite group G with Sylow p-subgroup S, we have $H^*(G; \mathbb{F}_p) \cong H^*(S; \mathbb{F}_p)^{\mathcal{F}_S(G)}$. We next show that a similar statement holds for profinite groups, proving Theorem 1.1.

Theorem 4.5 (Stable Elements Theorem for Pro-Fusion Systems)**.** *Let $\mathcal{F} = \varprojlim_{i \in I} \mathcal{F}_i$ be a pro-fusion system on a pro-p group S. Suppose that $\mathcal{F}^o$ is finitely generated, or that $\mathcal{F}$ is pro-saturated, or both. Then*

$$H_c^*(\mathcal{F}; \mathbb{F}_p) = H_c^*(S; \mathbb{F}_p)^{\mathcal{F}} = H_c^*(S; \mathbb{F}_p)^{\mathcal{F}^\circ} \cong \varinjlim_{i \in I} H^*(S_i; \mathbb{F}_p)^{\mathcal{F}_i}.$$

Remark 4.6. The conclusion of Theorem 4.5 using the assumption that $\mathcal{F}$ is pro-saturated was suggested by Peter Symonds [25].

Observe also that by Lemma 4.2, the above theorem holds under the assumption that $\mathcal{F}$ is finitely generated.

Proof. We have an isomorphism

$$H_c^*(S; \mathbb{F}_p) \cong \varinjlim_{i \in I} H^*(S_i; \mathbb{F}_p),$$

which is induced by the maps $\varphi_i^* \colon H^*(S_i; \mathbb{F}_p) \to H^*(S; \mathbb{F}_p)$. Since I is directed, every element of $\varinjlim_{i \in I} H^*(S_i; \mathbb{F}_p)$ can be represented by some $x_i \in H^*(S_i; \mathbb{F}_p)$ and two elements $x_i \in H^*(S_i; \mathbb{F}_p)$ and $x_j \in H^*(S_j; \mathbb{F}_p)$ represent the same element in $\varinjlim_{i \in I} H^*(S_i; \mathbb{F}_p)$ if and only if there exists some $k \in I$ with $k \geq i$, $k \geq j$ such that $\varphi_{ik}^*(x_i) = \varphi_{jk}^*(x_j)$.

First we check that the direct system $\{H^*(S_i; \mathbb{F}_p), \varphi_{ij}^*\}$ restricts to the subsystem $\{H^*(S_i; \mathbb{F}_p)^{\mathcal{F}_i}, \varphi_{ij}^*\}$; that is, if $j \geq i$, then $\varphi_{ij}^* \colon H^*(S_i; \mathbb{F}_p) \to H^*(S_j; \mathbb{F}_p)$ sends $H^*(S_i; \mathbb{F}_p)^{\mathcal{F}_i}$ into $H^*(S_j; \mathbb{F}_p)^{\mathcal{F}_j}$. To see this, let $x_i \in H^*(S_i; \mathbb{F}_p)^{\mathcal{F}_i}$. We want to show that $\varphi_{ij}^*(x_i) \in H^*(S_j; \mathbb{F}_p)^{\mathcal{F}_j}$. Let $\psi_j \colon P_j \to S_j$ be a morphism in $\mathcal{F}_j$. Then the commutative diagrams

$$\begin{tikzcd} S_j \arrow[r, "\varphi_{ij}"] & S_i \\ P_j \arrow[u, "\psi_j"] \arrow[r, "\varphi_{ij}|_{P_j}"] & P_i \arrow[u, "\psi_i"'] \end{tikzcd} \qquad \begin{tikzcd} S_j \arrow[r, "\varphi_{ij}"] & S_i \\ P_j \arrow[u, "\iota_{P_j}^{S_j}"] \arrow[r, "\varphi_{ij}|_{P_j}"] & P_i \arrow[u, "\iota_{P_i}^{S_i}"'] \end{tikzcd}$$

(where $P_i := \varphi_{ij}(P_j)$, $\psi_i := F_{ij}(\psi_j)$) induce commutative diagrams

$$\begin{tikzcd}
H^*(S_j;\mathbb{F}_p) \arrow[d, "\psi_j^*"'] & H^*(S_i;\mathbb{F}_p) \arrow[l, "\varphi_{ij}^*"'] \arrow[d, "\psi_i^*"] \\
H^*(P_j;\mathbb{F}_p) & H^*(P_i;\mathbb{F}_p) \arrow[l, "(\varphi_{ij}|_{P_j})^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H^*(S_j;\mathbb{F}_p) \arrow[d, "\mathrm{res}^{S_j}_{P_j}"'] & H^*(S_i;\mathbb{F}_p) \arrow[l, "\varphi_{ij}^*"'] \arrow[d, "\mathrm{res}^{S_i}_{P_i}"] \\
H^*(P_j;\mathbb{F}_p) & H^*(P_i;\mathbb{F}_p) \arrow[l, "(\varphi_{ij}|_{P_j})^*"']
\end{tikzcd}$$

Since $x_i \in H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i}$, the two images of x_i in $H^*(P_i;\mathbb{F}_p)$ coincide. Thus the two images of $\varphi_{ij}^*(x_i)$ in $H^*(P_j;\mathbb{F}_p)$ coincide too. This shows that $\varphi_{ij}^*(x_i) \in H^*(S_j;\mathbb{F}_p)^{\mathcal{F}_j}$, as desired. Thus it makes sense to consider $\varinjlim_{i\in I} H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i}$ as a subspace of $\varinjlim_{i\in I} H^*(S_i;\mathbb{F}_p)$.

