Flexible circuits in the d-dimensional rigidity matroid[*]

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Abstract

A bar-joint framework (G, p) in $\mathbb{R}^d$ is rigid if the only edge-length preserving continuous motions of the vertices arise from isometries of $\mathbb{R}^d$. It is known that, when (G, p) is generic, its rigidity depends only on the underlying graph G, and is determined by the rank of the edge set of G in the generic d-dimensional rigidity matroid $\mathcal{R}_d$. Complete combinatorial descriptions of the rank function of this matroid are known when $d = 1, 2$, and imply that all circuits in $\mathcal{R}_d$ are generically rigid in $\mathbb{R}^d$ when $d = 1, 2$. Determining the rank function of $\mathcal{R}_d$ is a long standing open problem when $d \geq 3$, and the existence of non-rigid circuits in $\mathcal{R}_d$ for $d \geq 3$ is a major contributing factor to why this problem is so difficult. We begin a study of non-rigid circuits by characterising the non-rigid circuits in $\mathcal{R}_d$ which have at most $d + 6$ vertices.

1 Introduction

A bar-joint *framework* (G, p) in $\mathbb{R}^d$ is the combination of a finite graph $G = (V, E)$ and a realisation $p : V \to \mathbb{R}^d$. The framework is said to be *rigid* if the only edge-length preserving continuous motions of its vertices arise from isometries of $\mathbb{R}^d$, and otherwise it is said to be *flexible*. The study of the rigidity of frameworks has its origins in the work of Cauchy and Euler on Euclidean polyhedra [5] and Maxwell [18] on frames.

Abbot [1] showed that it is NP-hard to determine whether a given d-dimensional framework is rigid whenever $d \geq 2$. The problem becomes more tractable for generic frameworks (G, p) since we can linearise the problem and consider 'infinitesimal rigidity' instead. We

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define the *rigidity matrix* $R(G,p)$ as the $|E| \times d|V|$ matrix in which, for $e = v_iv_j \in E$, the submatrices in row e and columns v_i and v_j are $p(v_i) - p(v_j)$ and $p(v_j) - p(v_i)$, respectively, and all other entries are zero. We say that (G,p) is *infinitesimally rigid* if $|V| \leq d+1$ and $\operatorname{rank} R(G,p) = \binom{|V|}{2}$ or $|V| \geq d+2$ and $\operatorname{rank} R(G,p) = d|V| - \binom{d+1}{2}$. Asimow and Roth [2] showed that infinitesimal rigidity is equivalent to rigidity for generic frameworks (and hence that generic rigidity depends only on the underlying graph of the framework).

The *d-dimensional rigidity matroid* of a graph $G = (V,E)$ is the matroid $\mathcal{R}_d(G)$ on E in which a set of edges $F \subseteq E$ is independent whenever the corresponding rows of $R(G,p)$ are independent, for some (or equivalently every) generic p. We denote the rank function of $\mathcal{R}_d(G)$ by r_d and put $r_d(G) = r_d(E)$. We say that G is: *$\mathcal{R}_d$-independent* if $r_d(G) = |E|$; *$\mathcal{R}_d$-rigid* if G is a complete graph on at most $d+1$ vertices or $r_d(G) = d|V| - \binom{d+1}{2}$; *minimally $\mathcal{R}_d$-rigid* if G is $\mathcal{R}_d$-rigid and $\mathcal{R}_d$-independent; and an *$\mathcal{R}_d$-circuit* if G is not $\mathcal{R}_d$-independent but $G - e$ is $\mathcal{R}_d$-independent for all $e \in E$.

It is not difficult to see that the 1-dimensional rigidity matroid of a graph G is equal to its cycle matroid. Landmark results of Pollaczek-Geiringer [16, 19], and Lovász and Yemini [17] characterise independence and the rank function in $\mathcal{R}_2$. These results imply that every $\mathcal{R}_d$-circuit is rigid when $d = 1,2$. This is no longer true when $d \geq 3$ (see Figures 1 and 2 below), and the existence of flexible $\mathcal{R}_d$-circuits is a fundamental obstuction to obtaining a combinatorial characterisation of independence in $\mathcal{R}_d$.

Previous work on flexible $\mathcal{R}_d$-circuits has concentrated on constructions, see Tay [20], and Cheng, Sitharam and Streinu [6]. We will adopt a different approach: that of characterising the flexible $\mathcal{R}_d$-circuits in which the number of vertices is small compared to the dimension. To state our theorem we need to define the following two families of graphs.

For $d \geq 3$ and $2 \leq t \leq d-1$, the graph $B_{d,t}$ is defined by putting $B_{d,t} = (G_1 \cup G_2) - e$ where $G_i \cong K_{d+2}$, $G_1 \cap G_2 \cong K_t$ and $e \in E(G_1 \cap G_2)$. Note that the graph $B_{3,2}$ is the well known flexible $\mathcal{R}_3$-circuit, commonly referred to as the "double banana". The family $\mathcal{B}^+_{d,d-1}$ consists of all graphs of the form $(G_1 \cup G_2) - \{e,f,g\}$ where: $G_1 \cong K_{d+3}$ and $e,f,g \in E(G_1)$; $G_2 \cong K_{d+2}$ and $e \in E(G_2)$; $G_1 \cap G_2 \cong K_{d-1}$; e,f,g do not all have a common end-vertex; if $\{f,g\} \subset E(G_1) \backslash E(G_2)$ then f,g do not have a common end-vertex. See Figure 1 for an illustration of the general construction and Figure 2 for specific examples.

Theorem 1. *Suppose G is a flexible $\mathcal{R}_d$-circuit with at most $d+6$ vertices. Then either*

(a) $d = 3$ and $G \in \{B_{3,2}\} \cup \mathcal{B}^+_{3,2}$ or

(b) $d \geq 4$ and $G \in \{B_{d,d-1},\ B_{d,d-2}\} \cup \mathcal{B}^+_{d,d-1}$.

Theorem 1 gives the following lower bound on the number of edges in a flexible $\mathcal{R}_d$-circuit. This is used in [11] to obtain an upper bound on $r_d(G)$ for all $1 \leq d \leq 11$.

Corollary 2. *Suppose $G = (V,E)$ is a flexible $\mathcal{R}_d$-circuit. Then $|E| \geq d(d+9)/2$, with equality if and only if $G = B_{d,d-1}$.*

Jordán [15] characterises $\mathcal{R}_d$-rigid graphs with at most $d+4$ vertices. He suggests in [15, Remark 1] that it may be possible to extend the characterisation to graphs on more than $d+4$ vertices, but notes that the simple degree condition given in his characterisation may not be sufficient because of the existence of the double banana. Theorem 1 implies the following characterisation of $\mathcal{R}_d$-rigid graphs with at most $d+6$ vertices. Our characterisation is in terms of *d-tight* subgraphs (which are defined in the next section).

