Automorphism groupoids in noncommutative projective geometry

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Abstract

We address a natural question in noncommutative geometry, namely the rigidity observed in many examples, whereby noncommutative spaces (or equivalently their coordinate algebras) have very few automorphisms by comparison with their commutative counterparts.

In the framework of noncommutative projective geometry, we define a groupoid whose objects are noncommutative projective spaces of a given dimension and whose morphisms correspond to isomorphisms of these. This groupoid is then a natural generalization of an automorphism group. Using work of Zhang, we may translate this structure to the algebraic side, wherein we consider homogeneous coordinate algebras of noncommutative projective spaces. The morphisms in our groupoid precisely correspond to the existence of a Zhang twist relating the two coordinate algebras.

We analyse this automorphism groupoid, showing that in dimension 1 it is connected, so that every noncommutative $\mathbb{P}^1$ is isomorphic to commutative $\mathbb{P}^1$. For dimension 2 and above, we use the geometry of the point scheme, as introduced by Artin-Tate-Van den Bergh, to relate morphisms in our groupoid to certain automorphisms of the point scheme.

We apply our results to two important examples, quantum projective spaces and Sklyanin algebras. In both cases, we are able to use the geometry of the point schemes to fully describe the corresponding component of the automorphism groupoid. This provides a concrete description of the collection of Zhang twists of these algebras.

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1 Introduction

It is well-known that affine algebraic varieties can have many automorphisms:

Theorem 1 ([Jun42], [vdK53]). *Let $k = \overline{k}$ be an algebraically closed field. The group of automorphisms of the affine plane $\mathbb{A}^2_k$ over k is isomorphic to $\mathrm{Aut}_{alg}(k[x,y])$, which is isomorphic to $(GL_2(k) \ltimes k^2) *_{S_2(k)} E_2(k)$ where*

$$E_2(k) = \{f : k[x,y] \to k[x,y] \mid f : (x,y) \mapsto (\alpha x + P(y), \beta y + \gamma), \alpha, \beta, \gamma \in k, P(y) \in k[y]\}$$

and $S_2(k)$ is the intersection of $GA_2(k) \stackrel{\text{def}}{=} GL_2 \ltimes k^2$ and $E_2(k)$, and $*$ denotes the amalgamated free product. □

Furthermore, informally,

Theorem 2 ([Nag72], [SU04]). $\mathrm{Aut}_{alg}(k[x,y,z])$ *contains wild automorphisms - that is, automorphisms that cannot be expressed in terms of elementary automorphisms.*

Of course, the situation is more tractable for projective rather than affine varieties:

Theorem 3. *Let $\mathbb{P}^n_k = (\mathbb{A}^{n+1}_k)/\sim$ be projective (n-)space. Then* $\mathrm{Aut}(\mathbb{P}^n_k) \cong \mathrm{Aut}_{\mathbb{Z}-gr}(k[x_0, \ldots, x_n]) \cong \mathrm{PGL}_{n+1}(k) = GL_{n+1}(k)/(k^*)^{n+1}$.

However, non-commutative deformations of these spaces typically have very few automorphisms:

Theorem 4 ([AC92]). *Let $q \in k^*$, q not a root of unity. Denote by $\mathcal{O}_q(\mathbb{P}^n_k)$ the k-algebra*

$$\mathcal{O}_q(\mathbb{P}^n_k) = k\langle x_0, \ldots, x_n\rangle / \langle x_i x_j = q x_j x_i \ \forall\ i < j\rangle.$$

Then $\mathrm{Aut}(\mathcal{O}_q(\mathbb{P}^n_k)) = \begin{cases} (k^*)^{n+1} & n \neq 2, \\ k \ltimes (k^*)^3 & n = 2. \end{cases}$ □

The $(k^*)^{n+1}$ is the torus action corresponding to the natural $\mathbb{Z}^{n+1}$-grading. The automorphisms that act by scaling corresponding to this torus preserve the total grading, which is the $\mathbb{Z}$-grading of interest.

The same phenomenon of a significantly smaller automorphism group has been observed for many other quantum algebras: quantized enveloping algebras and related algebras, quantum matrices, quantum Weyl algebras, Nichols algebras, and others. Contributors include Andruskiewitsch-Dumas [AD08], Fleury [Fle97], Goodearl-Yakimov [GY15], Joseph [Jos76], Launois-Lenagan [LL13], Rigal [Rig96] and Yakimov [Yak13],[Yak14].

This begs the (admittedly somewhat naive) question: where have all the classical automorphisms gone? The ultimate goal of the approach we will describe here is to understand how, given a projective variety X, one can study families of *noncommutative* projective spaces that are close to X in a suitable sense, and how one may lift some symmetries of X to symmetries of and between these noncommutative spaces.

We start with a brief discussion of the underlying philosophy we will adopt, with a view to making a more precise formulation of the previous paragraph.

1.1 Symmetries and groupoids

One classical notion of a geometry is that of a space equipped with an action of a set of invertible, structure-preserving transformations. This "space" can be a set, vector space, manifold, algebraic variety etc. and the set of transformations forms a group of symmetries of it.

In categorical language, a group is a category with one object and all of whose morphisms are invertible. An example relevant to our project is given by taking a $\mathbb{Z}$-graded commutative k-algebra A as the object and elements of the group of degree 0 automorphisms $\mathrm{Aut}_{\mathrm{gr}}(A)$ as the morphisms.

However, in addressing the question of recovering automorphisms from the commutative setting and of introducing a paradigm for studying automorphisms in noncommutative projective geometry it is natural, and necessary, to work with groupoids - that is, categories with more than one object but with all morphisms invertible.

Noncommutative algebras "close enough" to a given commutative algebra are of course not isomorphic. However, certain module categories associated to these algebras will be equivalent and even isomorphic - induced by twists of these algebras, explained below - and we wish to remember *how* these categories are equivalent. The morphisms between objects in a groupoid precisely retain this information. We want to say that certain objects are the same without identifying them. In Baez's phrasing, "groupoids are like sets with symmetries" ([Bae]). This point of view has also been expounded by Brown [Bro87] and Weinstein [Wei96] among others.

As such, we take a collection of noncommutative spaces that model our chosen classical space and, instead of allowing all possible morphisms between them, concentrate on just the isomorphisms between them.

That is, we do something like

- consider all spaces and all morphisms between them: a category of noncommutative schemes over k,
- throw away all objects not sufficiently like our chosen one and
- throw away any non-invertible morphisms.

This groupoid typically has many connected components. This will be the basis for the construction we describe. However, before that, we need to clarify what we will mean by a noncommutative space. We will adopt the approach of noncommutative projective geometry, the outline of which we will give below in Section 2 for the reader unfamiliar with this area.

1.2 Contents

In Section 2 we recall both a theorem of Serre providing an equivalence between the category of quasi-coherent sheaves on a projective scheme and a certain module category over its homogeneous coordinate ring, and a noncommutative analogue due to Artin and Zhang. These results motivate the notion that such module categories should be considered as noncommutative projective schemes. We define the class of algebras that we take to be homogeneous coordinate rings of noncommutative projective spaces and then recall the definition of a Zhang twist of a graded algebra. This is a family of automorphisms of the underlying vector space which induce a new associative multiplication, "twisting" the original multiplication. Finally, we recall the definition of the point scheme, a classical scheme which is an important geometric invariant of noncommutative projective schemes.

We introduce and define our main object of study, the groupoid $\mathscr{NC}(\mathbb{P}^n)$, in Section 3. This is a groupoid whose objects are certain module categories for the coordinate rings of noncommutative $\mathbb{P}^n$s and whose morphisms are particular equivalences of these categories induced by Zhang twists. That this is a groupoid is deduced from a result of Zhang, and that twisting preserves the properties of a noncommutative $\mathbb{P}^n$. We also briefly introduce slice categories associated to $\mathscr{NC}(\mathbb{P}^n)$, which allow us to study all the twists - that is, all of the "generalized automorphisms" - of a given noncommutative $\mathbb{P}^n$.

In Section 4 we begin studying the groupoid, considering the case $n = 1$. Here $\mathscr{NC}(\mathbb{P}^1)$ has only one connected component (which of course contains the object associated to the commutative polynomial ring).

We study $\mathscr{NC}(\mathbb{P}^2)$ in Section 5. Here, the behaviour is considerably more varied and subtle. As such, more sophisticated techniques are required to analyse the groupoid, in particular the geometry of the point scheme. Using results of Artin-Tate-Van den Bergh and Mori, relating twists of algebras in this dimension to isomorphisms between their point schemes (Proposition 35 and Theorem 36), we study the connected components of the commutative polynomial ring and those of certain quantum deformations of it. In the latter case, we recover some automorphisms not arising from graded algebra automorphisms of these deformations.

In Section 6 we consider the groupoid for general n. After introducing Mori's notion of a *geometric algebra*, we generalize Proposition 35, now connecting Zhang twists of such algebras with automorphisms of their associated point schemes, for arbitrary n. Let $\mathcal{A}$ be an object in $\mathscr{NC}(\mathbb{P}^n)$ associated to a noncommutative $\mathbb{P}^n$ which is geometric in Mori's sense. Let $E \subset \mathbb{P}^n$ be its associated point scheme, σ a certain automorphism of E and $\mathcal{L}$ the very ample line bundle giving the embedding into $\mathbb{P}^n$. The main result of the paper is the following, Theorem 42:

Theorem. *Let $\mathcal{A}$ be a geometric noncommutative $\mathbb{P}^n$ with homogeneous coordinate ring A and let $(E, \sigma, \mathcal{L})$ be its associated triple. Then*

$$\underline{\underline{\mathrm{Aut}}}(\mathcal{A}) \cong \mathrm{Aut}(E \uparrow \mathbb{P}^n).$$

Here, $\mathrm{Aut}(E \uparrow \mathbb{P}^n)$ is the group of automorphisms of the point scheme that extend to $\mathbb{P}^n$. The group $\underline{\underline{\mathrm{Aut}}}(\mathcal{A})$ is that obtained from the slice category associated to $\mathcal{A}$, modulo graded Morita equivalence. The theorem thus says that, up to graded Morita equivalence, the generalized automorphisms of a geometric noncommutative $\mathbb{P}^n$ are precisely controlled by the geometry of the point scheme. We conclude with two examples of the theorem: quantum deformations of polynomial rings in n variables; and Sklyanin algebras, an important class of noncommutative $\mathbb{P}^2$s, before briefly discussing an alternative approach to considering noncommutative $\mathbb{P}^2$s in an appendix.

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2 Preliminaries on noncommutative projective geometry

Let k be an algebraically closed field of characteristic 0 and let A be a right Noetherian $\mathbb{Z}$-graded k-algebra. Let $\mathrm{Gr}\, A$ denote the category of (not necessarily finitely generated) $\mathbb{Z}$-graded right A-modules with morphisms of degree 0. Henceforth, by any graded module category over a k-algebra A, we mean the category of right A-modules. We denote by (1) the grading shift functor on $\mathrm{Gr}\, A$. That is, for M a $\mathbb{Z}$-graded right A-module, and $n \in \mathbb{Z}$ denote by $M(n)$ the graded module whose degree m component is given by $M(n)_m = M_{n+m}$. Recall from [AZ94] that, for M in $\mathrm{Gr}\, A$, an element $x \in M$ is called *torsion* if $xA_{\geq s} = 0$ for some s. The set of torsion elements of M forms a graded A-submodule, $\tau(M)$. The module M is called torsion if $\tau(M) = M$ and torsion-free if $\tau(M) = 0$. The submodule $\tau(M)$ is the smallest such that the quotient $M/\tau(M)$ is torsion-free. The collection of all torsion modules forms a Serre subcategory of $\mathrm{Gr}\, A$ which we denote $\mathrm{Tors}(A)$.

Let $\pi : \mathrm{Gr}\, A \to \mathrm{Gr}\, A/\mathrm{Tors}(A)$ be the canonical quotient functor and define $\mathrm{QGr}\, A \overset{\mathrm{def}}{=} \mathrm{Gr}\, A/\mathrm{Tors}(A)$. One can roughly think of $\mathrm{QGr}\, A$ as the category whose objects are the same as those of $\mathrm{Gr}\, A$, but with torsion modules isomorphic to the zero module. More precisely, the objects of $\mathrm{QGr}\, A$ are those of $\mathrm{Gr}\, A$ and the morphisms can be described as

$$\mathrm{Hom}_{\mathrm{QGr}\, A}(\pi(M), \pi(N)) \cong \varinjlim \mathrm{Hom}_{\mathrm{Gr}\, A}(M', N/\tau(N))$$

where the limit runs over the quasi-directed category of submodules M' of M such that M/M' is torsion. See [AZ94] for more details.

To motivate the definition of a noncommutative projective space we want to use, we recall a classical result of Serre and its noncommutative analogue.

Let X be a projective scheme and $\mathcal{L}$ be a line bundle on X. To this data one can associate the homogeneous coordinate ring $B = B(X, \mathcal{L}) = \bigoplus_{n\in\mathbb{N}} \Gamma(X, \mathcal{L}^n)$. Given a quasi-coherent sheaf $\mathcal{M}$ on X we may similarly define $\Gamma_h(\mathcal{M}) \overset{\mathrm{def}}{=} \bigoplus_{n\in\mathbb{N}} \Gamma(X, \mathcal{M} \otimes \mathcal{L}^n)$, which is a graded B-module. Composing with π, we obtain a functor $\overline{\Gamma}_h \colon \mathrm{QCoh}(X) \to \mathrm{QGr}\, B$. Finally, an $\mathbb{N}$-graded k-algebra A is called *finitely graded* if $\dim_k A_i < \infty$ and *connected* if $A_0 = k$. By describing such an algebra as *connected graded*, we mean that it is both connected and finitely graded.

Then we have:

Theorem 5 ([Ser55])**.**

(i) *Let $\mathcal{L}$ be an ample line bundle on a projective scheme X. Then $\overline{\Gamma}_h$ defines an equivalence of categories between* $\mathrm{QCoh}(X)$ *and* $\mathrm{QGr}\, B$*.*

(ii) *If A is a commutative connected graded k-algebra generated in degree 1 then there exists a line bundle $\mathcal{L}$ over $X = \mathrm{Proj}(A)$ such that $A = B(X, \mathcal{L})$, up to a finite-dimensional vector space. Again,* $\mathrm{QGr}\, A \simeq \mathrm{QCoh}(X)$*.*

That is, in the commutative setting, studying the category of quasi-coherent sheaves on a projective variety is essentially the same as studying graded modules modulo torsion over a graded ring.

