TORSION FREE ENDOTRIVIAL MODULES FOR FINITE GROUPS OF LIE TYPE

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Abstract. In this paper we determine the torsion free rank of the group of endotrivial modules for any finite group of Lie type, in both defining and non-defining characteristic. Equivalently, we classify the maximal rank 2 elementary abelian ℓ-subgroups in any finite group of Lie type, for any prime ℓ. This classification may be of independent interest.

1. Introduction

Endotrivial modules play a significant role in the modular representation theory of finite groups; in particular, they are the invertible elements in the Green ring of the stable module category of finitely generated modules for the group algebra. Tensoring with an endotrivial module is a self equivalence of the stable module category and these operations generate the Picard group of self equivalences of Morita type in this category. The endopermutation modules, defined for finite groups of prime power order, are the sources of the irreducible modules for large classes of finite groups, and these endopermutation modules are built from the endotrivial modules.

Let G be a finite group and let k be a field of prime characteristic ℓ that divides the order of G. A finitely generated kG-module M is *endotrivial* if its k-endomorphism ring $\mathrm{Hom}_k(M,M)$ is the direct sum of a trivial module and a projective module. The isomorphism classes in the stable category of such modules form an abelian group $T(G)$ under the tensor product $\otimes_k$, where $M \otimes_k N$ is equipped with the diagonal G-action. The group has identity $[k]$ and the inverse to a class $[M]$ is the class $[M^*]$, where M^* is the k-dual of M. As $T(G)$ is finitely generated it is isomorphic to the direct sum of its torsion subgroup $TT(G)$, and a finitely generated torsion free group $TF(G) = T(G)/TT(G)$. We define the *torsion free rank* of $T(G)$ to be the rank of $TF(G)$ as a $\mathbb{Z}$-module. In [29], the second author used homotopy theory to describe $TT(G)$, tying the structure of $TT(G)$ to that of G itself, and in doing so, he also proved a conjecture by the first author and Thévenaz [16]. In a forthcoming article [12], we will provide a description of the torsion subgroup $TT(G)$ for G a finite group of Lie type for all primes, using homotopy theoretic methods. For more information on the

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history and applications of endotrivial modules, see the survey papers [11, 40], and the book by the third author [33].

We recall that, for any finite group G, there is a distinguished element in $T(G)$, namely the class of the shift of the trivial module, defined to be the kernel of the map from a projective cover of k to k. It is easily verified to be endotrivial. Moreover, by elementary homological algebra, the class of this element has infinite order in $TF(G)$ if and only if G contains a subgroup isomorphic to $\mathbb{Z}/\ell \times \mathbb{Z}/\ell$.

Our main theorem of this paper determines the rank of $TF(G)$ for G any finite group of Lie type of characteristic p. We show that it is generated by the class of the shift of the trivial module except in a few low-rank cases, that we describe explicitly. Before stating the precise version of the main theorem, we need to make clear what we mean by a finite group of Lie type.

Definition 1.1 (Finite group of Lie type)**.** By a *finite group of Lie type* in characteristic p we mean a group $G = \mathbb{G}^F$ for $\mathbb{G}$ a connected reductive algebraic group over an algebraically closed field of characteristic p, and F a Steinberg endomorphism, i.e., an endomorphism of $\mathbb{G}$ such that F^s is a standard Frobenius map F_q, for $q = p^r$ and some $s, r \geq 1$.

This definition is a bit more general than that of [32, Definition 21.6] in that we only assume $\mathbb{G}$ to be reductive instead of semisimple. For example, this includes the classical group $\mathrm{GL}_n(q)$. We now present our main theorem:

Theorem A. *Let G be a finite group of Lie type in characteristic p as in Definition 1.1. The group $TF(G)$ of torsion free endotrivial modules over a field of characteristic ℓ, with $\ell \mid |G|$, is zero or infinite cyclic generated by the class of the shift of the trivial module, except when G is on the following list:*

(1) *$\ell \neq p$ and $G \cong H \times K$, where $\ell \nmid |K|$, and H is either*
   (a) $\mathrm{PGL}_\ell(q)$ *with* $\ell \mid q-1$,
   (b) $\mathrm{PGU}_\ell(q)$ *with* $\ell \mid q+1$, *or*
   (c) ${}^3D_4(q)$ *with* $\ell = 3$.

(2) *$\ell = p$ and $G/Z(G)$ is either* $\mathrm{PSU}_3(p)$ *for* $p \geq 3$ *and* $3 \mid p+1$, $\mathrm{PSL}_3(p)$ *for* $p \geq 2$, $\mathrm{PGL}_3(p)$ *for* $p \geq 2$, $\mathrm{PSpin}_5(p)$ *for* $p \geq 5$, $\mathrm{SO}_5(p)$ *for* $p \geq 5$, *or* $G_2(p)$ *for* $p \geq 7$.

In case (1), $TF(G) \xrightarrow{\cong} TF(H)$ *has rank* 3 *if* $H \cong \mathrm{PGL}_\ell(q)$ *or* $\mathrm{PGU}_\ell(q)$ *and* $\ell > 2$, *and rank* 2 *if* $\ell = 2$ *or* $H \cong {}^3D_4(q)$*; see Theorems 3.1 and 6.1. In case* (2) *the ranks are listed in the tables in Section 7; see Theorem 7.1.*

The quotient groups $G/Z(G)$ occurring above as the classical groups $\mathrm{PSL}_3(p) = \mathrm{SL}_3(p)/C_3$, $\mathrm{PSU}_3(p) = \mathrm{SU}_3(p)/C_3$, and $\mathrm{PSpin}_5(p) = \mathrm{Spin}_5(p)/C_2$ are in fact not themselves finite groups of Lie type; see Remark 2.5 and Section 5 for more about this subtlety. Section 5 also contains analogous results for all groups of the form $\mathbb{G}^F/Z(\mathbb{G}^F)$, for simply connected simple $\mathbb{G}$, i.e., the *finite simple groups* associated to finite groups of Lie type. Special cases of the above results can be found in [13, 14, 15]. Note that the rank of $TF(G)$ depends on the characteristic ℓ of k, but not on the finer structure of k.

An elementary abelian ℓ-subgroup of G is a subgroup isomorphic to an $\mathbb{F}_\ell$-vector space. Its ℓ-rank is its $\mathbb{F}_\ell$-vector space dimension. The ℓ-rank of G, denoted $\mathrm{rk}_\ell(G)$, is the maximum of the ℓ-ranks of elementary abelian ℓ-subgroups of G. The groups in (1a) and (1b) of Theorem A have ℓ-rank $\ell - 1$ when ℓ is odd, while all other groups listed in (1) and (2) have ℓ-rank 2.

By a well-known correspondence, recalled in Theorem 1.2 below, our main result translates into a purely local group theoretic statement, Theorem B, which is in fact what we prove. Let $\mathcal{A}_\ell^{\geq 2}(G)$ denote the poset of noncyclic elementary abelian ℓ-subgroups of G, ordered by subgroup inclusion. We say that an elementary abelian ℓ-subgroup of G is maximal if it is maximal in $\mathcal{A}_\ell^{\geq 2}(G)$, i.e., if it is not properly contained in any other elementary abelian subgroup of G. The poset $\mathcal{A}_\ell^{\geq 2}(G)$ has a G-action by conjugation, and we can also consider the orbit space $\mathcal{A}_\ell^{\geq 2}(G)/G$. For any poset X, we can define its set of connected components $\pi_0(X)$, as equivalence classes of elements generated by the order relation, and note that, for a G-poset, $\pi_0(X)/G \xrightarrow{\cong} \pi_0(X/G)$. The following theorem states the correspondence.

Theorem 1.2 ([1, Theorem 4] [13, Theorem 3.1]). *For any finite group G and prime ℓ dividing the order of G, the rank of the group $TF(G)$ is equal to the number of connected components of the orbit space $\mathcal{A}_\ell^{\geq 2}(G)/G$. This number is* 0 *if* $\mathrm{rk}_\ell(G) = 1$*; it is equal to the number of conjugacy classes of maximal elementary abelian ℓ-subgroups in G if* $\mathrm{rk}_\ell(G) = 2$*; and it is equal to* 1 *more than the number of conjugacy classes of maximal elementary abelian ℓ-subgroups of rank* 2*, if* $\mathrm{rk}_\ell(G) > 2$.

The theorem above is Alperin's [1] original calculation of the torsion free rank of $T(G)$ in the case that G is a finite ℓ-group. The proof for arbitrary finite groups is given in [13] and uses very different methods. With this dictionary in place, we can state a local group theoretic version of our main result:

Theorem B. *Let G be a finite group of Lie type in characteristic p (see Definition 1.1) and ℓ an arbitrary prime.*

(1) *If* $\mathrm{rk}_\ell(G) > 2$*, then G does not have a maximal elementary abelian ℓ-subgroup of rank* 2*, unless $\ell > 3$, $\ell \neq p$, and G has the form given in Theorem A*(1a) *or* (1b) *(where* $\mathrm{rk}_\ell(G) = \ell - 1$*).*

(2) *If* $\mathrm{rk}_\ell(G) = 2$*, then all elementary abelian ℓ-subgroups of G of rank* 2 *are conjugate unless G has the form given in Theorem A*(2)*, in Theorem A*(1c)*, or in Theorem A*(1a)(1b)*, $\ell \leq 3$.*

To provide additional context to Theorem B, recall that G can only have a maximal elementary abelian ℓ-subgroup of rank 2 when $\mathrm{rk}_\ell(G) \leq \ell$ for ℓ odd, and $\mathrm{rk}_2(G) \leq 4$ when $\ell = 2$, by a theorem of Glauberman–Mazza [24] and MacWilliams [31] (restated as Theorem 2.3). Theorem B pins down exactly the cases where this does in fact occur for finite groups of Lie type. The study of elementary abelian ℓ-subgroups of $\mathbb{G}$ and $\mathbb{G}^F$ has a long history, with close relationship to cohomology and representation theory; see e.g., [6, 7, 34, 35, 39]. When $\ell \neq p$, conjugacy classes of elementary abelian ℓ-subgroups of $\mathbb{G}$ identify with those of the corresponding complex reductive algebraic group, or compact Lie group (see [3, Section 8]). In fact, they only depend on the ℓ-local structure as encoded in the ℓ-compact group $(B\mathbb{G})^\wedge_\ell$ obtained by ℓ-completing the classifying space $B\mathbb{G}$ in the sense of homotopy theory [28]. Similarly, the elementary abelian ℓ-subgroups of G are determined by $BG^\wedge_\ell$, an ℓ-local finite group [9] describable from the action of F on $B\mathbb{G}_{\hat{\ell}}$; see e.g., [30, Appendix C] for a summary. The question of existence of maximal rank 2 elementary abelian ℓ-subgroups can thus be asked more generally in the context of homotopy finite groups of Lie type, i.e., homotopy fixed-points of Steinberg endomorphisms on connected ℓ-compact groups [10, 30]. In fact we expect Theorem B to generalize to this setting, with the same conclusion, as simple ℓ-compact groups not coming from a compact connected Lie group are centerless and have a unique maximal elementary abelian ℓ-subgroup (see [3, Theorems 1.2 and 1.8] and [2, Theorem 1.1]). We do not pursue the details here, but see Remark 3.4.

One may similarly wonder if $TF(G)$ of Theorem A only depends on the ℓ-local structure in the stronger sense that if $H \to G$ induces an isomorphism of ℓ-fusion systems, is the *map* $TF(G) \to TF(H)$ an isomorphism? That question, however, has a negative answer in general, and we need to replace ℓ-fusion by a stronger ℓ-local invariant [5].

Structure of the paper. Section 2 collects background results needed later, including the aforementioned general Theorem 2.3 that gives conditions on $\mathrm{rk}_\ell(G)$ ensuring no maximal elementary abelian ℓ-subgroups of rank 2.

In Sections 3–7, we determine $TF(G)$ when $G = \mathbb{G}^F$, and $\mathbb{G}$ is simple. The cases when $3 \leq \ell \neq p$ are handled in Sections 3 and 4. In many cases it is known that the orbit space $\mathcal{A}_\ell^{\geq 2}(G)/G$ is connected (see [27, Section 4.10]). This allows us to reduce to examining some groups of small Lie rank, in Proposition 3.3, and these are then analyzed in Section 4. In Section 5, we extend the results of the previous sections to also compute $TF(G)$, for G a group closely associated to a group of Lie type such as $\mathrm{PSL}_n(q)$ or $\mathrm{PSp}_n(q)$, in the case that $\ell \geq 3$.

The case where $2 = \ell \neq p$ is handled in Section 6. Section 7 investigates the final case when $\ell = p$, extending work in [13]. In the case that $\ell = 2$ the associated groups are included in the analysis of Section 6.

Finally, in Section 8, we prove Theorems A and B in the general case where $\mathbb{G}$ is a connected reductive algebraic group.

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2. Preliminaries

> *Throughout the paper G is finite group (maybe subject to more assumptions, specified locally) and k is a field of some positive characteristic ℓ, dividing the order of G.*

In this section we provide some background material used throughout this paper.

Definition 2.1. A finitely generated kG-module M is *endotrivial* if $\mathrm{Hom}_k(M, M) \cong k \oplus P$ where P is a projective kG-module and k is the trivial kG-module. Thus, $\mathrm{Hom}_k(M, M) \cong k$ in the stable category of kG-modules modulo projectives. The set

$T(G)$ of stable isomorphism classes of endotrivial kG-modules forms a group under $-\otimes_k -$, called the *group of endotrivial kG-modules.*

Recall that in this context, $\mathrm{Hom}_k(M,M) \cong M^* \otimes_k M$ as kG-modules, and therefore the endotrivial modules are the invertible objects under tensor product in the stable module category of kG-modules modulo projectives.

The group $T(G)$ is a finitely generated abelian group ([13, Corollary 2.5]) hence $T(G) \cong TT(G) \oplus TF(G)$, for $TT(G)$ the torsion subgroup of $T(G)$, a finite group, and $TF(G) = T(G)/TT(G)$, a finitely generated free abelian group. In Theorem 1.2, the rank of $TF(G)$ is stated to be equal to the number of conjugacy classes of maximal elementary abelian ℓ-subgroups of G of rank 2 if $\mathrm{rk}_\ell(G) = 2$, or that number plus 1 in case $\mathrm{rk}_\ell(G) > 2$.