Corollary 3. *Let $G = (V, E)$ be a graph with $d+1 \le |V| \le d+6$. Then G is $\mathcal{R}_d$-rigid if and only if G has a d-tight, d-connected spanning subgraph H such that $B_{d,d-1}, B_{d,d-2} \not\subseteq H$.*

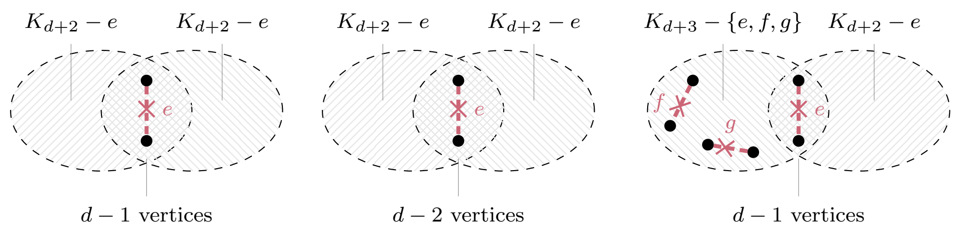

Figure 1: Graphs $B_{d,d-1}$ on the left, $B_{d,d-2}$ in the middle and $G \in \mathcal{B}^+_{d,d-1}$ on the right.

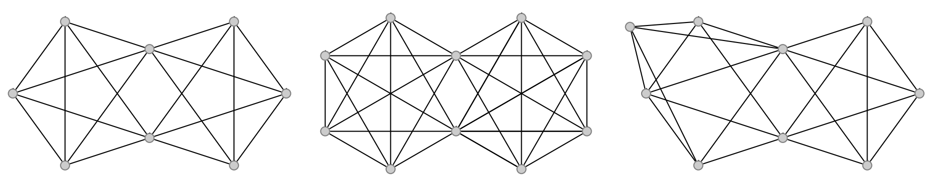

Figure 2: Graphs $B_{3,2}$ on the left, $B_{4,2}$ in the middle and $G \in \mathcal{B}^+_{3,2}$ on the right.

We will prove Theorem 1 and Corollaries 2 and 3 in Section 3.

2 Preliminary Lemmas

We will introduce some standard terminology and results from rigidity theory. We assume throughout this section that $d \ge 1$ is a fixed integer.

Given a vertex v in a graph $G = (V, E)$, we will use $d_G(v)$ and $N_G(v)$ to denote the degree and neighbour set respectively of v. For a set $V' \subseteq V$, we put $N_G(V') = \left(\bigcup_{v \in V'} N_G(v)\right) - V'$. We will use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degree, respectively, in G, and $\mathrm{dist}_G(x, y)$ to denote the length of a shortest path between two vertices $x, y \in V$. We will suppress the subscript in these notations whenever the graph is clear from the context. The graph G is *d-sparse* if $|E'| \leq d|V'| - \binom{d+1}{2}$ for all subgraphs $G' = (V', E')$ of G with $|V'| \geq d+2$. It is *d-tight* if it is d-sparse and has $d|V| - \binom{d+1}{2}$ edges. Our first result [23, Lemma 11.1.3] shows that every $\mathcal{R}_d$-independent graph is d-sparse.

Lemma 4. *Let $G = (V, E)$ be $\mathcal{R}_d$-independent with $|V| \geq d+2$. Then $|E| \leq d|V| - \binom{d+1}{2}$.*

The characterisations of $\mathcal{R}_d$-independence when $d \leq 2$ show that the converse of Lemma 4 holds for these values of d. The existence of flexible $\mathcal{R}_d$-circuits implies that the converse fails for all $d \geq 3$.

A graph G' is said to be obtained from another graph G by: a *(d-dimensional) 0-extension* if $G = G' - v$ for a vertex $v \in V(G')$ with $d_{G'}(v) = d$; or a *(d-dimensional) 1-extension* if $G = G' - v + xy$ for a vertex $v \in V(G')$ with $d_{G'}(v) = d+1$ and $x, y \in N_{G'}(v)$. The inverse operations of 0-extension and 1-extension are called *0-reduction* and *1-reduction*, respectively.

Lemma 5. *[23, Lemma 11.1.1, Theorem 11.1.7] Let G be $\mathcal{R}_d$-independent and let G' be obtained from G by a 0-extension or a 1-extension. Then G' is $\mathcal{R}_d$-independent.*

We can use Lemma 5 to show that an extension operation which adds a copy of K_3 preserves minimal rigidity.

Lemma 6. *Let $G = (V, E)$ be a graph, $\{V_1, V_2\}$ be a partition of V and put $G_i = G[V_i]$ for $i = 1, 2$. Suppose G_1 is minimally $\mathcal{R}_d$-rigid, $G_2 \cong K_3$, each vertex of G_2 has $d-1$ neighbours in G_1 and the set of all neighbours of the vertices of G_2 in G_1 has size at least d. Then G is minimally $\mathcal{R}_d$-rigid.*

Proof. Let $V(G_2) = \{x, y, z\}$ and N_x, N_y, N_z denote the set of neighbours of x, y, z in G_1, respectively. Since $|N_x \cup N_y \cup N_z| \geq d$ and $|N_x| = |N_y| = |N_z| = d-1$, at most two of the sets N_x, N_y, N_z can be the same. Therefore, we may assume that either the sets N_x, N_y, N_z are all pairwise distinct, or $N_x = N_y \neq N_z$ (by relabelling if necessary). This implies that the sets $N_z \setminus N_x$ and $N_y \setminus N_z$ are non-empty. Then G can be obtained from G_1 as follows. We first perform a 0-extension which adds x and edges from x to its $d-1$ neighbours in N_x as well as w for some $w \in N_z \setminus N_x$. We next perform a 1-extension which deletes xw, and adds z and the edges from z to its $d-1$ neighbours in N_z as well as to x and u for some $u \in N_y \setminus N_z$. Finally we perform one more 1-extension which deletes zu and adds y and the edges from y to its $d-1$ neighbours in N_y as well as x and z. See Figure 3. Hence, G is $\mathcal{R}_d$-independent by Lemma 5. Minimal $\mathcal{R}_d$-rigidity now follows by a simple edge count. □

A *(d-dimensional) vertex split* of a graph $G = (V, E)$ is the operation defined as follows: choose $v \in V$, $x_1, x_2, \ldots, x_{d-1} \in N_G(v)$ and a partition N_1, N_2 of pairwise disjoint sets N_1, N_2 with $N_1 \cup N_2 = N_G(v) \setminus \{x_1, x_2, \ldots, x_{d-1}\}$; then delete v from G and add two new vertices v_1, v_2 joined to N_1, N_2, respectively; finally add new edges $v_1v_2, v_1x_1, v_2x_1, v_1x_2, v_2x_2, \ldots, v_1x_{d-1}, v_2x_{d-1}$.

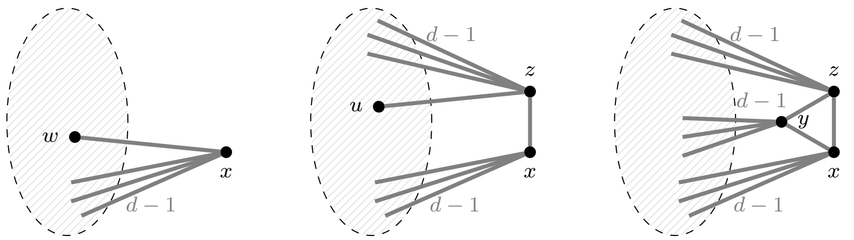

Figure 3: Construction of G in the proof of Lemma 6.

Lemma 7. *[22, Proposition 10] Let G be $\mathcal{R}_d$-independent and let G' be obtained from G by a vertex split. Then G' is $\mathcal{R}_d$-independent.*

Given a graph G, the *cone* G' of G is the graph obtained from G by adding a new vertex adjacent to every vertex of G.

