Scene analysis with symmetry

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1 Introduction

Given an incidence structure S and a straight line drawing of S in the plane, one may ask whether this drawing is the vertical projection of a spatial polyhedral scene. This is a well studied question in Discrete Geometry which has some beautiful connections to areas such as Geometric Rigidity Theory and Polytope Theory, see [5] for details. Moreover, this problem has important applications in Artificial Intelligence, Computer Vision and Robotics. In this paper we consider *symmetric* drawings and their vertical lifting properties.

1.1 Basic definitions and results

A *(polyhedral) incidence structure* S is an abstract set of vertices V, an abstract set of faces F, and a set of incidences $I \subseteq V \times F$.

A $(d-1)$-*picture* is an incidence structure S together with a corresponding location map $r : V \to \mathbb{R}^{d-1}$, and is denoted by $S(r)$. A *d-scene* $S(p, P)$ is an incidence structure $S = (V, F; I)$ together with a pair of location maps, $p : V \to \mathbb{R}^d$, and $P : F \to \mathbb{R}^d$, such that for each face F_j the vertices incident with F_j lie in a hyperplane. (Here P is an assignment of normal vectors to the faces.) A *lifting* of a $(d-1)$-picture $S(r)$ is a d-scene $S(p, P)$, with the vertical projection $\Pi(p) = r$.

A lifting $S(p, P)$ is *trivial* if all the faces lie in the same hyperplane. Further, $S(p, P)$ is *folded* (or *non-trivial*) if some pair of faces lie in different hyperplanes, and is *sharp* if each pair of faces sharing a vertex lie in distinct hyperplanes. A picture is called *sharp* if it has a sharp lifting. Moreover, a picture which has no non-trivial lifting is called *flat* (or *trivial*). A picture with a non-trivial lifting is called *foldable*.

Theorem 1 (Picture Theorem) *[4],[5] A generic $(d-1)$-picture of an incidence structure $S = (V, F; I)$ with at least two faces has a sharp lifting, unique up to lifting equivalence,* *if and only if* $|I| = |V| + d|F| - (d+1)$ *and* $|I'| \leq |V'| + d|F'| - (d+1)$ *for all subsets* I' *of incidences with at least two faces.*

The lifting matrix of a generic $(d-1)$*-picture* S *has independent rows if and only if for all non-empty subsets* I' *of incidences, we have* $|I'| \leq |V'| + d|F'| - d$.

1.2 Symmetric incidence structures and pictures

An *automorphism* of an incidence structure $S = (V, F; I)$ is a pair $\alpha = (\pi, \sigma)$, where π is a permutation of V and σ is a permutation of F such that $(v, f) \in I$ if and only if $(\pi(v), \sigma(f)) \in I$ for all $v \in V$ and $f \in F$. For simplicity, we will write $\alpha(v)$ for $\pi(v)$ and $\alpha(f)$ for $\sigma(f)$.

The automorphisms of S form a group under composition, denoted $\mathrm{Aut}(S)$. An *action* of a group Γ on S is a group homomorphism $\theta : \Gamma \to \mathrm{Aut}(S)$. The incidence structure S is called Γ-*symmetric* (with respect to θ) if there is such an action.

Let Γ be an abstract group, and let S be a Γ-symmetric incidence structure (with respect to θ). Further, suppose there exists a group representation $\tau : \Gamma \to O(\mathbb{R}^{d-1})$. Then we say that a picture $S(r)$ is Γ-*symmetric* (with respect to θ and τ) if

$$\tau(\gamma)(r_i) = r_{\theta(\gamma)(i)} \text{ for all } i \in V \text{ and all } \gamma \in \Gamma. \qquad (1)$$

In this case we also say that $\tau(\Gamma) = \{\tau(\gamma) | \, \gamma \in \Gamma\}$ is a *symmetry group* of $S(r)$.

