

Frameworks with coordinated edge motions

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Abstract

We develop a rigidity theory for bar-joint frameworks in Euclidean d -space in which specified classes of edges are allowed to change length in a coordinated fashion that requires differences of lengths to be preserved within each class. Rigidity for these coordinated frameworks is a generic property, and we characterize the rigid graphs in terms of redundant rigidity in the standard d -dimensional rigidity matroid. We also interpret our main results in terms of matroid unions.

1. Introduction

A (bar-joint) framework (G, p) is a graph $G = (V, E)$ and a map $p : V \rightarrow \mathbb{E}^d$. By identifying p with a vector in $\mathbb{E}^{d|V|}$ (using any order on V), we may also refer to p as a configuration of $|V|$ points in \mathbb{E}^d . Intuitively, we may think of a framework as a collection of fixed-length bars (corresponding to the edges of G) which are connected at their ends by joints (corresponding to the vertices of G) that allow bending in any direction of \mathbb{E}^d .

One fundamental question in rigidity theory is whether all edge-length preserving, continuous motions of a given framework are rigid body motions. In this case, a framework is called *rigid* and otherwise *flexible*. (See Section 2 for formal definitions.)

Generic rigidity In general, whether a framework (G, p) is rigid or flexible depends on both G and p ; that is, rigidity is a geometric property. However, there is a typical behavior as p varies and G is fixed.

Definition 1.1. A point configuration p of n points in dimension d is *generic* if the coordinates of the points $p(i)$ do not satisfy any polynomial equation with coefficients in \mathbb{Q} . Generic points are dense in the space of n -point configurations.

If G is a graph with n vertices, a framework (G, p) is called generic if p is generic.

Asimow and Roth proved the following fundamental result [1].

Theorem 1.2. *For every dimension d and graph G , either every generic framework (G, p) in \mathbb{E}^d is rigid or every generic framework (G, p) in \mathbb{E}^d is flexible.*

A consequence is that we may investigate the generic rigidity properties of a graph G .

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Definition 1.3. Let $d \in \mathbb{N}$ be a dimension and G a graph. Then G is *generically rigid* in dimension d if every generic framework (G, p) is rigid; otherwise every generic (G, p) is flexible and G is *generically flexible* in dimension d .

If G is generically rigid in dimension d , but no proper spanning subgraph of G is generically rigid, then G is *isostatic* in dimension d .

Rigidity matroids It is implicit in [1] and first explicitly observed and used by Lovász and Yemini in [13] that generic rigidity has a matroidal structure.

Definition 1.4. Fix a dimension d and let $n \geq d$. Let E^n be the edges of the complete graph K_n . The matroid on the ground set E^n of rank $dn - \binom{d+1}{2}$ that has as its bases the edge sets of the isostatic graphs with n vertices in dimension d is called the d -dimensional *rigidity matroid* of K_n and is denoted by $M_{d,n}$.

The restriction of $M_{d,n}$ to the edges of an n vertex graph G is the *rigidity matroid of G* , $M_d(G)$.

Lovász and Yemini define $M_{d,n}$ in terms of a linearization of rigidity called infinitesimal rigidity that we discuss in more detail in Section 2. This approach is now standard in the field; see, e.g., Whiteley’s survey [24] for an overview on the interplay between matroid theory and rigidity problems.

It is easy to check, using a randomized algorithm based on Gaussian elimination, whether a specific graph G is generically rigid in dimension d for any d and number of vertices n (a detailed analysis is in [8], but this is a folklore fact). On the other hand, except for dimensions $d = 1$, which is folklore, and $d = 2$, which is due to Pollaczek-Geiringer [18] (and later rediscovered by Laman [11]), a combinatorial characterization of the matroids $M_{d,n}$ is a notable open problem [20, Sec. 61.1.2, “Open problems”].

Coordinated rigidity: motivation and results In recent work, Nixon, Schulze, Tanigawa and Whiteley [16] defined a generalization of frameworks that enlarges the class of allowed motions. The vertices of (G, p) are partitioned into $k + 1$ different classes, V_0, V_1, \dots, V_k . The set of allowed configurations p is constrained so that for $j \geq 1$ all vertices $i \in V_j$ have the same distance to the origin (but this distance may change), and all the vertices i in V_0 lie on the unit sphere. The allowed motions are the continuous deformations in the space of allowed configurations. The motivation for studying these types of frameworks is to interpolate between rigidity in dimension d and dimension $d + 1$.

A model for this expanding spheres setup, which is present in [16], is based on Whiteley’s coning construction [23]. We first add a new vertex v_0 to G and fix it at the origin (the centre of the spheres) and then we connect v_0 to each of the vertices of G . The new edges joining v_0 with the vertices in V_0 must have fixed unit length, and the remaining new edges joining v_0 with vertices in $V_1 \cup \dots \cup V_k$ do not have fixed length, but all of the ones in the same class V_j must have the same length.

Inspired by [16] we consider frameworks in which not all of the bars are fixed-length in a more general fashion. We identify, in advance $k \in \mathbb{N}$ “coordination classes” of edges which are allowed to change their length, subject to edge length differences being preserved within each coordination class.

Our study of coordinated rigidity in such a general setup is also motivated by some recent results in condensed matter theory. In [7, 10, 19] it is shown that (nearly) minimally rigid frameworks can be “tuned” to have a number of interesting geometric and material properties. The

results we present here could potentially form the combinatorial part of a design methodology for such “meta-materials”. Other potential practical applications arise from the analysis of frameworks modelling engineering structures that are driven by a collection of pistons which are all connected to a central pump, and so will extend or contract based on the pressure across the whole system, or structures whose members are made of multiple different types of materials, which may expand at different rates when the structure is heated.

In this paper we develop the continuous and infinitesimal rigidity theory for coordinated frameworks and show that rigidity of coordinated frameworks is a generic property. Our main result, Theorem 4.1, shows that generic rigidity of a coordinated framework has a characterization in terms of the generic rigidity properties of a bar-joint framework with the same underlying graph. As a corollary, we identify the d -dimensional coordinated rigidity matroid for a fixed partition of the edges of the graph into coordination classes with a matroid union of the standard rigidity matroid and an associated partition matroid. In particular, this provides a deterministic polynomial time algorithm for checking generic coordinated rigidity in dimension 2 for any k .

Organization The structure of the paper is as follows. We start by briefly introducing the necessary definitions and results from standard (finite and infinitesimal) rigidity theory in Section 2. These definitions and results are then adapted in Section 3 to the coordinated context described above. Characterizations for generic coordinated rigidity in arbitrary dimension are then given in Section 4. Finally, in Section 5 we discuss algorithms for checking generic coordinated rigidity and outline some further directions of research.

2. Rigidity background

We start by introducing the definitions, notation, and basic results from rigidity theory that are required for the development of a rigidity theory for coordinated frameworks. (See [20], for example, for further details.)

2.1. Graphs

We denote graphs by $G = (V, E)$, where V is the set of vertices and E is the set of edges. In cases where G is not clear from the context, we write $V(G)$ for V and $E(G)$ for E . We usually use n and m to denote the number of vertices and edges, respectively, and write edges as unordered pairs $\{i, j\}$ of vertices. We also use the notation e for an edge when the endpoints aren’t important.

2.2. Point configurations

Fix a dimension d . A *d -dimensional configuration* p is an ordered tuple of n points $(p(1), \dots, p(n))$ in \mathbb{E}^d . Let $\text{Euc}(d)$ be the group of *rigid motions* of \mathbb{E}^d . We define configurations p and q to be *congruent* if there is a $T \in \text{Euc}(d)$ such that $q(i) = T(p(i))$ for all $1 \leq i \leq n$.