Lemma 8. *Let G' be the cone of a graph G. Then:*
(a) G is $\mathcal{R}_d$-rigid if and only if G' is $\mathcal{R}_{d+1}$-rigid [21];
(b) G is an $\mathcal{R}_d$-circuit if and only if G' is an $\mathcal{R}_{d+1}$-circuit [9].

Our next two lemmas concern the operation of 'gluing' two graphs together.

Lemma 9. *[23, Lemma 11.1.9] Let G_1, G_2 be subgraphs of a graph G and suppose that $G = G_1 \cup G_2$.*

(a) If $|V(G_1) \cap V(G_2)| \geq d$ and G_1, G_2 are $\mathcal{R}_d$-rigid then G is $\mathcal{R}_d$-rigid.

(b) If $G_1 \cap G_2$ is $\mathcal{R}_d$-rigid and G_1, G_2 are $\mathcal{R}_d$-independent then G is $\mathcal{R}_d$-independent.

(c) If $|V(G_1) \cap V(G_2)| \leq d-1$, $u \in V(G_1) - V(G_2)$ and $v \in V(G_2) - V(G_1)$ then $r_d(G+uv) = r_d(G)+1$.

Lemma 9(b) immediately implies that every $\mathcal{R}_d$-circuit $G = (V, E)$ is 2-connected and that, if $G - \{u, v\}$ is disconnected for some $u, v \in V$, then $uv \notin E$. Our next lemma gives more structural information for the case when $G - \{u, v\}$ is disconnected.

Given three graphs $G = (V, E)$, $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$, we say that G is a 2*-sum of G_1, G_2 along an edge e* if $G = (G_1 \cup G_2) - e$, $G_1 \cap G_2 = K_2$ and $e \in E_1 \cap E_2$. Our next result shows that the 2-sum of G_1, G_2 is an $\mathcal{R}_d$-circuit if and only if G_1, G_2 are both $\mathcal{R}_d$-circuits. Its proof relies on the matroid circuit elimination axiom (which states that if C_1, C_2 are distinct circuits in a matroid $\mathcal{M}$ and $e \in C_1 \cap C_2$ then $(C_1 \cup C_2) - e$ contains a circuit of $\mathcal{M}$).

Lemma 10. *Suppose that $G = (V, E)$ is the 2-sum of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then G is an $\mathcal{R}_d$-circuit if and only if G_1 and G_2 are both $\mathcal{R}_d$-circuits.*

Proof. We first prove necessity. Suppose that G is an $\mathcal{R}_d$-circuit. If G_1 and G_2 are both $\mathcal{R}_d$-independent then $G + uv$ is $\mathcal{R}_d$-independent by Lemma 9(b), a contradiction since G is an $\mathcal{R}_d$-circuit. If exactly one of G_1 and G_2, say G_1, is $\mathcal{R}_d$-independent then uv belongs to the unique $\mathcal{R}_d$-circuit contained in G_2. We may extend uv to a base of E_i, for $i = 1, 2$, and then apply Lemma 9(b) to obtain $r_d(G + uv) = r_d(G_1) + r_d(G_2) - 1$. Thus we have $r_d(G) = r_d(G + uv) = |E_1| + |E_2| - 2 = |E|$, a contradiction since G is an $\mathcal{R}_d$-circuit. Hence G_1 and G_2 are both $\mathcal{R}_d$-dependent. Then the matroid circuit elimination axiom combined with the fact that G is an $\mathcal{R}_d$-circuit imply that G_1 and G_2 are both $\mathcal{R}_d$-circuits.

We next prove sufficiency. Suppose that G_1 and G_2 are both $\mathcal{R}_d$-circuits. The circuit elimination axiom implies that G is $\mathcal{R}_d$-dependent and hence that G contains an $\mathcal{R}_d$-circuit $G' = (V', E')$. Since $G_i - uv$ is $\mathcal{R}_d$-independent for $i = 1, 2$, we have $E' \cap E_i \neq \emptyset$. This implies that G' is a 2-sum of $G'_1 = (G_1 \cap G') + uv$ and $G'_2 = (G_2 \cap G') + uv$. The proof of necessity in the previous paragraph now tells us that G'_1 and G'_2 are both $\mathcal{R}_d$-circuits. Since G_i is an $\mathcal{R}_d$-circuit and $G'_i \subseteq G_i$ we must have $G'_i = G_i$ for $i = 1, 2$ and hence $G = G'$. □

The special cases of Lemma 10 when $d = 2, 3$ were proved by Berg and Jordán [3] and Tay [20], respectively.

We next obtain some results on the graphs in $\{B_{d,d-1}\} \cup \{B_{d,d-1}\} \cup \mathcal{B}^+_{d,d-1}$. The *(d-dimensional) degree of freedom* of a graph $G = (V, E)$ with $|V| \geq d + 1$ is defined to be the number $d|V| - \binom{d+1}{2} - r_d(G)$, i.e. the minimum number of edges we need to add to G to make it $\mathcal{R}_d$-rigid. We may apply Lemma 10 to the $\mathcal{R}_3$-circuit K_5 to deduce that $B_{3,2}$ is an $\mathcal{R}_3$-circuit which has 18 edges and one degree of freedom. The same argument applied to the $\mathcal{R}_4$-circuit K_6 implies that $B_{4,2}$ is an $\mathcal{R}_4$-circuit with 28 edges and three degrees of freedom. We can now use Lemma 8(b) to deduce that $B_{d,d-1}$ is an $\mathcal{R}_d$-circuit with $\frac{d(d+9)}{2}$ edges and one degree of freedom and that $B_{d,d-2}$ is an $\mathcal{R}_d$-circuit with $d(d + 3)$ edges and three degrees of freedom, for all $d \geq 4$. Similarly, using the fact that $K_{d+3} - \{f, g\}$ is a rigid $\mathcal{R}_d$-circuit when f, g are non-adjacent, we may apply Lemma 10 to the $\mathcal{R}_3$-circuits K_5 and $K_6 - \{f, g\}$, for two non-adjacent edges f, g, to deduce that every graph in $\mathcal{B}^+_{3,2}$ is an $\mathcal{R}_3$-circuit with 21 edges and one degree of freedom. We can then use Lemma 8(b) to deduce that if a graph from $\mathcal{B}^+_{d,d-1}$ is obtained by coning, it is an $\mathcal{R}_d$-circuit unless f or g has an end-vertex in $V_1 \cap V_2$. Our next result extends this to all graphs in $\mathcal{B}^+_{d,d-1}$. Note that, since every graph in $\mathcal{B}^+_{d,d-1}$ has one more vertex and d more edges than $B_{d,d-1}$, the fact that the graphs in $\mathcal{B}^+_{d,d-1}$ are $\mathcal{R}_d$-circuits will imply that they each have $\frac{d(d+9)}{2} + d$ edges and one degree of freedom.

Lemma 11. *Every graph in $\{B_{d,d-1}, B_{d,d-2}\} \cup \mathcal{B}^+_{d,d-1}$ is an $\mathcal{R}_d$-circuit.*

Proof. We have already seen that $B_{d,d-1}$ and $B_{d,d-2}$ are $\mathcal{R}_d$-circuits. Let $G \in \mathcal{B}^+_{d,d-1}$ and suppose that $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and e, f, g are as in the definition of $\mathcal{B}^+_{d,d-1}$. Since G has $d|V(G)| - \binom{d+1}{2}$ edges and is not $\mathcal{R}_d$-rigid (since it is not d-connected), it is $\mathcal{R}_d$-dependent.