This suggests that we could naturally consider noncommutative projective schemes to be categories of the form $\mathrm{QGr}\, B$ for B in some suitable family of (not necessarily commutative) graded rings. For this definition to be reasonable and useful, one would like to consider rings for which an analogue of Serre's theorem holds - in particular, rings B for which $\mathrm{QGr}\, B$ is a Grothendieck category - and for this to form a sufficiently broad and interesting class of rings to study. The rings to which this analogue pertains are characterized in [AZ94, Theorem 4.5].

In [AZ94], a categorical version of the homogeneous coordinate ring is defined: namely, given the triple $(\mathcal{C}, \mathcal{O}, s)$ of a Grothendieck category $\mathcal{C}$, a Noetherian object $\mathcal{O}$ in $\mathcal{C}$ and an autoequivalence s, one can define the homogeneous coordinate ring

$$B = \Gamma_h(\mathcal{C}, \mathcal{O}, s) \stackrel{\text{def}}{=} \bigoplus_{i=0}^{\infty} \mathrm{Hom}(\mathcal{O}, s^i\mathcal{O})$$

with the triple $(\mathrm{QCoh}(X), \mathcal{O}_X, - \otimes \mathcal{L})$ for X a projective scheme and $\mathcal{L}$ a line bundle being the key example. Indeed, after giving a categorical definition for a pair $(\mathcal{O}, s)$ to be ample and, assuming certain additional technical conditions on the category $\mathcal{C}$ and on a right Noetherian, finitely graded k-algebra A, the authors show that:

- Given a triple $(\mathcal{C}, \mathcal{O}, s)$ as above, $\mathcal{C} \simeq \mathrm{QGr}(\Gamma_h(\mathcal{C}, \mathcal{O}, s))$
- For such a graded k-algebra A, $A \cong \Gamma_h(\mathrm{QGr}\, A, \pi A, (1))$ in sufficiently high degrees, and $(\pi A, (1))$ is ample.

(see [AZ94, §4] and [SVdB01, §2] for a more detailed exposition).

Remark 6. Note that in [AZ94, Theorem 4.5(1)] the first result is stated as an isomorphism of triples of the form $(\mathcal{C}, \mathcal{O}, s)$ as above. Such an isomorphism of triples requires only that the relevant categories, in our case $\mathcal{C}$ and $\mathrm{QGr}(\Gamma_h(\mathcal{C}, \mathcal{O}, s))$ are equivalent and not themselves isomorphic (see [AZ94, p.237]).

All of the k-algebras we consider henceforth satisfy the conditions of Artin-Zhang's theorem.

To further support the idea that studying module categories of the form $\mathrm{QGr}\, A$ is as fruitful as studying their associated rings or projective schemes - particularly with regard to our focus on automorphism groups - recall the following result of Bondal-Orlov:

Theorem 7 (cf. [BO01, Theorem 3.1])**.** *If X is a smooth irreducible projective variety with an ample canonical or anticanonical sheaf, then the group of isomorphism classes of exact autoequivalences is given by* $\mathrm{Aut}(D^b_{\mathrm{Coh}}(X)) \cong \mathrm{Aut}(X) \ltimes (\mathrm{Pic}(X) \oplus \mathbb{Z})$*.*

It follows that $\mathrm{Aut}(\mathrm{Coh}(X)) \cong \mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)$, so that if we wish to study invertible morphisms of our noncommutative spaces - meaning equivalences between categories of the form $\mathrm{QGr}\, B$ - then these do (essentially) correspond to considering automorphisms of varieties in the commutative case.

Indeed, as is shown in [AZ94, Corollary 6.9] and [AVdB90, Proposition 2.15], any automorphism of the category of coherent sheaves of $\mathcal{A}$-modules, for $\mathcal{A}$ a coherent sheaf of $\mathcal{O}_X$-algebras, is of the form $\sigma_*(-\otimes\mathcal{L})$ where $\sigma \in \mathrm{Aut}(X)$ and $\mathcal{L}$ is an invertible $(\mathcal{A}, \mathcal{A})$-bimodule. As such, following Artin-Zhang's theorem, one can study the so-called *twisted* homogeneous coordinate rings $B = \Gamma_h(\mathrm{QCoh}(X), \mathcal{O}_X, \sigma_*(- \otimes \mathcal{L}))$ and their module categories $\mathrm{QGr}\, B$ via an understanding of $\mathrm{Pic}(X)$ and of the automorphisms of X.

Given that projective varieties embed into an ambient projective space, we now restrict to considering noncommutative analogues of $\mathbb{P}^n$.

That is, we want to define the class of noncommutative k-algebras that we consider as giving rise to *noncommutative projective n-space* - pairs of the form $(\operatorname{QGr} A, \pi A)$ where the k-algebra A shares important homological and algebraic properties with a graded polynomial ring in $n+1$ variables. We adopt the definition given in section 4 of [Kee03]. We first recall some basic definitions giving noncommutative analogues of homological properties enjoyed by commutative polynomial rings. As such rings are Gorenstein, we make the following:

Definition 8. Let A be a connected graded k-algebra. Then A is called *Artin-Schelter Gorenstein* (AS-Gorenstein) if we have the following:

(i) A has finite left and right injective dimension n;

(ii) for some shift l,

$$\operatorname{Ext}^i_{\operatorname{Gr} A}(k, A) = \begin{cases} 0 & \text{if } i \neq n \\ k(l) & \text{if } i = n. \end{cases}$$

Remark 9. One reason for making this definition is that a Noetherian AS-Gorenstein ring A satisfies certain hypotheses required such that a noncommutative version of Serre duality holds for $(\operatorname{QGr} A, \pi A)$. That is, there exists an object $\omega^\circ \in D^b(\operatorname{QGr} A)$ and a natural isomorphism

$$\operatorname{RHom}(-, \omega^\circ) \cong \operatorname{RHom}(\pi A, -)^*$$

See [SVdB01, §7.4] and the references therein.

In keeping with the fact that polynomial rings also have finite global dimension we define:

Definition 10. Let A be a connected graded k-algebra. Then A is *Artin-Schelter regular* (AS-regular) of dimension n if:

(i) A has right and left global dimension n;

(ii) $\operatorname{GKdim}(A) < \infty$; and

(iii) A is AS-Gorenstein of injective dimension n.

We direct the reader to §8 of [SVdB01] for more detailed intuition and motivation behind this definition. With these notions in hand, we can define the class of algebras we are interested in.

Definition 11 (cf. [Kee03, Definition 4.3])**.** Let A be a connected graded k-algebra. We say that the pair $(\operatorname{QGr} A, \pi A)$ is a *noncommutative* $\mathbb{P}^n$ and that A is *the (homogeneous) coordinate ring of a noncommutative* $\mathbb{P}^n$ if

(i) A is a Noetherian domain;

(ii) A is AS-regular of dimension $n+1$;

(iii) if l is the shift from Definition 8, then $l = n+1$;

(iv) A is generated in degree 1; and

(v) A has Hilbert series $(1-t)^{-n-1}$.

As Keeler notes, his characterization is more restrictive than is found elsewhere in the literature.

Remark 12. Note that we have reordered the conditions in Keeler's definition so that A being Noetherian is placed first. In this case, A has equal left and right global dimensions, and equal left and right injective dimensions. Moreover, definitions 8 and 10 are left-right symmetric and so we can omit the conditions placed on A^{op} by Keeler.

Before proceeding further, we recall the definitions related to the notion of a Zhang twist, the relevance of which will become clear shortly.

Definition 13 ([Zha96, Definition 2.1])**.** Let $\mathbb{G}$ be a semigroup with identity e and let A be a $\mathbb{G}$-graded k-algebra. A *twisting system* for A is a set of graded k-linear A-automorphisms $\tau = \{\tau_g \mid g \in \mathbb{G}\}$ such that

$$\tau_g\left(y\tau_h\left(z\right)\right) = \tau_g\left(y\right)\tau_{gh}\left(z\right)$$

for all $g, h, l \in \mathbb{G}$, $y \in A_h$ and $z \in A_l$.

Any semigroup homomorphism $\mathbb{G} \to \mathrm{Aut}_{\mathbb{G}}\left(A\right)$ given by $g \mapsto \tau_g$ produces a twisting system $\{\tau_g \mid g \in \mathbb{G}\}$ where $\mathrm{Aut}_{\mathbb{G}}\left(A\right)$ denotes the group of graded algebra automorphisms of A. Twisting systems arising from semigroup homomorphisms as above are called *algebraic*. For example, for $\mathbb{G} = \mathbb{Z}$ and $f \in \mathrm{Aut}_{\mathbb{Z}}\left(A\right)$, we have the homomorphism $n \mapsto f^n$ and corresponding twisting system $\{f^n \mid n \in \mathbb{Z}\}$.

In the case of a connected $\mathbb{N}$-graded algebra $A = \bigoplus A_i$ generated in degree 1, the fundamental defining relations for a Zhang twist are equivalent to the following ([Zha96, p. 284]). A twisting system in this case is a set $\tau = \{\tau_n \mid n \in \mathbb{N}\}$ of graded linear isomorphisms satisfying

$$\tau_m(ab) = \tau_m(a)\tau_{m+n}\tau_n^{-1}(b)$$

for $a \in A_n$. Notice that if $\tau_{m+n}\tau_n^{-1}$ were equal to τ_m, we would have that τ_m was an algebra map (and in fact then an automorphism). So this formulation makes visible another way in which twisting systems are close to, but are more general than, algebra automorphisms.

As with other notions of twisting, such as twisting by automorphisms or by group 2-cocycles, given a twisting system τ for A, we can form a new algebra A^τ, whose underlying k-vector space is the same as that of A but whose multiplication is twisted by τ.

Definition 14 ([Zha96, Definition and Proposition 2.3])**.** Let A be a $\mathbb{G}$-graded k-algebra and $\tau = \{\tau_g \mid g \in \mathbb{G}\}$ be a twisting system for A. One defines the *twisted algebra of* A *by* τ, denoted A^τ, as the triple $(\oplus_g A_g, \star, 1_\tau)$ where $\star$ is an associative, graded multiplication given by

$$y \star z \stackrel{\text{def}}{=} y\tau_h\left(z\right)$$

for $y \in A_h$ and $z \in A_l$ and where $1_\tau \stackrel{\text{def}}{=} \tau_e^{-1}\left(1_A\right)$ is the identity in A^τ.

The following lemma concerns twisting systems for quadratic algebras and generalizes an observation of Zhang, made in examining Example 5.12 in [Zha96].

Lemma 15. *Let A be a connected $\mathbb{N}$-graded finitely generated algebra, generated in degree 1 and quadratic. Let $\tau = \{\tau_m \mid m \in \mathbb{N}\}$ be a twisting system. Suppose the graded linear isomorphism τ_1 is additionally an algebra automorphism and set $\tau' = \{\tau_1^m \mid m \in \mathbb{N}\}$. Then we have an isomorphism of algebras $A^\tau \cong A^{\tau'}$.*

Proof: Since A is generated in degree 1, we may present A as a quotient of the tensor algebra on A_1: let $\rho\colon T(A_1) \to A$ be the associated canonical map, so that $\ker\rho \cap A_2$ generates the ideal of (quadratic) relations defining A. Fix a basis $\{x_i \mid 1 \leq i \leq r\}$ for A_1.

Let $R \in A_2$ be a quadratic homogeneous relator, i.e. $R \in \ker\rho \cap A_2$. Write $R = \sum_{\underline{a} \in \mathbb{N}^2} \alpha_{\underline{a}} X^{\underline{a}}$, where $X^{\underline{a}} \stackrel{\text{def}}{=} x_1^{\underline{a}_1} x_2^{\underline{a}_2} \cdots x_r^{\underline{a}_r}$. Note that $\sum \underline{a}_i = 2$, since R is chosen to be quadratic.

For each monomial $X^{\underline{a}}$, let $s(\underline{a}) = \max\{i \mid a_i \neq 0\}$ and set $X_\tau^{\underline{a}} = x_1^{\underline{a}_1} \cdots x_{s(\underline{a})}^{\underline{a}_{s(\underline{a})}-1} * \tau_1^{-1}(x_{s(\underline{a})})$.

Arguing similarly to Example 5.12 of [Zha96], in A^τ we have

$$\begin{aligned} R^\tau &\stackrel{\text{def}}{=} \sum_{\underline{a}\in\mathbb{N}^r} \alpha_{\underline{a}} X_\tau^{\underline{a}} \\ &= \sum_{\underline{a}\in\mathbb{N}^r} \alpha_{\underline{a}} x_1^{\underline{a}_1} \cdots x_{s(\underline{a})}^{\underline{a}_{s(\underline{a})}-1} * \tau_1^{-1}(x_{s(\underline{a})}) \\ &= \sum_{\underline{a}\in\mathbb{N}^r} \alpha_{\underline{a}} x_1^{\underline{a}_1} \cdots x_{s(\underline{a})}^{\underline{a}_{s(\underline{a})}-1} \tau_1(\tau_1^{-1}(x_{s(\underline{a})})) \qquad (1) \\ &= \sum_{\underline{a}\in\mathbb{N}^r} \alpha_{\underline{a}} X^{\underline{a}} \\ &= 0. \end{aligned}$$

Here (1) is a consequence of R being quadratic, so that $x_1^{\underline{a}_1} \cdots x_{s(\underline{a})}^{\underline{a}_{s(\underline{a})}-1}$ has degree 1; recall that $y * z = y\tau_h(z)$ with $h = \deg y$.

We see that $A^{\tau'}$ is also generated by A_1 and $R^\tau = 0$ in $A^{\tau'}$; only the map τ_1 is involved in the above calculation. Hence A^τ is isomorphic as a graded algebra to $A^{\tau'}$. ☐

Remark 16. Note that the condition on τ_1 in the lemma is necessary, in the sense that $\tau' = \{\tau_1^m \mid m \in \mathbb{N}\}$ is a twisting system if and only if τ_1 is an algebra automorphism, under the other assumptions of the lemma. This follows from the fact that the defining property of a twisting system is equivalent to the above-stated identity,

$$\tau_m(yz) = \tau_m(y)\tau_{m+n}\tau_n^{-1}(z)$$

for all $y \in A_m$, $z \in A_n$.

If $\tau' = \{\tau_1^m \mid m \in \mathbb{N}\}$ is a twisting system, then $\tau'_{m+n} = \tau_1^{m+n}$ satisfies

$$\tau_m(yz) = \tau_m(y)\tau_m(z)$$

and in particular τ_1 is an algebra map. The converse is clear.