Fixing an affine structure and an origin, we can identify points in \mathbb{E}^d with their coordinates in \mathbb{R}^d , so we may regard p as a mapping $[n] \rightarrow \mathbb{R}^d$ or a vector in $(\mathbb{R}^d)^n \cong \mathbb{R}^{dn}$.

The tangent space to \mathbb{E}^d is \mathbb{R}^d at every point, so we have an identification between *velocity fields* p' supported on p and $(\mathbb{R}^d)^n \cong \mathbb{R}^{dn}$.

2.3. Frameworks, rigidity, and flexibility

We now define bar-joint frameworks formally.

Definition 2.1. A *d-dimensional (bar-joint) framework* (G, p) is defined by a graph G with n vertices and an n -point configuration p which assigns a point in \mathbb{R}^d to each vertex of G .

Continuous rigidity Underlying the notion of rigidity are the concepts of equivalence and congruence.

Definition 2.2. Two d -dimensional frameworks (G, p) and (G, q) are *equivalent* if

$$\|p(j) - p(i)\| = \|q(j) - q(i)\| \quad \text{for all } \{i, j\} \in E \quad (1)$$

Frameworks (G, p) and (G, q) are *congruent* if p and q are congruent.

Now we can define rigidity and flexibility.

Definition 2.3. A framework (G, p) is (locally) *rigid* if there is a neighborhood $U \subset \mathbb{R}^{dn}$ of p with the property that if $q \in U$ and (G, p) and (G, q) are equivalent, then they are congruent.

A *finite motion* of a framework (G, p) is a one-parameter family of frameworks (G, p_t) with $p_0 = p$ and (G, p_t) equivalent to (G, p) for all $t \in [0, 1]$. A finite motion is *non-trivial* if not all the (G, p_t) are congruent to (G, p) . A framework is *flexible* if it has a non-trivial finite motion.

The definitions of rigid and flexible are, a priori, not directly related. It is straightforward that a rigid framework is not flexible. Asimow and Roth [1] proved the strong converse.

Theorem 2.4. *Let (G, p) be a d -dimensional framework. Then if (G, p) is not rigid, it is flexible. Hence, for any d and G , every d -dimensional framework (G, p) is either rigid or flexible.*

Infinitesimal rigidity Theorem 2.4 holds unconditionally, and its proof relies on a difficult algebraic-geometric result (Milnor's curve selection lemma [14]). In the generic case, it suffices to study the linearization of rigidity.

Definition 2.5. An *infinitesimal motion* $p' \in \mathbb{R}^{dn}$ of a d -dimensional framework (G, p) is a velocity field supported on p such that

$$[p(j) - p(i)] \cdot [p'(j) - p'(i)] = 0 \quad \text{for all } \{i, j\} \in E \quad (2)$$

An infinitesimal motion is called *trivial* if it arises as the derivative of a rigid motion of \mathbb{E}^d , restricted to p . The dimension of the space of trivial infinitesimal motions of a framework in \mathbb{E}^d with at least d vertices is $\binom{d+1}{2}$.

A framework (G, p) in dimension d is *infinitesimally rigid* if every infinitesimal motion of it is trivial. Otherwise (G, p) is *infinitesimally flexible*.

Theorem 1.2 from the introduction follows from the following more specific statement.

Theorem 2.6 ([1]). *Fix a dimension d and let G be a graph with $n \geq d$ vertices. If a d -dimensional framework (G, p) is infinitesimally rigid, then it is rigid. If (G, p) is generic and infinitesimally flexible, then it is flexible.*

To study infinitesimal rigidity, we use the rigidity matrix.

Definition 2.7. Fix a dimension d and let G be a graph with $n \geq d$ vertices. The *rigidity matrix* $R(p)$ of a d -dimensional framework (G, p) is the $m \times dn$ matrix of the system (2), where p' is unknown. The space of infinitesimal motions $M(p)$ is the kernel of $R(p)$, and (G, p) is infinitesimally rigid if and only if $\text{rank} R(p) = dn - \binom{d+1}{2}$.

We are now ready to set up some important concepts relating to infinitesimal rigidity.

Definition 2.8. Let (G, p) be a framework in dimension d . A vector ω in the left kernel of the rigidity matrix $R(p)$ is called an *equilibrium stress* of (G, p) .

If (G, p) has no non-zero equilibrium stress, then (G, p) is *independent*. Otherwise (G, p) is *dependent*. A framework is independent if and only if its rigidity matrix has linearly independent rows.

An edge e of a graph G is called a *redundant edge of (G, p)* if there is an equilibrium stress ω of the framework (G, p) with $\omega(e) \neq 0$. (The edge e is called redundant because removing e from G does not change the rank of $R(p)$.)

A framework that is both infinitesimally rigid and independent is called *isostatic*.

2.4. The rigidity matroid

Throughout this paper we will use the standard matroid terminology of *rank*, *bases* and *independent sets* (see, e.g., [17] for an introduction to matroids).

A standard result that is implicit in [1] is

Lemma 2.9. Fix a dimension d and let G be a graph with $n \geq d$ vertices. If there is any configuration p of n points so that (G, p) is isostatic, then for every generic q , (G, q) is isostatic.

As a corollary we obtain a statement first made explicit in [13].

Lemma 2.10. The rigidity matroid $M_{d,n}$ from Definition 1.4 exists and is isomorphic to the linear matroid on the rows of the rigidity matrix $R(p)$ of the complete graph K_n for any generic choice of p .

Definition 2.11. Fix a dimension d and let $G = (V, E)$ be a graph with n vertices. The *rank* of G is the rank of E in the rigidity matroid $M_{d,n}$. An edge e of G is *redundant* if the rank of G and the rank of the graph $G \setminus \{e\}$ obtained from G by removing the edge e are the same in $M_{d,n}$.

A subset $E' = \{e_1, \dots, e_k\}$ of edges in G is redundant if $G \setminus E'$ has the same rank as G in $M_{d,n}$.

We will need the following lemmas.

Lemma 2.12. Fix a dimension d and let G be a graph with $n \geq d$ vertices. An edge e of G is redundant if and only if for any generic d -dimensional framework (G, p) there is an equilibrium stress ω with coordinate $\omega(e) \neq 0$.

Proof. Suppose there exists a generic d -dimensional framework (G, p) with the property that every equilibrium stress ω of (G, p) satisfies $\omega(e) = 0$. Then the row corresponding to e in $R(p)$ is outside the span of the other rows. Hence the rank of $R(p)$ will drop if we remove that row. By Lemma 2.10, e is not redundant.

Conversely, if for any generic p there is an equilibrium stress ω of (G, p) with $\omega(e) \neq 0$, then the row of $R(p)$ corresponding to e is a linear combination of some of the other rows. Hence, it is possible to pick a set of edges B of G not containing e so that the set of rows corresponding to B are a basis for the row space of $R(p)$. It follows that e is redundant. \square

Lemma 2.13. *Fix a dimension d and let G be a graph. Then a subset of edges $E' = \{e_1, \dots, e_k\}$ is redundant if and only if for any generic d -dimensional framework (G, p) there are equilibrium stresses $\omega_1, \dots, \omega_k$ so that $\omega_i(e_i) \neq 0$ for $i \in [k]$ and $\omega_i(e_j) = 0$ for $i \neq j \in [k]$.*

Proof. Define G_i to be the graph obtained by removing all the edges of E' except for e_i from G . Note that the set E' is redundant in G if and only if e_i is redundant in G_i for each i . Since an equilibrium stress ω_i of (G_i, p) is also an equilibrium stress of (G, p) , with $\omega_i(e_j) = 0$ for $i \neq j \in [k]$, applying Lemma 2.12 to each (G_i, p) completes the proof. \square

3. Coordinated frameworks and rigidity

The main objects of study in this paper are frameworks in which the edges are partitioned into coordination classes. We augment the allowed finite motions so that the edge lengths within each class may change, but the pairwise differences are preserved. Thus, the allowed motions are “coordinated” within each coordination class.

