# Scaling limits of random growth processes

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## Declaration

I declare that the work in this thesis has been done by myself and has not been submitted elsewhere for the award of any other degree. Chapter 2 presents a paper which has accepted for publication in Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques. Chapters 2 and 3 are based on joint work with my supervisor Amanda Turner but have been written by myself and I have contributed fully to every aspect of this work.
George Liddle
April 2021

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# Abstract


The topic of this thesis is random growth processes. These occur naturally in many real world settings such as in the growth of tumours and lightning strikes. As such we would like to model the processes so that we can effectively study their properties. In particular, we are interested in what the shape of the process is as it grows and so we wish to evaluate the scaling limits of the random processes.

In Chapter 1, we will provide the background material needed in order to study the random growth models. We will give examples of real world processes that we would like to study before describing the models used to study them. We then provide some previous results in the area to provide context for the independent research that follows.

Chapter 2 will follow [**LT21a**] closely. In this paper we evaluate a strongly regularised version of the Hastings-Levitov model HL($\alpha$) for $0 \leqslant \alpha < 2$. We consider the scaling limit of the model under capacity rescaling. We first consider the case where $\alpha = 0$ and show that the limiting structure of the cluster is not a disk, unlike in the small-particle limit. Then when $0 < \alpha < 2$ we show that under the same rescaling the cluster approaches a disk and we analyse the fluctuations.

In Chapter 3, we present results from a second paper [**LT21b**]. In this paper we study the anisotropic version of the Hastings-Levitov model AHL($\nu$). We consider the evolution of the harmonic measure on logarithmic timescales and show that there exists a logarithmic time window on which the harmonic measure flow, started from the unstable fixed point, moves stochastically from the unstable point towards a stable point.

Finally, in Chapter 4, we give the conclusions of this thesis and the scope for future work.

# Contents

CHAPTER 1

# Background

## 1.1. Introduction

A random growth process is an object that evolves over time as a succession of particles is attached to its boundary according to some underlying random structure. More explicitly, a random growth process is defined as a increasing sequence of compact sets $\{K_n\}_{n\geqslant 1}$ where $K_{n+1}$ is formed by attaching a particle at a random point on the boundary of $K_n$. These processes are regularly exhibited in the natural world [**Tur19**]. Examples include tumoral growth, lightning strikes and mineral aggregation. Often we would like to understand the growth of the underlying natural process. For example, consider tumoral growth, the following figures by [**GA10**] display simulations on the spatial distribution of cancer cells after 90 cell generations at different consumption rates. The value $k$ represents the consumption rate with a low consumption rate corresponding to a large amount of oxygen in the tissue surrounding the tumour. The authors model the process in two dimensions and show that by changing the amount of oxygen in the surrounding tissue exhibits a change in the shape of the cell growth [**GA10**].

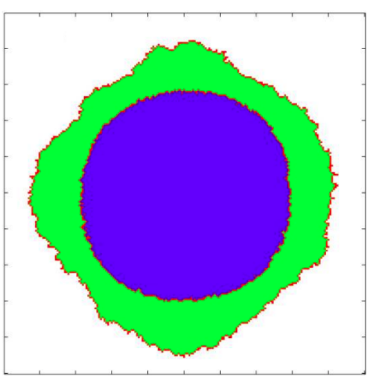

FIGURE 1. k = 2rc from [**GA10**]

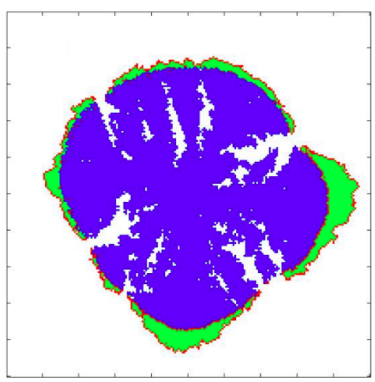

FIGURE 2. k = 5rc from [**GA10**]

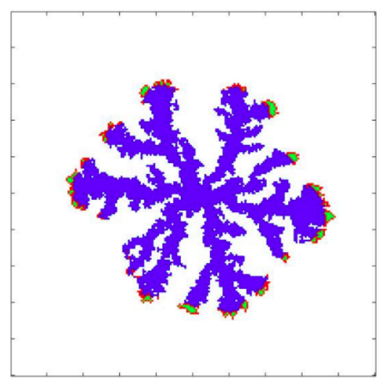

FIGURE 3. k = 10rc from [**GA10**]

In Figure 1, the low consumption rate produces growth in which the shape is close to a disk. However, as the amount of oxygen available to the tumour is reduced the growth becomes less disk-like and Figure 3 demonstrates growth to a branched diffusion process where the tumour only grows in directions in which oxygen becomes available. This experiment highlights a feature that will be demonstrated throughout the rest of this thesis; by changing a variable, such as the amount of oxygen, we see a phase transition from growth to a disk to growth to a non-disk.

Similar growth patterns have been exhibited in several other real world processes, examples include bacteria grown in a Petri dish and soot deposits within an engine [**Tur19**]. In order to study the behaviour of these real world processes we need to build mathematical models. Since the 1960's, models have been built in order to describe individual processes. The most famous examples include the Eden model [**Ede61**] and Diffusion Limited Aggregation (DLA) [**WS83**]. The Eden model is used to describe bacterial colony growth, whereas DLA describes mineral aggregation. Determining the shape of these random clusters as they increase in size is fundamental to understanding the real world processes the models describe. For example, understanding in which direction a tumour is likely to grow is vital for providing better medical treatment. This therefore poses both an interesting and challenging mathematical problem which is presented as the aim of this thesis; to discover the scaling limits of these models.

## 1.2. Lattice based models

Perhaps the most common way to model these random growth processes is to model them on a lattice. Modelling on a lattice has the benefit that the models are simple to define. Start with a particle at some point on the lattice, most often the origin, then attach the next particle at one of the unoccupied adjacent lattice points. Repeat this process iteratively, attaching one particle at each iteration to one of the unoccupied lattice points adjacent to the growing cluster. The choice of attaching point is made according to some probability measure specific to each individual model. This allows us to model a large class of real world growth models under this lattice based template.

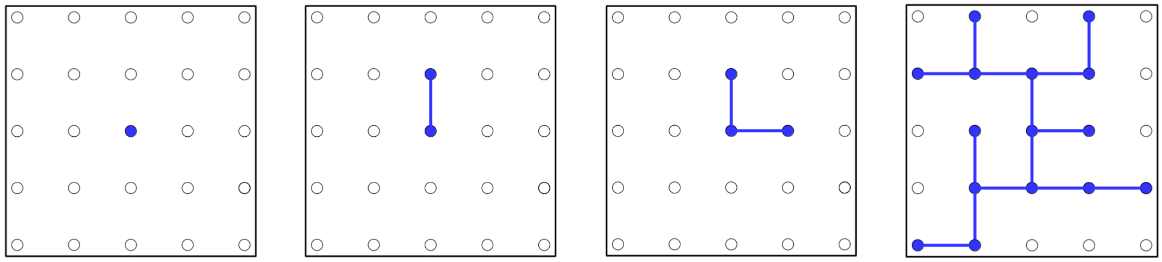

FIGURE 4. The evolution of a lattice based model.

One example of such a model is the Eden model used to model biological cell growth. This model describes a process where the organism starts at a point and then divides into daughter cells at each generation [**Ede61**]. The model is defined on the lattice $\mathbb{Z}^2$. Thus, first for a subset $A \subset \mathbb{Z}^2$ define the boundary of the set $A$ as

$$\partial A = \left\{ y \in \mathbb{Z}^2 \backslash A \ : \ \exists x \in A, y \sim x \right\}$$

where $y \sim x$ if $x$ is one of the four lattice points of $\mathbb{Z}^2$ adjacent to $y$. Then the Eden model is defined as a discrete Markov process $\{E_n\}_{n=0}^{\infty}$ where, $E_0 = \{0\}$ and $E_{n+1} = E_n \bigcup \{y_{n+1}\}$ for some $y_{n+1} \in \partial E_n$. In the case of the Eden model we wish to model cell growth, thus, the attaching points $y_{n+1}$ are chosen uniformly from $\partial E_n$. Note that this is proportional

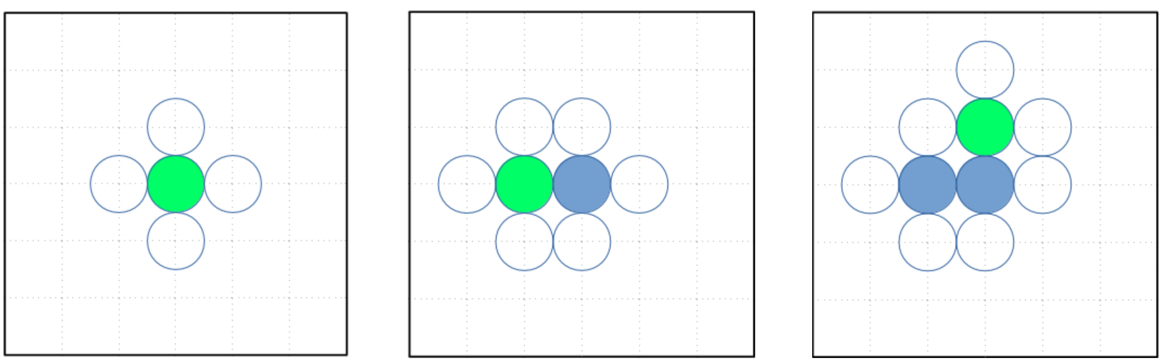

FIGURE 5. An example of how an Eden cluster may evolve.

to the number of unoccupied cells surrounding a particle on the cluster. Given this choice, one might expect that as the number of particles tends to infinity the growth would become isotropic. However, in contrast, simulations and consequent results have shown that the Eden cluster grows anisotropically as the number of particles converges to infinity (see [**Ede61**], [**Ric73**] and [**Kes93**]).

Another well studied growth model is Diffusion Limited Aggregation (DLA) introduced in [**WS83**]. In this case the model is used to describe mineral aggregation. The model is a variation of the Eden model. As above, we will define the model on $\mathbb{Z}^2$. First, the definition of harmonic measure on $\mathbb{Z}^2$, as seen from infinity, is provided as follows.

DEFINITION 1.2.1 (Harmonic measure on $\mathbb{Z}^2$). *Let $\{S_x(n) : n \in \mathbb{N}\}$ be a 2-dimensional simple random walk started at some point $x \in \mathbb{Z}^2$ and fix a closed set $A \subset \mathbb{Z}^2$. Define a measure $\mathcal{H}_A(.)$ by*

$$\mathcal{H}_A(y) := \lim_{\|x\|\to\infty} \mathbb{P}(S_x(\tau) = y, \tau < \infty)$$

*for $y \in A$ where $\tau = \inf\{n \geqslant 0 : S_x(n) \in \partial A\}$ and $\|.\|$ is the Euclidean norm.*

Similar to the construction of the Eden model, DLA is constructed as the discrete random process $\{D_n\}_{n=0}^{\infty}$ where, $D_0 = \{0\}$ and $D_{n+1} = D_n \bigcup \{y_{n+1}\}$ for some $y_{n+1} \in \partial D_n$. However, instead of attaching a particle uniformly, in order to model mineral aggregation the attaching points are chosen according to harmonic measure, $\mathbb{P}\left(y_{n+1} = y\right) = \mathcal{H}_{\partial D_n}(y)$.

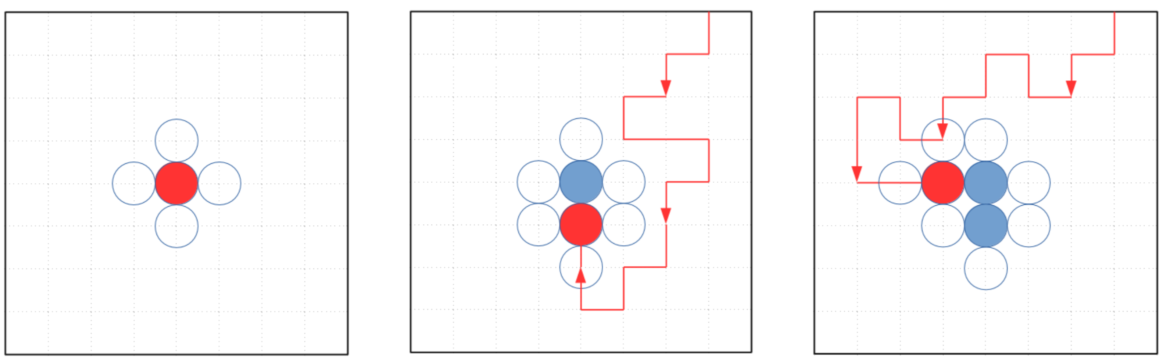

FIGURE 6. An example of how a DLA cluster may evolve.

Note that we have provided definitions for the Eden model and DLA in two dimensions but it is possible to define both models in higher dimensions. However, more care is needed when defining DLA in higher dimensions since the random walk is transient for $d \geqslant 3$.

Since its introduction in 1981 DLA has been widely studied yet there have been very few subsequent results, highlighting the difficulty of the problem. One major result is the following result by Kesten [**Kes90**] which provides a bound on the maximum distance from the origin for a lattice point belonging to the cluster. Let $K_n$ be the DLA cluster with $n$

particles. Then if we let

$$r(n) = \max\{|x| : x \in K_n\}$$

the following result holds.

THEOREM 1.2.2 (Kesten 1990). *There exist constants $C(d)$ such that with probability 1*

$$r(n) \leqslant C(d) n^{\frac{2}{d+1}}$$

*eventually if $d \geqslant 2$ but $d \neq 3$,*

$$r(n) \leqslant C(3)(n \log(n))^{\frac{1}{2}}$$

*eventually if $d = 3$.*

Kesten's upper bound proves that we do not converge to a one dimensional line, however, it does not rule out the possibility of convergence to a disk. Nevertheless, this is perhaps the only truly significant result on DLA since its introduction. Whilst the use of lattice based models is advantageous in that they are simple to define, this highlights one of the disadvantages of lattice based models, they are often very difficult to study because they do not provide many techniques for us to use in order to analyse the cluster. Furthermore, under this restriction, the models do not correspond to many real world examples.

## 1.3. Conformal models

One way we can combat the restrictions of lattice-based models is to form models using conformal maps instead. This method has the benefit of allowing us to use complex analysis techniques, amongst others, in order to study the processes. These models are described as follows. We start by defining the conformal map that attaches a particle to the boundary of the unit disk in the complex plane $\mathbb{C}$ at a particular angle. We then compose several of the maps in order to form a cluster.

Define $\Delta$ as the exterior unit disk in the complex plane, $\Delta = \{z \in \mathbb{C} \ : |z| > 1\}$. Let $P \subset \overline{\Delta}$ be a compact set such that $P \cap \Delta$ is non-empty and $\Delta^c \cup P$ is simply connected. We call $P$ a particle. Then by the Riemann mapping theorem there exists a unique conformal map $f : \Delta \to \Delta \backslash P$ of the form

$$f(z) = e^c z + \mathcal{O}(1)$$

as $z \to \infty$ for some real valued $c > 0$. The value $e^c$ is called the logarithmic capacity of the union $\Delta^c \cup P$. In the planar aggregation literature it has become standard to refer to $P$ as a particle of capacity $c$. In addition, although a slight abuse of notation, for any conformal map $f : \Delta \to \mathbb{C}$ it is convenient to refer to the capacity of the map to be,

$$\lim_{z\to\infty} \log\left(f'(z)\right) := \log f'(\infty).$$

The capacity provides a notion of size, in particular, for particles $P_1$ and $P_2$ with corresponding capacities $c_1$ and $c_2$ respectively, it follows that if $P_1 \subset P_2$ then $c_1 < c_2$. Furthermore, as $c \to 0$ the map converges locally uniformly to the identity map (see for example Proposition 3.55 in [**Law08**]), encapsulating that the particle size shrinks to 0 as $c \to 0$. The explicit formula for the map $f : \Delta \to \Delta \backslash [1, 1 + d(c)]$ that attaches a radial slit of length $d = d(c)$ to the boundary at $z = 1$ is given by [**STV19**],

$$f(z) = \frac{e^c}{2z}\left(z^2 + 2(1 - e^{-c})z + 1 + (z + 1)\sqrt{z^2 + 2(1 - 2e^{-c})z + 1}\right) \tag{1.1}$$

with a continuous branch of the square root taken on $\Delta$, which is possible because the roots of the quadratic inside the square root lie on the unit circle. The relation between the capacity $c$ and length $d$ in this case is,

$$e^c = 1 + \frac{d^2}{4(1 + d)}.$$

Hence, $c = \frac{d^2}{4} + o(1)$. Thus, now that we have a way to describe the size of the conformal maps we can define the single particle mapping. Define

$$f_c(z) : \Delta \to \Delta \backslash P$$

as the map which takes $\Delta$ to itself minus a particle $P$ of capacity $c > 0$ on the boundary at $z = 1$. Thus, given a sequence of attaching angles $\{\theta_n\}_{n=1}^{\infty}$ and capacities $\{c_n\}_{n=1}^{\infty}$ we can define a sequence of maps $\{f_n\}_{n=1}^{\infty}$ with the n$^{\text{th}}$ particle map defined as,

$$f_n(z) = e^{i\theta_n} f_{c_n}(z e^{-i\theta_n})$$

where $\theta_n$ is the attaching angle and $c_n$ is the capacity of the n$^{\text{th}}$ particle map $f_{c_n}(z)$. By continuity we can extend this definition to the boundary of the disk by defining $f_{c_n}(e^{i\theta_n}) = \lim_{r\to 1} f_{c_n}(re^{i\theta_n})$.

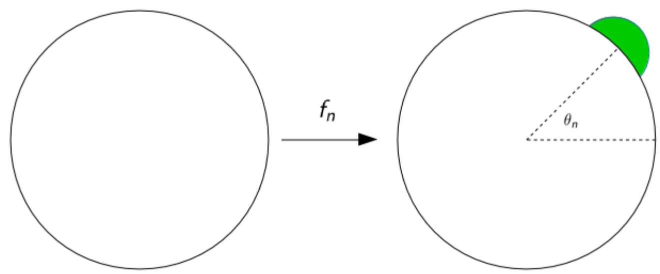


FIGURE 7. Mapping a single particle.

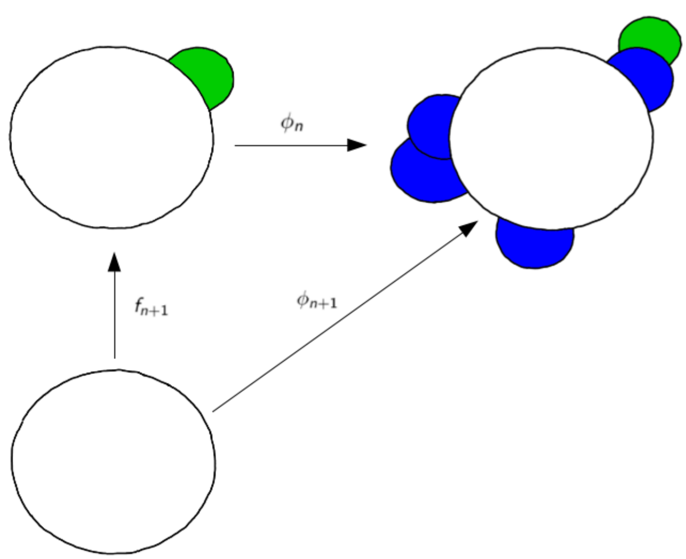


FIGURE 8. Mapping a cluster.

Now we can define the growing cluster. Define the map $\phi_{n+1}$ inductively,

$$\phi_{n+1} = f_1 \circ f_2 \circ .... \circ f_{n+1} = \phi_n \circ f_{n+1}.$$

Then by our assumptions on the particle $P$, this forms a growing sequence of compact sets $\{K_n\}_{n\geqslant 1}$, such that $\phi_n : \Delta \to \mathbb{C}\backslash K_n$, which we call a cluster. By varying the size, shape and attaching angle of the particles this general method allows us to form a wide range of models.

## 1.4. Construction of conformal models from lattice based models

Whilst the models that are constructed on a lattice present challenges we do not wish to discard them completely. In fact they remain highly significant because we still want to understand the properties of the underlying real world processes that they were introduced to model. As such, we want to construct off-lattice versions of the models using the conformal map method described in the last section so that we can study the properties of these models more easily. In this section we will describe how an off-lattice version of DLA is constructed, to do so we will closely follow the method of Turner in [**Tur19**].

To construct DLA using conformal maps we will use the same set up as in Section 1.3. The distinguishing feature is how the attaching angles and capacities are chosen to be distributed. Recall, in the DLA model, at each generation, a random walk is started on the lattice sufficiently far away from the origin and is run until the walk reaches one of the unoccupied particles on the boundary of the cluster, then this particle becomes part of the cluster and the process is repeated. Thus, instead of performing a random walk on lattice we now want to choose the attaching angles $\theta_n$ so that $\phi_{n-1}(e^{i\theta_n})$ shares the same distribution as the hitting distribution of Brownian motion on the boundary of the cluster $K_n$ started at infinity. The following definition of harmonic measure, as seen from infinity, will be used extensively throughout this thesis.

Definition 1.4.1 (Harmonic measure). *Let $\{B_x(t) : t \geqslant 0\}$ be a 2-dimensional Brownian motion started at some point $x$ and fix a compact and non-polar set $A \subset \mathbb{C}$. Define a measure $\mathcal{H}_A(.)$ by*

$$\mathcal{H}_A(B) := \lim_{\|x\| \to \infty} \mathbb{P}(B_x(\tau) \in B, \tau < \infty)$$

*for $B \subset A$ Borel where $\tau = \inf\{t \geqslant 0 : B_x(t) \in A\}$ and $\|.\|$ is the Euclidean norm.*

Thus, we want the local growth rate to be chosen according to harmonic measure. Under the image of the map $z \to \frac{1}{z}$ this is equivalent to requiring the distribution of $e^{i\theta_n}$ to be the hitting distribution of $\phi_{n-1}^{-1}\left(\frac{1}{B_t}\right)$ on the unit disk where $B_t$ is a Brownian motion started at 0. But since Brownian motion is conformally invariant, $\phi_{n-1}^{-1}\left(\frac{1}{B_t}\right)$ is a time change of Brownian motion. Then, by the symmetry properties of Brownian motion, the hitting distribution of a Brownian motion on a disk is uniform. So in this case we distribute $\theta_n \sim \mathrm{Unif}[0, 2\pi]$.