We start with a few elementary but useful observations.

Lemma 2.2. *Let P be a finite ℓ-group.*

(a) *If P has a normal elementary abelian ℓ-subgroup H of ℓ-rank $\ell+1$ or more, then P has no maximal elementary abelian subgroups of rank 2.*
(b) *If P has ℓ-rank 2 and the center of P is not cyclic, then P has exactly one maximal elementary abelian subgroup with ℓ-rank 2.*
(c) *If P has ℓ-rank at least 3 and the center of P is not cyclic, then P has no maximal elementary abelian subgroups of ℓ-rank 2.*

Proof. The proofs of parts (b) and (c) are straightforward. For (a), let x be a noncentral element of P of order ℓ. If $x \in H$, then $C_P(x) \geq H$ has ℓ-rank at least 3 by assumption and the statement holds. If $x \notin H$, then the conjugation action of x on H can be regarded as a linear action on an $\mathbb{F}_\ell$-vector space of dimension at least $\ell+1$, and therefore must have at least two linearly independent eigenvectors for the eigenvalue 1. That is, conjugation by x fixes two nontrivial distinct generators of H in some suitable generating set, and since $x \notin H$, we conclude that the subgroup of P generated by x and these two elements is elementary abelian of rank 3. So x is not contained in a maximal elementary abelian subgroup of P of rank 2, and part (a) follows. □

For our analysis, we employ results of Glauberman–Mazza and MacWilliams that guarantee, under suitable conditions on the ℓ-rank of the finite group G, that the group has no maximal elementary abelian ℓ-subgroups of rank 2. The sectional ℓ-rank of a group G is the maximal ℓ-rank of any section of G. A section of G is the quotient of a subgroup of G by a normal subgroup of that subgroup.

Theorem 2.3. *Let G be a finite group and let ℓ be a prime.*

(a) [24, Theorem A] *If $\ell \geq 3$ and $\mathrm{rk}_\ell(G) \geq \ell+1$, then G has no maximal elementary abelian ℓ-subgroups of rank 2.*
(b) [31, Four Generator Theorem] *Suppose that G has sectional 2-rank at least 5. Then a Sylow 2-subgroup of G has a normal elementary abelian subgroup with 2-rank 3. In such a case G has no maximal elementary abelian 2-subgroup of rank 2.*

Part (b) in Theorem 2.3 is a reformulation, which better suits our analysis, of [31, Four Generator Theorem]. The theorem (which was part of the program to classify finite simple groups) asserts that, in a finite 2-group G with no normal elementary abelian subgroup of rank 3, every subgroup can be generated by at most four elements. Thus, if the sectional 2-rank of a 2-group G is 5 or more, then some Frattini quotient $P/\Phi(P)$, for P a subgroup of G, has 2-rank 5 or more. By the theorem, G has a normal elementary abelian subgroup with 2-rank 3, implying that G has no maximal elementary abelian subgroup of rank 2, by Lemma 2.2. Our interpretation follows because, for any ℓ, the sectional ℓ-rank of a finite group is equal to that of its Sylow ℓ-subgroups.

We also record the following result, which is used to relate the torsion free ranks of groups of endotrivial modules of finite groups of Lie type arising from isogenous algebraic groups.

Proposition 2.4. *Let*

$$1 \longrightarrow Z \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1$$

be an exact sequence of finite groups where Z and K have order prime to ℓ, and Z central in H. Then the induced map $\mathcal{A}_\ell^{\geq 2}(H)/H \twoheadrightarrow \mathcal{A}_\ell^{\geq 2}(G)/G$ is a surjection, which is an isomorphism of posets if the image of H in G controls ℓ-fusion in G. In particular $TF(H) \cong \mathbb{Z}$ implies $TF(G) \cong \mathbb{Z}$, with the converse also true if the image of H in G controls ℓ-fusion in G (e.g., if $K = 1$).

Proof. Since K and Z have orders that are prime to ℓ, the map $H \to G$ induces a bijection of ℓ-subgroups. Furthermore, conjugacy in H implies conjugacy in G, with the converse also being true if the image of H in G controls ℓ-fusion in G. Note that the image of H in G is isomorphic to H/Z. The statement about torsion free ranks follows using the standard translation by Theorem 1.2. ☐

We conclude this section with a discussion of our conventions for finite groups of Lie type.

Remark 2.5 (Finite groups of Lie type)**.** As stated in Definition 1.1 we take a finite group of Lie type to mean a group of the form $G = \mathbb{G}^F$, for $\mathbb{G}$ a connected reductive algebraic group over an algebraically closed field of positive characteristic p, and F a Steinberg endomorphism. We refer to [32], or the original [38], for a thorough description of properties of such groups, but quickly go through a few key points to aid to the reader: A connected reductive algebraic group $\mathbb{G}$ over an algebraically closed field is classified by its root datum $\mathbb{D}$ (which is field independent). The action of F on $\mathbb{G}$ (up to inner automorphisms) is also determined by its effect on $\mathbb{D}$ (up to Weyl group conjugation) allowing for a "combinatorial" classification of finite groups of Lie type $\mathbb{G}^F$. It is most explicit when $\mathbb{G}$ is further assumed simple, see [32, Table 22.1]. In this case $\mathbb{G}^F$ is "close" to being simple, in the following sense: A formula of Steinberg [38, Corollary 12.6(b)] says that $G/O^{p'}(G) \xrightarrow{\cong} \pi_1(\mathbb{G})_F$, the coinvariants of the action of F on the fundamental group $\pi_1(\mathbb{G})$. (As usual $O^{p'}(-)$ denotes the smallest normal subgroup of p' index, and $O_{p'}(-)$ denotes the largest normal subgroup of p' order.) Thus, subgroups H with $O^{p'}(G) \leq H \leq G$ can be parametrized by "Lie theoretic" data consisting of $\mathbb{G}$, F, and a subgroup of $\pi_1(\mathbb{G})_F$. They are hence "close" to finite groups of Lie type, though, e.g., the order formula [32, Corollary 24.6] does not hold —

some books dealing with finite *simple* groups, e.g., [27, Definition 2.2.1], instead refer to groups of the form $O^{p'}(\mathbb{G}^F)$ as finite groups of Lie type. Dual to p'-quotients we have that

$$Z(G) = O_{p'}(G) = Z(\mathbb{G})^F \tag{2.1}$$

(see [32, Lemma 24.12]). Normal p'–subgroups and quotients are related, as

$$\mathbb{G}_{sc}^F/Z(\mathbb{G}_{sc}^F) \xrightarrow{\cong} O^{p'}((\mathbb{G}/Z(\mathbb{G}))^F), \tag{2.2}$$

for $\mathbb{G}_{sc}$ the simply connected cover of $\mathbb{G}$ (see [32, Proposition 24.21]). With a few small exceptions [32, Theorem 24.17], this is a finite simple group, if $\mathbb{G}$ is simple. For example $\mathrm{PSL}_n(q) \cong O^{p'}(\mathrm{PGL}_n(q))$ is simple unless (n, q) is $(2, 2)$ or $(2, 3)$. We determine $TF(H)$ for for such groups H in Section 5.

3. When $\mathbb{G}$ is simple, $3 \le \ell \ne p$: Generic case

In this section G is a finite group of Lie type as in Definition 1.1, where we furthermore assume that the ambient algebraic group $\mathbb{G}$ is simple (and hence determined by an irreducible root system and an isogeny type). The aim of Sections 3 and 4 is to prove the following.

Theorem 3.1. *Let $G = \mathbb{G}^F$ be a finite group of Lie type where $\mathbb{G}$ is a simple algebraic group. Assume that $3 \le \ell \ne p$ and that $\mathrm{rk}_\ell(G) \ge 2$. Then $TF(G) \cong \mathbb{Z}$ except in the following cases:*

(a) *$\ell \ge 3$ and G is isomorphic to either $\mathrm{PGL}_\ell(q)$ with ℓ dividing $q-1$ or $\mathrm{PGU}_\ell(q)$ with ℓ dividing $q+1$. In these cases, $TF(G) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.*

(b) *$\ell = 3$ and G is isomorphic to ${}^3D_4(q)$. In this case, $TF(G) \cong \mathbb{Z} \oplus \mathbb{Z}$.*

The proof of Theorem 3.1 entails a reduction, accomplished in this section, to some cases of small rank and specific types. The analysis of the small rank cases is done in Section 4.

The following is taken from [27, Theorem 4.10.3].

Theorem 3.2. *Let $G = \mathbb{G}^F$ be a finite group of Lie type arising from a simple algebraic group $\mathbb{G}$ with a Steinberg endomorphism F, and $\ell \ne p$, and write $\mathbb{G} \cong \mathbb{G}_{sc}/Z$ for a finite central subgroup Z. Assume that*

(i) *the prime ℓ does not divide the order of Z^F. This is true if $\ell \nmid |Z(\mathbb{G}_{sc})^F|$.*

(ii) *the prime ℓ is odd and good for $\mathbb{G}$ (meaning that $\ell > 3$ if the type of $\mathbb{G}$ is E_6, E_7, F_4 or G_2, $\ell > 5$ if the type of $\mathbb{G}$ is E_8).*

Then any elementary abelian ℓ-subgroup A of G is contained in an elementary abelian ℓ-subgroup of maximal rank. Also, any two elementary abelian ℓ-subgroups of maximal rank are conjugate except possibly if $\ell = 3$ and $G \cong {}^3D_4(q)$.

Proof. Assume first that $\mathbb{G}$ is simply connected, i.e., Z is trivial. Under condition (ii), [27, Theorem 4.10.3(e)] says that every elementary abelian ℓ-subgroup of G is contained in an elementary abelian ℓ-subgroup of maximal rank. Finally [27, Theorem 4.10.3(f)] implies that all maximal elementary abelian ℓ-subgroups of G are conjugate, unless $G \cong {}^3D_4(q)$, again using (ii). This proves the theorem in the simply connected case.

Because $|Z^F|$ is assumed prime to ℓ, the conclusion for G follows from that of G_{sc} by Proposition 2.4 applied the exact sequence

$$1 \longrightarrow Z^F \longrightarrow G_{sc} \longrightarrow G \longrightarrow Z_F \longrightarrow 1\ , \tag{3.1}$$

of [32, Lemma 24.20], where $|Z^F| = |Z_F|$ and $|G_{sc}| = |G|$ by [32, Corollary 24.6]. □

The next proposition builds on Theorem 3.2 and handles many of the cases in Theorem 3.1, with the rest being postponed to the next section. In the proof we employ the non-standard notation, where e.g., $B_2(p)$ without subscript "sc" or "ad", denotes *any* group arising from a simple algebraic group $\mathbb{G}$ over an algebraically closed field of characteristic p with root system B_2, and $F = F_p$ is the standard Frobenius given by raising to the pth power.

Proposition 3.3. *Let ℓ be an odd prime, $\ell \neq p$. Suppose that $G = \mathbb{G}^F$ is a finite group of Lie type where $\mathbb{G}$ is a simple algebraic group and F is a Steinberg endomorphism. Assume that the ℓ-rank of G is at least 2, and G does not have one of the forms $A_{n-1}(q)$ with ℓ dividing both $q-1$ and n, ${}^2A_{n-1}(q)$ with ℓ dividing both $q+1$ and n, or ${}^3D_4(q)$ with $\ell = 3$. Then $TF(G) \cong \mathbb{Z}$.*

Proof. Let $Z = Z(\mathbb{G}_{sc})$, whose order is given in [32, Table 9.2] (the order of "$\Lambda(\Phi)$"). The order of $Z^F = Z(G_{sc})$ is given in [32, Table 24.2]. It follows from Theorem 3.2 that $TF(G) \cong \mathbb{Z}$ if ℓ is odd and good for $\mathbb{G}$, $\ell \nmid |Z^F|$, and G is not isomorphic to ${}^3D_4(q)$. Consequently, it remains to discuss the cases that either (i) ℓ divides $|Z^F|$, (ii) $\ell = 3$ and $\mathbb{G}$ has exceptional type or (iii) $\ell = 5$ and $\mathbb{G}$ has type E_8. We show, by explicit arguments, that in those cases there are also no maximal elementary abelian ℓ-subgroups of rank 2, unless the ℓ-rank of the group is 2, in which case there is a unique one. This shows that $TF(G) \cong \mathbb{Z}$ by Theorem 1.2.

First note that case (i) is basically ruled out by the hyphotheses. That is, if $\mathbb{G}$ has type B_n, C_n or D_n, then $|Z|$ is a power of 2 and hence is not divisible by ℓ. If $\mathbb{G}$ has type A_{n-1} then the only cases where $\ell \mid |Z^F|$ are exactly the ones we exclude in our formulation of the proposition. Finally if $\mathbb{G}$ is of exceptional type and $\ell \mid |Z|$, then the only possibility is $\mathbb{G}$ having type E_6 and $\ell = 3$, which is covered under (ii) below.

This leaves (ii) and (iii), i.e., the exceptional types with $\ell = 3$ and E_8 with $\ell = 5$. In other words, by the classification of Steinberg endomorphisms [32, Theorem 22.5], the groups we need to consider are $G_2(q)$, $F_4(q)$, ${}^2F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ and $E_8(q)$ at $\ell = 3$ and $E_8(q)$ at $\ell = 5$. (Note that ${}^2F_4(q)$ only exists in characteristic 2 and ${}^2G_2(q)$ does not appear on the list as we assume $q \neq 3$.) We handle these groups on a case-by-case basis:

$F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, and $E_8(q)$ with $\ell = 3$: We claim that in all these cases, there is an elementary abelian 3–subgroup of rank at least 4, in fact inside a maximal torus, which then shows $TF(G) \cong \mathbb{Z}$ by Theorem 2.3(a). When $\ell \nmid |Z^F|$ it is enough to see that the multiplicity of the cyclotomic polynomials Φ_1 and Φ_2 in the order polynomial of the complete root datum ${}^d\mathbb{D}$ is (at least) 4, by [27, Theorem 4.10.3(b)]. (Recall that a complete root datum ${}^d\mathbb{D}$ consists of a root datum $\mathbb{D}$ together with the twisting "d", see [32, Definition 22.10] and [27, Definition 2.2.4].) This follows by inspecting [26, Part I, Table 10:2]. The only cases where we can have $\ell \mid |Z^F|$ are (again by [32, Table 24.2]) when either $E_6(q)$ with $q \equiv 1 \pmod 3$ or ${}^2E_6(q)$ with

$q \equiv -1 \pmod 3$. But as the multiplicity of Φ_1, respectively Φ_2, in the order polynomial of the complete root datum E_6, respectively ${}^2E_6$, is 6, we have that the ℓ-rank of G_{sc} is (at least) 6 for these groups (again by [27, Theorem 4.10.3(b)]), and hence the ℓ-rank of G is at least 5.

