Weak c-ideals of a Lie algebra

Zekiyé CILOGLU SAHIN¹; David Anthony TOWERS²

¹Department of Mathematics, Faculty of Arts and Sciences, Suleyman Demirel University, Isparta, Turkey.
²Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster, United Kingdom.

Received: .201  •  Accepted/Published Online: .201  •  Final Version: ..201

Abstract: A subalgebra $B$ of a Lie algebra $L$ is called a weak c-ideal of $L$ if there is a subideal $C$ of $L$ such that $L = B + C$ and $B \cap C \leq B_L$, where $B_L$ is the largest ideal of $L$ contained in $B$. This is analogous to the concept of weakly $c$-normal subgroups, which has been studied by a number of authors. We obtain some properties of weak $c$-ideals and use them to give some characterisations of solvable and supersolvable Lie algebras. We also note that one-dimensional weak $c$-ideals are $c$-ideals.

Key words: Weak $c$-ideal, Frattini ideal, Lie algebras, Nilpotent, Solvable, Supersolvable.

1. Introduction

Throughout $L$ will denote a finite-dimensional Lie algebra over a field $F$. If $B$ is a subalgebra of $L$ we define $B_L$, the core (with respect to $L$) of $B$ to be the largest ideal of $L$ contained in $B$. We say that a subalgebra $B$ of $L$ is a weak $c$-ideal of $L$ if there is a subideal $C$ of $L$ such that $L = B + C$ and $B \cap C \leq B_L$. This is a generalisation of the concept of a $c$-ideal which was studied in [9]. It is analogous to the concept of weakly $c$-normal subgroups as introduced by Zhu, Guo and Shum in [15]; this concept has since been further studied by a number of authors, including Zhong and Yang ([14]), Zhong, Yang, Ma and Lin ([13]), Tashtoush ([7]) and Jehad ([4]) who called them $c$-subnormal subgroups.

The maximal subalgebras of a Lie algebra $L$ and their relationship to the structure of $L$ have been studied extensively. It is well known that $L$ is nilpotent if and only if every maximal subalgebra of $L$ is an ideal of $L$ (see [1]). A further result is that if $L$ is solvable then every maximal subalgebra of $L$ has codimension one in $L$ if and only if $L$ is supersolvable (see [2]). In [9] similar characterisations of solvable and supersolvable Lie algebras were obtained in terms of $c$-ideals. The purpose here is to generalise these results to ones relating to weak $c$-ideals.

In section two we give some basic properties of weak $c$-ideals; in particular, it is shown that weak $c$-ideals inside the Frattini subalgebra of a Lie algebra $L$ are necessarily ideals of $L$. In section three we first show that all maximal subalgebras of $L$ are weak $c$-ideals of $L$ if and only if $L$ is solvable and that $L$ has a solvable maximal subalgebra that is a weak $c$-ideal if and only if $L$ is solvable. Unlike the corresponding results for $c$-ideals, it is necessary to restrict the underlying field to characteristic zero, as is shown by an example. Finally we have that if all maximal nilpotent subalgebras of $L$ are weak $c$-ideals, or if all Cartan subalgebras of $L$ are...
weak c-ideals and $F$ has characteristic zero, then $L$ is solvable.

In section four we show that if $L$ is a solvable Lie algebra over a general field and every maximal subalgebra of each maximal nilpotent subalgebra of $L$ is a weak c-ideal of $L$ then $L$ is supersolvable. If each of the maximal nilpotent subalgebras of $L$ has dimension at least two then the assumption of solvability can be removed. Similarly if the field has characteristic zero and $L$ is not three-dimensional simple then this restriction can be removed. In the final section we see that every one-dimensional subalgebra is a weak c-ideal if and only if it is a c-ideal.

If $A$ and $B$ are subalgebras of $L$ for which $L = A + B$ and $A \cap B = 0$ we will write $L = A \oplus B$. The ideals $L^{(k)}$ and $L^k$ are defined inductively by $L^{(1)} = L^1 = L$, $L^{(k+1)} = [L^{(k)}, L^{(k)}]$, $L^{k+1} = [L, L^k]$ for $k \geq 1$. If $A$ is a subalgebra of $L$, the centralizer of $A$ in $L$ is $C_L(A) = \{ x \in L : [x, A] = 0 \}$.

