THE PARABOLIC ALGEBRA REVISITED

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Abstract. The parabolic algebra $\mathcal{A}_p$ is the weakly closed operator algebra on $L^2(\mathbb{R})$ generated by the unitary semigroup of right translations and the unitary semigroup of multiplication by the analytic exponential functions $e^{i\lambda x}, \lambda \geq 0$. It is reflexive, with an invariant subspace lattice $Lat\mathcal{A}_p$ which is naturally homeomorphic to the unit disc (Katavolos and Power, 1997). The structure of $Lat\mathcal{A}_p$ is used to classify strongly irreducible isometric representations of the partial Weyl commutation relations. A formal generalisation of Arveson's notion of a *synthetic* commutative subspace lattice is given for general subspace lattices, and it is shown that $Lat\mathcal{A}_p$ is not synthetic relative to the $H^\infty(\mathbb{R})$ subalgebra of $\mathcal{A}_p$. Also, various new operator algebras, derived from isometric representations and from compact perturbations of $\mathcal{A}_p$, are defined and identified.

1. Introduction

Let M_λ and D_μ be the unitary operators on the Hilbert space $L^2(\mathbb{R})$ given by

$$M_\lambda f(x) = e^{i\lambda x} f(x), \quad D_\mu f(x) = f(x - \mu)$$

where μ, λ are real. As is well-known, the 1-parameter unitary groups $\{D_\mu, \mu \in \mathbb{R}\}$ and $\{M_\lambda, \lambda \in \mathbb{R}\}$ provide an irreducible representation of the Weyl commutation relations $M_\lambda D_\mu = e^{i\lambda\mu} D_\mu M_\lambda$, and the weak operator topology closed operator algebra that they generate is the von Neumann algebra $B(L^2(\mathbb{R}))$ of all bounded operators. See Taylor [25], for example. On the other hand Katavolos and Power [11] considered the weakly closed operator algebra generated by the unitary semigroups, for $\mu \geq 0$ and $\lambda \geq 0$, and showed it to be a proper subalgebra, containing no self-adjoint operators, other than real multiples of the identity, and no nonzero finite rank operators. Moreover this operator algebra, the *parabolic algebra* $\mathcal{A}_p$, was shown to be reflexive, in the sense that $\mathcal{A}_p = AlgLat\mathcal{A}_p$, with the set of invariant subspaces naturally homeomorphic to a closed disc.

In what follows we revisit the parabolic algebra and its noncommutative invariant subspace lattice and we examine associated operator algebras arising from semigroups of isometries and from compact perturbations. Also, an isometries generalisation of the Stone-von Neumann uniqueness theorem is obtained by making use of the identification of $Lat\mathcal{A}_p$.

Recall that the Stone-von Neumann theorem provides a complete classification of pairs of strongly continuous unitary groups acting on a separable Hilbert space satisfying the Weyl commutation relations [24], [15]. Specifically, there is one irreducible class, modelled

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by translation and multiplication operators on $L^2(\mathbb{R})$, and the finite and countable direct sums of this representation determine the other unitary equivalence classes. Rosenberg [21] has given an interesting historical perspective on the origins of this result, whose strict proof was completed by von Neumann [15] in 1931. It is possibly well-known that a strongly continuous isometric representation of the (partial) Weyl commutations relations for two semigroups of isometries may be dilated to a unique minimal strongly continuous unitary representation of the (full) Weyl commutation relations. A proof is given in Theorem 4.2. However we are not aware of a unitary equivalence class classification for such pairs of semigroups and we obtain partial results here. We show that the classes which are *strongly irreducible*, in the sense that their unique minimal unitary dilations are irreducible, are parametrised by the closed unit disc with a boundary point removed.

Recall also, that an invariant subspace lattice $\mathcal{L}$ of an operator algebra is a commutative subspace lattice (CSL) if its associated orthogonal projections form a commuting family. Arveson [1] has defined such a lattice to be *synthetic* if $Alg\mathcal{L}$ coincides with a certain minimal weak*-closed algebra $\mathcal{A}_{\min}$ constructed directly from pseudo-integral operators associated with $\mathcal{L}$. Less technical than this is the equivalent property that $\mathcal{L}$ is synthetic if and only if $Alg\mathcal{L}$ is the unique weak*-closed algebra $\mathcal{A}$ such that $Lat\mathcal{A} = \mathcal{L}$ and $\mathcal{A}$ contains a maximal abelian self-adjoint algebra associated with $\mathcal{L}$. Arveson showed, moreover, that this notion of synthesis is related to sets of spectral synthesis in harmonic analysis, and that CSLs failing to be synthetic could be constructed in terms of sets failing spectral synthesis. On the other hand the continuous projection nest $\mathcal{N}_v$, for $L^2(\mathbb{R})$, and indeed any complete projection nest, is synthetic. See also Davidson [3], [4] and Shulman and Turowska [23]. In Section 3 we introduce an analogous notion of synthesis for a noncommutative reflexive subspace lattice $\mathcal{L}$, namely synthesis relative to a maximal abelian subalgebra of $Alg\mathcal{L}$, and we show that $Lat\mathcal{A}_p$ is not synthetic relative to the maximal abelian subalgebra $M_{H^\infty(\mathbb{R})}$ of $\mathcal{A}_p$.

In Sections 5, 6 we examine, respectively, the weakly closed operator algebras determined by the restrictions of $\mathcal{A}_p$ to an invariant subspace, and the *quasicompact algebra* of $\mathcal{A}_p$, which is the C^*-algebra

$$Q\mathcal{A}_p = (\mathcal{A}_p + \mathcal{K}) \cap (\mathcal{A}_p^* + \mathcal{K}).$$

In particular we show that $Q\mathcal{A}_p$ strictly contains $(\mathcal{A}_p \cap \mathcal{A}_p^*) + \mathcal{K} = \mathbb{C}I + \mathcal{K}$.

Understanding the algebraic and geometric structure of the parabolic algebra presents some interesting challenges and in the final section we discuss four natural problems.

2. The parabolic algebra

We start by recalling basic facts and notation concerning the parabolic algebra $\mathcal{A}_p$, its subspace of Hilbert-Schmidt operators and its invariant subspaces. Also, in Section 2.1 we indicate the unitarily equivalences with $\mathcal{A}_p$ of the operator algebras associated with (λ, μ)-cones.

The Volterra nest $\mathcal{N}_v$ is the nest of subspaces $L^2([\lambda, +\infty))$, for $\lambda \in \mathbb{R}$, together with the trivial subspaces $\{0\}, L^2(\mathbb{R})$. The analytic nest $\mathcal{N}_a$ is the unitarily equivalent nest $F^*\mathcal{N}_v$, where F is Fourier transform with

$$Ff(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-itx} dt.$$

By the Paley-Wiener theorem the analytic nest consists of the chain of subspaces

$$e^{isx}H^2(\mathbb{R}), \quad s \in \mathbb{R},$$

together with the trivial subspaces. These nests determine the Volterra nest algebra $\mathcal{A}_v = Alg\mathcal{N}_v$ and the analytic nest algebra $\mathcal{A}_a = Alg\mathcal{N}_a$, where $Alg\mathcal{S}$ denotes the algebra of operators that leave invariant the subspaces in a set $\mathcal{S}$ of closed subspaces. In particular, by simple duality, both algebras are *reflexive* operator algebras in the sense that $\mathcal{A} = AlgLat\mathcal{A}$ where $Lat\mathcal{A}$ denotes the lattice of closed subspaces left invariant by each operator of $\mathcal{A}$.

Define also the reflexive *Fourier binest algebra* $\mathcal{A}_{FB} = Alg(\mathcal{N}_a \cup \mathcal{N}_v) = \mathcal{A}_a \cap \mathcal{A}_v$. The union $\mathcal{N}_a \cup \mathcal{N}_v$ is a continuous complete lattice of subspaces with noncommuting subspace projections, and is known as the *Fourier binest* [11].

The antisymmetry property $\mathcal{A}_{FB} \cap \mathcal{A}_{FB}^* = \mathbb{C}I$ follows readily, since $\mathcal{A}_v \cap \mathcal{A}_v^*$ is the algebra of multiplication operators M_ϕ with $\phi \in L^\infty(\mathbb{R})$, and these operators must leave $H^2(\mathbb{R})$ invariant. Also $\mathcal{A}_{FB}$ contains no non-zero finite rank operators. This follows from the structure of such operators in a nest algebra (see Davidson [3]) and the fact that a pair of proper subspaces from $\mathcal{N}_a$ and $\mathcal{N}_v$ have trivial intersection.

Consider now the *parabolic algebra* $\mathcal{A}_p$. This is the weak operator topology closed operator algebra generated by the unitary semigroups of operators $\{M_\lambda, \lambda \geq 0\}$ and $\{D_\mu, \mu \geq 0\}$. Since the generators of $\mathcal{A}_p$ leave the binest invariant, we have $\mathcal{A}_p \subseteq \mathcal{A}_{FB}$. That these two algebras are equal was shown in Katavolos and Power [11]. We recall this here by repeating the argument of Levene [13].

Write $\mathcal{C}_2$ for the space of Hilbert-Schmidt operators on $L^2(\mathbb{R})$ and recall that every such operator has the form $Intk$ for some square-summable function $k(x, y)$ in $L^2(\mathbb{R}^2)$ where $Intk$ denotes the Hilbert-Schmidt operator acting on $L^2(\mathbb{R})$ given by

$$(Intk\, f)(x) = \int_{\mathbb{R}} k(x, y)f(y)dy.$$

The following identification of the Hilbert-Schmidt operators in the Fourier binest algebra is a straightforward argument using the Fourier-Plancherel transform. The tensor product $H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)$ is the Hilbert space tensor product.

Proposition 2.1. *For $k \in L^2(\mathbb{R}^2)$ let $\Theta_p(k)(x, t) = k(x, x - t)$. Then*

$$\mathcal{A}_{FB} \cap \mathcal{C}_2 \subseteq \{Intk \,|\, \Theta_p(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)\}.$$

Now, given $h \in H^\infty(\mathbb{R}) \cap H^2(\mathbb{R}), \phi \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$, let $h \otimes \phi$ denote the function $(x, y) \mapsto h(x)\phi(y)$. The integral operator $Intk$, that is induced by the function $k = \Theta_p^{-1}(h \otimes \phi)$, lies in the parabolic algebra. Indeed, we have $Intk = M_h\Delta_\phi$, where M_h is the bounded multiplication operator for h and where Δ_ϕ is the bounded operator defined by the sesquilinear form

$$\langle \Delta_\phi f, g\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(t)D_t f(x)\overline{g(x)}dxdt, \text{ where } f, g \in L^2(\mathbb{R}).$$

When ϕ is a step function then Δ_ϕ is a linear combination of translation operators D_μ with $\mu \geq 0$, and by standard approximation and dominated convergence arguments it follows that in general Δ_ϕ lies in the weak*-closed span of these translation operators. Alternatively, this may be deduced from the fact that Δ_ϕ commutes with the unitary semigroup of translations.