Now we check that the image of $\varinjlim_{i\in I} H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i}$ in $H^*_c(S;\mathbb{F}_p)$ is contained in $H^*_c(S;\mathbb{F}_p)^{\mathcal{F}}$. Let $x_i \in H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i}$, representing an element of $\varinjlim_{i\in I} H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i}$. If P is a closed subgroup of S and $\psi\colon P \to S$ is an $\mathcal{F}$-morphism, then the commutative diagrams

$$\begin{tikzcd}
S \arrow[r, "\varphi_i"] & S_i \\
P \arrow[u, "\psi"] \arrow[r, "\varphi_i|_P"] & P_i \arrow[u, "\psi_i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "\varphi_i"] & S_i \\
P \arrow[u, "\iota_P^S"] \arrow[r, "\varphi_i|_P"] & P_i \arrow[u, "\iota_{P_i}^{S_i}"']
\end{tikzcd}$$

(where $P_i := \varphi_i(P)$, $\psi_i := F_i(\psi)$) induce commutative diagrams

$$\begin{tikzcd}
H^*_c(S;\mathbb{F}_p) \arrow[d, "\psi^*"'] & H^*(S_i;\mathbb{F}_p) \arrow[l, "\varphi_i^*"'] \arrow[d, "\psi_i^*"] \\
H^*_c(P;\mathbb{F}_p) & H^*(P_i;\mathbb{F}_p) \arrow[l, "(\varphi_i|_P)^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H^*_c(S;\mathbb{F}_p) \arrow[d, "\mathrm{res}^S_P"'] & H^*(S_i;\mathbb{F}_p) \arrow[l, "\varphi_i^*"'] \arrow[d, "\mathrm{res}^{S_i}_{P_i}"] \\
H^*_c(P;\mathbb{F}_p) & H^*(P_i;\mathbb{F}_p) \arrow[l, "(\varphi_i|_P)^*"']
\end{tikzcd}$$

Since $x_i \in H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i}$, the two images of x_i in $H^*(P_i;\mathbb{F}_p)$ coincide. Thus the two images of $\varphi_i^*(x_i)$ in $H^*_c(P;\mathbb{F}_p)$ also coincide. This shows that the image of $\varinjlim_{i\in I} H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i}$ in $H^*_c(S;\mathbb{F}_p)$ is contained in $H^*_c(S;\mathbb{F}_p)^{\mathcal{F}}$. Clearly $H^*_c(S;\mathbb{F}_p)^{\mathcal{F}}$ is contained in $H^*_c(S;\mathbb{F}_p)^{\mathcal{F}^\circ}$. Thus it remains to show that $H^*_c(S;\mathbb{F}_p)^{\mathcal{F}^\circ}$ is contained in the image of $\varinjlim_{i\in I} H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i}$.

Suppose first that $\mathcal{F}^o$ is finitely generated. By [24, Lemma 3.3], we may assume that $\mathcal{F}_i = \langle F_i(\mathcal{F})\rangle$ for all $i \in I$ (see also [24, Proof of Proposition 3.7]). In particular, to determine the stable subring it is enough to consider $F_i(\mathcal{F})$, i.e., $H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i} = H^*(S_i;\mathbb{F}_p)^{F_i(\mathcal{F})}$ for all $i \in I$. Now, assume that that $x \in H^*_c(S;\mathbb{F}_p)^{\mathcal{F}^\circ}$. Let P be an open subgroup of S and let $\psi\colon P \to S$ be an $\mathcal{F}^\circ$-morphism. Since I is directed, there is some $i \in I$ and $x_i \in H^*(S_i;\mathbb{F}_p)$ such that $x = \varphi_i^*(x_i)$. Since $\psi^*(x) = \mathrm{res}^S_P(x)$, the last two previous diagrams show that $\psi_i^*(x_i)$ and $\mathrm{res}^{S_i}_{P_i}(x_i)$ have the same image under $(\varphi_i|_P)^*$. Since $H^*_c(P;\mathbb{F}_p) \cong \varinjlim_{i\in I} H^*(P_i;\mathbb{F}_p)$ with the isomorphism induced by the $(\varphi_i|_P)^*$, the elements $\psi_i^*(x_i)$ and $\mathrm{res}^{S_i}_{P_i}(x_i)$ in $H^*(P_i;\mathbb{F}_p)$ have the same image under $(\varphi_{ij}|_P)^*$ for some $j \geq i$. Thus by replacing x_i by $\varphi_{ij}^*(x_i)$ we may assume that $\psi_i^*(x_i) = \mathrm{res}^{S_i}_{P_i}(x_i)$. We can do this for the finite number of generators ψ of $\mathcal{F}^\circ$. Since I is directed, there exist $i \in I$ and $x_i \in H^*(S_i;\mathbb{F}_p)$ such that $x = \varphi_i^*(x_i)$ and $\psi_i^*(x_i) = \mathrm{res}^{S_i}_{P_i}(x_i)$ for all ψ. Since $H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i} = H^*(S_i;\mathbb{F}_p)^{F_i(\mathcal{F})}$, it follows that

$x_i \in H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i}$, and hence x is in the image of $\varinjlim_{i\in I} H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i}$, finishing the proof in the case when $\mathcal{F}^o$ is finitely generated.