$G_2(q)$ with $\ell = 3$: We give a direct argument that all elementary abelian 3-subgroups of rank 2 are conjugate. By [4, Lemma 4], the commutator subgroup of the centralizer of the center of a Sylow 3-subgroup of G is isomorphic to $\mathrm{SL}_3(q)$ if $q \equiv 1 \pmod 3$, respectively to $\mathrm{SU}_3(q)$ if $q \equiv -1 \pmod 3$. In either case, any two elementary abelian 3-subgroups of rank 2 are conjugate by Theorem 3.2.

${}^2F_4(2^{2a+1})$ with $\ell = 3$: It follows from [26, Proofs of (10-1) and (10-2), p. 118] that ${}^2F_4(2^{2a+1})$ contains $\mathrm{SU}_3(2^{2a+1})$ of index prime to 3. All rank 2 elementary abelian 3-subgroups are conjugate in $\mathrm{SU}_3(2^{2a+1})$ by Theorem 3.2, and hence this holds for ${}^2F_4(2^{2a+1})$ as well.

$E_8(q)$ with $\ell = 5$: From [26, Proofs of (10-1) and (10-2), p. 118] we see that $E_8(q)$ contains $\mathrm{SU}_5(q^2)$ as a subgroup of index prime to 5 (the coefficients are in $\mathbb{F}_{q^4}$). Hence, every elementary abelian 5-subgroup of G is contained in one of rank 4 by Theorem 3.2. Consequently, there are no maximal elementary abelian 5-subgroups of rank 2. □

Remark 3.4. For the interested reader, we briefly sketch how Proposition 3.3 (and Theorem 3.2) could alternatively be obtained via homotopy theory. If ℓ does not divide the order of the fundamental group of a connected ℓ-compact group BG, then every elementary abelian ℓ–subgroup of rank at most 2 is conjugate into a torus by [3, Theorem 1.8], generalizing Borel and Steinberg's classical theorem [39, Theorem 2.27]. The homotopical Lang square of Friedlander–Quillen [10, (1)] now relates elementary abelian ℓ–subgroups in BG to those in the homotopical finite group of Lie type BG^{hF}. When F is the standard Frobenius with q congruent to 1 modulo ℓ this shows that the centralizer of every element of order ℓ in BG^{hF} has ℓ-rank at least the Lie rank of the ℓ-compact group BG. For general F one first uses untwisting [30, Theorem C.8] to reduce to the previous case, now inside another ℓ-compact group. Note that untwisting assumes that the order of the twisting is prime to ℓ, which explains why ${}^3D_4(q)$ when $\ell = 3$ needs to be treated separately. Indeed the conclusion that $TF(G)$ has rank 2 in this case shows that this is not only a technical limitation.

4. When $\mathbb{G}$ is simple, $3 \le \ell \ne p$: Specific cases

In this section, we examine the cases not covered by Proposition 3.3, thereby completing the proof of Theorem 3.1. The analysis is case by case, and we assume $\ell \ne p$ throughout.

Proof of Theorem 3.1. First consider $G = {}^3D_4(q)$, with $\ell = 3 \nmid q$. By [26, Part I, 10-1(4)], a Sylow 3-subgroup S of G has the form $(C_{3^{a+1}} \times C_{3^a}) \rtimes C_3$, where $3^a = |q^2 - 1|_3$. From [21, Theorem 5.10], we also know that $S \cong B(3, 2(a+1); 0, 0, 0)$ is a 3-group of maximal nilpotency class of 3-rank 2 and order 3^{2a+2}. Let A be the maximal subgroup of S of the form $C_{3^{a+1}} \times C_{3^a}$, let B be the subgroup of A formed by the elements of order 3, and let V_1 be any non-normal maximal elementary abelian subgroup of S (necessarily of rank 2). The subgroups B and V_1 are those denoted likewise in [21]. In [21, Theorem 5.10], the authors prove that all the non-normal maximal elementary abelian subgroups of S are G-conjugate. They also show that V_1 is the Sylow 3-subgroup of $C_G(V_1)$, and from the description of S, it is clear that B is not a Sylow 3-subgroup of $C_G(B)$. Therefore, B and V_1 cannot be G-conjugate, and it follows that $TF(G) \cong \mathbb{Z} \oplus \mathbb{Z}$.

For the remainder of the proof assume that G has type either $A_{n-1}(q)$ with $\ell \geq 3$ and $\ell \mid q-1$ or ${}^2A_{n-1}(q)$ with $\ell \geq 3$ and $\ell \mid q+1$. We assume also that ℓ divides the order of Z^F and thus n is a multiple of ℓ. If $n > \ell$, then $TF(G) \cong \mathbb{Z}$ by Theorem 2.3(a). Thus, we are reduced to consider the cases $G = A_{\ell-1}(q)$ with $q \equiv 1 \pmod{\ell}$, and $G = {}^2A_{\ell-1}(q)$ with $q \equiv -1 \pmod{\ell}$. Because ℓ is prime there are exactly two distinct isogeny types. If $\mathbb{G}$ is simply connected, the asserted result follows by Theorem 3.2. We are left with the cases $G = \mathrm{PGL}_\ell(q)$ and $G = \mathrm{PGU}_\ell(q)$ with the appropriate congruences of q modulo ℓ. Because the ℓ-local structures of the two groups are almost identical, we consider only $G = \mathrm{PGL}_\ell(q)$.

Let $\widehat{G} = \mathrm{GL}_\ell(q)$ with ℓ dividing $q-1$. We choose a Sylow ℓ-subgroup of $\widehat{G}$ to be a subgroup of the normalizer of a maximal torus of diagonal matrices (see Theorem 3.2). The normalizer of the torus is a wreath product, of the form $N \cong \mathrm{GL}_1(q)^{\times \ell} \rtimes \mathfrak{S}_\ell$, where $\mathfrak{S}_\ell$ is the symmetric group on ℓ letters. That is, it is the subgroup of diagonal matrices with an action by the group of permutation matrices. Let ζ be a primitive ℓ^{th} root of unity in $\mathbb{F}_q$. Let γ be a generator for the Sylow ℓ-subgroup of $\mathrm{GL}_1(q)$, so that $\zeta = \gamma^{\ell^{s-1}}$ for some s and $\gamma^{\ell^s} = 1$. Let x be the $\ell \times \ell$ permutation matrix

$$x = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix},$$

let y be the diagonal matrix (of size ℓ) with diagonal entries $\gamma, 1, \dots, 1$, and let $z = \gamma I$ be the scalar matrix. A Sylow ℓ-subgroup $\widehat{S}$ of $\widehat{G}$ is generated by x and y. Then a Sylow ℓ-subgroup of G is $S \cong \widehat{S}/\langle z \rangle$. The subgroup $\widehat{S}$ has a maximal subgroup $T = \langle y, xyx^{-1}, \dots, x^{\ell-1}yx^{1-\ell} \rangle$, which is abelian.

Let $\phi : \widehat{S} \to S$ be the quotient map. We note that two subgroups E and F in S are conjugate in G if and only if their inverse images $\phi^{-1}(E)$ and $\phi^{-1}(F)$ are conjugate in $\widehat{G}$. Consequently, to find the maximal elementary abelian subgroups of rank 2 in S, it suffices to look for the subgroups E of order ℓ^{s+2} in $\widehat{S}$ that contain z and have the property that $E/\langle z \rangle$ is elementary abelian. For the sake of this proof, call such a group *$Q2$-elementary.*

For our analysis, we identify three subgroups. Let $a = y^{\ell^{s-1}}$ and let b be the diagonal matrix with diagonal entries $1, \zeta, \zeta^2, \dots, \zeta^{\ell-1}$. Notice that $xbx^{-1}b^{-1} = \zeta \cdot I = z^{\ell^{s-1}}$. Let

$$E_1 = \langle a, xax^{-1}, \dots, x^{\ell-1}ax^{1-\ell}, z \rangle, \qquad E_2 = \langle x, b, z \rangle, \qquad \text{and} \quad E_3 = \langle ax, b, z \rangle.$$

We claim that every $Q2$-elementary subgroup of $\widehat{S}$ is either conjugate to one of E_2 or E_3 or is conjugate to a subgroup of E_1. Note that E_1 is abelian whereas the other two are not. Also, every element of order ℓ in E_2 has determinant 1, but this is not true of E_3. Hence, E_2 and E_3 are not conjugate, and neither is conjugate to a subgroup of E_1.

Note first that any $Q2$-elementary subgroup of T must be contained in E_1 as E_1 is a direct product of ℓ cyclic subgroups of order ℓ and $\langle z\rangle$ is a direct factor. In particular, $E_1/\langle z\rangle$ contains all elements of order ℓ in $T/\langle z\rangle$. Suppose that H is a $Q2$-elementary subgroup that is not in T. Then H contains an element of the form tx for some $t \in T$. By a direct calculation, we notice that the centralizer in $T/\langle z\rangle$ of the class of x is a direct factor of $T/\langle z\rangle$ that is cyclic of order ℓ^s. It is generated by the image in $T/\langle z\rangle$ of diagonal matrix u with entries $1, \gamma, \dots, \gamma^{\ell-1}$. The subgroup of elements of order ℓ in this group is generated by $b = u^{\ell^{s-1}}$. So we can assume that $H = \langle tx, b, z\rangle$.

It remains to find the conjugacy classes. Suppose that $w \in T$ is diagonal with entries $w_1, \dots, w_\ell$. Then $wxw^{-1} = vx$ where v has diagonal entries $w_1w_2^{-1}, w_2w_3^{-1}, \dots, w_\ell w_1^{-1}$. In other words, x is conjugate in $\widehat{S}$ to vx for v any diagonal matrix with entries $v_1, \dots, v_\ell$ satisfying the condition that the product $v_1 \cdots v_\ell = 1$. It follows that any possible conjugacy class of $Q2$-elementary subgroups not in T has a representative of the form $H = \langle a^i x, b, z\rangle$ for $i = 1, \dots, \ell^s - 1$. Now, $(a^i x)^\ell = z^i$. If $i = m\ell$ for some $m \geq 1$, then $v = a^i x z^{-m}$ has the property that $v^\ell = 1$. In this case $v = tx$ where $t \in T$ has the property that the product of its (diagonal) entries is 1. Thus, v is conjugate to x by an element in T, and H is conjugate to $\langle x, b, z\rangle$.

So we are down to the situation that $H = \langle a^i x, b, z\rangle$, for $i = 0, 1, \dots, \ell - 1$. But now notice that x is conjugate to x^j for $j = 1, \dots, \ell - 1$ by a permutation matrix, an ℓ-cycle, that centralizes a and normalizes $\langle b, z\rangle$. It follows that if $i \neq 0$, then $a^i x$ is conjugate to $a^i x^{-i}$ and $H = \langle a^i x, b, z\rangle$ is conjugate to E_3. This proves the claim.

Recall that $E_1/\langle z\rangle$ has ℓ-rank $\ell \geq 3$. It follows that $E_1/\langle z\rangle$, $E_2/\langle z\rangle$ and $E_3/\langle z\rangle$ are in three distinct connected components of the orbit poset $\mathcal{A}_\ell^{\geq 2}(G)/G$ of noncyclic elementary abelian ℓ-subgroups and that there are no other components containing subgroups of rank 2. In other words, $TF(G)$ has rank 3. □

We now establish the rank of $TF(G)$ in some specific cases that are useful in Section 5.

Proposition 4.1. *Suppose that $\ell \geq 3$, and either $G \cong \mathrm{PSL}_\ell(q)$ with $q \equiv 1 \pmod{\ell}$, or $G \cong \mathrm{PSU}_\ell(q)$ with $q \equiv -1 \pmod{\ell}$. Assume that if $\ell = 3$, then $q \equiv 1 \pmod 9$ in the first case and $q \equiv -1 \pmod 9$ in the second. Then $TF(G)$ has rank $\ell + 1$.*

Proof. The ℓ-local structures of $\mathrm{PSL}_\ell(q)$ with ℓ dividing $q - 1$ and $\mathrm{PSU}_\ell(q)$ with ℓ dividing $q + 1$ are very similar. We give the proof only in the case that $G = \mathrm{PSL}_\ell(q)$. The proof in the case of $\mathrm{PSU}_\ell(q)$ follows by the same line of reasoning. We include a complete analysis, though much of the information in the proof is in the more general paper [20].

We continue mostly with the notation introduced in the proof of Theorem 3.1 for $G = A_{\ell-1}(q)$, except that we let $H = \mathrm{SL}_\ell(q)$ and $G = \mathrm{PSL}_\ell(q) = H/\langle z\rangle$ where $z = \zeta I$ generates the center of H (not the same z as in the previous proof). A Sylow ℓ-subgroup of H has the form $S = T \rtimes \langle x\rangle$, where T is the collection of diagonal ℓ-elements having determinant 1. Any element of S that is not in T is a power of an element of the form ax for some $a \in T$. We note that the diagonal element y as above, with entries $\gamma, 1, \dots, 1$, is not in H. The subgroup S is generated by x and $w = x^{-1}y^{-1}xy$ which is diagonal with entries $\gamma, \gamma^{-1}, 1, \dots, 1$, and T is generated by the conjugates of w by powers of x.