2. Preliminary Results

Definition 2.1 Let $I$ be a subalgebra of $L$. We call $I$ a subideal of $L$ if there is a chain of subalgebras

$I = I_0 < I_1 < ... < I_n = L$,

where $I_j$ is an ideal of $I_{j+1}$ for each $0 \leq j \leq n - 1$.

Definition 2.2 A subalgebra $B$ of a Lie algebra $L$ is a weak c-ideal of $L$ if there exists a subideal $C$ of $L$ such that

$L = B + C$ and $B \cap C \leq B_L$,

where $B_L$, the core of $B$, is the largest ideal of $L$ contained in $B$.

Definition 2.3 A Lie algebra $L$ is called weak c-simple if $L$ does not contain any weak c-ideals except the trivial subalgebra and $L$ itself.

Lemma 2.4 Let $L$ be a Lie algebra. Then the following statements hold:

1. Let $B$ be a subalgebra of $L$. If $B$ is a c-ideal of $L$ then $B$ is a weak c-ideal of $L$.

2. $L$ is weak c-simple if and only if $L$ is simple.

3. If $B$ is a weak c-ideal of $L$ and $K$ is a subalgebra with $B \leq K \leq L$, then $B$ is a weak c-ideal of $K$.

4. If $I$ is an ideal of $L$ and $I \leq B$, then $B$ is a weak c-ideal of $L$ if and only if $B/I$ is a weak c-ideal of $L/I$.

Proof (1) By the definition every ideal is a c-ideal and every c-ideal is a weak c-ideal so the proof is obvious.

(2) Suppose first that $L$ is simple and let $B$ be a weak c-ideal with $B \neq L$. Then

$L = B + C$ and $B \cap C \leq B_L$

where $C$ is a subideal of $L$. But, since $L$ is simple, $B_L$ must be 0. Moreover, $C \neq 0$ so $C = L$. Hence $B = 0$ and $L$ is weak c-simple.

Conversely, suppose $L$ is weak c-simple. Then, since every ideal of $L$ is a weak c-ideal, $L$ must be simple.

(3) If $B$ is a weak c-ideal of $L$ then there exists a subideal $C$ of $L$ such that

$L = B + C$ and $B \cap C \leq B_L$
Then $K = K \cap L = K \cap (B + C) = B + (K \cap C)$. Since $C$ is a subideal of $L$ there exists a chain of subalgebras

$$C = C_0 < C_1 < \ldots < C_n = L$$

where $C_j$ is an ideal of $C_{j+1}$ for each $0 \leq j \leq n - 1$. If we intersect this chain with $K$ we get

$$C \cap K = C_0 \cap K < C_1 \cap K < \ldots < C_n \cap K = L \cap K = K$$

and obviously $C_j \cap K$ is an ideal of $C_{j+1} \cap K$ for each $0 \leq j \leq n - 1$. Hence $C \cap K$ is a subideal of $K$. Also,

$$B \cap (C \cap K) \leq B_K$$

so that $B$ is a weak c-ideal of $L$.

(4) Suppose first that $B/I$ is a weak c-ideal of $L/I$. Then there exists a subideal $C/I$ of $L/I$ such that

$$L/I = B/I + C/I \text{ and } B/I \cap C/I \leq (B/I)_{L/I} = B_L/I$$

It follows that $L = B + C$ and $B \cap C \leq B_L$ where $C$ is a subideal of $L$.

Suppose conversely that $I$ is an ideal of $L$ with $I \leq B$ and $B$ is a weak c-ideal of $L$. Then there exists a $C$ subideal of $L$ such that

$$L = B + C \text{ and } B \cap C \leq B_L.$$

Since $I$ is an ideal and $I \leq B$ the factor algebra

$$L/I = (B + C)/I = B/I + (C + I)/I$$

where $(C + I)/I$ is a subideal of $L/I$ and

$$(B/I) \cap (C + I)/I = (B \cap (C + I))/I = (I + B \cap C)/I \leq B_L/I = (B/I)_{L/I}$$

so $B/I$ is a weak c-ideal of $L/I$. \hfill \Box

The Frattini subalgebra of $L$, $F(L)$, is the intersection of all of the maximal subalgebras of $L$. The Frattini ideal, $\varphi(L)$, of $L$ is $F(L)_L$. The next result is a generalisation of [9, Proposition 2.2]. The same proof works but we will include it for completeness.