For the capacities, we will instead consider the diameter $d_n$ of each particle. For a slit this means the $n^{th}$ attached particle is $P_n = e^{i\theta_n}(1, 1 + d_n]$. The map $\theta \to e^{i\theta}$ maps the interval to $P_n$. Specifically, it maps $\theta_n$ to the tip of the particle and $\theta + \mu_n$, for some $\mu_n \in [0, 2\pi]$, to the base of the particle. Thus there exists an interval $[\theta_n, \theta_n + \mu_n]$ such that the map $f_n$ maps the interval to $P_n$. The length of attached particle is distorted by the map $\phi_{n-1}$ and given by,

$$\int_0^{\mu_n} |\phi_{n-1}^{'}(f_n(e^{i(\theta_n+\theta)}))||(f_n'(e^{i(\theta_n+\theta)})|d\theta.$$

Therefore the length of the attached particle is

$$d_n|\phi_{n-1}'(x_n e^{i\theta_n})|$$

for some $x_n \in [1, 1+d_n]$. In the real world models, including DLA, we often want the particles to be roughly the same size at each attachment. Therefore, using that $c_n = \frac{d_n^2}{4} + o(1)$ the capacity $c_n$ of the added particle is chosen as,

$$c_n = c|\phi_{n-1}'(e^{i\theta_n})|^{-2}$$

where $0 < c < \infty$ is the capacity of the first particle. With this choice each particle is approximately the same size. This demonstrates how the off-lattice version of DLA is constructed using the conformal mappings method. It is possible to do the same for the Eden model and various other lattice based models [**Tur19**].

## 1.5. Hastings-Levitov model

Diffusion Limited Aggregation is an example of a random growth process where the local growth rate is determined by harmonic measure. The class of growth processes that satisfy this condition are said to demonstrate Laplacian growth [**HL98**] and they occur regularly within the real world. Therefore, we would like a collection of models that allows us to study this class of processes as whole. The Hastings-Levitov model HL($\alpha$) introduced in [**HL98**] is a collection of models used to describe Laplacian growth and is formed by using conformal mappings as described above. It is particularly useful because it allows us to vary between previously well known models such as DLA and Eden simply by varying the parameter $\alpha$.

The structure of the model is the same as the general conformal model described in the previous section. All that remains is to choose how the attaching angles and capacities are distributed on the maps $\{f_n\}_{n\geqslant 0}$. As in the construction of an off-lattice version of DLA we want to model Laplacian growth with the local growth rate determined by harmonic measure. Thus, choose the angles $\theta_n$ to be independently distributed uniformly on the unit disk.

The capacities are chosen as,

$$c_n = c|\phi'_{n-1}(e^{i\theta_n})|^{-\alpha}$$

for some $c > 0$. This choice allows us to vary between the off-lattice models and, as seen above, provides an off-lattice version of DLA when $\alpha = 2$. Although the physical construction differs, Hastings and Levitov put forward numerical evidence to argue that HL(1) corresponds to an off-lattice version of the Eden model. In very recent work [**NST21**] Turner et al show how this is satisfied explicitly in a regularised setting.

The final element to consider in construction of the Hastings-Levitov model is the shape of the attaching particle. The choice we make is determined by which real world process we are trying to model. Hastings and Levitov introduce both the strike and bump mappings in [**HL98**]. The bump map attaches a non-empty interior on the boundary whereas the strike map attaches a slit. An explicit form of the strike map is provided above in equation (1.1). In most cases we fix our choice our particle before evaluating the scaling limits of the models, however, we do not want to produce results that are dependent on the choice of particle, thus we often use a general family of particles (see for example [**LT21a**]) that allow us to recover all of the classical maps.

An extension of the Hastings-Levitov model exists in the form of the Aggregate Loewner Evolution (ALE) model ALE($\alpha, \eta$) introduced in [**STV19**]. In this model the attaching angles are chosen proportional to the density of harmonic measure on the cluster boundary, raised to some power $\eta$. We will focus on the Hastings-Levitov model in this thesis corresponding to ALE$(\alpha, 0)$ but interested readers should see [**NST19**] and [**Hig20**].

Another variation of the Hastings-Levitov model is the anisotropic version introduced in [**JVST12**] as AHL$(\nu)$. This model is constructed in the same way as the Hastings-Levitov model with $\alpha = 0$ but instead of attaching uniformly the attaching angles are i.i.d distributed randomly on the unit circle according to a non-uniform measure $\nu$. This model is analysed in the second paper presented in Chapter 3.

## 1.6. Scaling limits

Now we have all we need in order to start evaluating the models. Several natural questions arise when studying the models including,

- Does a scaling limit exist as $n \to \infty$?
- What is the shape of the cluster in its scaling limit?
- What is the behaviour of the fluctuations in this limit?
- Each particle comes with a natural notion of ancestry determined by which particle it attached to. This particle also has a direct ancestor and so on. We can repeat the process of considering the direct ancestor of a succession of particles in order to trace an ancestral path of a particle on the boundary. Thus a natural question is, what is the ancestral path of a particle attached on the boundary of the cluster?

This thesis will attempt to answer some of these questions. In order to answer the first two questions it is necessary to define the scaling limit in this context. There are two natural ways that have previously been used to evaluate the scaling limit of the clusters formed using conformal maps. The first, and perhaps the most natural, is known as the small-particle limit. This method was first used to evaluate HL$(\alpha)$ in [**NT12**] when $\alpha = 0$. Under this scaling limit we send the capacity, and hence the size, of the attached particle to zero as $n \to \infty$, with $nc \approx t$ for some fixed value $t$. Most of the research into the Hastings-Levitov model has been done in the small-particle limit and in the next section we will highlight the results which will be most relevant to our own research.

The second way in which we can take the scaling limit is known as the limit under capacity rescaling. Using this method, rather than sending the particle size to zero, the particle size is fixed. Then we rescale the whole cluster by its total logarithmic capacity at each stage so that it is contained inside the unit disk and then evaluate the shape of the rescaled cluster as we send the number of particles to infinity. This method was introduced in [**RZ05**], the details of which are described in the next section, and will be the focus of our first paper in Chapter 2.

Once we have shown the existence of a scaling limit and evaluated it we can then evaluate the fluctuations on this limit. Consider for example the Strong Law of Large Numbers, whilst this is a strong result, the Central Limit Theorem allows us to fully understand the distribution. Similarly, we will need to establish a shape theorem and evaluate the fluctuations in order to understand the scaling limit of the Hastings-Levitov model.

Finally, once we understand the shape of the growing cluster we may also want to evaluate how the particles are attached at each stage so that we can understand the ancestry of each attached particle. To do so we will analyse the harmonic measure on the boundary of the cluster.

## 1.7. Existing results

In this section we will describe the existing results relevant to the independent research that follows in later sections. We will split this into two subsections. In the first subsection we will discuss previous work in the small-particle limit. Most of the previous research into the scaling limits of the Hastings-Levitov model and its variants has analysed the limit in the small-particle limit and, as such, this first subsection will consist of a summary of three papers ([**NT12**], [**JVST15**] and [**JVST12**]) most relevant to this thesis but the reader is directed to [**STV19**] and [**TT20**] amongst others for further discussion. In the second subsection we consider previous results using the capacity rescaling limit. There has been little work in this area and therefore we consider in detail a paper by Rohde and Zinsmeister [**RZ05**] which introduces the method.

### 1.7.1. Results in the small particle limit.

1.7.1.1. *Hastings-Levitov Aggregation in the small-particle limit.* We start by describing the results of a paper by Norris and Turner [**NT12**]. This paper is of particular significance because it is the first to use the small-particle limit to evaluate the Hastings-Levitov model. The model is described as in Section 1.5 and the authors evaluate the model for $\alpha = 0$. They establish a shape theorem and furthermore evaluate the harmonic measure on the boundary of the cluster.

We have seen above that the capacities of the Hastings-Levitov model are given by,

$$c_n = c|\phi'_{n-1}(e^{i\theta_n})|^{-\alpha}$$

for some $c > 0$ and a parameter $\alpha \geqslant 0$. When $\alpha > 0$ the capacities have a non-trivial dependence on the past which makes the process very hard to analyse. However, when $\alpha = 0$ the capacities are given by a deterministic value $c_n = c$ and therefore the total capacity of the map $\phi_n$ at infinity is $cn$. This greatly increases the accessibility of the problem. Thus, this paper first evaluates the model in the case where $\alpha = 0$ before $\alpha > 0$ is tackled under regularisation in later papers.

We start by describing the shape theorem for $\alpha = 0$. One of the defining and most useful features of the $\alpha = 0$ case is that the process $(\phi_n^{-1}(z))_{n \geqslant 0}$ is a Markov process for all $z \in (\mathbb{C} \cup \infty)\backslash K_0$ [**NT12**]. As a result the authors are able to use fluid limit analysis on the random maps. This produces the following shape theorem.

Theorem 1.7.1. *Let $\tilde{P}_n = K_n \backslash K_{n-1}$. Then consider for $\epsilon \in (0,1]$ and $m \in \mathbb{N}$ the event $\Omega[m,\epsilon]$ specified by the following conditions: for all $n \leqslant m$ and all $n' > m+1$,*

$$|z - e^{cn - i\theta_n}| \leqslant \epsilon e^{cn} \quad \textit{for all } z \in \tilde{P}_n$$

*and*

$$dist(w, K_n) \leqslant \epsilon e^{cn} \quad \textit{whenever } |w| \leqslant e^{cn}$$

*and*

$$|z| > (1-\epsilon)e^{cm} \quad \textit{for all } z \in \tilde{P}_{n'}.$$

*Then if we assume that* $\epsilon = c^{\frac{1}{3}}(\log(\frac{1}{c}))^8$ *and* $m = \lfloor c^{-3} \rfloor$. *Then* $\mathbb{P}(\Omega[m,\epsilon]) \to 1$ *as* $c \to 0$.

Informally, this result tells us that the cluster grows like an expanding disk of radius $e^{cn}$. Note that the powers in the theorem above are crucial and the authors state that some effort was made in order to maximise the power $\frac{1}{3}$ [**NT12**].

This theorem provides an understanding of what the shape of the $\alpha = 0$ cluster looks like in the small-particle limit. However, we often want to understand the underlying structure of the cluster and thus it is necessary to understand the ancestry of each of the particles. To do so the authors evaluate how the harmonic measure evolves on the boundary of the cluster. When $\alpha = 0$, this concept can be explained as follows. For a point $x \in (0,1)$, define,

$$\gamma(x) = \frac{1}{2\pi i}\log(f_c^{-1}(e^{2\pi i x}))$$

choosing the branch of logarithm which results in $x = \frac{1}{2}$ being fixed. This can be extended to the real line as follows, if $x = k + a$ where $a \in (0,1]$ then define $\gamma(x) = k + \gamma(a)$. Then for all $x \in \mathbb{R}$ define

$$\gamma_n(x) = \gamma(x - \theta_n) + \theta_n.$$

Observe that using this definition $\gamma_n(x) = \frac{1}{2\pi i}\log(f_n^{-1}(e^{2\pi i x}))$ with the branch of logarithm inferred from the definition above. Figure 9 demonstrates how the function $\gamma_n(x)$ describes the change in angle of a point $x$ on the boundary under the transformation $f_n(x)$ and thus $\gamma_n(x)$ tells us how the harmonic measure evolves under the map $f_n(x)$.

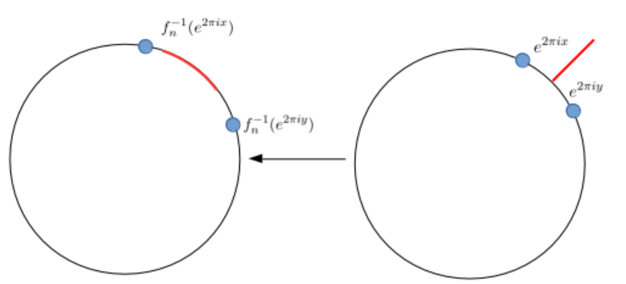


FIGURE 9. How the harmonic measure evolves on the boundary of the cluster

Let $\tilde{\gamma}(x) = \gamma(x) - x$ then define the discrete harmonic measure flow under the map $\phi_n$ for $x \in \mathbb{R}$ and $n > m$ as,

$$X_{m,n}(x) = X_{m,n-1}(x) + \tilde{\gamma}(X_{m,n-1}(x) - \theta_n) \tag{1.2}$$

with $X_{m,m}(x) = x$. Therefore,

$$X_n(x) = \gamma_n(X_{n-1}(x)).$$

Thus if $\Gamma_{m,n}(x) = f_n^{-1} \circ ... f_m^{-1}(x)$ then

$$\begin{aligned} X_{m,n}(x) &= \frac{1}{2\pi i} \log(f_n^{-1}(e^{2\pi i X_{m,n-1}(x)})) \\ &= \frac{1}{2\pi i} \log(f_n^{-1}(\Gamma_{m,n-1}(e^{2\pi i x}))) \\ &= \frac{1}{2\pi i} \log(\Gamma_{m,n}(e^{2\pi i x})). \end{aligned}$$

Note that $X_{m,n}(x)$ is defined in this way to make sure the branch of the logarithm respects the composition structure. Then the harmonic measure flow can be rewritten as

$$X_{m,n}(x) = \sum_{i=1}^{n} \tilde{\gamma}(X_{m,i-1}(x) - \theta_i) + x.$$

This details how the cluster is evolving as each particle is attached and hence the ancestry of each particle. We will study this concept in more detail when we analyse the harmonic measure on the boundary of the anisotropic Hastings-Levitov cluster in Chapter 3.

The authors make the following assumptions in order to analyse the harmonic measure

$$d \in \left(0, \frac{1}{3}\right) \text{ and } P \subset \{z \in \mathbb{C} : |z-1| \leqslant d\} \text{ and } 1+d \in P \text{ and } P = \{\bar{z} : z \in \tilde{P}\}.$$

Furthermore, in order to see a non-trivial limit the authors rescale time, thus, let

$$\bar{X}_{s,t}(x) = X_{\bar{n}(s),\bar{n}(t)}(x)$$

where $\bar{n}(t) = \lfloor tc^{-\frac{3}{2}} \rfloor$. With this definition, and the assumptions above the authors reach the following result on the convergence of the rescaled harmonic measure flows $\left\{\bar{X}_{s,t}(x)\right\}_{0 \leqslant s < t}$.

THEOREM 1.7.2. *Assume that the basic particle $P$ satisfies the conditions above. Then the rescaled harmonic measure flow converges to the Brownian web, uniformly in $P$, as $c \to 0$.*

In addition, the rescaled Harmonic measure flow also converges to the Brownian web on the line [**NT12**]. Therefore, this paper provides an understanding of both the shape of the Hastings-Levitov cluster when $\alpha = 0$ and its underlying structure. We would like to understand the same properties for $\alpha > 0$ and in order to do so a regularisation to the model is introduced.

1.7.1.2. *Scaling limits in a regularised setting.* We now summarise [**JVST15**] where a regularised version of the Hastings-Levitov model is analysed for $\alpha \geqslant 0$. We will consider a similar regularisation in the independent work presented in Chapter 2. The authors show that in the small-particle limit the Hastings-Levitov model converges to a growing disk provided the regularisation is sufficient. Then they analyse the harmonic measure flow on the boundary of the cluster and show that by changing the rate at which $\alpha \to 0$ the harmonic measure flows converge to either the identity map or a version of the Brownian web on the circle.

As seen above, with capacities defined as

$$c_n = c|\phi'_{n-1}(e^{i\theta_n})|^{-\alpha}$$

it is clear that for $\alpha > 0$ the growth of the cluster is strongly dependent on its history. This makes the model very difficult to study and thus the introduction if a regularisation on the capacities is necessary in order to analyse the shape of the cluster in its limit. Therefore, Turner et al [**JVST15**] introduce the regularisation factor $\sigma$ so that the capacities are redefined as

$$c_n = \frac{c}{|\phi'_{n-1}(e^{\sigma + i\theta_n})|^{\alpha}}$$

with $\sigma > 0$. This allows us to move away from the bad behaviour of $\phi'_{n-1}$ near the boundary of the unit disk and to deduce estimates on $c_n$. Notice that as $\sigma \to 0$ the model converges back to the original Hastings-Levitov model HL($\alpha$).

The authors then introduce the deterministic sequence,

$$c_n^* = \frac{c}{1 + \alpha c(n-1)}$$

for $n = \mathbb{N}$. They show that provided $\sigma$ does not converge to zero too quickly then the capacities $c_n$ are close to $c_n^*$ with high probability. This is provided in the form of the following result [**JVST15**].

THEOREM 1.7.3. *Let $\sigma \gg (\log(c^{-1}))^{\frac{-1}{2}}$ and let $N \in \mathbb{N}$ be fixed. Then there exists some absolute constant $\beta > 0$ such that*

$$\mathbb{P}\left(\sup_{n \leqslant N} \left|\log\left(\frac{c_n}{c_n^*}\right) > \alpha c^\beta\right|\right) \to 0$$

*as $c \to 0$.*

With the $c_n^*$ notation defined, it is possible to introduce equivalent notation $f_n^*(z)$ for the conformal maps defined with the star capacities and then analyse the cluster formed using these conformal maps. Then the authors use two key facts [**JVST15**] in order to transition between the different models and prove a shape theorem for the $\sigma$ regularised version of the Hastings-Levitov model. The first is that the convergence $\sup_k \frac{|c_k - c_k^*|}{c} \to 0$ implies weak convergence of driving measures for the Loewner representation of the growth process. Secondly, weakly convergent driving measures lead to sequences of conformal maps that converge in the sense of Caratheodory. With these two facts the authors state the following shape theorem [**JVST15**].

THEOREM 1.7.4. *Let $T > 0$ and $\alpha > 0$ be fixed. Then suppose $n = \lfloor \frac{T}{c} \rfloor$ and $\sigma \gg (\log(c^{-1}))^{\frac{-1}{2}}$. Then, as $c \to 0$ the laws of the maps $\phi_n(z)$ converge weakly with respect to uniform convergence on compact subsets to a point mass at $(1 + \alpha T)^{\frac{1}{\alpha}} z$.*

Thus, the shape of the regularised cluster converges in the small-particle limit to a growing disk in the sense of Caratheodory. Whilst this regularisation means that the authors are not studying the true Hastings-Levitov model for $\alpha > 0$ they argue that the regularised model with $\alpha = 2$ is consistent with a model where all particles are attached of the same size, as in the case of DLA. Similarly, when $\alpha = 1$ the result is consistent with a model which exhibits growth proportional to local arc length as in the case of the Eden model.

However, as in the previously studied paper [**NT12**] the authors would like to study the evolution of harmonic measure on the cluster boundary in order to better understand the underlying structure of the cluster. As in the previous section consider the rescaled harmonic measure flow

$$\bar{X}_{s,t}(x) = X_{\bar{n}(s),\bar{n}(t)}(x)$$

where $\bar{n}(t) = \lfloor tc^{-\frac{3}{2}} \rfloor$. In this case, the following result [**JVST15**] demonstrates the three possible cases that can occur dependent on the behaviour of $\alpha = \alpha(c)$ as $c \to 0$.

THEOREM 1.7.5. *Suppose that $\sigma \geqslant \log(c^{-1})^{\frac{-1}{2}}$. Then as $c \to 0$, on timescales of order $c^{\frac{-3}{2}}$, one of the following three situations arises.*

- *If $\alpha c^{-\frac{1}{2}} \to 0$, the rescaled harmonic measure flow converges to the Brownian web.*
- *If $\alpha c^{-\frac{1}{2}} \to \infty$(sufficiently slowly), the rescaled harmonic measure flow converges to the identity flow.*
- *If $\alpha c^{-\frac{1}{2}} \to a \in (0, \infty)$, the rescaled harmonic measure flow converges to a time-change of the Brownian web, stopped at a finite time that is decreasing in $a$.*

The interpretation of this result is given as follows [**JVST15**]. For $\alpha \ll c^{\frac{1}{2}}$ the harmonic measure flow converges to the Brownian web. However, as all Brownian motion on a circle starting from a fixed time coalesce to a single Brownian motion eventually this tells us that the points on the boundary of the regularised cluster arriving after a certain time all share the same ancestor. However, when $\alpha \gg c^{\frac{1}{2}}$ the number of infinite branches becomes unbounded in the limit as $c \to 0$. Finally, if $\alpha c^{\frac{3}{2}} \to a$ then there exists a random number of infinite branches in the regularised cluster.

1.7.1.3. *Anisotropic growth.* Finally, we describe the results of the paper [**JVST12**] which introduced the anisotropic version of the Hastings-Levitov model. This model will be the focus of Chapter 3. The model is formulated in the same way as the Hastings-Levitov model introduced in Section 1.5 with $\alpha = 0$, however, rather than choosing the attaching angles uniformly, instead, choose the angles $\theta_i$ to be independent identically distributed random variables on the unit circle with common law $\nu$. More explicitly, throughout the remainder of this subsection denote

$$\phi_{n+1} = \phi_n \circ f_{n+1} = f_1 \circ f_2 \circ .... \circ f_{n+1}$$

with $f_n(z) = e^{i\theta_n} f_{c_n}(ze^{-i\theta_n})$ where $c_n = c$ for some fixed $c$ and the angles $\theta_n$ are i.i.d distributed randomly on the unit circle according to some non-uniform probability measure $\nu$ .

The paper is split into two main sections, in the first section the authors consider the scaling limit of the cluster in the small particle limit and show a shape theorem exists. In the second section the authors consider the evolution of the harmonic measure on the cluster boundary and show that in compact time it converges to the solution of an ordinary differential equation before studying its fluctuations. In Chapter 3, I will extend these results by considering the harmonic measure flow on logarithmic timescales.