A $Q2$-elementary subgroup, if it is not contained in T, must have the form $J_a = \langle ax, b, z\rangle$ for some a in T. That is, these are the nonabelian subgroups J such that $J/\langle z\rangle$ is elementary abelian of rank 2. Note that $J_a = J_{a'}$ if and only if $a'a^{-1} \in \langle b, z\rangle$. So there are $|T|/\ell^2$ such subgroups. A direct calculation shows that $N_S(J_a)$ has order $|S|/\ell^4$. Thus, there are exactly ℓ S-conjugacy classes of such subgroups. Let $E_i = \langle w^i x, b, z\rangle$, for $i = 0, \dots, \ell-1$. All of these subgroups are conjugate in $\widehat{G} = \mathrm{GL}_\ell(q)$ by some power of the element y. Our purpose is to show, however, that no two of them are conjugate in H. The theorem then follows, because our observation implies that the classes $E_i/\langle z\rangle$ for $0 \le i < \ell$ are distinct conjugacy classes of maximal elementary abelian ℓ-subgroups of $\mathrm{PSL}_\ell(q)$ of rank 2. The subgroup $T/\langle z\rangle$ also has a maximal elementary abelian subgroup $E/\langle z\rangle$, and none of the E_i's is conjugate to a subgroup of E since the latter is abelian.

Consider the subgroup $N = N_H(E_0)$, the normalizer in $\mathrm{SL}_\ell(q)$ of $E_0 = \langle x, b, z\rangle$. The subgroup E_0 is an extraspecial group of order ℓ^3 and exponent ℓ. Its outer automorphism group is isomorphic to $\mathrm{GL}_2(\ell)$ (see the discussion in [41]). Because the centralizer of E_0 in H is the center of H, N is an extension

$$1 \longrightarrow E_0 \longrightarrow N \longrightarrow U \longrightarrow 1$$

where U is isomorphic to a subgroup of $\mathrm{SL}_2(\ell)$ since it must also centralize $\langle z\rangle$.

Observe that E_0 is a proper subgroup of $N_S(E_0)$. In particular, there is an element u of T whose class generates the center of $S/\langle b, z\rangle$ that is in $N_S(E_0)$. Hence, U has an element of order ℓ. Moreover, $N_H(T)/T$ is isomorphic to the symmetric group on ℓ letters. This group has an $\ell-1$ cycle that normalizes the subgroup generated by the class of the element x. It must also normalize $\langle b, z\rangle$ and $\langle u, b, z\rangle$. Consequently, U contains the subgroup B of upper triangular matrices in $\mathrm{SL}_2(\ell)$. Because B is a maximal subgroup of $\mathrm{SL}_2(\ell)$, we need only show that U has at least one element that is not in B to conclude that $U \cong \mathrm{SL}_2(\ell)$.

Let v be the Vandermonde matrix

$$v = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \zeta & \zeta^2 & \dots & \zeta^{\ell-1} \\ 1 & \zeta^2 & \zeta^4 & \dots & \zeta^{2(\ell-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \zeta^{\ell-1} & \zeta^{2(\ell-1)} & \dots & \zeta^{(\ell-1)^2} \end{bmatrix} \qquad \text{so that} \qquad v^2 = \begin{bmatrix} \ell & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \ell \\ 0 & 0 & \dots & \ell & 0 \\ \vdots & \vdots & \dots & & \vdots \\ 0 & \ell & \dots & 0 & 0 \end{bmatrix}.$$

Note that the columns (and also the rows) are eigenvectors for the matrix x with corresponding eigenvalues $1, \zeta, \zeta^2, \dots, \zeta^{\ell-1}$. Thus, we have that $xv = vb$. The computation of the matrix v^2 is straightforward as each row is orthogonal (under the usual dot product) to all but one of the columns.

Next we note that the determinant of v^2 is $\varepsilon\ell^\ell = (\varepsilon\ell)^\ell$ where $\varepsilon = \pm 1$, the sign depending on the parity of $(\ell-1)/2$. Because the group $\mathbb{F}_q^\times$ is cyclic and ℓ is prime to 2, the determinant of v is also an ℓ^{th}-power. That is, there is some μ in $\mathbb{F}_q^\times$ such that $\mathrm{Det}(v) = \mu^\ell$ and $\mu^2 = \varepsilon\ell$. Let h be the product of v with the scalar matrix $\mu^{-1}I$. Then $\mathrm{Det}(h) = 1$, $h \in H$ and $xh = hb$. In addition, h^2 has the same form as v^2 except that the nonzero entries that are equal to ℓ in v^2 are replaced by ε in h^2. That is,

$h^2 = (1/\varepsilon\ell)v^2$. So we find that $h^2xh^{-2} = x^{-1}$ by direct calculation. Also, we have that $h^{-1}xh = b$ and $h^{-1}bh = x^{-1}$. So h is in N and its class in U, identified in a subgroup of $\mathrm{SL}_2(\ell)$, is the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This element is not in the subgroup B, and hence we have shown that $U \cong \mathrm{SL}_2(\ell)$.

Because $N_H(E_0)/E_0$ is the outer automorphism group of E_0 we have that $N_{\widehat{G}}(E_0) = N_H(E_0)\widehat{Z}$, where $\widehat{Z}$ denotes the center of $\widehat{G} = \mathrm{GL}_\ell(q)$. The same holds if we replace E_0 by E_i since they are conjugate in $\widehat{G}$. Thus, we have that if $g \in N_{\widehat{G}}(E_i)$, then the determinant of g is an ℓ^{th} power of some element in $\mathbb{F}_q^\times$.

Finally, suppose that there is an element g in H such that $gE_ig^{-1} = E_j$ for $i < j$. We know also that $y^{j-i}E_iy^{i-j} = E_j$. Therefore, $y^{i-j}g \in N_{\widehat{G}}(E_i)$. However, this is not possible. The reason is that γ is a generator of the Sylow ℓ-subgroup of the multiplicative group $\mathbb{F}_q^\times$ and $0 < i - j < \ell$, the determinant of $y^{i-j}g$, which is equal to γ^{i-j}, is not an ℓ^{th} power. Hence, if $i \neq j$, then E_i is not H-conjugate to E_j and then $E_i/\langle z\rangle$ is not G-conjugate to $E_j/\langle z\rangle$. This proves the proposition. □

5. Groups associated to finite groups of Lie type for $\ell \geq 3$

In this section we are interested in some of the groups associated to finite groups of Lie type. Suppose that $G_0 = G_{sc}$ is a finite group of Lie type arising from a simply connected simple algebraic group $\mathbb{G}$. If $G_0 = \mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$, let $G_1 = \mathrm{GL}_n(q)$, or $\mathrm{GU}_n(q)$, respectively. If $\mathbb{G}$ is symplectic or orthogonal, take G_1 to be the conformal group of that type (cf. [32, pp. 7-8] and [27, Section 2.7]). For example, if $G_0 = \mathrm{Sp}_{2n}(q)$, then $G_1 = \mathrm{CSp}_{2n}(q)$, the group of all $2n \times 2n$-matrices X with the property that $XfX^{tr} = af$ for some $a \in \mathbb{F}_q$, where f it the matrix of the symplectic form. If $G_0 = \mathrm{Spin}^+_{2n}(q)$, then G_1 is the conformal group $\mathrm{CSpin}^+_{2n}(q)$.

We see below that if G_0, the fixed points of a simply connected algebraic group under a Steinberg endomorphism, has trivial center, then we may assume that $G_0 = G_1$ and any associated group is a direct product of G_0 with some abelian group. For that reason we concentrate on the classical groups. For the groups of type E_6, ${}^2E_6$ and E_7, we have the following. This applies also in the case that $\ell = 2$.

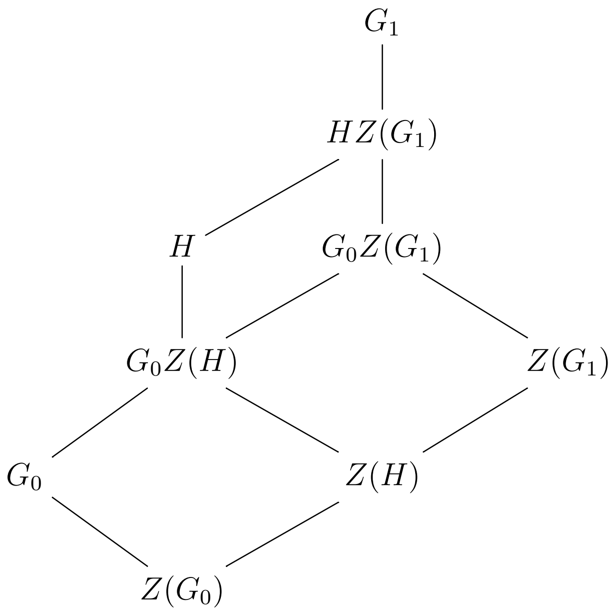
Proposition 5.1. *Suppose that G is the simple finite group of type E_6, ${}^2E_6$ or E_7. Then for any prime ℓ we have that $TF(G) \cong \mathbb{Z}$ provided G has ℓ-rank at least 2.*

Proof. In the case that the group has type E_6 or ${}^2E_6$, the center of G_{sc}, coming from the simply connected algebraic group of the same type, has order 1 or 3. If $\ell \neq 3$, then any inflation of an endotrivial kG-module to G_{sc} is also endotrivial, and the proposition follows from known results. If $\ell = 3$, then the 3-rank of G is greater than 4 and we are done by Theorem 2.3. The center of the group G_{sc} of type E_7 has order 1 or 2. The same argument as above works in this case. □

For the remainder of the section, assume that $G_0 = G_{sc}$ is a classical group, thus having one of the types $A_n, {}^2A_n, B_n, C_n, D_n$ or ${}^2D_n$. We see from Tits' Theorem [32, Theorem 24.17]) that G_0 is a perfect group, unless G_0 is isomorphic to one of $\mathrm{SL}_2(2)$,

$\mathrm{SL}_2(3)$, $\mathrm{SU}_3(2)$ or $\mathrm{Sp}_4(2)$. Moreover, except in those cases, $|G_1/G_0| = |Z(G_1)|$, and because G_1/G_0 is abelian, $G_0 = [G_1, G_1]$.

By an *associated group* of G_0, we mean a group $G = H/J$, where $G_0 \leq H \leq G_1$ and $J \leq Z(H) \leq Z(G_1)$ such that G contains the group $G_0/Z(G_0)$ as a section. For example, in type A_{n-1}, an associated group is a quotient $G = H/J$ where $\mathrm{SL}_n(q) \leq H \leq \mathrm{GL}_n(q)$ and $J \leq Z(H) \leq Z(\mathrm{GL}_n(q))$. The simple group $\mathrm{PSL}_n(q)$ is an example. In any type, a diagram for such groups has the form



where the associated group is $G = H/J$ for J some subgroup of $Z(H)$. Note that J may or may not contain $Z(G_0)$.

Our analysis will entail understanding the structure of G, and will benefit substantially from knowing when G is isomorphic to a product of groups.

Lemma 5.2. *In addition to the above notation, assume that $G_0 = [G_1, G_1]$ is a perfect group. Let π be the set of primes that divide the order of $Z(G_0)$. Let $G = H/J$ be a section of G_1 as above so that $G_0 \leq H$, $J \leq Z(G_1) \cap H$. Then there exist subgroups $H' \leq H$, $J' \leq Z(H)$ and $V \leq Z(H/J)$ such that*

$$G = H/J \cong \hat{G} \times V$$

where $\hat{G} \cong H'/J'$, $Z(\hat{G})$ and $\hat{G}/[\hat{G}, \hat{G}]$ are π-groups and V is a π'-group.

Proof. Write $G_1/G_0 \cong U_1 \times V_1$ and $Z(G_1) \cong U_0 \times V_0$ where U_i is a π-group and V_i is a π'-group for $i = 0, 1$. Let $\phi : G_1 \to V_1$ be the quotient by G_0 composed with the projection onto V_1. Let X denote the kernel of ϕ. Note that $G_0 \cap V_0 = \{1\}$ since $Z(G_0)$ is a π-group. Moreover, since $|G_1/G_0| = |Z(G_1)|$, we have that $|V_0| = |V_1|$. Consequently, the restriction of ϕ to V_0 gives an isomorphism from V_0 to V_1, and $G_1 \cong X \times V_0$.

The subgroup H contains G_0, and hence it is the inverse image under the quotient map $G_1 \to G_1/G_0$ of a subgroup $U_1' \times V_1'$ for $U_1' \leq U_1$, $V_1' \leq V_1$. Thus, $H \cong H' \times V_0'$ where H' is the inverse image under ϕ of U_1' and $V_0' \cong V_1'$ is the inverse image of V_1' under the restriction of ϕ to V_0. It follows that $Z(H) = Z(H') \times V_0'$ where $Z(H') \leq Z(X)$

is a π-group. Thus, $J = J' \times V_0''$ for $J' \le Z(H')$ and $V_0'' \le V_0'$. The lemma follows by letting $V = V_0'/V_0''$. □

The main aim of the section is to prove the following theorem.

Theorem 5.3. *Let $G_0 = \mathbb{G}^F$ be a finite group of Lie type, where $\mathbb{G}$ is a classical, simple and simply connected algebraic group. Let G be one of the associated finite groups of G_0. Assume that $\ell \ge 3$ does not divide p and that the ℓ-rank of G is at least 2. Then $TF(G) \cong \mathbb{Z}$ except in the following cases.*

(a) *If $G \cong \mathrm{PSL}_\ell(q)$ with $q \equiv 1 \pmod{\ell}$ if $\ell > 3$, and with $q \equiv 1 \pmod 9$ if $\ell = 3$, then $TF(G)$ has rank $\ell + 1$.*

(b) *If $G \cong \mathrm{PSU}_\ell(q)$ with $q \equiv -1 \pmod{\ell}$ if $\ell > 3$, and with $q \equiv -1 \pmod 9$ if $\ell = 3$, then $TF(G)$ has rank $\ell + 1$.*

(c) *If $\ell = 3$ and $G \cong {}^3D_4(q)$, then $TF(G)$ has rank 2.*

Proof. The last case (c) was treated in Section 4 (see also Theorem 3.1).

Assume that the group has the form $G = H/J$ as in the previous notation of the section. We prove the theorem for groups of Lie type B_n, C_n, D_n and ${}^2D_n$, by noticing that $G_0 = G_{sc}$ has center that has order either 2 or 4 (see [32, Table 24.2]). Consequently, if ℓ divides the order of $Z(G) = Z(H)/J$ then G has a direct factor that is a cyclic ℓ-group. In such a case the center of a Sylow ℓ-subgroup of G has ℓ-rank at least 2 and we are done. On the other hand, if ℓ does not divide the order of $Z(G)$, then by Lemma 5.2, a Sylow ℓ-subgroup of G is isomorphic to that of G_0. These cases have already been considered.