**Proposition 2.5** Let $B, C$ be subalgebras of $L$ with $B \leq F(C)$. If $B$ is a weak c-ideal of $L$ then $B$ is an ideal of $L$ and $B \leq \varphi(L)$.

**Proof** Suppose that $L = B + K$ where $K$ is a subideal of $L$ and $B \cap K \leq B_L$. Then $C = C \cap L = C \cap (B + K) = B + C \cap K = C \cap K$ since $B \leq F(C)$. Hence $B \leq C \leq K$, giving $B = B \cap K \leq B_L$ and $B$ is an ideal of $L$. It then follows from [8, Lemma 4.1] that $B \leq \varphi(L)$. \hfill \Box

An ideal $A$ is complemented in $L$ if there is a subalgebra $U$ of $L$ such that $L = A + U$ and $A \cap U = 0$.

We adapt this to define a complemented weak c-ideal as follows.

**Definition 2.6** Let $L$ be a Lie algebra and $B$ is a weak c-ideal of $L$. A weak c-ideal $B$ is complemented in $L$ if there is a subideal $C$ of $L$ such that $L = B + C$ and $B \cap C = 0$.

Then we can give the following lemma:
Lemma 2.7 If $B$ is a weak c-ideal of a Lie algebra $L$, then $B/B_L$ has a subideal complement in $L/B_L$, i.e., there exists a subideal subalgebra $C/B_L$ of $L/B_L$ such that $L/B_L$ is semidirect sum of $C/B_L$ and $B/B_L$. Conversely, if $B$ is a subalgebra of $L$ such that $B/B_L$ has a subideal complement in $L/B_L$ then $B$ is a weak c-ideal of $L$.

Proof Let $B$ be a weak c-ideal of $L$. Then there exists a subideal $C$ of $L$ such that $B + C = L$ and $B \cap C \leq B_L$. If $B_L = 0$ then $B \cap C = 0$ and so that $C$ is a subideal complement of $B$ in $L$. Assume that $B_L \neq 0$, then we can construct the factor algebras $B/B_L$ and $(C + B_L)/B_L$. If we intersect these two factor algebras we have

\[
\frac{B}{B_L} \cap \frac{C + B_L}{B_L} = \frac{B \cap (C + B_L)}{B_L} = \frac{B_L + (B \cap C)}{B_L} = \frac{B_L}{B_L} = 0
\]

Hence, $(C + B_L)/B_L$ is a subideal complement of $B/B_L$ in $L/B_L$. Conversely, if $K$ is a subideal of $L$ such that $K/B_L$ is a subideal complement of $B/B_L$ in $L/B_L$ then we have that

\[
L/B_L = (B/B_L) + (K/B_L) \text{ and } (B/B_L) \cap (K/B_L) = 0
\]

Then $L = B + K$ and $B \cap K \leq B_L$. Therefore $B$ is a weak c-ideal of $L$. \qed

3. Some characterisations of soluble algebras

We will use the following Lemma which is due to Stewart [6, Lemma 4.2.5]

Lemma 3.1 Let $L$ be a Lie algebra over any field having two subideals $H$ and $K$ such that $K$ is simple and not abelian. Suppose that $H \cap K = 0$. Then $[H, K] = 0$.

Theorem 3.2 Let $L$ be a Lie-algebra over a field $F$ of characteristic zero and let $B$ be an ideal of $L$. Then $B$ is soluble if and only if every maximal subalgebra of $L$ not containing $B$ is a weak c-ideal of $L$.