In this section we define coordinated frameworks. To do this, we need to describe the combinatorial and geometric data describing a coordinated framework and the allowed finite motions. We then define infinitesimal motions of coordinated frameworks and derive an associated rigidity matrix. The development runs in parallel to Section 2.

3.1. Combinatorial data

Fix a parameter $k \in \mathbb{N}$, which we call the number of *coordination classes*, and let $G = (V, E)$ be a graph. A *coordination map* is a function $c : E \rightarrow \{0, 1, \dots, k\}$. The underlying combinatorial structure of a coordinated framework is a pair (G, c) . For convenience, we define the notation $E_i := c^{-1}(i)$ for $i \in \{0, 1, \dots, k\}$, where E_0 is the set of *uncoordinated edges*, and E_i for $i \in [k]$ is the i -th *coordination class*. Throughout this paper, we assume that $E_i \neq \emptyset$ for all $i = 1, \dots, k$.

We call a pair (G, c) , where G is a graph and $c : E \rightarrow \{0, 1, \dots, k\}$ is a coordination map, a *k-coordinated graph*.

3.2. Coordinated frameworks and rigidity

Let (G, c) be a k -coordinated graph with n vertices. A *placement* (p, r) of (G, c) is given by a point configuration p of n points in dimension d and a vector $r \in \mathbb{R}^k$. Two placements (p, r) and (q, s) are *congruent* if p and q are congruent.

A *coordinated framework* (G, c, p, r) is given by a k -coordinated graph (G, c) and a placement (p, r) . Two frameworks (G, c, p, r) and (G, c, q, s) are *equivalent* if

$$\|p(j) - p(i)\| = \|q(j) - q(i)\| \quad \text{for all } \{i, j\} \in E_0 \quad (3)$$

$$\|p(j) - p(i)\| + r(\ell) = \|q(j) - q(i)\| + s(\ell) \quad \text{for all } \{i, j\} \in E_\ell, \text{ with } \ell \in [k] \quad (4)$$

and they are *congruent* if they are equivalent and the placements are congruent. Figure 1 shows two equivalent, but not congruent, realizations of a 2-coordinated graph (K_4, c) .

A coordinated framework (G, c, p, r) is *rigid* if there is a neighborhood $U \subset \mathbb{R}^{dn} \times \mathbb{R}^k \cong \mathbb{R}^{dn+k}$ of (p, r) with the property that if $(q, s) \in U$ and (G, c, q, s) is equivalent to (G, c, p, r) , then the two frameworks are congruent. A coordinated framework is *generic* if p is generic.

A *finite motion* of a coordinated framework (G, c, p, r) is a one-parameter family (G, c, p_t, r_t) with $(p_0, r_0) = (p, r)$ and all the (G, c, p_t, r_t) are equivalent to (G, c, p, r) , for $t \in [0, 1]$. A finite motion is non-trivial if not all the (G, c, p_t, r_t) are congruent to (G, c, p, r) . A coordinated framework is *flexible* if it has a non-trivial finite motion.

Remark 3.1. Geometrically, what is maintained over a finite motion is the differences in length between pairs of edges $\{i, j\}$ and $\{u, v\}$ in the same coordination class E_ℓ , since

$$\|p_t(j) - p_t(i)\| + r_t(\ell) - \|p_t(v) - p_t(u)\| - r_t(\ell) = \|p_t(j) - p_t(i)\| - \|p_t(v) - p_t(u)\|$$

does not depend on $r_t(\ell)$, so it must be constant over the motion. \diamond

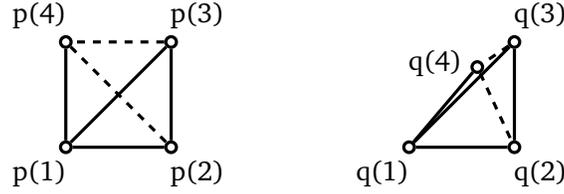


Figure 1: Two equivalent but non-congruent coordinated frameworks (K_4, c, p, r) and (K_4, c, q, s) in the plane with $k = 1$, where edges in E_1 are denoted by dashed lines. The coordinates of the points are $p(1) = q(1) = (0, 0)$, $p(2) = q(2) = (1, 0)$, $p(3) = q(3) = (1, 1)$, $p(4) = (0, 1)$ and $q(4)$ has coordinates close to $(0.64767, 0.761921)$. Also, $r = 0$ and s is an algebraic number close to 0.574773

3.3. Coordinated infinitesimal rigidity

Assuming that there are no zero-length edges in a coordinated framework (G, c, p, r) , the Jacobian matrix of the system (3)–(4) is the linear system

$$\frac{[p(j) - p(i)] \cdot [p'(j) - p'(i)]}{\|p(j) - p(i)\|} = 0 \quad \text{for all } \{i, j\} \in E_0 \quad (5)$$

$$\frac{[p(j) - p(i)] \cdot [p'(j) - p'(i)]}{\|p(j) - p(i)\|} + r'(\ell) = 0 \quad \text{for all } \{i, j\} \in E_\ell, \text{ with } \ell \in [k] \quad (6)$$

so a (p', r') satisfying (5)–(6) preserves the coordinated framework's constraints to first order. Because it will be easier to work with combinatorially, we instead define an *infinitesimal motion* of a coordinated framework (G, c, p, r) to be a pair (p', r') consisting of a velocity field p' supported on p and a vector $r' \in \mathbb{R}^k$ such that

$$[p(j) - p(i)] \cdot [p'(j) - p'(i)] = 0 \quad \text{for all } \{i, j\} \in E_0 \quad (7)$$

$$[p(j) - p(i)] \cdot [p'(j) - p'(i)] + r'(\ell) = 0 \quad \text{for all } \{i, j\} \in E_\ell, \text{ with } \ell \in [k] \quad (8)$$

We justify this as follows:

Lemma 3.2. *Let (G, c, p, r) be a coordinated framework such that the endpoints of every edge are distinct. Then there is a non-zero solution to (5)–(6) if and only if there is a non-zero solution to (7)–(8).*

Proof. Suppose that (p', r') satisfies (5)–(6). Clearing the denominator in (6), we get

$$[p(j) - p(i)] \cdot [p'(j) - p'(i)] + \|p(j) - p(i)\| r'(\ell) = 0$$

for all edges $\{i, j\} \in E_\ell$. If we define the vector

$$r'' = (\|p(j) - p(i)\| r'(\ell))_{\ell=1}^k$$

it then follows that (p', r'') satisfies (7)–(8). The other direction is similar. \square

Define $\mathbb{1}(c)$ to be the $m \times k$ matrix that has as its columns the characteristic vectors of the E_ℓ . Then (7)–(8) is equivalent to

$$R(p)p' + \mathbb{1}(c)r' = 0 \tag{9}$$

where $R(p)$ is the rigidity matrix of the *underlying framework* (G, p) . Note that r does not appear in (9), so infinitesimal rigidity of (G, c, p, r) depends only on p . Thus, for analysing infinitesimal rigidity, we may assume that $r = 0$.