Since the linear span of the functions $h\otimes\phi$ is dense in the Hilbert space $H^2(\mathbb{R})\otimes L^2(\mathbb{R}_+)$, it follows from Proposition 2.1 that

$$\mathcal{A}_{FB}\cap\mathcal{C}_2\subseteq\{Intk\,|\,\Theta_p(k)\in H^2(\mathbb{R})\otimes L^2(\mathbb{R}_+)\}\subseteq\mathcal{A}_p\cap\mathcal{C}_2\subseteq\mathcal{A}_{FB}\cap\mathcal{C}_2$$

and so $\mathcal{A}_{FB}\cap\mathcal{C}_2=\mathcal{A}_p\cap\mathcal{C}_2$.

Theorem 2.2. *The parabolic algebra coincides with the Fourier binest algebra and is a reflexive operator algebra.*

Proof. Let $h_n(x)=ni/(x+ni)$ so that $h_n\in H^\infty(\mathbb{R})$ with $|h_n(x)|\leq 1$ for all x and $h_n(x)\to 1$ uniformly on compact sets. Then $M_{h_n}\to I$ boundedly in the strong operator topology. Consider a sequence of operators $K_n=M_{h_n}\Delta_{\phi_n}$, where, similarly, $|\phi_n(x)|\leq 1$ for all x and $\phi_n(x)\to 1$ uniformly on compact sets. This is an operator norm bounded sequence of Hilbert-Schmidt operators which tends to the identity operator in the strong operator topology. Thus if $X\in\mathcal{A}_{FB}$ then X is the strong operator topology limit of the Hilbert-Schmidt operators XK_n, and so X belongs to $\mathcal{A}_p$. □

We remark that the sequence (K_n) in the proof above is a bounded approximate identity for $\mathcal{A}_{FB}$ with respect to the strong operator topology. The existence of such sequences, consisting of compact operators, plays a key role in Section 6.

Let us now consider the invariant subspace lattice $Lat\mathcal{A}_p$. In [11] a cocycle argument with inner functions and unimodular functions on the line is used to obtain the following identification:

$$Lat\mathcal{A}_p=\{K_{\lambda,s}|\lambda\in\mathbb{R},s\geq 0\}\cup\mathcal{N}_v$$

where

$$K_{\lambda,s}=M_\lambda M_{\phi_s}H^2(\mathbb{R})\quad\text{and}\quad\phi_s(x)=e^{-isx^2/2}.$$

In fact we employ a similar cocycle argument in Section 3. Note that we continue to write M_λ as shorthand for the multiplication operator $M_{e^{i\lambda x}}$.

For $s>0$ we have the nest $\mathcal{N}_s=M_{\phi_s}\mathcal{N}_a$, and for distinct values of $s\geq 0$ these are disjoint, except for the trivial subspaces. If we view $Lat\mathcal{A}_p$ as a set of projections endowed with the strong operator topology, then it is homeomorphic to the closed unit disc and the topological boundary is the Fourier binest. See Figure 1.

It follows in particular that the Fourier binest lattice is not reflexive. That is, the lattice $LatAlg(\mathcal{N}_a\cup\mathcal{N}_v)$ strictly contains $\mathcal{N}_a\cup\mathcal{N}_v$.

We note that similar results have been obtained by Kastis [8] for the strong operator topology closed operator algebras on $L^q(\mathbb{R})$ generated by the corresponding shift isometries and multiplication isometries of $L^q(\mathbb{R})$, for $1<q<\infty, q\neq 2$. However, it is not known whether there is a similar homeomorphism between their invariant subspace lattices and the closed unit disc.

2.1. **The parabolic algebra and** (λ,μ)**-cones.** The parabolic algebra is defined as the weak*-closed linear span of the unitary operators $M_\lambda D_\mu$ associated with parameters (λ,μ) in the first quadrant cone in $\mathbb{R}^2$. Other parameter cones in $\mathbb{R}^2$ similarly determine operator algebras, in view of the Weyl commutation relations. However we now show that that these algebras, for simple cones, are canonically unitarily equivalent to $\mathcal{A}_p$.

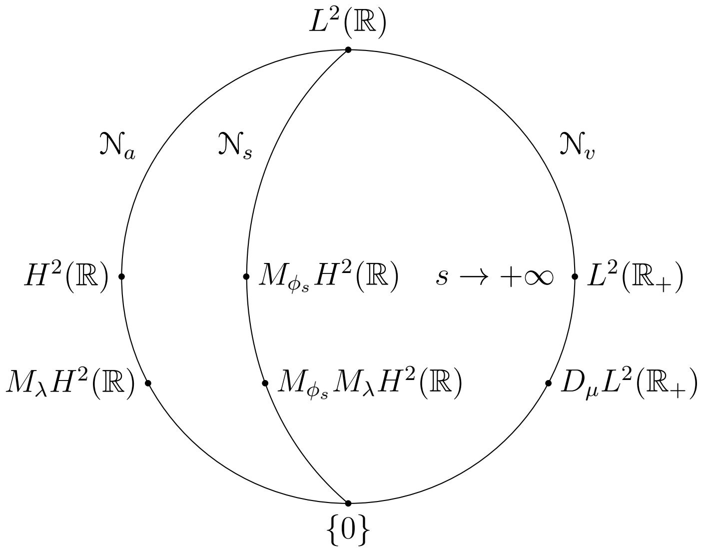

FIGURE 1. Parametrising $Lat\mathcal{A}_p$ by the unit disc.

Let $\mathfrak{S}$ be a cone in $\mathbb{R}^2$, that is, an additive semigroup containing 0 with the additional property that $r(\lambda, \mu) \in \mathfrak{S}$, for all $r > 0$ and $(\lambda, \mu) \in \mathfrak{S}$. Define $\mathcal{A}_{\mathfrak{S}}$ to be the w^*-closed linear span of the unitaries $M_\lambda D_\mu$, for $(\lambda, \mu) \in \mathfrak{S}$. This is an operator algebra and is equal to $\mathcal{A}_p$ if $\mathfrak{S} = \mathbb{R}^2_+$. We show that for any simple cone $\mathfrak{S}$, that is, one determined by an acute or obtuse angle, there is an explicit unitary equivalence between $\mathcal{A}_{\mathfrak{S}}$ and $\mathcal{A}_p$. The following parametrisation will be convenient. With $s_1, s_2 \in \mathbb{R}$ let $\mathfrak{S}_{s_1,s_2}$ be the additive cone for distinct rays, $(x, s_1x), x \geq 0$, and $(x, s_2x), x \geq 0$, and let $\mathcal{A}_{s_1,s_2} = \mathcal{A}_{\mathfrak{S}_{s_1,s_2}}$. Also we write Ad_Z for the map $X \to ZXZ^*$ determined by a unitary operator Z and an associated domain of operators.

Theorem 2.3. *Let $\mathfrak{S}$ be a simple cone in $\mathbb{R}^2$. Then $\mathcal{A}_{\mathfrak{S}}$ is unitarily equivalent to the parabolic algebra.*

Proof. Since $Ad_F D_\mu = M_{-\mu}$, without loss of generality we consider cones $\mathfrak{S}_{0,s}$ and $\mathfrak{S}_{s_1,s_2}$, with $s, s_1 > 0$. In particular, it suffices to prove that the corresponding algebras $\mathcal{A}_{0,s}$ and $\mathcal{A}_{s_1,s_2}$ (see Figure 2) are unitarily equivalent to the parabolic algebra.

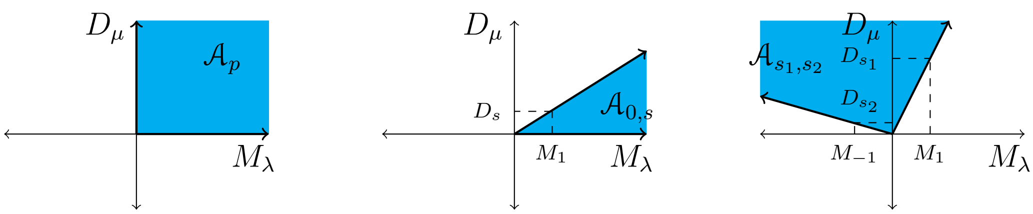

FIGURE 2. Parameter cones for the algebras $\mathcal{A}_p$, $\mathcal{A}_{0,s}$ and $\mathcal{A}_{s_1,s_2}$.

Let $s \in \mathbb{R}$. Given $\mu \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$, compute

$$\begin{aligned}(D_\mu M_{\phi_s} f)(x) &= (M_{\phi_s} f)(x-\mu) = e^{-is(x-\mu)^2/2} f(x-\mu) \\ &= e^{-is\mu^2/2} e^{-isx^2/2} e^{is\mu x} f(x-\mu) = e^{-is\mu^2/2} (M_{\phi_s} M_{s\mu} D_\mu f)(x),\end{aligned}$$

which implies

$$Ad_{M_{\phi_s}}(M_{s\mu} D_\mu) = e^{is\mu^2/2} D_\mu. \tag{2.1}$$

Also, applying Ad_F to this equation we obtain

$$Ad_{D_{\phi_s}}(D_{s\mu} M_{-\mu}) = e^{-is\mu^2/2} M_{-\mu},$$

where, for a function ψ, we write D_ψ for the operator $F M_\psi F^*$. With the substitutions $s \to -s^{-1}, \mu \to -\mu s$ we obtain

$$Ad_{D^*_{\phi_{s^{-1}}}}(D_\mu M_{s\mu}) = e^{-is\mu^2/2} M_{s\mu}. \tag{2.2}$$

Thus, $\mathcal{A}_{0,s}$ is unitarily equivalent to the parabolic algebra, since, by equation (2.1), the map $Ad_{M_{\phi_s}}$ sends $\mathcal{A}_{0,s}$ onto $\mathcal{A}_p$. In the general case, let $s_1 > s_2$. In view of the equation (2.2), the map Ad_Z, with $Z = D^*_{\phi_{s_2^{-1}}}$, restricted to the domain $\mathcal{A}_{s_1,s_2}$, is an isomorphism onto $\mathcal{A}_{0,s_1-s_2}$, and so the theorem follows. □

3. Synthetic lattices and Lat$\mathcal{A}_p$.

Let $\mathcal{L}$ be a reflexive subspace lattice in the usual sense of Halmos, so that $\mathcal{L} = LatAlg\mathcal{L}$, and let $\mathcal{B}$ be a maximal abelian subalgebra of $Alg\mathcal{L}$. We define $\mathcal{L}$ to be *synthetic relative to* $\mathcal{B}$ if whenever $\mathcal{A}$ is a weak*-closed operator algebra with $\mathcal{B} \subseteq \mathcal{A}$ and $Lat\mathcal{A} = \mathcal{L}$ then $\mathcal{A} = Alg\mathcal{L}$.