Proof Suppose every maximal subalgebra of $L$ not containing $B$ is a weak c-ideal of $L$. Then we need to show $B$ is soluble. Assume that this is false and let $L$ be a minimal counter-example. Let $A$ be a minimal ideal of $L$ and assume that $M/A$ is a maximal subalgebra of $L/A$ such that $(B + A)/A \not\subseteq M/A$. Then $M$ is a maximal subalgebra of $L$ with $B \not\subseteq M$, so $M$ is a weak c-ideal of $L$. It follows that $M/A$ is a weak c-ideal of $L/A$, and hence that $(B + A)/A$ is solvable. If $B \cap A = 0$, then $B \cong B/B \cap A \cong (B + A)/A$ is solvable. So we can assume that every minimal ideal of $L$ is contained in $B$. Moreover, $B/A$ is soluble for each such minimal ideal. If $L$ has two distinct minimal ideals $A_1$ and $A_2$ then $B \cong B/A_1 \cap A_2$ is solvable, so $L$ is monolithic with monolith $A$, say.

If $A$ is abelian then $B$ is soluble, so we must have that $A$ is simple. Clearly, $B \not\subseteq \varphi(L)$, since $\varphi(L)$ is nilpotent, so there is a maximal subalgebra $M$ of $L$ such that $B \not\subseteq M$. Then $M$ must be a weak c-ideal of $L$, so there is a subideal $C$ of $L$ such that $L = M + C$ and $M \cap C \subseteq M_L$. Since $B \not\subseteq M_L$ we have that $M_L = 0$. 

1
It follows that $L$ is primitive of type 2 and hence that $C_L(A) = 0$, by [10, Theorem 1.1]. But $[C, A] = 0$ by Lemma 3.1, so $C = 0$, a contradiction. Hence $B$ is solvable. So suppose now that $B$ is solvable and let $M$ be a maximal ideal of $L$ not containing $B$. Then there exists $k \in \mathbb{N}$ such that $B^{(k+1)} \subseteq M$, but $B^{(k)} \not\subseteq M$.

Clearly $L = M + B^{(k)}$ and $B^{(k)} \cap M$ is an ideal of $L$, so $B^{(k)} \cap M \subseteq M_L$. It follows that $M$ is a $c$-ideal and hence a weak $c$-ideal of $L$.

**Corollary 3.3** Let $L$ be a Lie algebra over a field $F$ of characteristic zero. Then $L$ is solvable if and only if every maximal subalgebra of $L$ is a weak $c$-ideal of $L$.

Unlike the corresponding results for $c$-ideals, the above two results do not hold in characteristic $p > 0$, as the following example shows.

**Example 3.4** Let $L = sl(2) \otimes O_1 + 1 \otimes F(\frac{\partial}{\partial x} + x \frac{\partial}{\partial y})$, where $O_1 = F[x]$ with $x^p = 0$ is the truncated polynomial algebra in 1 indeterminate and the ground field, $F$, is algebraically closed of characteristic $p > 2$.

Then $A = sl(2) \otimes O_1$ is the unique minimal ideal of $L$. Put $S = sl(2) = Fu_{-1} + Fu_0 + Fu_1$ with $[u_{-1}, u_0] = u_{-1}$, $[u_{-1}, u_1] = u_0$, $[u_0, u_1] = u_1$ and let $M = (Fu_0 + Fu_1) \otimes O_1 + 1 \otimes F(\frac{\partial}{\partial x} + x \frac{\partial}{\partial y})$. This is a maximal subalgebra of $L$ which doesn’t contain $A$. Suppose that it is a weak $c$-ideal of $L$. Then there is a subideal $C$ of $L$ such that $L = C + M$ and $C \cap M \subseteq M_L = 0$.

Let $C = C_0 < C_1 < \ldots < C_n = L$

where $C_j$ is an ideal of $C_{j+1}$ for each $0 \leq j \leq n - 1$. Then $A \subseteq C_{n-1}$, so $A = C_{n-1}$ or $C_{n-1} = A + 1 \otimes F \frac{\partial}{\partial x}$.