The authors start by providing a shape theorem. In order to do so they write the sequence of conformal maps that form the cluster as a Loewner chain driven by the measure $\nu$. The construction is as follows; start by supposing we have a growing sequence of compact sets $\{K_t\}_{t\geqslant 0}$ as in the construction of the Hastings-Levitov model above. Now let $\mathbb{D}$ be the unit disk in the complex plane. Then there exists a family of conformal maps

$$f_t : \mathbb{D} \to \mathbb{C}\backslash K_t$$

which fixes infinity and where $f_t'(\infty) > 0$. This family of maps is called a Loewner chain and can be parametrised with respect to a family of measures $\{\nu_t\}_{t\geqslant 0}$. The conformal maps then satisfy the Loewner-Kufarev equation [**JVST12**],

$$\partial_t f_t(z) = z f_t'(z) \int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} d\nu_t(\zeta).$$

The authors then use this construction to show that in the small particle limit the the cluster converges to the solution of the Loewener-Kufarev equation [**JVST12**].

THEOREM 1.7.6. *Let $\phi$ denote the solution to the Loewner-Kufarev equation driven by the measures $\{\nu_t\}_{t\geqslant 0}$ and evaluated at time $T$ for some fixed $T \in (0,\infty)$ and $\nu_t = \nu$ for all $t$.*

*Set* $n = \lfloor \frac{T}{c} \rfloor$, *and define the conformal map as above* $\phi_n = f_1 \circ f_2 \circ .... \circ f_n$. *Then* $\phi_n$ *converges to* $\phi$ *uniformly on compacts almost surely as* $c \to 0$.

Therefore, with this shape theorem the authors describe how the cluster behaves away from the boundary. But as in the previous two papers they want to understand the underlying structure of the cluster and so need to analyse the harmonic measure flow. As in Section 1.7.1.1, for $x \in \mathbb{R}$ and $m < n$, define the harmonic measure flow as

$$X_{m,n}(x) = \frac{1}{2\pi i} \log(\Gamma_{m,n}(e^{2\pi i x}))$$

where $\Gamma_{n,m}$ is defined as the restriction to the boundary,

$$\Gamma_{n,m} = f_n^{-1} \circ ... \circ f_m^{-1} | \partial K_0.$$

The harmonic measure flow can be embedded into continuous time as follows. Suppose $0 < T_1 < T_2 < ...$ are times of a Poisson process, independent of the attaching angles $\theta_i$ with rate $\frac{1}{c}$. Then for an interval $[s,t] \subset [0,\infty)$, define $\widetilde{\Gamma}_{s,t} = \Gamma_{m,n}$ where $m$ and $n$ are the smallest and largest integers such that both $T_m \in [s,t]$ and $T_n \in [s,t]$ [**JVST12**]. Then, as in the discrete case above, for $s < t$, define the continuous harmonic measure flow

$$\widetilde{X}_{s,t}(x) = \frac{1}{2\pi i} \log(\widetilde{\Gamma}_{s,t}(e^{2\pi i x})).$$

Note that some of this notation is not consistent with that in Chapter 3, where instead we consider the discrete harmonic measure flow, and this should be read independently. However, as in Chapter 3, define $b(x)$ as the Hilbert transform of the measure $\nu$,

$$b(x) = c_0 h_\nu(x) + \frac{1}{2\pi} \int_0^1 \cot(\pi z)(h_\nu(x-z) - h_\nu(x)) dz,$$

for some constant $c_0$, where $h_\nu$ is the density of $\nu$ on $\mathbb{R}$. Furthermore, for $t > s$, define $\psi_{s,t}(x)$ as solution to the ordinary differential equation

$$\dot{\psi}_{s,t}(x) = b(\psi_{s,t}(x))$$

for $x \in \mathbb{R}$ and $\psi_{s,s}(x) = x$. With these definitions in mind the authors state the following result [**JVST12**].

THEOREM 1.7.7. *For $\widetilde{X}_{s,t}$ and $\psi_{s,t}$ defined above, then for for a fixed $T$ and any $\epsilon > 0$,*

$$\lim_{c\to 0} \mathbb{P}\left(\sup_{s<t<T} |\widetilde{X}_{s,t}(x) - \psi_{s,t}(x)| > \epsilon\right) = 0.$$

Finally, the authors consider the fluctuations on this convergence. They show they are of order $c^{\frac{1}{4}}$ so as in [**JVST12**] for fixed $(s,x) \in (0,\infty) \times \mathbb{R}$ define,

$$Z_t^P(x) = c^{-\frac{1}{4}}(\widetilde{X}_{s,t}(x) - \psi_{s,t}(x))$$

and let $Z_t(x)$ be the solution to the linear stochastic differential equation

$$dZ_t(x) = \sqrt{h_\nu(x)\psi_{(s,t]}(x)}dB_t + b'(\psi_{(s,t]}(x))Z_t(x)dt$$

for $t \geqslant s$ starting from $Z_s(x) = 0$, where $B_t$ is a standard Brownian motion. This leads us to the following result [**JVST12**].

THEOREM 1.7.8. *As $c \to 0$, the processes $Z_t^P \to Z_t$ in distribution.*

Therefore, this paper classifies both the convergence of the harmonic measure and the fluctations up to a bounded time. In Chapter 3, we will extend these results to consider the harmonic measure on logarithmic timescales and show that there exists a critical logarithmic time window in which the harmonic measure flow, started from the unstable point of the ordinary differential equation, moves stochastically from an unstable trajectory towards a stable trajectory.

**1.7.2. Results under capacity rescaling.** In this section we outline the existing results on the scaling limits of HL($\alpha$) under capacity rescaling. In this setting, rather than sending the size of the particle to zero as we send the number of particles to infinity, instead, we keep the value $c$ fixed and rescale the cluster by its total logarithmic capacity at each iteration before sending the number of particles to infinity. This method has been less studied than the small particle limit and the results consist of those introduced in a paper by Rohde and Zinsmeister [**RZ05**].

The paper studies the Hastings-Levitov model introduced in the previous sections under capacity rescaling and is split into two parts. First, the authors show that when $\alpha = 0$ the logarithmic capacity of the limiting cluster grows exponentially and under capacity rescaling the rescaled cluster has finite length. In the second half of the paper the case when $\alpha > 0$ is considered under a regularisation and it is shown that the logarithmic capacity of the growing cluster at each iteration grows fast but not exponentially.

As described in the previous sections, if $\alpha > 0$ the Hastings-Levitov model is very difficult to study due to correlations between the capacities. Thus, in order to study the model the authors first introduce a regularisation on the model. This regularised version of the model is denoted RHL($\alpha$) and is defined as follows [**RZ05**]. Let be $d$ be the length of a slit with capacity $c$. The relation is then given by $c = \log\left(1 + \frac{d^2}{4(1+d)}\right)$. Let $u_n = e^{i\theta_n}$. Then the regularisation is chosen as,

$$d_n = d^{1-\frac{\alpha}{2}} \epsilon_n(u_n, d)^{\frac{\alpha}{2}}$$

where

$$\epsilon_n(u, d) = \inf\{\epsilon > 0 \ : \ \epsilon|\phi_n'((1+\epsilon)u)| = d\}$$

and $\phi_{n+1} = \phi_n \circ f_{n+1} = f_1 \circ f_2 \circ .... \circ f_{n+1}$ with $f_n(z) = e^{i\theta_n} f_{c_n}(ze^{-i\theta_n})$ as above. This regularisation provides a 'dual' relationship between RHL(0) and RHL(2) [**RZ05**]. With this choice of regularisation the article is split into three main cases that determine how the total capacity is affected when the value of $0 \leqslant \alpha < 2$ is changed. This in turn allows us to make statements about the dimension of the limiting cluster for different values of $\alpha$. Start with the case where $\alpha = 0$. Note that we will evaluate this case further in Chapter 2. When $\alpha = 0$ the choice of regularisation means that $d_n = d$ for a fixed $d$. Therefore, $\phi_n = f_1 \circ f_2 \circ ... \circ f_n$ with $f_k(z) = e^{i\theta_k} f_c(ze^{-i\theta_k})$ and $f_c : \Delta \to \Delta \backslash [1, 1+d]$ for some fixed $d > 0$. Furthermore,

$$f_c(z) = e^c z + \mathcal{O}(1)$$

at $\infty$ with $c = \log\left(1 + \frac{d^2}{4(1+d)}\right)$. So the total logarithmic capacity of the cluster $K_n$ at infinity is given by $C_n = e^{cn}$. Denote $\sum_0$ as the space of univalent normalised functions on $\Delta$. Then we have a measure $\widetilde{P}_n$ induced on $\sum_0$, given by

$$\widetilde{P}_n = (\sigma_n)_*(l^n)$$

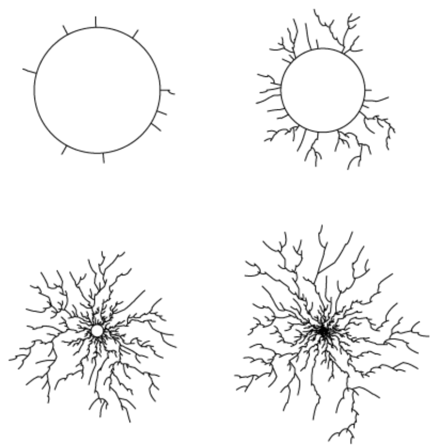

FIGURE 10. Examples of growing cluster for n=10,100,500,1000, from [**RZ05**].

where $l^n$ is just the product of Lebesgue measures [**RZ05**] under the map

$$\sigma_n : (\theta_1, ..., ...\theta_n) \to e^{-cn}\phi_n.$$

The authors then go on to prove that when $\alpha = 0$ the scaling limit exists via the following theorem [**RZ05**].

THEOREM 1.7.9. *There is a probability measure $\widetilde{P}_\infty$ on $\Sigma_0$ such that the sequence of measures $\widetilde{P}_n$ converges weakly to $\widetilde{P}_\infty$.*

So now we know the scaling limit exists the authors want to deduce properties of this limit. Before they do, they provide the following definitions [**RZ05**]. First let $l_n$ be defined as follows;

$$l_n = \int_1^{1+d} |\phi_n'(ru_n)|dr.$$

Then define $L_n$ to be the length of $K_n$ so that

$$l_{n+1} = L_{n+1} - L_n.$$

Finally, define the rescaled cluster $\widetilde{K_n}$ as

$$\widetilde{K_n} = \frac{K_n - A_n}{C_n}$$

where $A_n = \lim_{z\to\infty} \phi_n(z) - z$. Similarly,

$$\widetilde{L_n} = \frac{L_n}{C_n}.$$

Using these definitions the authors prove the following two theorems [**RZ05**]. Firstly in the case where $d$ is large the following theorem holds,

THEOREM 1.7.10. *There is a constant $d_0$ such that for $d > d_0$ and all choices of $\theta_1, \theta_2....\theta_n$*

$$L_n \leqslant C(d)C_n$$

*for some constant $C(d) > 0$ and thus,*

$$\widetilde{L_n} \leqslant C(d).$$

Then for small $d > 0$ the following theorem holds.

THEOREM 1.7.11. *There exists a constant $C = C(d)$ such that*

$$\mathbb{E}(L_n) \leqslant C(d)C_n.$$

*and thus,*

$$\mathbb{E}(\widetilde{L_n}) \leqslant C(d).$$

Note that the difference here is that in Theorem 1.7.10 the inequality holds for all $n$, whereas, in the case of Theorem 1.7.11 the bound is on the expected value of $L_n$. Now we move on to the case where $0 < \alpha < 1$. First the authors adapt the regularisation. Fix $d < c$, then redefine the component of regularisation $\epsilon$ as follows [**RZ05**]. Decompose the unit circle into dyadic intervals. Now for an interval with order $k$ define $x_I$ as its centre and set

$$z_I = (1 + 2^{-k})x_I.$$

Then, let $u$ be a point on the circle, and suppose $k$ is the order of the minimal dyadic interval containing $u$ such that $2^{-k}|\phi_n'(z_I)| \geqslant d$ then set

$$\epsilon(u, d) = 2^{-k}.$$

Of course, as $\alpha$ is no longer equal to zero the regularisation no longer gives us a constant slit length. The authors manage to show that, with $\alpha$ in this region, the total capacity $C_n$ grows fast, as a polynomial in $n$ but not exponentially unlike the case where $\alpha = 0$. We combine results from the paper to give the result in the following form [**RZ05**].

THEOREM 1.7.12. *If* $0 < \alpha < 1$ *is small enough, then almost surely there exists a constant* $K > 0$ *such that for every* $n \geqslant 0$,

$$K n^{\frac{1}{\alpha}(1-7\sqrt{\alpha})} \leqslant C_n \leqslant C(d) n^{\frac{2}{\alpha}}$$

*where* $C(d)$ *is the constant dependent on* $d$ *from Theorem 1.7.10.*

The authors use this theorem to gain an inequality involving both the length and logarithmic capacity of the cluster [**RZ05**].

THEOREM 1.7.13. *For small* $\alpha > 0$ *and for* $\beta > \alpha$, *almost surely there exists a constant* $K > 0$ *such that for every* $n \geqslant 0$,

$$\widetilde{L_n} \leqslant K C_n^{\frac{2\beta}{1-7\sqrt{\beta}}}$$

Therefore, by combining all of the theorems above we can summarise as follows. We first see that there is a phase transition that occurs when $\alpha = 0$. In addition, when $\alpha = 0$ the capacity grows exponentially but when $\alpha > 0$ the growth is only polynomial. Finally, the bounds provided by Rohde and Zinsmeister leave open the possibility of another phase transition at $\alpha = 1$ but it is unclear whether or not this holds.

## 1.8. Outline of thesis

We are now in a position to present the independent research for the award of this thesis. This will be presented in the form of two papers. In the first paper we evaluate a strongly regularised version of the Hastings-Levitov model HL($\alpha$) for $0 \leqslant \alpha < 2$. We first consider the case where $\alpha = 0$ and show that under capacity rescaling, the limiting structure of the cluster is not a disk, unlike in the small-particle limit. Then we consider the case where $0 < \alpha < 2$ and show that under the same rescaling the cluster approaches a disk. We also evaluate the fluctuations and show that, when represented as a holomorphic function, they behave like a Gaussian field dependent on $\alpha$. In the second paper we study the anisotropic version of the Hastings-Levitov model AHL($\nu$). We consider the evolution of the harmonic measure and first show that we have convergence of the harmonic measure flow up to a logarithmic time. We then evaluate the fluctuations on compact time and show that their behaviour is stochastic. Finally we show there exists a critical logarithmic time window where on this timescale the harmonic measure flow, started from the unstable point, moves from an unstable trajectory towards a stable trajectory. Presenting the research in this way may lead to some repetition, particularly in the introduction to each paper, however, we believe that as independent bodies of work they should be presented as such.

CHAPTER 2

# Scaling limits and fluctuations for random growth under capacity rescaling

In this chapter we present our first paper [**LT21a**]. We evaluate a strongly regularised version of the Hastings-Levitov model HL($\alpha$) for $0 \leqslant \alpha < 2$. Previous results have concentrated on the small-particle limit where the size of the attaching particle approaches zero in the limit. However, we consider the case where we rescale the whole cluster by its logarithmic capacity before taking limits, whilst keeping the particle size fixed. We first consider the case where $\alpha = 0$ and show that under capacity rescaling, the limiting structure of the cluster is not a disk, unlike in the small-particle limit. Then we consider the case where $0 < \alpha < 2$ and show that under the same rescaling the cluster approaches a disk. We also evaluate the fluctuations and show that, when represented as a holomorphic function, they behave like a Gaussian field dependent on $\alpha$. Furthermore, this field becomes degenerate as $\alpha$ approaches 0 and 2, suggesting the existence of phase transitions at these values.

## 2.1. Introduction

Random growth occurs in many real world settings, for example we see it exhibited in the growth of tumours and bacterial growth. We would like to be able to model such processes to determine their behaviour in their scaling limits. Since the 1960's, models have been built in order to describe individual processes. Perhaps the most famous examples of such models are the Eden model [**Ede61**] and Diffusion Limited Aggregation (DLA) [**WS83**]. The Eden model is used to describe bacterial colony growth, whereas, DLA describes mineral aggregation (see for example [**RZ05**]).

In their 1998 paper [**HL98**], Hastings and Levitov introduced a one parameter family of conformal maps HL($\alpha$) which can be used to model Laplacian growth processes and allows us to vary between the previous models by varying the parameter $\alpha$. In contrast to many well studied lattice based models, HL($\alpha$) is formed by using conformal mappings [**HL98**]. We can then use complex analysis techniques to evaluate the growth. We consider a regularised version of this model and show that at certain values of $\alpha$ a phase transition on the scaling limits occurs.

**2.1.1. Outline of the model.** In order to define our model we start by defining the single particle map. Define $\Delta$ as the exterior of the unit disk in the complex plane, $\Delta = \{|z| > 1\}$. For any conformal map $f : \Delta \to \mathbb{C}$ we define the capacity of the map to be,

$$\lim_{z\to\infty} \log\left(f'(z)\right) := \log f'(\infty).$$

For each $c > 0$, we then choose a general single particle mapping $f_c : \Delta \to \mathbb{C}\backslash K$ which takes the exterior of the unit disk to itself minus a particle of capacity $c > 0$ at $z = 1$. Note that we can then rescale and rotate the mapping $f_c(z)$ to allow any attaching point on the boundary of the unit disk by letting $f_n(z) = e^{i\theta_n} f_{c_n}(ze^{-i\theta_n})$ where $\theta_n$ is the attaching angle and $c_n$ is the capacity of the $n^{\text{th}}$ particle map $f_{c_n}(z)$.

We can now form the cluster by composing the single particle maps. Let $K_0 = \Delta^c = \{|z| \leqslant 1\}$. Suppose that we have some compact set $K_n$ made up of $n$ particles. We can find a bi-holomorphic map which fixes $\infty$ and takes the exterior of the unit disk to the complement of $K_n$ in the complex plane, $\phi_n : \Delta \to \mathbb{C}\backslash K_n$. We then define the map $\phi_{n+1}$ inductively;

$$\phi_{n+1} = \phi_n \circ f_{n+1} = f_1 \circ f_2 \circ .... \circ f_{n+1}.$$

There are several possible choices for the family of maps $\{f_c\}_{c>0}$. The choice we make is determined by what shape we would like the attaching particles to have. Hastings and Levitov introduce both the strike and bump mappings in [**HL98**]. The strike map attaches a single slit onto the boundary at $z = 1$ whereas the bump map attaches a particle with non-empty interior. We would like results to exhibit some universality in the specific choice of particle shapes. However, we do need to impose some restrictions on how the particles localise around the attachment points as $c \to 0$. The specific condition we require is

$$f_c(z) = e^c z \exp\left(\frac{2c}{z-1} + \delta_c(z)\right) \tag{2.1}$$

where $\delta_c(z)$ is some function of $z$ with $|\delta_c(z)| < \frac{\tilde{\lambda} c^{\frac{3}{2}} |z|}{|z-1|(|z|-1)}$ and $\tilde{\lambda} \in [0, \infty)$ is a constant that depends only on the family of particles and not on $c$ nor $z$. In [**NST19**], Norris et al show there exists some absolute constant $c_0$ such that, provided $0 \leqslant c < c_0$, families of slit maps, bump maps and indeed many other natural choices, satisfy the condition. Therefore, we take our single particle mappings from a class of particles satisfying (2.1) for fixed $\tilde{\lambda}$. In the proofs that follow it will become clear that our results do not depend on the precise value of $\delta_c(z)$.

Now it just remains to define how the attaching points $\theta_n$ and capacities $c_n$ are chosen. We want to model Laplacian growth and so we choose the $\theta_n$ to be uniformly distributed, independent for each $n$, on the circle. This choice is made because after renormalisation of $\phi_n$, the Lebesgue measure of the unit circle under the image of $\phi_n$ is harmonic measure as seen from infinity [**RZ05**], and the harmonic measure of a portion of the unit circle is just the arclength of that portion rescaled by $2\pi$.

Finally, we must choose how the capacities $c_n$ are distributed. Hastings and Levitov [**HL98**] introduced a parameter $\alpha$ in order to distinguish between the various individual models they would like to encode within this one model for Laplacian growth. They choose,

$$c_n = c|\phi'_{n-1}(e^{i\theta_n})|^{-\alpha}$$

for some $c > 0$. This gives an off-lattice version of the Eden model when $\alpha = 1$ and DLA when $\alpha = 2$. In Section 2.3, we show that the total logarithmic capacity, $\phi'_n(\infty)$ is well approximated by $(1+\alpha cn)^{\frac{1}{\alpha}}$. Therefore, if we define a version of HL($\alpha$) using the very strong regularisation $\tilde{c}_n = c|\phi'_{n-1}(\infty)|^{-\alpha}$, we show in Proposition 2.3.1 that $\tilde{c}_n$ is approximately given by

$$c_n^* := \frac{c}{1 + \alpha c(n-1)}. \tag{2.2}$$

Therefore, for a lot of computations we do the analysis using the deterministic sequence $c_n^*$ rather than $\tilde{c}_n$. In particular, in what follows, we denote $\phi_n = f_1 \circ ... \circ f_n$ where $f_n(z) = e^{i\theta_n} f_{c_n^*}(ze^{-i\theta_n})$ with $\theta_n$ i.i.d uniform on $[0, 2\pi]$. Throughout the paper we keep $c$ fixed. Occasionally we may require $c$ to be bounded by some constant which may depend on $\alpha$ but, crucially, not on $n$. We then rescale the cluster by its total logarithmic capacity and evaluate the shape of the rescaled map $e^{-\sum_{i=1}^n c_i^*} \phi_n$ as $n \to \infty$.