For a commutative subspace lattice $\mathcal{L}$, being synthetic relative to a maximal abelian self-adjoint subalgebra of $Alg\mathcal{L}$ is equivalent to Arveson's definition of synthetic. In this more technical formulation $Alg\mathcal{L}$ must coincide with a certain *minimal algebra* generated by so-called pseudo-integral operators constructed with the help of a coordinatisation of $\mathcal{L}$. (See also Theorem 5.3 of [4].) These generating operators play a role analogous to the Hilbert-Schmidt operators in a nest algebra.

As we have seen, the parabolic algebra is the weak*-closure of its Hilbert-Schmidt integral operators [11]. Despite this regularity property we now show that its invariant subspace lattice fails to be synthetic relative to $M_{H^\infty(\mathbb{R})}$, the maximal abelian subalgebra of analytic multiplication operators.

Theorem 3.1. *The subspace lattice $Lat\mathcal{A}_p$ is not synthetic relative to $M_{H^\infty(\mathbb{R})}$.*

Proof. Let $\mathcal{A}_0$ be the subalgebra of $\mathcal{A}_p$ generated by the operators M_λ, for $\lambda \geq 0$, and the products $M_\lambda D_\mu$ where $\lambda \geq 2$ (and $\mu \geq 0$) or $\mu \geq 2$ (and $\lambda \geq 0$), and let $\mathcal{A}$ be the weak operator topology closure of $\mathcal{A}_0$. In the first part of the proof we show that $Lat\mathcal{A} = Lat\mathcal{A}_p$ by means of the inner function cocycle argument similar to that used for the determination of $Lat\mathcal{A}_p$ in [11]. In the second part of the proof we show that $\mathcal{A} \neq \mathcal{A}_p$.

Let $K \in Lat\mathcal{A}$. Since $M_\lambda K \subseteq K$ for all $\lambda \geq 0$, it follows from Beurling's theorem that $K = uH^2(\mathbb{R})$, for some unimodular function u, or $K = L^2(E)$, where E is a Borel set in $\mathbb{R}$. If $K = L^2(E)$ then, since it is an invariant subspace for $D_\mu M_\lambda$, for $\mu \geq 0, \lambda \geq 2$, it follows that $E = [t, \infty)$, for some $t \in \mathbb{R}$.

Suppose now that $K = uH^2(\mathbb{R})$. Fix $\lambda > 2$. Then $M_\lambda D_\mu$ is in $\mathcal{A}_0$, for all $\lambda \geq 0$, and $M_\lambda D_\mu K \subseteq K$. Also $M_\lambda D_\mu K$ is invariant under $M_{\lambda'}$, for all $\lambda' \geq 0$, and so, by Beurling's theorem, $D_\mu M_\lambda K = \omega_\mu u H^2(\mathbb{R})$, for some unimodular function ω_μ. Note that ω_{μ_2} divides ω_{μ_1} for all $0 < \mu_2 < \mu_1$. Moreover, calculating directly we have $D_\mu M_\lambda K = u(x-\mu)e^{i\lambda x}H^2(\mathbb{R})$. Hence we obtain $u(x-\mu)e^{i\lambda x} = c_\mu \omega_\mu u(x)$, for some unimodular constant. Redefining ω_μ we may assume that $c_\mu = 1$. Thus

$$\omega_\mu = \frac{u(x-\mu)}{u(x)} e^{i\lambda x}.$$

Therefore, we get the cocycle equation

$$\omega_{\mu_1+\mu_2} = \frac{u(x-\mu_1-\mu_2)}{u(x-\mu_1)} \frac{u(x-\mu_1)}{u(x)} e^{i\lambda x} = \omega_{\mu_2}(x-\mu_1)\omega_{\mu_2}(x)e^{-i\lambda(x-\mu_1)}.$$

Thus $\omega_{\mu_2}(x-r)$ divides ω_{μ_1} for all $0 < r < \mu_1 - \mu_2$. Fix μ_1 and μ_2 with $0 < \mu_2 < \mu_1$. If ω_{μ_2} has any zeros in the upper half plane, then those zeros and all their translates by r, with $0 < r < \mu_1 - \mu_2$, must be zeros of ω_{μ_1}. However, this would imply that the analytic function ω_{μ_1} is identically zero, a contradiction, so ω_{μ_2} admits a trivial Blaschke product. Hence ω_{μ_2} can be written in the form

$$\omega_{\mu_2}(z) = \alpha\, e^{i\beta z} \exp\left\{ i \int_{\mathbb{R}} \frac{sz+1}{s-z} \frac{1}{s^2+1} d\mu(s) \right\},\ \operatorname{Im} z > 0,$$

for some unimodular α, some real β and some singular measure μ. Let ω_{μ_1} be associated with the triple α', β' and ν. Again, $\omega_{\mu_2}(x-r)$ divides ω_{μ_1}, which yields that all the translates of μ by r, with $0 < r < \mu_1 - \mu_2$, are dominated by the singular measure ν, and so $\mu = 0$. Thus

$$\omega_{\mu_2}(x) = \alpha_{\mu_2} e^{i\beta(\mu_2)x},$$

where α is unimodular and β is strictly increasing. By the cocycle equation, we have

$$\alpha(\mu_1+\mu_2)e^{i\beta(\mu_1+\mu_2)x} = \alpha(\mu_2)e^{i\beta(\mu_2)(x-\mu_1)}\alpha(\mu_2)e^{i\beta(\mu_2)x}e^{-i\lambda(x-\mu_1)},$$

which implies that $\alpha(\mu_1+\mu_2) = \alpha(\mu_1)\alpha(\mu_2)e^{-i\beta(\mu_2)\mu_1}e^{i\lambda\mu_1}$ and $\beta(\mu_1+\mu_2) = \beta(\mu_1)+\beta(\mu_2) - \lambda$. Thus $\beta(\mu_2) = \rho\mu_2 + \lambda$, for some $\rho > 0$. Therefore, $\alpha(\mu_2) = e^{i\sigma\mu_2}e^{-i\rho\mu_2^2/2}$. Hence

$$\omega_{\mu_2} = e^{i\sigma\mu_2}e^{-i\rho\mu_2^2/2}e^{i(\rho\mu_2+\lambda)x} = \frac{u(x-\mu_2)}{u(x)}e^{i\lambda x}.$$

Therefore, fixing $x = x_0$ we have

$$u(x_0-\mu_2) = u(x_0)e^{i(-\rho\mu_2^2/2+\sigma\mu_2+\rho\mu_2 x_0)}.$$

Thus

$$u(y) = ce^{i(-\rho y^2/2+\sigma\mu_2)}$$

and the first part of the proof is complete.

We now give a separation argument showing that $D_1 \notin \mathcal{A}_p$. An intuitive argument for this is also given in Remark 3.2.

Let $f_n, g_n \in L^2[0,2]$, with $\|f_n\|_2 = \|g_n\|_2 = 1$, let (α_n) be a summable sequence, and define the associated weak*-continuous functional ω on $B(L^2(\mathbb{R}))$ with

$$\omega(T) = \sum_{n\in\mathbb{N}} \alpha_n \langle TM_{\phi_1}F^{-1}f_n, M_{\phi_1}F^{-1}g_n\rangle.$$

Consider a finite sum $\sum_m c_m M_{\lambda_m} D_{\mu_m}$ in $\mathcal{A}_0$, with $\lambda_m + \mu_m \geq 2$ for every m, and h in $H^\infty(\mathbb{R})$. We have

$$\omega(M_h + \sum_m c_m M_{\lambda_m} D_{\mu_m} + D_1) = \sum_{n\in\mathbb{N}} \alpha_n \langle (M_h + \sum_m c_m M_{\lambda_m} D_{\mu_m} + D_1) M_{\phi_1} F^{-1} f_n, M_{\phi_1} F^{-1} g_n \rangle$$

$$= \sum_{n\in\mathbb{N}} \alpha_n \langle M_{\phi_1} F^{-1}(D_h + \sum_m c_m e^{-m^2/2} D_{\lambda_m} D_{\mu_m} M_{-\mu_m} + e^{i/2} D_1 M_{-1}) f_n, M_{\phi_1} F^{-1} g_n \rangle$$
$$= \sum_{n\in\mathbb{N}} \alpha_n \langle M_{\phi_1} F^{-1}(D_h + \sum_m c_m e^{-m^2/2} D_{\lambda_m+\mu_m} M_{-\mu_m} + e^{i/2} D_1 M_{-1}) f_n, M_{\phi_1} F^{-1} g_n \rangle$$
$$= \sum_{n\in\mathbb{N}} \alpha_n \langle (D_h + e^{-i/2} D_1 M_{-1}) f_n, g_n \rangle.$$

The last equality follows from the fact that for all $f, g \in L^2[0,2]$ we have $\langle D_\lambda f, g\rangle = 0$, when $\lambda \geq 2$.

We may write the previous equality as

$$\omega(M_h + \sum_m c_m M_{\lambda_m} D_{\mu_n} + D_1) = \omega_{[0,2]}(D_h + e^{-i/2} D_1 M_{-1})$$

where $\omega_{[0,2]}$ denotes the w^*-continuous functional on $B(L^2[0,2])$ determined by the sequences (f_n) and (g_n), and we note that every w^*-continuous functional on $B(L^2[0,2])$ is of this form.