Since r' can be the zero vector, (9) is homogeneous. Thus, the infinitesimal motions form a vector space that contains a $\binom{d+1}{2}$ -dimensional subspace of motions $(p', \mathbf{0})$, with p' a trivial infinitesimal motion of (G, p) . This is the space of trivial infinitesimal motions of (G, c, p, r) . We define (G, p, c, r) to be *infinitesimally rigid* if these are the only infinitesimal motions, and *infinitesimally flexible* otherwise. The construction in Lemma 3.2 maps vectors with $r' = 0$ to vectors with $r'' = 0$ and vice versa, so the definition of infinitesimal rigidity by (5)–(6) is the same as the one here.

Examples of an infinitesimally flexible and an infinitesimally rigid coordinated framework with $k = 1$ and $d = 2$ are shown in Figure 2. Note that if G is generically flexible (or even isostatic) in dimension d , then (G, c, p, r) can never be infinitesimally rigid for any d -dimensional configuration p .

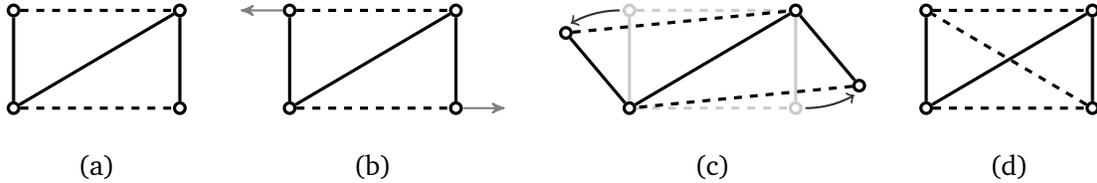


Figure 2: A coordinated framework with $k = 1$ in the plane, where the edges in E_1 are shown dashed (a). This framework has a non-trivial infinitesimal motion shown in (b) which extends to a non-trivial finite motion, as indicated in (c). Adding another bar to the framework in (a) yields an infinitesimally rigid framework, as shown in (d).

We define the space of *infinitesimal motions* $M^+(p)$ of (G, c, p, r) to be the space of solutions to (9). By rearranging, we see that $M^+(p)$ is the kernel of the $m \times (dn + k)$ matrix $R^+(p) := (R(p), \mathbb{1}(c))$, which we call the *coordinated rigidity matrix*.

Example 3.3. The framework (K_4, c, p, r) with $k = 1$ shown in Figure 1 has the following coordinated rigidity matrix:

$$R^+(p) = \begin{bmatrix} p(1) - p(2) & p(2) - p(1) & \mathbf{0} & \mathbf{0} & 0 \\ p(1) - p(3) & \mathbf{0} & p(3) - p(1) & \mathbf{0} & 0 \\ p(1) - p(4) & \mathbf{0} & \mathbf{0} & p(4) - p(1) & 0 \\ \mathbf{0} & p(2) - p(3) & p(3) - p(2) & \mathbf{0} & 0 \\ \mathbf{0} & p(2) - p(4) & \mathbf{0} & p(4) - p(2) & 1 \\ \mathbf{0} & \mathbf{0} & p(3) - p(4) & p(4) - p(3) & 1 \end{bmatrix}$$

where each $p(i)$ is considered a 2-dimensional row vector. The rank of this matrix is 6 and hence (K_4, c, p, r) is infinitesimally rigid.

Note that for any p , the row rank of this matrix is clearly at most 6. For a 2-dimensional framework on four vertices with $k = 2$ to be infinitesimally rigid, we would require a column rank of 7.

Theorem 3.4. *Let (G, c, p, r) be a d -dimensional coordinated framework. If (G, c, p, r) is infinitesimally rigid, then it is rigid. If (G, c, p, r) is generic and infinitesimally flexible, then it is flexible.*

Theorem 3.4 can be established using differential-geometric arguments along the lines of [1]. We omit the (standard) details, which can be found in [21]. The combinatorial perspective on this is:

Corollary 3.5. *For every k -coordinated graph (G, c) and dimension d , either every d -dimensional generic coordinated framework (G, c, p, r) is rigid or every d -dimensional generic coordinated framework (G, c, p, r) is flexible.*

In light of Corollary 3.5, we define (G, c) to be *generically rigid* in dimension d if there is a generic d -dimensional framework (G, c, p, r) that is rigid, and otherwise *generically flexible*.

We say that a coordinated framework (G, c, p, r) is *independent* if the coordinated rigidity matrix $R^+(p)$ has independent rows. Moreover, (G, c, p, r) is *isostatic* if it is infinitesimally rigid and independent. Similarly, (G, c) is *generically independent (isostatic)* in dimension d if there is a generic d -dimensional framework (G, c, p, r) that is independent (isostatic).

4. Generic coordinated rigidity

In this section we develop the generic theory for coordinated rigidity and show that in all dimensions d , generic coordinated rigidity can be characterized in terms of redundant rigidity in the standard d -dimensional rigidity matroid.

4.1. Main theorem

Given a family of sets, \mathcal{E} , a *transversal* of \mathcal{E} is a set containing exactly one element from each member of \mathcal{E} .

The following is our main theorem.

Theorem 4.1. *For $d \geq 1$ and $k \geq 1$, (G, c) is generically rigid in dimension d if and only if G is generically rigid in dimension d and some transversal $\{e_1, \dots, e_k\}$ of the coordination classes E_1, \dots, E_k is redundant in $M_d(G)$.*

Theorem 3.4 implies that generic coordinated rigidity is a matroidal property, and we may use Theorem 4.1 to describe the coordinated rigidity matroid. Before we give the proof of Theorem 4.1, we reformulate this result in terms of matroid unions. We need the following definitions.

Let E be a finite set and let $\mathcal{E} = \{E_1, \dots, E_k\}$ be a collection of disjoint subsets of E . The *transversal matroid* $T_E(\mathcal{E})$ on E induced by the E_i has as its bases the sets

$$\{\{e_1, \dots, e_k\} : e_i \in E_i \text{ for all } i \in [k]\}$$

See, e.g., [3, 17].

If M_1 and M_2 are two matroids on a common ground set E , then the *matroid union* $M_1 \vee M_2$ is defined as the matroid on E with the property that a subset F is independent in $M_1 \vee M_2$ if and only if it has the form $F = F_1 \cup F_2$, where F_i is independent in M_i for $i = 1, 2$.

With these definitions, we have the following matroidal formulation of Theorem 4.1.

Theorem 4.2. *Let (G, c) be a k -coordinated graph, and let $T_E(\mathcal{E})$ be the transversal matroid on E induced by the coordination classes $\mathcal{E} = \{E_1, \dots, E_k\}$. Then the d -dimensional k -coordinated rigidity matroid of (G, c) is the union $M_d(G) \vee T_E(\mathcal{E})$ of the d -dimensional rigidity matroid of G and the matroid $T_E(\mathcal{E})$.*

Proof. We show that the two matroids have the same bases. The generically isostatic k -coordinated spanning subgraphs of (G, c) form the bases of the d -dimensional coordinated rigidity matroid of (G, c) . Let (H, c') be such a graph, with $H = (V, F)$ and $c' = c|_F$. By Theorem 4.1, H must have exactly $dn - \binom{d+1}{2} + k$ edges, including a transversal $F' = \{e_1, \dots, e_k\}$ of \mathcal{E} that is redundant in $M_d(H)$. Thus $F \setminus F'$ is independent in $M_d(G)$, and hence a basis of $M_d(G)$, and $\{e_1, \dots, e_k\}$ is a basis of $T_E(\mathcal{E})$. Conversely, Theorem 4.1 implies that any k -coordinated graph arising in this way is a basis of the d -dimensional k -coordinated rigidity matroid of (G, c) .