In the latter case it is straightforward to check that $C_{n-2} \subseteq A$. In either case, $C$ must be inside a proper ideal of $A$, and hence inside $S \oplus O_1^+$, where $O_1^+$ is spanned by $x, x^2, \ldots, x^{p-1}$. But now $u_{-1} \otimes 1 \not\subseteq C + M$. Hence $M$ is not a weak $c$-ideal of $L$.

**Lemma 3.5** Let $L = U + C$ be a Lie algebra, where $U$ is a solvable subalgebra of $L$ and $C$ is a subideal of $L$. Then there exists $n_0 \in \mathbb{N}$ such that $L^{(n_0)} \subseteq C$.

**Proof** Let $C = C_0 < C_1 < \ldots < C_k = L$ where $C_i$ is an ideal of $C_{i+1}$ for $0 \leq i \leq k - 1$. Then $L/C_{k-1}$ is solvable and so there exists $n_{k-1}$ such that $L^{(n_{k-1})} \subseteq C_{k-1}$. Suppose that $L^{(n_i)} \subseteq C_i$ for some $0 \leq i \leq k - 1$.

Now $C_i/C_{i-1}$ is solvable, and so there is $r_i$ such that $C_i^{(r_i)} \subseteq C_{i-1}$. Hence $L^{(n_i + r_i)} = (L^{(n_i)})^{(r_i)} \subseteq C_{i-1}$. Put $n_{i-1} = n_i + r_i$. The result now follows by induction.

**Theorem 3.6** Let $L$ be a Lie algebra over a field $F$ of characteristic zero. Then $L$ has a solvable maximal subalgebra that is a weak $c$-ideal of $L$ if and only if $L$ is solvable.

**Proof** Suppose first that $L$ has a solvable maximal subalgebra $M$ that is a weak $c$-ideal of $L$. We show that $L$ is solvable. Let $L$ be a minimal counter-example. Then there is a subideal $K$ of $L$ such that $L = M + K$ and $M \cap K \subseteq M_L$. If $M_L \neq 0$ then $L/M_L$ is solvable, by the minimality assumption, and $M_L$ is solvable, whence $L$ is solvable, a contradiction. It follows that $M_L = 0$ and $L = M + K$. If $R$ is the solvable radical of $L$ then $R \subseteq M_L = 0$, so $L$ is semisimple. But now, for all $n \geq 1$, $L = L^{(n)} \subseteq K \neq L$, by Lemma 3.5, a contradiction. The result follows. The converse follows from Corollary 3.3.
Theorem 3.7 Let \( L \) be a Lie algebra over a field of characteristic zero such that all maximal nilpotent subalgebras are weak \( c \)-ideals of \( L \). Then \( L \) is solvable.

Proof Suppose that \( L \) is not solvable but that all maximal nilpotent subalgebras of \( L \) are weak \( c \)-ideals of \( L \). Let \( L = R \oplus S \) be the Levi decomposition of \( L \), where \( S \neq 0 \). Let \( B \) be a maximal nilpotent subalgebra of \( S \) and \( U \) be a maximal nilpotent subalgebra of \( L \) containing it. Then there is a subideal \( C \) of \( L \) such that \( L = U + C \) and \( U \cap C \subseteq U_L \). It follows from Lemma 3.5 that \( S = S^{(n_0)} \subseteq L^{(n_0)} \subseteq C \), and so \( B \subseteq U \cap C \subseteq U_L \), whence \( S \cap U_L \neq 0 \). But \( S \cap U_L \) is an ideal of \( S \) and so is semisimple. Since \( U \) is nilpotent this is a contradiction. \( \square \)

Theorem 3.8 Let \( L \) be a Lie algebra, over a field \( F \) of characteristic zero, in which every Cartan subalgebra of \( L \) is a weak \( c \)-ideal of \( L \). Then \( L \) is solvable.