Now the operator $D_1 M_{-1}$ does not lie in the w^*-closure of the compression of the algebra $\{D_h : h \in H^\infty(\mathbb{R})\}$ to $L^2[0,2]$. Indeed, the compression algebra is commutative while the compression of $D_1 M_{-1}$ does not commute with the compression of $D_{1/2}$. Thus, by the topological Hahn-Banach theorem, there exists a w^*-continuous functional $\omega_{[0,2]}$ on $B(L^2[0,2])$, such that $|\omega_{[0,2]}(e^{-i/2} D_1 M_{-1})| = 1$, while $\omega_{[0,2]}(D_h) = 0$ for every $h \in H^\infty(\mathbb{R})$. However, from the calculations above, this implies that there exists a w^*-continuous functional ω on $B(L^2(\mathbb{R}))$, such that $\omega(M_h + \sum_m a_m M_{\lambda_m} D_{\mu_n}) = 0$ and $|\omega(D_1)| = 1$. Thus $D_1 \notin \mathcal{A}$ and the proof is complete. $\square$

Remark 3.2. Note that $\mathcal{A}$ is also the closure of the subalgebra $M_{H^\infty(\mathbb{R})} + M_2\mathcal{A}_p + D_2\mathcal{A}_p$ and this is contained in $\mathcal{A}_v$, the Volterra nest algebra of operators with "lower triangular support". The following rough argument based on support sets shows that $\mathcal{A} \neq \mathcal{A}_p$ and provides insight for the more explicit separation argument above, showing that $D_1 \notin \mathcal{A}_p$. The support of D_μ, with $0 < \mu < 2$, is a line parallel to the main diagonal. This main diagonal is the support set for $M_{H^\infty(\mathbb{R})}$ and the support set for $D_2\mathcal{A}_p$ is empty "above" the support for D_2. It follows from this that if D_μ is in $\mathcal{A}$ then it must be in the weak operator topology closure of $M_2\mathcal{A}_p$. Since $M_2\mathcal{A}_p$ is closed in this topology it follows that $D_\mu \in M_2\mathcal{A}_p$. However, this cannot be true for all such μ since this implies that $I \in M_2\mathcal{A}_p$, a contradiction.

Remark 3.3. While synthesis relative to a maximal abelian algebra generalises the CSL notion of synthetic we note that for a noncommutative lattice $\mathcal{L}$ there may be no counterpart to the notion of a minimal weak*-closed subalgebra. To see this consider the subalgebras $\mathcal{A}_t = M_{H^\infty(\mathbb{R})} + M_t\mathcal{A}_p + D_t\mathcal{A}_p$, for $t > 0$, which form a decreasing chain of weak*-closed subalgebras with intersection equal to $M_{H^\infty(\mathbb{R})}$. The proof above shows that $Lat\mathcal{A}_t = Lat\mathcal{A}_p$, for all t, and yet $LatM_{H^\infty(\mathbb{R})} \neq Lat\mathcal{A}_p$.

We remark that the stronger synthesis property for a reflexive lattice $\mathcal{L}$, which requires that $\mathcal{A} = Alg\mathcal{L}$ for *every* weak*-closed unital subalgebra $\mathcal{A}$ with $Lat\mathcal{A} = Alg\mathcal{L}$ is a distinctly stronger notion. Indeed, the discrete nest and the Volterra nest fail to have this uniqueness property since these lattices can be attained by a single operator, and hence by an abelian weak*-closed unital subalgebra. See [3], [19] for such unicellular operators.

On the other hand we remark that $H^\infty(\mathbb{R})$, as an operator algebra on $L^2(\mathbb{R})$, or even as the Toeplitz operator algebra on $H^2(\mathbb{R})$, does have this property by virtue of being *hereditarily reflexive.* This in turn is a consequence of the fact that dual space functionals are implementable by rank one operators. See, for example, Davidson [2] and Hadwin [7].

4. Isometric Representations of the WCR

Let us define an *isometric representation* of the Weyl commutation relations to be a pair of strongly continuous semigroups of isometries $U_\lambda, V_\mu, \lambda, \mu \geq 0$, acting on a separable Hilbert space $\mathcal{K}$, with

$$U_\lambda V_\mu = e^{i\lambda\mu} V_\mu U_\lambda, \quad \lambda, \mu \geq 0.$$

An isometric representation is *irreducible* if there is no proper reducing closed subspace for the representation, and we say that it is *strongly irreducible* if it has a minimal unitary dilation on a separable Hilbert space which is irreducible.

For $\lambda \in \mathbb{R}, s > 0$, let $\rho_{\lambda,s}$ be the isometric representation arising from the restriction of the analytic multiplication semigroup and the right translation semigroup to the invariant subspace $K_{\lambda,s}$. Also, for $\lambda \in \mathbb{R}$, let ρ_λ be the restriction representation given by restriction to $M_\lambda H^2(\mathbb{R})$, and for $\mu \in \mathbb{R}$ let ρ^μ be given by restriction to $D_\mu L^2(\mathbb{R}_+)$. Finally, let ρ_{id} be the identity representation.

Recall that a semigroup of isometries is said to be *pure* if the intersection of the ranges of the isometries is the zero subspace. We say that the isometric representation ρ is of type uu, up, pu, or pp if the semigroups of isometries $\{U_\lambda\}, \{V_\mu\}$ are, respectively, (i) unitary semigroups, (ii) a unitary semigroup and a pure semigroup, (iii) a pure semigroup and unitary semigroup, and (iv) pure semigroups. In particular ρ is of type pp if the intersection of the spaces $U_\lambda\mathcal{K}$ for $\lambda \geq 0$ is the zero subspace, and the intersection of the spaces $V_\mu\mathcal{K}$ for $\mu \geq 0$ is the zero subspace.

The Stone-von Neumann uniqueness theorem for unitary groups satisfying the Weyl commutation relations ensures that the strongly irreducible isometric representations ρ of type uu are unitarily equivalent to ρ_{id}. More generally we obtain the following classification.

Theorem 4.1. *Let ρ be an isometric representation of the Weyl commutation relations which is strongly irreducible and which is not of type uu. Then ρ satisfies one of the following 3 equivalences.*

(i) ρ is of type pu and is unitarily equivalent to ρ_λ for some $\lambda \in \mathbb{R}$,
(ii) ρ is of type up and is unitarily equivalent to ρ^μ for some $\mu \in \mathbb{R}$,
(iii) ρ is of type pp and is unitarily equivalent to $\rho_{\lambda,s}$ for some $\lambda \in \mathbb{R}, s > 0$.

Moreover, no pair of distinct representations from the set of representations

$$\rho_\lambda, \rho^\mu, \rho_{\lambda,s}, \quad \textit{for} \quad \lambda, \mu \in \mathbb{R}, s > 0,$$

is a unitarily equivalent pair.

The proof has 3 ingredients, namely, the unitary dilation of isometric representations of the Weyl commutation relations, the Stone-von Neumann theorem, and the nature of the closed invariant subspaces for the model representation ρ_{id}. The following dilation theorem, for the first step here, is perhaps well-known. The proof we give is a simple variation of the proof of the dilation of commuting isometries given in Paulsen's book [16].

Theorem 4.2. *Let $\{S_\lambda : \lambda \geq 0\}, \{T_\mu : \mu \geq 0\}$ be an isometric representation of the Weyl commutation relations on the Hilbert space $\mathcal{H}$. Then there is a Hilbert space $\mathcal{K}$ and a unitary representation of the Weyl commutation relations given by $\{U_\lambda : \lambda \in \mathbb{R}\}$ and $\{V_\mu : \mu \in \mathbb{R}\}$ acting on $\mathcal{K}$, with $P_\mathcal{H} U_\lambda\big|_\mathcal{H} = S_\lambda$ and $P_\mathcal{H} V_\mu\big|_\mathcal{H} = T_\mu$, for $\lambda, \mu \geq 0$.*

Proof. By Naimark's theorem ([16] Theorems 4.8 and 5.4), there is a Hilbert space $\mathcal{K}_1$ containing $\mathcal{H}$ and a strong operator topology continuous (SOT-continuous) unitary group $\{\tilde{S}_\lambda : \lambda \in \mathbb{R}\}$ such that $P_\mathcal{H} \tilde{S}_\lambda\big|_\mathcal{H} = S_\lambda$ for every $\lambda \geq 0$. We may assume that this is a minimal dilation of $\{S_\lambda : \lambda \geq 0\}$, that is, the linear span of $\{\tilde{S}_\lambda h : \lambda \in \mathbb{R},\, h \in \mathcal{H}\}$ is dense in $\mathcal{K}_1$.

Fix some $\mu \geq 0$ and define :

$$\tilde{T}_\mu \left(\sum_{n=1}^{N} \tilde{S}_{\lambda_n} h_n \right) = \sum_{n=1}^{N} e^{i\lambda_n \mu} \tilde{S}_{\lambda_n} T_\mu h_n.$$

We claim that $\tilde{T}_\mu$ is well-defined and isometric. To see this note that given $h = \sum_{n=1}^{N} \tilde{S}_{\lambda_n} h_n$ we have

$$\begin{aligned}
\|\tilde{T}_\mu h\|^2 &= \left\langle \tilde{T}_\mu \left(\sum_{n=1}^{N} \tilde{S}_{\lambda_n} h_n \right), \tilde{T}_\mu \left(\sum_{m=1}^{N} \tilde{S}_{\lambda_m} h_m \right) \right\rangle \\
&= \left\langle \sum_{n=1}^{N} e^{i\lambda_n \mu} \tilde{S}_{\lambda_n} T_\mu h_n, \sum_{m=1}^{N} e^{i\lambda_m \mu} \tilde{S}_{\lambda_m} T_\mu h_m \right\rangle \\
&= \sum_{\lambda_n \geq \lambda_m} \left\langle e^{i\lambda_n \mu} \tilde{S}_{\lambda_n} T_\mu h_n, e^{i\lambda_m \mu} \tilde{S}_{\lambda_m} T_\mu h_m \right\rangle + \sum_{\lambda_m > \lambda_n} \left\langle e^{i\lambda_n \mu} \tilde{S}_{\lambda_n} T_\mu h_n, e^{i\lambda_m \mu} \tilde{S}_{\lambda_m} T_\mu h_m \right\rangle \\
&= \sum_{\lambda_n \geq \lambda_m} \left\langle e^{i(\lambda_n - \lambda_m)\mu} \tilde{S}_{\lambda_n - \lambda_m} T_\mu h_n, T_\mu h_m \right\rangle + \sum_{\lambda_m > \lambda_n} \left\langle T_\mu h_n, e^{i(\lambda_m - \lambda_n)\mu} \tilde{S}_{\lambda_m - \lambda_n)} T_\mu h_m \right\rangle \\
&= \sum_{\lambda_n \geq \lambda_m} \left\langle e^{i(\lambda_n - \lambda_m)\mu} S_{\lambda_n - \lambda m} T_\mu h_n, T_\mu h_m \right\rangle + \sum_{\lambda_m > \lambda_n} \left\langle T_\mu h_n, e^{i(\lambda_m - \lambda_n)\mu} S_{\lambda_m - \lambda_n} T_\mu h_m \right\rangle \\
&= \sum_{\lambda_n \geq \lambda_m} \left\langle T_\mu S_{\lambda_n - \lambda_m} h_n, T_\mu h_m \right\rangle + \sum_{\lambda_m > \lambda_n} \left\langle T_\mu h_n, T_\mu S_{\lambda_m - \lambda_n} h_m \right\rangle .
\end{aligned}$$

Thus

$$\begin{aligned}\|\tilde{T}_\mu h\|^2 &= \sum_{\lambda_n \geq \lambda_m} \langle S_{\lambda_n - \lambda_m} h_n, h_m\rangle + \sum_{\lambda_m > \lambda_n} \langle h_n, S_{\lambda_m - \lambda_n} h_m\rangle \\ &= \sum_{\lambda_n \geq \lambda_m} \langle S_{\lambda_n - \lambda_m} h_n, h_m\rangle + \sum_{\lambda_m > \lambda_n} \langle h_n, S_{\lambda_m - \lambda_n} h_m\rangle \\ &= \left\langle \sum_{n=1}^{N} \tilde{S}_{\lambda_n} h_n, \sum_{m=1}^{N} \tilde{S}_{\lambda_m} h_m \right\rangle = \|h\|^2.\end{aligned}$$

Also, it follows that $\{\tilde{T}_\mu : \mu \geq 0\}$ is a SOT-continuous semigroup of isometries and that the unitary operators $\tilde{S}_\lambda$ and the isometries $\tilde{T}_\mu$ satisfy the WCR for all real λ and for $\mu \geq 0$.