**2.1.2. Previous work.** With the model now defined we can outline the work already done in this area. Most work has been done in the small-particle limit. This method involves evaluating the limiting cluster $\phi_n$ as we send the particle capacity $c \to 0$ while sending $n \to \infty$ with $nc \sim t$ for some $t$. Using this method Turner and Norris show that for $\alpha = 0$ the limiting cluster in the small particle case behaves like a growing disk [**NT12**]. Furthermore, Turner, Viklund and Sola show that in the small particle limit the shape of the cluster in a regularised setting approaches a circle for all $\alpha \geqslant 0$ provided the regularisation is sufficient [**JVST15**]. Moreover, Silvestri [**Sil17**] shows that the fluctuations on the boundary, for HL(0), in this small particle limit can be characterised by a log-correlated Gaussian field.

A different approach to that of the small-particle limit is to not let $c \to 0$ as $n \to \infty$, but instead, the limit of the cluster is found by rescaling the whole cluster by the logarithmic capacity of the cluster at time $n$, before taking limits as the number of particles tends to infinity. Rohde and Zinsmeister show that in the case of $\alpha = 0$ the rescaled cluster converges to a (random) limit with respect to the topology of normalised exterior Riemann maps [**RZ05**].

Our work will follow the second approach. We will use results and ideas from the papers listed above, and in particular methods from [**NST19**], in order to characterise the limiting shape of the cluster in a regularised setting for $0 \leqslant \alpha < 2$ and then evaluate the fluctuations. Our results break down for $\alpha \geqslant 2$. This will be the subject of future work.

**2.1.3. Statement of results.** We first consider the case where $\alpha = 0$ and show that under capacity rescaling, if the the limiting rescaled cluster exists then it can not be a disk. This comes in the form of the following theorem[1] appearing later as Theorem 2.2.1.

THEOREM. *Given any sequence $\{\theta_k\}_{1\leqslant k\leqslant n}$ of angles between $0$ and $2\pi$ and $c > 0$, set $\Psi_n = f_1 \circ ... \circ f_n$ where $f_k(z) = e^{i\theta_k} f_c(e^{-i\theta_k} z)$ and let $f_c(z)$ be any fixed capacity map such that $f_n(z) \neq e^c z$. There exists an $\epsilon > 0$ such that for all $r > 1$ and $c > 0$,*

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| > \epsilon.$$

*In particular if $\{\theta_k\}_{1\leqslant k\leqslant n}$ are i.i.d uniform on $[0, 2\pi]$ and $f_c(z)$ is a fixed capacity map in the class of particles given by (2.1) then $\Psi_n$ is the $HL(0)$ process and the statement above shows that $HL(0)$ does not converge to a disk under capacity rescaling.*

This result is particularly interesting because it is independent of our choice of angles. If we have a constant capacity map of the right form then there is no possible way to choose the angles so that under capacity rescaling the limiting cluster (should it exist) looks like a disk.

Next we consider the case where $0 < \alpha < 2$ and show that under capacity rescaling the $HL(\alpha)$ cluster approaches a disk. We then evaluate the fluctuations and show that they behave like a Gaussian field dependent on $\alpha$. Our two main results, appearing later as Theorem 2.5.1 and Theorem 2.6.10 respectively, are stated as follows.

THEOREM. *For $0 < \alpha < 2$, let the map $\phi_n$ be defined as above with $c_n^*$ as defined in (2.2) and $\theta_n$ i.i.d uniform on $[0, 2\pi]$. Then for any $r > 1$,*

$$\mathbb{P}\left(\limsup_{n\to\infty}\left\{\sup_{|z|\geqslant r} |e^{-\sum_{i=1}^{n} c_i^*}\phi_n(z) - z| > \frac{\log n}{\sqrt{n}}\right\}\right) = 0.$$

This result tells us we have uniform convergence of our cluster in the exterior disk to a disk. The following result shows that the fluctuations behave like a Gaussian field.

[1]Note that we make no assumption on the choice of angles $\{\theta_k\}_{1\leqslant k\leqslant n}$ in this theorem and so we use the notation $\Psi_n$ to differentiate from $\phi_n$ where the angles are chosen uniformly.

THEOREM. *Let $0 < \alpha < 2$ and $\phi_n$ be defined as in Theorem 2.5.1. Then as $n \to \infty$,*

$$\sqrt{n}\left(e^{-\sum_{i=1}^{n} c_i^*}\phi_n(z) - z\right) \to \mathcal{F}(z)$$

*in distribution on $\mathcal{H}$, where $\mathcal{H}$ is the space of holomorphic functions on $|z| > 1$, equipped with a suitable metric $\boldsymbol{d}_{\mathcal{H}}$ defined later, and where*

$$\mathcal{F}(z) = \sum_{m=0}^{\infty} (A_m + iB_m) z^{-m}$$

*with $A_m$, $B_m \sim \mathcal{N}\left(0, \frac{2}{\alpha(2m+2-\alpha)}\right)$ and $A_m$, $B_k$ independent for all choices of $m$ and $k$.*

Notice that it is clear this result does not hold for $\alpha = 0$ or $\alpha = 2$. This is in contrast to **[JVST15]** where results hold for all $\alpha \geqslant 0$ and suggests a phase transition at these values.

**2.1.4. Outline of the paper.** The outline of the paper is as follows. In Section 2.2 we will show that for clusters formed by composing maps of constant capacity and of a certain form, we can not pick a sequence of angles so that the limiting cluster under capacity rescaling approaches a disk. In particular, under capacity rescaling HL(0) is not a growing disk. Then in Section 2.3 we will show that our choice of capacities is a good approximation to the regularisation of HL($\alpha$) at $\infty$. In Section 2.4, we show that the pointwise limit of the cluster for $0 < \alpha < 2$ is a disk and then in Section 2.5 we will use a Borel-Cantelli argument to show we have uniform convergence on the exterior disk. Finally, in Section 2.6 we will evaluate the fluctuations for $0 < \alpha < 2$ and show that they are distributed according to a Gaussian field dependent on $\alpha$.

## 2.2. The case where $\alpha = 0$

We want to evaluate the limiting shape of our random cluster. We first deal with the case where $\alpha = 0$. We will show in this section that in the limit HL(0) does not approach a disk. Furthermore, we will prove a stronger statement that for clusters formed by composing maps of constant capacity, in the class of particles defined in (2.1), we can not approach a disk under capacity rescaling. We note that in the case where $\alpha = 0$ our regularisation does not effect the model, so this result holds for HL(0) under no regularisation. One might expect that the scaling limit is a growing disk, this would agree with the result in the small particle limit [**NT12**]. However, the following theorem proves this does not hold.

THEOREM 2.2.1. *Given any sequence $\{\theta_k\}_{1\leqslant k\leqslant n}$ of angles between 0 and $2\pi$ and $c>0$, set $\Psi_n = f_1 \circ ... \circ f_n$ where $f_k(z) = e^{i\theta_k} f_c(e^{-i\theta_k}z)$ and let $f_c(z)$ be any fixed capacity map such that $f_n(z) \neq e^c z$. There exists an $\epsilon > 0$ such that for all $r > 1$ and $c > 0$,*

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| > \epsilon.$$

*In particular if $\{\theta_k\}_{1\leqslant k\leqslant n}$ are i.i.d uniform on $[0,2\pi]$ and $f_c(z)$ is a fixed capacity map in the class of particles given by (2.1) then $\Psi_n$ is the HL$(0)$ process and the statement above shows that HL$(0)$ does not converge to a disk under capacity rescaling.*

PROOF. Under our assumptions we know that for all $r>1$ there exists $\epsilon_r > 0$ such that

$$\sup_{|z|>r} \left|e^{-c} f_n(z) - z\right| = \epsilon_r.$$

Suppose for a contradiction that for all $\epsilon > 0$,

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| < \epsilon.$$

In particular, under this assumption,

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| < \frac{\epsilon_r}{2}.$$

Then we can write,

$$|e^{-cn}\Psi_n(z) - z| = \left|\left(e^{-c}f_n(z) - z\right) + e^{-c}\left(e^{-c(n-1)}\Psi_{n-1}(f_n(z)) - f_n(z)\right)\right|$$

which we can bound below for all $|z|>r$ as follows,

$$|e^{-cn}\Psi_n(z) - z| \geqslant |e^{-c}f_n(z) - z| - \sup_{|z|>r} |e^{-c}||e^{-c(n-1)}\Psi_{n-1}(f_n(z)) - f_n(z)|.$$

We can then take the supremum of both sides, and by the Schwarz lemma we can use that $|f_n(z)| > r$ for all $|z| > r$, to reach the following bound on the supremum,

$$\sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| \geqslant \sup_{|z|>r} |e^{-c}f_n(z) - z| - \sup_{|z|>r} |e^{-c(n-1)}\Psi_{n-1}(z) - z|. \tag{2.3}$$

Therefore,

$$\limsup_{n\to\infty} \sup_{|z|>r} |e^{-cn}\Psi_n(z) - z| \geqslant \frac{\epsilon_r}{2}$$

a contradiction. □

This is a strong result because it proves that if we have a cluster which is composed of functions of the right form, no matter how we pick our sequence of attaching angles $\{\theta_n\}$ the limiting structure of the cluster, when rescaled by its logarithmic capacity, does not approach a disk.

## 2.3. Regularisation

The aim of this section is to provide some justification for the choice of $c_n^*$ as an approximation to the regularisation of HL($\alpha$) at $\infty$. Recall that we choose,

$$c_n^* = \frac{c}{1+\alpha c(n-1)}.$$

We start by providing some notation used throughout the remainder of the paper. Let $\phi_k$ and $c_i^*$ be defined as above, then we denote

$$C_{k,n}^* = \sum_{i=k}^{n} c_i^*.$$

**2.3.1. Error term evaluation.** In order to more easily apply complex analysis methods to our cluster we would like to write the sum $C_{1,n}^*$ in a simplified form. We do so by providing the following approximation on the sum, subject to an error term which converges to 0, uniformly in $k$, as $n \to \infty$.

Lemma 2.3.1. *For $c_n^* = \frac{c}{1+\alpha c(n-1)}$ we have the following equality;*

$$C_{k,n}^* = \frac{1}{\alpha}\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)(1+\epsilon_{k,n})$$

*where*

$$0 < \epsilon_{k,n} < \frac{\alpha^2 c^2 (n-k+1)}{(1+\alpha c(k-1))(1+\alpha cn)\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)} \leqslant \frac{\alpha c}{\log(1+\alpha cn)}.$$

*Therefore,* $\epsilon_{k,n} \to 0$*, uniformly in* $k$*, as* $n \to \infty$*.*

Proof. We will approximate the sum with

$$\frac{1}{\alpha}\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right) = \int_k^{n+1} \frac{c}{1+\alpha c(x-1)} dx.$$

Then

$$\begin{aligned} C^*_{k,n} - \frac{1}{\alpha}\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right) &= \sum_{i=k}^{n}\left(c^*_i - \int_i^{i+1}\frac{c}{1+\alpha c(x-1)}dx\right) \\ &\leqslant \sum_{i=k}^{n}\left(c^*_i - c^*_{i+1}\right) \\ &= \frac{\alpha c^2(n-k+1)}{(1+\alpha c(k-1))(1+\alpha cn)}. \end{aligned}$$

Thus,

$$0 < \epsilon_{k,n} < \frac{\alpha^2 c^2(n-k+1)}{(1+\alpha c(k-1))(1+\alpha cn)\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)}.$$

So we consider,

$$\begin{aligned} &\sup_{k\leqslant n} \frac{\alpha^2 c^2(n-k+1)}{(1+\alpha c(k-1))(1+\alpha cn)\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)} \\ &= \frac{\alpha^2 c^2}{1+\alpha cn}\sup_{k\leqslant n}\frac{n-k+1}{(1+\alpha c(k-1))\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)}. \end{aligned}$$

So let us find,

$$\sup_{k\leqslant n}\frac{n-k+1}{(1+\alpha c(k-1))\log\left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)}.$$

Let $x = 1+\alpha c(k-1)$ and find the derivative

$$\frac{d}{dx}\left(\frac{1+\alpha cn - x}{x\log\left(\frac{1+\alpha cn}{x}\right)}\right) = \frac{(1+\alpha cn) - (1+\alpha cn)\log\left(\frac{1+\alpha cn}{x}\right) - x}{x^2\left(\log\left(\frac{1+\alpha cn}{x}\right)\right)^2}.$$

The numerator in this fraction is increasing and from this it is clear that the derivative is negative. Therefore the maximum occurs when $k = 1$. Thus,

$$0 \leqslant \epsilon_{k,n} \leqslant \frac{\alpha^2 c^2}{1+\alpha cn} \frac{n}{\log(1+\alpha cn)} \leqslant \frac{\alpha c}{\log(1+\alpha cn)}.$$

Furthermore, taking the limit as $n \to \infty$ we have $\epsilon_{k,n} \to 0$, uniformly in $k$, as claimed. □

The following corollary provides a nice bound on $(1+\alpha ck)^{1+\epsilon_{k,n}}$ which will make computations in later sections easier.

Corollary 2.3.2. *Let $\epsilon_{k,n}$ be defined as in Lemma 2.3.1. Then for $1 \leqslant k \leqslant n$ and $\alpha \geqslant 0$ the following bound holds,*

$$(1+\alpha ck)^{1+\epsilon_{k,n}} \leqslant (1+\alpha c e^{\alpha c})(1+\alpha ck).$$

Proof. We can write

$$(1+\alpha ck)^{1+\epsilon_{k,n}} = (1+\alpha ck)(1+\alpha ck)^{\epsilon_{k,n}} = (1+\alpha ck)(1+(1+\alpha ck)^{\epsilon_{k,n}}-1).$$

So let $\delta_{k,n} = (1+\alpha ck)^{\epsilon_{k,n}} - 1$, then

$$\delta_{k,n} = (e^{\epsilon_{k,n}\log(1+\alpha ck)} - 1) \leqslant \epsilon_{k,n}\log(1+\alpha ck)e^{\epsilon_{k,n}\log(1+\alpha ck)}.$$

We have just shown that

$$|\epsilon_{k,n}| \leqslant \frac{\alpha c}{\log(1+\alpha cn)}.$$

So,

$$0 \leqslant |\delta_{k,n}| \leqslant \alpha c e^{\alpha c}.$$

Therefore,

$$(1+\alpha ck)^{1+\epsilon_{k,n}} \leqslant (1+\alpha ck)(1+\alpha c e^{\alpha c}).$$

□

**2.3.2. Regularisation approximation.** With the estimates provided above we can now provide justification for our choice of $c_n^*$. We start by providing some more notation. For each $n \in \mathbb{N}$, $c$ defined as above we denote $\phi_n^\infty = \phi_{n-1}^\infty \circ f_n^\infty$ where $f_n^\infty(z) = e^{i\theta_n} f_{\tilde{c}_n}(ze^{-i\theta_n})$

with $\theta_n$ i.i.d uniform on $[0, 2\pi]$ and

$$\tilde{c}_n = \frac{c}{\left|\left(\phi_{n-1}^{\infty}\right)'(\infty)\right|^{\alpha}}.$$

Furthermore, we define,

$$\tilde{C}_{k,n} = \sum_{i=k}^{n} \tilde{c}_i.$$

The maps $\phi_n^{\infty}$ correspond to the true model for HL($\alpha$) regularised at $\infty$. The aim of the remainder of this section will be to prove the following theorem.

PROPOSITION 2.3.3. *For $C_{1,n}^*$ and $\tilde{C}_{1,n}$ defined as above, the following inequality holds,*

$$\left|C_{1,n}^* - \tilde{C}_{1,n}\right| \leqslant 12c$$

*Furthermore,*

$$\tilde{c}_n = c_n^*(1 + \epsilon_n^{\infty})$$

*where $\epsilon_n^{\infty} \to 0$, uniformly in $n$, as $c \to 0$.*

Therefore if we choose our $c$ sufficiently small we see that our regularisation is a good approximation to regularisation at infinity. In order to prove Proposition 2.3.3 we first form a difference equation on $C_{1,n}^*$.

LEMMA 2.3.4. *With $C_{1,n}^*$ defined as above the following equality holds*

$$C_{1,n}^* = C_{1,n-1}^* + ce^{-\alpha C_{1,n-1}^*} + \kappa_n$$

*where $0 \leqslant \kappa_n \leqslant \frac{2\alpha c^2}{1+\alpha c(n-1)}$.*

PROOF. Let

$$\kappa_n = \left(C_{1,n}^* - C_{1,n-1}^*\right) - ce^{-\alpha C_{1,n-1}^*}.$$

Then by the definition of $C_{1,n}^*$,

$$\kappa_n = c_n^* - ce^{-\alpha C_{1,n-1}^*}.$$

Thus, using the approximation from Lemma 2.3.1,

$$\begin{aligned}
\kappa_n &= c_n^* - \frac{c}{(1+\alpha c(n-1))^{1+\epsilon_{1,n-1}}} \\
&= \frac{c}{1+\alpha c(n-1)}\left(1 - \frac{1}{(1+\alpha c(n-1))^{\epsilon_{1,n-1}}}\right) \\
&= \frac{c}{1+\alpha c(n-1)}\left(1 - \exp\left(-\epsilon_{1,n-1}\log(1+\alpha c(n-1))\right)\right).
\end{aligned}$$

Since $\epsilon_{1,n-1}$ is small for small $c$ we can Taylor expand the exponential to get,

$$\kappa_n = \frac{c}{1+\alpha c(n-1)}\left(\epsilon_{1,n-1}\log(1+\alpha c(n-1)) - r(n,c)\right).$$

where $r(n,c)$ is the remainder term in the Taylor expansion. From Lemma 2.3.1 we know $0 \leqslant \epsilon_{1,n-1} \leqslant \frac{\alpha c}{\log(1+\alpha c(n-1))}$. Moreover, $0 \leqslant r(n,c) \leqslant e^{\alpha c}(\epsilon_{1,n-1}\log(1+\alpha c(n-1)))^2$, so for $c$ sufficiently small,

$$0 \leqslant \kappa_n \leqslant \frac{2\alpha c^2}{1+\alpha c(n-1)}.$$

□

We can now show that $C_{1,n}^*$ and $\tilde{C}_{1,n}$ are sufficiently close by proving Proposition 2.3.3.

Proof of Proposition 2.3.3. We will prove the statement inductively. By definition, $C_{1,1}^* - \tilde{C}_{1,1} = 0$. So assume,

$$\left|C_{1,n-1}^* - \tilde{C}_{1,n-1}\right| \leqslant 12c.$$

Then note that since,

$$\tilde{c}_n = \frac{c}{\left|\left(\phi_{n-1}^{\infty}\right)'(\infty)\right|^{\alpha}}$$

then

$$\tilde{c}_n = \frac{c}{\left(e^{\tilde{C}_{1,n-1}}\right)^{\alpha}}.$$

Furthermore,

$$\tilde{C}_{1,n} = \tilde{C}_{1,n-1} + \tilde{c}_n.$$

Therefore,

$$\tilde{C}_{1,n} = \tilde{C}_{1,n-1} + \frac{c}{\left(e^{\tilde{C}_{1,n-1}}\right)^{\alpha}}.$$

Thus, by Lemma 2.3.4,

$$\begin{aligned} C^*_{1,n} - \tilde{C}_{1,n} &= \left(C^*_{1,n-1} - \tilde{C}_{1,n-1}\right) + c\left(e^{-\alpha C^*_{1,n-1}} - e^{-\alpha \tilde{C}_{1,n-1}}\right) + \kappa_n \\ &= \left(C^*_{1,n-1} - \tilde{C}_{1,n-1}\right) + ce^{-\alpha C^*_{1,n-1}}\left(1 - e^{\alpha\left(C^*_{1,n-1} - \tilde{C}_{1,n-1}\right)}\right) + \kappa_n. \end{aligned}$$

Taylor expanding the $e^{\alpha\left(C^*_{1,n-1} - \tilde{C}_{1,n-1}\right)}$ term gives,

$$\begin{aligned} C^*_{1,n} - \tilde{C}_{1,n} &= \left(C^*_{1,n-1} - \tilde{C}_{1,n-1}\right) + c\alpha e^{-\alpha C^*_{1,n-1}}\left(\tilde{C}_{1,n-1} - C^*_{1,n-1} - r(n,c)\right) + \kappa_n \\ &= \left(C^*_{1,n-1} - \tilde{C}_{1,n-1}\right)\left(1 - c\alpha e^{-\alpha C^*_{1,n-1}}\right) + \left(\kappa_n - r(n,c)c\alpha e^{-\alpha C^*_{1,n-1}}\right). \end{aligned}$$

where $r(n,c)$ is the Taylor remainder term. We know $r(n,c) = \frac{e^{\xi}}{2}\alpha\left(C^*_{1,n-1} - \tilde{C}_{1,n-1}\right)^2$ for some $\xi$ between 0 and $\alpha\left(C^*_{1,n-1} - \tilde{C}_{1,n-1}\right)$. Thus, under our assumption that $|C^*_{1,n-1} - \tilde{C}_{1,n-1}| \leqslant 12c$, we have,

$$0 \leqslant r(n,c)c\alpha e^{-\alpha C^*_{1,n-1}} \leqslant \frac{144c^3\alpha^2 e^{12\alpha c}}{1 + \alpha c(n-1)}.$$

Then if $c$ is small enough,

$$-r(n,c)c\alpha e^{-\alpha C^*_{1,n-1}} \geqslant \frac{-2\alpha c^2}{1 + \alpha c(n-1)}.$$