Finally, suppose that $\mathcal{F}$ is pro-saturated. By [24, Proposition 4.5], we may assume that $\mathcal{F}_i = \mathcal{F}/N_i$ and hence the functors $F_i\colon \mathcal{F}\to\mathcal{F}_i$ and $F_{ij}\colon \mathcal{F}_j\to\mathcal{F}_i$ are all surjective on objects and morphisms. Moreover, by [24, Proposition 4.4], the restriction of F_i to the full subcategory $\mathcal{F}^o$ on the open subgroups, $F_i\colon \mathcal{F}^o\to\mathcal{F}_i$, is also surjective on objects and morphisms. Suppose that $x\in H^*_c(S;\mathbb{F}_p)^{\mathcal{F}^\circ}$. Since I is directed, there is some $i\in I$ and $x_i\in H^*(S_i;\mathbb{F}_p)$ such that $x=\varphi_i^*(x_i)$. Now, each morphism $\psi_i\in \operatorname{Hom}_{\mathcal{F}_i}(P_i,S_i)$ arises from some morphism $\psi\in\operatorname{Hom}_{\mathcal{F}^o}(P,S)$ with

$$N_i \subseteq P. \tag{8}$$

Since $\psi^*(x) = \operatorname{res}^S_P(x)$, the last two previous diagrams show that $\psi_i^*(x_i)$ and $\operatorname{res}^{S_i}_{P_i}(x_i)$ have the same image under $(\varphi_i|_P)^*$. Then

$$(\varphi_{ij}|_{P_j})^*(\psi_i^*(x_i) - \operatorname{res}^{S_i}_{P_i}(x_i)) = \psi_j^*(x_j) - \operatorname{res}^{S_j}_{P_j}(x_j) = 0 \in H^*(P_j;\mathbb{F}_p) \tag{9}$$

for some $j\geq i$, where $x_j=\varphi_{ij}^*(x_i)$. Repeating this argument for the finitely many morphisms of $\mathcal{F}_i$, we find $k\geq i$ such that equation (9) holds for $j=k$ and for every morphism $\psi_i\in\mathcal{F}_i$ and, in addition, for each $\psi_i\in\mathcal{F}_i$ we have chosen P satisfying (8). To finish, we check that $x_k\in H^*(S_k;\mathbb{F}_p)^{\mathcal{F}_k}$. Any morphism $\chi_k\in\operatorname{Hom}_{\mathcal{F}_k}(Q_k,S)$ arises from some morphism $\chi\in\operatorname{Hom}_{\mathcal{F}}(Q,S)$. Then $\chi_i\in\mathcal{F}_i$ and, from the choice of P for χ_i satisfying (8), we have that $Q_k\subseteq P_k$. From this and Equation (9) applied to $j=k$ and $\chi_i\in\mathcal{F}_i$, we find that $\psi_k^*(x_k)-\operatorname{res}^{S_k}_{P_k}(x_k)=0$. ☐

The next result proves Theorem 1.2.

Corollary 4.7. *Let G be a profinite group with Sylow pro-p subgroup S. Then we have*

$$H^*_c(G;\mathbb{F}_p)\cong H^*_c(S;\mathbb{F}_p)^{\mathcal{F}_S(G)} = H^*_c(S;\mathbb{F}_p)^{\mathcal{F}_S(G)^\circ}.$$

Proof. Write $G=\varprojlim_{i\in I} G_i$ and $S=\varprojlim_{i\in I} S_i$. Then we have

$$H^*_c(G;\mathbb{F}_p)\cong \varinjlim_{i\in I} H^*(G_i;\mathbb{F}_p)\cong \varinjlim_{i\in I} H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_{S_i}(G_i)},$$

where the second isomorphism is due to the stable elements theorem for finite groups. Thus, as $\mathcal{F}_S(G)$ is pro-saturated, by Theorem 4.5 it follows that

$$\varinjlim_{i\in I} H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_{S_i}(G_i)}\cong H^*_c(S;\mathbb{F}_p)^{\mathcal{F}_S(G)} = H^*_c(S;\mathbb{F}_p)^{\mathcal{F}_S(G)^\circ},$$

finishing the proof. ☐

We give an easy condition on a profinite group that ensures that the pro-fusion system that it induces on a Sylow pro-p subgroup is finitely generated, and hence stable elements are determined by a finite number of conditions.

Lemma 4.8. *Let G be a profinite group with an open Sylow pro-p subgroup S. Then $\mathcal{F}_S(G)$ is finitely generated.*

Proof. Let T be a (finite) set of representatives of the right cosets of S in G. We claim that the finite set $\{c_t\colon S\cap S^t\to S\cap {}^tS \mid t\in T\}$ generates $\mathcal{F}_S(G)$. To see this, suppose that $c_g\colon P\to Q$ is an $\mathcal{F}_S(G)$-morphism given by some $g\in G$. There exist $t\in T$ and $s\in S$ such that $g=st$. So $c_g=c_sc_t$, where $c_t\colon P\to {}^tP$ is a restriction of $c_t\colon S\cap S^t \to S\cap {}^tS$ and $c_s\colon {}^tP\to Q$ is a restriction of $c_s\in \mathrm{Inn}(S)$. This proves the lemma. □

Corollary 4.7 for a profinite group with an open Sylow pro-p subgroup can also be proven directly using the restriction and transfer maps for cohomology of profinite groups and their open subgroups (see [22, §6.7]), in exactly the same way as the stable elements theorem is proven for finite groups.

We end this section with the construction of a spectral sequence involving a strongly closed subgroup of a pro-saturated fusion system on a pro-p group. The idea is based on the spectral sequence for a saturated fusion system on a finite p-group [9]. The next result proves Theorem 1.3.