Given a coordinate ring of a noncommutative $\mathbb{P}^n$ (that is, an algebra A satisfying the conditions in Definition 11), Theorem 1.3 in [Zha96] states that if A is a Noetherian domain, then so is any twist of it. Clearly, the degree of an element is preserved by twisting, and hence any twist of A is generated in degree 1 with Hilbert series $(1-t)^{-n}$. From Theorem 5.11 of [Zha96], we have that any twist of A will be AS-regular of the same dimension. Finally, we have that for some twisted algebra $B \cong A^\tau$, $\mathrm{Ext}^i_{\mathrm{Gr}\,A}(k, A) \cong \mathrm{Ext}^i_{\mathrm{Gr}\,B}(k^\tau, B)$ where k is the A-module $A/A_{\geq 1}$. Hence $k^\tau \cong A^\tau/A^\tau_{\geq 1} \cong B/B_{\geq 1} \cong k$ as B-modules. The class of connected graded k-algebras defined above is thus closed under Zhang twisting. With this in mind, we make the following definition.

Definition 17. Let A and B be $\mathbb{G}$-graded k-algebras. We say that A and B are twist-equivalent if there exists a twisting system τ for A such that $B \cong A^\tau$ as graded k-algebras.

As in various incarnations of noncommutative algebraic geometry, it is useful to define classical commutative objects which capture some of the information of their noncommutative counterparts. We finish this section by recalling an important invariant of noncommutative projective schemes - the point scheme - introduced by Artin, Tate and Van den Bergh. To do so, we wish to define a noncommutative analogue of a point on a projective scheme and so study irreducible objects of QGr A. As such, we make the following:

Definition 18. Let A satisfy the conditions of Definition 11. A *point module* for A is a graded cyclic module with Hilbert series $1/(1-t)$.

Definition 19. Let A be as above and let $d \in \mathbb{N}$. A *truncated point module* of length $d+1$ is a graded cyclic A-module with Hilbert series $\sum_{i=0}^d t^i$.

Before defining the point scheme, it remains to introduce the notion of a *multilinearization* of an element in the defining ideal of A.

We adopt the shorthand $\overline{\{n\}} = \{0, 1, \ldots, n\}$ and $\underline{i} = \{i_0, \ldots, i_{d-1}\}$ for a vector in $\overline{\{n-1\}}^d$. Let $A = T(V)/I$ be as before, where $V = kx_0 + \cdots + kx_{n-1}$ and let $f \in I_d$ for some $d \in \mathbb{N}$. Thus, $f = \sum_I \alpha_{\underline{i}} x_{i_0} \ldots x_{i_{d-1}}$ for some $\alpha_{\underline{i}} \in k$.

Definition 20 ([ATVdB90])**.** Define $\widetilde{f}$ to be the element of the homogeneous coordinate ring of $\left(\mathbb{P}^{n-1}\right)^{\times d}$ given by

$$\sum_I \alpha_{\underline{i}} x_{i_0,0} \ldots x_{i_{d-1},d-1}$$

where the coordinate ring of the $(j+1)^{st}$ copy of $\mathbb{P}^{n-1}$ is given by $k\left[x_{0,j}, \ldots, x_{n-1,j}\right]$. The element $\widetilde{f}$ is the *multilinearization* of f.

As explained in [ATVdB90, §3], truncated point modules of length $d+1$ are parametrized by a projective scheme Γ_d - that is, Γ_d represents the functor of taking flat families of truncated point modules and is thus independent of the chosen presentation of A. By sending a truncated point module of length $d+1$ to one of length d by factoring out the highest degree component, a morphism of schemes $\Gamma_d \to \Gamma_{d-1}$ is induced, corresponding to the forgetting of the last component in the projection $\left(\mathbb{P}^{n-1}\right)^{\times d} \to \left(\mathbb{P}^{n-1}\right)^{\times d-1}$. Finally, we come to:

Definition 21 ([ATVdB90])**.** The *point scheme* of A, $\Gamma(A)$, is defined as the inverse limit of the diagram:

$$\Gamma_0 \leftarrow \Gamma_1 \leftarrow \cdots \leftarrow \Gamma_{d-1} \leftarrow \Gamma_d \leftarrow \cdots$$

The significance of the point scheme in our framework will become apparent later.

Remark 22. In general, $\Gamma(A)$ is a pro-scheme, representing the functor of taking point modules of A. However, it is often the case that the projection $\Gamma_d \to \Gamma_{d-1}$ is an isomorphism for some d. In that case, the inverse system is constant for $i \geq d-1$ and $\Gamma(A) \cong \Gamma_{d-1}$ is a scheme. This is true of regular algebras of dimension 3, considered in [ATVdB90], and of the algebras we are concerned with here.

3 Definition of $\mathscr{N}\mathscr{C}\left(\mathbb{P}^n\right)$

The above description of Zhang twisting is rather concrete, which has advantages when one wants to explicitly calculate specific twists, as we shall later. However, even demonstrating some elementary properties, such as showing that twist-equivalence is actually an equivalence relation, can be cumbersome when using this definition.

Along with the definition, the other key insight of Zhang was that twist-equivalence of A and B is equivalent to the existence of a well-behaved functor between certain corresponding categories of graded modules for A and B. From this, it is easy to see that twisting gives an equivalence relation. We briefly outline the results of Zhang in this direction, taken from [Zha96], in order to motivate the definition of the groupoid we wish to study.

Assume that A is a connected $\mathbb{Z}$-graded k-algebra. In Definition 11, we assume a lot more about algebras that are associated to noncommutative $\mathbb{P}^n$s, but even this is already reasonably strong.

Let $\mathrm{GR}\, A$ denote the category of $\mathbb{Z}$-graded A-modules with morphisms being graded morphisms of any degree - that is, $\mathrm{Hom}_{\mathrm{GR}\, A}(M, N) = \bigoplus_{r \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}\, A}(M, N(r))$ (note that Zhang writes $\underline{\mathrm{Hom}}(M, N)$).

We also note that $\mathrm{Hom}_{\mathrm{GR}\, A}(M, M)$, denoted $\Gamma(M)$ by Zhang, is important in a key theorem of Zhang ([Zha96, Theorem 3.3]), most notably in the case $M = A_A$. We define $\mathrm{GrEnd}(A_A) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathrm{GR}\, A}(A_A, A_A)$, the graded endomorphism algebra of A_A (we avoid the notation Γ, which we used earlier for the point scheme of A). By the proof of [Zha96, Theorem 3.4], with the above assumptions, $\mathrm{GrEnd}(A_A) \cong A$; that is, by analogy with the ungraded theory, A is isomorphic to its graded endomorphism algebra.

We say that A and B are graded Morita equivalent if $\mathrm{GR}\, A$ is equivalent to $\mathrm{GR}\, B$ by a graded functor, i.e. a functor which induces a map of graded rings on Hom spaces. For connected graded algebras, A and B are graded Morita equivalent if, and only if, they are isomorphic.

As previously, we say A and B are twist-equivalent if B is isomorphic to a Zhang twist of A. By combining results of Artin-Zhang and Zhang, as detailed in the proof, we obtain the following.

Proposition 23. *Let A and B be coordinate rings of noncommutative $\mathbb{P}^n$s, so that $(\mathrm{QGr}\,A, \pi A)$ and $(\mathrm{QGr}\,B, \pi B)$ are noncommutative $\mathbb{P}^n$s. The following are equivalent:*

(i) *A and B are twist-equivalent.*

(ii) *$\mathrm{Gr}\,A$ is isomorphic to $\mathrm{Gr}\,B$.*

(iii) *$\mathrm{QGr}\,A$ is equivalent to $\mathrm{QGr}\,B$ via a functor $\mathcal{F}$ and*

$$\mathcal{F}((\pi A)(n)) \cong (\pi B)(n) \text{ for all } n \tag{SSS}$$

(the "preserves shifts of the structure sheaf" condition).

Furthermore A and B are graded Morita equivalent, i.e. $\mathrm{GR}\,A$ is equivalent to $\mathrm{GR}\,B$ via a graded functor if, and only if, $B \cong A^\tau \cong A$.

Proof: Consider the following assertions:

(1) B is isomorphic to a Zhang twist of A;

(2) $\mathrm{Gr}\,A$ is equivalent to $\mathrm{Gr}\,B$;

(3) $\mathrm{Gr}\,A$ is isomorphic to $\mathrm{Gr}\,B$;

(4) $\mathrm{Gr}\,A$ is equivalent to $\mathrm{Gr}\,B$ via a functor F and

$$F(A(n)) \cong B(n) \text{ for all } n; \tag{GrSSS}$$

(5) $\mathrm{Gr}\,A$ is isomorphic to $\mathrm{Gr}\,B$ and (GrSSS) holds;

(6) $\mathrm{QGr}\,A$ is equivalent to $\mathrm{QGr}\,B$;

(7) $\mathrm{QGr}\,A$ is equivalent to $\mathrm{QGr}\,B$ and (SSS) holds for πA and πB;

(8) $\mathrm{QGr}\,A$ is isomorphic to $\mathrm{QGr}\,B$ and (SSS) holds for πA and πB

Clearly,

- (4) implies (2),
- (5) implies (3) implies (2) and
- (8) implies (7) implies (6).

The first fundamental result of Zhang's paper is [Zha96, Theorem 3.1], which proves that (1) implies (3). Next, [Zha96, Theorem 3.4] proves that (1) holds if and only if (4). Notice that in the course of proving (4) implies (1), one has that $\mathrm{GrEnd}(A_A) \cong A$, $\mathrm{GrEnd}(B_B) \cong B$ and $\mathrm{GrEnd}(B) \cong \mathrm{GrEnd}(A)^\tau$, and hence $B \cong A^\tau$.

Now, by [Zha96, Theorem 3.5], recalling that we are assuming that A and B are connected graded, we have (1) if and only if (2). The proof of [Zha96, Theorem 3.5] (Case 1) also tells us that (2) implies (4), that is, the condition (SSS) holds automatically for connected $\mathbb{Z}$-graded algebras. Correspondingly [Zha96, Theorem 3.5] works for $\mathbb{N}$-graded algebras also.

Assuming further that A, B are $\mathbb{N}$-graded right Noetherian AS-Gorenstein algebras - as we do, as these are among the conditions of Definition 11 - we have the following. The AS-Gorenstein condition implies, by Artin-Zhang ([AZ94]), that

$$\mathrm{GrEnd}(\pi A) \stackrel{\text{def}}{=} \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{QGr}\,A}(\pi A, (\pi A)(n)) \cong A$$

which very much need not be the case in general.

Then by the proof of [Zha96, Theorem 3.7], one has

- (5) implies (7) and, most importantly,
- (1) if and only if (7).

The key point is that [Zha96, Theorem 3.7(2)] gives that the existence of an equivalence of $\mathrm{QGr}\, A$ with $\mathrm{QGr}\, B$ satisfying (SSS) implies $\mathrm{GrEnd}(\pi A)$ twist-equivalent to $\mathrm{GrEnd}(\pi B)$, for not necessarily AS-Gorenstein algebras. As noted above, having AS-Gorenstein as well yields $A \cong \mathrm{GrEnd}(\pi A)$, $B \cong \mathrm{GrEnd}(\pi B)$ and hence twist-equivalence of A and B, that is, (7) implies (1).

That is,

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (7)$$

and (5) and (8) both imply all of these equivalent statements. In particular we have that (1) if and only if (3) if and only if (7), as claimed.

The final claim of the proposition is immediate from the discussion preceding its statement. □

We wish to highlight that the list of conditions in Definition 11 is extremely strong. In particular, as noted in Example 3.10 of [Zha96], for a $\mathbb{Z}$-graded algebra A with right progenerator P, A and $B \overset{\text{def}}{=} \oplus_n \mathrm{Hom}_{\mathrm{Gr}\, A}(P, P(n))$ are graded Morita equivalent, and hence $\mathrm{Gr}\, A$ is equivalent to $\mathrm{Gr}\, B$. However, in this generality B is not always isomorphic to a twist of A whereas in our setting, where we have $\mathbb{N}$-graded AS-Gorenstein algebras, these statements are equivalent.

The proof of Proposition 23 uses the following result due to Artin and Zhang ([AZ94])

Proposition 24. *Let A be a coordinate ring of a noncommutative $\mathbb{P}^n$ and let $(\mathrm{QGr}\, A, \pi A)$ be the associated noncommutative projective space.*

The graded endomorphism algebra of πA in $\mathrm{QGr}\, A$,

$$\mathrm{GrEnd}(\pi A) \overset{\text{def}}{=} \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{QGr}\, A}(\pi A, (\pi A)(n))$$

is isomorphic as graded k-algebras to A. □

Hence, in our setting the algebra A can be recovered from the data $(\mathrm{QGr}\, A, \pi A)$. We mention this result separately to highlight that regarding the pair $(\mathrm{QGr}\, A, \pi A)$ as a noncommutative projective space mirrors Grothendieck's insight that to study a scheme, one should study its category of quasi-coherent sheaves.

We thus take condition (iii) in Proposition 23, $\mathrm{QGr}\, A$ being equivalent to $\mathrm{QGr}\, B$ and (SSS) holding, as the most appropriate formulation of a noncommutative projective space of the three equivalent conditions of the proposition, as it is closest to the statement in Serre's theorem. That is, given Grothendieck's philosophy and Serre's result, we regard $\mathrm{QGr}\, A$ as the category of quasi-coherent sheaves on a noncommutative space and regard such spaces as being equivalent if the categories $(\mathrm{QGr}\, A, \pi A)$ and $(\mathrm{QGr}\, B, \pi B)$ are equivalent.

It is useful to phrase certain statements in terms of properties of the algebras that are coordinate rings of noncommutative $\mathbb{P}^n$s, as we have being doing, only because not all of the conditions of Definition 11 have a convenient expression internal to $\mathrm{QGr}\, A$.

Given that an algebra A and its appropriate module category determine one another, we now define our main object of study, the groupoid $\mathscr{NC}(\mathbb{P}^n)$:

Definition 25. Let $\mathscr{NC}(\mathbb{P}^n)$ be the category whose objects are the pairs $(\mathrm{QGr}\, A, \pi A)$ where A is an algebra satisfying the conditions of definition 11 and whose morphisms are equivalences of categories $\mathcal{F} : \mathrm{QGr}\, A \to \mathrm{QGr}\, B$ such that $\mathcal{F}(\pi A(m)) \cong \pi B(m)$ for all $m \in \mathbb{N}$.