Now consider the minimal unitary dilation V_μ of the semigroup of isometries $\tilde{T}_\mu$ on a Hilbert space $\mathcal{K}$ containing $\mathcal{K}_1$, so that $P_{\mathcal{K}_1} V_\mu \big|_{\mathcal{K}_1} = \tilde{T}_\mu$ for $\mu \geq 0$. We have $\mathcal{K} = \overline{span\{V_\mu k : \mu \in \mathbb{R}, k \in \mathcal{K}_1\}}^{\|\cdot\|}$ and we may define the unitary group $\{U_\lambda : \lambda \in \mathbb{R}\}$ by

$$U_\lambda \left(\sum_{n=1}^{N} V_{\mu_n} h_n \right) = \sum_{n=1}^{N} e^{i\lambda\mu_n} T_{\mu_n} \tilde{S}_\lambda h_n.$$

That the operators U_λ are unitaries follows from the argument above and it follows that $\{U_\lambda\}$ and $\{V_\mu\}$ give the required unitary dilation. □

Proof of Theorem 4.1. By the previous theorem and the Stone von Neumann theorem every strongly irreducible isometric representation ρ is unitarily equivalent to a representation $\rho_{\mathcal{H}}$ obtained from the restriction of $\{M_\lambda : \lambda \geq 0\}$ and $\{D_\mu : \mu \geq 0\}$ to an invariant subspace $\mathcal{H}$. In particular $\mathcal{H}$ is a nonzero subspace in $Lat\mathcal{A}_p$ and by the description of this lattice in Section 2 the subspace $\mathcal{H}$ takes one of the 4 types, (i) $L^2(\mathbb{R})$, (ii) $M_\lambda H^2(\mathbb{R})$, (iii) $D_\mu L^2(\mathbb{R}_+)$, (iv) $K_{\lambda,s}$, for some $\lambda \in \mathbb{R}$ and $s > 0$.

Let $\rho_{\mathcal{H}_1}, \rho_{\mathcal{H}_2}$ be any two representations, of the same type, (ii), (iii) or (iv), which are unitarily equivalent. Then there is a unitary $Z : \mathcal{H}_1 \to \mathcal{H}_2$ such that $Z \circ \rho_{\mathcal{H}_1} \circ Z^* = \rho_{\mathcal{H}_2}$. In particular

$$ZM_\lambda|_{\mathcal{H}_1} = M_\lambda|_{\mathcal{H}_2} Z, \quad \text{for } \lambda \geq 0.$$

By the intertwining form of the lifting theorem for the commutant of a continuous semigroup of isometries it follows that $Z = P_{\mathcal{H}_2} \tilde{Z}|_{\mathcal{H}_1}$ where $\tilde{Z}$ is an operator in the commutant of $\{M_\lambda : \lambda \geq 0\}$. (The single isometry variant of this, from which this semigroup lifting theorem may be deduced, is due to Sarason [22].) Thus $\tilde{Z} = M_h$ for some unimodular analytic function $h(z)$. Since Z is isometric it follows, in each of the cases (ii)-(iv), that the operator $Z = P_{\mathcal{H}_2} M_h|_{\mathcal{H}_1}$ is equal to the restriction operator $M_h|_{\mathcal{H}_1}$. We now have

$$M_h D_\mu|\mathcal{H}_1 = D_\mu M_h|\mathcal{H}_1, \quad \text{for } \mu \geq 0,$$

from which it follows that $D_\mu h = h$ for all μ. Thus h is a constant function and $\mathcal{H}_1$ is equal to $\mathcal{H}_2$. □

Remark 4.3. We note that the assumption of strong irreducibility is necessary in Theorem 4.1. Let $\mathcal{H}$ be the subspace of $L^2(\mathbb{R}^2)$ given by

$$\mathcal{H} = \{f \in L^2(\mathbb{R}^2) : f(x, y) = 0, \text{ for a.e. } (x, y) \in \mathbb{R}^2_-\}$$

and consider the strongly continuous isometric semigroups with $U_\lambda = M_\lambda \otimes D_\lambda$ and $V_\mu = D_\mu \otimes I$. This gives an isometric representation of the partial Weyl commutation relations which is irreducible. However, their joint minimal unitary dilation is given by the pair of semigroups $M_\lambda \otimes D_\lambda$ and $D_\mu \otimes I$, acting on $L^2(\mathbb{R}^2)$. This representation is not irreducible, since the space $L^2(\mathbb{R}) \otimes H^2(\mathbb{R})$ reduces both unitary groups, and in fact is a representation with infinite multiplicity.

5. Restriction algebras

Let us refer to an operator algebra of the form $\mathcal{A}|_K$, with K in $Lat\mathcal{A}$, as a *restriction algebra* for $\mathcal{A}$, and refer to the weak*-closure, $(\mathcal{A}|_K)^{-w^*}$, as a *closed restriction algebra.* In Theorem 4.1 we showed that the strongly isometric representations of the partial Weyl commutation relations were in bijective correspondence with the restriction representations, ρ_K say, for nonzero subspaces K in $Lat\mathcal{A}_p$. We now show that nevertheless, the closed restriction algebras of $\mathcal{A}_p$, for K a proper subspace, are all unitarily equivalent, and in fact are unitary equivalent to the Volterra nest algebra for the half-line.

We obtain this uniqueness of closed restriction algebra, for K proper, in two steps. In the first step, Lemma 5.1, we see that within the 3 types for K, a pair of closed restriction algebras are unitary equivalent in a canonical way in terms of explicit operators, including dilation unitaries V_t. In the second step we show that there are noncanonical unitary equivalences with the half-line Volterra nest algebra. The arguments for this rely on the density property in Lemma 5.3.

Lemma 5.1. *Let K_1, K_2 be proper closed subspaces which both belong to one of the following three subsets of $Lat\mathcal{A}_p$: (i) $\mathcal{N}_v$, (ii) $\mathcal{N}_a$, (iii) the subspaces $K_{\lambda,s}$, for $\lambda \in \mathbb{R}, s > 0$. Then the restriction algebras for K_1 and K_2 are unitarily equivalent.*

Proof. It is straightforward to check that the restriction algebras of (i) (resp. (ii)) are unitarily equivalent to the restriction algebra $\mathcal{A}_p|_{L^2(\mathbb{R}_+)}$ (resp. $\mathcal{A}_p|_{H^2(\mathbb{R})}$) by means of a unitary of the form D_μ (resp. M_λ). For the algebras in (iii) we first introduce the dilation unitaries $V_t, t \geq 0$, defined by $(V_t f)(x) = e^{t/2} f(e^t x)$. (This abuses earlier notation for WCR isometries but is consistent with the M_λ, D_μ, V_t notation of [10], [11], [12].) We have the commutation relations

$$V_t M_\lambda = M_{\lambda e^t} V_t, \quad V_t D_\mu = D_{\mu e^{-t}} V_t.$$

In particular the unitary automorphism Ad_{V_t} of $B(L^2(\mathbb{R}))$ restricts to a unitary automorphism of $\mathcal{A}_p$. Also, for $s_1, s_2 > 0$ and $t = \frac{1}{2} \log \frac{s_1}{s_2}$, we have $V_t M_{\phi_{s_2}} H^2(\mathbb{R}) = M_{\phi_{s_1}} H^2(\mathbb{R})$. It follows routinely from this that the algebras of (iii) are unitarily equivalent to the algebra $\mathcal{A}_p|_{M_{\phi_1} H^2(\mathbb{R})}$ by means of unitaries of the form $M_\lambda D_\mu V_t$. □

The next lemma is well-known and is a consequence of the weak*-density of the algebra of analytic trigonometric polynomials in $L^\infty(\mathbb{R}_+)$. For completeness we give a proof which is also a prelude to the separation argument for the proof of Lemma 5.5.

Lemma 5.2. *Let P_+ be the orthogonal projection on $L^2(\mathbb{R}_+)$. Then the operator algebra $P_+ M_{H^\infty}|_{L^2(\mathbb{R}_+)}$ is w^*-dense in the maximal abelian von Neumann algebra $M_{L^\infty(\mathbb{R}_+)}$.*

Proof. If $P_+ M_{H^\infty} P_+$ is not dense in $M_{L^\infty(\mathbb{R}_+)}$, then there exists an essentially bounded function ϕ supported on $\mathbb{R}_+$ and a w^*-continuous functional $\omega : B(L^2(\mathbb{R})) \to \mathbb{C}$, such that

$\omega(P_+M_fP_+) = 0$, for every bounded analytic function f, and $\omega(M_\phi) = 1$. On the other hand, the restriction of ω to the multiplication algebra $M_{L^\infty(\mathbb{R})}$ induces a w^*-continuous functional on $L^\infty(\mathbb{R})$, which we also denote by ω. Hence there exist $h \in L^1(\mathbb{R})$, such that

$$\omega(f) = \int_{\mathbb{R}} f(x)h(x)dx.$$

Take $f(x) = e^{i\lambda x}$. Then

$$\omega(P_+M_fP_+) = 0 \quad \Leftrightarrow \quad \int_{\mathbb{R}} e^{i\lambda x}\chi_{\mathbb{R}_+}(x)h(x)dx = 0.$$

Since this is true for all $\lambda \geq 0$ it follows that $\chi_{R_+}h$ lies in $H^1(\mathbb{R})$ and so is equal to the zero function. Therefore the essential support of the function h is contained in $\mathbb{R}_-$. Hence, given any function $\phi \in L^\infty(\mathbb{R}_+)$

$$\omega(M_\phi) = \int_{\mathbb{R}} \phi(x)h(x)dx = 0,$$

which is the desired contradiction. □

The Volterra nest algebra $\mathcal{A}_{v+}$ on $L^2(\mathbb{R}_+)$ is defined as the algebra of operators on this space which leaves invariant each of the subspaces $L^2[t,\infty)$, for $t \geq 0$.