We will complete the proof by showing that $G - h$ is $\mathcal{R}_d$-independent for all edges h of G. If h is incident with a vertex $x \in V_2 \setminus V_1$, then we can reduce $G - h$ to $G_1 - \{e, f, g\}$ by recursively deleting vertices of degree at most d (starting from x). Since $G_1 - \{e, f, g\}$ is $\mathcal{R}_d$-independent, Lemma 5 and the fact that edge deletion preserves independence now imply that $G - h$ is $\mathcal{R}_d$-independent. Thus we may assume that $h \in E_2$.

Suppose that f, g, h do not have a common end-vertex. Choose a vertex $x \in V_2 \setminus V_1$ and let $H = G - h - x + e$ be the graph obtained by applying a 1-reduction at x. We can reduce H to $G_1 - \{f, g, h\}$ by recursively deleting vertices of degree at most d. Since f, g, h do not have a common end-vertex, $G_1 - \{f, g, h\}$ is $\mathcal{R}_d$-independent. We can now use Lemma 5 to deduce that $G - h$ is $\mathcal{R}_d$-independent.

Hence we may assume that f, g, h have a common end-vertex u. The definition of $\mathcal{B}^+_{d,d-1}$ now implies that at least one of f and g, say f, is an edge of $G_1 \cap G_2$. Since e, f, g do not have a common end-vertex, e is not incident with u and hence e, g, h do not have a common end-vertex. We can now apply the argument in the previous paragraph with the roles of e and f reversed to deduce that $G - h$ is $\mathcal{R}_d$-independent. $\square$

Lemma 12. *Let G be a graph obtained from $B_{d,d-1}$ by a 1-extension operation. Then either G is $\mathcal{R}_d$-rigid or $G \in \mathcal{B}^+_{d,d-1}$.*

Proof. Let v be the new vertex added by the 1-extension and consider $B_{d,d-1} = (G_1 \cup G_2) - e$ where $G_i \cong K_{d+2}$, $G_1 \cap G_2 \cong K_{d-1}$ and $e \in E(G_1 \cap G_2)$. If $N_G(v) \subseteq V(G_i)$ for some $i = 1, 2$, then $G \in \mathcal{B}^+_{d,d-1}$.

Hence, we may assume that there exist vertices $v_1 \in N_G(v) \cap (V(G_1) \setminus V(G_2))$ and $v_2 \in N_G(v) \cap (V(G_2) \setminus V(G_1))$. Note that as v_1 and v_2 are on different sides of the cut set $V(G_1) \cap V(G_2)$ of $B_{d,d-1}$, we have $v_1v_2 \notin E(B_{d,d-1})$. Let f be the edge of $B_{d,d-1}$ deleted by the 1-extension. We may use Lemma 9(c) to obtain

$$r_d(B_{d,d-1} - f + v_1v_2) = r_d(B_{d,d-1} + v_1v_2) = r_d(B_{d,d-1}) + 1.$$

Since $B_{d,d-1}$ has one degree of freedom, this implies that $B_{d,d-1} - f + v_1v_2$ is $\mathcal{R}_d$-rigid. We may now use the fact that G can be obtained from $B_{d,d-1} - f + v_1v_2$ by a 1-extension operation on the edge v_1v_2 and Lemma 5 to conclude that G is $\mathcal{R}_d$-rigid. $\square$

Our last two lemmas are rather technical results which we will need in our proof of Theorem 1.

Lemma 13. *(a) Every 6-regular graph on 10 vertices is $\mathcal{R}_4$-independent.*

(b) *Every 12-regular graph on 15 vertices is $\mathcal{R}_9$-independent.*

Proof. There are 21 6-regular graphs on 10 vertices (see OEIS [12] sequence A165627 for the count and references to lists for download). The number of 12-regular graphs on 15 vertices is 17. These can be obtained from the fact that the complement of a 12-regular graph on 15 vertices is a 2-regular graph on 15 vertices, i. e. a graph consisting of disjoint cycles.

Now we need to show that these graphs are indeed $\mathcal{R}_d$-independent in the stated dimensions. We can do so with the help of any computer algebra system. For each graph, we choose a vector $p \in \mathbb{R}^{d|V|}$ and compute the rank of $R(G, p)$. We know that as soon as we find a p such that rank $R(G, p) = |E(G)|$, we will have rank $R(G, q) = |E(G)|$ for all generic q. We did this by taking a random choice for p and checking that rank $R(G, p) = |E(G)|$. (Due to generic rigidity, almost every random choice will do.) □

Lemma 14. *Suppose that $G = (V, E)$ is a graph with $|V| \geq 11$, minimum degree two and maximum degree three. Then there exist vertices $x, y \in V$ with $d(x) = 2$, $d(y) = 3$ and* $\mathrm{dist}(x, y) \geq 3$.

Proof. Assume $G = (V, E)$ is a counterexample to the lemma. Choose a vertex $v \in V$ of degree 2. Then there are at most 6 vertices at distance 1 or 2 from v. Hence G has at most 6 vertices of degree 3. Now choose a vertex $u \in V$ of degree 3. Each neighbour of u is either a vertex of degree 2 which has at most one other neighbour of degree 2 or a vertex of degree 3 which has at most two other neighbours of degree 2. Therefore G has at most 6 vertices of degree 2. If there does not exist 6 vertices of degree 3 in G then the number of vertices of degree 3 in G is at most 4 by parity, and we would have $|V| \leq 10$. Hence there are exactly 6 vertices of degree 3 and v is adjacent to two vertices of degree 3. Since v is an arbitrary vertex of degree two, every vertex of degree 2 is adjacent to two vertices of degree 3. Now choose w to be a vertex of degree 3 at distance 2 from v and a vertex $y \neq v$, of degree 2, not adjacent to w. Then $\mathrm{dist}(w, y) \geq 3$. □

3 Main results

We will prove Theorem 1, Corollary 2 and Corollary 3.

3.1 Proof of Theorem 1

We proceed by contradiction. Suppose the theorem is false and choose a counterexample $G = (V, E)$ such that d is as small as possible and, subject to this condition, $|V|$ is as small as possible. Since all $\mathcal{R}_d$-circuits are $\mathcal{R}_d$-rigid when $d \leq 2$, we have $d \geq 3$. Since G is an $\mathcal{R}_d$-circuit, $G - v$ is $\mathcal{R}_d$-independent for all $v \in V$, and we can now use the fact that

0-extension preserves $\mathcal{R}_d$-independence (by Lemma 5) to deduce that $\delta(G) \geq d+1$. Since G is a flexible $\mathcal{R}_d$-circuit, G is d-sparse by Lemma 4.

Case 1: $d(v) = d+1$ for some $v \in V$.