By Proposition 23, the groupoid $\mathscr{NC}(\mathbb{P}^n)$ partitions into connected components corresponding precisely to all of the Zhang twists of any given algebra A whose pair $(\mathrm{QGr}\, A, \pi A)$ is in that component. For example, for $n \geq 2$, $\mathcal{O}(\mathbb{P}^n) = k[x_0, x_1, \ldots, x_n]$ and $\mathcal{O}_q(\mathbb{P}^n_k)$ correspond to pairs in different connected components. Henceforth, we call the component containing $(\operatorname{QGr}\mathcal{O}(\mathbb{P}^n), \pi\mathcal{O}(\mathbb{P}^n))$ the *commutative component*.

Our initial project is to attempt to understand these connected components for the cases $n = 1$ and $n = 2$ and, as such, to systematically introduce the study of generalized automorphisms into this strand of noncommutative geometry, in keeping with the classical literature of Artin, Stafford, Tate, Van den Bergh, Zhang et al. and the more recent work of Pym ([Pym15]).

By considering the groupoid formed from noncommutative spaces and those equivalences between them induced by Zhang twists, we can give categorical interpretations of the twists of a given algebra A germane to our project of defining and studying an object that generalizes the graded automorphism group of A. As such, we will soon consider the slice category associated to $(\operatorname{QGr} A, \pi A)$, which allows us to focus attention on the connected component of $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ containing $(\operatorname{QGr} A, \pi A)$.

This object will not be a group but in the two approaches we outline it can be thought of as either a groupoid or a functor. This object does not form a group because in general taking a Zhang twist of a given algebra does not produce an isomorphic algebra. Composing Zhang twists thus involves passing between different domains and codomains, and so between different module categories. However, the groupoid precisely encodes the compositions that are defined.

Remark 26. The proof of Proposition 23 shows that we do not have a lot of scope to relax our definition of $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$, to - for example - obtain a groupoid with fewer components. (We will see later that for $n > 1$, $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ has many components.) In particular, the condition $\operatorname{Gr} A$ isomorphic to $\operatorname{Gr} B$ is seen to imply and be implied by $\operatorname{Gr} A$ equivalent to $\operatorname{Gr} B$, so weakening "isomorphic" to "equivalent" has no effect here. We consider it to be an interesting but wide open question as to whether there is a weaker form of equivalence than twist-equivalence yielding fewer equivalence classes but retaining sufficient strength to obtain results comparable with our main theorem below.

Remark 27. An additional property on the algebras we consider that one might wish for, with its roots in geometry, is the Calabi-Yau property. In work of Reyes-Rogalski-Zhang ([RRZ14],[RRZ17]), a definition of a skew Calabi-Yau A is given (the term twisted Calabi-Yau being used elsewhere), part of which asks for the existence of an algebra automorphism μ_A of A, known as the Nakayama automorphism. Then the algebra is Calabi-Yau if μ_A is an inner automorphism; the original definition of Calabi-Yau is due to Ginzburg [Gin06].

We refer the reader to [RRZ14] for a comprehensive discussion but note that our assumptions in Definition 11 in particular imply that homogeneous coordinate rings of noncommutative $\mathbb{P}^n$s are skew Calabi-Yau ([RRZ14, Lemma 1.2]). In [RRZ14], the authors establish a relationship between the Nakayama automorphism of a skew Calabi-Yau algebra and that of a twist by a graded algebra automorphism, noting that not every skew Calabi-Yau algebra can be twisted by an automorphism to a Calabi-Yau one.

Given that we have additional assumptions on our algebras and also that we wish to consider all Zhang twists, not just those coming from automorphisms, the question of whether or not each connected component of $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ contains a noncommutative projective space whose homogeneous coordinate ring is Calabi-Yau remains open. We note that Pym [Pym15] has an analogous statement in the setting of deformations, but the analytic argument used there is not available in our framework.

We also note that in [RRZ17], the authors prove that we can detect whether the algebras we consider are Calabi-Yau by examining the algebra $E = \operatorname{Ext}^{\bullet}_A(k, k)$, specifically showing that A is Calabi-Yau if and only if E is graded-symmetric.

We now adopt the calligraphic font $\mathcal{A} \stackrel{\text{def}}{=} (\operatorname{QGr} A, \pi A)$ for objects of $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$. Given an object $\mathcal{A}$, we wish to focus attention on the morphisms to $\mathcal{A}$. The connected components of $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ do not in general have a terminal object but, for a given $\mathcal{A}$, we can form the slice category $\mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A}$, whose terminal object is the morphism $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. The objects of $\mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A}$ are maps $\mathcal{B} \to \mathcal{A}$ and the morphisms are the triangles:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{f} & \mathcal{C} \\ & \searrow \; \swarrow & \\ & \mathcal{A} & \end{array}$$

where f is a morphism in $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$.

The slice category $\mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A}$ comes with a forgetful functor Φ to $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$, given on objects by taking the domain, $\Phi(\mathcal{B} \to \mathcal{A}) = \mathcal{B}$ and on morphisms by taking the map f in the triangle above. In our situation, since $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ is a groupoid, one can easily show that this functor is full. However, it is not faithful; $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ and its slices are closely related but do not hold identical information. Part of the reason for this is that "unique" is a very strong statement in this set-up whereas "unique up to isomorphism" is very weak.

Remark 28. The slice category has a close relationship with the functor $\mathrm{Hom}_{\mathscr{N}\mathscr{C}(\mathbb{P}^n)}(-, \mathcal{A})$, the latter also being a natural way to focus on morphisms in $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ whose codomain is $\mathcal{A}$. The relationship between these is akin to that between a function $f \colon X \to Y$ and its graph $\Gamma(f) \subseteq X \times Y$. In this instance we will find that statements regarding the structure of the slice category (and hence of twists of A) are easier to state for the category than for the associated functor.

However at other times it is fruitful to consider the functor $\mathrm{Hom}_{\mathscr{N}\mathscr{C}(\mathbb{P}^n)}(-, \mathcal{A})$, not least because the form of this functor permits its interpretation as a (representable) presheaf. We note many potentially interesting morphisms between noncommutative projective spaces are not contained in the groupoid $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ and as such this presheaf is rather special.

The presheaf model and the corresponding Yoneda embedding $Y \colon \mathscr{N}\mathscr{C}(\mathbb{P}^n) \to [\mathscr{N}\mathscr{C}(\mathbb{P}^n)^{\mathrm{op}}, \mathrm{Set}]$ has a "test space" interpretation: $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ is our category of noncommutative projective spaces and the presheaf $Y(\mathcal{A}) = \mathrm{Hom}_{\mathscr{N}\mathscr{C}(\mathbb{P}^n)}(-, \mathcal{A})$ captures the information of all of the noncommutative spaces *isomorphic to* $\mathcal{A}$ and how they are isomorphic. That is, we are "testing for sameness", which as we suggested earlier should correspond to the generalized notion of symmetry that we need in this generality.

By Proposition 23, morphisms $\mathcal{F} : (\mathrm{QGr}\, B, \pi B) \to (\mathrm{QGr}\, A, \pi A)$ in $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ correspond to twists τ such that $B \cong A^\tau$. As such, the slice category $\mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A}$ gives a categorical interpretation of Zhang twists. Indeed, we could define $\mathrm{Twists}(A) \stackrel{\mathrm{def}}{=} \mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A}$, as the twists of the algebra A corresponding precisely to the morphisms in $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ with codomain $(\mathrm{QGr}\, A, \pi A)$. That is, $\mathrm{Twists}(A) = \mathrm{Hom}_{\mathscr{N}\mathscr{C}(\mathbb{P}^n)}(-, (\mathrm{QGr}\, A, \pi A))$.

This makes clear the fact that Zhang twists under composition form a groupoid, as the slice of the groupoid $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ is a groupoid. This supports the contention referred to earlier that the correct replacement of automorphisms in the commutative setting should be morphisms in a groupoid. More precisely, we have the following elementary results.

Lemma 29. *Let A, B be coordinate rings of noncommutative $\mathbb{P}^n$s and let $f \colon B \to A$ be a graded algebra isomorphism. Then the pushforward $f_\star \colon \mathrm{QGr}\, B \to \mathrm{QGr}\, A$ is an equivalence of categories satisfying $f_\star(\pi B(m)) \cong (\pi A)(m)$ for all $m \in \mathbb{N}$, and hence defines an object in $\mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A}$, where $\mathcal{A} = (\mathrm{QGr}\, A, \pi A)$.* □

Remark 30. Note that if B is isomorphic to A, then certainly B is isomorphic to a twist of A, namely the identity twist. For connected graded algebras, isomorphism is equivalent to graded Morita equivalence, as explained above.

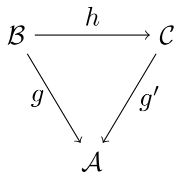
In general, not every twist will arise from an automorphism of algebras, but certainly every automorphism of algebras gives rise to an algebraic twist (p. 7). If we take $B = A$ in the above - that is, f an automorphism - then the pushforward functor $f_\star$ yields an automorphism of $\mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A}$, in the following sense, this being a categorical version of the algebraic twist construction.

For a category $\mathcal{C}$, let $\mathrm{Iso}(\mathcal{C})$ denote the group of auto-isomorphisms of $\mathcal{C}$.

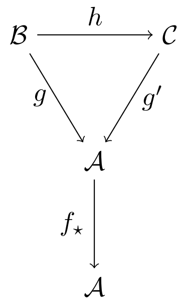
Proposition 31. *Let $\mathrm{Aut}_{\mathrm{gr}}(A)$ denote the group of k-algebra automorphisms of A that are $\mathbb{Z}$-graded of degree 0. For each coordinate ring of a noncommutative $\mathbb{P}^n$, A, there is an injection*

$$\mathrm{Aut}_{\mathrm{gr}}(A) \hookrightarrow \mathrm{Iso}(\mathcal{A}) \hookrightarrow \mathrm{Iso}\left(\mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A}\right)$$

given by $f \mapsto f_\star \circ -$ which on objects is given by $f_\star \circ - : (\mathcal{B} \to \mathcal{A}) \to \left(\mathcal{B} \to \mathcal{A} \xrightarrow{f_\star} \mathcal{A}\right)$, where $f_\star$ is the equivalence of module categories induced via push-forward along an isomorphism of algebras. On morphisms, one sends the diagram



to the diagram



Proof: Let us briefly abuse notation and write simply $f_\star$, as opposed to $f_\star \circ -$. By the properties of composition of homomorphisms, $f_\star$ is clearly a functor. In the same manner, it can also be readily verified that $x_\star \circ y_\star = (x \circ y)_\star$, that $(f_\star)^{-1} = \left(f^{-1}\right)_\star$, and that $(1_{\mathcal{A}})_\star = \mathrm{Id}_{\mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A}}$. Injectivity follows by definition. □

Note that we could have used the pull-back of modules along a morphism $f^\star$ in the previous proposition. The pull-back applies more generally to any morphism of modules, though would produce a contravariant functor. As we are concerned only with push-forward and pull-back under isomorphisms of modules, we choose to work with the push-forward. Accepting the various notions of a group action on a category, in our context, the above definition is sufficient.

We also note that the above injection is not usually surjective. For in the commutative case, the Picard group also acts by auto-equivalences on $\operatorname{Coh} X$, by the result of Bondal and Orlov recalled earlier.

We now turn our attention to a more detailed analysis for $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ for small n, beginning with $n = 1$.

4 Noncommutative $\mathbb{P}^1$s

We now consider the case of $\mathscr{N}\mathscr{C}\left(\mathbb{P}^1\right)$. To describe the algebras A such that $(\operatorname{QGr} A, \pi A) \in \mathscr{N}\mathscr{C}\left(\mathbb{P}^1\right)$, we mimic the discussion in Section 1 of [AS87] and more specifically the proof of their Theorem 1.5. We are concerned with a special case of their results but we consider it helpful to include a detailed exposition here for the particular case at hand, where in some parts we can be more explicit.

Let A be a connected graded k-algebra generated in degree 1, and write

$$A \cong k\langle x_1 \ldots, x_{r_1}\rangle / \left(f_1, \ldots, f_{r_2}\right)$$

where r_1 is the minimal number of generators and r_2 is the minimal number of homogeneous relations. Without loss of generality, we can write the relations as

$$f_i = \sum_{j=1}^{r_1} m_{ij} x_j.$$

We then define an exact sequence of left A-modules

$$A^{r_2} \xrightarrow{M} A^{r_1} \to A \to_A k \to 0 \tag{2}$$

where $M = (m_{ij})$ with $m_{ij} \in k\langle x_1, \ldots, x_{r_1}\rangle$.

Proposition 32 (cf. [AS87],[Zha96])**.** *Let A be a coordinate ring of a noncommutative $\mathbb{P}^1$. Then we have $r_1 = 2$ and $r_2 = 1$ so that A has a single quadratic relation f. Hence, one obtains a graded resolution of left A-modules of the form*

$$0 \to A(-2) \xrightarrow{M} A(-1) \oplus A(-1) \to A \to_A k \to 0 \tag{3}$$

Proof: Consider a minimal resolution of $_A k$

$$0 \to A^{r_2} \to A^{r_1} \to A \to_A k \to 0 \tag{4}$$

with r_1 and r_2 as before. The ranks of the free modules are given by $r_i = \dim_k \operatorname{Tor}_i^A(k,k)$. From the AS-Gorenstein property with shift $l = 2$ (Definition 8), we know that applying the functor $\operatorname{Hom}(-,A)$ to this resolution will produce a resolution of the right A module $k_A = \operatorname{Ext}_A^2(k,A)$

$$0 \leftarrow k_A \leftarrow A^{r_2} \leftarrow A^{r_1} \leftarrow A \leftarrow 0 \tag{5}$$

and this is minimal, since (4) is minimal. Since $\dim_k \operatorname{Tor}_i^A(k,k)$ is symmetric in both left and right, we conclude that $r_2 = 1$. Henceforth set $r_1 = r$.

We have therefore obtained a resolution

$$0 \to A(-s) \xrightarrow{M} (A(-1))^{\oplus r} \to A \to_A k \to 0 \tag{6}$$

where s is the degree of f. Setting $a_n \stackrel{\text{def}}{=} \dim_k A_n$, the above exact sequence gives rise to the recurrence relation $a_n - ra_{n-1} + a_{n-s} = 0$. The characteristic polynomial of this relation is $p(t) = t^s - rt^{s-1} + 1$.