Lemma 5.3. *The restriction algebra $\mathcal{A}_p|_{L^2(\mathbb{R}_+)}$ is w^*-dense in $\mathcal{A}_{v+}$.*

Proof. As noted in the introduction, the weak*-closed linear span of the products $M_\lambda D_\mu$, for $\lambda, \mu \in \mathbb{R}$, is equal to $B(L^2(\mathbb{R}))$. Let P_t and Q_t be the orthogonal projections onto the subspaces $L^2[0,t]$ and $L^2[t,\infty)$ respectively. Then $Q_tM_\lambda D_\mu P_t = 0$, for all $\mu \leq 0$. It follows that the weak*-closed linear span, $\mathcal{A}_t$ say, of the products $Q_tM_\lambda D_\mu P_t$, for $\mu \geq 0, \lambda \in \mathbb{R}$, is equal to the space $Q_tB(L^2(\mathbb{R}))P_t$. Also, by Lemma 5.2 the spaces $\mathcal{A}_t$ belong to the weak*-closure of $\mathcal{A}_p|_{L^2(\mathbb{R}_+)}$. On the other hand, every finite rank operator of $\mathcal{A}_{v+}$ necessarily lies in a finite sum of the spaces $\mathcal{A}_t$ [3], and so belongs to this weak*-closure. Also, the finite rank operators of $\mathcal{A}_{v+}$ are weak*-dense in $\mathcal{A}_{v+}$, and so the proof is complete. □

Proposition 5.4. *The restriction algebra $\mathcal{A}_p|_{H^2(\mathbb{R})}$ is w^*-dense in the nest algebra $Alg\mathcal{N}_{a+}$ on $H^2(\mathbb{R})$ for the nest*

$$\mathcal{N}_{a+} = \{e^{i\lambda x}H^2(\mathbb{R}) : \lambda \geq 0\} \cup \{0\},$$

and is unitarily equivalent to $\mathcal{A}_{v+}$.

Proof. Let $F : H^2(\mathbb{R}) \to L^2(\mathbb{R}_+)$ be the restriction of the Fourier-Plancherel transform. Then,

$$F\mathcal{A}_p\big|_{H^2(\mathbb{R})}F^*P_+ = \{FAP_{H^2(\mathbb{R})}F^*P_+ : A \in \mathcal{A}_p\} = \{FAF^*P_+ : A \in \mathcal{A}_p\}.$$

Also, $FM_\lambda F^* = D_\lambda$ and $FD_\mu F^* = M_{-\mu}$. As in the proof of Lemma 5.2 the algebra generated by the semigroup $\{M_{-\lambda}P_+ : \lambda \geq 0\}$ is dense in $M_{L^\infty(\mathbb{R}_+)}$. Therefore it follows that $F\mathcal{A}_p\big|_{H^2(\mathbb{R})}F^*|_{L^2(\mathbb{R}_+)}$ is w^*-dense in $\mathcal{A}_{v+}$. Thus

$$(\mathcal{A}_p\big|_{H^2(\mathbb{R})})^{-w^*} = F^*(\mathcal{A}_{v+})F\big|_{H^2(\mathbb{R})} = Alg\mathcal{N}_a\big|_{H^2(\mathbb{R})},$$

as required. □

We now fix $s > 0$ and consider the case $K = K_s = M_{\phi_s} H^2(\mathbb{R})$. Recall from the proof of Theorem 2.3 that $M_{\phi_s} M_{s\mu} D_\mu M_{\phi_s}^* = e^{-is\mu^2/2} D_\mu$. Applying $Ad(M_{\phi_s}^*)$ to $\mathcal{A}_p\big|_{K_s}$ we have the following identification of operator algebras on $H^2(\mathbb{R})$:

$$\begin{aligned} M_{\phi_s}^* \mathcal{A}_p\big|_{K_s} M_{\phi_s} &= \{M_{\phi_s}^* A P_{K_s} M_{\phi_s} : A \in \mathcal{A}_p\} \\ &= \{M_{\phi_s}^* A M_{\phi_s} P_{H^2(\mathbb{R})} : A \in \mathcal{A}_p\} \\ &= \{A P_{H^2(\mathbb{R})} : A \in \mathcal{A}_{sp}\} \end{aligned}$$

where $\mathcal{A}_{sp}$ is the weak*-closed algebra generated by $\{M_\lambda, M_{s\mu} D_\mu : \lambda, \mu \geq 0\}$.

Applying the Fourier transform we obtain that the algebra

$$\mathcal{A}_{sp}^{\mathcal{F}}\big|_{L^2(\mathbb{R}_+)} := F^* \mathcal{A}_{\text{sp}} F\big|_{H^2(\mathbb{R})}$$

is generated as a weak*-closed algebra by the set of isometries

$$\{D_\lambda\big|_{L^2(\mathbb{R}_+)}, D_{s\mu} M_{-\mu}\big|_{L^2(\mathbb{R}_+)} : \lambda, \mu \geq 0\}.$$

Lemma 5.5. *The algebra $\mathcal{A}_{sp}^{\mathcal{F}}\big|_{L^2(\mathbb{R}_+)}$ is dense in the Volterra nest algebra $\mathcal{A}_{v+}$.*

Proof. Fix some $s, \mu > 0$. Let ω be a w^*-continuous functional

$$\omega : B(L^2(\mathbb{R})) \to \mathbb{R} : T \mapsto \sum_k \langle T h_k, g_k \rangle$$

for some $h_k, g_k \in L^2(\mathbb{R})$. Suppose that it annihilates $P_+ \mathcal{A}_{sp}^{\mathcal{F}} P_+$, and therefore the operators $P_+ D_{s\mu} M_{-\lambda} P_+$ for all $\lambda \in [0, \mu]$. Then for these parameters we have

$$\omega(P_+ D_{s\mu} M_{-\lambda} P_+) = 0.$$

Define the bounded linear functional

$$\omega_\mu : B(L^2(\mathbb{R})) \to \mathbb{R} : T \mapsto \omega(P_+ D_{s\mu} T P_+).$$

Identify the restriction of ω_μ on the multiplication algebra $M_{L^\infty(\mathbb{R})}$ with the w^*-continuous functional $\omega_\mu(f) = \int_{\mathbb{R}} f(x) h_\mu(x) dx$, where h_μ is an $L^1(\mathbb{R})$ function. Then it follows from the definition of ω_μ that h_μ is zero on $\mathbb{R}_-$.

Thus

$$0 = \omega(D_{s\mu} M_{-\lambda} P_+) = \omega_\mu(M_{-\lambda}) = \int_{\mathbb{R}} e^{-i\lambda x} h_\mu(x) dx.$$

Since $\int_{\mathbb{R}} e^{-i\lambda x} h_\mu(x) dx = 0$, it follows that $\hat{h}_\mu$ vanishes on the interval $(0, \mu)$. On the other hand, $\hat{h}_\mu \in H^\infty(\mathbb{R}) \cup C_0(\mathbb{R})$ and so $h_\mu = 0$. Thus for $f \in L^\infty(\mathbb{R}_+)$ we have

$$\omega(M_f) = \lim_{\mu \to 0} \omega(P_+ D_{s\mu} M_f P_+) = \lim_{\mu \to 0} \omega_\mu(M_f) = 0.$$

It follows, by the usual separation principle, that the multiplication algebra for $L^\infty(\mathbb{R}_+)$ must lie in the w^*-closure. Since the right shift operators also lie in the algebra it follows by standard arguments that $\mathcal{A}_{v+}$ is contained in the closure, completing the proof. □

Combining the results of this section, we have the following,

Theorem 5.6. *Let K be a proper invariant subspace of the parabolic algebra. Then $(\mathcal{A}_p|_K)^{-w^*}$ is unitarily equivalent to the Volterra nest algebra $\mathcal{A}_{v+}$.*

6. Quasicompact algebras

Let $\mathcal{A}$ be a weak*-closed operator algebra on the Hilbert space $\mathcal{H}$ and $\mathcal{K} = \mathcal{K}(\mathcal{H})$ the ideal of the compact operators. Define the *quasicompact algebra* of $\mathcal{A}$ to be the C*-algebra $Q\mathcal{A}$ where

$$Q\mathcal{A} = (\mathcal{A} + \mathcal{K}) \cap (\mathcal{A}^* + \mathcal{K}).$$

Analogous algebras have been studied systematically in the theory of function spaces, the principal example being the nonseparable algebra of quasicontinuous functions,

$$QC(\mathbb{T}) = (H^\infty(\mathbb{T}) + C(\mathbb{T})) \cap \left(\overline{H^\infty(\mathbb{T})} + C(\mathbb{T})\right),$$

where $C(\mathbb{T})$ is the algebra of continuous functions on the unit circle (see [5]). Determining the structure of $Q\mathcal{A}$ and whether it differs from $\mathcal{A} \cap \mathcal{A}^* + \mathcal{K}$ seems to be a rather deep problem in general. However for $\mathcal{A} = \mathcal{A}_v$ the following is well-known.

Theorem 6.1. *The quasicompact algebra $Q\mathcal{A}_v$ is not equal to $\mathcal{A}_v \cap \mathcal{A}_v^* + \mathcal{K}$.*

The proof of this theorem has two main ingredients. The first of these is that the triangular truncation operator with respect to the Volterra nest,

$$\mathcal{P}_v : \mathcal{C}_2(L^2(\mathbb{R})) \to \mathcal{A}_v \cap \mathcal{C}_2(L^2(\mathbb{R})).$$

is a contractive projection in the space $\mathcal{C}_2(L^2(\mathbb{R}))$ of Hilbert-Schmidt operators which is an unbounded operator with respect to the operator norm. (See Chapter 4 of [3] for example.) The second ingredient is to use this fact to create unit norm finite rank operators A_k in $\mathcal{A}_v$, with orthogonal domains and ranges and with operator norms $\|A_k - A_k^*\|$ tending to zero. Then the infinite sum A of the A_k is a compact perturbation of A^* which, furthermore, does not belong to $M_{L^\infty(\mathbb{R})} + \mathcal{K}$.