Theorem 4.9. *Let $\mathcal{F}$ be a pro-saturated fusion system on a pro-p group S and let $T\leq S$ be a strongly $\mathcal{F}$-closed subgroup. Then there is a first quadrant cohomological spectral sequence with second page*

$$E_2^{n,m}=H^n_c(S/T;H^m_c(T;\mathbb{F}_p))^{\mathcal{F}}\cong H^n_c(S/T;H^m_c(T;\mathbb{F}_p))^{\mathcal{F}^o}$$

and converging to $H^*_c(\mathcal{F};\mathbb{F}_p)=H^*_c(S;\mathbb{F}_p)^{\mathcal{F}}=H^*_c(S;\mathbb{F}_p)^{\mathcal{F}^o}$.

Proof. The $\mathcal{F}$-stable elements of the second page in the statement are taken with respect to the functor $\mathcal{F}\to\mathsf{Ab}$ that sends each subgroup $P\leq S$ to the second page

$$H^n_c(P/P\cap T;H^m_c(P\cap T;\mathbb{F}_p))$$

of the Lyndon-Hochshild-Serre spectral sequence $\{E_k^{*,*}(P),d_k\}_{k\geq 2}$ associated to the following short exact sequence of profinite groups (see Equation (2)):

$$P\cap T\to P\to P/P\cap T\cong PT/P.$$

Fix $i\in I$ and write $S_i=S/N_i$. Then $T_i=TN_i/N_i$ is a strongly $\mathcal{F}_i$-closed subgroup of S_i and, according to [9, Theorem 1.1], there is a first quadrant spectral sequence with second page

$$E_2^{n,m}=H^n(S_i/T_i;H^m(T_i;\mathbb{F}_p))^{\mathcal{F}_i}$$

and converging to $H(S_i;\mathbb{F}_p)^{\mathcal{F}_i}$. Each subgroup $P_i\leq S_i$ is of the form $P_i=P/N_i$ for $N_i\trianglelefteq P\leq S$ and the $\mathcal{F}_i$-stable elements in the last display are taken with respect to the functor that sends P_i to the second page,

$$H^n(P_i/P_i\cap T_i;H^m(P_i\cap T_i;\mathbb{F}_p)),$$

of the Lyndon-Hochshild-Serre spectral sequence $\{E_k^{*,*}(P_i),d_k\}_{k\geq 2}$ associated to the following short exact sequence of finite groups,

$$P_i\cap T_i\to P_i\to P_i/P_i\cap T_i\cong P_iT_i/P_i.$$

Now, for each $P\leq S$, we have $P\cap T=\varprojlim_{i\in I}P_i\cap T_i$ and, by [26, Theorem 1.2.5(a)], $P/P\cap T=\varprojlim_{i\in I}P_i/P_i\cap T_i$. For fixed non-negative integers n and m, the collection $\{H^m(P_i\cap T_i;\mathbb{F}_p)\}_{i\in I}$ is a direct system of discrete abelian groups and hence, by [26, Theorem 9.7.2(e)], we have a ring isomorphism

$$H^n_c(P/P\cap T;H^m_c(P\cap T;\mathbb{F}_p))\cong\varinjlim_{i\in I}H^n(P_i/P_i\cap T_i;H^m(P_i\cap T_i;\mathbb{F}_p)).$$

In other words, using the above notation,

$$E_2^{*,*}(P)=\varinjlim_{i\in I}E_2^{*,*}(P_i). \tag{10}$$

By induction on $k \geq 2$, we obtain ring isomorphisms

$$\begin{aligned} E_{k+1}^{*,*}(P) =& H^*(E_k^{*,*}(P), d_k) \cong H^*(\varinjlim_{i\in I} E_k^{*,*}(P_i), d_k) \cong \\ \cong& \varinjlim_{i\in I} H^*(E_k^{*,*}(P_i), d_k) \cong \varinjlim_{i\in I} E_{k+1}^{*,*}(P_i), \end{aligned}$$

because the direct limit over a directed set is an exact functor, it commutes with the cohomology functor. Now, the functoriality of short exact sequences, the functoriality of the Lyndon-Hochshild-Serre spectral sequence on short exact sequences, and the arguments in the proof of Theorem 4.5, prove that

$$E_k^{*,*}(S)^{\mathcal{F}} = E_k^{*,*}(S)^{\mathcal{F}^o} \cong \varinjlim_{i\in I} E_k^{*,*}(S_i)^{\mathcal{F}_i},$$

for all $k \geq 2$. In particular, as $E_{k+1}^{*,*}(S_i)^{\mathcal{F}_i} = H^*(E_k^{*,*}(S_i)^{\mathcal{F}_i}, d_k)$ for each i, we have:

$$\begin{aligned} E_{k+1}^{*,*}(S)^{\mathcal{F}} =& \varinjlim_{i\in I} E_{k+1}^{*,*}(S_i)^{\mathcal{F}_i} = \varinjlim_{i\in I} H^*(E_k^{*,*}(S_i)^{\mathcal{F}_i}, d_k) \cong \\ \cong& H^*(\varinjlim_{i\in I} E_k^{*,*}(S_i)^{\mathcal{F}_i}, d_k) = H^*(E_k^{*,*}(S)^{\mathcal{F}}, d_k), \end{aligned}$$

where we have used again that direct limits over direct sets are exact functors. Hence, $\{E_k^{*,*}(S)^{\mathcal{F}}, d_k\}_{k\geq 2}$ is a sub-spectral sequence of $\{E_k^{*,*}(S), d_k\}_{k\geq 2}$. The fact that it converges to $H_c(S;\mathbb{F}_p)^{\mathcal{F}}$ is proven by considering filtrations as in [9, Proof of Theorem 4.1]. □

In the above theorem, if $\mathcal{F} = \mathcal{F}_S(G)$ for a profinite group G with a Sylow pro-p subgroup S and $T \trianglelefteq G$, then the spectral sequence coincides with the LHS s.s. (2).