We know that A has Hilbert series $1/(1-t)^2 = 1 + 2t + 3t^2 + 4t^3 + \dots$ Setting $(r,s) = (2,2)$ gives the correct dimensions and it is clear that $r, s \geq 2$. We claim that this is the only solution.

To show this, we demonstrate that if $r + s > 4$, then $p(t)$ has a real root strictly greater than 1. If $s \geq 2$ and $r > 2$, $p(1) = 2 - r < 0$ but $p(t) > 0$ for $t \gg 1$ and we are done.

Otherwise, if $r = 2$ and $s > 2$, $p(1) = 0$. Since $p(t) > 0$ for $t \gg 1$ still, it suffices to show that the derivative

$$p'(t) = t^{s-2}(st - r(s-1))$$

is negative at $t = 1$. But for $r = 2$, $p'(1) = s - 2(s-1) = 2 - s < 0$ and so we are done in this case too. □

We have shown that $A \cong k\langle x,y\rangle/(f)$ where $\deg(f) = 2$. One can show that, in order for A to satisfy the condition on $\operatorname{Ext}^i_{\operatorname{Gr} A}(k,A)$ in Definition 8, f must be of the form $x(ax+by) - y(cx+dy) = 0$ for $ad - bc \neq 0$. By Example 3.6 of [Zha96], these are precisely the algebras which are Zhang twists of $k[x,y]$ and two such algebras are isomorphic if, and only if, the linear automorphisms corresponding to two relations f_1 and f_2 are in the same $\mathrm{PGL}_2(k)$-conjugacy class. Thus, we have that $\mathscr{NC}(\mathbb{P}^1)$ has one connected component, whose morphisms are controlled by $\mathrm{PGL}_2(k)$-conjugacy class.

More categorically, via Serre's theorem, we then see that every $\operatorname{QGr} A \in \mathscr{NC}(\mathbb{P}^1)$ is equivalent to $\operatorname{QCoh} \mathbb{P}^1$. We may interpret this as saying that there are no non-trivial noncommutative $\mathbb{P}^1$'s, and the trivial ones all have the classical automorphism group as their twists.

The above theoretical considerations are complemented by the fact that in low dimensions we can proceed very concretely.

In the case of a connected $\mathbb{N}$-graded algebra $A = \bigoplus A_i$ generated in degree 1, recall that the fundamental defining relations for a Zhang twist are equivalent to the following ([Zha96, p. 284]): a twisting system in this case is a set $\tau = \{\tau_n \mid n \in \mathbb{N}\}$ of graded linear isomorphisms satisfying

$$\tau_m(ab) = \tau_m(a)\tau_{m+n}\tau_n^{-1}(b)$$

for $a \in A_n$.

Since A is assumed to be generated in degree 1, we can view the above as giving us a way to inductively construct twisting systems. Let τ_m^i denote $\tau_m|_{A_i} \colon A_i \to A_i$. By results of Zhang, without loss of generality τ_0 may be taken to be the identity, so we start by choosing $\tau_1^1 \colon A_1 \to A_1$([Zha96, Proposition 2.4]). This is equivalent to choosing an invertible matrix acting on a set of generators of the algebra.

Then since τ_1^2 will need to respect the relations of our algebra we may try to define τ_1^2 by solving to find τ_2^1 such that the twisting relation above is satisfied. If we can do this, we can then try to find τ_2^2, τ_3^1 and so on, building up τ piece by piece.

If we know more about our algebra, such as that A is quadratic, then we might expect this process to terminate and the higher components of the maps to be determined. Note though that we will need to specify τ_m^1 for all m.

Conversely, we may view the above process as a series of obstructions for a collection of linear maps to form a twisting system. A failure of the existence of solutions to the relevant equations at some point can indicate that a particular choice of τ_1^1 cannot lead to a twisting system, for example. That is, we can consider twisting as a type of deformation problem. In the case of noncommutative $\mathbb{P}^n$s corresponding to *geometric* algebras (see Definition 37), we can partially resolve this problem by computing certain twists via geometric data (see Theorem 42).

By writing down a matrix T representing the projection from the tensor algebra over A_1 to A, we can produce the following linear algebra computations corresponding to this process, and for dimension 1 actually solve these, as follows.

Example 33. We consider first $A = \mathcal{O}(\mathbb{P}^1) = k[x, y]$, with $\deg x = \deg y = 1$.

Let

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

whose kernel is spanned by $v = (0\ 1\ \ -1\ 0)^T$ corresponding to $x \otimes y - y \otimes x$, and

$$S = A \otimes BA^{-1} = \lambda^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \alpha d - \beta c & \beta a - \alpha b \\ \gamma d - \delta c & \delta a - \gamma b \end{pmatrix}$$

where $A = \tau_1^1$, $B = \tau_2^1$ and $\lambda = \det A$. Then A and B will extend to a twisting system precisely if $\ker TS \subseteq \ker T$: we may calculate and see that

$$TSv = 0 \qquad \Leftrightarrow \qquad A^2 \begin{pmatrix} \beta & \delta \\ -\alpha & -\gamma \end{pmatrix} = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} \qquad \Leftrightarrow \qquad \text{up to scaling, } B = A^2$$

By means of this computation, we see that all twists of the commutative polynomial ring are algebraic: they come from automorphisms as $\{\tau_m = f^m\}$.

Example 34. Consider next $A = \mathcal{O}_q(\mathbb{P}^1)$ (p.2), also with $\deg x = \deg y = 1$.

Let $q \in k^*$ and let

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

whose kernel is spanned by $v = (0\ 1\ \ -q\ 0)^T$ corresponding to $x \otimes y - qy \otimes x$, and

$$S = A \otimes BA^{-1} = \lambda^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \alpha d - \beta c & \beta a - \alpha b \\ \gamma d - \delta c & \delta a - \gamma b \end{pmatrix}$$

as before. Then A and B will extend to a twisting system precisely if $\ker TS \subset \ker T$: we may calculate and see that

$$TSv = 0 \qquad \Leftrightarrow \qquad A \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} A \begin{pmatrix} \beta & \delta \\ -\alpha & -\gamma \end{pmatrix} = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} \qquad \Leftrightarrow \qquad \text{up to scaling, } B = A \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} A.$$

That is, $S = \nu A \otimes A \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$. This ought not to be a surprise: $\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$ is precisely what we twist the polynomial ring $\mathcal{O}(\mathbb{P}^1) = k[x, y]$ by to get $\mathcal{O}_q(\mathbb{P}^1)$. So we see that $\mathcal{O}_q(\mathbb{P}^1)$ "has the same twists" as $\mathcal{O}(\mathbb{P}^1)$ - which we knew from a more theoretical analysis above - and we see this concretely, as a combination of a general twist of the polynomial ring and the particular twist taking us to the quantum plane.

However, two significant differences occur in higher dimensions, even dimension 2. Firstly, it is straightforward to show explicitly that the polynomial ring and the algebra $\mathcal{O}_q(\mathbb{P}^2)$ (often less formally called "quantum $\mathbb{P}^2$") are no longer twist-equivalent. Secondly, the dimension of A_1 and thus the size of the matrices T, S increases significantly (from $(1+1)^2 = 4$ to $(2+1)^2 = 9$) and the dimension of the kernel of T becomes 3 rather than just 1. So the above explicit linear-algebraic approach becomes less viable.

Fortunately in the case of $\mathbb{P}^2$ - that is, algebras that are AS-regular of dimension 3, so of global dimension 3 - we have the seminal work of Artin-Tate-Van den Bergh ([ATVdB90],[ATVdB91]) available to us, in which the authors introduced point schemes and associated concepts. In the next section, we will utilize this to analyse the groupoid $\mathscr{N}\mathscr{C}\left(\mathbb{P}^2\right)$.

5 Noncommutative $\mathbb{P}^2$s

In [ATVdB90] and [ATVdB91] the authors study AS-regular algebras A of dimension 3, among which are the noncommutative analogues of $\mathbb{P}^2$ we wish to consider. To such an algebra A they associate a triple $\mathscr{T} = (E, \sigma, \mathcal{L})$ where E is the point scheme of A (Definition 21), σ is an automorphism of E and $\mathcal{L} \stackrel{\text{def}}{=} j^*\mathcal{O}(1)$ where j is the embedding of E in $\mathbb{P}^2$.

Let r be the size of a minimal generating set for A. Let $\mathscr{T}$ be a triple associated to a regular algebra as above. In [ATVdB90, Theorem 2] it is shown that these triples satisfy

$$(\sigma - 1)^2 [\mathcal{L}] = 0 \quad \text{if } r = 3 \quad \text{and} \quad \left(\sigma^2 - 1\right)(\sigma - 1)[\mathcal{L}] = 0 \quad \text{if } r = 2$$

where $[\mathcal{L}]$ is the class of $\mathcal{L}$ in $\operatorname{Pic}(E)$ and such triples are also called regular. Let s be the degree of the defining relations of A.

For regular algebras of dimension 3 generated in degree 1, there are two possibilities: A is generated by three elements and has three quadratic relations; or A is generated by two elements and has two cubic relations. As noted in [AS87, p. 175], A has the same Hilbert series as the polynomial ring if and only if the algebra has three generators and is quadratic, so henceforth we will focus on this case, where $\mathcal{L}^{(\sigma-1)^2} \cong \mathcal{O}_E$ (here $\mathcal{L}^\rho$ denotes the pullback $\rho^*\mathcal{L}$ along an automorphism ρ).

In §8 of [ATVdB91] the authors define the notion of twisting a regular $\mathbb{Z}$-graded algebra A by a graded A-automorphism. This twisting process was later generalized by Zhang to that described above. They then define an action of graded A-automorphisms on the set of regular triples and demonstrate a correspondence between twists of a given algebra and images of its regular triple under the action of A-automorphisms. For completeness, we give a slight generalization of their argument ([ATVdB91, Proposition 8.9]) in the case $r = 3$ for Zhang twists.

Proposition 35. *Let A be a regular algebra of dimension three determined by the regular triple $\mathscr{T} = (E, \sigma, \mathcal{L})$ and let τ be a twisting system on A. By dualization and projectivization, there is an automorphism $\overline{\tau_1}^*$ of $\mathbb{P}(A_1)$ induced by τ_1 and $\overline{\tau_1}^*(E) = E$. Denote by τ_1^E the induced automorphism of E and let $\mathscr{T}_{\tau_1^E}$ be the triple $\left(E, \tau_1^E\sigma, \mathcal{L}\right)$. Then A^τ is the algebra determined (up to isomorphism) by $\mathscr{T}_{\tau_1^E\sigma}$.*

Proof: Since Zhang twisting preserves regularity, A^τ is regular. Let $\varphi = (1, \tau)$ and let $\Gamma' \stackrel{\text{def}}{=} \varphi(\Gamma(E))$. Since $\Gamma(E)$ is the locus of common zeroes of the multilinearized relations of A, $\left\{\widetilde{f_i}\right\}$, we have that Γ' is the locus of common zeroes of the multilinearized twisted relations $\left\{\widetilde{f_{\tau,i}}\right\}$ and hence, $\Gamma' = \Gamma(A^\tau)$.

Since $s = 2$, we have that $E = \mathrm{pr}_1\Gamma'$. Since A^τ is regular, Γ' defines an automorphism σ' of E. Hence, A^τ defines a triple $\mathscr{T}' = (E, \sigma', \mathcal{L})$. By analysing the construction, one can see that $\sigma' = \tau_1^E\sigma$ and so $\mathscr{T}' = \mathscr{T}_{\tau_1^E\sigma} = \left(E, \tau_1^E\sigma, \mathcal{L}\right)$. □

To proceed further, we will use a combination of algebraic and geometric methods, as were illustrated for $\mathscr{N}\mathscr{C}\left(\mathbb{P}^1\right)$. On the geometric side, we have the following theorem of Mori ([Mor06]); the version below has been adapted to the specific case at hand.

Theorem 36 (cf. [Mor06, Theorem 4.7])**.** *Let A and A' be coordinate rings of noncommutative $\mathbb{P}^2$s, with associated triples $\mathscr{T} = (E, \sigma, \mathcal{L})$ and $\mathscr{T}' = (E', \sigma', \mathcal{L}')$ respectively. Then A is twist-equivalent to A' if and only if there exists a sequence of automorphisms $\{\rho_m \mid m \in \mathbb{N}\} \subset \operatorname{Aut}\mathbb{P}^2$ such that, setting $\rho_m^E \stackrel{\text{def}}{=} \rho_m|_E$, we have $\rho_m^E(E) \cong E'$ and $\rho_{n+1}^E\sigma = \sigma'\rho_n^E$ for all $n \in \mathbb{N}$.*

Diagrammatically,

$$\begin{array}{ccccccccccc}
E & \xrightarrow{\sigma} & E & \xrightarrow{\sigma} & E & \xrightarrow{\sigma} & E & \xrightarrow{\sigma} & E & \xrightarrow{\sigma} & \\
\downarrow{\scriptstyle \rho_0^E} & & \downarrow{\scriptstyle \rho_1^E} & & \downarrow{\scriptstyle \rho_2^E} & & \downarrow{\scriptstyle \rho_3^E} & & \downarrow{\scriptstyle \rho_4^E} & & \\
E' & \xrightarrow[\sigma']{} & E' & \xrightarrow[\sigma']{} & E' & \xrightarrow[\sigma']{} & E' & \xrightarrow[\sigma']{} & E' & \xrightarrow[\sigma']{} &
\end{array}$$

Mori's original version of the result makes fewer assumptions on A and A', including that the algebras concerned are only $\mathbb{Z}$-graded and not necessarily $\mathbb{N}$-graded. By use of [Zha96, Proposition 2.8] for the $\mathbb{N}$-graded case and an examination of Mori's proof, one sees that for $\mathbb{N}$-graded algebras, a family $\{\rho_m \mid m \in \mathbb{N}\}$ is sufficient, rather than a family $\{\rho_m \mid m \in \mathbb{Z}\}$ as Mori requires.

Also, the conclusion of the theorem is expressed as $\operatorname{Gr} A$ being isomorphic to $\operatorname{Gr} A'$ if and only if $\{\rho_m \mid m \in \mathbb{N}\}$ exists with the stated properties. Of course, we have already seen that for coordinate rings of noncommutative projective spaces, this is the same as twist-equivalence. Indeed, Mori's proof is not categorical but goes via the existence of a Zhang twist, for which Mori has sufficiently many assumptions.