Hence, the d -dimensional coordinated rigidity matroid has the claimed matroid union structure. \square

To prove Theorem 4.1 we need the following specialized fact from matroid theory. This can be found in [3, Proposition 7.6.14], for example.

Lemma 4.3. *Let M_1 and M_2 be two linearly representable matroids (over \mathbb{R}) on the same ground set. Then the matroid union $M_1 \vee M_2$ is also linearly representable, and a representation may be obtained by a matrix of the form (A, DB) where the rows of A represent M_1 , the rows of B represent M_2 , and D is a diagonal matrix of algebraically independent transcendentals.*

We also need some specialized lemmas regarding equilibrium stresses. Let (G, p) be a framework with at least d vertices. We say that the *tyed down* rigidity matrix $R^{i_1, \dots, i_d}(p)$ is the matrix obtained from $R(p)$ by throwing away $d - j + 1$ columns associated with $p(i_j)$. White and Whiteley [22] showed that ω is an equilibrium stress of (G, p) if and only if $\omega R^{i_1, \dots, i_d}(p) = 0$ for any choice of the i_j .

Lemma 4.4 ([22]). *Suppose that (G, p) is a generic framework so that $(G \setminus \{e\}, p)$ is isostatic (and hence e is redundant). Then (G, p) has a unique equilibrium stress ω where $\omega(e) = 1$ and for all other edges f , we have*

$$\omega(f) = (\det R_{e \rightarrow x}^{i_1, \dots, i_d}(p))^{-1} \det(R_{f \rightarrow e}^{i_1, \dots, i_d}(p))$$

where the i_j are any tie-down vertices and $R_{f \rightarrow e}^{i_1, \dots, i_d}$ is obtained by removing the row corresponding to e and then replacing the row corresponding to f with it and $R_{e \rightarrow x}^{i_1, \dots, i_d}$ by simply dropping the row corresponding to e .

Proof. White and Whiteley [22] show that ω is independent of this kind of standard tie down, and then observe that, in this case, Cramer's rule gives the claimed form, since the row corresponding to e is in the span of the others, which are a row basis. \square

Next we show that if the length of the edge e gets very small, then the stress coefficient corresponding to e becomes very large compared to all other coefficients.

Lemma 4.5. *Suppose that (G, p) is a generic framework so that $(G \setminus \{e\}, p)$ is isostatic (and hence e is redundant). Let $e = \{i_1, i_2\}$ and define p^t to be like p except $p^t(i_2) = tp(i_2) + (1-t)p(i_1)$. Let ω^t be the equilibrium stress of (G, p^t) with $\omega^t(e) = 1$. Then for all other edges f , we have for generic p ,*

$$|\omega^t(f)| \rightarrow 0$$

as $t \rightarrow 0$.

Proof. Since $(G \setminus \{e\}, p)$ is isostatic, (G, p) has a unique equilibrium stress, up to a scalar. Moreover, the stress coefficient of e is non-zero. Note that for any polynomially defined property, such as the rank of the rigidity matrix or the stress coefficients being non-zero, all but finitely many of the frameworks (G, p^t) have this property. So there exists a t_0 so that for all $t < t_0$, all frameworks (G, p^t) are isostatic and hence have a unique equilibrium stress (up to a scalar) that is non-zero on e . In the following we assume that $t < t_0$.

We fix the stress coefficient $\omega^t(e)$ to be 1 for all t , so that each (G, p^t) has a unique equilibrium stress. Consider the stress coefficients of all edges f other than e . A formula for $\omega^t(f)$ is given in Lemma 4.4. Since the lengths and directions of the edges of $G \setminus \{e\}$ only change a bounded amount as $t \rightarrow 0$, and the length of the edge e approaches zero as $t \rightarrow 0$, the determinant in the numerator in the formula in Lemma 4.4 approaches zero as $t \rightarrow 0$.

Let us now consider the determinant in the denominator in the formula in Lemma 4.4. If the limit framework $(G \setminus \{e\}, p^0)$ is isostatic, then the denominator is bounded away from zero for all t and the result follows. If $(G \setminus \{e\}, p^0)$ is not isostatic, then both the numerator and the denominator may converge to zero, but we will show that in this case the numerator approaches zero at a faster rate than the denominator, which then again gives the result.

Without loss of generality, we may assume that $p^t(i_1)$ lies at the origin and $p^t(i_2) = (t, 0, \dots, 0)$, since the frameworks (G, p^t) are then still 'quasi-generic', that is, congruent to generic frameworks. We may further assume that all the coordinates of $p^t(i_1)$ and all except the first coordinate of $p^t(i_2)$ are tied down (so the associated columns in the tied down rigidity matrices in Lemma 4.4 are missing) and the first coordinates of all vertices adjacent to i_2 are not tied down.

Suppose first that $f = \{i_2, i_3\}$ and let x_3 be the first coordinate of $p^t(i_3)$ (which is independent of t). By genericity, $x_3 \neq 0$. Let us first consider the matrix $R_{f \rightarrow e}^{i_1, \dots, i_d}(p^t)$ in the numerator. We may assume that the first row of this matrix corresponds to e and hence is the vector $(t, 0, \dots, 0)$. Let the matrix obtained from $R_{f \rightarrow e}^{i_1, \dots, i_d}(p^t)$ by removing the first row be denoted by A , and let $C(t)$ be the cofactor of the $(1, 1)$ -entry of $R_{f \rightarrow e}^{i_1, \dots, i_d}(p^t)$. Then $\det(R_{f \rightarrow e}^{i_1, \dots, i_d}(p^t)) = tC(t)$. Let us now consider the matrix $R_{e \rightarrow x}^{i_1, \dots, i_d}(p^t)$ in the denominator. Since a reordering of the rows and columns does not

change the degrees of the monomials in the determinants, we may assume that the first row corresponds to the edge f (and hence the $(1, 1)$ -entry is $(t - x_3)$) and that the matrix obtained from $R_{e \rightarrow x}^{i_1, \dots, i_d}(p^t)$ by removing the first row is again A . Then $\det(R_{e \rightarrow x}^{i_1, \dots, i_d}(p^t)) = (t - x_3)C(t) + D(t)$ for some polynomial $D(t)$. Since this polynomial contains the expression $x_3 C(t)$, where x_3 is a non-zero constant, it follows that the lowest degree of the non-zero terms in the denominator is strictly smaller than the degree of any of the non-zero terms in the numerator. Thus, $|\omega^t(f)| \rightarrow 0$ for every edge $f \neq e$ that is incident with i_2 .

Suppose next that there is an edge f of G that is not incident with i_2 and has the property that $|\omega^t(f)|$ is bounded away from zero as $t \rightarrow 0$. Then, by continuity, the limit framework $(G \setminus \{e\}, p^0)$ has an equilibrium stress ω^0 that is non-zero on f and – by the argument above – zero on all edges that are incident with i_2 . But this says that the framework obtained from (G, p^0) by removing the vertex i_2 and its incident edges has a non-zero equilibrium stress. This is a contradiction because the vertices of this framework are in quasi-generic position (since they remained fixed as $t \rightarrow 0$) and its underlying graph $G \setminus \{i_2\}$ is a subgraph of the isostatic graph $G \setminus \{e\}$.