A similar thing happens in types A_n and ${}^2A_n$. That is, if ℓ does not divide the order of $Z(G_0)$, then regardless of whether ℓ divides $|Z(G)|$, we are done by the same arguments as above. Consequently, we can assume that ℓ divides the order of $Z(G_0)$, requiring that ℓ divides both $n+1$ and $q-1$ in type A_n, and that ℓ divides both $n+1$ and $q+1$ in type ${}^2A_n$.

For the untwisted type A_n, we need to consider the case when ℓ divides both $n+1$ and $q-1$. However, by Theorem 2.3, if $n+1 > \ell$, then the ℓ-rank of G is greater than ℓ, and therefore G cannot have any maximal elementary abelian ℓ-subgroup of rank 2. So it remains to consider the case $\ell = n+1$ with $q \equiv 1 \pmod{\ell}$. Similarly, in the twisted case ${}^2A_n$, we may assume that $\ell = n+1$ with $q \equiv -1 \pmod{\ell}$. In addition, by Lemma 5.2, we may assume that the orders of J and H/G_0 are powers of ℓ.

If $J = \{1\}$, then $G \le \mathrm{GL}_\ell(q)$ or $G \le \mathrm{GU}_\ell(q)$. In either case, an eigenvalue argument tells us that any element of order ℓ is conjugate to an element of the diagonal torus. Hence, we are done in this case, and we may assume that $J \ne \{1\}$.

If $J \ne Z(H)$, then there exists an element x in $Z(H)$ such that $x \notin J$ but $x^\ell \in J$. Also, because J is not trivial, there exists an element of order ℓ in the diagonal torus in H whose class in H/J is central in a Sylow ℓ-subgroup. Thus, in such a case, the center of a Sylow ℓ-subgroup of H/J has ℓ-rank 2 and we are done by Lemma 2.2. So assume that $J = Z(H)$. Thus, G is a subgroup of $\mathrm{PGL}_\ell(q)$ or $\mathrm{PGU}_\ell(q)$.

In the untwisted situation, we are down to two possibilities. First if H/G_0 is a Sylow ℓ-subgroup of G_1/G_0 then J is a Sylow ℓ-subgroup of $Z(G_1)$. In such a case $G = H/J \cong \mathrm{PGL}_\ell(q)$. This case has been treated in Section 4. In the other case, that

$J < Z(G_1)$, we have that $G \cong \mathrm{PSL}_\ell(q)$ and ℓ divides $q-1$. Similarly, in the twisted case we are down to the situation that $G \cong \mathrm{PSU}_\ell(q)$ and ℓ divides $q+1$.

Observe that if $\ell = 3$, with 3 dividing $q-1$ and 9 not dividing $q-1$, then a Sylow 3-subgroup of $\mathrm{PSL}_3(q)$ is elementary abelian of order 9. The same holds for $\mathrm{PSU}_3(q)$ if 3 divides $q+1$ and 9 does not divide $q+1$. Hence, $TF(G)$ has rank 1 in both of these cases. Thus, it remains to calculate the ranks of $TF(G)$ in the cases (a) and (b) of the theorem. These cases are covered by Proposition 4.1. □

6. When $\mathbb{G}$ is simple, $2 = \ell \neq p$

The goal of this section is to establish Theorems 6.1 and 6.2. Some results of this section will also be used in Section 8.

Theorem 6.1. *Let G be a finite group of Lie type (see Definition 1.1) with the ambient group $\mathbb{G}$ a simple algebraic group. Suppose $\ell = 2 \neq p$ and that $TF(G)$ has rank greater than 1. Then G has nonabelian dihedral Sylow 2-subgroups, $G \cong \mathrm{PGL}_2(q) \cong \mathrm{PGU}_2(q)$ for q odd, and $TF(G) \cong \mathbb{Z} \oplus \mathbb{Z}$*

We also calculate the ranks of $TF(G)$ when G is one of the associated groups in the case that $\ell = 2$ is not the defining characteristic of the group. The notion of an associated group was introduced in Section 5. We adopt the notation used at the beginning of Section 5. In particular, G_1 is one of the general linear or conformal group such as $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$ or $\mathrm{CSp}_n(q)$ and $G_0 = G_{sc}$. The group $G = H/J$ is a section of G_1 such that $G_0 \leq H \leq G_1$ and $J \leq Z(H)$.

The groups of endotrivial modules for the associated groups of type A_n are determined in the paper [15]. Our aim in this section is to take a more conceptual and less technical approach. For this reason some arguments from [15] are included here. In particular, exceptional cases occur when $G_0 \cong \mathrm{SL}_2(q)$, and some additional explanation is provided.

Our main theorem to address the associated groups is the following.

Theorem 6.2. *Let $G \cong H/J$ be an associated group of a finite group of Lie type as defined above with q odd, and let $\ell = 2$. Then $TF(G) \cong \mathbb{Z}$ is cyclic except in the following cases.*

(a) *$G = \mathrm{SL}_2(q) \cong \mathrm{SU}_2(q)$.*
(b) *$G = \mathrm{PSL}_2(q) \times C \cong \mathrm{PSU}_2(q) \times C$ with $q \equiv \pm 1 \pmod 8$ and C a cyclic group of odd order. (See Lemma 5.2.)*
(c) *$G = \mathrm{PGL}_2(q) \times C \cong \mathrm{PGU}_2(q) \times C$, where C is a cyclic group of odd order.*

In case (a), a Sylow 2-subgroup of G is quaternion and $TF(G) = \{0\}$. In cases (b) and (c), $Z(H)/J$ has odd order, a Sylow 2-subgroup of G is (nonabelian) dihedral and $TF(G) \cong \mathbb{Z} \oplus \mathbb{Z}$.

In the proof, we first show that the theorem holds for groups of large Lie rank. The groups of small Lie rank are considered on a case by case inspection. The main reduction theorem is taken from [25].

Theorem 6.3. *Let $\widehat{G} = \mathbb{G}^F$ be a finite group of Lie type in odd characteristic, with $\mathbb{G}$ simple and simply connected, and set $\ell = 2$. Then $TF(G) \cong \mathbb{Z}$, for G any associated group to $\widehat{G}$, as defined above, provided that $\widehat{G}$ is not one of the following types.*

(a) $A_1(q)$, $A_2(q)$, ${}^2A_2(q)$,
(b) $A_3(q)$ *for* $q \not\equiv 1 \pmod 8$,
(c) $A_4(q)$ *for* $q \equiv -1 \pmod 4$,
(d) ${}^2A_3(q)$ *for* $q \not\equiv 7 \pmod 8$,
(e) ${}^2A_4(q)$ *for* $q \equiv 1 \pmod 4$,
(f) $B_2(q)$,
(g) ${}^3D_4(q)$,
(h) $G_2(q)$, *or* ${}^2G_2(q)$.

Proof. Recall that by Tits' theorem [32, Theorem 24.17] $\widehat{G}/Z(\widehat{G})$ is simple, except in a few cases which are among the cases excluded above. In [25, Main Theorem], all finite simple groups having sectional 2-rank at most 4 are listed. If the finite simple group associated to $\widehat{G}$ is not on the above list, then G has sectional 2-rank greater than 4. (See [19, Section 3.5] or [27, Theorem 2.2.10] for a list of isomorphisms between finite groups of Lie type.) So G has no maximal elementary abelian 2-subgroups of rank 2, by Theorem 2.3(b) as desired. □

We may now complete the proofs of the main theorems of this section. For the proof, recall that if $G \cong A \times B$, with B of order prime to ℓ, then $TF(G) \cong TF(A)$, by Proposition 2.4.

Proof of Theorems 6.1 and 6.2. By Theorem 6.3, we need only deal with the groups listed. The Sylow 2-subgroups of finite groups of Lie type are known to be cyclic only when G is associated to a finite group of Lie type $A_1(2)$. The groups $\mathrm{SL}_2(q) \cong \mathrm{SU}_2(q)$ have quaternion Sylow 2-subgroups, and hence $TF(G) \cong \{0\}$ in those cases.

Recall that for any finite group G with (nonabelian) dihedral Sylow 2-subgroup we have $TF(G) \cong \mathbb{Z} \oplus \mathbb{Z}$ as it is not possible for the two S-conjugacy classes of elementary abelian subgroups of order 4 in S to fuse in G (cf. [33, Section 3.7]). The Sylow 2-subgroups of the groups in Theorem 6.2(b) are nonabelian dihedral. Note that if $q \equiv \pm 3 \pmod 8$ then the Sylow 2-subgroups of $\mathrm{PSL}_2(q)$ are elementary abelian of order 4, and $TF(\mathrm{PSL}_2(q)) \cong \mathbb{Z}$. It is easily verified that the Sylow 2-subgroups of $\mathrm{PGL}_2(q) \cong \mathrm{PGU}_2(q)$ are dihedral and not abelian. So $TF(G) \cong \mathbb{Z} \oplus \mathbb{Z}$ in this case.

An eigenvalue argument tells us that any involution in H for either $\mathrm{SL}_2(q) \leq H \leq \mathrm{GL}_2(q)$ or $\mathrm{SU}_2(q) \leq H \leq \mathrm{GU}_2(q)$ is conjugate to a diagonal matrix. In the unitary case, note that the eigenspaces of an involution are orthogonal to each other, so that we can construct a change of basis matrix that is unitary. Hence, $TF(G) \cong \mathbb{Z}$ if J has odd order. Therefore, for the proof for groups of type A_1, we need only consider quotients $G = H/J$ where J has even order.

Note that $\mathrm{GL}_2(q)$ is not isomorphic to $\mathrm{GU}_2(q)$. However, arguments for these cases are almost identical. That is, we can find q' with $q' \equiv -q \pmod 4$ such that $\mathrm{SL}_2(q')$ or $\mathrm{GL}_2(q')$ have isomorphic Sylow 2-subgroups to those of $\mathrm{SU}_2(q)$ or $\mathrm{GU}_2(q)$, respectively (cf. [18, Section 1]). So we prove only the linear case.

If $q \equiv 3 \pmod 4$, then 4 does not divide the order of $Z(\mathrm{GL}_2(q))$. By our assumptions, $Z(H)/J$ has odd order, and hence, by Lemma 5.2, $Z(H)/J$ is a direct factor of H/J and we are done. So we may assume that $q \equiv 1 \pmod 4$ and that $Z(H)/J$ has even order. Then there is an element z in $Z(H)$ that represents a nontrivial involution in

H/J. In addition, the diagonal matrix with entries 1 and -1 is an involution whose image in H/J is central in a Sylow 2-subgroup and distinct from the image of z. Thus, the center of a Sylow 2-subgroup of H/J has 2-rank equal to 2 and $TF(H/J) \cong \mathbb{Z}$ by Lemma 2.2.

Types A_2, A_4, ${}^2A_2$ and ${}^2A_4$. The proofs that $TF(G) \cong \mathbb{Z}$ for groups of type A_2 and A_4 are given in [15, Sections 6 and 9]. The structure of the Sylow 2-subgroups are very similar for the twisted and untwisted cases [18]. Hence, we leave the proofs of the twisted cases, ${}^2A_2$ and ${}^2A_4$, to the reader. We note that centers for all finite groups G_{sc} of these types have odd order. Consequently, by Lemma 5.2, the Sylow 2-subgroup of $Z(H)/J$ of these types is a direct factor, which can be assumed to be trivial for the purposes of the proof.

Types A_3, ${}^2A_3$ and B_2. We prove the results only for groups of type A_3 and B_2, because the proofs for groups of type ${}^2A_3$ are very similar to those of type A_3 (in the ${}^2A_3$ case, we take the matrix of the hermitian form to be the identity matrix). Following the notation introduced at the beginning of Section 6, let G_0 be $\mathrm{SL}_4(q)$ or $\mathrm{Sp}_4(q) \cong \mathrm{Spin}_5(q)$ in type A_3 or B_2, respectively. Let $G_1 = \mathrm{GL}_4(q)$ in the first case and $G_1 = \mathrm{CSp}_4(q)$ in the second. Here, $\mathrm{CSp}_4(q)$ is the group of 4×4 matrices X with entries in $\mathbb{F}_q$ having the property that $X^{tr} f X = af$ for some $a \in \mathbb{F}_q^\times$, f being the matrix of the symplectic form. For the purposes of this proof assume that the symplectic form is given as

$$f = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Let $G = H/J$ be a group associated to G_0. That is, $G_0 \leq H \leq G_1$ and $J \leq Z(H)$. Then a Sylow 2-subgroup $S = S_G$ of G is a section of a Sylow 2-subgroup S_{G_1} of G_1. Indeed, a Sylow 2-subgroup S_H of H is subgroup of a Sylow 2-subgroup R of $\mathrm{GL}_4(q)$. The group R is isomorphic to a wreath product $R = (U_1 \times U_2) \rtimes C_2$ where U_1, U_2 are Sylow 2-subgroups of $\mathrm{GL}_2(q)$ [18]. In particular, we use the following notation:

$$s(A,B) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \qquad t(A,B) = \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix} = ws(A,B),$$

where these are matrices of 2×2 blocks, A and B are elements of $\mathrm{GL}_2(q)$ and $w = t(I,I)$. Then R is generated by all $s(A,B)$ for A and B in $S_{\mathrm{GL}_2(q)}$ and the element $t(I,I)$ where I is the 2×2 identity matrix. Note that an element of J must be a scalar matrix $s(\zeta I, \zeta I)$ for some J. Because of the choices of the form, there are Sylow 2-subgroups of $\mathrm{CSp}_4(q)$ that respect this structure

Note that there exist subgroups D_J and M_H of $\mathbb{F}_q^\times$ that determine J and H. That is, J is the set of all scalar matrices with diagonal entry in D_J. In type A_3, H is the subgroup of all elements in $\mathrm{GL}_4(q)$ with determinant in M_H. In type B_2, H is the subgroup of all X with $X^{tr} f X = af$ for some $a \in M_H$.

Suppose that J has odd order. Then, by an eigenvalue argument (cf. [14, Lemma 3.3]), any involution in H is conjugate to a diagonal matrix. Note that in type B_2 (and ${}^2A_3$), the eigenspaces V_1 and V_{-1} corresponding to the eigenvalues 1 and -1 of an involution u are orthogonal to each other. Consequently, there exists a change of basis matrix that conjugates u into a diagonal matrix and also preserves the form, and it is an element of H. It follows that every elementary abelian 2-subgroup in G is conjugate to a subgroup of the image modulo J of the group of diagonal elements of order 2 in H. Hence, in this case we are finished. For the rest of the proof assume that J has even order.