Let $\tilde{\kappa}_n = \kappa_n - r(n,c)c\alpha e^{-\alpha C^*_{1,n-1}}$, then $\frac{-2\alpha c^2}{1+\alpha c(n-1)} \leqslant \tilde{\kappa}_n \leqslant \frac{2\alpha c^2}{1+\alpha c(n-1)}$. Hence,

$$C^*_{1,n} - \tilde{C}_{1,n} = \left(C^*_{1,n-1} - \tilde{C}_{1,n-1}\right)\rho_{n-1} + \tilde{\kappa}_n$$

where $\rho_{n-1} = 1 - c\alpha e^{-\alpha C^*_{1,n-1}}$. So,

$$\begin{aligned} C^*_{1,n} - \tilde{C}_{1,n} &= \left(C^*_{1,n-2} - \tilde{C}_{1,n-2}\right)\rho_{n-2}\rho_{n-1} + \tilde{\kappa}_{n-1}\rho_{n-1} + \tilde{\kappa}_n \\ &= \left(C^*_{1,1} - \tilde{C}_{1,1}\right)\prod_{i=1}^{n-1}\rho_i + \sum_{j=2}^{n-1}\left(\tilde{\kappa}_j \prod_{k=j}^{n-1}\rho_k\right) + \tilde{\kappa}_n \end{aligned}$$

but since $\left(C^*_{1,1} - \tilde{C}_{1,1}\right) = 0$,

$$C^*_{1,n} - \tilde{C}_{1,n} = \sum_{j=2}^{n-1}\left(\tilde{\kappa}_j \prod_{k=j}^{n-1}\rho_k\right) + \tilde{\kappa}_n.$$

We first analyse $\prod_{k=j}^{n-1}\rho_k$,

$$\begin{aligned} \prod_{k=j}^{n-1}\rho_k &= \prod_{k=j}^{n-1}\left(1 - c\alpha e^{-\alpha C^*_{1,k-1}}\right) \\ &= \prod_{k=j}^{n-1}\left(1 - \frac{\alpha c}{(1+\alpha c(k-1))^{1+\epsilon_{1,k-1}}}\right) \\ &= \exp\left(\sum_{k=j}^{n-1}\log\left(1 - \frac{\alpha c}{(1+\alpha c(k-1))^{1+\epsilon_{1,k-1}}}\right)\right). \end{aligned}$$

Using the Taylor expansion of $\log\left(1 - \frac{\alpha c}{(1+\alpha c(k-1))^{1+\epsilon_{1,k-1}}}\right)$ we have,

$$\prod_{k=j}^{n-1}\rho_k = \exp\left(\sum_{k=j}^{n-1}\frac{-\alpha c}{(1+\alpha c(k-1))^{1+\epsilon_{1,k-1}}}\right)\exp\left(\sum_{k=j}^{n-1}\tilde{r}(k,c)\right)$$

where $\tilde{r}(k,c)$ is the Taylor remainder term. But since for each $2 \leqslant k \leqslant n-1$, $\sum_{k=j}^{n-1}\tilde{r}(j,c) \leqslant 0$ in the expansion of $\log\left(1 - \frac{\alpha c}{(1+\alpha c(k-1))^{1+\epsilon_{1,k-1}}}\right)$,

$$0 \leqslant \prod_{k=j}^{n-1}\rho_k \leqslant \exp\left(\sum_{k=j}^{n-1}\frac{-\alpha c}{(1+\alpha c(k-1))^{1+\epsilon_{1,k-1}}}\right). \tag{2.4}$$

By Corollary 2.3.2,

$$0 \leqslant (1+\alpha c(k-1))^{1+\epsilon_{1,k-1}} \leqslant (1+\alpha c e^{\alpha c})(1+\alpha c(k-1)).$$

Therefore,

$$\frac{\alpha c}{(1+\alpha c(k-1))^{1+\epsilon_{1,k-1}}} \geqslant \frac{\alpha c}{(1+\alpha c e^{\alpha c})(1+\alpha c(k-1))}.$$

Thus,

$$\begin{aligned}
\sum_{k=j}^{n-1} \frac{\alpha c}{(1+\alpha c(k-1))^{1+\epsilon_{1,k-1}}} &\geqslant \frac{1}{(1+\alpha c e^{\alpha c})} \sum_{k=j}^{n-1} \frac{\alpha c}{(1+\alpha c(k-1))} \\
&\geqslant \frac{1}{(1+\alpha c e^{\alpha c})} \int_j^n \frac{\alpha c}{(1+\alpha c(x-1))} dx \\
&= \frac{1}{(1+\alpha c e^{\alpha c})} \log\left(\frac{1+\alpha c(n-1)}{1+\alpha c(j-1)}\right).
\end{aligned}$$

where the second inequality follows using a Riemann sum approximation. Hence by (2.4),

$$0 \leqslant \prod_{k=j}^{n-1} \rho_k \leqslant \left(\frac{1+\alpha c(j-1)}{1+\alpha c(n-1)}\right)^{\frac{1}{(1+\alpha c e^{\alpha c})}} \leqslant \left(\frac{1+\alpha c(j-1)}{1+\alpha c(n-1)}\right)^{\frac{1}{2}}$$

for $c$ chosen sufficiently small. Finally we see that,

$$\begin{aligned}
\sum_{j=2}^{n-1}\left(\tilde{\kappa}_j \prod_{k=j}^{n-1} \rho_k\right) &\leqslant \frac{2\alpha c^2}{(1+\alpha c(n-1))^{\frac{1}{2}}} \sum_{j=2}^{n-1} \frac{1}{(1+\alpha c(j-1))^{\frac{1}{2}}} \\
&\leqslant 6c
\end{aligned}$$

where the last inequality follows by approximating with a Riemann integral. Thus for $c$ small enough,

$$\left|\sum_{j=2}^{n-1}\left(\tilde{\kappa}_j \prod_{k=j}^{n-1} \rho_k\right)\right| \leqslant 6c.$$

Hence, since for all $n$, and $c$ sufficiently small,

$$|\tilde{\kappa}_n| \leqslant 6c$$

it follows that,

$$\left|C^*_{1,n} - \tilde{C}_{1,n}\right| \leqslant 12c. \tag{2.5}$$

Finally, consider $\frac{\tilde{c}_n}{c_n^*}$,

$$\frac{\tilde{c}_n}{c_n^*} = \frac{(1+\alpha c(n-1))}{e^{\alpha \tilde{C}_{1,n}}}$$
$$= e^{\alpha\left(\frac{1}{\alpha}\log(1+\alpha c(n-1)) - C_{1,n}^*\right)} e^{\alpha\left(C_{1,n}^* - \tilde{C}_{1,n}\right)}$$

Thus by (2.5),

$$\frac{\tilde{c}_n}{c_n^*} \leqslant e^{\alpha\left(\frac{1}{\alpha}\log(1+\alpha c(n-1)) - C_{1,n-1}^*\right)} e^{12\alpha c}.$$

Therefore, using the bound in Lemma 2.3.1,

$$\frac{\tilde{c}_n}{c_n^*} \leqslant e^{-\epsilon_{1,n-1}\log(1+\alpha c(n-1))} e^{12\alpha c}$$
$$\leqslant e^{12\alpha c}.$$

Thus,

$$\tilde{c}_n = c_n^*(1+\epsilon_n^\infty)$$

where $\epsilon_n^\infty \to 0$ uniformly as $c \to 0$. □

Now define the following measures on the space $S = [0, 2\pi] \times [0, \infty)$ ,

$$d\mu_c^*(\theta, t) = \delta_{\xi_c^*(t)} dt, \quad d\tilde{\mu}_c(\theta, t) = \delta_{\tilde{\xi}_c(t)} dt$$

where,

$$\xi_c^*(t) = \exp\left(i \sum_{k=1}^{\infty} \theta_k \mathbb{1}_{[C_{1,k-1}^*, C_{1,k}^*]}(t)\right), \quad \tilde{\xi}_c(t) = \exp\left(i \sum_{k=1}^{\infty} \theta_k \mathbb{1}_{[\tilde{C}_{1,k-1}, \tilde{C}_{1,k}]}(t)\right).$$

Using the theory of Loewner chains (see, for example, Section 7 of [**JVST15**]), $\phi_n$ is a good approximation to $\phi_n^\infty$ provided the measures $\mu_c^*$ and $\tilde{\mu}_c$ are close in the sense stated in Corollary 2.3.5 below. For a function $g$ and a measure $\mu$, denote,

$$\langle g, \mu \rangle = \int_S g(\theta, t) d\mu(\theta, t).$$

It follows that, for the measures $\mu_c^*(\theta,t)$, $\tilde{\mu}_c(\theta,t)$ defined above,

$$\langle g,\mu_c^*\rangle = \sum_{k=1}^{\infty}\int_{C^*_{1,k-1}}^{C^*_{1,k}} g(\theta_k,t)dt, \quad \langle g,\tilde{\mu}_c\rangle = \sum_{k=1}^{\infty}\int_{\tilde{C}_{1,k-1}}^{\tilde{C}_{1,k}} g(\theta_k,t)dt.$$

Then the following corollary holds.

Corollary 2.3.5. *Let $g : S \to \mathbb{R}$ be a continuous function with compact support. Then,*

$$|\langle g,\mu_c^*\rangle - \langle g,\tilde{\mu}_c\rangle| \to 0$$

*uniformly as $c \to 0$.*

Proof. Since $g$ has compact support, there exists some $0 < T < \infty$ such that $g(x,t) = 0$ whenever $t > T$. Thus,

$$|\langle g,\mu_c^*\rangle - \langle g,\tilde{\mu}_c\rangle| = \left|\sum_{k=1}^{k_T}\int_{C^*_{1,k-1}}^{C^*_{1,k}} g(\theta_k,t)dt - \sum_{k=1}^{k_T}\int_{\tilde{C}_{1,k-1}}^{\tilde{C}_{1,k}} g(\theta_k,t)dt\right|.$$

where $k_T = \inf\{k : C^*_{1,k} \wedge \tilde{C}_{1,k} > T\}$. By the continuity of the function $g$ there exists $s_k^* \in [C^*_{1,k-1}, C^*_{1,k}]$ and $\tilde{s}_k \in [\tilde{C}_{1,k-1}, \tilde{C}_{1,k}]$ such that,

$$|\langle g,\mu_c^*\rangle - \langle g,\tilde{\mu}_c\rangle| \leqslant \sum_{k=1}^{k_T} |c_k^* g(\theta_k, s_k^*) - \tilde{c}_k g(\theta_k,\tilde{s}_k)|\,.$$

We can bound the term in the summation as follows,

$$\begin{aligned}|c_k^* g(\theta_k, s_k^*) - \tilde{c}_k g(\theta_k,\tilde{s}_k)| \leqslant & c_k^* |g(\theta_k,s_k^*) - g(\theta_k,\tilde{s}_k)| + |c_k^* - \tilde{c}_k|\,|g(\theta_k,\tilde{s}_k)| \\ \leqslant & c_k^* \left(\sup_{|s-t|<14c} |g(\theta_k,s) - g(\theta_k,t)| + \epsilon_k^\infty \|g\|_\infty\right)\end{aligned}$$

where $\epsilon_k^\infty$ is the uniform bound from Proposition 2.3.3. Therefore, since bounded continuous functions on compact time are uniformly continuous we can find a uniform bound on the first term and hence a uniform bound on the sum,

$$\begin{aligned}|\langle g,\mu_c^*\rangle - \langle g,\tilde{\mu}_c\rangle| \leqslant & \left(\sum_{k=1}^{k_T} c_k^*\right)\left(\sup_{|s-t|<14c,\theta\in[0,2\pi]} |g(\theta,s) - g(\theta,t)| + \sup_{0\leqslant k<\infty} \epsilon_k^\infty \|g\|_\infty\right) \\ \leqslant & (T+12c)\left(\sup_{|s-t|<14c,\theta\in[0,2\pi]} |g(\theta,s) - g(\theta,t)| + \sup_{0\leqslant k<\infty} \epsilon_k^\infty \|g\|_\infty\right)\end{aligned}$$

which converges to 0 uniformly as $c \to 0$.

□

As a remark we note that it is straightforward to prove the almost sure version of Corollary 2.3.5. For notational simplicity all subsequent results are proved for $\phi_n$, however, it is straightforward to verify that $c$ can be chosen sufficiently small such that analogous results hold for $\phi_n^\infty$.

## 2.4. Pointwise convergence for $0 < \alpha < 2$

**2.4.1. Estimates.** In this section we will provide estimates for several variables which we will then call on throughout the rest of the paper. Whilst this work is an essential part of the analysis, we advise that the reader may skip the proofs of this section if they are only interested the main results of the paper.

We start by providing some notation used throughout the remainder of the paper. Let $\phi_k$ and $c_i^*$ be defined as above. Recall, we denote $C_{k,n}^* = \sum_{i=k}^n c_i^*$. Then for any $z \in \mathbb{C}$ we define our increments $X_{k,n}(z)$ as;

$$X_{k,n}(z) := e^{-C_{1,n}^*}\left(\phi_k\left(e^{C_{k+1,n}^*}z\right) - \phi_{k-1}\left(e^{C_{k,n}^*}z\right)\right). \tag{2.6}$$

Let $\mathcal{F}_{k-1}$ be the $\sigma$-algebra, $\sigma\left(\theta_i\,;1 \leqslant i \leqslant k-1\right)$. We first show that for all $0 < k \leqslant n$,

$$\mathbb{E}(X_{k,n}(z)|\mathcal{F}_{k-1}) = 0.$$

This is shown in the following lemma and highlights the power of using conformal maps.

LEMMA 2.4.1. *Define the sequence $\{X_{k,n}(z)\}_{k=0}^n$ and corresponding filtration $(\mathcal{F}_k)_{k=0}^n$ as above. For each $z \in \mathbb{C}$, the following property is satisfied for all $0 < k \leqslant n$,*

$$\mathbb{E}(X_{k,n}(z)|\mathcal{F}_{k-1}) = 0.$$

PROOF. We first show;

$$\int_0^{2\pi} \phi_{k-1}(e^{i\theta} f_{c_k^*}(e^{-i\theta}z))\frac{d\theta}{2\pi} = \phi_{k-1}(e^{c_k^*}z).$$

Let $w = e^{i\theta}$, then the integral can be rewritten as

$$\int_0^{2\pi} \phi_{k-1}(e^{i\theta} f_{c_k^*}(e^{-i\theta}z))\frac{d\theta}{2\pi} = \frac{1}{2\pi i}\int_C \frac{\phi_{k-1}(wf_{c_k^*}(z/w))}{w}dw$$

where $C$ is the unit circle centered at 0. The map $w \longmapsto \phi_{k-1}(wf_{c_k^*}(z/w))$ is analytic on the disk of radius $\frac{1}{|z|}$ with a removable singularity at 0 and so by Cauchy's integral formula,

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{\phi_{k-1}(wf_{c_k^*}(z/w))}{w}dw &= \lim_{w\to 0}\phi_{k-1}(wf_{c_k^*}(z/w)) \\ &= \phi_{k-1}(\lim_{w\to 0} wf_{c_k^*}(z/w)) \\ &= \phi_{k-1}\left(\lim_{w\to 0}\left(e^{c_k^*}z + a_0 w + a_1\frac{w^2}{z^2} + ...\right)\right)\end{aligned}$$

for some complex number sequence of $a_i$'s. Thus,

$$\int_0^{2\pi} \phi_{k-1}(e^{i\theta} f_{c_k^*}(e^{-i\theta}z))\frac{d\theta}{2\pi} = \phi_{k-1}(e^{c_k^*}z)$$

as required. So now let us consider $\mathbb{E}(\phi_k(z)|\mathcal{F}_{k-1})$. This can be rewritten as

$$\mathbb{E}(\phi_k(z)|\mathcal{F}_{k-1}) = \mathbb{E}(\phi_{k-1}(e^{i\theta_k} f_{c_k^*}(e^{-i\theta_k}z))|\mathcal{F}_{k-1}).$$

The only randomness here comes from $\theta_k$, the $c_k^*$ are pre-determined, and so,

$$\mathbb{E}(\phi_k(z)|\mathcal{F}_{k-1}) = \int_0^{2\pi} \phi_{k-1}(e^{i\theta} f_{c_k^*}(e^{-i\theta}z))\frac{d\theta}{2\pi} = \phi_{k-1}(e^{c_k^*}z).$$

Therefore,

$$\mathbb{E}(\phi_k(e^{C^*_{k+1,n}}z)|\mathcal{F}_{k-1}) = \phi_{k-1}(e^{C^*_{k,n}}z).$$

Thus,

$$\mathbb{E}(X_{k,n}|\mathcal{F}_{k-1}) = e^{-C^*_{1,n}}\left(\mathbb{E}(\phi_k(e^{C^*_{k+1,n}}z)|\mathcal{F}_{k-1}) - \phi_{k-1}(e^{C^*_{k,n}}z)\right) = 0$$

as required.

$\square$

We also define the sum,

$$M_n(z) := \sum_{k=1}^{n} X_{k,n}(z) = e^{-C^*_{1,n}}\phi_n(z) - z. \tag{2.7}$$

Lemma 2.4.1 shows that for each fixed $n$, we have for $k = 0, ..., n$ that $\mathbb{E}(M_n|\mathcal{F}_k) = \sum_{j=1}^{k} X_{j,n}$ which is a martingale with respect to $k$. We will also need to define the variance

$$T_n(z) := \sum_{k=1}^{n} \mathbb{E}(|X_{k,n}(z)|^2 \mid \mathcal{F}_{k-1}). \tag{2.8}$$

Our aim is to show that we approach a disk pointwise, equivalently, for a fixed value $z$, $|M_n(z)| \to 0$ as $n \to \infty$. Throughout we use $\lambda$ to denote strictly positive, unless stated otherwise, constants which may change from line to line. Where these constants depend on parameters from the model we indicate these explicitly.

We will find pointwise bounds on $X_{k,n}(z)$ and $T_n(z)$. By definition;

$$\begin{aligned}|X_{k,n}(z)| &= e^{-C^*_{1,n}}|\phi_k(e^{C^*_{k+1,n}}z) - \phi_{k-1}(e^{C^*_{k,n}}z)| \\ &= e^{-C^*_{1,n}}|\phi_{k-1}(e^{i\theta_k} f_{c^*_k}(e^{-i\theta_k} e^{C^*_{k+1,n}}z)) - \phi_{k-1}(e^{C^*_{k,n}}z)|.\end{aligned}$$

We start by showing that for $|z| > r$, for some $r > 1$, we can bound $\delta_c\left(e^{-i\theta_k} e^{C^*_{k+1,n}}z\right)$ by a constant via the following lemma.

Lemma 2.4.2. *For $C^*_{k,n}$ and $\delta_c(z)$ defined as above, and for $|z| > r$ for some $r > 1$, the following bound holds,*

$$|\delta_{c^*_k}\left(e^{-i\theta_k} e^{C^*_{k+1,n}}z\right)| < \lambda(\alpha, c, r)k^{\frac{1}{\alpha}-\frac{3}{2}}n^{-\frac{1}{\alpha}} \leqslant \lambda(\alpha, c, r)k^{-\frac{3}{2}} < \lambda(\alpha, c, r)$$

*where $\lambda(\alpha, c, r)$ is a positive constant dependent on $\alpha$, $c$ and $r$.*

Proof. From equation (2.1) we know

$$|\delta_{c^*_k}(z)| \leqslant \frac{\tilde{\lambda}(c^*_k)^{\frac{3}{2}}|z|}{|z-1|(|z|-1)}$$

where $\tilde{\lambda}$ is some constant. Therefore,

$$\left|\delta_{c_k^*}\left(e^{-i\theta_k}e^{C^*_{k+1,n}}z\right)\right| \leqslant \frac{\tilde{\lambda}(c_k^*)^{\frac{3}{2}}|e^{C^*_{k+1,n}}z|}{|e^{-i\theta_k}e^{C^*_{k+1,n}}z-1|(|e^{C^*_{k+1,n}}z|-1)}.$$

Since $|z| > r$,

$$\left|\delta_{c_k^*}\left(e^{-i\theta_k}e^{C^*_{k+1,n}}z\right)\right| \leqslant \frac{\tilde{\lambda}(c_k^*)^{\frac{3}{2}}e^{C^*_{k+1,n}}r}{(e^{C^*_{k+1,n}}r-1)^2}.$$

Note that $\tilde{\lambda}$ could equal zero here. So using the estimates on $e^{C^*_{k+1,n}}$ and $\epsilon_{k,n}$ from Lemmas 2.3.1 and Corollary 2.3.2 respectively we have the following bound,

$$\begin{aligned}\left|\delta_{c_k^*}\left(e^{-i\theta_k}e^{C^*_{k+1,n}}z\right)\right| &\leqslant \lambda(\alpha,c,r)k^{\frac{1}{\alpha}-\frac{3}{2}}n^{-\frac{1}{\alpha}}\\ &\leqslant \lambda(\alpha,c,r)k^{-\frac{3}{2}} < \lambda(\alpha,c,r)\end{aligned}$$

where $\lambda(\alpha,c,r)$ is a constant dependent on $\alpha$, $c$ and $r$. ☐

Note that we will need the intermediate bound in a later proof. We introduce the following parametrisation.

Definition 2.4.3. *For each $n \in \mathbb{N}$, $z \in \mathbb{C}$, $k \leqslant n$ and $\delta_c(z)$ defined as in (2.1), we define the following parametrisation for $0 \leqslant s \leqslant 1$,*

$$\eta_{k,n}(s,z) = e^{C^*_{k,n}}z\exp\left(s\left(\frac{2c_k^*}{e^{-i\theta_k}e^{C^*_{k+1,n}}z-1}+\delta_c\left(e^{-i\theta_k}e^{C^*_{k+1,n}}z\right)\right)\right).$$

Using this parametrisation we see,

$$\eta_{k,n}(0,z) = e^{C^*_{k,n}}z, \qquad\qquad \eta_{k,n}(1,z) = e^{i\theta_k}f_{c_k^*}(e^{-i\theta_k}e^{C^*_{k+1,n}}z)$$

where $f_{c_k^*}(z)$ is defined as in Section 2.1. Therefore,

$$|X_{k,n}(z)| = e^{-C^*_{1,n}}|\phi_{k-1}(\eta_{k,n}(1,z)) - \phi_{k-1}(\eta_{k,n}(0,z))|.$$

Before finding pointwise bounds on $X_{k,n}(z)$ and $T_n(z)$, we first find pointwise bounds on elements of $\eta_{k,n}(s,z)$ and its derivative with respect to $s$ which we denote by $\dot{\eta}_{k,n}(s,z)$.