In the proof of Theorem 6.3 we adopt a similar strategy. However, in $\mathcal{A}_p$ there are no finite rank operators and we must make use of compact operators for the A_k. Also the orthogonality of domains and ranges must be replaced by an approximate form of this.

Lemma 6.2. *The restriction of the triangular truncation operator $\mathcal{P}_v\big|_{\mathcal{A}_p+\mathcal{A}_p^*}$ is unbounded.*

Proof. Let p_n be a real coefficient polynomial on $\mathbb{T}$ with supremum norm 1, such that the polynomials $f_n(z) = p_n(z) - \overline{p_n(z)}$ satisfy the property $\|f_n\|_\infty \to 0$. For example, take

$$p_n(z) = c_n \sum_{k=1}^{n} \frac{1}{k} z^k.$$

for appropriate constants c_n. Let Z be a unitary operator in $\mathcal{A}_p$ with full spectrum, such as M_1. Take (F_n) to be a strong operator topology approximate identity of Hilbert-Schmidt operators in the unit ball of $\mathcal{A}_p$, as in the remarks following Theorem 2.2.

By the functional calculus $\|p_n(Z)\| = \|p_n\| = 1$ and there exists a sequence (ξ_n) in the unit sphere of $L^2(\mathbb{R})$, such that $\|p_n(Z)\xi_n\| > 2/3$, for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Then $p_n(ZF_m)\xi_n \to p_n(Z)\xi_n$ as $m \to \infty$. Choose inductively a subsequence (F_{m_n}), denoted (F_n), such that

$$\|p_n(ZF_n)\xi_n\| > 1/2.$$

Since $p_n(ZF_n)$ belongs to $\mathcal{A}_p$, we have $\langle K, p_n(ZF_n)^*\rangle_{H-S} = 0$, for every Hilbert-Schmidt operator $K \in \mathcal{A}_v$ and $n \in \mathbb{N}$. Thus

$$\begin{aligned}\|\mathcal{P}_v(p_n(ZF_n) - p_n(ZF_n)^*)\| =& \|\mathcal{P}_v(p_n(ZF_n)) - \mathcal{P}_v(p_n(ZF_n)^*)\| = \\ &= \|p_n(ZF_n)\| \geq \|p_n(ZF_n)\xi_n\| > 1/2.\end{aligned}$$

On the other hand, since ZF_n is a contraction for every $n \in \mathbb{N}$ we have yields

$$\|p(ZF_n) + q(ZF_n)^*\| \leq \|p + \overline{q}\|$$

for all polynomials p, q in the disc algebra. (See Theorem 2.6 and Corollary 2.8 of [16].) Taking $p = p_n$ and $q = -p_n$, it follows

$$\|p_n(ZF_n) - p_n(ZF_n)^*\| \leq \|p_n - \overline{p_n}\| \to 0,$$

which completes the proof. □

Theorem 6.3. *The algebra $\mathcal{A}_p \cap \mathcal{A}_p^* + \mathcal{K} = \mathbb{C}I + \mathcal{K}$ is a proper subalgebra of the quasicompact algebra $Q\mathcal{A}_p$.*

Proof. Recall the operators $p_n(ZF_n)$ from the proof of Lemma 6.2, where $Z \in \mathcal{A}_p$ is unitary with spectrum the unit circle, (F_n) is the bounded approximate identity in $\mathcal{A}_p$ of Hilbert-Schmidt operators, and $p_n(z)$ are polynomials in $C(\mathbb{T})$, of unit $\|\cdot\|_\infty$-norm, with real parts converging uniformly to zero. Also we have $1/2 \leq \|p_n(ZF_n)\| \leq 1$.

Since these operators are compact, there exist compact intervals K_n such that for any $f \in L^2(\mathbb{R})$,

$$\|p_n(ZF_n)f\| \leq \|P_{K_n}f\| + \frac{1}{2^n}\|f\| \tag{6.1}$$

and

$$\|P_{\mathbb{R}\setminus K_n}p_n(ZF_n)f\| \leq \frac{1}{2^n}\|f\|, \tag{6.2}$$

where P_{K_n} is the projection on $L^2(K_n)$. We can also arrange that for all n,

$$\|\left(p_n(ZF_n) - p_n(ZF_n)^*\right)f\| \leq \|p_n(ZF_n) - p_n(ZF_n)^*\| \left(\|P_{K_n}f\| + \frac{1}{2^n}\|f\|\right) \tag{6.3}$$

and

$$\|P_{\mathbb{R}\setminus K_n}\left(p_n(ZF_n) - p_n(ZF_n)^*\right)f\| \leq \frac{1}{2^n}\|p_n(ZF_n) - p_n(ZF_n)^*\|\,\|f\|. \tag{6.4}$$

We now choose (t_n) to be an increasing sequence so that the translates $\Lambda_n = K_n + t_n$ are disjoint sets, with $\max \Lambda_n < \min \Lambda_{n+1}$ for all n. We also write $\Lambda_0 = \mathbb{R}\setminus \cup_{n=1}^\infty \Lambda_n$. Since the projection of triangular truncation with respect to the binest commutes with Ad_{D_t}, it follows that the operators

$$A_n = D_{t_n}(p_n(ZF_n))D_{t_n}^*$$

lie in $\mathcal{A}_p$.

Claim 1: Given $f \in L^2(\mathbb{R})$, the sequence $(\sum_{k=1}^{n} A_k f)_n$ is convergent.

To prove this claim, it suffices to show that the given sequence is a Cauchy sequence. Let C be the compact set $\cup_{m=n}^{N}\Lambda_m$. Then

$$\left\|\sum_{k=n}^{N} A_k f\right\|^2 = \int_{\mathbb{R}\setminus C}\left|\sum_{k=n}^{N} A_k f\right|^2 + \int_C \left|\sum_{k=n}^{N} A_k f\right|^2.$$

Estimating the integrals we have

$$\begin{aligned}\int_{\mathbb{R}\setminus C}\left|\sum_{k=n}^{N} A_k f\right|^2 = \left\|P_{\mathbb{R}\setminus C}\sum_{k=n}^{N} A_k f\right\|^2 &\le \left(\sum_{k=n}^{N}\|P_{\mathbb{R}\setminus C}A_k f\|\right)^2\\ &\le \left(\sum_{k=n}^{N}\frac{1}{2^k}\|f\|\right)^2 = \left(\sum_{k=n}^{N}\frac{1}{2^k}\right)^2\|f\|^2.\end{aligned}$$

Also,

$$\int_C\left|\sum_{k=n}^{N} A_k f\right|^2 = \int_{\mathbb{R}}\left|\sum_{m=n}^{N} P_{\Lambda_m}\sum_{k=n}^{N} A_k f\right|^2 \le 2\int_{\mathbb{R}}\left|\sum_{m=n}^{N} P_{\Lambda_m}A_m f\right|^2 + 2\int_{\mathbb{R}}\left|\sum_{m=n}^{N}\sum_{\substack{k=n\\k\neq m}}^{N} P_{\Lambda_m}A_k f\right|^2.$$

The first term here gives

$$\begin{aligned}\int_{\mathbb{R}}\left|\sum_{m=n}^{N} P_{\Lambda_m}A_m f\right|^2 = \sum_{m=n}^{N}\|P_{\Lambda_m}A_m f\|^2 &\le \sum_{m=n}^{N}(\|P_{\Lambda_m}f\| + \frac{1}{2^m}\|f\|)^2\\ &\le \sum_{m=n}^{N}\left(\|P_{\Lambda_m}f\|^2 + \left(\frac{2}{2^m}+\frac{1}{2^{2m}}\right)\|f\|^2\right)\\ &\le \|P_C f\|^2 + \left(\sum_{m=n}^{N}\left(\frac{2}{2^m}+\frac{1}{2^{2m}}\right)\right)\|f\|^2.\end{aligned}$$

Note that for every $\epsilon_1 > 0$, we can choose n_0 big enough so that $\|P_A f\| \le \epsilon_1\|f\|$, where $A = \cup_{m=n_0}^{\infty}\Lambda_m$.

For the second term, it follows by relation (6.2) that

$$\begin{aligned}\int_{\mathbb{R}}\left|\sum_{m=n}^{N}\sum_{\substack{k=n\\k\neq m}}^{N} P_{\Lambda_m}A_k f\right|^2 &= \left\|\sum_{m=n}^{N}\sum_{\substack{k=n\\k\neq m}}^{N} P_{\Lambda_m}A_k f\right\|^2 = \left\|\sum_{k=n}^{N}\sum_{\substack{m=n\\m\neq k}}^{N} P_{\Lambda_m}A_k f\right\|^2\\ &= \left\|\sum_{k=n}^{N} P_{C\setminus\Lambda_k}A_k f\right\|^2 \le \left(\sum_{k=n}^{N}\|P_{C\setminus\Lambda_k}A_k f\|\right)^2\\ &\le \left(\sum_{k=n}^{N}\frac{1}{2^k}\|f\|\right)^2 = \left(\sum_{k=n}^{N}\frac{1}{2^k}\right)^2\|f\|^2.\end{aligned}$$

Combining the above estimates we get

$$\Big\|\sum_{k=n}^{N} A_k f\Big\|^2 \leq \left(3\left(\sum_{k=n}^{N}\frac{1}{2^k}\right)^2 + 2\sum_{m=n}^{N}\left(\frac{2}{2^m}+\frac{1}{2^{2m}}\right) + 2\epsilon_1^2\right)\|f\|^2.$$

Hence, there exists $n_0 \in \mathbb{N}$ such that $\|\sum\limits_{k=n}^{N} A_k f\|^2 \leq \epsilon$, for all $n, N > n_0$, proving the first claim.

Claim 2: The sequence $\{\sum\limits_{k=1}^{n} A_k\}_n$ is uniformly bounded.