So, as claimed, the magnitude of all stress coefficients of ω^t other than the one for e approach zero as $t \rightarrow 0$. \square

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let (G, c, p, r) be generic. Recall that for analysing infinitesimal rigidity, we may assume that $r = 0$. We first prove necessity, so we suppose that $(G, c, p, 0)$ is infinitesimally rigid. By Lemma 4.3, the matrix $(R(p), D\mathbb{1}(c))$ is a linear representation for the matroid union $M_d(G) \vee T_E(\mathcal{E})$ of the d -dimensional generic rigidity matroid of G and the transversal matroid on E induced by the coordination classes $\mathcal{E} = \{E_1, \dots, E_k\}$, where D is an $m \times m$ diagonal matrix of algebraically independent transcendentals. Since the coordinated rigidity matrix $R^+(p)$ of $(G, c, p, 0)$ has the form $(R(p), \mathbb{1}(c))$ it follows that any independent set in the d -dimensional k -coordinated rigidity matroid of (G, c) must also be independent in $M_d(G) \vee T_E(\mathcal{E})$. So the rank of $R^+(p)$ is upper-bounded by

$$\max_{E' \subset E} \left\{ \text{rank}_{M_d(G)}(E \setminus E') + \text{rank}_{T_E(\mathcal{E})}(E') \right\} \leq dn - \binom{d+1}{2} + k. \quad (10)$$

Since $(G, c, p, 0)$ is infinitesimally rigid, we have equality throughout. Let E' be a subset of E that gives the maximum $dn - \binom{d+1}{2} + k$ in (10), and let G' be the graph induced by $E \setminus E'$. Since

$$\text{rank}_{M_d(G)}(E \setminus E') = dn - \binom{d+1}{2}$$

the framework (G', p) is infinitesimally rigid. This makes the edges in E' redundant. Since $\text{rank}_{T_E(\mathcal{E})}(E') = k$, E' contains a transversal of \mathcal{E} .

For sufficiency, we suppose that there is a spanning subgraph H of G so that H is generically isostatic and $F = E \setminus E(H)$ contains a transversal of the coordination classes \mathcal{E} . Call the transversal $T = \{e_1, \dots, e_k\}$, and let G' be the graph obtained from H by adding the edges of T to H . We show that (G', c) is generically isostatic by constructing a coordinated framework $(G', c, p, 0)$ that is isostatic. The intuitive idea is that we want to find a generic position p so that, if we consider the equilibrium stresses $\omega_1, \dots, \omega_k$ of (G', p) from the statement of Lemma 2.13, the entries not in the left-hand $k \times k$ block of the $k \times m$ matrix W that has the ω_i as rows are very small. In this case $X = W\mathbb{1}(c)$ is diagonally dominant, and so full rank. The idea for doing this latter step is to make

the lengths of the edges e_1, \dots, e_k , one after the other, all very short. By Lemma 4.5, this makes the stress coefficient corresponding to ω_i on e_i relatively large for each i . The transversal structure is important, since the stresses ω_i don't need the other transversal edges at all. This leaves only the problem that, as we make a transversal edge e_i short, the stress coefficients corresponding to the equilibrium stresses $\omega_1, \dots, \omega_{i-1}$ for some non-transversal (but coordinated) edges may increase again. We get around that by using Lemma 4.5 iteratively, so that e_1 is much shorter than e_2 and so on. What follows gives the details.

We start with a generic framework (H, p) . Set $H_1 = H \cup \{e_1\}$. By genericity, e_1 is redundant in (H_1, p) . Let $e_1 = \{i_1, i_2\}$. By Lemma 4.5, for every $\varepsilon_1 > 0$, we may shrink the length $|e_1|$ of the edge e_1 by moving the vertex i_2 towards i_1 (while fixing the positions of all other vertices), until we obtain a generic configuration p^t with $|e_1| = t$ so that the unique equilibrium stress ω_1^t of (H_1, p^t) has $\omega_1^t(e_1) = 1$ and coefficients on all other edges of magnitude at most ε_1 . For each $i = 1, \dots, k$, let κ_i be the size of the i th coordination class of the graph $(H, c|_{E(H)})$. Then, since we may assume that ε_1 is sufficiently small, the first entry of $\omega_1^t \mathbb{1}(c)$ has a magnitude of at least $1 - \kappa_1 \varepsilon_1$, and the other entries of $\omega_1^t \mathbb{1}(c)$ have a respective magnitude of at most $\kappa_i \varepsilon_1$ for $i = 2, \dots, k$. So as t approaches zero, the magnitude of the first entry of $\omega_1^t \mathbb{1}(c)$ stays bounded away from 0, whereas the magnitudes of the other entries approach 0.

Now we repeat the process on the graph $H_2 = H \cup \{e_2\}$ starting from (H, p^t) . Let $i_3 \neq i_1, i_2$ be a vertex incident with e_2 . Then, again by Lemma 4.5, for every $\varepsilon_2 > 0$, we may shrink the length $|e_2|$ of the edge e_2 by moving the vertex i_3 (while fixing the positions of all other vertices) until we obtain a generic configuration $p^{t,s}$ with $|e_2| = s$ so that the unique equilibrium stress $\omega_2^{t,s}$ of $(H_2, p^{t,s})$ has $\omega_2^{t,s}(e_2) = 1$ and coefficients on all other edges of magnitude at most ε_2 . In fact, by continuity of the stress coefficients, if $|e_1| = t$ was chosen sufficiently small, then there exists a generic configuration $p^{t',s}$ with $|e_2| = s$ so that for all $t' \leq t$, the generic configuration $p^{t',s}$ with $|e_1| = t'$ has the property that the unique equilibrium stress $\omega_2^{t',s}$ of $(H_2, p^{t',s})$ has $\omega_2^{t',s}(e_2) = 1$ and coefficients on all other edges of magnitude at most ε_2 . (Note that since e_1 is not an edge of H_2 , a small perturbation of $|e_1| = t$ gives rise to a bounded perturbation of the stress coefficients of $\omega_2^{t,s}$.)

Because of the transversal structure, as s approaches zero, the magnitude of the second entry of $\omega_2^{t,s} \mathbb{1}(c)$ stays bounded away from 0, whereas the magnitudes of the other entries approach 0. Let ε_2 be chosen sufficiently small so that the magnitude of the second entry of $\omega_2^{t,s} \mathbb{1}(c)$ is larger than the sum of the magnitudes of the other entries. So the vector $\omega_2^{t,s} \mathbb{1}(c)$ has the desired structure, and, as mentioned above, we may assume that t is sufficiently small so that the vector $\omega_2^{t',s} \mathbb{1}(c)$ also has the desired structure for all $t' \leq t$.

As we carry out this second step, the equilibrium stress ω_1^t changes continuously to $\omega_1^{t,s}$, and the magnitude of the stress coefficients on the edges of H may be increasing again (while the stress coefficient on e_1 is held fixed at 1). So to keep the desired structure for the vector $\omega_1^{t,s} \mathbb{1}(c)$ we may need to choose a t' that is smaller than t .

At the end of the second step, when the configuration $p^{t,s}$ is reached, the magnitude of each of the stress coefficients of ω_1^t on the edges of H will have changed by at most some positive factor $\Delta_{t,s}$. This factor changes as t is replaced with $t' \leq t$, but by continuity of the stress coefficients, there also exists a constant $\Delta_s > 0$ such that for all $t' \leq t$ we have $|\omega_1^{t',s}(f)| \leq \Delta_s |\omega_1^{t,s}(f)|$ for all edges f of H . Since we may assume that ε_1 is sufficiently small, the first entry of the vector $\omega_1^{t,s} \mathbb{1}(c)$ has a magnitude of at least $1 - \kappa_1 \Delta_s \varepsilon_1$ at the end of the second step, and the other entries have a respective magnitude of at most $\kappa_i \Delta_s \varepsilon_1$, $i = 2, \dots, k$.