LEMMA 2.4.4. *For $\eta_{k,n}(s,z)$ defined in (2.4.3), for each $z \in \mathbb{C}$ with $|z| > r$ and each $0 \leqslant s \leqslant 1$, the following pointwise bound holds,*

$$\left|\exp\left(s\left(\frac{2c_k^*}{e^{-i\theta_k}e^{C_{k+1,n}^*}z-1}+\delta_{c_k^*}\left(e^{-i\theta_k}e^{C_{k+1,n}^*}z\right)\right)\right)\right| \leqslant \lambda(\alpha,c,r)$$

*where $\lambda(\alpha,c,r)$ is a constant dependent on $\alpha$, $c$ and $r$. Furthermore,*

$$|\dot{\eta}_{k,n}(s,z)| \leqslant \lambda(\alpha,c,r)\left|\frac{c_k^* e^{C_{k,n}^*}z}{e^{-i\theta_k}e^{C_{k+1,n}^*}z-1}\right| \leqslant \lambda(\alpha,c,r)\frac{c_k^* e^{C_{k,n}^*}}{e^{C_{k+1,n}^*}r-1}.$$

PROOF. Let $\lambda(\alpha,c,r)$ be some constant that we allow to vary throughout the proof. First notice that since $c_k^* < c$ and $e^{C_{k+1,n}^*}|z| > r$ it follows that

$$\left|s\left(\frac{2c_k^*}{e^{-i\theta_k}e^{C_{k+1,n}^*}z-1}\right)\right| \leqslant \frac{2c}{r-1}.$$

Therefore as,

$$\begin{aligned}&\left|\exp\left(s\left(\frac{2c_k^*}{e^{-i\theta_k}e^{C_{k+1,n}^*}z-1}+\delta_{c_k^*}\left(e^{-i\theta_k}e^{C_{k+1,n}^*}z\right)\right)\right)\right| \\ &\leqslant \exp\left(\left|\frac{2c_k^*}{e^{-i\theta_k}e^{C_{k+1,n}^*}z-1}\right|+\left|\delta_{c_k^*}\left(e^{-i\theta_k}e^{C_{k+1,n}^*}z\right)\right|\right)\end{aligned}$$

we use the bound above along with Lemma 2.4.2 to reach the following bound

$$\begin{aligned}\left|\exp\left(s\left(\frac{2c_k^*}{e^{-i\theta_k}e^{C_{k+1,n}^*}z-1}+\delta_{c_k^*}\left(e^{-i\theta_k}e^{C_{k+1,n}^*}z\right)\right)\right)\right| &\leqslant \exp\left(\frac{2c}{r-1}+\lambda(\alpha,c,r)\right) \\ &= \lambda(\alpha,c,r).\end{aligned}$$

Now consider $\dot{\eta}_{k,n}(s,z)$. Recalling that

$$\eta_{k,n}(s,z) = e^{C_{k,n}^*}z\exp\left(s\left(\frac{2c_k^*}{e^{-i\theta_k}e^{C_{k+1,n}^*}z-1}+\delta_{c_k^*}\left(e^{-i\theta_k}e^{C_{k+1,n}^*}z\right)\right)\right)$$

we see that

$$|\dot{\eta}_{k,n}(s,z)| \leqslant \left|\left(\frac{2c_k^*}{e^{-i\theta_k}e^{C_{k+1,n}^*}z-1}+\delta_{c_k^*}\left(e^{-i\theta_k}e^{C_{k+1,n}^*}z\right)\right)\right||\eta_{k,n}(s,z)|.$$

Then using the bound we found above,

$$|\dot{\eta}_{k,n}(s,z)| \leqslant \lambda(\alpha,c,r)|e^{C_{k,n}^*}z|\left(\left|\frac{2c_k^*}{e^{-i\theta_k}e^{C_{k+1,n}^*}z-1}\right|+\left|\delta_{c_k^*}\left(e^{-i\theta_k}e^{C_{k+1,n}^*}z\right)\right|\right)$$

where $\lambda(\alpha, c, r)$ is some constant. Now using the fact that $|z| > r$ and the bound from the proof of Lemma 2.4.2 we see that

$$|\dot{\eta}_{k,n}(s,z)| \leqslant \lambda(\alpha, c, r)\left|\frac{2c_k^* e^{C_{k,n}^*} z}{e^{-i\theta_k} e^{C_{k+1,n}^*} z - 1}\right| \leqslant \lambda(\alpha, c, r)\frac{2c_k^* e^{C_{k,n}^*}}{e^{C_{k+1,n}^*} r - 1}.$$

where the second inequality follows by using that $|z| > r$ again. □

Now we can use the bounds above to give us a pointwise bound on $X_{k,n}(z)$. We will use the following distortion theorem in the proof [**Pom75**].

THEOREM 2.4.5. *For a function from the exterior disc into the complex plane $F : \Delta \to \mathbb{C}$ that is univalent except for a simple pole at $\infty$ and Laurent expansion of the form*

$$F(z) = z + a_0 + \sum_{n=1}^{\infty} a_n z^{-n}$$

*we have the estimate*

$$\frac{|z|^2 - 1}{|z|^2} \leqslant |F'(z)| \leqslant \frac{|z|^2}{|z|^2 - 1} \leqslant \frac{|z|}{|z| - 1} \qquad z \in \Delta.$$

Our bound on $X_{k,n}(z)$ is given by the following lemma.

LEMMA 2.4.6. *For the sequence $\{X_{k,n}(z)\}_{k=0}^{n}$ and corresponding filtration $\mathcal{F}_k$ defined as above, and for a fixed $|z| > r$, the following property is satisfied for all $0 < k \leqslant n$;*

$$|X_{k,n}(z)| < \lambda(\alpha, c, r)\frac{c_k^*}{e^{C_{k+1,n}^*} r - 1}$$

*where $\lambda(\alpha, c, r)$ is a constant dependent on $\alpha$, $c$ and $r$. Furthermore, for $0 < \alpha \leqslant 1$,*

$$\sup_{k \leqslant n} |X_{k,n}(z)| < \lambda(\alpha, c, r)\frac{1}{n}$$

*and for $\alpha > 1$,*

$$\sup_{k \leqslant n} |X_{k,n}(z)| < \lambda(\alpha, c, r)\frac{1}{n^{\frac{1}{\alpha}}}.$$

PROOF. By definition

$$|X_{k,n}(z)| = e^{-C_{1,n}^*}|\phi_{k-1}(\eta_{k,n}(1,z)) - \phi_{k-1}(\eta_{k,n}(0,z))|.$$

Hence,

$$\begin{aligned}|X_{k,n}(z)| &= e^{-C^*_{1,n}} \left| \int_0^1 \phi'_{k-1}(\eta_{k,n}(s,z))\ \dot{\eta}_{k,n}(s,z) ds \right| \\ &\leqslant e^{-C^*_{1,n}} \int_0^1 \left|\phi'_{k-1}(\eta_{k,n}(s,z))\right| |\dot{\eta}_{k,n}(s,z)|\ ds.\end{aligned}$$

Using Lemma 2.4.4 we have,

$$|\dot{\eta}_{k,n}(s,z)| \leqslant \lambda(\alpha,c,r)\frac{2c^*_k e^{C^*_{k,n}} r}{e^{C^*_{k+1,n}} r - 1}.$$

where $\lambda(\alpha, c, r)$ is a non-zero constant that will vary throughout this proof. Moreover, we can find a bound on $\int_0^1 \left|\phi'_{k-1}(\eta_{k,n}(s,z))\right|\ ds$ using Theorem 2.4.5,

$$\int_0^1 \left|\phi'_{k-1}(\eta_{k,n}(s,z))\right|\ ds < e^{C^*_{1,k-1}} \sup_{0<s<1} \frac{|\eta_{k,n}(s,z)|}{|\eta_{k,n}(s,z)| - 1}.$$

Note that in order to apply the distortion theorem to our function $\phi_{k-1}$ we had to rescale by a factor of $e^{C^*_{1,k-1}}$. It is easy to show that $\inf_{0\leqslant s\leqslant 1} |\eta_{k,n}(s,z)| \geqslant |z|$ and therefore for $|z| > r$,

$$\int_0^1 \left|\phi'_{k-1}(\eta_{k,n}(s,z))\right|\ ds < e^{C^*_{1,k-1}} \frac{r}{r-1}.$$

Thus, by compiling the bounds above,

$$\begin{aligned}|X_{k,n}(z)| &< \lambda(\alpha,c,r) e^{-C^*_{1,n}} \frac{e^{C^*_{1,k-1}} r}{r-1} \frac{2c^*_k e^{C^*_{k,n}} r}{e^{C^*_{k+1,n}} r - 1} \\ &< \lambda(\alpha,c,r)\frac{c^*_k}{e^{C^*_{k+1,n}} r - 1}.\end{aligned}$$

Using the estimates in Lemma 2.3.1 and Corollary 2.3.2 we have,

$$|X_{k,n}(z)| < \lambda(\alpha,c,r) k^{\frac{1}{\alpha}-1} n^{-\frac{1}{\alpha}}.$$

First consider the case where $0 < \alpha \leqslant 1$. Then $\frac{1-\alpha}{\alpha} \geqslant 0$. Hence, it is clear that the maximum occurs when $k = n$ and thus

$$\sup_{k\leqslant n} |X_{k,n}(z)| < \lambda(\alpha,c,r)\frac{1}{n}.$$

However, when $\alpha > 1$, $k^{\frac{1-\alpha}{\alpha}} < 1$ , so

$$\sup_{k \leqslant n} |X_{k,n}(z)| < \lambda(\alpha, c, r) \frac{1}{n^{\frac{1}{\alpha}}}$$

where $\lambda(\alpha, c, r)$ is a constant dependent on $\alpha$, $c$ and $r$. □

It is now clear to see that as $n$ approaches infinity the bound on $X_{k,n}(z)$ approaches zero pointwise.

Corollary 2.4.7. *For $X_{k,n}(z)$ defined as above;*

$$\lim_{n \to \infty} \sup_{k \leqslant n} |X_{k,n}(z)| = 0.$$

Now we want to calculate a bound on the variation $T_n(z) = \sum_{k=1}^{n} \mathbb{E}(|X_{k,n}(z)|^2 | \mathcal{F}_{k-1})$. This is given by the following lemma.

Lemma 2.4.8. *The following inequality holds for sufficiently large $n$. If $0 < \alpha < 2$,*

$$T_n(z) \leqslant \lambda(\alpha, c, r) \frac{1}{n}$$

*where $\lambda(\alpha, c, r) > 0$ is some constant.*

Proof. First let us look at $|X_{k,n}(z)|^2$. As before we can bound

$$|X_{k,n}(z)|^2 < e^{-2C^*_{1,n}} \left| \int_0^1 \phi'_{k-1}(\eta_{k,n}(s,z))\, ds \right|^2 \sup_{0 \leqslant s \leqslant 1} |\dot{\eta_{k,n}}(s)|^2.$$

Therefore,

$$\mathbb{E}(|X_{k,n}(z)|^2 \mid \mathcal{F}_{k-1}) \leqslant e^{-2C^*_{1,n}} \mathbb{E} \left( \left| \int_0^1 \phi'_{k-1}(\eta_{k,n}(s,z))\, ds \right|^2 \sup_{0 \leqslant s \leqslant 1} |\dot{\eta_{k,n}}(s)|^2 \mid \mathcal{F}_{k-1} \right).$$

We can find an upper bound on the integral using a distortion theorem again and then remove it from the expectation. By Theorem 2.4.5 above,

$$\left| \int_0^1 \phi'_{k-1}(\eta_{k,n}(s,z))\, ds \right|^2 < e^{2C^*_{1,k-1}} \frac{r^2}{(r-1)^2}.$$

So all that remains to calculate is $\mathbb{E}(\sup_{0\leqslant s\leqslant 1}|\dot{\eta}_{k,n}(s,z)|^2|\mathcal{F}_{k-1})$. Firstly by Lemma 2.4.4, for all $0 \leqslant s \leqslant 1$,

$$|\dot{\eta}_{k,n}(s,z)| \leqslant \lambda(\alpha,c,r)\frac{|c_k^* \, e^{c_k^*} w|}{|e^{-i\theta_k}w - 1|}.$$

where $w = e^{C^*_{k+1,n}}r$. Moreover, since the $c_k^*$ are predetermined, the only randomness here comes from the $\theta_k$ and thus,

$$\mathbb{E}(\sup_{0\leqslant s\leqslant 1}|\dot{\eta}_{k,n}(s,z)|^2 \mid \mathcal{F}_{k-1}) \leqslant 4(c_k^*)^2 e^{2c_k^*}\int_0^{2\pi}\frac{|w|^2}{|e^{-i\theta}w-1|^2}d\theta.$$

It is easily shown that for $w \in \mathbb{C}$,

$$\int_0^{2\pi}\frac{|w|^2}{|e^{-i\theta}w-1|^2}d\theta \leqslant \frac{6|w|}{|w|-1}.$$

Therefore,

$$\mathbb{E}(\sup_{0\leqslant s\leqslant 1}|\dot{\eta}_{k,n}(s,z)|^2|\mathcal{F}_{k-1}) \leqslant 24(c_k^*)^2 e^{2c_k^*}\frac{re^{C^*_{k+1,n}}}{re^{C^*_{k+1,n}}-1}.$$

It is clear for all $k \leqslant n$, $c_k^* < c$, therefore,

$$\mathbb{E}(\sup_{0\leqslant s\leqslant 1}|\dot{\eta}_{k,n}(s,z)|^2 \mid \mathcal{F}_{k-1}) \leqslant 24e^{2c}(c_k^*)^2\frac{re^{C^*_{k,n}}}{re^{C^*_{k+1,n}}-1}.$$

Finally we can use the bound

$$\frac{1}{re^{C^*_{k+1,n}}-1} \leqslant \frac{1}{re^{C^*_{k,n}}-1}\frac{e^c r}{r-1}$$

and bring together the previous bounds to reach the following bound on $T_n(z)$. Let $\lambda(\alpha,c,r) > 0$ be some constant that will vary throughout. Then,

$$\begin{aligned}
T_n(z) &\leqslant \lambda(\alpha,c,r)\sum_{k=1}^n\left(e^{-2C^*_{1,n}}\;\; e^{2C^*_{1,k-1}}\;\; (c_k^*)^2\frac{e^{C^*_{k,n}}}{re^{C^*_{k,n}}-1}\right)\\
&\leqslant \lambda(\alpha,c,r)\sum_{k=1}^n(c_k^*)^2\frac{e^{-C^*_{k,n}}}{\left(e^{C^*_{k,n}}r-1\right)}\\
&\leqslant \lambda(\alpha,c,r)\sum_{k=1}^n(c_k^*)^2 e^{-2C^*_{k,n}}.
\end{aligned}$$

We can substitute in the known values for $c_k^*$ and $C_{k,n}^*$to reach the following bound on $T_n(z)$,

$$T_n(z) \leqslant \lambda(\alpha, c, r) \sum_{k=1}^{n} \left( \frac{c}{1 + \alpha c(k-1)} \right)^2 \left( \frac{1 + \alpha c(k-1)}{1 + \alpha cn} \right)^{\frac{2}{\alpha}(1+\epsilon_{k,n})}.$$

Let $x = \frac{1+\alpha c(k-1)}{1+\alpha cn}$. Then,

$$T_n(z) \leqslant \lambda(\alpha, c, r) \frac{1}{1 + \alpha cn} \int_{\frac{1}{1+\alpha cn}}^{1} x^{\frac{2}{\alpha}-2} dx.$$

This integral is bounded above by a constant if $0 < \alpha < 2$. Therefore we bound above by

$$T_n(z) \leqslant \lambda(\alpha, c, r) \frac{1}{n}.$$

□

Moreover since $T_n(z) \geqslant 0$, we have the following corollary.

COROLLARY 2.4.9. *For $0 < \alpha < 2$,*

$$\lim_{n \to \infty} T_n(z) = 0.$$

**2.4.2. Results.** We are now in a position to analyse the limiting structure of the map $\phi_n$ as $n \to \infty$ for $0 < \alpha < 2$. Our aim is to use the bounds on the increments $X_{k,n}(z)$ and $T_n(z)$ found in the previous section to produce a pointwise estimate on the difference between the cluster map and the disk of logarithmic capacity $e^{C_{k,n}^*}$. In order to do so we will apply the following theorem which is an immediate consequence of Proposition 2.1 in [**Fre75**].

THEOREM 2.4.10. *Let $n$ be any positive integer. Suppose $X_{k,n}$ is $\mathcal{F}_k$-measurable and $\mathbb{E}\{X_{k,n} \mid \mathcal{F}_{k-1}\} = 0$ and define $M_n = \sum_{k=1}^{n} X_{k,n}$ and $T_n = \sum_{k=1}^{n} \mathrm{Var}\{X_{k,n} \mid \mathcal{F}_{k-1}\}$. Let $M$ be a positive real number and suppose $\mathbb{P}\{|X_{k,n}| \leqslant M$ for all $k \leqslant n\} = 1$. Then for all positive numbers $a$ and $b$,*

$$\mathbb{P}\{M_n \geqslant a \;\; and \;\; T_n \leqslant b\} \leqslant \exp\left[ \frac{-a^2}{2(Ma + b)} \right].$$

*Consequently, for all positive numbers $a$ and $b$,*

$$\mathbb{P}\{|M_n| \geqslant a \ \text{ and } \ T_n \leqslant b\} \leqslant 2\exp\left[\frac{-a^2}{2(Ma+b)}\right].$$

Recall, $e^{-C^*_{1,n}}\phi_n(z) - z = \sum_{k=1}^n X_{k,n}(z)$. Hence, we can now apply Theorem 2.4.10 to our cluster to obtain pointwise results for $0 < \alpha < 2$.

THEOREM 2.4.11. *Let $c_i^*$ and $\phi_k$ be defined as above. Then for $0 < \alpha < 2$, and any sufficiently small positive real number $a$ and $n$ sufficiently large,*

$$\mathbb{P}\left(|e^{-C^*_{1,n}}\phi_n(z) - z| > a\right) \leqslant 4e^{-\frac{a^2 n}{\lambda(\alpha,c,r)}}$$

*for some strictly positive constant $\lambda(\alpha, c, r)$. Therefore, for all $0 < \alpha < 2$ if we let $a(n) = \frac{\log(n)}{\sqrt{n}}$ then for all $z \in \mathbb{C}$ with $|z| > 1$,*

$$\lim_{n\to\infty} \mathbb{P}\left(|e^{-C^*_{1,n}}\phi_n(z) - z| > \frac{\log(n)}{\sqrt{n}}\right) = 0.$$

PROOF. First note, we have shown in Lemma 2.4.1, $\mathbb{E}(X_{k,n}(z)|\mathcal{F}_{k-1}) = 0$ where

$$X_{k,n}(z) = e^{-C^*_{1,n}}\left(\phi_k\left(e^{C^*_{k+1,n}}z\right) - \phi_{k-1}\left(e^{C^*_{k,n}}z\right)\right).$$

Recall, $M_n(z) = \sum_{k=1}^n X_{k,n}(z)$, and note that we can split $M_n$ into real and imaginary parts, thus,

$$\mathbb{P}\left(\left|M_n\right| > a\right) \leqslant \mathbb{P}\left(\Re(M_n) > \frac{a}{\sqrt{2}}\right) + \mathbb{P}\left(\Im(M_n) > \frac{a}{\sqrt{2}}\right).$$

Moreover,

$$\sup_{k\leqslant n} \Re(X_{k,n}(z)) < \sup_{k\leqslant n} |X_{k,n}(z)|$$
$$\sup_{k\leqslant n} \Im(X_{k,n}(z)) < \sup_{k\leqslant n} |X_{k,n}(z)|.$$

It is easy to see that both $\Re(X_{k,n}(z))$ and $\Im(X_{k,n}(z))$ satisfy the property that the expectation with respect to the filtration is zero and so by applying Theorem 2.4.10 with $M = b_X(n)$

and $b = b_T(n)$, for any positive real number $a$,

$$\mathbb{P}\left(\left|\sum_{k=1}^{n} X_{k,n}(z)\right| \geqslant a\right) \leqslant \mathbb{P}\left(\Re\left(\sum_{k=1}^{n} X_{k,n}(z)\right) \geqslant \frac{a}{\sqrt{2}}\right) + \mathbb{P}\left(\Im\left(\sum_{k=1}^{n} X_{k,n}(z)\right) \geqslant \frac{a}{\sqrt{2}}\right)$$
$$\leqslant 4\exp\left[\frac{-a^2}{4(b_X(n)\frac{a}{\sqrt{2}} + b_T(n))}\right]$$

where $b_X(n)$, $b_T(n)$ are the bounds on $|X_{k,n}(z)|$ and $T_n(z)$ from Lemma 2.4.6 and Lemma 2.4.8 respectively. We first deal with the case that $0 < \alpha \leqslant 1$. In Lemma 2.4.6 we have seen

$$\sup_{k \leqslant n} |X_{k,n}(z)| < \lambda_1(\alpha, c, r)\frac{1}{n}$$

for some positive constant $\lambda_1(\alpha, c, r)$ and by Lemma 2.4.8,

$$T_n(z) \leqslant \lambda_2(\alpha, c, r)\frac{1}{n}$$

for some positive constant $\lambda_2(\alpha, c, r)$. Therefore,

$$\mathbb{P}\left(|e^{-\sum_{i=1}^{n} c_i^*}\phi_n(z) - z| > a\right) \leqslant 4\exp\left(\frac{-a^2 n}{4\left(\lambda_1(\alpha, c, r)\frac{a}{\sqrt{2}} + \lambda_2(\alpha, c, r)\right)}\right).$$

But for $a$ sufficiently small, $\lambda_1(\alpha, c, r)\frac{a}{\sqrt{2}} \leqslant \lambda_2(\alpha, c, r)$ so let $\lambda(\alpha, c, r) = 8\lambda_2(\alpha, c, r)$ then

$$\mathbb{P}\left(|e^{-\sum_{i=1}^{n} c_i^*}\phi_n(z) - z| > a\right) \leqslant 4\exp\left(\frac{-a^2 n}{\lambda(\alpha, c, r)}\right).$$

Now for $1 < \alpha < 2$,

$$\sup_{k \leqslant n} |X_{k,n}(z)| < \lambda_1(\alpha, c, r)\frac{1}{n^{\frac{1}{\alpha}}}$$

for some positive constant $\lambda_1(\alpha, c, r)$ and

$$T_n(z) \leqslant \lambda_2(\alpha, c, r)\frac{1}{n}$$

for some positive constant $\lambda_2(\alpha, c, r)$. Therefore,

$$\mathbb{P}\left(|e^{-\sum_{i=1}^{n} c_i^*}\phi_n(z) - z| > a\right) \leqslant 4\exp\left(\frac{-a^2 n^{\frac{1}{\alpha}}}{4\left(\lambda_1(\alpha, c, r)\frac{a}{\sqrt{2}} + \lambda_2(\alpha, c, r)n^{\frac{1-\alpha}{\alpha}}\right)}\right).$$

Now for $a = a(n) = \frac{\log(n)}{\sqrt{n}}$, and $n$ sufficiently large, $\lambda_1(\alpha, c, r)\frac{a}{\sqrt{2}} \leqslant \lambda_2(\alpha, c, r)n^{\frac{1-\alpha}{\alpha}}$. Therefore, using the same $\lambda(\alpha, c, r)$ as above, for all $0 < \alpha < 2$,

$$\mathbb{P}\left(|e^{-\sum_{i=1}^{n} c_i^*}\phi_n(z) - z| > a\right) \leqslant 4\exp\left(\frac{-a^2 n}{\lambda(\alpha, r, c)}\right).$$

Thus, for $a(n) = \frac{\log(n)}{\sqrt{n}}$,

$$\lim_{n\to\infty} \mathbb{P}\left(|e^{-C_{1,n}^*}\phi_n(z) - z| > \frac{\log(n)}{\sqrt{n}}\right) = 0.$$

□

## 2.5. Uniform convergence in the exterior disk for $0 < \alpha < 2$

In the previous section we showed that the rescaled functions $e^{-C_{1,n}^*}\phi_n(z)$ converge pointwise in probability to the identity. Now we need to show that the maps converge locally uniformly. Thus, our aim of this section will be to prove the following theorem.