In particular, Mori's result works in higher dimensions: he gives a definition of a *geometric algebra*, this being a quadratic algebra that satisfies some conditions relating to the point scheme. For dimension 2, it is known that every coordinate ring of a noncommutative projective $\mathbb{P}^2$ is geometric (see the references given in [Mor06]).

It is natural to ask whether this is true for all noncommutative $\mathbb{P}^n$s with $n > 2$; given what is known of the rich landscape of algebras, with quite diverse examples, we expect that the answer is no in general. Specifically, a generic AS-regular algebra of dimension 4 has a finite point scheme, whereas geometricity asks that the point scheme is sufficiently large to encode the relations of the algebra (see [Van15, §1.4]).

However, Mori's results do show that if we have a geometric algebra then any twist of this is also geometric. So provided we know that one algebra in a given connected component of $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ is geometric, we immediately know this for all algebras in the component. For analysis of twists of some particular algebra, this can allow us to use Mori's results.

Staying with dimension 2, let us use Mori's result to analyse twist-equivalence for the 2-dimensional algebras analogous to $\mathcal{O}(\mathbb{P}^1) = k[x, y]$ and $\mathcal{O}_q(\mathbb{P}^1)$ that we treated in the previous section.

Consider the component of $\mathscr{N}\mathscr{C}\left(\mathbb{P}^2\right)$ containing $(\operatorname{QGr} A, \pi A)$ for $A = \mathcal{O}(\mathbb{P}^2)$, whose associated triple is $(\mathbb{P}^2, \mathrm{id}, \mathcal{O}(1))$. Any triple for an algebra twist-equivalent to A must necessarily have point scheme $\mathbb{P}^2$, so without loss of generality we may let A' be an algebra associated to the triple $(\mathbb{P}^2, \sigma, \mathcal{O}(1))$ for $\sigma \in \mathrm{PGL}_3(k)$ arbitrary.

In this situation, Mori's theorem may be rephrased (as in [Mor06, Remark 4.8]) to say that A and A' are twist-equivalent if and only if there exists $\rho_0 \in \mathrm{PGL}_3(k)$ such that

- $\rho_0(\mathbb{P}^2) = \mathbb{P}^2$ and
- $\rho_m \stackrel{\text{def}}{=} \sigma^m \rho_0(\mathrm{id})^{-m}$ can be extended to an automorphism of $\mathbb{P}^2$ for all $m \in \mathbb{N}$.

But $\rho_0 = \mathrm{id}$ clearly satisfies this, so we deduce that $(\operatorname{QGr} A', \pi A')$ is also in the connected component and $A' \cong A^\sigma$, the twist of A by the automorphism induced by σ.

Mori ([Mor06, Remark 4.9]) also shows that such A and A' are isomorphic if and only if $\{\rho_m\}$ can be taken to be constant, i.e. $\rho_m = \rho$ for all m. Then the condition $\rho_{m+1}\sigma = \sigma' \rho_m$ for all $m \in \mathbb{N}$ reduces to conjugacy:

$$A^\sigma \cong A^{\sigma'} \quad \Leftrightarrow \quad \exists\, \rho \text{ such that } \sigma' \rho = \rho \sigma \quad \Leftrightarrow \quad \sigma' \text{ is conjugate to } \sigma$$

This generalizes the earlier statement for noncommutative $\mathbb{P}^1$s, where there is only one component whose morphisms are controlled by $\mathrm{PGL}_2(k)$ conjugacy.

For every $(\operatorname{QGr} A', \pi A')$ in the commutative component of $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$, we have that $\operatorname{QGr} A' \simeq \operatorname{QCoh}(\mathbb{P}^n)$. Furthermore, by exactly the same argument as above for $\mathbb{P}^2$, every element of $\mathrm{PGL}_{n+1}(k)$ gives rise to a twist, and graded Morita equivalence is controlled by conjugacy in $\mathrm{PGL}_{n+1}(k)$.

We may use this method to analyse other components too. The following is an extended presentation of [Mor06, Example 4.10], which we include both for the reader's convenience, as an illustration of the method, and also in preparation for its extension later.

Consider the algebra $\mathcal{O}_Q(\mathbb{P}^2)$ for $Q = (\alpha, \beta, \gamma) \in (k^*)^3$ with presentation

$$\mathcal{O}_Q(\mathbb{P}^2) \stackrel{\text{def}}{=} k\langle x, y, z\rangle / \langle zy = \alpha yz, xz = \beta zx, yx = \gamma xy\rangle.$$

This is the natural multi-parameter generalization of $\mathcal{O}_q(\mathbb{P}^2) = \mathcal{O}_{(q^{-1},q,q^{-1})}(\mathbb{P}^2)$.

Provided $\alpha\beta\gamma \notin \{0, 1\}$, all algebras of the form $\mathcal{O}_Q(\mathbb{P}^2)$ have the same point scheme $E \subset \mathbb{P}^2$, namely the union of the three lines in $\mathbb{P}^2$ given by the vanishing of the coordinates x, y and z; let $l_1 = \mathbb{V}(x)$ etc. The associated $\sigma = \sigma(Q)$ is given by $\sigma(Q) = \mu(\alpha, \beta, \gamma)$ where

$$\begin{aligned} \mu|_{l_1}(0, b, c) &= (0, \alpha b, c) \\ \mu|_{l_2}(a, 0, c) &= (a, 0, \beta c) \\ \mu|_{l_3}(a, b, 0) &= (\gamma a, b, 0). \end{aligned}$$

One may easily check that such a $\mu(\alpha, \beta, \gamma)$ extends to all of $\mathbb{P}^2$ if and only if $\alpha\beta\gamma = 1$.

Furthermore, the subgroup of elements of $\mathrm{PGL}_3(k)$ that preserve this triangle E is isomorphic to $(k^*)^2 \rtimes S_3$, where the 2-torus action precisely comes from the action of maps of the above form $\mu(\alpha, \beta, \gamma)$ with $\alpha\beta\gamma = 1$ and the S_3 permutes the three lines.

In [Mor06, Example 4.10], Mori uses his theorem to give necessary and sufficient conditions for the twist-equivalence of A and A' associated to $(E, \sigma = \mu(\alpha, \beta, \gamma))$ and $(E, \sigma' = \mu(\alpha', \beta', \gamma'))$.

For example, take $\rho_0 = (1\ 2) \in S_3$. Then one may straightforwardly check that

$$(\sigma'\rho_0\sigma^{-1})|_{l_1}(0, b, c) = (\alpha^{-1}b, 0, \beta' c) = (b, 0, \alpha\beta' c) \in l_2$$

with similar expressions for $\sigma'\rho_0\sigma^{-1}$ on l_2 and l_3. From these, one sees that $\sigma'\rho_0\sigma^{-1} = \mu(\alpha'\beta, \alpha\beta', \gamma\gamma')\circ\rho_0$.

Now ρ_0 extends to $\mathbb{P}^2$ by definition, hence $\sigma'\rho_0\sigma^{-1}$ does if and only if $\mu(\alpha'\beta, \alpha\beta', \gamma\gamma')$ does, which happens if and only if $\alpha'\beta'\gamma' = (\alpha\beta\gamma)^{-1}$. Furthermore this condition suffices to ensure that $(\sigma')^m\rho_0(\sigma)^{-m}$ extends to $\mathbb{P}^2$ for all m and hence A and A' are twist-equivalent.

This phenomenon - of the sufficiency of one condition, independent of m, to establish twist-equivalence - can also be seen in other examples (Theorems 5.2 and 5.4 of [Mor06]) and we will return to this later.

Iterating through the other choices for ρ_0, one may show that $A = \mathcal{O}_{(\alpha,\beta,\gamma)}(\mathbb{P}^2)$ and $A' = \mathcal{O}_{(\alpha',\beta',\gamma')}(\mathbb{P}^2)$ are twist equivalent if and only if $\alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}$.

Recalling that $Q = (q^{-1}, q, q^{-1})$ gives the single parameter quantum $\mathbb{P}^2$, $\mathcal{O}_q(\mathbb{P}^2)$, we see as a special case that $\mathcal{O}_q(\mathbb{P}^2)$ is not twist equivalent to $\mathcal{O}_{q'}(\mathbb{P}^2)$ for all but one other choice of $q' \in k^*$:

$$((q')^{-1}) \cdot q' \cdot ((q')^{-1}) = (q^{-1}) \cdot q \cdot (q^{-1})^{\pm 1} \qquad \Leftrightarrow \qquad q' = q^{\pm 1}$$

This makes it very clear that $\mathscr{N}\mathscr{C}\left(\mathbb{P}^2\right)$ has very many connected components. It also shows that in the case of quantum algebras defined with respect to a parameter $q \in k^*$ (or indeed, multiple parameters) varying q does not give rise to twist equivalences. So while twist equivalence has some features of a deformation-theoretic problem, it is too strong and somewhat too discrete to be related to smooth variation of deformation parameters and the associated geometry of the latter.

This completely answers the question of when two algebras of the form $\mathcal{O}_Q(\mathbb{P}^2)$, that is, multi-parameter quantum $\mathbb{P}^2$s, have associated objects in $\mathscr{N}\mathscr{C}\left(\mathbb{P}^2\right)$ in the same connected component. However, more work is needed to completely describe these components in full: they contain objects associated to algebras A not of the form $\mathcal{O}_Q(\mathbb{P}^2)$.

The above example shows that there exist non-trivial twists of the standard quantum $\mathbb{P}^2$, $\mathcal{O}_q(\mathbb{P}^2)$, induced by automorphisms of the point scheme and associated to elements of S_3, which is not a subgroup of $\mathrm{Aut}(\mathcal{O}_q(\mathbb{P}^2)) = k \ltimes (k^*)^3$ (see Theorem 4, recalled earlier). That is, we have recovered some classical automorphisms.

6 Geometric noncommutative $\mathbb{P}^n$s

To proceed further, we use a careful analysis of Mori's proof of the above theorem to generalize Proposition 35. First we give the precise definition of a geometric algebra.

Definition 37 ([Mor06, Definition 4.3])**.** A quadratic algebra $A = T(V)/R$ is called geometric if there is a pair (E, σ) where $j: E \to \mathbb{P}(V^*)$ is an embedding of E as a closed k-subscheme and σ is a k-automorphism of E such that

1. $\Gamma_2 = \mathcal{V}(R) \subset \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ is the graph of E under σ, and
2. setting $\mathcal{L} = j^*\mathcal{O}_{\mathbb{P}(V^*)}(1)$, the map

$$\mu: H^0(E, \mathcal{L}) \otimes H^0(E, \mathcal{L}) \to H^0(E, \mathcal{L} \otimes_{\mathcal{O}_E} \sigma^*\mathcal{L})$$

defined by $v \otimes w \mapsto v \otimes (w \circ \sigma)$ has $\ker \mu = R$, with the identification

$$H^0(E, \mathcal{L}) = H^0(\mathbb{P}(V^*), \mathcal{O}(\mathbb{P}(V^*)(1)) = V$$

as k-vector spaces.

If A is geometric as above, we associate the triple $(E, \sigma, \mathcal{L})$ to both A and the pair $\mathrm{QGr}\, A, \pi A)$.

Then we may give Mori's original theorem in its greater generality (with some adjustments in notation to fit with our own).

Theorem 38 ([Mor06, Theorem 4.7])**.** *Let $A = T(V)/I$ and $A' = T(V)/I'$ be graded algebras finitely generated in degree 1 over k.*

1. *If $A = \mathcal{A}(E, \sigma)$ is geometric and $\mathrm{Gr}\, A \cong \mathrm{Gr}\, A'$ then $A' = \mathcal{A}(E', \sigma')$ is also geometric and there is a sequence of automorphisms $\{\tau_n^*\}$ of $\mathbb{P}(V^*)$, each of which sends E isomorphically onto E' such that $(\tau_{n+1}^*|_E)\sigma = \sigma'(\tau_n^*|_E)$ for every $n \in \mathbb{Z}$.*
2. *Conversely if $A = \mathcal{A}(E, \sigma)$ and $A' = \mathcal{A}(E', \sigma')$ are geometric and there is a sequence of automorphisms $\{\tau_n^*\}$ of $\mathbb{P}(V^*)$, each of which sends E isomorphically onto E' such that $(\tau_{n+1}^*|_E)\sigma = \sigma'(\tau_n^*|_E)$ for every $n \in \mathbb{Z}$, then $\mathrm{Gr}\, A \cong \mathrm{Gr}\, A'$.*

We note that if A, A' are $\mathbb{N}$-graded, the index n may be taken to range over $\mathbb{N}$ rather than $\mathbb{Z}$, with no change to the conclusions.

We will insert the adjective "geometric" into our terminology for noncommutative projective spaces in the natural way, referring to $(\mathrm{QGr}\, A, \pi A)$ as a geometric noncommutative $\mathbb{P}^n$ if $(\mathrm{QGr}\, A, \pi A)$ is a noncommutative $\mathbb{P}^n$ for a geometric algebra A, and similarly refer to such an A as a coordinate ring for a geometric noncommutative $\mathbb{P}^n$.

Proposition 39. *Let A be a homogeneous coordinate ring for a geometric noncommutative $\mathbb{P}^n$ and let $(E, \sigma, \mathcal{L})$ be its associated (regular) triple. Let τ be a twisting system for A.*

By dualization and projectivization, there is an automorphism $\overline{\tau_1}^*$ of $\mathbb{P}(A_1^*)$ induced by τ_1 and $\overline{\tau_1}^*(E) = E$. Denote by τ_1^E the induced automorphism of E and let $\mathscr{T}_{\tau_1^E\sigma}$ be the triple $\left(E, \tau_1^E\sigma, \mathcal{L}\right)$. Then A^τ is the algebra determined (up to isomorphism) by $\mathscr{T}_{\tau_1^E\sigma}$.

That is, by comparison with Proposition 35, we are able to remove the assumption on the dimension at the expense of requiring geometricity.

Proof: By Theorem 38, A^τ is geometric and has an associated triple $(E', \sigma', \mathcal{L}')$. Now since A^τ is a twist of A, there are graded linear isomorphisms $\phi_n: A^\tau \to A$ such that $\phi_n(a *_{A^\tau} b) = \phi_n(a) *_A \phi_{n+l}(b)$ for $a \in A_l^\tau$, where $*_{A^\tau}$ and $*_A$ are the multiplications in A^τ and A respectively. These graded linear isomorphisms are obtained via [Zha96, Proposition 2.8(1)] and a careful examination of the proof of that result - which is stated for an algebra B *isomorphic to* a twist of A - shows that if B is *equal* to a twist of A, i.e. we take $B = A^\tau$ in Zhang's proposition, then in fact $\phi_n = \tau_n$. (In general, $\phi_n = \tau_n \circ f$ for $f: B \to A^\tau$.)