So if the vector $\omega_1^{t,s}\mathbb{1}(c)$ no longer has the desired structure, then we may choose a suitably smaller ε_1 so that for a corresponding suitably small $t' < t$ the vector $\omega_1^{t',s}\mathbb{1}(c)$ does have the desired structure, i.e., the magnitude of the first entry of the vector $\omega_1^{t',s}\mathbb{1}(c)$ is still larger than the sum of the magnitudes of the other entries at the end of the second step. So we can guarantee that both the vector $\omega_1^{t',s}\mathbb{1}(c)$ and the vector $\omega_2^{t',s}\mathbb{1}(c)$ have the desired structure for the configuration $p^{t',s}$ at the end of the second step.

Iterating this process, we see that we may choose constants $\varepsilon_1 \ll \varepsilon_2 \ll \dots \ll \varepsilon_k$ so that eventually we arrive at a configuration p^k which is generic and has X diagonally dominant.

We conclude that, generically, the matrix

$$[R(p) \quad \mathbb{1}(c)]$$

has empty co-kernel, and hence rank $dn - \binom{d+1}{2} + k$. This completes the proof of sufficiency. \square

4.2. Examples

Figure 3 shows an example of a rigid coordinated framework with two coordination classes in dimension 2. The edges e_1 and e_2 form a redundant transversal of the coordination classes E_1 and E_2 and certify generic rigidity by Theorem 4.1. The edges f_1 and f_2 are another transversal of E_1 and E_2 , but they are not redundant.

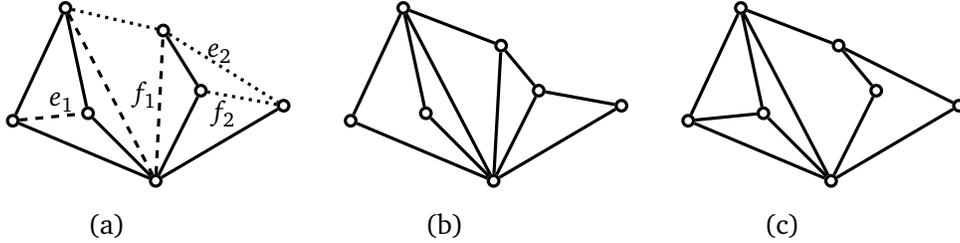


Figure 3: A coordinated framework in the plane with $k = 2$, where edges in E_1 and E_2 are indicated by dashed and dotted lines, respectively (a). Removing the redundant edge pair e_1, e_2 results in a graph that is rigid as an uncoordinated framework (b), satisfying the conditions of Theorem 4.1 for the coordinated framework in (a) to be rigid. Removing the pair f_1, f_2 , results in the flexible uncoordinated framework (c).

Figure 4 shows two examples of 2-coordinated graphs that yield flexible frameworks for generic configurations in dimension 2. In both cases, there is no transversal of the coordination classes which is redundant.

Finally, Figure 5 shows an example of a flexible coordinated framework (G, c, p, r) with three coordination classes in dimension 3 whose underlying framework (G, p) is rigid. There is only one edge in each coordination class (the edges joining p_3 with p_6, p_7 and p_8) and if we remove these three edges, we clearly have a flexible uncoordinated framework, as the line through p_4 and p_5 acts like a hinge. Thus, the three edges in $E_1 \cup E_2 \cup E_3$ are not redundant.

We note that p_3 is a cone vertex in this framework, as it is joined to all other vertices. Thus, as described in the introduction, we may think of this framework as a spherical framework, with

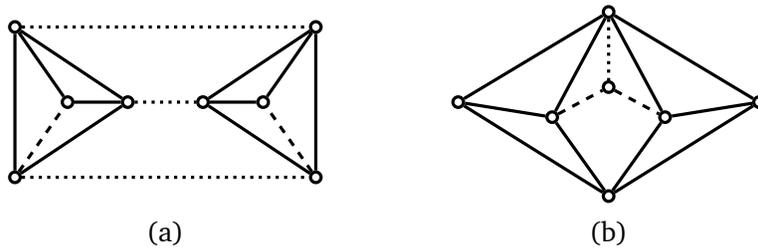


Figure 4: A pair of generically flexible 2-coordinated graphs, where edges in E_1 and E_2 are indicated by dashed and dotted lines, respectively. In the graph shown in (a) none of the edges in E_2 is redundant in $M_2(G)$. In the graph shown in (b), each of the edges in E_1 and E_2 is redundant in $M_2(G)$, but no pair of them is.

p_3 fixed at the centre of the sphere. In this model, the uncoordinated edges incident to p_3 are assumed to have unit length, and the other three edges incident to p_3 are allowed to have arbitrary length. This example was given in [16, Figure 1] to show that the sparsity counts established there are not always sufficient for the generic rigidity of such 3-dimensional spherical frameworks. However, as described above, it follows immediately from Theorem 4.1 that (G, c) is generically flexible, and hence we may deduce that the (non-generic) spherical framework must at least be infinitesimally flexible.

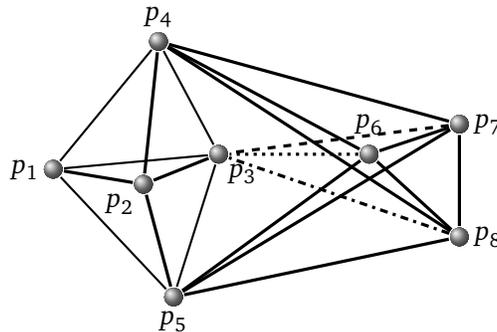


Figure 5: A coordinated framework in 3-space with $k = 3$. The edges in E_1, E_2 and E_3 are indicated by dashed, dotted and dashed-dotted lines, respectively. The framework is flexible by Theorem 4.1, since the three edges in $E_1 \cup E_2 \cup E_3$ are not redundant.

5. Closing remarks and further work

In this paper we set up a rigidity model for a type of coordinated edge motions and characterized the coordinated rigid graphs in terms of redundant rigidity in the standard rigidity matroid. To finish up, we discuss some outstanding issues and open questions.

5.1. Algorithms

Note that Theorem 4.2 implies that in dimensions $d = 1, 2$, there is a deterministic, polynomial time algorithm to check whether a k -coordinated graph (G, c) is generically rigid in dimension d . Since we have deterministic independence oracles for the matroids $M_{d,n}$ when $d = 1, 2$ [2, 12] and $T_E(\mathcal{E})$, Edmonds's algorithm [6] yields a deterministic polynomial time algorithm for $M_{d,n} \vee T_E(\mathcal{E})$ for these d and any k . Combining Edmonds's algorithm [6] and the pebble game [12] in a naive way results in a running time bound that is $\omega(n^3)$. This means that the deterministic algorithm is slower than a randomized algorithm based on checking the rank of the rigidity matrix over a finite field (see, e.g., [8]).

For $d \geq 3$, no deterministic polynomial time algorithm for checking independence in the generic rigidity matroid $M_{d,n}$ is known. Since we need a randomized algorithm as a subroutine, using Edmonds's algorithm does not reduce the complexity.

A more refined complexity analysis in dimensions $d = 1, 2$ or algorithms for finding rigid components in coordinated frameworks would be interesting.