Proof Suppose that every Cartan subalgebra of \( L \) is a weak \( c \)-ideal of \( L \), and that \( L \) has a non-zero Levi factor \( S \). Let \( H \) be a Cartan subalgebra of \( S \) and let \( B \) be a Cartan subalgebra of its centralizer in the solvable radical of \( L \). Then \( C = H + B \) is a Cartan subalgebra of \( L \) (see [3]) and there is a subideal \( K \) of \( L \) such that \( \bar{L} = \bar{C} + \bar{K} \) and \( \bar{C} \cap \bar{K} \leq \bar{C}_L \). Now there is an \( r \geq 2 \) such that \( L^{(r)} \leq K \), by Lemma 3.5. But \( S \leq L^{(r)} \leq K \), so \( C \cap S \leq C \cap K \leq C_L \) giving \( C \cap S \leq C_L \cap S = 0 \), a contradiction. It follows that \( S = 0 \) and hence that \( L \) is solvable. \( \square \)

4. Some characterisations of supersolvable algebras

The following is proved in [9, Lemma 4.1]

Lemma 4.1 Let \( L \) be a Lie algebra over any field \( F \), let \( A \) be an ideal of \( L \) and let \( U/A \) be a maximal nilpotent subalgebra of \( L/A \). Then \( U = C + A \), where \( C \) is a maximal nilpotent subalgebra of \( L \).

We will also need the following result.

Lemma 4.2 Let \( L \) be a Lie algebra over any field \( F \) and suppose that \( L = B + K \), where \( B \) is a nilpotent subalgebra and \( K \) is a subideal of \( L \). Then there exists \( s \in \mathbb{N} \) such that \( L^s \subseteq K \). Moreover, if \( A \) is a minimal ideal of \( L \) then either \( A \subseteq K \) or \( [L, A] = 0 \).

Proof Since \( K \) is a subideal of \( L \), there exists \( r \in \mathbb{N} \) such that \( L \) \((\text{ad } K)^r \subseteq K \). As \( B \) is nilpotent, there exists \( s \in \mathbb{N} \) such that \( L^s = (B + K)^s \subseteq K \). Now \([L, A] = A \) or \([L, A] = 0 \) and the former implies that \( A \subseteq L^s \subseteq K \). \( \square \)

Lemma 4.3 Let \( L \) be a Lie algebra, over any field \( F \), in which every maximal subalgebra of each maximal nilpotent subalgebra of \( L \) is a weak \( c \)-ideal of \( L \), and let \( A \) be a minimal abelian ideal of \( L \). Then every maximal subalgebra of each maximal nilpotent subalgebra of \( L/A \) is a weak \( c \)-ideal of \( L/A \).

Proof Suppose that \( U/A \) is a maximal nilpotent subalgebra of \( L/A \). Then \( U = C + A \) where \( C \) is a maximal nilpotent subalgebra of \( L \) by Lemma 4.1. Let \( B/A \) be a maximal subalgebra of \( U/A \). Then \( B = B \cap (C + A) = B \cap C + A = D + A \) where \( D \) is a maximal subalgebra of \( C \) with \( B \cap C \leq D \). Now \( D \) is a weak \( c \)-ideal of \( L \) so there is a subideal \( K \) of \( L \) with \( L = D + K \) and \( D \cap K \leq D_L \).
If \( A \leq K \) we have
\[
\frac{L}{A} = \frac{D + K}{A} = \frac{D + A}{A} + \frac{K}{A} = \frac{B}{A} + \frac{K}{A},
\]
and
\[
\frac{B \cap K}{A} = \frac{B \cap K}{A} = \frac{(D + A) \cap K}{A} = \frac{D \cap K + A}{A} \leq \frac{D_L + A}{A} \leq \left( \frac{B}{A} \right)_{L/A}.
\]

So suppose that \( A \not\leq K \). Then Lemma 4.2 shows that \([L, A] = 0\). It follows that \( A \leq C \) and \( B = D \). We have \( L = B + K \) and \( B \cap K \leq B_L \), so
\[
\frac{L}{A} = \frac{B}{A} + \frac{K + A}{A}
\]
and
\[
\frac{B \cap (K + A)}{A} = \frac{B \cap (K + A)}{A} = \frac{B \cap K + A}{A} \leq \frac{B_L + A}{A} \leq \left( \frac{B}{A} \right)_{L/A}.
\]

\( \square \)

**Lemma 4.4** Let \( L \) be a Lie algebra over any field \( F \), in which every maximal nilpotent subalgebra of \( L \) is a weak c-ideal of \( L \), and suppose that \( A \) is a minimal abelian ideal of \( L \) and \( M \) is a core-free maximal subalgebra of \( L \). Then \( A \) is one dimensional.