Consequently, in Mori's theorem (38), the key to the proof of the first part is the dualization of the maps $\tau_n|_{A_1}$, namely $\overline{\tau_n}^*: \mathbb{P}(A_1^*) \to \mathbb{P}(A_1^*)$. (Since the $\tau_n|_{A_1}$ are isomorphisms, it follows that the maps $(\tau_n|_{A_1})^*$ are also isomorphisms and hence, being injective, they descend to the projectivization of A_1^*.)

Then Mori shows that $E' = (\overline{\tau_0}^*|_E)(E)$ and $\sigma' = (\overline{\tau_1}^*|_E) \circ \sigma \circ (\overline{\tau_0}^*|_E)^{-1}$. As above, set $\tau_1^E = \overline{\tau_1}^*|_E$.

In principle, τ_0 can be any graded k-linear automorphism of A and then these claims would be best possible. However, by [Zha96, Proposition 2.4], we may if necessary - with an acceptable loss of generality - replace τ by a twisting system τ' for which $\tau'_0 = \mathrm{id}$, and have $A^\tau \cong A^{\tau'}$.

Doing this, and bearing in mind that we have potentially introduced a "hidden" isomorphism in the course of doing so, we conclude that the triples $(E', \sigma', \mathcal{L}')$ and $(E, \tau_1^E \circ \sigma, \mathcal{L})$, with $\tau_1^E \in \operatorname{Aut} E$, yield isomorphic algebras, one of these being A^τ. $\square$

Remark 40. Mori's theorem also tells us how to construct the full twisting system. For as in [Mor06, Remark 4.8], the condition $(\overline{\tau_{n+1}}^*|_E)\sigma = \sigma'(\overline{\tau_n}^*|_E)$ implies that given $\overline{\tau_1}^*$, we may inductively define

$$\overline{\tau_n}^* \stackrel{\text{def}}{=} ((\sigma')^n \sigma^{-n})^* = ((\tau_1^E \sigma)^n \sigma^{-n})^*$$

and dualize and extend these from automorphisms of $\mathbb{P}(A_1^*)$ to automorphisms of $T(A_1)$, which will define a twisting system τ on A.

We note two consequences. The first is that the twisting system τ obtained is completely determined by τ_1, corresponding to A being generated in degree 1.

Secondly, if τ_1^E commutes with σ then the twisting system τ is algebraic: $\overline{\tau_n}^* = (\tau_1^E)^n$ and hence $\tau_n = \tau_1^n$ for all n. As in [ATVdB91, Proposition 8.8] this is precisely the situation of A^τ being a twist by an automorphism. Conversely, non-algebraic twists arise when we can find some τ_1^E that does not commute with σ in $\operatorname{Aut}(E)$.

Let E be a closed subscheme of $\mathbb{P}^n$ and define the following two subgroups:

$$\operatorname{Aut}(E \uparrow \mathbb{P}^n) \stackrel{\text{def}}{=} \{\sigma \in \operatorname{Aut} E \mid (\exists \hat{\sigma} \in \operatorname{Aut} \mathbb{P}^n)(\hat{\sigma}|_E = \sigma)\} \subseteq \operatorname{Aut} E,$$
$$\operatorname{Aut}(\mathbb{P}^n \downarrow E) \stackrel{\text{def}}{=} \{\rho \in \operatorname{Aut} \mathbb{P}^n \mid \rho(E) = E\} \subseteq \operatorname{Aut} \mathbb{P}^n.$$

Restriction defines a surjective group homomorphism $\operatorname{Res}_E \colon \operatorname{Aut}(\mathbb{P}^n \downarrow E) \to \operatorname{Aut}(E \uparrow \mathbb{P}^n)$.

We may then interpret the previous proposition as follows. Given a geometric noncommutative $\mathbb{P}^n$ with associated coordinate ring A, we have a triple $(E, \sigma, \mathcal{L})$ with $\sigma \in \operatorname{Aut} E$. If τ is a twisting system for A, then $\tau_1^E = \operatorname{Res}_E(\overline{\tau_1}^*) \in \operatorname{Aut}(E \uparrow \mathbb{P}^n)$ is such that $(E, \tau_1^E \sigma, \mathcal{L})$ determines A^τ up to isomorphism.

Furthermore, by symmetry, every twist is of this form.

So we conclude that

Corollary 41. *Let A be a coordinate ring of a geometric noncommutative $\mathbb{P}^n$ with associated triple $(E, \sigma, \mathcal{L})$. The isomorphism classes of algebras twist-equivalent to A are in bijection with the elements of the coset* $\operatorname{Aut}(E \uparrow \mathbb{P}^n)\sigma \subset \operatorname{Aut} E$. $\square$

The coset $\operatorname{Aut}(E \uparrow \mathbb{P}^n)\sigma$ is not a group, unless the identity of $\operatorname{Aut} E$ is contained in $\operatorname{Aut}(E \uparrow \mathbb{P}^n)\sigma$, in which case we must in fact have $E = \mathbb{P}^n$, $\operatorname{Aut}(E \uparrow \mathbb{P}^n) = \operatorname{Aut} E = \operatorname{Aut} \mathbb{P}^n = \operatorname{Aut}(\mathbb{P}^n \downarrow E)$ and A twist-equivalent to $\mathcal{O}(\mathbb{P}^n)$.

Note that there is a bijection between any two cosets of $\operatorname{Aut}(E \uparrow \mathbb{P}^n)$ so that any two algebras having the same point scheme E, twist-equivalent or not, can be viewed as having the same number of twists.

The group $\operatorname{Aut}(E \uparrow \mathbb{P}^n)$ has a natural left action on $\operatorname{Aut}(E \uparrow \mathbb{P}^n)\sigma$, giving rise to a groupoid structure on $\operatorname{Aut}(E \uparrow \mathbb{P}^n)\sigma$ by defining the set of objects to be the elements of $\operatorname{Aut}(E \uparrow \mathbb{P}^n)\sigma$ and the morphisms to be pairs $(\tau', \tau\sigma) \colon \tau\sigma \to \tau'\tau\sigma$. This is precisely the action groupoid $\operatorname{Aut}(E \uparrow \mathbb{P}^n)\sigma \mathbin{/\!/} \operatorname{Aut}(E \uparrow \mathbb{P}^n)$, also known as the weak quotient.

It is well-known that the action groupoid gives rise to a faithful functor

$$\mathcal{G} \colon \operatorname{Aut}(E \uparrow \mathbb{P}^n)\sigma \mathbin{/\!/} \operatorname{Aut}(E \uparrow \mathbb{P}^n) \to \operatorname{Aut}(E \uparrow \mathbb{P}^n),$$

the latter being the group $\operatorname{Aut}(E \uparrow \mathbb{P}^n)$ considered as a one-object groupoid. The functor $\mathcal{G}$ is given by sending every object to the unique object of $\operatorname{Aut}(E \uparrow \mathbb{P}^n)$ and each morphism $(\tau', \tau\sigma)$ to τ'. In the instance at hand, with the action groupoid arising from the left action of a group on a coset, we have a transitive action, or equivalently a connected groupoid. Then $\mathcal{G}$ is the functor giving rise to the equivalence of a connected groupoid with a group.

That is, while it is conceptually helpful to consider $\operatorname{Aut}(E \uparrow \mathbb{P}^n)\sigma \mathbin{/\!/} \operatorname{Aut}(E \uparrow \mathbb{P}^n)$, as it keeps the role of σ explicit, this groupoid is in fact equivalent to the group $\operatorname{Aut}(E \uparrow \mathbb{P}^n)$.

We fix some $\mathcal{A} = (\mathrm{QGr}\, A, \pi A)$ with A having associated triple $(E, \sigma, \mathcal{L})$. We may now define a functor $\mathcal{F}\colon \mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A} \to \mathrm{Aut}(E \uparrow \mathbb{P}^n)\sigma$, by $\mathcal{F}(\mathcal{B}) = \sigma'$ for $(E, \sigma', \mathcal{L})$ the triple associated to B. We may see by examining the definitions of morphisms in the two categories that given a twisting system τ, we have the associated map $\tau_1^E \sigma = \sigma'$, and by the proposition (asserting that twists precisely correspond to $\sigma \mapsto \tau_1^E \sigma$), this is functorial.

However while this functor $\mathcal{F}$ is both full and essentially surjective, it not an equivalence: it fails to be faithful because we may have graded Morita equivalences in $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$, which give rise to isomorphic twists that the triples cannot detect. We saw this when we noted that we were suppressing some isomorphisms in the proof of Proposition 39 above - indeed, precisely those coming from graded Morita equivalences.

Since $\mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A}$ is a connected groupoid, it too is equivalent to a group, which we denote $\underline{\mathrm{Aut}}(\mathcal{A})$. Note that this group is isomorphic to the same such group for any object in the connected component of $\mathcal{A}$. Via $\mathcal{G}$, it follows that $\mathcal{F}$ induces a surjection $\underline{\mathrm{Aut}}(\mathcal{A}) \twoheadrightarrow \mathrm{Aut}(E \uparrow \mathbb{P}^n)$.

From this, we can obtain an isomorphism by first taking the quotient of $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ by graded Morita equivalence, which we denote $\overline{\mathscr{N}\mathscr{C}(\mathbb{P}^n)}$. The corresponding functors $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ then induce an equivalence $\overline{\mathscr{N}\mathscr{C}(\mathbb{P}^n)}/\overline{\mathcal{A}} \simeq \mathrm{Aut}(E \uparrow \mathbb{P}^n)$.

Let us denote by $\underline{\underline{\mathrm{Aut}}}(\mathcal{A})$ the group obtained from the connected groupoid $\overline{\mathscr{N}\mathscr{C}(\mathbb{P}^n)}/\overline{\mathcal{A}}$ which is, by definition, the automorphism group of $\overline{\mathcal{A}}$ in $\overline{\mathscr{N}\mathscr{C}(\mathbb{P}^n)}/\overline{\mathcal{A}}$.

This gives our main result, which identifies the Zhang twists of homogeneous coordinate rings of noncommutative $\mathbb{P}^n$s (up to graded Morita equivalence) with the automorphisms of the associated point scheme that extend to $\mathbb{P}^n$.

Theorem 42. *Let $\mathcal{A} = (\mathrm{QGr}\, A, \pi A)$ be a geometric noncommutative $\mathbb{P}^n$ with homogeneous coordinate ring A and let $(E, \sigma, \mathcal{L})$ be its associated triple. Then*

$$\underline{\underline{\mathrm{Aut}}}(\mathcal{A}) \cong \mathrm{Aut}(E \uparrow \mathbb{P}^n).$$

That is, the automorphism group $\underline{\underline{\mathrm{Aut}}}(\mathcal{A})$ of $\overline{\mathcal{A}}$ in the connected component $\overline{\mathscr{N}\mathscr{C}(\mathbb{P}^n)}/\overline{\mathcal{A}}$ containing the equivalence class of $\mathcal{A}$ of the groupoid of noncommutative projective spaces up to graded Morita equivalence, $\overline{\mathscr{N}\mathscr{C}(\mathbb{P}^n)}$, is isomorphic to the group $\mathrm{Aut}(E \uparrow \mathbb{P}^n)$ of automorphisms of the point scheme E of A that extend to $\mathbb{P}^n$. □

We reiterate that the group $\mathrm{Aut}(E \uparrow \mathbb{P}^n)$ and hence the group $\underline{\underline{\mathrm{Aut}}}(\mathcal{A})$ depends only on E, so that any two components of $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$ with the same associated point schemes have the same twists.

Also note that the group on the left hand side is defined entirely algebraically, whereas the group on the right hand side is defined entirely geometrically.

Remark 43. Some care needs to be taken with respect to twists by graded algebra automorphisms of A, $\mathrm{Aut}_{\mathrm{gr}}(A)$. As we saw in Proposition 31, $\mathrm{Aut}_{\mathrm{gr}}(A)$ embeds in $\mathrm{Iso}(\mathscr{N}\mathscr{C}(\mathbb{P}^n)/\mathcal{A})$, but this does not imply that $\mathrm{Aut}_{\mathrm{gr}}(A)$ embeds in $\underline{\underline{\mathrm{Aut}}}(\mathcal{A})$. To form the latter group, we take a quotient by graded Morita equivalences, or equivalently by twists yielding isomorphic algebras. A twist by a graded algebra automorphism may or may not yield an isomorphic algebra, so that in general we only know that a quotient of $\mathrm{Aut}_{\mathrm{gr}}(A)$ embeds into $\underline{\underline{\mathrm{Aut}}}(\mathcal{A})$ (this quotient possibly being the trivial group).

Thus, Theorem 42 does not necessarily give any information about the graded automorphism groups of homogeneous coordinate rings of geometric noncommutative $\mathbb{P}^n$s.

Zhang twists of a given algebra are rarely isomorphic to one another. The group $\underline{\underline{\mathrm{Aut}}}(\mathcal{A})$ characterizes the generalized automorphisms arising as morphisms in the groupoid $\mathscr{N}\mathscr{C}(\mathbb{P}^n)$, and not those coming from isomorphisms of the algebra A itself. That is, $\underline{\underline{\mathrm{Aut}}}(\mathcal{A})$ describes the non-algebraic twists of A, without containing the classical automorphism group $\mathrm{Aut}(A)$ as a subgroup.

We conclude with two examples, the first being $\mathcal{O}_q(\mathbb{P}^n)$, generalizing Example 34 and the discussion in Section 5. The second example concerns 3-dimensional Sklyanin algebras, whose point schemes are smooth elliptic curves.

Example 44. Recall that we define quantum projective n-space to be the k-algebra

$$\mathcal{O}_q(\mathbb{P}^n) = k\langle x_0, \ldots, x_n \rangle / \langle x_i x_j = q x_j x_i \ \forall\ i < j \rangle$$

where $q \in k^*$ is assumed not a root of unity.

Define $\begin{bmatrix} n+1 \\ 2 \end{bmatrix} = \{I \subseteq \{0, \dots, n\} \mid |I| = 2\}$. By work of De Laet-Le Bruyn[1] ([DLLB15, Proposition 1]), the point scheme of $\mathcal{O}_q(\mathbb{P}^n)$ is the union of the lines

$$\ell_I \stackrel{\text{def}}{=} \mathbb{V}(\{x_j \mid j \notin I\}), \text{ for } I \in \begin{bmatrix} n+1 \\ 2 \end{bmatrix}$$

in $\mathbb{P}^n$. Set $E = \bigcup_{I \in \begin{bmatrix} n+1 \\ 2 \end{bmatrix}} \ell_I$.