A symmetric picture is called $\tau(\Gamma)$-generic if the vertex positions are "as generic as possible", that is, the only correspondence among the coordinates of the vertices is implied by the symmetry group $\tau(\Gamma)$.

2 Liftings with incidental symmetry

Now we summarise results regarding the effect of symmetry on the lifting properties of $(d-1)$-pictures. It was proven in [1] that the number of vertices, faces and incidences fixed by the elements of Γ play a key role in the foldability of symmetric pictures. For every symmetry group of the plane a necessary condition for minimal flatness was given.

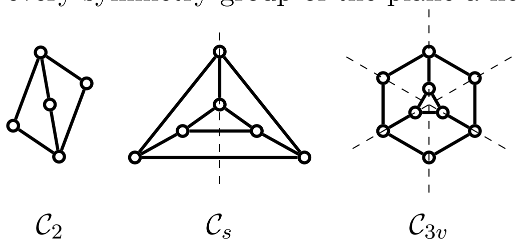

Figure 1: Some symmetric 2-pictures with a (sharp) symmetry-induced lifting with 2-fold rotational, reflectional and dihedral symmetry (where all interior regions are faces). All of these structures are flat in a generic non-symmetric position.

In the next two results $\mathcal{C}_3$ is the 3-fold rotational group and V_3 and I_3 denote the set of vertices and incidences fixed by the 3-fold rotation, see [1] for a detailed definition.

Theorem 2 *[2] A* $\mathcal{C}_3$*-symmetric incidence structure* $S = (V, F; I)$ *is* $\mathcal{C}_3$*-generically minimally flat if and only if* $|I| = |V| + 3|F| - 3$, $|I'| \leq |V'| + 3|F'| - 3$ *for every subset of incidences* $|I'|$ *with at least one face and* $|I_3(S)| = |V_3(S)|$.

Theorem 3 *[2] Let* $S = (V, F, I)$ *be a* $\mathcal{C}_3$*-symmetric incidence structure with* $|I'| \leq |V'| + 3|F'| - 4$ *for every substructure of* S *with at least two faces.*

1. *If* $|V_3(S)| = 0$ *then* S *is* $\mathcal{C}_3$*-generically sharp.*

2. *If* $|V_3(S)| = |I_3(S)| = 1$ *and* $|I'| \leq |V'| + 3|F'| - 6$ *holds for every* $\mathcal{C}_3$*-symmetric substructure of* S *with at least two faces, then* S *is* $\mathcal{C}_3$*-generically sharp.*

3 Liftings with forced symmetry

In this section we consider the case where the resulting d-scene is required to "extend" the symmetry into a higher dimension.

We first give an example of a symmetric $(d-1)$-picture that is foldable, but none of its folded liftings "extends" the symmetry of the $(d-1)$-picture. Consider the 2-picture in Figure 2. Using Theorem 1 it is easy to see that this 2-picture has a non-trivial lifting as it does not have enough incidences to be flat since $|I| = |V| + 3|F| - 4 = 16$. On the other hand consider a lifting of the same 2-picture which admits a 4-fold rotational symmetry around the z-axis. Such a symmetry forces the vertices belonging to the same vertex orbit to lie in a plane orthogonal to the z-axis. But then the constraints corresponding to the faces force every vertex to lie in the same plane, so the 3-scene must be flat.

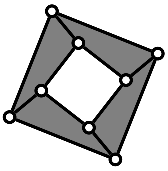

Figure 2: A 2-picture with 4-fold rotational symmetry around the origin that has a non-trivial lifting but has no non-trivial symmetric lifting which admits 4-fold rotational symmetry around the z axis. The 2-scene consists of 8 vertices which belong to two vertex orbits and four faces (shown is gray colour) which belong to the same face orbit.

3.1 Formal definitions

Let $S(r)$ be a Γ-symmetric $(d-1)$-picture with symmetry group $\tau(Gamma)$ and let $\tau' : \Gamma \to O(\mathbb{R}^d)$ be a representation of Γ so that:

1. the hyperplane of $S(r)$ is invariant under $\tau'(\Gamma)$;

2. the restriction of $\tau'(\Gamma)$ to the hyperplane of $S(r)$ is $\tau(\Gamma)$.