Since G does not contain the rigid $\mathcal{R}_d$-circuit K_{d+2}, v has two non-adjacent neighbours v_1, v_2. If $H = G - v + v_1v_2$ was $\mathcal{R}_d$-independent then G would be $\mathcal{R}_d$-independent by Lemma 5. Hence H contains an $\mathcal{R}_d$-circuit C. Since C has at most $d+5$ vertices, the minimality of G implies that either C is $\mathcal{R}_d$-rigid, or $C = B_{d,d-1}$ and C is a spanning subgraph of H. If the latter alternative occurs then Lemma 12 would imply that G contains a circuit $C' \in \mathcal{B}^+_{d,d-1}$ and we would contradict the choice of G. Hence C is $\mathcal{R}_d$-rigid and G contains the minimally $\mathcal{R}_d$-rigid subgraph $C - v_1v_2$. Since C is a rigid $\mathcal{R}_d$-circuit, we have $|V(C)| \geq d+2$. Let G' be a minimally $\mathcal{R}_d$-rigid subgraph of G with at least $d+2$ vertices, which is maximal with respect to inclusion, and put $X = V(G) \backslash V(G')$. Then $1 \leq |X| \leq 4$. If some vertex in X had at least d neighbours in G', then we could create a larger $\mathcal{R}_d$-rigid subgraph by performing a 0-extension. Hence each $x \in X$ has at most $d-1$ neighbours in G'. Since G has minimum degree at least $d+1$, each $x \in X$ has at least two neighbours in X and we have $3 \leq |X| \leq 4$.

Subcase 1.1: $|X| = 3$. Then $G[X] = K_3$. In addition, G' is a minimally rigid graph on $d+2$ or $d+3$ vertices so either $G' = K_{d+2} - e$ for some edge e, or $G' = K_{d+3} - \{e, f, g\}$ for some edges e, f, g which are not all incident with the same vertex. If $|N_G(X)| \geq d$ then we could construct an $\mathcal{R}_d$-rigid spanning subgraph of G by Lemma 6. Hence $|N_G(X)| = d-1$. Since G does not contain a copy of K_{d+2}, at least one edge, say e, with its end-vertices in $N_G(X)$ is missing from G. This gives $G = B_{d,d-1}$ when $G' = K_{d+2} - e$, so we must have $G' = K_{d+3} - \{e, f, g\}$. Since $G \notin \mathcal{B}^+_{d,d-1}$, f and g have a common end-vertex u, and are both have at least one endvertex in $V(G') \backslash N_G(X)$. Since $\delta(G) \geq d+1$, we must have $u \in N_G(X)$. Then the graph obtained from G by deleting all the edges from u to its neighbours in $V(G') \setminus N_G(X)$ is a copy of $B_{d,d-2}$ in G. This contradicts the choice of G

Subcase 1.2: $|X| = 4$. Then $C_4 \subseteq G[X] \subseteq K_4$ and $|V(G')| = d+2$. Since G' is minimally rigid, we have $G' = K_{d+2} - e$ for some edge e.

Claim 15. $N_G(X) = V(G')$.

Proof of claim. Suppose, for a contradiction, that $N_G(X) \neq V(G')$. Let $Y = X \cup N_G(X)$. Then $G[Y]$ is a proper subgraph of G so is $\mathcal{R}_d$-independent. If $G[N_G(X)]$ was complete, then G would be $\mathcal{R}_d$-independent by Lemma 9(b), since $G = G' \cup G[Y]$, G' and $G[Y]$ are $\mathcal{R}_d$-independent, and $G' \cap G[Y] = G[N(X)]$ is complete. Hence $G[N_G(X)]$ is not complete. Since $G' = K_{d+2} - e$, this implies that both end-vertices of e belong to $N_G(X)$. Choose a vertex $w \in V(G') \backslash N_G(X)$ and an edge f of G' which is incident with w. Consider the graph $G'' = G + e - f$.

Suppose $G''[Y]$ is $\mathcal{R}_d$-independent. Since $G''[N_G(X)]$ induces a complete graph, we can use Lemma 9(b) as above to deduce that G'' is $\mathcal{R}_d$-independent. Then $G' + e = K_{d+2}$ is the unique $\mathcal{R}_d$-circuit in $G'' + f$ and hence $G = G'' + f - e$ is $\mathcal{R}_d$-independent. This contradicts the choice of G. Hence $G''[Y]$ is $\mathcal{R}_d$-dependent.

Let C be an $\mathcal{R}_d$-circuit in $G''[Y]$. Since $G'' - e = G - f$ is a proper subgraph of G and G is an $\mathcal{R}_d$-circuit, we have $e \in E(C)$. We also have $w \notin V(C)$ since $w \notin Y$.

Suppose $C = B_{d,d-1}$. Then $V(C) = V(G'') \setminus \{w\} = V(G) \setminus \{w\}$. Since $E(C) \setminus E(G) = \{e\}$, we may apply a 1-extension to C by adding w and its $d + 1$ neighbours in G and deleting e, to obtain a spanning subgraph of G. By Lemma 12, this spanning subgraph of G is either $\mathcal{R}_d$-rigid (implying that G is $\mathcal{R}_d$-rigid) or it is a member of $\mathcal{B}^+_{d,d-1}$. Both of these possibilities contradict the choice of G. Thus $C \neq B_{d,d-1}$ and the minimality of G now implies that C is rigid.

Since $G' + e = K_{d+2}$ and $e \in E(C) \cap E(G' + e)$, the matroid circuit elimination axiom implies that $(C-e) \cup G'$ is $\mathcal{R}_d$-dependent. Since $(C-e) \cup G' \subseteq G$, we must have $(C-e) \cup G' = G$. This implies that X and all edges of G incident to X are contained in C. Thus $N_G(X) \subset V(C)$. If $|N_G(X)| \geq d$, then $G = G' \cup (C - e)$ would be rigid by Lemma 9(a). Hence $|N_G(X)| \leq d - 1$. If $|N_G(X)| = d - 2$, then $C = K_{d+2}$ and $G = B_{d,d-2}$. Hence $|N_G(X)| = d - 1$. Then $C = K_{d+3} - f - g$ for two non-adjacent edges f, g and $G \in \mathcal{B}^+_{d,d-1}$. This contradicts the choice of G and completes the proof of the claim. □

Suppose $G[X] = C_4$. Since $\delta(G) = d + 1$ and no vertex of X has more than $d - 1$ neighbours in G', each vertex of X has degree $d + 1$ in G. Claim 15 and the facts that $|N_G(X)| = |V(G')| = d + 2$ and each vertex of X has $d - 1$ neighbours in $V(G')$, imply that there exists a vertex $u \in X$ such that $|N_G(X - u) \cap V(G')| \geq d$. We can perform a 1-reduction of G which deletes u and adds an edge between the two neighbours of u in X. We can now apply Lemma 6 to the resulting graph H on $d + 5$ vertices to deduce that H is $\mathcal{R}_d$-rigid. This implies that G is $\mathcal{R}_d$-rigid and contradicts the choice of G. Hence $G[X] \neq C_4$.