The associated automorphism of E is defined on each line as

$$\sigma|_{\ell_{\{i_1,i_2\}}}(0 : \cdots : 0 : p_{i_1} : 0 : \cdots : 0 : p_{i_2} : 0 : \cdots : 0) = (0 : \cdots : 0 : p_{i_1} : 0 : \cdots : 0 : qp_{i_2} : 0 : \cdots : 0),$$

corresponding to the relator $x_{i_1} \otimes x_{i_2} - qx_{i_2} \otimes x_{i_1}$ via the correspondence indicated in the definition of geometricity (37). Indeed, the existence of such data (E, σ) affirms that $\mathcal{O}_q(\mathbb{P}^n)$ is geometric.

This automorphism does not extend to all of $\mathbb{P}^n$. Indeed, let $\underline{\lambda} = (\lambda_I)_{I \in \begin{bmatrix} n+1 \\ 2 \end{bmatrix}} \in (k^*)^{\begin{bmatrix} n+1 \\ 2 \end{bmatrix}}$ and define $\mu(\underline{\lambda}) \in \operatorname{Aut} E$ by

$$\mu(\underline{\lambda})|_{\ell_{\{i_1,i_2\}}}((0 : \cdots : 0 : p_{i_1} : 0 : \cdots : 0 : p_{i_2} : 0 : \cdots : 0) = (0 : \cdots : 0 : \lambda_{\{i_1,i_2\}} p_{i_1} : 0 : \cdots : 0 : p_{i_2} : 0 : \cdots : 0).$$

Then one may check that a purported extension of $\mu(\underline{\lambda})$ to $\mathbb{P}^n$ would have to be represented by the image of the diagonal matrix

$$\operatorname{diag}(\prod_{k=0}^{n-1} \lambda_{k(k+1)}, \dots, \prod_{k=i}^{n-1} \lambda_{k(k+1)}, \dots, \lambda_{(n-1)n}, 1)$$

in $\mathrm{PGL}_{n+1}(k)$ subject to the conditions

$$\lambda_{i_1,i_2} = \prod_{k=i_1}^{i_2-1} \lambda_{k(k+1)}$$

for all $i_1 < i_2$. Now $\sigma = \mu(\underline{\lambda})$ with $\lambda_{i_1,i_2} = q^{-1}$ for all i_1, i_2, which therefore does not extend to $\mathbb{P}^n$ since q is not a root of unity.

Noting that any element ν of $\mathrm{PGL}_{n+1}(k)$ preserving $E \subseteq \mathbb{P}^n$ must be projectively linear, send each line ℓ_I to another line ℓ_J and each intersection $e_i \stackrel{\text{def}}{=} \ell_{i,i'} \cap \ell_{i,i''}$ to another such intersection, we see that ν is determined by its values on the intersections e_i, $i \in \{0, \dots, n\}$. From a consideration of the remaining possibilities, it follows that

$$\operatorname{Aut}(\mathbb{P}^n \downarrow E) \cong \operatorname{Aut}(E \uparrow \mathbb{P}^n) \cong S_{n+1} \ltimes ((k^*)^{n+1}/k^*)$$

with S_{n+1} acting on $(k^*)^{n+1}$ by the natural permutation action $\triangleright$: that is, for $\sigma \in S_{n+1}$, $\mu \in (k^*)^{n+1}/k^*$ and $\underline{p} \in \mathbb{P}^n$ we have $(\sigma \triangleright \mu)(\underline{p}) = (\sigma \circ \mu \circ \sigma^{-1})(\underline{p})$.

We therefore conclude that the automorphism group $\underline{\underline{\operatorname{Aut}}}(\operatorname{QGr} \mathcal{O}_q(\mathbb{P}^n), \pi\mathcal{O}_q(\mathbb{P}^n))$ is isomorphic to the group $\operatorname{Aut}(E \uparrow \mathbb{P}^n) \cong S_{n+1} \ltimes ((k^*)^{n+1}/k^*)$ of automorphisms of the point scheme $E = \bigcup_{I \in \begin{bmatrix} n+1 \\ 2 \end{bmatrix}} \ell_I$ that extend to $\mathbb{P}^n$.

Notice in particular that elements of $(S_{n+1}, \underline{0}) \subseteq S_{n+1} \ltimes ((k^*)^{n+1}/k^*)$ do not induce graded algebra automorphisms of $\mathcal{O}_q(\mathbb{P}^n)$ and therefore correspond to non-algebraic twists.

For example, the transposition $(0\,1)$ gives rise to the triple $(E, (0\,1)\sigma, \mathcal{L})$ and we can compute the relations in its associated algebra as per Definition 37. For example, on ℓ_{01}, $(0\,1)\sigma$ maps $(p_0 : p_1 : 0 : \cdots : 0)$ to $(qp_1 : p_0 : 0 : \cdots : 0)$, for which

$$(x_0 \otimes x_0 - qx_1 \otimes x_1)((p_0 : p_1 : 0 : \cdots : 0), (qp_1 : p_0 : 0 : \cdots : 0)) = p_0(qp_1) - qp_1p_0 = 0,$$

leading to the relation $x_0^2 = qx_1^2$.

Continuing in this way yields the algebra

$$k\langle x_0, \dots, x_n \rangle / \langle x_0^2 = qx_1^2,\ x_0x_j = qx_jx_1,\ x_ix_j = qx_jx_i\ \forall\ 1 \le i < j \le n \rangle.$$

[1] See also Belmans-De Laet-Le Bruyn, [BDLLB16].

Example 45. Let k be of characteristic not equal to 2 or 3. Consider the 3-dimensional Sklyanin algebras

$$\mathrm{Skl}_3(a,b,c) \stackrel{\mathrm{def}}{=} k\langle x_0,x_1,x_2\rangle/\langle ax_ix_{i+1}+bx_{i+1}x_i+cx_{i+2}^2\rangle$$

where all indices are taken modulo 3. For all but a known finite set of points (a,b,c), this algebra is a geometric noncommutative $\mathbb{P}^2$ with associated point scheme the smooth elliptic curve

$$E\colon abc(x^3+y^3+z^3)-(a^3+b^3+c^3)xyz=0$$

and automorphism σ given by translation by a point (with respect to the group law on E).

A summary of relevant results and references concerning the 3-dimensional Sklyanin algebras (and their relationship with mathematical physics) may be found in work of Walton ([Wal12]). For completeness, we also note that De Laet ([DL17]) has identified an action by graded algebra automorphisms of the Heisenberg group H_3 of order 27 on Sklyanin algebras.

By general results on elliptic curves (see for example, [Sil09, §III]), every automorphism of E is the composition of a translation and an isogeny of the curve with itself. Thus $\mathrm{Aut}(E)$ is the semidirect product of the curve itself with the auto-isogeny group, the isogenies preserving a chosen base point and the translations changing the base point.

The auto-isogeny groups of elliptic curves are well-known: depending on the j-invariant of the curve, the auto-isogeny group is cyclic of order $n=2$, 4 or 6, realized as μ_n acting as $[\zeta](x:y:z)=(\zeta^2x:\zeta^3y:z)$ (cf. [Sil09, Corollary III.10.2]). It is common to call the auto-isogeny group $\mathrm{Aut}(E)$, but we shall not; rather we will denote it $\mathcal{I}(E)$. Note that the aforementioned action is projectively linear and hence $\mathcal{I}(E)\subseteq \mathrm{Aut}(E\uparrow\mathbb{P}^2)$.

Furthermore, it is known when a translation extends to an automorphism of the ambient $\mathbb{P}^2$. This is given explicitly in [Mor06, Lemma 5.3], where it is shown that a translation τ_p by a point p extends to $\mathbb{P}^2$ if and only if p is 3-torsion, i.e. τ_p has order 3. The group $E[3]$ of 3-torsion elements of an elliptic curve E is also known: it is isomorphic to $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$, a group of order 9 ([Sil09, Corollary III.6.4]).

Combining these results, we conclude that $\mathrm{Aut}(E\uparrow\mathbb{P}^2)\cong(\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z})\rtimes\mathcal{I}(E)$, a finite group of order $9n$. One may interpret this as saying that Sklyanin algebras are extremely rigid, or very noncommutative, as they have very few twists.

Note that the auto-isogenies act as graded algebra automorphisms of $\mathrm{Skl}_3(a,b,c)$; indeed $\mathrm{Skl}_3(a,b,c)^{[\zeta]}=\mathrm{Skl}_3(\zeta^2a,\zeta^3b,c)$. Furthermore, for a point p, $\tau_p\sigma$ is again a translation by a point, so that twists of Sklyanin algebras by translations are again Skylanin algebras. Thus, we see that $\mathscr{N}\mathscr{C}(\mathbb{P}^2)$ contains countably many connected components which, modulo graded Morita equivalence, are finite and consist of Sklyanin algebras.

Appendix

A An alternative approach to analysing noncommutative $\mathbb{P}^2$s

As explained in the introduction to [ATVdB91] and explored in more detail in §7 of that work, there is another approach to analysing noncommutative $\mathbb{P}^2$s that one can take, by studying the "open complement" of the "closed subscheme" defined by the point scheme of a noncommutative $\mathbb{P}^2$. We briefly recall the general construction and then give some more details in the specific case of $\mathcal{O}_Q(\mathbb{P}^2)$. We will not prove the assertions we make: proofs may be found in [ATVdB91].

Given a homogeneous coordinate ring of a noncommutative $\mathbb{P}^2$, A say, we have seen the construction of its point scheme E, parameterizing point modules in $\mathrm{QGr}\,A$, and the associated triple $(E,\sigma,\mathcal{L}=\mathcal{O}_E(1))$. The coordinate ring of E is in fact given by a quotient of A by a unique (up to scalar) normal regular degree 3 element $g\in A_3$. Specifically, $B=A/gA$ is a twisted homogeneous coordinate ring $B=\bigoplus_{n\geq 0}\mathrm{Hom}(\mathcal{O}_E,\mathcal{L}\otimes\mathcal{L}^\sigma\otimes\cdots\otimes\mathcal{L}^{\sigma^{n-1}})$.

Furthermore, $\mathrm{QGr}\,B\simeq\mathrm{QCoh}\,E$ and g annihilates point modules in $\mathrm{QGr}\,A$. Indeed, via the surjection $A\twoheadrightarrow B=A/gA$, we have $\mathrm{QGr}\,B\subset\mathrm{QGr}\,A$. By a natural extension of the usual terminology, we regard $(\mathrm{QGr}\,B,\pi B)$ as defining a closed subscheme of $(\mathrm{QGr}\,A,\pi A)$.

It is then also natural to ask about the open complement. This is given by the degree zero part Λ_0 of the graded localized algebra $\Lambda = A[g^{-1}]$.

One has a natural functor $\operatorname{Gr} A \to \operatorname{Gr} \Lambda_0$, given by localization: $M \mapsto M[g^{-1}]_0$. Furthermore, the category of finite-dimensional Λ_0-modules is equivalent to the category of g-torsion-free normalized A-modules (see [ATVdB91, p.371] for details).

Then the dichotomy between g-torsion and g-torsion-free A-modules corresponds to the complementarity of the closed subscheme and its open complement.

Lastly, we note that it follows from [Zha96, Proposition 5.4] that one may directly generalize [ATVdB91, Proposition 8.12], to show that if A^τ is a twist of A then $\Lambda_0 = A[g^{-1}]_0 \cong (A^\tau)[g_\tau^{-1}]_0$. That is, twisting is compatible with the decomposition of $\operatorname{QGr} A$ into the closed subscheme (corresponding to the point scheme) and its open complement (given by Λ_0).

For the case of $A = \mathcal{O}_Q(\mathbb{P}^2)$, we may calculate Λ_0 explicitly, as follows.

Recall that we have

$$\mathcal{O}_Q(\mathbb{P}^2) \stackrel{\text{def}}{=} k\langle x, y, z\rangle / \langle zy = \alpha yz, xz = \beta zx, yx = \gamma xy\rangle$$

so that $g = (1 - \alpha\beta\gamma)xyz$ is the canonical normalizing element described above. As previously, we assume that $\alpha\beta\gamma \notin \{0, 1\}$.

From the quasi-commutation relations in $\mathcal{O}_Q(\mathbb{P}^2)$, it is straightforward to verify the following relations in Λ:

$$\begin{aligned} g x^i y^j z^k &= \alpha^{i-j} \beta^{i-k} \gamma^{j-k} x^i y^j z^k g \\ g^{-1} x^i y^j z^k &= \alpha^{j-i} \beta^{k-i} \gamma^{k-j} x^i y^j z^k g^{-1} \\ (x^i y^j z^k g^{-1})(x^l y^m z^n g^{-1}) &= \alpha^{-\left|\begin{smallmatrix} i-1 & j-1 \\ l-1 & m-1 \end{smallmatrix}\right|} \beta^{-\left|\begin{smallmatrix} i-1 & k-1 \\ l-1 & n-1 \end{smallmatrix}\right|} \gamma^{-\left|\begin{smallmatrix} j-1 & k-1 \\ m-1 & n-1 \end{smallmatrix}\right|} (x^l y^m z^n g^{-1})(x^i y^j z^k g^{-1}) \end{aligned}$$

Indeed, we note that computationally it is easiest to work in the fully-localized algebra $A[x^{-1}, y^{-1}, z^{-1}]$, into which Λ naturally embeds.

A generating set for Λ_0 is given by the cubic monomials in x, y and z that span A_3, multiplied by the inverse of g:

$$\{x^3 g^{-1}, y^3 g^{-1}, z^3 g^{-1}, x^2 y g^{-1}, x^2 z g^{-1}, y^2 z g^{-1}, x y^2 g^{-1}, x z^2 g^{-1}, y z^2 g^{-1}, xyz g^{-1} = (1 - \alpha\beta\gamma)^{-1}\}.$$

From the above relations, we see that every pair of these satisfies a quasi-commutation relation and since $\mathcal{O}_Q(\mathbb{P}^2)$ is quadratic, we deduce that Λ_0 is in this case isomorphic to what is commonly referred to in the literature as a quantum polynomial algebra on nine generators. This contrasts starkly with the commutative case, where $g = 0$, $E = \mathbb{P}^2$, $B = A$ and there is an empty open complement.

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