Next suppose that $S_J \neq S_{Z(H)}$. That is, suppose that there is an element of the center of H whose order is a power of 2, and that is not in J. In particular there exists a scalar element of H whose square is in J. In addition, because the order of J is even, the element $s(I, -I)$ is central in $S = S_G$. Thus, $Z(S)$ has 2-rank 2 and we are done by Lemma 2.2.

We have reduced the proof to the situation in which $S_J = S_{Z(H)}$. Our aim is to show that the centralizer of every involution in S has 2-rank at least 3. This will complete the proof in the cases of types A_3 and B_2 (and ${}^2A_3$).

First consider involutions represented modulo J by a matrix of the form $s(A, B)$ in the case that $q \equiv 1 \pmod 4$ and the type is A_3 or B_2. (The argument in the case or type ${}^2A_3$ with $q \equiv 3 \pmod 4$ is very similar.) In this case, a Sylow 2-subgroup of $\mathrm{GL}_2(q)$ is generated by the elements

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad X_\zeta = \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix}$$

for ζ a generator of the Sylow 2-subgroup of $\mathbb{F}_q^\times$. Let T be the subgroup of $S_{\mathrm{GL}_2(G)}$ generated by the scalar matrices of the form $WX_{\zeta^m}WX_{\zeta^m}$ for any m. If the class of $u = s(A, B) \in H$ is an involution in H/J, then $A^2 = B^2 = \mu I$ for some $\mu \in \mathbb{F}_q^\times$. The quotient $S_{\mathrm{GL}_2(q)}/T$ is a dihedral group generated by the classes of W and X_ζ. An involution in this group must be represented by either W or X_{ζ^m} for some m. Then if the class of $u = s(A, B)$ is an involution in H/J, it has either the form $s(X_{\zeta^m}, X_{\zeta^m})$ or $s(A, B)$ with A and B in the subgroup $V = \langle X_{-1}, W\rangle$. Now notice that the subgroup generated by w and all $s(A, B)$ with $A, B \in V$ is elementary abelian of 2-rank at least 3. If $u = s(X_{\zeta^m}, X_{\zeta^m})$ is in H, then so also is w and $s(I, -I)$, and the classes of these elements generate a subgroup of H/J having 2-rank 3. So we are done in this case.

Next suppose that the class of $s(A, B)$ is an involution in H/J, in the case that $q \equiv 3 \pmod 4$ and the type is A_3 or B_2. (The same argument works when the type is ${}^2A_3$ with $q \equiv 1 \pmod 4$.) In this case $J = Z(\mathrm{GL}_4(q))$ has order 2 and is generated by $-I_4$, where I_4 is the 4×4 identity matrix. A Sylow 2-subgroup $S_{\mathrm{GL}_2(q)}$ is semidihedral. In this case one of two things can happen. The first is that A and B are actual involutions. If A is a noncentral involution, the subgroup generated by the classes of w, $s(A, A)$ and $s(I, -I)$ has 2-rank 3 in H/J. The other possibility is that A and B have order 4 and commute modulo J. The only possibility here is that A and B are contained in a quarternionic subgroup of order 8 in $S_{\mathrm{GL}_2(q)}$. If A is not contained in the subgroup generated by B then the classes of w, $s(A, B)$, and $s(B, A)$ generate an elementary abelian subgroup in H/J of order 8. Otherwise, let X be another generator of the quaternionic subgroup. Then the classes of w, $s(A, B)$ and $s(X, X)$ generate an elementary abelian subgroup of order 8. So we are done in this case.

Finally, suppose that the class of $u = t(A, B) = ws(A, B)$ is an involution in H/J. It must be that $AB = BA = \mu I$ for some $\mu \in \mathbb{F}_q^\times$. That is, $B = \mu A^{-1}$. In the case that the type is A_3, then $s(A, I)^{-1}t(I, \mu I)s(A, I) = t(A, B)$. So every such involution is conjugate to one of the form $y_\mu = t(I, \mu I)$. In turn, any y_μ commutes with any involution $s(A, A)$ for A not central in $S_{\mathrm{GL}_2(q)}$. Thus, in type A_3, the centralizer of u has 2-rank at least 3, and we are done.

So suppose the type is B_2. We have that $ufu^{tr} = \mu f$ implying that $AYA^{tr} = Y$, as expected. A set of representatives of the generators of $S_{\mathrm{GL}_2(q)}$ can be chosen so that their product with their transpose is a scalar matrix (see the above descriptions in addition to [18]). The implication is that $v = t(y, y)$ commutes with u. Thus, the centralizer of u has 2-rank at least 3, as it contains the image in H/J of $\langle u, j, t(-I, I)\rangle$.

To summarize, we have proved that the centralizers of the involutions in a group associated to a finite group of Lie type A_3, ${}^2A_3$ and B_2 have 2-rank at least 3, and so there are no maximal elementary abelian 2-subgroups of rank 2.

Types ${}^3D_4$, G_2 and ${}^2G_2$. Fong and Milgram [22] studied in great detail the 2-local structure of G in the case that G has type ${}^3D_4$ or G_2, and described the structure of the centralizers of the Klein four groups in a fixed Sylow 2-subgroup of G. They proved that these split into two conjugacy classes and that their centralizers both have 2-rank 3. While they assumed that $q \equiv 1 \pmod 4$, the Sylow 2-subgroups are isomorphic to those in the case where $q \equiv 3 \pmod 4$. So the same conclusion is reached. A detailed description in the general case is in the paper by Fong and Wong [23]. Note that $G_2(q)$ embeds in ${}^3D_4(q)$ as a subgroup of odd index, and hence their Sylow 2-subgroups are isomorphic (see also [23, Theorem]). We are left with the case of the groups ${}^2G_2(3^{2n+1})$. By [27, Theorem 4.10.2(e)] (see also [36, Theorem 8.5]), a Sylow 2-subgroup of ${}^2G_2(3^{2n+1})$ is elementary abelian of order 8, and so there are no maximal elementary abelian 2-subgroups of rank 2.

This completes the proof of Theorems 6.1 and 6.2. □

7. When $\mathbb{G}$ is simple, $\ell = p$

When $\ell = p$, the structure of a Sylow ℓ–subgroup of G does not depend on the isogeny type. However, $TF(G)$ can and does depend on the isogeny type because of the fusion of ℓ-subgroups. The following theorem summarizes the calculation of $TF(G)$ in the defining characteristic.

Theorem 7.1. *Let G be a finite group of Lie type, as in Definition 1.1. Assume that the ambient algebraic group $\mathbb{G}$ is simple, and $\ell = p$. Then $TF(G) \cong \mathbb{Z}$, provided G is not one of the following types.*

- (a) $A_1(p)$,
- (b) ${}^2A_2(p)$,
- (c) ${}^2B_2(2^{2a+1})$ *(for $a \geq 1$)*,
- (d) ${}^2G_2(3^{2a+1})$ *(for $a \geq 0$)*,
- (e) $A_2(p)$,
- (f) $B_2(p)$ *and*
- (g) $G_2(p)$.

In these exceptions, $TF(G)$ is given in Tables 7.1 and 7.2.

We proceed to justify this result. For the simple algebraic group $\mathbb{G}$ fix an F-stable maximal split torus $\mathbb{T}$. Let Φ be the root system associated to $(\mathbb{G},\mathbb{T})$. The positive (resp. negative) roots are Φ^+ (resp. Φ^-), and Δ is a base consisting of simple roots.

Let $\mathbb{B}$ be an F-stable Borel subgroup containing $\mathbb{T}$ corresponding to the positive roots, and $\mathbb{U}$ be the unipotent radical of $\mathbb{B}$. Then $\mathbb{B} = \mathbb{U} \rtimes \mathbb{T}$ with $\mathbb{B}$ and $\mathbb{U}$ being F-stable. Set $B = \mathbb{B}^F$ and $U = \mathbb{U}^F$.

There are three kinds of finite groups of Lie type G according to the type of F: (i) the untwisted groups, (ii) the twisted (Steinberg) groups and (iii) the very twisted groups (cf. [13, Section 4], [27, Section 2.3]). In case (ii), F involves a nontrivial graph automorphism τ of order d of the underlying Dynkin diagram, as well as the Frobenius map. The automorphism τ induces a map from Φ to the *twisted root system* $\widetilde{\Phi}$ of G. Furthermore, we can define an equivalence relation on $\widetilde{\Phi}$ by identifying positive colinear roots, and let $\widehat{\Phi}$ be the set of equivalence classes. Therefore, we have mappings $\Phi \to \widetilde{\Phi} \to \widehat{\Phi}$. Let $\widehat{\Delta}$ be the image of Δ under this composition of maps and $\widetilde{\Delta}$ be the image of Δ under $\Phi \to \widetilde{\Phi}$. There are root subgroups of G and these are indexed by the elements of $\widehat{\Phi}$. In the case that G is untwisted then $\Phi = \widetilde{\Phi} = \widehat{\Phi}$. In case G is a Steinberg group but not ${}^2A_{2m}(q)$ we have $\widetilde{\Phi} = \widehat{\Phi}$ (cf. [27, Section 2.3] for more details).

As stated in the proof of [32, Proposition 24.21], there is a short exact sequence of groups

$$1 \longrightarrow Z^F \longrightarrow G_{sc} \longrightarrow G \longrightarrow Z_F \longrightarrow 1 .$$

In the case that $\ell = p$, U is a Sylow p-subgroup of G. From [32, Table 24.2], p does not divide $|Z^F|$. Therefore, the Sylow p-subgroups of G_{sc} and of G are isomorphic for any isogeny type, and so $TF(U_{sc}) \cong TF(U)$.

Given a finite group of Lie type G where the underlying algebraic group is simple when $\ell = p$, one can make reductions to analyzing $TF(G)$ in specific cases as follows. First, $TF(G) \cong \mathbb{Z}$ when $|\widehat{\Delta}| \geq 3$ by [13, Theorems 7.3 and 7.5]. Note that the proofs of these results depend only on the structure of the Sylow ℓ-subgroups. In the case when $|\widehat{\Delta}| = 2$, by [13, Theorems 7.3 and 7.5], $TF(G) \cong \mathbb{Z}$ unless G is $A_2(p)$, $B_2(p)$ or $G_2(p)$. (Recall that we use the non-standard notation that e.g., $B_2(p)$ without any subscript denotes *any* group in this isogeny class.) The computation for $TF(G)$ for these groups is given in Table 7.1.

Table 7.1: $|\widehat{\Delta}| = 2$

G		rank $TF(G)$
$A_2(p)_{sc}$	$p = 2$	2
$A_2(p)_{sc}$	$p \geq 3$, $p \not\equiv 1 \pmod 3$	3
$A_2(p)_{sc}$	$p \geq 3$, $p \equiv 1 \pmod 3$	5
$A_2(p)_{ad}$	$p = 2$	2
$A_2(p)_{ad}$	$p \geq 3$	3
$B_2(p)$	$p = 2, 3$	1
$B_2(p)$	$p \geq 5$	2
$G_2(p)$	$p = 2, 3, 5$	1
$G_2(p)$	$p \geq 7$	2

Finally, in the case that $|\widehat{\Delta}| = 1$, the Sylow ℓ-subgroups are trivial intersection subgroups. The groups G with $|\widehat{\Delta}| = 1$ are $A_1(q)$, ${}^2A_2(q)$, ${}^2B_2(2^{2a+1})$, and ${}^2G_2(3^{2a+1})$. If $G = A_1(q)$ or ${}^2A_2(q)$ with $q > p$, the Sylow p-subgroups of G have a noncyclic center, and therefore $TF(G) \cong \mathbb{Z}$ by Theorem 1.2. For the rest of the cases when $|\widehat{\Delta}| = 1$, $TF(G)$ is given in Table 7.2 (cf. [13, Section 5]).

Table 7.2: $|\widehat{\Delta}| = 1$

G		rank $TF(G)$
$A_1(p)$	$p \geq 2$	0
${}^2A_2(p)_{sc}$	$p = 2$	0
${}^2A_2(p)_{sc}$	$p \geq 3$, $p \not\equiv -1 \pmod 3$	1
${}^2A_2(p)_{sc}$	$p \geq 3$, $p \equiv -1 \pmod 3$	3
${}^2A_2(p)_{ad}$	$p = 2$	0
${}^2A_2(p)_{ad}$	$p \geq 3$	1
${}^2B_2(2)$		0
${}^2B_2(2^{2a+1})$	$a > 0$	1
${}^2G_2(3^{2a+1})$	$a \geq 0$	1

There is still some explanation needed to justify the data in the tables. We rely on some of the computations in [13] in cases where there is one isogeny type. The results in [13] were only stated for the finite groups of Lie type arising from groups of adjoint isogeny type. Our new result, Theorem 7.1, extends to all finite groups of Lie type. We now proceed to dissect the cases when there is more than one isogeny type.

For $A_1(p)$ a Sylow p-subgroup is cyclic of order p, and so $TF(G)$ does not depend on the isogeny type. For $B_2(p) = C_2(p)$, we can use the calculations in [13, Section 8] which handle $B_2(p)_{sc}$ and $B_2(p)_{ad}$.

Next we consider the case of $A_2(p)$ where there are two isogeny types. Let $U \cong U_{sc} \cong U_{ad}$ denote a Sylow p-subgroup in either type. The Sylow p-subgroup U of G is an extraspecial p-group of order p^3 and exponent p, if $p > 2$. Moreover, if $p = 2$ then $\mathrm{SL}_3(2) \cong \mathrm{PSL}_2(7)$ so U is a dihedral group of order 8, and has two maximal elementary abelian 2-subgroups which are not conjugate in U or in G. Consequently, $TF(G) \cong \mathbb{Z} \oplus \mathbb{Z}$.