5. MOD-p COHOMOLOGY OF $\mathrm{GL}_2(\mathbb{Z}_p)$

Let G be a profinite group with a Sylow pro-p subgroup S that contains a strongly $\mathcal{F}_S(G)$-closed subgroup. Then the spectral sequence described in Theorem 4.9 can be used to compute the mod-p cohomology ring structure of G. The computations are easier when $\mathcal{F}_S(G)$ is a finitely generated pro-fusion system, in particular when S is an open subgroup of G by Lemma 4.8. A widely studied class of profinite groups with open Sylow p-subgroups is the class of *compact p-adic analytic groups* (see [11, Corollary 8.34]). In this section, we compute the mod-p cohomology ring structure of $\mathrm{GL}_2(\mathbb{Z}_p)$ for any odd prime p, and the outcome is stated in Theorem 1.4.

Following [11, Section 5.1], if $G = \mathrm{GL}_2(\mathbb{Z}_p)$, then $G = \varprojlim_{i\geq 1} G/K_i$, where the subgroups $K_i = \{x \in G \mid x \equiv I_2 \pmod{p^i}\}$ are the *congruence subgroups,* for $i \geq 1$. We know that K_1 is a uniform pro-p group of rank 4 (see [11, Section 4.1 and Theorem 5.2]). Let us set the Sylow pro-p subgroup of G,

$$S = \{g \in \mathrm{GL}_2(\mathbb{Z}_p) | \; g \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{p}\}.$$

In particular K_1 is a strongly $\mathcal{F}_S(G)$-closed subgroup of S. Moreover we have a group extension

$$1 \longrightarrow K_1 \longrightarrow S \longrightarrow \mathbb{Z}/p \longrightarrow 1\ , \tag{11}$$

and the spectral sequence of Theorem 4.9 takes the form

$$E_2^{n,m} = H_c^n(\mathbb{Z}/p; H_c^m(K_1;\mathbb{F}_p))^{\mathcal{F}_S(G)} \Rightarrow H_c^{n+m}(\mathrm{GL}_2(\mathbb{Z}_p);\mathbb{F}_p). \tag{12}$$

In Equation (11), the group $\mathbb{Z}_p$ is the subgroup of G generated by $h = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. For $p = 3$, G contains 3-torsion and the extension (11) splits, while G has no p-torsion for $p > 3$, and hence the extension does not split.

5.1. **Spectral sequence for S.** We start by describing the second page of the spectral sequence (12).

We write

$$H^*(\mathbb{Z}/p) = \Lambda(u) \otimes \mathbb{F}_p[v] \text{ and } H_c^*(K_1) \cong \Lambda(y_{11}, y_{12}, y_{21}, y_{22}),$$

where the generators have degrees $|u| = 1$, $|v| = 2$ and $|y_{ij}| = 1$. Recall that the cohomology of K_1 is known because it is a uniform pro-p group (see [26, Theorem 11.6.1] and the subsequent discussion). The generators y_{ij} are the continuous homomorphisms $y_{ij} \in \mathrm{Hom}_c(K_1, \mathbb{F}_p)$ mapping an element $g = \left(\begin{smallmatrix}1+pa & pb\\ pc & 1+pd\end{smallmatrix}\right) \in K_1$ to

$$y_{11}(g) = \bar{a},\ y_{12}(g) = \bar{b},\ y_{21}(g) = \bar{c},\ y_{22}(g) = \bar{d},$$

where $\bar{\ }$ denotes the projection map $\mathbb{Z}_p \to \mathbb{Z}/p$. We calculate the action of h^{-1} on $H_c^1(K_1)$, and we obtain

$$y_{11} \mapsto y_{11} - y_{21}\ ,\ y_{12} \mapsto y_{11} + y_{12} - y_{21} - y_{22}\ ,\ y_{21} \mapsto y_{21} \text{ and } y_{22} \mapsto y_{21} + y_{22}. \tag{13}$$

Using the exterior algebra structure of $H_c^*(K_1)$, the above action can be extended to all of $H_c^*(K_1)$ as follows:

$$H_c^m(K_1) \cong \begin{cases} J^1 & \text{for } m = 0, 4\\ J^1 \oplus J^3 & \text{for } m = 1, 3\\ J^3 \oplus J^3 & \text{for } m = 2,\end{cases} \tag{14}$$

where J^n denotes the $\mathbb{Z}/p$-module of dimension n associated to the Jordan block of size $n \times n$. We use the periodic $\mathbb{F}_p[\mathbb{Z}/p]$-free resolution in [7, §I.6] to compute the cohomology groups $E_2^{n,m} = H_c^n(\mathbb{Z}/p; H_c^m(K_1))$. Set $y_1 = y_{11} + y_{22}$, $y_2 = y_{21}$, $y_3 = y_{11}y_{21}$ and $y_4 = y_{11}y_{12}y_{21} - y_{12}y_{21}y_{22}$. Then we have,

$$H_c^n(\mathbb{Z}/p; J^1) = \mathbb{F}_p\{j\} \quad \text{with } j = \begin{cases} 1 & m = 0,\\ y_1 & m = 1,\\ y_4 & m = 3,\\ y_1y_4 & m = 4,\end{cases}$$