Let $f \in L^2(\mathbb{R})$. Then

$$\Big\|\sum_{k=1}^{n} A_k f\Big\|^2 = \lim_{M\to\infty}\Big\|\sum_{m=0}^{M} P_{\Lambda_m}\sum_{k=1}^{n} A_k f\Big\|^2.$$

Also

$$\Big\|\sum_{m=0}^{M} P_{\Lambda_m}\sum_{k=1}^{n} A_k f\Big\|^2 \leq 2\Big\|\sum_{k=1}^{n} P_{\Lambda_k}A_k f\Big\|^2 + 2\Big\|\sum_{k=1}^{n}\sum_{\substack{m=0\\ m\neq k}}^{M} P_{\Lambda_m}A_k f\Big\|^2.$$

Applying now the relation (6.1) we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^{n} P_{\Lambda_k}A_k f\Big\|^2 &\leq \sum_{k=1}^{n}\left(\|P_{\Lambda_k} f\| + \frac{1}{2^k}\|f\|\right)^2\\ &= \sum_{k=1}^{n}\|P_{\Lambda_k} f\|^2 + \sum_{k=1}^{n}\left(\frac{2}{2^k}+\frac{1}{2^{2k}}\right)\|f\|^2 \leq 4\|f\|^2.\end{aligned}$$

Moreover

$$\begin{aligned}\Big\|\sum_{k=1}^{n}\sum_{\substack{m=0\\ m\neq k}}^{M} P_{\Lambda_m}A_k f\Big\|^2 &\leq \left(\sum_{k=1}^{n}\Big\|\sum_{\substack{m=0\\ m\neq k}}^{M} P_{\Lambda_m}A_k f\Big\|\right)^2 \leq \left(\sum_{k=1}^{n}\|P_{\mathbb{R}\setminus\Lambda_k}A_k f\|\right)^2 \leq \left(\sum_{k=1}^{n}\frac{1}{2^k}\|f\|\right)^2\\ &= \left(\sum_{k=1}^{n}\frac{1}{2^k}\right)^2\|f\|^2 \leq \|f\|^2.\end{aligned}$$

Hence the norms $\|\sum\limits_{k=1}^{n} A_k\|$ are indeed uniformly bounded, as required.

Let $M = \sup_n \|\sum\limits_{k=1}^{n} A_k\| < \infty$ and define the linear operator A acting on $L^2(\mathbb{R})$ by the formula

$$Af = \lim_n \sum_{k=1}^{n} A_k f.$$

By the first claim A is well-defined and by the second claim, $\|Af\| \leq M\|f\|$, for $f \in L^2(\mathbb{R})$. Thus A is a bounded operator equal to the SOT-limit of the sequence $\{\sum_{k=1}^{n} A_k\}_n$, and so lies in $\mathcal{A}_p$. Note also that A is not a compact operator since $\|A_k\| \geq 1/2$ for all k.

Define now the Hilbert-Schmidt operators

$$X_n = A_n - A_n^* = D_{t_n}(p_n(ZF_n) - p_n(ZF_n)^*)D_{t_n}^*$$

and note that $\|X_n\| \to 0$. By the calculations above we see that the sequence of the partial sums of $\sum_{n=1}^{\infty} X_n$ is Cauchy with respect to the operator norm and so the norm limit $X = \sum_{n=1}^{\infty} X_n$ is a compact operator on $L^2(\mathbb{R})$. Since involution is continuous in the weak operator topology, we see that $A - A^* = X$. Thus A lies in the quasicompact algebra $Q\mathcal{A}_p$ and it remains to show that $A \notin \mathbb{C}I + \mathcal{K}$.

Assume that this is not true. Since the algebra $\mathbb{C}I + \mathcal{K}$ is norm closed we have $A = cI + K$ with $c \in \mathbb{C}$ and $K \in \mathcal{K}$. Left multiply both sides by the projection P_{Λ_0}. Since multiplication is separately SOT-continuous we see that $P_{\Lambda_0}A$ is the SOT-limit of the operators $\{\sum_{k=1}^{n} P_{\Lambda_0}A_k\}_n$. Moreover we have the estimates

$$\|P_{\Lambda_0}A_k\| \leq \frac{1}{2^k}, \text{ for every } k \in \mathbb{N},$$

so the above sequence converges uniformly to $P_{\Lambda_0}A$. Therefore $P_{\Lambda_0}A$ is a compact operator, and so cP_{Λ_0} is also compact. However, Λ_0 is a union of nonempty intervals and so $c = 0$. But this implies that A is compact, which gives the desired contradiction. □

Remark 6.4. We remark that Q$\mathcal{A}_p$, like QC and also Q$\mathcal{A}_v$, is a nonseparable C*-algebra. Indeed one can determine a well-separated uncountable set by considering operators of the form $\sum_{k=1}^{\infty} b_k A_k$ where (b_k) is a 0-1 sequence and (A_k) is a sufficiently approximately orthogonal sequence of unit norm operators in $\mathcal{A}_p$ as in the proof above.

Remark 6.5. As we noted in Section 2, the parabolic algebra $\mathcal{A}_p$ is equal to the intersection $\mathcal{A}_v \cap \mathcal{A}_a$. It seems plausible, given the proof above, that the *essential parabolic algebra* $\mathcal{A}_p/\mathcal{K}$ is a proper subalgebra of the intersection algebra $(\mathcal{A}_v/\mathcal{K}) \cap (\mathcal{A}_a/\mathcal{K})$ however we do not know if this is the case. This is equivalent to determining whether $(\mathcal{A}_v + \mathcal{K}) \cap (\mathcal{A}_a + \mathcal{K})$ is strictly larger than $\mathcal{A}_p + \mathcal{K}$.

7. Further directions

The parabolic algebra sits between the very well-understood algebra $H^\infty(\mathbb{R})$ and the equally well-understood Volterra nest algebra $\mathcal{A}_v$. Determining further algebraic and geometric properties of $\mathcal{A}_p$ is likely to require methods from both commutative and noncommutative perspectives. In particular, the following 3 problems seem to be rather deep requiring both perspectives.

Question 1. *How are the weakly closed ideals of $\mathcal{A}_p$ characterised?* The weakly closed ideals of a nest algebra $\mathcal{A}$ are determined by left-continuous order homomorphisms from $Lat\mathcal{A}$ to $Lat\mathcal{A}$ ([6], [3]). Similar such boundary functions, from $Lat\mathcal{A}_p$ to $Lat\mathcal{A}_p$, are likely to play a role in resolving this question.

Question 2. *What is the Jacobson radical of $\mathcal{A}_p$?* In particular is there some kind of analogue of Ringrose's characterisation ([20], [3])?

Question 3. *Is there a variant of Arveson's distance formula for $\mathcal{A}_p$?* Arveson's distance formula ([3], [17]) for a nest algebra is closely related to distance formulae for $H^\infty(\mathbb{R})$ so in some ways this is a natural problem for $\mathcal{A}_p$. Also it would lead to the hyperreflexivity of $\mathcal{A}_p$, a property known for both $\mathcal{A}_v$ and for $H^\infty(\mathbb{R})$ as an operator algebra on $H^2(\mathbb{R})$ (Davidson [2]).

Question 4. *Does $\mathcal{A}_p$ have zero divisors?* We suspect that this intriguing question [18] is less deep and indeed we believe that there are zero divisors even in the norm-closed algebra that is generated by the 2 semigroups. This algebra is analysable as a norm-closed semi-crossed product [9] and elements admit natural generalised Fourier series. First let us remark that the unclosed algebra has no divisors of zero simply because operators in the algebra have generalised finite Fourier series of the form

$$\sum_{\lambda_k \in \mathbb{R}_+} A_k M_{\lambda_k}$$

where each A_k is a finite linear combination of the D_μ for $\mu \in \mathbb{R}_+$. It follows that a product of nonzero elements has a nonzero first Fourier coefficient and so is nonzero. On the other hand is it possible to construct 2 absolutely norm-convergent generalised Fourier series (with no first nonzero coefficient) such that the product is zero?

References

[1] W. B. Arveson, Operator algebras and invariant subspaces, Ann. of Math., 100 (1974), 433-532.
[2] K. R. Davidson, The distance to the analytic Toeplitz operators, Illinois J. Math., 31 (1987), 265-273.
[3] K. R. Davidson, *Nest Algebras*, Pitman Research Notes in Math., 191, Longman 1988.
[4] K. R. Davidson, The mathematical legacy of William Arveson, J. Operator Th., 68 (2012), 101-127.
[5] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Springer, New York, (1998).
[6] J. A. Erdos and S. C. Power, Weakly closed ideals of nest algebras, J. Operator Th., 7 (1982), 219-235.
[7] D. Hadwin, A general view of reflexivity, Trans. Amer. Math. Soc., 344 (1994), 325-360.
[8] E. Kastis, The parabolic algebra on L^p-spaces, J. Math. Anal. and Appl., 455 (2017), 698-713.
[9] E. Kastis, The norm-closed triple semigroup algebra, Israel J. Math., 230 (2019), 855-894.
[10] E. Kastis and S. C. Power, The operator algebra generated by the translation, dilation and multiplication semigroups, Journal of Funct. Anal. 269 (2015), 3316-3335.
[11] A. Katavolos and S. C. Power, The Fourier binest algebra, Math. Proc. Cambridge Philos. Soc., 122 (1997), 525–539.
[12] A. Katavolos and S. C. Power, Translation and dilation invariant subspaces of $L^2(\mathbb{R})$, J. Reine Angew. Math., 552 (2002), 101–129.
[13] R. H. Levene, Lie semigroup operator algebras, Ph.D. Thesis, Lancaster University, 2004.
[14] R. H. Levene and S. C. Power, Reflexivity of the translation-dilation algebras on $L^2(\mathbb{R})$, International J. Math., 14 (2003), 1081–1090.
[15] J. von Neumann, Die Eindentigkeit der Schrodingerschen Operatoren, Math. Ann. 104 (1931), 570-578.
[16] V. I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Studies in Advanced Mathematics, 78. Cambridge University Press, Cambridge, (2002).
[17] S. C. Power, The distance to upper triangular operators, Math. Proc. Camb. Phil. Soc., 88 (1980), 327-329.
[18] S. C. Power, Invariant subspaces of translation semigroups, Proceedings of the First Advanced Course in Operator Theory and Complex Analysis, Ed. Afonso Montes Rodríguez, University of Seville, 2006.

[19] H. Radjavi and P. Rosenthal, *Invariant subspaces*, Dover, 2003.

[20] J. R. Ringrose, On some algebras of operators. Proc. London Math. Soc., 15 (1965), 61-83.

[21] J. A. Rosenberg, A selective history of the Stone-von Neumann theorem, Operator algebras, quantization, and noncommutative geometry, Contemp. Math., 365, p 331-353, Amer. Math. Soc., Providence, RI, 2004.

[22] D. Sarason, Generalized interpolation in H^∞, Trans. Amer. Math. Soc. 127 (1967), 179-203.

[23] V. Shulman and L. Turowska, Operator synthesis. I. Synthetic sets, bilattices and tensor algebras, Journal of Functional Analysis, 209 (2004), 293-331.

[24] M. H. Stone, Linear transformations in Hilbert space. III. Operational methods and group theory, Proc. Nat. Acad. Sci. U.S.A. 16 (1930), 172-175.

[25] M. E. Taylor, Noncommutative Harmonic Analysis, Math. Surveys and Monographs, vol 22, Amer. Math. Soc., 1986.

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