Suppose $G[X] = C_4 + f$ for some edge $f = wx$. Then w and x have degree $d + 1$ or $d + 2$ in G and the vertices in $X \setminus \{w, x\}$ have degree $d + 1$. If $d_G(w) = d_G(x) = d + 2$ then G would have more than $d|V| - \binom{d+1}{2}$ edges, so could not be a flexible $\mathcal{R}_d$-circuit. Hence we may assume that $d_G(w) = d + 1$. Construct H from G by performing a 1-reduction which deletes w and adds an edge between the two non-adjacent neighbours of w in X. If $d_G(x) = d+1$, then x would have degree d in H and we could reduce H to G' by recursively deleting the remaining three vertices of X beginning with x, so that every deleted vertex has degree at most d. Since G' is $\mathcal{R}_d$-independent this would imply that G is $\mathcal{R}_d$-independent and contradict the choice of G. Hence $d_G(x) = d + 2$. We can now apply Lemma 6 to deduce that either H is $\mathcal{R}_d$-rigid or $|N_G(X - w) \cap V(G')| = d - 1$ and H is $B_{d,d-1}$. The first alternative would imply that G is $\mathcal{R}_d$-rigid, and the second alternative would imply that either G is $\mathcal{R}_d$-rigid or $G \in \mathcal{B}^+_{d,d-1}$ by Lemma 12. Both alternatives contradict the choice

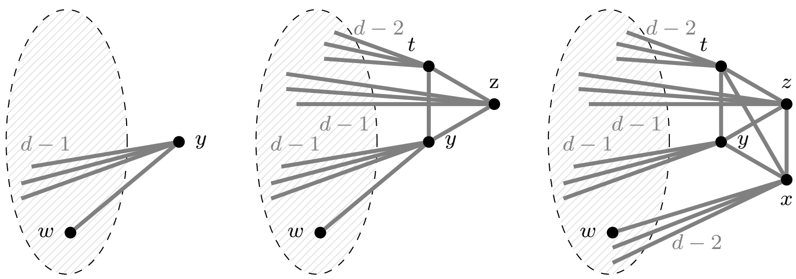

Figure 4: Construction of $\hat{G}$ in the proof of Case 1.

of G.

Hence $G[X] \neq C_4$. Then each vertex in X has degree $d+1$ or $d+2$ in G. In addition, at most two vertices of X can have degree $d+2$ in G, otherwise G would have more than $d|V| - \binom{d+1}{2}$ edges and could not be a flexible circuit. Let $\hat{G}$ be obtained from G by adding edges from vertices in X to vertices in G' in such a way that X has exactly two vertices of degree $d+1$ and exactly two vertices of degree $d+2$ in $\hat{G}$. We will show that G is $\mathcal{R}_d$-independent by proving that $\hat{G}$ is minimally $\mathcal{R}_d$-rigid.

Since $N_{\hat{G}}(X) = V(G')$ by Claim 15, we may choose vertices $x, y \in X$ such that x has degree $d+1$, y has degree $d+2$ and some vertex $w \in V(G')$ is a neighbour of x in $\hat{G}$ but not y. Let $X = \{x, y, z, t\}$ where z has degree $d+2$ and t has degree $d+1$ in $\hat{G}$. We can construct $\hat{G}$ from G' by first performing a 0-extension which adds y and all edges from y to its neighbours in G' as well as to w, then add z and then t by successive 0-extensions, and finally add x by a 1-extension which removes the edge yw. see Figure 4. Since G' is minimally $\mathcal{R}_d$-rigid this implies that $\hat{G}$ is also minimally $\mathcal{R}_d$-rigid. This contradicts the fact that G is an $\mathcal{R}_d$-circuit and completes the proof of Case 1.

Case 2: $\boldsymbol{\delta(G) \geq d+2}$.

Choose $v \in V$ with $d(v) = \Delta(G)$. If $G - v$ was $\mathcal{R}_{d-1}$-independent then G would be $\mathcal{R}_d$-independent by Lemma 8. This is impossible since G is an $\mathcal{R}_d$-circuit. Hence $G - v$ contains an $\mathcal{R}_{d-1}$-circuit C. By the minimality of d, C is $\mathcal{R}_{d-1}$-rigid or $C \in \{B_{d-1,d-2}, B_{d-1,d-3}\} \cup \mathcal{B}^+_{d-1,d-2}$.

Claim 16. *$G - v$ is $\mathcal{R}_{d-1}$-rigid.*

Proof of Claim. We first consider the case when $C \in \{B_{d-1,d-3}\} \cup \mathcal{B}^+_{d-1,d-2}$. Then C has $d+5$ vertices so is a spanning subgraph of $G - v$. We have seen that every graph in $\mathcal{B}^+_{d-1,d-2}$ has one degree of freedom and has three vertices of degree d on the smaller side of its $(d-2)$-separation, and that $B_{d-1,d-3}$ has three degrees of freedom and has four vertices of degree d on each side of its $(d-3)$-separation. These observations and the facts that

$\delta(G-v) \geq d+1$ and each nontrivial infinitesimal motion of a generic realisation of C is an infinitesimal rotation about the afffine subspace which contains its separating set of size $d-2$, respectively $d-3$, imply that we can add edges of $G-v$ to C which cross the separating set to make it $\mathcal{R}_{d-1}$-rigid. Hence $G-v$ is $\mathcal{R}_{d-1}$-rigid.

We next consider the case when $C = B_{d-1,d-2}$. If C is a spanning subgraph of G then we can procceed as in the previous paragraph to deduce that $G-v$ is $\mathcal{R}_{d-1}$-rigid. So we may assume that this is not the case. Then $(G-v) \setminus C$ has exactly one vertex u. Since $d_{G-v}(u) \geq d+1$, $G-v$ is $\mathcal{R}_{d-1}$-rigid unless all neighbours of u belong to the same copy of $K_{d+1}-e$ in $B_{d-1,d-2}$. Suppose the second alternative occurs and let H be the spanning subgraph of $G-v$ obtained by adding u and all its incident edges to $B_{d-1,d-2}$. Then H has one degree of freedom and the smaller side of the $(d-2)$-separation of H contains vertices which have degree d in H and degree at least $d+1$ in $G-v$. We can now add an edge of $G-v$ to H which crosses its $(d-2)$-separator to make it $\mathcal{R}_{d-1}$-rigid. Hence $G-v$ is $\mathcal{R}_{d-1}$-rigid.

It remains to consider the case when C is $\mathcal{R}_{d-1}$-rigid. Then $|V(C)| \geq d+1$. Let H be a maximal $\mathcal{R}_{d-1}$-rigid subgraph of $G-v$ containing C. Suppose $H \neq G-v$ and note that $(G-v)-H$ has at most 4 vertices. Since each vertex of $(G-v)-H$ has at most $d-2$ neighbours in H and $\delta(G-v) \geq d+1$ we have $(G-v)-H = K_4$ and $H = C = K_{d+1}$. We can now apply Lemma 6 to a minimally rigid spanning subgraph of H, and to each K_3 in $(G-v)-H$, in order to deduce that all vertices of $(G-v)-H$ are adjacent to the same set of $d-2$ vertices of H. This cannot occur since every vertex of H which is not joined to a vertex of $G-v-H$ would have degree at most $d+1$ in G, contradicting the assumption of Case 2. Hence $H = G-v$ and $G-v$ is $\mathcal{R}_{d-1}$-rigid. $\square$

Let $(G-v)^*$, respectively C^*, be obtained from $G-v$, respectively C, by adding v and all edges from v to the vertices of $G-v$, respectively C. Then $(G-v)^*$ is $\mathcal{R}_d$-rigid by Claim 16 and Lemma 8, and, when C is $\mathcal{R}_{d-1}$-rigid, C^* is an $\mathcal{R}_d$-circuit by Lemma 8(b).

Let S be the set of all edges of G^* which are not in G. Since C^* is rigid or $C^* \in \{B_{d,d-1}, B_{d,d-2}\} \cup \mathcal{B}^+_{d,d-1}$, C^* is not an $\mathcal{R}_d$-circuit in G. Hence $E(C^*) \cap S \neq \emptyset$. If $|S| = 1$, say $S = \{f\}$, then $G = (G-v)^* - f$ would be $\mathcal{R}_d$-rigid since $(G-v)^*$ is $\mathcal{R}_d$-rigid and $f \in E(C^*)$. Hence $|S| \geq 2$ and $\Delta(G) = d(v) \leq |V| - 3$. Let $\bar{G}$ be the complement of G.