**Proof** We have that \( L = A + M \) and \( A \) is the unique minimal ideal of \( L \), by [10, Theorem 1.1]. Let \( C \) be a maximal nilpotent subalgebra of \( L \) with \( A \leq C \). If \( A = C \), choose \( B \) to be a maximal subalgebra of \( A \), so that \( A = B + Fa \) and \( B_L = 0 \). Then \( B \) is a weak c-ideal of \( L \). So there is a subideal of \( K \) of \( L \) with \( L = B + K \) and \( B \cap K \leq B_L = 0 \). Now \( L = B + K = B + K^L = K^L \), since \( B \leq A \leq K^L \). It follows that \( K = L \), whence \( B = 0 \) and \( A = Fa \) is one dimensional.

So suppose that \( C \neq A \). Then \( C = A + M \cap C \). Let \( B \) be a maximal subalgebra of \( C \) containing \( M \cap C \). Then \( B \) is a weak c-ideal of \( L \), so there is a subideal \( K \) of \( L \) with \( L = B + K \) and \( B \cap K \leq B_L \). If \( A \leq B_L \leq B \), we have \( C = A + M \cap C \leq B \), a contradiction. Hence \( B_L = 0 \) and \( L = B + K \). Now \( C = B + C \cap K \) and \( B \cap C \cap K = B \cap K = 0 \). As \( C \) is nilpotent this means that \( \dim(C \cap K) = 1 \). If \( A \subseteq K \) we have that \( A \leq C \cap K \), so \( \dim A = 1 \), as required. Otherwise, \([L, A] = 0\), by Lemma 4.2 and again \( \dim A = 1 \).

We can now prove our main result.

**Theorem 4.5** Let \( L \) be a solvable Lie algebra over any field \( F \) in which every maximal subalgebra of each maximal nilpotent subalgebra of \( L \) is a weak c-ideal of \( L \). Then \( L \) is supersolvable.

**Proof** Let \( L \) be a minimal counter-example and let \( A \) be a minimal abelian ideal of \( L \). Then \( L/A \) satisfies the same hypothesis by Lemma 4.3 We thus have that \( L/A \) is supersolvable and it remains to show that \( \dim A = 1 \).

If there is another minimal ideal \( I \) of \( L \), then
\[
A \cong (A + I)/I \leq L/I
\]
which is supersolvable and so \( \dim A = 1 \). So we can assume that \( A \) is the unique minimal ideal of \( L \). Also, if \( A \leq \varphi(L) \), we have that \( L/\varphi(L) \) is supersolvable, whence \( L \) is supersolvable by [2, Theorem 7]. We therefore, further assume that \( A \not\leq \varphi(L) \). It follows that \( L = A + M \), where \( M \) is a core-free maximal subalgebra of \( L \). The result now follows from Lemma 4.4. \( \square \)
If $L$ has no one-dimensional maximal nilpotent subalgebras, we can remove the solvability assumption from the above result provided that $F$ has characteristic zero.

**Corollary 4.6** Let $L$ be a Lie algebra over a field $F$ of characteristic zero in which every maximal nilpotent subalgebra has dimension at least two. If every maximal subalgebra of each maximal nilpotent subalgebra of $L$ is a weak $c$-ideal of $L$, then $L$ is supersolvable.

**Proof** Let $N$ be the nilradical of $L$, and let $x \notin N$. Then $x \in C$ for some maximal nilpotent subalgebra $C$ of $L$. Since $\dim C > 1$, there is a maximal subalgebra $B$ of $C$ with $x \in B$. Then there is a subideal $K$ of $L$ such that $L = B + K$ and $B \cap K \subseteq B_L \leq C_L \leq N$. Clearly, $x \notin K$, since otherwise $x \in B \cap K \leq N$. Moreover, $L' \subseteq K$ for some $r \in \mathbb{N}$, by Lemma 4.2. We have shown that if $x \notin N$ there is a subideal $K$ of $L$ with $x \notin K$ and $L' \subseteq K$.