We say that $S(r)$ is $\tau'(\Gamma)$*-symmetry-forced flat* if it has no non-trivial $\tau'(\Gamma)$-symmetric liftings. Otherwise it is $\tau'(\Gamma)$*-symmetry-forced foldable.* If it has a $\tau'(\Gamma)$-symmetric sharp lifting then it is $\tau'(\Gamma)$*-symmetry-forced sharpe.*

In order to state our results we also need to define a quotient incidence structure. We choose a set of representatives $\mathcal{O}_V = \{v_1, \dots, v_n\}$, one for each vertex orbit. Similarly, let $\mathcal{O}_F = \{f_1, \dots, f_m\}$ and $\mathcal{O}_I = \{i_1, \dots, i_k\}$ be the sets of representatives of F and I, respectively. If $i_l = (\gamma_1 v_i, \gamma_2 f_j) \in I$ where $i_l \in \mathcal{O}_i$, $v_i \in \mathcal{O}_V$, $f_j \in \mathcal{O}_F$ and $\gamma_1, \gamma_2 \in \Gamma$ then we assign $\gamma_1^{-1}\gamma_2$ to i_l. We will use the notation $\psi(i_l) = \gamma_1^{-1}\gamma_2$.

The *gain bipartite graph* (G_S, ψ) of a Γ-symmetric incidence structure S is an edge-labeled bipartite directed multigraph constructed as follows. The two vertex classes are $\mathcal{O}_V$ and $\mathcal{O}_F$ and there is an edge with label γ between v_i and f_j for each possible group element γ for which $i_l = (v_i, \gamma f_l)$. The edges are oriented towards $\mathcal{O}_F$.

The gain of a closed (not directed) walk $e_1, e_2, e_3, \dots, e_k$ that starts at a vertex in $\mathcal{O}_V$ is $\psi(e_1)\psi(e_2)^{-1}\psi(e_3)\dots\psi(e_k)^{-1}$. (Note that every other edge is used in the reverse direction; for these the inverse of their edge label is taken.) The *gain group* of a connected edge set K and a vertex v spanned by K is defined by taking the set of gains of every closed walk in K starting with v. (Further investigations show that the choice of v can be arbitrary.) A connected edge set is *balanced*, if its gain group is the trivial group. Otherwise it is *unbalanced*. A not connected edge set is balanced, if it does not have an unbalanced component.

3.2 Necessary sparsity conditions for $d = 2$

Consider the special case when $d = 2$. Let $S(r)$ be a reflection-symmetric 1-picture. There are two choices for Γ', namely $\mathcal{C}_2$ (half-turn) and $\mathcal{C}_s$ (reflection). For these two symmetry groups we can give necessary conditions for the constraints to be independent.

Let (G_S, ψ) be the gain-bipartite graph of the incidence structure S. In order to determine independent constraints, every connected subgraph $G'_S = (V_1, F_1; E_1)$ of G_S has to satisfy the following two properties (for both $\mathcal{C}_2$ and $\mathcal{C}_s$):

1. for balanced sets $|E_1| \leq |V_1| + 2|F_1| - 2$;
2. for unbalanced sets we have $|E_1| \leq |V_1| + \sum_{f_j \in F_1} c_j - 1$ where $c_j = 1$ if $(v_i, f_j) \in I$ and $(\gamma(v_i), f_j) \in I$ for some i and $\gamma \neq \mathrm{id}$ and $c_j = 2$ otherwise.

4 Further work

We expect that similar necessary conditions for forced symmetric liftings can also be established for higher dimensions. To obtain combinatorial characterisations, it is natural to consider inductive Henneberg-type construction moves. The results in [3] may also provide useful tools. These investigations are left for a future paper.

Acknowledgements

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