Suppose $|V| \leq d+5$. Then, since $d+2 \leq \delta(G) \leq \Delta(G) \leq |V|-3$, we have $|V| = d+5$ and G is $(d+2)$-regular. This implies that $\bar{G}$ is a 2-regular graph on $d+5 \geq 8$ vertices and we may choose two non-adjacent vertices v_1, v_2 with no common neighbours in $\bar{G}$. Then $v_1v_2 \in E$ and $|N_G(v_1) \cap N_G(v_2)| = d-1$. We can use the facts that G is d-sparse, $(d+2)$-regular and $|V| = d+5$ to deduce that G/v_1v_2 is d-sparse. (If not, then some set $X \subseteq V(G/v_1v_2)$ induces more that $d|X| - \binom{d+1}{2}$ edges. Then $|X| \geq d+2$ and the fact that each vertex of $V(G/v_1v_2) \setminus X$ has degree at least $d+1$ implies that G/v_1v_2 has more that $d|V(G/v_1v_2)| - \binom{d+1}{2}$ edges. This contradicts the fact that G is a flexible $\mathcal{R}_d$-circuit so has at most $d|V(G)| - \binom{d+1}{2}$ edges.) Since $|V(G/v_1v_2)| = d+4$, G/v_1v_2 has no flexible $\mathcal{R}_d$-circuits by the minimality of G. Hence G/v_1v_2 is $\mathcal{R}_d$-independent. Since $|N_G(v_1)\cap N_G(v_2)| = d-1$, we can now use Lemma 7 to deduce that G is $\mathcal{R}_d$-independent and contradict the choice of G.

Hence $|V| = d+6$. Since $\delta(G) \geq d+2$ and $\Delta(G) \leq d+3$ we have $\delta(\bar{G}) \geq 2$ and $\Delta(\bar{G}) \leq 3$. We can now complete the proof of the theorem by considering three subcases.

Subcase 2.1: $\boldsymbol{\delta(\bar{G}) = 2}$ and $\boldsymbol{\Delta(\bar{G}) = 3}$. In this case there exist two vertices $x, y \in V$ with $d_{\bar{G}}(x) = 2$, $d_{\bar{G}}(y) = 3$ and $\mathrm{dist}_{\bar{G}}(x, y) \geq 3$ by Lemma 14. Hence $|N_G(x)\cap N_G(y)| = d-1$ and we can deduce as in the previous paragraph that G/xy is d-sparse.

Suppose G/xy contains an $\mathcal{R}_d$-circuit. Then $G/xy = B_{d,d-1}$ by the minimality and d-sparsity of G. Since $B_{d,d-1}$ has $d-3$ vertices of degree $d+4$ and only the vertex obtained by contracting xy has degree $d+4$ in G/xy we must have $d = 4$. And when $d = 4$, the fact that $\delta(G) = 6$ would imply that x and y are adjacent in G to each of the six vertices of degree five in G/xy. Since they are also adjacent to each other this contradicts the fact that $d_G(y) = d+2 = 6$.

Hence G/v_1v_2 is $\mathcal{R}_d$-independent. Since $|N_G(v_1) \cap N_G(v_2)| = d-1$, we can now use Lemma 7 to deduce that G is $\mathcal{R}_d$-independent and contradict the choice of G.

Subcase 2.2: $\boldsymbol{\bar{G}}$ is 2-regular. In this case we have $|S| = 2$ and G is $(d+3)$-regular. The fact that $(G-v)^*$ is $\mathcal{R}_d$-rigid and contains at least two $\mathcal{R}_d$-circuits (G and C^*) tells us that $|E((G-v)^*)| \geq d|V(G)| - \binom{d+1}{2} + 2$. Since $|E| = |E((G-v)^*)| - |S|$ and G is d-sparse this gives

$$\frac{(d+3)(d+6)}{2} = |E| = d|V(G)| - \binom{d+1}{2} = \frac{d(d+11)}{2}.$$

This implies that $d = 9$ and $|V| = 15$. We can now use Lemma 13(b) to deduce that G is $\mathcal{R}_9$-independent, contradicting the fact that G is an $\mathcal{R}_9$-circuit.

Subcase 2.3: $\boldsymbol{\bar{G}}$ is 3-regular. In this case we have $|S| = 3$ and G is $(d+2)$-regular. Since $(G-v)^*$ is $\mathcal{R}_d$-rigid and contains at least two $\mathcal{R}_d$-circuits we can deduce as in the previous subcase that $|E(G)| \geq d|V| - \binom{d+1}{2} - 1$. The fact that G is d-sparse now gives

$$\frac{(d+2)(d+6)}{2} = |E| = d|V| - \binom{d+1}{2} - \alpha = \frac{d(d+11)}{2} - \alpha$$

for some $\alpha = 0, 1$. This implies that $\alpha = 0$, $d = 4$ and $|V| = 10$. We can now use Lemma 13(a) to deduce that G is $\mathcal{R}_4$-independent, contradicting the fact that G is an $\mathcal{R}_4$-circuit. □

3.2 Proof of Corollary 2

The corollary follows immediately from Theorem 1 if $|V| \leq d+6$. Since $\delta(G) \geq d+1$ we have $|E| > d(d+9)/2$ when either $|V| \geq d+8$, or $|V| = d+7$ and $\delta(G) \geq d+2$. Hence we may assume that $|V| = d+7$ and $\delta(G) = d+1$. Choose a vertex v with $d(v) = d+1$. Then v

has two non-adjacent neighbours v_1, v_2 since otherwise G would contain the rigid $\mathcal{R}_d$-circuit K_{d+2}. Let $H = G - v + v_1v_2$. If H was $\mathcal{R}_d$-independent then G would be $\mathcal{R}_d$-independent by Lemma 5. Hence H contains an $\mathcal{R}_d$-circuit C. If C is flexible then Theorem 1 implies that $C \in \{B_{d,d-1}, B_{d,d-2}\} \cup \mathcal{B}^+_{d,d-1}$ and hence $|E| > |E(C)| \geq d(d+9)/2$. Thus we may assume that C is $\mathcal{R}_d$-rigid. Then $C - v_1v_2$ is an $\mathcal{R}_d$-rigid subgraph with at least $d+2$ vertices. Let $X = V(G)\backslash V(C)$ and let $E(X, V\backslash X)$ be the set of edges with one endvertex in X and one in $V\backslash X$. Then $1 \leq |X| \leq 5$. Since $\delta(G) = d+1$ and $|X| \leq 5$ we have

$$\begin{aligned}
|E| &= |E(C - v_1v_2)| + |E(X)| + |E(X, V\backslash X)| \\
&\geq d|V\backslash X| - \binom{d+1}{2} + \binom{|X|}{2} + |X|(d+1-|X|+1) \\
&= d|V| - \binom{d+1}{2} - \frac{|X|(|X|-3)}{2} \\
&\geq \frac{d(d+13)}{2} - 5.
\end{aligned}$$

We can now use the fact that $d \geq 3$ to deduce that $|E| > d(d+9)/2$. □

3.3 Proof of Corollary 3

Suppose $G = (V, E)$ is $\mathcal{R}_d$-rigid. Then $r_d(G) = d|V| - \binom{d+1}{2}$. Let $H = (V, F)$ be a maximal $\mathcal{R}_d$-independent subgraph of G. Then $|F| = r_d(H) = d|V| - \binom{d+1}{2}$, H is d-tight and $B_{d,d-1}, B_{d,d-2} \not\subseteq H$ by Lemma 11. In addition H is d-connected by Lemma 9(c).