THEOREM 2.5.1. *For $0 < \alpha < 2$, let the map $\phi_n$ be defined as above with $c_n^*$ as defined in (2.2) and $\theta_n$ i.i.d, uniform on $[0, 2\pi]$. Then for any $r > 1$ we have the following inequality*

$$\mathbb{P}\left(\sup_{|z|\geqslant r} |e^{-\sum_{i=1}^{n} c_i^*}\phi_n(z) - z| > \frac{\log(n)}{\sqrt{n}}\right) < \lambda_1(\alpha, c, r)e^{-\frac{\log(n)^2}{\lambda_2(\alpha,c,r)}}$$

*where $\lambda_1(\alpha, c, r)$, $\lambda_2(\alpha, c, r) > 0$ are constants. Hence, by Borel-Cantelli,*

$$\mathbb{P}\left(\sup_{|z|\geqslant r} \left|e^{-\sum_{i=1}^{n} c_i^*}\phi_n(z) - z\right| = o\left(\frac{\log n}{\sqrt{n}}\right) \text{ as } n \to \infty\right) = 1.$$

The proof of the theorem will be constructed as follows. We will show that for a finite number of equally spaced points along the circle $|z| = r$ the inequality holds. Then we will show that between these points the probability that the difference between the maps when evaluated at these points is sufficiently small. First define

$$M_n(z, w) := M_n(z) - M_n(w)$$

with $M_n(z)$ defined in equation (2.7). Then we must choose the spacing between the finite set of points. With the choice of $\alpha$ and $c$ fixed we choose points, on a radius $|z| = r$, to be equally spaced at angles $\frac{2\pi}{L_{r,n}}$ where

$$L_{r,n} = \gamma(\alpha, c, r) n^{\frac{3}{2}}$$

and $\gamma(\alpha, c, r)$ is a constant,

$$\gamma(\alpha, c, r) = 4\pi r \frac{1}{c}(e^c + 1)(1 + \alpha c)(1 + \alpha e^{\alpha c}) \left( \log\left( \frac{r}{r-1} \right) + 1 \right) (\log(1 + \alpha c) + 1) \,. \tag{2.9}$$

The reason for this choice of spacing will become clear in the proof of the lemmas that follow. We start by proving that we can find a finite number of equally spaced points, with the above spacing along the circle $|z| = r$, such that the inequality in Theorem 2.5.1 holds.

Lemma 2.5.2. *Let $\{z_i\}_{i=1}^{L_{r,n}}$ be defined as finite set of points on the boundary of the unit circle of radius $|z| = r$ with equally spaced at angles $\frac{2\pi}{L_{r,n}}$ and $L_{r,n}$ defined as above. Then, for sufficiently large $n$, we have the following inequality*

$$\mathbb{P}\left( \exists i : |M_n(z_i)| > \frac{\log n}{2\sqrt{n}} \right) \leqslant \lambda_1(\alpha, c, r) e^{-\frac{(\log(n))^2}{\lambda_2(\alpha, c, r)}}$$

*where $\lambda_1(\alpha, c, r), \lambda_2(\alpha, c, r) > 0$ are constants.*

Proof. We have shown using Theorem 2.4.11 that for $0 < \alpha < 2$ and for any $1 \leqslant i \leqslant L_{r,n}$

$$\mathbb{P}\left( |M_n(z_i)| > \frac{\log n}{2\sqrt{n}} \right) \leqslant 4 e^{-\frac{\log(n)^2}{\lambda(\alpha, c, r)}}$$

for some constant $\lambda(\alpha, c, r) > 0$. Therefore,

$$\mathbb{P}\left( \exists i : |M_n(z_i)| > \frac{\log n}{2\sqrt{n}} \right) < 4 \sum_{k=1}^{L_{r,n}} e^{-\frac{\log(n)^2}{\lambda(\alpha, c, r)}}.$$

The terms in the sum have no dependence on $k$ and as such we can find an upper bound,

$$\begin{aligned} \mathbb{P}\left( \exists i : |M_n(z_i)| > \frac{\log n}{4\sqrt{n}} \right) &\leqslant 4 L_{r,n} e^{-\frac{\log(n)^2}{\lambda(\alpha, c, r)}} \\ &= 4\gamma(\alpha, c, r) n^{\frac{3}{2}} e^{-\frac{\log(n)^2}{\lambda(\alpha, c, r)}} \end{aligned}$$

where $\gamma(\alpha, c, r) > 0$ is the constant defined in equation (2.9). Let $\lambda_1(\alpha, c, r) = 4\gamma(\alpha, c, r)$, then

$$\mathbb{P}\left(\exists i : |M_n(z_i)| > \frac{\log n}{2\sqrt{n}}\right) \leqslant \lambda_1(\alpha, c, r) e^{\frac{3}{2}\log n - \frac{\log(n)^2}{\lambda(\alpha,c,r)}}.$$

For sufficiently large $n > e^{3\lambda(\alpha,c,r)}$,

$$\frac{\frac{3}{2}\log n}{\frac{\log(n)^2}{\lambda(\alpha,c,r)}} \leqslant \frac{1}{2}.$$

Therefore, let $\lambda_2(\alpha, c, r) = 2\lambda(\alpha, c, r)$ and then for $n$ sufficiently large,

$$\mathbb{P}\left(\exists i : |M_n(z_i)| > \frac{\log n}{2\sqrt{n}}\right) \leqslant \lambda_1(\alpha, c, r) e^{-\frac{(\log(n))^2}{\lambda_2(\alpha,c,r)}}$$

with $\lambda_1(\alpha, c, r), \lambda_2(\alpha, c, r) > 0$. □

We now prove that for points $w \in \mathbb{C}$ in between the points in the set $\{z_i\}_{i=1}^{L_{r,n}}$ the difference $M_n(z_i, w)$ is negligible.

Lemma 2.5.3. *For $|z| = |w| = r$ with $\arg(z) = \theta_z$, $\arg(w) = \theta_w$ and $|\theta_z - \theta_w| < \frac{2\pi}{L_{r,n}}$ and $L_{r,n}$ defined as above we have the following bound;*

$$|M_n(z, w)| \leqslant \frac{\log(n)}{2\sqrt{n}}$$

*and hence,*

$$\mathbb{P}\left(\exists w, z \in \mathbb{C} : |z| = |w| = r,\ |\arg(z) - \arg(w)| < \frac{2\pi}{L_{r,n}},\ |M_n(z, w)| > \frac{\log(n)}{2\sqrt{n}}\right) = 0.$$

Proof. We want to find a bound on $|M_n(z, w)|$ so we first find a bound on $|X_{k,n}(z, w)| = |X_{k,n}(z) - X_{k,n}(w)|$.

$$\begin{aligned}
&|X_{k,n}(z, w)| \\
&= e^{-\sum_{k=1}^{n} c_i^*} \left| \left(\phi_k\left(e^{C^*_{k+1,n}} z\right) - \phi_{k-1}\left(e^{C^*_{k,n}} z\right)\right) - \left(\phi_k\left(e^{C^*_{k+1,n}} w\right) - \phi_{k-1}\left(e^{C^*_{k,n}} w\right)\right)\right|.
\end{aligned}$$

Let $0 \leqslant s, t \leqslant 1$ and then

$$\begin{aligned}
\tau_{k,n}(s) &= e^{C^*_{k+1,n}} |z| e^{i(\theta_z s + \theta_w(1-s))} \\
\rho_{k,n}(t) &= e^{C^*_{k,n}} |z| e^{i(\theta_z t + \theta_w(1-t))}.
\end{aligned}$$

Thus,

$$|X_{k,n}(z,w) \leqslant |\phi_k(\tau_{k,n}(1)) - \phi_k(\tau_{k,n}(0))| + |\phi_{k-1}(\rho_{k,n}(1)) - \phi_{k-1}(\rho_{k,n}(0))|.$$

If we consider the $\tau$ terms in the upper bound, we have

$$|\phi_k(\tau_{k,n}(1)) - \phi_k(\tau_{k,n}(0))| \leqslant \int_0^1 \left|\phi_k'(\tau_{k,n}(s))\right| |\dot{\tau_{k,n}}(s)| ds.$$

Using Theorem 2.4.5,

$$|\phi_k(\tau_{k,n}(1)) - \phi_k(\tau_{k,n}(0))| \leqslant e^{C^*_{1,k}} \sup_{0\leqslant s\leqslant 1} \frac{|\tau_{k,n}(s)|}{|\tau_{k,n}(s)|-1} e^{C^*_{k+1,n}} |\theta_z - \theta_w||z|.$$

Therefore,

$$|\phi_k(\tau_{k,n}(1)) - \phi_k(\tau_{k,n}(0))| \leqslant e^{C^*_{1,n}} |z|^2 |\theta_z - \theta_w| \frac{e^{C^*_{k+1,n}}}{e^{C^*_{k+1,n}}|z| - 1}.$$

By a similar argument

$$|\phi_{k-1}(\rho_{k,n}(1)) - \phi_{k-1}(\rho_{k,n}(0))| \leqslant e^{C^*_{1,n}} |z|^2 |\theta_z - \theta_w| \frac{e^c e^{C^*_{k+1,n}}}{e^{C^*_{k+1,n}}|z| - 1}.$$

Therefore using the fact $|z| = r$,

$$|X_{k,n}(z,w)| \leqslant r^2(e^c+1)|\theta_z - \theta_w| \frac{e^{C^*_{k+1,n}}}{e^{C^*_{k+1,n}} r - 1}.$$

We can therefore use the approximation $e^{C^*_{k,n}} \approx \left(\frac{1+\alpha cn}{1+\alpha c(k-1)}\right)^{\frac{1}{\alpha}}$ and take the sum to write

$$|M_n(z,w)| \leqslant r^2(e^c+1)|\theta_z - \theta_w| \left| \sum_{k=1}^{n} \left( \frac{\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{k,n}}{\alpha}}}{r\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{k,n}}{\alpha}} - 1} \right) \right|$$

where $\epsilon_{k,n}$ is the same error term from Section 2.2. We can use the bound from Corollary 2.3.2 to remove the $\epsilon_{k,n}$ term,

$$\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1}{\alpha}} < \left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{k,n}}{\alpha}} \leqslant (1+\alpha c e^{\alpha c}) \left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1}{\alpha}}$$

Then $x = \left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1}{\alpha}}$ and integrating between $x = \left(\frac{1+\alpha cn}{1+\alpha c}\right)^{\frac{1}{\alpha}}$ and $x = 1$ gives

$$\left|\sum_{k=1}^{n}\left(\frac{\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{k,n}}{\alpha}}}{r\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{k,n}}{\alpha}}-1}\right)\right| \leqslant \frac{1}{c}\left|\int_1^{\left(\frac{1+\alpha cn}{1+\alpha c}\right)^{\frac{1}{\alpha}}}\frac{1+\alpha ck}{rx-1}dx\right|$$
$$\leqslant \frac{1}{c}(1+\alpha cn)\left|\int_1^{\left(\frac{1+\alpha cn}{1+\alpha c}\right)^{\frac{1}{\alpha}}}\frac{1}{rx-1}dx\right|.$$

Thus,

$$\left|\sum_{k=1}^{n}\left(\frac{\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{k,n}}{\alpha}}}{r\left(\frac{1+\alpha cn}{1+\alpha ck}\right)^{\frac{1+\epsilon_{k,n}}{\alpha}}-1}\right)\right| \leqslant \frac{1}{cr}(1+\alpha cn)\left|\log\left(\frac{r-1}{r\left(\frac{1+\alpha cn}{1+\alpha c}\right)^{\frac{1}{\alpha}}-1}\right)\right|$$
$$\leqslant \frac{1}{cr}(1+\alpha cn)\log\left(\frac{r\left(1+\alpha cn\right)^{\frac{1}{\alpha}}}{r-1}\right).$$

Therefore,

$$|M_n(z,w)| \leqslant \frac{\gamma(\alpha,c,r)}{4\pi}|\theta_z-\theta_w|n\log n$$

where $\gamma(\alpha, c, r)$ is the constant defined in equation (2.9). Then we use the fact that $|\theta_z-\theta_w| = \frac{2\pi}{L_{r,n}}$ and write

$$|M_n(z,w)| \leqslant \frac{\log n}{2\sqrt{n}}.$$

So,

$$\mathbb{P}\left(\exists w,z\in\mathbb{C} : |z|=|w|=r,\ |\arg(z)-\arg(w)|<\frac{2\pi}{L_{r,n}},\ |M_n(z,w)|>\frac{\log(n)}{2\sqrt{n}}\right)=0.$$

□

So we can combine these two lemmas to give our proof of Theorem 2.5.1.

Proof of Theorem 2.5.1. As in the previous two lemmas we separate the circle into points $\frac{2\pi}{L_{r,n}}$ apart. We can then form the following bound;

$$\begin{aligned}
&\mathbb{P}\left(\sup_{|z|=r} |e^{-C^*_{1,n}}\phi_n(z) - z| > \frac{\log n}{\sqrt{n}}\right) \\
&\leqslant \mathbb{P}\left(\exists i : |M_n(z_i)| > \frac{1}{2}\frac{\log n}{\sqrt{n}}\right) \\
&+ \mathbb{P}\left(\exists w, z \in \mathbb{C} : |\theta_z - \theta_w| < \frac{2\pi}{L_{r,n}},\ M_n(z,w) > \frac{1}{2}\frac{\log n}{\sqrt{n}}\right).
\end{aligned}$$

Using Lemmas 2.5.2 and 2.5.3 we see,

$$\mathbb{P}\left(\sup_{|z|=r} |e^{-C^*_{1,n}}\phi_n(z) - z| > \frac{\log n}{\sqrt{n}}\right) \leqslant \lambda_1(\alpha, c, r)e^{-\frac{(\log(n))^2}{\lambda_2(\alpha,c,r)}}$$

where $\lambda_1(\alpha, c, r), \lambda_2(\alpha, c, r) > 0$ are constants. Then using the maximum modulus principle we see that that the maximum occurs on the boundary and so,

$$\mathbb{P}\left(\sup_{|z|\geqslant r} |e^{-C^*_{1,n}}\phi_n(z) - z| > \frac{\log n}{\sqrt{n}}\right) \leqslant \lambda_1(\alpha, c, r)e^{-\frac{(\log(n))^2}{\lambda_2(\alpha,c,r)}}.$$

It is clear to see the upper bound is summable and hence by a Borel-Cantelli argument,

$$\mathbb{P}\left(\limsup_{n\to\infty} \sup_{|z|\geqslant r} |e^{-C^*_{1,n}}\phi_n(z) - z| > \frac{\log n}{\sqrt{n}}\right) = 0.$$

□

## 2.6. Fluctuations for $0 < \alpha < 2$

**2.6.1. Discarding the lower order terms.** In the previous sections we have shown that the rescaled functions $e^{-C^*_{1,n}}\phi_n(z)$ converge locally uniformly to the identity with probability one. It immediately follows that the image domain almost surely converges to $\Delta$ in the Carathéodory topology. Now we would like to see how much we fluctuate from the disk. To do so we aim to produce a central limit theorem that will tell us what the distribution of the fluctuations is. Up until this point we have used

$$X_{k,n}(z) = e^{-C^*_{1,n}}\left(\phi_k\left(e^{C^*_{k+1,n}}z\right) - \phi_{k-1}\left(e^{C^*_{k,n}}z\right)\right).$$

We aim to prove that the fluctuations of $M_n(z) = \left|e^{-C^*_{1,n}}\phi_n(z) - z\right|$ are of order $\frac{1}{\sqrt{n}}$. First, we want to establish the leading order behaviour of the increments $X_{k,n}(z)$ in order to simplify the calculation of the fluctuations. Therefore, we introduce the quantity,

$$\mathcal{X}_{k,n}(z) = \frac{2c^*_k\sqrt{n}z}{e^{-i\theta_k}e^{C^*_{k+1,n}}z - 1}.$$

Using similar methods as in the proof of Lemma 2.4.1 it is simple to show that for all $0 < k \leqslant n$,

$$\mathbb{E}(\mathcal{X}_{k,n}(z)|\mathcal{F}_{k-1}) = 0.$$

The following lemma shows that $\mathcal{X}_{k,n}(z)$ is a good approximation to $\sqrt{n}X_{k,n}(z)$.

LEMMA 2.6.1. *Let $Y_{k,n}(z) = \sqrt{n}X_{k,n}(z) - \mathcal{X}_{k,n}(z)$. Then if $0 < \alpha < 2$, for any $\epsilon > 0$ and $r > 1$,*

$$\mathbb{P}\left(\limsup_{n\to\infty}\sup_{|z|>r}\left|\sum_{k=1}^n Y_{k,n}(z)\right| > \epsilon\right) = 0$$

PROOF. Fix some $r > 1$. Then in Theorem 2.5.1 we showed that,

$$\mathbb{P}\left(\limsup_{n\to\infty}\left\{\sup_{|z|\geqslant r}|e^{-C^*_{1,n}}\phi_n(z) - z| > \frac{\log n}{\sqrt{n}}\right\}\right) = 0.$$

Denote the event,

$$\omega(r) = \left\{\liminf_{n\to\infty}\left\{\sup_{|z|\geqslant r}|e^{-C^*_{1,n}}\phi_n(z) - z| \leqslant \frac{\log n}{\sqrt{n}}\right\}\right\}.$$

Now choose $r' = \frac{r+1}{2}$. We have shown that $\mathbb{P}(\omega(r')) = 1$. Therefore,

$$\mathbb{P}\left(\limsup_{n\to\infty}\sup_{|z|>r}\left|\sum_{k=1}^n Y_{k,n}(z)\right| < \epsilon\right) = \mathbb{P}\left(\limsup_{n\to\infty}\sup_{|z|>r}\left|\sum_{k=1}^n Y_{k,n}(z)\right| < \epsilon \;\middle|\; \omega(r')\right).$$

For $|z| > r'$ on the event $\omega(r')$ there exists an integer $k_0 \geqslant 2$ such that if $k \geqslant k_0$ then,

$$|e^{-C^*_{1,k-1}}\phi_{k-1}(z) - z| \leqslant \frac{2\log(k-1)}{\sqrt{k-1}}. \tag{2.10}$$

Thus, we split into two cases. First consider the case where $k \leqslant k_0$. Then by Lemma 2.4.6 for $0 < \alpha \leqslant 1$,

$$\sup_{k\leqslant n}|\sqrt{n}X_{k,n}(z)| < \lambda(\alpha, c, r)n^{-\frac{1}{2}}$$

and for $\alpha > 1$,

$$\sup_{k \leqslant n} |X_{k,n}(z)| < \lambda(\alpha, c, r) n^{\frac{1}{2} - \frac{1}{\alpha}}$$

for some constant $\lambda(\alpha, c, r) > 0$. Similarly, using the definition of $\mathcal{X}_{k,n}(z)$ and bounds from Lemma 2.3.1 and Corollary 2.3.2, for $0 < \alpha \leqslant 2$

$$|\mathcal{X}_{k,n}(z)| \leqslant \lambda(\alpha, c, r) k^{\frac{1}{\alpha} - 1} n^{\frac{1}{2} - \frac{1}{\alpha}}$$

for some constant $\lambda(\alpha, c, r) > 0$. It follows that for $k \leqslant k_0$, if $0 < \alpha \leqslant 1$,

$$\sup_{k \leqslant n} |Y_{k,n}(z)| < \lambda(\alpha, c, r) n^{-\frac{1}{2}}$$

and if $1 < \alpha < 2$,

$$\sup_{k \leqslant n} |Y_{k,n}(z)| < \lambda(\alpha, c, r) n^{\frac{1}{2} - \frac{1}{\alpha}}$$

for some constant $\lambda(\alpha, c, r) > 0$. Thus, if $0 < \alpha < 2$, for any $\epsilon > 0$ and $r > 1$,

$$\mathbb{P}\left( \limsup_{n \to \infty} \sup_{|z| > r} \left| \sum_{k=1}^{k_0} Y_{k,n}(z) \right| > \epsilon \right) = 0 \tag{2.11}$$