and

$$H_c^n(\mathbb{Z}/p; J^3) = \begin{cases} \mathbb{F}_3\{j\} & n = 0,\ p = 3,\\ 0 & n > 0,\ p = 3,\\ \mathbb{F}_p\{j\} & p > 3,\ n \text{ even},\\ \mathbb{F}_p\{\bar{j}\} & p > 3,\ n \text{ odd},\end{cases} \quad \text{with } j = \begin{cases} y_2 & m = 1,\\ y_3 & m = 2,\\ y_2y_1 + y_3 & m = 2,\\ y_1y_3 & m = 3.\end{cases}$$

and with

$$\begin{aligned} \overline{y_2} &= y_{12} - y_{11}\ , & \overline{y_3} &= y_{12}y_{22} - y_{12}y_{21}\ ,\\ \overline{y_2y_1 + y_3} &= y_{11}y_{12}\ , & \overline{y_1y_3} &= y_{11}y_{12}y_{21} - y_{11}y_{12}y_{22}.\end{aligned}$$

To denote a class in $E_2^{n,m}$, we use the representative in $H_c^n(\mathbb{Z}/p; H_c^m(K_1))$ defined above multiplied with $uv^{\frac{n-1}{2}}$ or $v^{\frac{n}{2}}$ according to the parity of n.

The cup products in $E_2^{*,*}$ may be computed using the diagonal approximation (see [7, Exercise, p. 108]): if α and α' are representatives in $H_c^m(K_1)$ and $H_c^{m'}(K_1)$ of classes in $E_2^{n,m}$ and $E_2^{n',m'}$ respectively, then their product is represented by

$$\begin{cases} \alpha\alpha' & \text{if } nn' \text{ is even, and} \\ \sum_{0 \leq i < j < p} h^i(\alpha)h^j(\alpha') & \text{if } nn' \text{ is odd.} \end{cases}$$

For $p = 3$, we obtain that $E_2^{*,*}$ is generated by the classes u, v, y_1, y_2, y_3, y_4 and it has the following presentation as a bigraded $\mathbb{F}_3$-algebra,

$$\mathbb{F}_3[v] \otimes \Lambda[u, y_1, y_2, y_3, y_4]/\{y_2u = y_2v = y_3u = y_3v = y_2y_3 = y_2y_4 = y_3y_4 = 0\}. \tag{15}$$

For $p > 3$, we obtain that $E_2^{*,*}$ is generated by the classes $u, v, y_1, y_2, y_3, y_4, u\overline{y_2}, u\overline{y_3}$, because of the relations

$$u\overline{y_1y_3} = \frac{1}{2}uy_4 - y_1u\overline{y_3} \quad \text{and} \quad u\overline{y_1y_2} = -y_1u\overline{y_2}.$$

In both cases, multiplication by v is an isomorphism from $E_2^{n,m}$ to $E_2^{n+2,m}$ for $n \geq 1$ and $m \geq 0$. Figure 1 describes the bottom left corner of the spectral sequences in (12), for $p = 3$ and $p > 3$ separately. In this and the subsequent tables, we leave an entry blank if it is zero.

4	y_1y_4	uy_1y_4	vy_1y_4
3	y_4, y_1y_3	uy_4	vy_4
2	y_3, y_1y_2		
1	y_1, y_2	uy_1	vy_1
0	1	u	v
	0	1	2

4	y_1y_4	uy_1y_4	vy_1y_4
3	y_4, y_1y_3	$uy_4, u\overline{y_1y_3}$	vy_1y_3, vy_4
2	y_3, y_1y_2	$u\overline{y}_3, u\overline{y_1y_2}$	vy_3, vy_1y_2
1	y_1, y_2	$uy_1, u\overline{y}_2$	vy_1, vy_2
0	1	u	v
	0	1	2

FIGURE 1. The corner of E_2 for $p = 3$ (left) and $p > 3$ (right).

5.2. **Continuous mod-3 cohomology ring of** $\mathrm{GL}_2(\mathbb{Z}_3)$**.** We set $p = 3$ and we compute the ring $H_c^*(\mathrm{GL}_2(\mathbb{Z}_3); \mathbb{F}_3)$.

Theorem 5.1. *The spectral sequence* (12) *for* $p = 3$ *satisfies that* $E_2^{\mathcal{F}_S(G)}$ *is the free bigraded algebra with generators*

$E_2^{0,m}$	y_1		y_4
m	1	2	3

$E_2^{n,0}$			uv	v^2
n	1	2	3	4

and, in addition, $d_i(y_1) = d_i(y_4) = d_i(uv) = d_i(v^2) = 0$ *for all* $i \geq 2$*. Moreover,* $H_c^*(\mathrm{GL}_2(\mathbb{Z}_3)) \cong \mathbb{F}_3[X] \otimes \Lambda(Z_1, Z_2, Z_3)$*, where* Z_1, Z_2, Z_3 *and* X *are the liftings of* y_1, uv, y_4 *and* v^2*, respectively; of degrees* $|Z_1| = 1$*,* $|Z_2| = |Z_3| = 3$ *and* $|X| = 4$*.*