If $p > 2$ when G is of type $A_2(p)$, then all the elements of U have order p, and the maximal elementary abelian p-subgroups have rank 2. Set

$$x_{\alpha+\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad x_\alpha^i x_\beta^j = \begin{bmatrix} 1 & 0 & 0 \\ i & 1 & 0 \\ 0 & j & 1 \end{bmatrix}$$

The maximal elementary abelian p-subgroups of B all contain the central subgroup generated by $x_{\alpha+\beta}$, and one can choose as the other generator an element of the form $x_\alpha^i x_\beta^j$ (i.e., elements in the Frattini quotient of U, $U/\Phi(U)$).

Since $B \cong U \rtimes T$ stabilizes the central subgroup of U, it follows that the B-conjugacy classes of maximal elementary abelian p-subgroups are in one to one correspondence with the T-conjugacy classes on $X = U/\Phi(U)$.

Consider the action by conjugation of the group $T = \{t_{a,b,c} \mid \; a, b, c \in \mathbb{F}_p^\times\}$ where $t_{a,b,c}$ is the 3×3 diagonal matrix with entries a, b, c. Let $|X/T|$ be the number of T-conjugacy classes on X. Then by a well-known lemma stated by Burnside (due to Frobenius):

$$|X/T| = \frac{1}{|T|}\sum_{t\in T}|X^t|.$$

where $X^t = \{x \in X \mid t.x = x\}$. In this case, a direct computation shows that

$$|X^{t_{a,b,c}}| = \begin{cases} 0 & a \neq b \text{ and } b \neq c, \\ p^2-1 & a = b = c, \\ p-1 & [a = b \text{ and } b \neq c] \text{ or } [a \neq b \text{ and } b = c]. \end{cases} \tag{7.1}$$

By keeping track of the number of elements that occurs in each case of (7.1), it follows that

$$|X/T| = \frac{1}{(p-1)^3}[(p-1)(p^2-1) + 2(p-1)(p-2)(p-1)] = 3.$$

Consequently, for $G = \mathrm{GL}_3(p)$, $TF(B) = \mathbb{Z}^{\oplus 3}$. The argument can be easily adapted to also show that for $G = \mathrm{PGL}_3(p)$, and for $\mathrm{SL}_3(p)$ when $p \not\equiv 1 \pmod 3$, one has $|X/T| = 3$, and $TF(B) = \mathbb{Z}^{\oplus 3}$.

Now, set $T = \{t_{a,b,c} \mid abc = 1\}$ and consider $\mathrm{SL}_3(p)$ for $p \equiv 1 \pmod 3$. Then (7.1) yields

$$|X/T| = \frac{1}{(p-1)^2}[3(p^2-1) + 2(p-4)(p-1)] = 5.$$

Consequently, $TF(B) = \mathbb{Z}^{\oplus 5}$. Finally, for all the cases when $G = A_2(p)$ one has $TF(G) \cong TF(B)$ by using the Bruhat decomposition.

Next we consider the case of ${}^2A_2(p)$. When $p = 2$, U is a quaternion group and the 2-rank of U is 1. Therefore, in this case $TF(G) = \{0\}$.

Now assume that $p \geq 3$. The case where $G = \mathrm{SU}_3(p)$ was done in [13, Section 5]. This corresponds to ${}^2A_2(p)_{sc}$ (not ${}^2A_2(p)_{ad}$ which is incorrectly stated in [13, Section 5]).

Now consider $G = \mathrm{PGU}_3(p)$ for $p \geq 3$. We will use explicit matrices in $\mathrm{GU}_3(p)$ and the conventions in [13, Section 5]. As in the untwisted case we consider $D = \{t_{a,b,c} \mid a, b, c \in \mathbb{F}_{p^2}^\times\}$, and $D \cap \mathrm{GU}_3(p)$. The relations we obtain by intersecting are $ac^p = 1$, $b^{p+1} = 1$, and $ca^p = 1$. In U there are $p + 1$ elementary abelian p-subgroups of p-rank 2 given by $E_i = \langle x_i, z\rangle$, $1 \leq i \leq p + 1$. Let t be a generator for $\mathbb{F}_{p^2}^\times$. The elements x_i and z are defined by

$$x_i = \begin{pmatrix} 1 & 0 & 0 \\ t^i & 1 & 0 \\ b_i & t^{ip} & 1 \end{pmatrix} \quad \text{with } b_i + b_i^p = t^{i(p+1)}\ , \tag{7.2}$$

$$z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & 0 & 1 \end{pmatrix} \quad \text{where } u \in \mathbb{F}_{p^2} \text{ satisfies } \; u + u^p = 0.$$

For any j, we can find $a \in \mathbb{F}_{p^2}^{\times}$ and b, c such that $a^{-1}b = t^j$ satisfying the aforementioned relations as follows. Set $a = t^{(p-1)-j}$, $b = t^{p-1}$ and $c = t^{-((p-1)-j)p}$. Then

$$t_{a,b,c} x_i t_{a,b,c}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a^{-1}bt^i & 1 & 0 \\ a^{-1}cb_i & b^{-1}ct^{ip} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ t^{i+j} & 1 & 0 \\ a^{-1}cb_i & t^{(i+j)p} & 1 \end{bmatrix}. \tag{7.3}$$

One can verify that $a^{-1}cb_i$ satisfies the equation in (7.2) with i replaced with $i+j$. This shows that under conjugation by elements in $D \cap \mathrm{GU}_3(p)$, there is a single conjugacy class among $\{E_i \mid 1 \leq i \leq p+1\}$. Hence, for $G = \mathrm{PGU}_3(p)$ with $p \geq 3$, $TF(G) \cong \mathbb{Z}$.

8. Extending the results from simple to reductive groups

Let $G = \mathbb{G}^F$ be a finite group of Lie type arising from a connected reductive algebraic group $\mathbb{G}$ and a Steinberg endomorphism F of $\mathbb{G}$. In this section, we show that the torsion free rank of the group of endotrivial modules of G can be obtained by considering the components of the decomposition of $\mathbb{G}$ as a product of simple algebraic groups. Our detailed analysis completes the proofs of Theorems A and B.

From [17, 1.8], we have that $\mathbb{G} = [\mathbb{G}, \mathbb{G}] \cdot \mathbb{S}$ where the derived subgroup $[\mathbb{G}, \mathbb{G}]$ is semisimple and $\mathbb{S} = Z(\mathbb{G})^0$ is the connected center of $\mathbb{G}$. The intersection of these groups $Z = [\mathbb{G}, \mathbb{G}] \cap \mathbb{S}$ is a finite group. Therefore, we have an exact sequence

$$1 \longrightarrow Z \longrightarrow [\mathbb{G}, \mathbb{G}] \times \mathbb{S} \longrightarrow \mathbb{G} \longrightarrow 1. \tag{8.1}$$

Set $G = \mathbb{G}^F$ and $G_{ss} = [\mathbb{G}, \mathbb{G}]^F$. Upon taking fixed points, one obtains an exact sequence (cf. [32, Lemma 24.20])

$$1 \longrightarrow Z^F \longrightarrow G_{ss} \times \mathbb{S}^F \xrightarrow{\psi} G \longrightarrow Z_F \longrightarrow 1 \tag{8.2}$$

with Z_F denoting co-invariants. Here, ψ is injective on restriction to both G_{ss} and $\mathbb{S}^F$.

Since $[\mathbb{G}, \mathbb{G}]$ is semisimple one can express $[\mathbb{G}, \mathbb{G}] = \mathbb{H}_1 \cdots \mathbb{H}_s$ where each $\mathbb{H}_i$ is a central product of n_i isomorphic simple algebraic groups $\mathbb{K}_i$ where F preserves $\mathbb{H}_i$ and $\mathbb{H}_i^F \cong \mathbb{K}_i^{F^{n_i}}$ [27, Proposition 2.2.11], the fixed points of $\mathbb{K}_i$ under F^{n_i}. So there is an exact sequence

$$1 \longrightarrow A \longrightarrow \mathbb{H}_1 \times \cdots \times \mathbb{H}_s \longrightarrow [\mathbb{G}, \mathbb{G}] \longrightarrow 1 \tag{8.3}$$

for a finite abelian group A of order prime to p. Once again, we apply [32, Lemma 24.20] to get the exact sequence

$$1 \longrightarrow A^F \longrightarrow \mathbb{H}_1^F \times \cdots \times \mathbb{H}_s^F \longrightarrow G_{ss} \longrightarrow A_F \longrightarrow 1. \tag{8.4}$$

For each i, set $H_i = \mathbb{H}_i^F \leq G_{ss}$. In addition, we have the following statements.

(i) $|Z_F| = |Z^F|$ and $|A_F| = |A^F|$.

(ii) Suppose that x is an element in G that is not in G_{ss}. For any i, conjugation by x preserves H_i. Moreover, if H_i is isomorphic to $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$ or $\mathrm{Sp}_n(q)$, then x induces on H_i an automorphism that coincides with conjugation by an element in (respectively) $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$ or $\mathrm{CSp}_n(q)$.

The equalities in (i) follow from the fact that the order of a finite group of Lie type is independent of the isogeny type, which is a consequence of the order formula [32, Corollary 24.6]. For (ii), let $x \in G$ with $x \notin G_{ss}$. From (8.1), $x = gz$ where $g \in [\mathbb{G}, \mathbb{G}]$ and $z \in \mathbb{S}$ with $z \neq 1$. Here $F(x) = x$, so that $g^{-1}F(g) = zF(z^{-1})$. Moreover, from (8.3), $g = h_1 h_2 \dots h_s$ with $h_j \in \mathbb{H}_j$ for $j = 1, 2, \dots, s$. Because z is central and $H_1 \cdots H_s$ is a central product, action of conjugation by x on H_i is the same as conjugation by h_i. Thus, h_i is an element of $\mathbb{H}_i$ that normalizes H_i. As explained in [27, Proposition 2.5.9(b)], this means that h_i lies in the preimage of $(\mathbb{H}_i/Z)^F$ in $\mathbb{H}_i$, with Z a central subgroup of $\mathbb{H}_i$. Now, if H_i is $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$ or $\mathrm{Sp}_n(q)$, then we can without restriction assume that $\mathbb{K}_i$ is either SL_n or Sp_n. Let $\tilde{\mathbb{K}}_i$ be GL_n and CSp_n respectively, and let $\tilde{\mathbb{H}}_i$ be the corresponding central product, constructed as for $\mathbb{H}_i$. Note that $\mathbb{H}_i \leq \tilde{\mathbb{H}}_i$, that the central subgroup $\tilde{Z}$ of $\tilde{\mathbb{H}}_i$ is connected, and that $(\mathbb{H}_i/Z)^F \cong (\tilde{\mathbb{H}}_i/\tilde{Z})^F$. The preimage of $(\tilde{\mathbb{H}}_i/\tilde{Z})^F$ in $\tilde{\mathbb{H}}_i$ equals $\tilde{\mathbb{H}}_i^F \tilde{Z}$, as $\tilde{Z}$ is connected, so $h_i \in \tilde{\mathbb{H}}_i^F \tilde{Z}$. Hence, h_i, and therefore x, induce the same conjugation on H_i as an element in $\tilde{\mathbb{H}}_i^F$, which is what we claimed in (ii). The main theorem of this section is the following.

Theorem 8.1. *Suppose that G is a finite group of Lie type with $G = \mathbb{G}^F$ for $\mathbb{G}$ a connected reductive algebraic group over an algebraically closed field of characteristic p, and F a Steinberg endomorphism. Assume that $TF(G)$ has rank greater than 1.*

If $\ell \neq p$ then $G \cong U \times V$ where V has order prime to ℓ and $TF(G) \cong TF(U)$. Moreover,

(a) *if $2 < \ell \neq p$ then U is one of the groups listed in Theorem 3.1, and*
(b) *if $\ell = 2 \neq p$ then U is one of the groups listed in Theorem 6.1 and V is abelian.*

In the event that $\ell = p$, then $G/Z(G) \cong H/Z(H)$, where H is one of the groups in Tables 7.1 and 7.2.

The proof is divided into three cases. First we deal with $\ell = p$, and then with $\ell \neq p$, which is again divided into two steps depending on whether ℓ is odd or even. Throughout the proof we employ the conventions introduced prior to the theorem.

Observe first that if $G = U \times V$, and ℓ does not divide $|V|$, then the restriction map provides an isomorphism $TF(G) \xrightarrow{\cong} TF(U)$. This is because, in this case, any endotrivial kU-module becomes an endotrivial kG-module on inflation, so the restriction map $T(G) \to T(U)$ is surjective; and it has finite kernel, again because the index of U in G is prime to ℓ.

Proof of Theorem 8.1 when $\ell = p$. In this case the groups Z^F and Z_F have order relatively prime to ℓ. Hence, ψ induces an isomorphism on Sylow ℓ-subgroups. Note that, as we are in the defining characteristic, ℓ divides the order of each H_i. However, then $s = 1$ in (8.4), as otherwise a Sylow ℓ-subgroup S of G would split as a non-trivial direct product implying $TF(G) \cong \mathbb{Z}$ by Lemma 2.2. This also means that $A = 1$, and $G_{ss} = H_1$. We have a central extension $1 \to \mathbb{S} \to \mathbb{G} \to \mathbb{G}/\mathbb{S} \to 1$ producing on fixed-points another central extension

$$1 \to \mathbb{S}^F \to \mathbb{G}^F \to (\mathbb{G}/\mathbb{S})^F \to 1$$

where $(\mathbb{G}/\mathbb{S})^F \cong \mathbb{K}^{F^{n_1}}$ for some simple algebraic group $\mathbb{K}$ by [27, Proposition 2.2.11]. Now set $H = \mathbb{K}^{F^{n_1}}$ so that $G/Z(G) \cong H/Z(H)$. Observe that $TF(G) \xrightarrow{\cong} TF(H)$ by Proposition 2.4. Hence, Theorem 7.1 says that H is one of the groups listed in Tables 7.1 and 7.2. □

Proof of Theorem 8.1 when $3 \leq \ell \neq p$. Assume that $TF(G)$ is not cyclic.