5.2. Recursive constructions

Recursive graph construction moves (such as the Henneberg construction moves) which preserve generic rigidity play an important role in combinatorial rigidity theory (see, e.g., [15]). These moves can be used to prove key theorems such as the theorem of Laman and Pollaczek-Geiringer, to analyze graphs for generic rigidity, and to generate classes of generically rigid or isostatic graphs in all dimensions.

In the case when $d = 2$, a "Henneberg-type" characterization of generically isostatic coordinated graphs with one coordination class is given in [21]. Moreover, it is shown in [21] that the special case of Theorem 4.1 when $d = 2$ and $k = 1$ can be proved directly using these modified Henneberg moves.

A more complex Henneberg-type characterization of generic coordinated rigidity when $d = 2$ and $k = 2$ is also established in [21]. However, in contrast to the $d = 2$ and $k = 1$ case, the proof relies on Theorem 4.1. We refer the reader to [21] for further details and conjectures.

5.3. Coordination classes maintaining sums or ratios of edge lengths

It is natural to try to extend our work to coordinated frameworks in which the edges in each class have to coordinate their edge lengths changes in a different fashion, say by maintaining the sum of the edge lengths or the pairwise ratios of the edge lengths. The rigidity analysis of such coordinated frameworks seems more complex than the one considered in this paper.

For example, in dimension 2 it is well known that the standard rigidity matroid is equivalent to the parallel drawing matroid (see [24], for example), and it is easy to see that if the lengths of two edges of (G, p) are allowed to change so that the ratio of their lengths is maintained, then in the corresponding parallel drawing matroid these edges are allowed to change their direction, but their angle is maintained. The rigidity analysis of frameworks with angle constraints, however, is known to be very difficult in general.

5.4. Global coordinated rigidity

This paper deals exclusively with local rigidity of coordinated frameworks. A stronger condition than local rigidity is *global rigidity*, which would, in the coordinated setting, require that *any* (G, c, q, s) that is equivalent to a coordinated framework (G, c, p, r) be congruent to it (as opposed to just the (G, c, q, s) with (q, s) lying in a small neighborhood of (p, r)). See Figure 1 for an example of a coordinated framework in the plane which is locally rigid, but not globally rigid. Whether global rigidity is a generic property for coordinated frameworks is open.

For bar-joint frameworks, that global rigidity is a generic property is a deep result of Gortler, Healy and Thurston [8], who built on work of Connelly [4, 5] and Hendrickson [9]. It would be interesting to know how much of the theory underlying [8] carries over to the coordinated setting.

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References

- [1] L. Asimow and B. Roth. The rigidity of graphs. *Trans. Amer. Math. Soc.*, 245:279–289, 1978. doi: 10.2307/1998867.
- [2] A. R. Berg and T. Jordán. Algorithms for graph rigidity and scene analysis. In *Algorithms—ESA 2003*, volume 2832 of *Lecture Notes in Comput. Sci.*, pages 78–89. Springer, Berlin, 2003. doi: 10.1007/978-3-540-39658-1_10.
- [3] T. Brylawski. *Constructions. in Theory of Matroids*, N. White, editor, Cambridge University Press, 1986.
- [4] R. Connelly. Rigidity and energy. *Invent. Math.*, 66(1):11–33, 1982. doi: 10.1007/BF01404753.
- [5] R. Connelly. Generic global rigidity. *Discrete Comput. Geom.*, 33(4):549–563, 2005. doi: 10.1007/s00454-004-1124-4.
- [6] J. Edmonds. Minimum partition of a matroid into independent subsets. *J. Res. Nat. Bur. Standards Sect. B*, 69B:67–72, 1965. ISSN 0160-1741.
- [7] W. G. Ellenbroek, V. F. Hagh, A. Kumar, M. F. Thorpe, and M. van Hecke. Rigidity loss in disordered systems: Three scenarios. *Phys. Rev. Lett.*, 114:135501, Apr 2015. doi: 10.1103/PhysRevLett.114.135501.
- [8] S. J. Gortler, A. D. Healy, and D. P. Thurston. Characterizing generic global rigidity. *Amer. J. Math.*, 132(4):897–939, 2010. ISSN 0002-9327. doi: 10.1353/ajm.0.0132.

- [9] B. Hendrickson. Conditions for unique graph realizations. *SIAM J. Comput.*, 21(1):65–84, 1992. ISSN 0097-5397. doi: 10.1137/0221008.
- [10] D. Hexner, A. J. Liu, and S. R. Nagel. Role of local response in manipulating the elastic properties of disordered solids by bond removal. *Soft Matter*, 14:312–318, 2018. doi: 10.1039/C7SM01727H.
- [11] G. Laman. On graphs and rigidity of plane skeletal structures. *J. Engrg. Math.*, 4:331–340, 1970. doi: 10.1007/BF01534980.
- [12] A. Lee and I. Streinu. Pebble game algorithms and sparse graphs. *Discrete Math.*, 308(8): 1425–1437, 2008. doi: 10.1016/j.disc.2007.07.104.
- [13] L. Lovász and Y. Yemini. On generic rigidity in the plane. *SIAM J. Algebraic Discrete Methods*, 3(1):91–98, 1982. ISSN 0196-5212. doi: 10.1137/0603009.
- [14] J. Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.
- [15] A. Nixon and E. Ross. One brick at a time: a survey of inductive constructions in rigidity theory. In *Rigidity and symmetry*, volume 70 of *Fields Inst. Commun.*, pages 303–324. Springer, New York, 2014. doi: 10.1007/978-1-4939-0781-6_15.
- [16] A. Nixon, B. Schulze, S.-I. Tanigawa, and W. Whiteley. Rigidity of frameworks on expanding spheres. *SIAM J. Discrete Math.*, 32(4):2591–2611, 2018. ISSN 0895-4801. doi: 10.1137/17M1116088.
- [17] J. Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011. doi: 10.1093/acprof:oso/9780198566946.001.0001.
- [18] H. Pollaczek-Geiringer. Über die Gliederung Ebener Fachwerke. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 7(1): 58–72, 1927. doi: 10.1002/zamm.19270070107.
- [19] D. R. Reid, N. Pashine, J. M. Wozniak, H. M. Jaeger, A. J. Liu, S. R. Nagel, and J. J. de Pablo. Auxetic metamaterials from disordered networks. *Proceedings of the National Academy of Sciences*, 115(7):E1384–E1390, 2018. doi: 10.1073/pnas.1717442115.
- [20] B. Schulze and W. Whiteley. Rigidity and Scene Analysis. in *Handbook of Discrete and Computational Geometry*, C.D. Toth, J. O’Rourke, J.E. Goodman, editors, Third Edition, Chapman & Hall CRC, 2017.
- [21] H. Serocold. Rigidity of frameworks with coordinated constraint relaxations. Ph.D. thesis, Lancaster University, 2019.
- [22] N. L. White and W. Whiteley. The algebraic geometry of stresses in frameworks. *SIAM J. Algebraic Discrete Methods*, 4(4):481–511, 1983. doi: 10.1137/0604049.
- [23] W. Whiteley. Cones, infinity and 1-story buildings. *Structural Topology*, (8):53–70, 1983. ISSN 0226-9171. With a French translation.

- [24] W. Whiteley. Some Matroids from Discrete Applied Geometry. In *Matroid theory (Seattle, WA, 1995)*, volume 197 of *Contemp. Math.*, pages 171–311. Amer. Math. Soc., Providence, RI, 1996. doi: 10.1090/conm/197/02540.