Suppose that $L$ is not solvable. Then there is a semisimple Levi factor $S$ of $L$. Choose $x \in S$. Then $x \in S = S' \subseteq K$, a contradiction. Thus $L$ is solvable and the result follows from Theorem 4.5. □

If $L$ has a one-dimensional maximal nilpotent subalgebra, then we can also remove the solvability assumption from Theorem 4.4, provided that underlying field $F$ has again characteristic zero and $L$ is not three-dimensional simple.

**Corollary 4.7** Let $L$ be a Lie algebra over a field $F$ of characteristic zero. If every maximal nilpotent subalgebra of each maximal nilpotent subalgebra of $L$ is a weak $c$-ideal of $L$, then $L$ is supersolvable or three dimensional simple.

**Proof** If every maximal nilpotent subalgebra of $L$ has dimension at least two, then $L$ is supersolvable by Corollary 4.6. So we need only consider the case where $L$ has a one-dimensional maximal nilpotent subalgebra, say $Fx$. Suppose first that $L$ is semisimple, so $L = S_1 \oplus \ldots \oplus S_n$, where $S_i$ is a simple ideal of $L$ for $1 \leq i \leq n$.

Let $n > 1$. If $x \in S_i$, then choosing $s \in S_j$ with $j \neq i$, we have that $Fx + Fs$ is a two dimensional abelian subalgebra, which contradicts the maximality of $Fx$. If $x \notin S_i$ for every $1 \leq i \leq n$, then $x$ has nonzero projections in at least two of the $S_k$’s, say $s_i \in S_i$ and $s_j \in S_j$. But then $Fx + Fs_i$ is a two-dimensional abelian subalgebra, a contradiction again. It follows that $L$ is simple. But then $Fx$ is a Cartan subalgebra of $L$, which yields that $L$ has rank one and thus is three dimensional.

So now let $L$ be a minimal-counter example. We have seen that $L$ is not semisimple, so it has a minimal abelian ideal $A$. By Lemma 4.3, $L/A$ is supersolvable or three-dimensional simple. In the former case, $L$ is solvable and so is supersolvable, by Theorem 4.5.

In the latter case, $L = A \oplus S$ where $S$ is three-dimensional simple, and so a core-free maximal subalgebra of $L$. It follows from Lemma 4.4 that $\dim A = 1$. But now $C_L(A) = A$ or $L$. In the former case $S \cong L/A = L/C_L(A) \cong Inn(A)$, a subalgebra of $Der(A)$, which is impossible. Hence $L = A \oplus S$, where $A$ and $S$ are both ideals of $L$ and again $L$ has no one-dimensional maximal nilpotent subalgebras. □

5. One dimensional weak $c$-ideals

**Lemma 5.1** Let $L$ be a Lie algebra over any field $F$. Then the one-dimensional subalgebra $Fx$ of $L$ is a weak $c$-ideal of $L$ if and only if it is a $c$-ideal of $L$.

**Proof** Let $Fx$ be a weak $c$-ideal of $L$. Then there is a subideal $K$ of $L$ such that $L = Fx + K$ and $Fx \cap K \leq (Fx)_L$. Since either $K = L$ or $K$ has codimension one in $L$, it is an ideal of $L$ and $Fx$ is a $c$-ideal
We say that $L$ is almost abelian if $L = L^2 \oplus Fx$, where $L^2$ is abelian and $[x, y] = y$ for all $y \in L^2$. Then
the following result follows from Lemma 5.1 and [9, Theorem 5.2].

**Theorem 5.2** Let $L$ be a Lie algebra over any field $F$. Then all one-dimensional subalgebras of $L$ are weak c-ideals of $L$ if and only if:

(i) $L^3 = 0$; or

(ii) $L = A \oplus B$, where $A$ is an abelian ideal of $L$ and $B$ is an almost abelian ideal of $L$.

**Acknowledgment**

The authors would like to thank the referees for their valuable comments.

**References**