STEP 1: We prove first that the prime ℓ does not divide $|H_i|$ for more than one i. Assume that $TF(G)$ is not cyclic and that there is more than one H_i whose order is divisible by ℓ. Note that ℓ has to divide $|Z(H_i)|$ every time it divides $|H_i|$, since otherwise a Sylow ℓ-subgroup S of G splits as a non-trivial direct factor implying that $Z(S)$ has ℓ-rank at least 2. This means that we are done by Lemma 2.2. The tables of centers of the finite groups of Lie type (cf. [32, Table 24.2]) show that if ℓ divides $|Z(H_i)|$, then H_i has one of the types: $A_{n-1}(q)$ for $\ell \mid (n, q-1)$, ${}^2A_{n-1}(q)$ for $\ell \mid (n, q+1)$, $E_6(q)$ with $\ell = 3$, or ${}^2E_6(q)$ with $\ell = 3$. Hence, we can assume that H_i is one of these types when ℓ divides $|Z(H_i)|$. The two last cases, involving the groups of type E, can furthermore be eliminated, using Theorem 2.3, as the 3-ranks of $E_6(q)$ and ${}^2E_6(q)$ are 6.

We now deal with the groups of type A. Because ℓ divides n, the ℓ-ranks of these groups are at least $\ell - 1$. Therefore, if we have more than one H_i of order divisible by ℓ, and none of the groups splits off as a direct factor, the ℓ-rank of the resulting group will be at least $(\ell-1)+(\ell-1)-1 = 2\ell-3$. This number has to be at most ℓ by Theorem 2.3. So we conclude that the only possibility is that $\ell = 3$ and $n = 2$, assuming that ℓ divides the order of the center of H_i.

Note that if there is an H_i whose order is not divisible by 3, then H_i is a Suzuki group (Lie type ${}^2B_2$), and these groups have trivial centers. So for the purposes of our argument, we may assume that there are exactly two components H_1 and H_2 both having order divisible by 3. Moreover, because $Z(H_1)$ and $Z(H_2)$ are not trivial we have that these groups must be the finite groups arising from the simply connected algebraic groups: $H_i = \mathrm{SL}_3(q_i)$ where 3 divides $q_i - 1$, or $H_i = \mathrm{SU}_3(q_i)$ with 3 dividing $q_i + 1$. Let 3^{t_i} be the highest power of 3 dividing $q_i - 1$ in the first case and dividing $q_i + 1$ in the second.

In the exact sequence (8.4), the image of the group A^F is central in $H_1 \times H_2$ and hence it must have order either 1 or 3. Similarly in sequence (8.2), the image of Z^F in $H_1H_2 = G_{ss}$ is central and its order is either 1 or 3. We claim first that if $A^F = \{1\}$, then we are done. The reason is that then $G_{ss} \cong H_1 \times H_2$ which has 3-rank 4. The map ψ is injective on G_{ss}, so that G also has 3-rank 4, and we are finished by Theorem 2.3(a). Hence, $G_{ss} = H_1H_2$ is the central product of H_1 and H_2 over a central subgroup of order 3.

Let S_i be a Sylow 3-subgroup of H_i and S a Sylow 3-subgroup of G. Each S_i can be chosen to have a maximal toral subgroup $T_i = C_{3^{t_i}} \times C_{3^{t_i}}$ of diagonal matrices with an element of order 3 in the form of a permutation matrix acting on it. Thus, its center has order 3^{t_i}.

Suppose that $|Z^F| = 1$. In the event that both t_1 and t_2 are greater than 1, there are elements $y_1 \in Z(S_1)$ and $y_2 \in Z(S_2)$ having order 9 such that $y_1^3 = z_1$ and $y_2^3 = z_2$ are the central elements in H_1 and H_2 that are identified when A^F is factored out. Thus, the classes of $y_1y_2^{-1}$ and z_2 modulo A^F are in the center of S and the center of S has 3-rank equal to 2. Consequently, we are done in this case and we may assume that $t_1 = 1$.

Still assuming that $|Z^F| = 1$, we are down to the situation that S_1 is an extraspecial group of order 27 and exponent 3. If the class of $(x, y) \in S_1 \times S_2$ modulo A^F has order 3, then $(x, y)^3 = (1, y^3) \in A^F$ and y has order 3. Thus, the class of (x, y) modulo A^F commutes with those of $(x, 1)$ and $(1, y)$. In this way we see that the centralizer of every element of order 3 in S has 3-rank at least 3, and we are done with this case.

We conclude that $|Z^F| = 3$ and we can assume that S is an extension:

$$1 \longrightarrow S_1S_2 \longrightarrow S \longrightarrow Z_F \longrightarrow 1$$

where Z_F is cyclic of order 3. From the above arguments, we know that the centralizers of elements of order 3 in S_1S_2 have 3-rank 3. For the purposes of this proof, assume that $H_i \cong \mathrm{SL}_3(q_i)$. Let $x \in S$ be an element of order 3 that is not in S_1S_2. Then x must act on S_1 as conjugation by an element $\hat{x}$ of $\mathrm{GL}_3(q_1)$. So $\hat{x}$ is conjugate (by an element $\mathrm{SL}_3(q_1)$) to an element of the diagonal torus. Therefore, its centralizer K_1 in $H_1 \cong \mathrm{SL}_3(q_1)$ has 3-rank 2. The same happens for the centralizer K_2 of its action on H_2. By a similar argument, the same condition holds when H_1 or H_2 is isomorphic to $\mathrm{SU}_3(q)$. It follows that the subgroup of G generated by x, K_1 and K_2 has 3-rank at least 4. Hence, G has 3-rank at least 4 and we are done by Theorem 2.3(a). This completes the first step.

Step 2: In this step we complete the proof assuming that ℓ divides $|H_1|$ and does not divide $|H_i|$ for $i > 1$. Assume that $TF(G)$ has rank greater than 1. We wish to show that G has the form $U \times V$, where V has order prime to ℓ and U is one of the groups listed in Theorem 3.1.

If $\ell \nmid |Z(H_1)|$, then a Sylow ℓ-subgroup of H_1 is a direct factor in some Sylow ℓ-subgroup of G. As the ℓ-part of the center of a Sylow ℓ-subgroup of G is cyclic if the rank of $TF(G)$ is greater than one, we conclude that $|\mathbb{S}^F|$ is prime to ℓ. Hence, G has the same ℓ-local structure as H_1. Theorem 3.1 now shows that H_1 is isomorphic to one of the groups listed in that theorem. In particular $Z(H_1) = 1$, so $G \cong H_1 \times V$ for some ℓ'-group V, as asserted.

Next suppose that $\ell \mid |Z(H_1)|$. Our aim is to prove that there are no groups with $TF(G)$ having rank greater than 1 that can occur, thus finishing the proof in the case that $\ell \geq 3$. First note that, with our assumptions, G has the same ℓ-local structure as $(\mathbb{G}/(\mathbb{H}_2 \cdots \mathbb{H}_s))^F$, and that the ℓ-part of $\mathbb{S}^F$ is cyclic as the ℓ-part of $Z(G)$ is. The rank argument from Step 1 shows that H_1 must have Lie type A. More precisely, we must have $H_1 \cong \mathrm{SL}_\ell(q)$ with $\ell \mid (q-1)$ or $H_1 \cong \mathrm{SU}_\ell(q)$, with $\ell \mid (q+1)$. The sequence (8.2) shows that the ℓ-local structure of G must agree with that of a central product $\langle H_1, \zeta\rangle\Delta$ where ζ is an element with determinant of order ℓ inside $\mathrm{GL}_\ell(q)$ or $\mathrm{GU}_\ell(q)$, Δ is cyclic of order ℓ^t, for some t, and $\langle H_1, \zeta\rangle \cap \Delta$ has order ℓ. However, such a group has the same poset of conjugacy classes of elementary abelian ℓ-subgroup as $\langle H_1, \zeta\rangle$, which is an associated group as defined in Section 5. Hence, the torsion free rank of the group of endotrivial modules cannot be larger than 1, as the group does not appear in Theorem 5.3. $\square$

Proof of Theorem 8.1 when $2 = \ell \neq p$. Assume first that $s > 1$ and that $TF(G)$ has rank greater than 1. We want to show that this case cannot occur. Observe first that every factor H_i, being a nonabelian finite group of Lie type, has even order, as does $H_i/Z(H_i)$. In addition, the order of the center of any factor must be even, as otherwise a Sylow 2-subgroup of H_i is a direct factor of some Sylow 2-subgroup of G and hence its center has 2-rank greater than 1. As a result we can assume that every H_i has type A_n, for n odd, B_n, C_n, D_n or E_7 by the table of orders of centers in [32, Table 24.2].

Recall that by Theorem 2.3, the sectional 2-rank of G can not be 5 or more. The group G contains the direct product $H_1/Z(H_1) \times \cdots \times H_s/Z(H_s)$ as a section. From the proof of Theorem 6.2, we know that the sectional 2-rank of a group of type A_1 or ${}^2A_1$ is 2, while the sectional 2-rank of a group of type A_n or ${}^2A_n$ for $n \geq 3$ is at least 3. In addition, the sectional 2-ranks for groups of types B_n, C_n, D_n and E_7 are at least 3. As a result, the only possible situation with sectional 2-rank less than 5 occurs when there are exactly two components H_1 and H_2 both of type A_1 or ${}^2A_1$. We henceforth assume that this is the situation.

Because ψ is injective on restriction to $\mathbb{S}^F$. It must be that Z^F is either trivial or has order 2. In addition, the image W of the inclusion of Z^F into $G_{ss} \times \mathbb{S}^F$ followed by the projection onto $\mathbb{S}^F$ must be the Sylow 2-subgroup of $\mathbb{S}^F$. The reason is that otherwise, the quotient group $G_{ss}/Z(G_{ss}) \times \mathbb{S}^F/W$, which is a section of G, has sectional 2-rank 5 and by Theorem 2.3(b), $TF(G) \cong \mathbb{Z}$. If Z^F is trivial, then so is Z_F and a Sylow 2-subgroup S of G is either a direct product or a central product of quaternion groups. In the first case, $Z(S)$ has 2-rank 2 and we are done by Lemma 2.2. A direct calculation shows that the all maximal elementary abelian 2-subgroups of a central product of quaternion groups have 2-rank 3.

Hence, we may assume that Z^F has order 2 and that S is an extension (cf. the exact sequence (8.2))

$$1 \longrightarrow S_1S_2 \longrightarrow S \longrightarrow C_2 \longrightarrow 1$$

where S_1, S_2 are normal quaternion subgroups and S_1S_2 is a central product. We have noted already that the centralizer of any involution in S_1S_2 has 2-rank 3. We need only show the same for any involution x not in S_1S_2. The involution x must act on each S_i as an element of $\mathrm{GL}_2(q)$, which means that it must normalize, but not centralize, some (necessarily cyclic, since S_i are quaternion) subgroup $\langle y_1 \rangle$ of order 4 in S_1 and another $\langle y_2 \rangle$ in S_2. But then $y_1^2 = y_2^2$ is the nontrivial central element in S_1S_2, and hence y_1y_2 is a noncentral involution in the centralizer of x. So we have shown $c_G(x)$ has 2-rank at least 3. Therefore, we have reduced ourselves to situation where $s = 1$.

Now assume that $s = 1$. We follow the pattern of Step 2 of the proof in the case that $p \neq \ell \geq 3$. As shown in that proof, we may assume that $\ell = 2$ divides the order of $Z(H_1)$, as otherwise $G \cong H_1 \times V$ where H_1 is one of the listed groups. In addition we may assume that H_1 has sectional 2-rank at most 4. The combination of the conditions that $2 \mid |Z(H_1)|$ and that the sectional rank be less than 5, means that H_1 must have one of the types A_1, ${}^2A_1$, A_3, ${}^2A_3$ or B_2 (see Theorem 6.3 and [32, Table 24.2]). Then as in Step 2 of the odd characteristic case, the 2-local structure of H_1 is that of a central product. Note that in the case that H_1 has type B_2 and $H_1 = \mathrm{Sp}_4(q)$, then the element ζ has order 2 in $\mathrm{CSp}_4(q)$. We note also that if H_i has type A_3, and $q \equiv 1$

modulo 4, then a Sylow 2-subgroup of H_1 has a rank 3 torus that is a characteristic subgroup. It follows that $TF(G) \cong \mathbb{Z}$, as we have seen before. The same happens if H_1 has type ${}^2A_3$ and $q \equiv 3 \pmod 4$. Hence, the only possibilities are that H_1 is one of $\mathrm{SL}_2(q) \cong \mathrm{SU}_2(q)$, $\mathrm{SL}_4(q)$ with $q \equiv 3 \pmod 4$, $\mathrm{SU}_4(q)$ with $q \equiv 1 \pmod 4$ or $\mathrm{Sp}_4(q)$. As before we conclude that the group G has the same poset of conjugacy classes of elementary abelian 2-subgroups as an associated group to H_1 as defined in Section 5. In the case that $\ell = 2$ these groups were treated in Section 6. In particular, Theorem 6.2 is sufficient to finish the proof. □

This finishes the proof of Theorem 8.1. We now verify that this indeed proves the main theorems.

Proof of Theorems A and B. First recall that Theorem B is equivalent to Theorem A by Theorem 1.2, where in Theorem B we have sorted the list by ℓ-rank instead of by prime. To verify Theorem A, suppose that $TF(G)$ has rank greater than 1.

If $\ell \neq p$ and $\ell > 2$, then Theorem 8.1(a) says that $G \cong H \times K$ where $\ell \nmid |K|$ and H is listed in Theorem 3.1, which is the list in Theorem A(1) with $\ell \neq 2$.

If $\ell \neq p$ and $\ell = 2$ then Theorem 8.1(b) tells us that $G \cong H \times K$ with $\ell \nmid |K|$ and $H \cong \mathrm{PGL}_2(q) \cong \mathrm{PGU}_2(q)$, which is the list in Theorem A(1) with $\ell = 2$.

Now suppose that $\ell = p$. Then the last part of Theorem 8.1 says that $G/Z(G) \cong H/Z(H)$, where H is one of the groups in Theorem 7.1 with the rank of $TF(H)$ greater than 1. An inspection of Tables 1 and 2 now shows that H is either ${}^2A_2(p)_{sc}$ with $3 \mid p+1$, $A_2(p)_{sc}$, $A_2(p)_{ad}$, $B_2(p)_{sc}$ with $p \geq 5$, $B_2(p)_{ad}$ with $p \geq 5$, or $G_2(p)$ with $p \geq 7$. This produces the list for $G/Z(G) \cong H/Z(H)$ given in Theorem A(2), by translating into classical group notation.

The theorems and tables quoted in Theorem A give the indicated ranks, finishing the proof of that theorem. □

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