Conversely, suppose that G has a spanning subgraph $H = (V, F)$ which satisfies the hypotheses of the statement. Since H is d-tight, it is d-sparse and hence does not contain any $\mathcal{R}_d$-rigid circuits. If $C \subseteq H$ for some $\mathcal{B}^+_{d,d-1}$ then we would have $C = H$ since C is d-tight and $|V(C)| = d+6 \geq |V(G)|$. This would contradict the hypothesis that H is d-connected so H contains no graph in $\mathcal{B}^+_{d,d-1}$. Theorem 1 combined with the hypothesis that $B_{d,d-1}, B_{d,d-2} \not\subseteq H$ now implies that H does not contain any flexible $\mathcal{R}_d$-circuits. Hence H is $\mathcal{R}_d$-independent. Since H is d-tight, it is $\mathcal{R}_d$-rigid. This and the hypothesis that H is a spanning subgraph of G imply that G is $\mathcal{R}_d$-rigid. □

4 Closing Remarks

We briefly consider some possible extensions of our results.

4.1 Generalised 2-sums

Let $G = (V, E)$, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We say that G is a *t-sum of G_1, G_2 along an edge e* if $G = (G_1 \cup G_2) - e$, $G_1 \cap G_2 = K_t$ and $e \in E_1 \cap E_2$. We conjecture that Lemma 10 can be extended to t-sums.

Conjecture 17. *Suppose that G is a t-sum of G_1, G_2 along an edge e for some $2 \leq t \leq d+1$. Then G is an $\mathcal{R}_d$-circuit if and only if G_1, G_2 are $\mathcal{R}_d$-circuits.*

Our proof technique for Lemma 10 gives the following partial result.

Lemma 18. *Let $G = (V, E)$, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that G is a t-sum of G_1, G_2 along an edge e for some $2 \leq t \leq d+1$.*

(a) If G is an $\mathcal{R}_d$-circuit, then G_1 and G_2 are both $\mathcal{R}_d$-circuits.

(b) If G_1 and G_2 are both $\mathcal{R}_d$-circuits, then G contains a unique $\mathcal{R}_d$-circuit G' and $E \backslash (E_1 \cap E_2) \subseteq E(G')$.

Proof. (a) If G_1 and G_2 are both $\mathcal{R}_d$-independent, then Lemma 9(b) implies that $G_1 \cup G_2$ is $\mathcal{R}_d$-independent. This contradicts the facts that G is an $\mathcal{R}_d$-circuit and $G \subseteq G_1 \cup G_2$. If exactly one of G_1 and G_2, say G_1, is $\mathcal{R}_d$-independent then e belongs to the unique $\mathcal{R}_d$-circuit in G_2 and Lemma 9(b) gives $r_d(G) = r_d(G + e) = |E_1| + |E_2| - \binom{t}{2} - 1 = |E|$. This again contradicts the hypothesis that G is an $\mathcal{R}_d$-circuit. Hence G_1 and G_2 are both $\mathcal{R}_d$-dependent. Then the matroid circuit elimination axiom combined with the fact that G is an $\mathcal{R}_d$-circuit imply that G_1 and G_2 are both $\mathcal{R}_d$-circuits.

(b) The circuit elimination axiom implies that G is $\mathcal{R}_d$-dependent and hence that G contains an $\mathcal{R}_d$-circuit $G' = (V', E')$. Since $G_i - e$ is $\mathcal{R}_d$-independent for $i = 1, 2$, we have $E' \backslash E_i \neq \emptyset$. Let G'_i be obtained from $G_i \cap G'$ by adding an edge between every pair of non-adjacent vertices in $V' \cap V_1 \cap V_2$. If G'_i is a proper subgraph of G_i for $i = 1, 2$ then each G'_i is $\mathcal{R}_d$-independent and we can use Lemma 9(b) to deduce that $G'_1 \cup G'_2$ is $\mathcal{R}_d$-independent. This gives a contradiction since $G' \subseteq G'_1 \cup G'_2$. Relabelling if necessary we have $G'_1 = G_1$. If $G'_2 \neq G_2$ then we may deduce similarly that $G'_1 \cup G'_2 - e$ is independent. This again gives a contradiction since $G' \subseteq G'_1 \cup G'_2 - e$. Hence $G'_2 = G_2$. It remains to show uniqueness. For $i = 1, 2$, let B_i be a base of $\mathcal{R}_d(G_i)$ which contains $E(G_1) \cap E(G_2)$. Then $|B_i| = |E_i| - 1$ and Lemma 9(b) gives

$$r_d(G) = r_d(G_1 \cup G_2 - e) = r_d(G_1 \cup G_2) = |B_1| + |B_2| - \binom{t}{2} = |E| - 1.$$

Hence, G contains a unique $\mathcal{R}_d$-circuit. □

Conjecture 17 holds when $t = d + 1$ and G_1, G_2 are both globally rigid in $\mathbb{R}^d$ by a result of Connelly [8]. It also holds when $d = 2$ and $t = 3$ by a result of Jordán [14, Theorem 3.6.15].

4.2 Highly connected flexible circuits

Bolker and Roth [4] determined $r_d(K_{s,t})$ for all complete bipartite graphs $K_{s,t}$. Their result implies that $K_{d+2,d+2}$ is a $(d+2)$-connected $\mathcal{R}_d$-circuit for all $d \geq 3$ and is flexible when $d \geq 4$, see [10, Theorem 5.2.1]. We know of no $(d+3)$-connected flexible $\mathcal{R}_d$-circuits and it is tempting to conjecture that they do not exist.

For the case when $d = 3$, Tay [20] gives examples of 4-connected flexible $\mathcal{R}_3$-circuits and Jackson and Jordán [13] conjecture that all 5-connected $\mathcal{R}_3$-circuits are rigid. An analogous statement has recently been verified for circuits in the closely related C_2^1-cofactor matroid by Clinch, Jackson and Tanigawa [7].

4.3 Extending Theorem 1

We saw in the previous subsection that $K_{d+2,d+2}$ is a flexible $\mathcal{R}_d$-circuit with $2d+4$ vertices for all $d \geq 4$. We can obtain a $(d+2)$-connected flexible $\mathcal{R}_d$-circuit on $d+8$ vertices by recursively applying the coning operation to the flexible $\mathcal{R}_4$-circuit $K_{6,6}$ and then applying Lemma 8. This suggests that it may be difficult to extend Theorem 1 to graphs on $d+8$ vertices, but it is conceivable that all flexible $\mathcal{R}_d$-circuits on $d+7$ vertices have the form $(G_1 \cup G_2) - S$ where $G_i \in \{K_{d+2}, K_{d+3}, K_{d+4}\}$, $G_1 \cap G_2 \in \{K_{d-3}, K_{d-2}, K_{d-1}\}$ and S is a suitably chosen set of edges.

For the case when $d = 3$, Tay [20] gives examples of 3-connected flexible $\mathcal{R}_3$-circuits with 13 vertices but it is possible that all flexible circuits on at most 12 vertices can be obtained by taking 2-sums of rigid circuits on at most 9 vertices.

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