Proof. By Lemma 4.8, $\mathcal{F}_S(G)$ is generated by the finite set

$$\mathcal{M} = \{c_{g_1}, c_{g_2} : K_1 \to K_1\} \cup \{c_{g_t}, c_{g_z} : S \to S\},$$

where, the conjugation morphisms are given by the following elements,

$$g_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad g_2 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \quad g_t = \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad g_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right),$$

with $t, z \in \{1,2\}$. A straightforward computation shows that the elements of $E_2^{0,*}$ that restrict to $H_c^*(K_1)^{\langle g_1, g_2\rangle}$ are the following,

$E_2^{0,m}$	y_1		y_4	y_1y_4
m	1	2	3	4

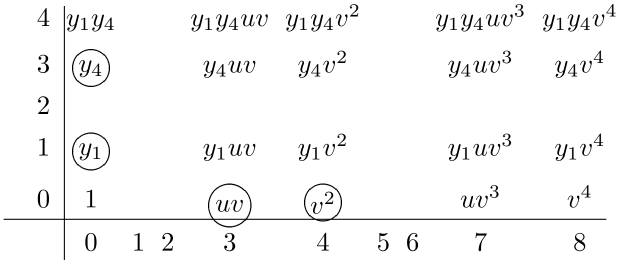
It remains to find the stable elements in $E_2^{*,*}$ under the action of g_t,

$$u \mapsto t^{-1}u,\ v \mapsto t^{-1}v,\ y_1 \mapsto y_1,\ \ y_2 \mapsto ty_2,\ y_3 \mapsto ty_3,\ \ y_4 \mapsto y_4,$$

and the action of g_z,

$$u \mapsto zu, v \mapsto zv, y_1 \mapsto y_1,\ \ y_2 \mapsto z^{-1}y_2,\ y_3 \mapsto z^{-1}y_3,\ \ y_4 \mapsto y_4.$$

A short computation shows that $E_2^{\mathcal{F}_S(G)}$ is generated by the elements y_1, uv, y_4, v^2, which are circled in the next figure.



4	y_1y_4			y_1y_4uv	$y_1y_4v^2$			$y_1y_4uv^3$	$y_1y_4v^4$
3	y_4			y_4uv	y_4v^2			y_4uv^3	y_4v^4
2									
1	y_1			y_1uv	y_1v^2			y_1uv^3	y_1v^4
0	1			uv	v^2			uv^3	v^4
	0	1	2	3	4	5	6	7	8

Moreover, by Equation (15), $E_2^{\mathcal{F}_S(G)}$ is the free bigraded algebra on these generators. Clearly, $d_i(y_1) = d_i(uv) = d_i(v^2) = 0$ for all $i \geq 2$. To deduce the differentials $d_i(y_4)$, $i \geq 2$, consider the quotient group $Q = S/K_2$, where $K_2 = \{x \in \mathrm{GL}_2(\mathbb{Z}_3) \mid x \equiv I_2 \pmod{3^2}\}$ denotes the second congruence subgroup.

This yields a commutative diagram of split extensions,

$$\begin{tikzcd}
1 \arrow[r] & K_1 \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & \mathbb{Z}/3 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & K \arrow[r] & Q \arrow[r] & \mathbb{Z}/3 \arrow[r] & 1.
\end{tikzcd}$$

We have $Q = \langle k_{11}, k_{12}, k_{21}, k_{22}, h\rangle$ with $K \cong (\mathbb{Z}/3)^4$ generated by the elements

$$k_{11} = \begin{pmatrix} 1+3 & 0 \\ 0 & 1 \end{pmatrix}, \quad k_{12} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad k_{21} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad \text{and} \quad k_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1+3 \end{pmatrix}.$$

The action of h permutes the first three elements of the following basis of K cyclically, and h fixes the last one:

$$\{k_{11} - k_{12} + k_{21} - k_{22},\ k_{21},\ -k_{11} - k_{12} + k_{21} + k_{22},\ k_{11} + k_{22}\}.$$

So $Q = (\mathbb{Z}/3 \wr \mathbb{Z}/3) \times \mathbb{Z}/3$ and the LHS spectral sequence associated to Q collapses at the second page (see [1, IV.1, Theorem 1.7]). Moreover, it is straightforward to see that y_4 is in the image of the map $H^*(K) \to H_c^*(K_1)$, and so $d_i(y_4) = 0$ for $i \geq 2$. Finally, the only possible lift of a free bigraded algebra is the free graded algebra on the same total degrees [18, §1.5, Example 1.K]. The description of $H_c^*(\mathrm{GL}_2(\mathbb{Z}_3); \mathbb{F}_3)$ follows. $\square$

5.3. **Continuous mod-p cohomology ring of** $\mathrm{GL}_2(\mathbb{Z}_p)$ **for** $p > 3$**.** We now compute the ring structure of $H_c^*(\mathrm{GL}_2(\mathbb{Z}_p);\mathbb{F}_p)$ for $p > 3$.

Theorem 5.2. *The spectral sequence* (12) *for* $p > 3$ *satisfies that* $E_\infty^{\mathcal{F}_S(G)}$ *is a free bigraded algebra with generators* y_1, vy_2 *with bigraded degrees* $|y_1| = (0,1)$ *and* $|vy_2| = (2,1)$*. In addition,* $H_c^*(\mathrm{GL}_2(\mathbb{Z}_p);\mathbb{F}_p) \cong \Lambda(Z_1, Z_2)$*, where* Z_1 *and* Z_2 *are the liftings of* y_1 *and* vy_2*, respectively; of total degrees* $|Z_1| = 1$*,* $|Z_2| = 3$*.*

Proof. We follow the proof of Theorem 5.1 and the notation there. The action of g_t on the remaining generators of E_2 for $p > 3$ (see Figure 1) is determined by the following equations,

$$\begin{array}{rclcrcl} \overline{y_2} & \mapsto & \frac{t^{-1}-1}{2}y_1 + t^{-1}\overline{y_2}, & & \overline{y_3} & \mapsto & t^{-1}\overline{y_3}, \\ \overline{y_1y_2} & \mapsto & t^{-1}\overline{y_1y_2}, & & \overline{y_1y_3} & \mapsto & \frac{1-t^{-1}}{2}y_4 + t^{-1}\overline{y_1y_3} \end{array}$$

and the action of g_z by the next set of expressions,

$$\overline{y_2} \mapsto \frac{z-1}{2}y_1 + z\overline{y_2}, \quad \overline{y_3} \mapsto z\overline{y_3}, \quad \overline{y_1y_2} \mapsto z\overline{y_1y_2}, \quad \overline{y_1y_3} \mapsto \frac{1-z}{2}y_4 + z\overline{y_1y_3}.$$

A tedious computation shows that $E_2^{\mathcal{F}_S(G)}$ is generated by

$$y_1,\ y_4,\ vy_2,\ vy_3,\ \tfrac{1}{2}uv^{p-3}y_1 + uv^{p-3}\overline{y_2},\ uv^{p-3}\overline{y_1y_2},$$
$$uv^{p-3}\overline{y_3},\ -\tfrac{1}{2}uv^{p-3}y_4 + uv^{p-3}\overline{y_1y_3},\ uv^{p-2},\ v^{p-1}$$

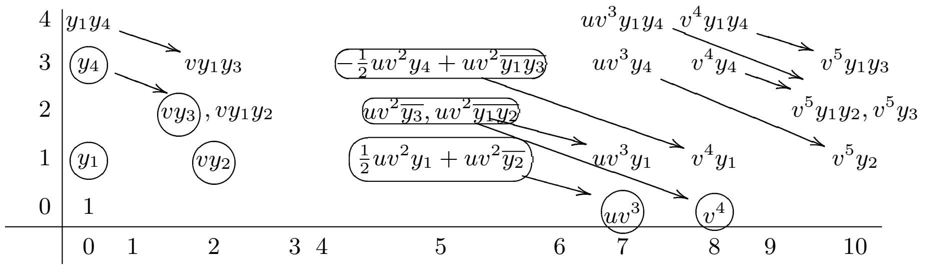
with bigraded degrees

$$\begin{aligned} |y_1| = (0,1), |y_4| = (3,0), |vy_2| = (2,1), |vy_3| = (2,2), |\tfrac{1}{2}uv^2y_1 + uv^2\overline{y_2}| = (2p-5,1), \\ |uv^2\overline{y_1y_2}| = (2p-5,2), |uv^{p-3}\overline{y_3}| = (2p-5,2), \\ |-\tfrac{1}{2}uv^{p-3}y_4 + uv^{p-3}\overline{y_1y_3}| = (2p-5,3), |uv^{p-2}| = (2p-3,0), |v^{p-1}| = (2p-2,0). \end{aligned}$$

Multiplication by v^{p-1} gives an isomorphism

$$(E_2^{n,m})^{\mathcal{F}_S(G)} \cong (E_2^{n+2(p-1),m})^{\mathcal{F}_S(G)} \tag{16}$$

for all $n \geq 1$ and $m \geq 0$. The following figure is the E_2 corner for the $p = 5$ case and the circled elements are the generators listed above. The only possibly non-trivial differentials are d_2 and d_3 and they are also depicted in the figure.



Since G is a compact p-adic analytic group, its cohomology ring is finitely generated (see [16]) and, since G has no p-torsion, we conclude that $H_c^*(G;\mathbb{F}_p)$ is a finite $\mathbb{F}_p$-algebra from Quillen's F-isomorphism Theorem [20]. In particular, the elements in the columns $2p-5+2(p-1)k$, $2p-3+2(p-1)k$, $2p-2+2(p-1)k$ and $2p+2(p-1)k$ must vanish for all $k \geq k_0$ for some $k_0 \geq 0$. The periodicity in Equation (16) and the Leibniz rule force $k_0 = 0$, and it follows that all the differentials between non-trivial elements of these columns are nonzero. In turn, this implies that $d_2(y_4) = \alpha v y_3 + \beta v y_1 y_2$ for some $\alpha, \beta \in \mathbb{F}_p$ with $\alpha \neq 0$. Consequently the only non-trivial elements in $E_\infty^{\mathcal{F}_S(G)} = E_4^{\mathcal{F}_S(G)}$ are y_1, vy_2 and $\overline{\alpha' v y_3 + \beta' v y_1 y_2}$ with α' and β' in $\mathbb{F}_p$ such that the determinant $\begin{vmatrix} \alpha & \alpha' \\ \beta & \beta' \end{vmatrix} \neq 0$. Also, the product vy_1y_2 is equal to $\overline{\alpha' v y_3 + \beta' v y_1 y_2}$ because $\begin{vmatrix} \alpha & 0 \\ \beta & 1 \end{vmatrix} = \alpha \neq 0$. Thus, $E_\infty^{\mathcal{F}_S(G)}$ is the free bigraded $\mathbb{F}_p$-algebra with generators y_1 and vy_2 of bigraded degrees $|y_1| = (0,1)$ and $|vy_2| = (2,1)$. Again the only lift is the free graded $\mathbb{F}_p$-algebra with generators Z_1 and Z_2 of degrees $|Z_1| = 1$ and $|Z_2| = 3$. ☐

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