Now we consider when $k \geqslant k_0$ and calculate a bound on $|Y_{k,n}(z)|$ in this case. Let

$$\widetilde{X}_{k,n}(z) = \sqrt{n} \left( e^{-C^*_{k,n}} f_k(e^{C^*_{k+1,n}} z) - z \right) = \sqrt{n} e^{-C^*_{k,n}} \int_0^1 \dot{\eta}_{k,n}(s, z) ds$$

where $\eta_{k,n}(s, z)$ is defined as in Section 2.3. Note that in the case where $k = 1$, $\widetilde{X}_{1,n}(z) = X_{1,n}(z)$. Then,

$$\sqrt{n} X_{k,n}(z) - \widetilde{X}_{k,n}(z) = \sqrt{n} e^{-C^*_{1,n}} \left( \int_0^1 \dot{\eta}_{k,n}(s, z) \left( \phi'_{k-1}(\eta_{k,n}(s, z)) - e^{C^*_{1,k-1}} \right) ds \right).$$

Let $g(z) = e^{-C^*_{1,k-1}} \phi_{k-1}(z) - z$. For fixed $z$, the function $g(\zeta)$ is holomorphic on the closed disc of radius $R := |z| - r'$ with centre $z$. So by Cauchy's theorem for $0 < \alpha < 2$,

$$g'(z) = \frac{1}{2\pi i} \int_{C_R} \frac{g(\zeta)}{(\zeta - z)^2} d\zeta$$

where $C_R$ is the circle of radius $R$ centred at $z$. Therefore, using the bound from equation (2.10),

$$|g'(z)| \leqslant \frac{2 \log(k-1)}{(|z| - r')\sqrt{k-1}}.$$

So on $\omega(r')$,

$$
\begin{aligned}
&|\sqrt{n}X_{k,n}(z) - \widetilde{X}_{k,n}(z)| \\
&\leqslant 2\sqrt{n}e^{-C^*_{1,n}}\left(\int_0^1 \dot{\eta}_{k,n}(s,z)\left(\frac{1}{(|\eta_{k,n}(s,z)| - r')}\frac{e^{C^*_{1,k-1}}\log(k-1)}{\sqrt{k-1}}\right)ds\right).
\end{aligned}
$$

Then since, $\inf_{0\leqslant k\leqslant n}|\eta_{k,n}(s,z)| \geqslant |z|$,

$$
\begin{aligned}
|\sqrt{n}X_{k,n}(z) - \widetilde{X}_{k,n}(z)| &\leqslant \frac{2\sqrt{n}}{r-r'}e^{-C^*_{k,n}}\frac{\log(k-1)}{\sqrt{k-1}}\int_0^1 |\dot{\eta}_{k,n}(s,z)|ds \\
&\leqslant \lambda(\alpha,c,r)\sqrt{n}e^{-C^*_{k,n}}\frac{\log(k-1)}{\sqrt{k-1}}\frac{c^*_k e^{C^*_{k,n}}}{e^{C^*_{k+1,n}}r-1} \\
&\leqslant \lambda(\alpha,c,r)\frac{\sqrt{n}}{n^{\frac{1}{\alpha}}}\frac{\log(k)k^{\frac{1}{\alpha}}}{k^{\frac{3}{2}}},
\end{aligned}
$$

where the second inequality follows from Lemma 2.4.4. Now consider,

$$
\begin{aligned}
&|\widetilde{X}_{k,n}(z) - \mathcal{X}_{k,n}(z)| \\
&\leqslant \sqrt{n}\left|\left(\frac{2c^*_k}{e^{-i\theta_k}e^{C^*_{k+1,n}}z-1}\right)\left(e^{-C^*_{k,n}}\int_0^1 \eta_{k,n}(s,z)ds - z\right)\right| \\
&+ \sqrt{n}\left|\left(e^{-C^*_{k,n}}\int_0^1 \eta_{k,n}(s,z)ds\right)\delta_{c^*_k}\left(e^{-i\theta_k}e^{C^*_{k+1,n}}z\right)\right| \\
&\leqslant \sqrt{n}\left(\left(\frac{2c^*_k}{e^{C^*_{k+1,n}}r-1}\right)\left(r\int_0^1\left|e^{x_{k,n}(s)}-1\right|ds\right) + \lambda(\alpha,c,r)\left|\delta_{c^*_k}\left(e^{-i\theta_k}e^{C^*_{k+1,n}}z\right)\right|\right)
\end{aligned}
$$

where $\lambda(\alpha,c,r)$ is some positive constant that we will vary and

$$
x_{k,n}(s) = s\left(\frac{2c^*_k}{e^{-i\theta_k}e^{C^*_{k+1,n}}z-1} + \delta_{c^*_k}\left(e^{-i\theta_k}e^{C^*_{k+1,n}}z\right)\right).
$$

Furthermore,

$$
|e^{x_{k,n}(s)} - 1| \leqslant \lambda(\alpha,c,r)|x_{k,n}(s)| \leqslant \lambda(\alpha,c,r)k^{\frac{1}{\alpha}-1}n^{-\frac{1}{\alpha}}
$$

where the second inequality follows from Lemmas 2.3.1 and 2.4.2 and Corollary 2.3.2. Hence by using the bound on $\delta_c$ from Lemma 2.4.2 we see that,

$$
|\widetilde{X}_{k,n}(z) - \mathcal{X}_{k,n}(z)| \leqslant \lambda(\alpha,c,r')\sqrt{n}\left(\left(k^{\frac{1}{\alpha}-1}n^{-\frac{1}{\alpha}}\right)^2 + k^{\frac{1}{\alpha}-\frac{3}{2}}n^{-\frac{1}{\alpha}}\right).
$$

Since $k^{\frac{1}{\alpha}} \leqslant n^{\frac{1}{\alpha}}$ we have

$$|\widetilde{X}_{k,n}(z) - \mathcal{X}_{k,n}(z)| \leqslant \lambda(\alpha, c, r) k^{\frac{1}{\alpha}-\frac{3}{2}} n^{\frac{1}{2}-\frac{1}{\alpha}}.$$

Therefore,

$$|Y_{k,n}(z)| \leqslant \lambda(\alpha, c, r) \log(n) n^{\frac{1}{2}-\frac{1}{\alpha}} k^{\frac{1}{\alpha}-\frac{3}{2}}.$$

Then we split into cases, if $0 < \alpha \leqslant \frac{2}{3}$,

$$\sup_{k \leqslant n} |Y_{k,n}(z)| \leqslant \lambda(\alpha, c, r) \frac{\log(n)}{n} \to 0$$

as $n \to \infty$. However, if $\frac{2}{3} \leqslant \alpha < 2$ then

$$\sup_{k \leqslant n} |Y_{k,n}(z)| \leqslant \lambda(\alpha, c, r) \log(n) n^{\frac{1}{2}-\frac{1}{\alpha}} \to 0$$

as $n \to \infty$. Moreover,

$$\mathbb{E}(|Y_{k,n}(z)|^2 \, |\mathcal{F}_{k-1}) \leqslant \lambda(\alpha, c, r) \frac{n}{n^{\frac{2}{\alpha}}} \frac{\log(n)^2 k^{\frac{2}{\alpha}}}{k^3}.$$

Thus if $0 < \alpha \leqslant 1$,

$$\sum_{k=k_0}^{n} \mathbb{E}(|Y_{k,n}(z)|^2 \, |\mathcal{F}_{k-1}) \leqslant \lambda(\alpha, c, r) \frac{\log(n)^3}{n} \to 0$$

as $n \to \infty$. If $1 < \alpha < 2$,

$$\sum_{k=k_0}^{n} \mathbb{E}(|Y_{k,n}(z)|^2 \, |\mathcal{F}_{k-1}) \leqslant \lambda(\alpha, c, r) \frac{\log(n)^2 n}{n^{\frac{2}{\alpha}}} \to 0$$

as $n \to \infty$. Therefore, since $Y_{k,n}(z)$ is also a martingale difference array we can use these bounds to apply the same methods to the difference $Y_{k,n}(z)$ as we did to $X_{k,n}(z)$ in Sections 2.4 and 2.5 along with a Borel-Cantelli argument to show that

$$\mathbb{P}\left( \limsup_{n \to \infty} \sup_{|z| > r} \sum_{k=k_0}^{n} |Y_{k,n}(z)| > \epsilon \right) = 0.$$

Therefore, combining this with equation (2.11) gives,

$$\mathbb{P}\left(\limsup_{n\to\infty}\sup_{|z|>r}\sum_{k=0}^{n}|Y_{k,n}(z)|>\epsilon\right)=0.$$

$\square$

**2.6.2. Laurent coefficients.** In the previous section we showed that we could discard the lower order terms of $X_{k,n}(z)$. We now wish to calculate the Laurent coefficients of the remaining higher order terms $\mathcal{X}_{k,n}(z)$ and hence evaluate the fluctuations of the cluster. We first notice that

$$\mathbb{E}(\mathcal{X}_{k,n}(z)|\mathcal{F}_{k-1})=0$$

and therefore $\mathcal{X}_{k,n}(z)$ is also a martingale difference array. We aim to use the following result of Mcleish [**McL74**] to produce a central limit theorem. Whilst Mcleish's result is more powerful than we need in this paper, it provides a framework to use similar techniques even when we do not have a nice decomposition.

THEOREM 2.6.2 (McLeish). *Let $(D_{k,n})_{1\leqslant k\leqslant n}$ be a martingale difference array with respect to the filtration $\mathcal{F}_{k,n}=\sigma(D_{1,n},D_{2,n},...,D_{k,n})$. Let $M_n=\sum_{i=1}^{n}D_{i,n}$ and assume that;*

*(I) for all $\rho>0$, $\sum_{k=1}^{n}D_{k,n}^2\ \mathbb{1}(|D_{k,n}|>\rho)\to 0$ in probability as $n\to\infty$.*
*(II) $\sum_{k=1}^{n}D_{k,n}^2\to s^2$ in probability as $n\to\infty$ for some $s^2>0$.*

*Then $M_n$ converges in distribution to $\mathcal{N}(0,s^2)$.*

Note that condition (I) in Theorem 2.6.2 combines two conditions in [**McL74**]. Theorem 2.6.2 only applies to real valued random variables and as such we will split $\mathcal{X}_{k,n}(z)$ into real and imaginary parts. We start by calculating the Laurent coefficients.

$$\mathcal{X}_{k,n}(z)=\frac{2c_k^*\sqrt{n}}{e^{-i\theta_k}e^{C_{k+1,n}^*}}\left(\frac{1}{1-\frac{1}{e^{-i\theta_k}e^{C_{k+1,n}^*}z}}\right).$$

We can choose $|z|>r$ such that $\left|\frac{1}{e^{-i\theta_k}e^{C_{k+1,n}^*}z}\right|<1$, then

$$\mathcal{X}_{k,n}(z)=\sum_{m=0}^{\infty}\frac{2c_k^*\sqrt{n}}{(e^{-i\theta_k}e^{C_{k+1,n}^*})^{m+1}}\frac{1}{z^m}.$$

So the $m^{\text{th}}$ coefficient is dependent on $n$ and $k$ and we can rewrite $\mathcal{X}_{k,n}$ as

$$\mathcal{X}_{k,n}(z) = \sum_{m=0}^{\infty} a_{k,n}(m) \frac{1}{z^m}$$

where $a_{k,n}(m) = \frac{2c_k^* \sqrt{n}}{(e^{C_{k+1,n}^*})^{m+1}} e^{i\theta_k(m+1)}$. So we can calculate real and imaginary parts of these coefficients,

$$\Re(a_{k,n}(m)) = \frac{2c_k^* \sqrt{n}}{(e^{C_{k+1,n}^*})^{m+1}} \cos(\theta_k(m+1)),$$

$$\Im(a_{k,n}(m)) = \frac{2c_k^* \sqrt{n}}{(e^{C_{k+1,n}^*})^{m+1}} \sin(\theta_k(m+1)).$$

In order to use Theorem 2.6.2 we need to calculate the second moments of the coefficients. We will just consider the case of the real coefficients here but the imaginary coefficients give the same results. Thus, we calculate,

$$\begin{aligned} \mathbb{E}((\Re(a_{k,n}(m)))^2 | \mathcal{F}_{k-1}) &= \frac{4(c_k^*)^2 n}{(e^{C_{k+1,n}^*})^{2(m+1)}} \frac{1}{2\pi} \int_0^{2\pi} \cos^2(\theta(m+1)) d\theta \\ &= \frac{2(c_k^*)^2 n}{(e^{C_{k+1,n}^*})^{2(m+1)}}. \end{aligned}$$

It is clear to see here why we have the same expected value of the imaginary coefficients. So now we can take the sum over $n$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}((\Re(a_{k,n}(m)))^2 | \mathcal{F}_{k-1}) = \lim_{n\to\infty} \left( 2n \sum_{k=1}^{n} \frac{(c_k^*)^2}{\left(e^{C_{k+1,n}^*}\right)^{2(m+1)}} \right).$$

Recall that $c_k^* = \frac{c}{1+\alpha c(k-1)}$ and we have shown we can approximate the term in the denominator in the following way;

$$e^{C_{k+1,n}^*} = \left( \frac{1 + \alpha cn}{1 + \alpha ck} \right)^{\frac{1+\epsilon_{k+1,n}}{\alpha}}$$

where $\epsilon_{k+1,n}$ is the error defined in Lemma 2.3.1. Therefore, we can write

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}((\Re(a_{k,n}(m)))^2 | \mathcal{F}_{k-1}) = \lim_{n\to\infty} \left( 2nc^2 \sum_{k=1}^{n} \frac{(1 + \alpha ck)^{\left(\frac{(1+\epsilon_{k+1,n})(2(m+1))}{\alpha}\right) - 2}}{(1 + \alpha cn)^{\left(\frac{(1+\epsilon_{k+1,n})(2(m+1))}{\alpha}\right)}} \right)$$

We know $\epsilon_{k+1,n} \to 0$ so our aim is to show that this term in the sum is insignificant. We define the function $h : \mathbb{R} \to \mathbb{R}$ as the term inside the sum;

$$h(x) := \frac{(1+\alpha ck)^{\left(\frac{(1+x)(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(1+x)(2(m+1))}{\alpha}\right)}}.$$

Our aim is to show,

$$\left|\lim_{n\to\infty} 2nc^2 \sum_{k=1}^{n} \left(h(\epsilon_{k+1,n}) - h(0)\right)\right| = 0.$$

If we can show this then we can just ignore the $\epsilon_{k,n}$ and find the limit,

$$\lim_{n\to\infty} 2nc^2 \sum_{k=1}^{n} h(0)$$

which we will show converges to a real number. We provide this in the form of the following lemma.

LEMMA 2.6.3. *With $h : \mathbb{R} \to \mathbb{R}$ defined as above we have*

$$\left|\lim_{n\to\infty} 2nc^2 \sum_{k=1}^{n} \left(h(\epsilon_{k+1,n}) - h(0)\right)\right| = 0$$

PROOF. Consider

$$|h(\epsilon_{k+1,n}) - h(0)| = \left| \frac{(1+\alpha ck)^{\left(\frac{(1+\epsilon_{k+1,n})(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(1+\epsilon_{k+1,n})(2(m+1))}{\alpha}\right)}} - \frac{(1+\alpha ck)^{\left(\frac{(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}} \right|.$$

Then let $y_{k,n} = \left(\frac{1+\alpha ck}{1+\alpha cn}\right)^{\frac{2m+2}{\alpha}}$, thus we can write

$$|h(\epsilon_{k+1,n}) - h(0)| = \frac{1}{(1+\alpha ck)^2} |y_{k,n}| \left| y_{k,n}^{\epsilon_{k+1,n}} - 1 \right|.$$

Furthermore, since $\log(y_{k,n}) < 1$,

$$\left| y_{k,n}^{\epsilon_{k+1,n}} - 1 \right| = \left| e^{\epsilon_{k+1,n} \log y_{k,n}} - 1 \right| \leqslant |\epsilon_{k+1,n}| |\log y_{k,n}|.$$

So using the first bound on $\epsilon_{k,n}$ from Lemma 2.3.1 we have,

$$
\begin{aligned}
|h(\epsilon_{k+1,n}) - h(0)| &\leqslant \frac{1}{(1+\alpha ck)^2}\left(\frac{1+\alpha ck}{1+\alpha cn}\right)^{\frac{2m+2}{\alpha}} \frac{\alpha\left(\alpha c^2(n-k)\right)\left|\log\left(\left(\frac{1+\alpha ck}{1+\alpha cn}\right)^{\frac{2m+2}{\alpha}}\right)\right|}{(1+\alpha ck)(1+\alpha cn)\log\left(\frac{1+\alpha cn}{1+\alpha ck}\right)} \\
&\leqslant (2m+2)\alpha c^2 n \frac{(1+\alpha ck)^{\frac{2m+2}{\alpha}-3}}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}}.
\end{aligned}
$$

Now we take the sum over $k$,

$$
2nc^2 \sum_{k=1}^{n} |h(\epsilon_{k+1,n}) - h(0)| \leqslant 4n^2(m+1)\alpha c^4 \frac{1}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}} \sum_{k=1}^{n} (1+\alpha ck)^{\frac{2m+2}{\alpha}-3}.
$$

Which we can approximate with a Riemann integral;

$$
2nc^2 \sum_{k=1}^{n} |h(\epsilon_{k+1,n}) - h(0)| \leqslant 4n^2(m+1)\alpha c^4 \frac{1}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}} \int_0^n (1+\alpha cx)^{\frac{2m+2}{\alpha}-3} dx.
$$

Now we need to consider cases, firstly in the case where we have $\frac{2m+2}{\alpha} - 3 \neq -1$ and so

$$
\begin{aligned}
&\left|\lim_{n\to\infty} 2nc^2 \sum_{k=1}^{n} (h(\epsilon_{k+1,n}) - h(0))\right| \\
&\leqslant \lim_{n\to\infty} 4n^2(m+1)\alpha c^4 \frac{1}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}} \left[\frac{1}{\alpha c\left(\frac{2m+2}{\alpha}-2\right)}(1+\alpha cx)^{\frac{2m+2}{\alpha}-2}\right]_0^n \\
&= \lim_{n\to\infty} \left(\frac{2(m+1)\alpha c^3}{m+1-\alpha}\left(\frac{n^2}{(1+\alpha cn)^3} - \frac{n^2}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}}\right)\right).
\end{aligned}
$$

Hence, since $0 < \alpha < 2$,

$$
\left|\lim_{n\to\infty} 2nc^2 \sum_{k=1}^{n} (h(\epsilon_{k+1,n}) - h(0))\right| = 0.
$$

Now consider the case where $\frac{2m+2}{\alpha} - 3 = -1$ and so

$$\begin{aligned}
&\left| \lim_{n\to\infty} 2nc^2 \sum_{k=1}^{n} (h(\epsilon_{k+1,n}) - h(0)) \right| \\
&\leqslant \lim_{n\to\infty} 4n^2(m+1)\alpha c^4 \frac{1}{(1+\alpha cn)^{\frac{2m+2}{\alpha}+1}} \left[ \frac{1}{\alpha c} \log(1+\alpha cx) \right]_0^n \\
&= \lim_{n\to\infty} 4n^2 c^3 \frac{\log(1+\alpha cn)}{(1+\alpha cn)^3} \\
&= 0.
\end{aligned}$$

Therefore in all cases we have

$$\left| \lim_{n\to\infty} 2nc^2 \sum_{k=1}^{n} (h(\epsilon_{k+1,n}) - h(0)) \right| = 0.$$

□

Hence by using the above lemma we can ignore the $\epsilon_{k+1,n}$ term in our summation. We now want to check the conditions of Theorem 2.6.2. We introduce the notation,

$$A_m^n = \sum_{k=1}^{n} \Re(a_{k,n}(m)), \qquad B_m^n = \sum_{k=1}^{n} \Im(a_{k,n}(m)).$$

We aim to apply Theorem 2.6.2 to show convergence of the finite dimensional distributions of $(A_i^n, B_j^n)_{i,j\geqslant 0}$ to some multivariate Gaussian distribution. The Cramér-Wold Theorem (see for example [**Dur19**]) tells us that it is sufficient to show convergence in distribution of all finite linear combinations of $A_i^n$, $B_j^n$. Therefore, let

$$\mathfrak{X}_{k,n} = \sum_{i=1}^{p} \mu_i \Re(a_{k,n}(i)) + \sum_{j=1}^{q} \nu_j \Im(a_{k,n}(j))$$

for some $1 \leqslant p, q < \infty$ and sequences of scalars $(\mu_i)_{1\leqslant i\leqslant p}$, $(\nu_j)_{1\leqslant j\leqslant q}$. It follows that $\mathfrak{X}_{k,n}$ is also a martingale difference array. Therefore, we will apply Theorem 2.6.2 to $\mathfrak{X}_{k,n}$ to show that we have convergence in distribution of finite linear combinations and hence joint convergence in distribution to a multivariate distribution. We start by checking condition (II) of Theorem 2.6.2 holds.

LEMMA 2.6.4. *Assume $m \geqslant 0$ and $0 < \alpha < 2$. Then*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Re(a_{k,n}(m)))^2|\mathcal{F}_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Im(a_{k,n}(m)))^2|\mathcal{F}_{k-1}) = \frac{2}{\alpha(2m+2-\alpha)}.$$

*Furthermore, for any $m_1, m_2 \geqslant 0$,*

$$\mathrm{Cov}(\Re(a_{k,n}(m_1), \Im(a_{k,n}(m_2))) = 0$$

*and if $m_1 \neq m_2$,*

$$\mathrm{Cov}(\Re(a_{k,n}(m_1), \Re(a_{k,n}(m_2))) = \mathrm{Cov}(\Im(a_{k,n}(m_1), \Im(a_{k,n}(m_2))) = 0.$$

PROOF. We have shown above that, in the case of the real coefficients, calculating $\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Re(a_{k,n}(m)))^2|\mathcal{F}_{k-1})$ reduces to calculating the expression

$$\lim_{n\to\infty}\left(2nc^2\sum_{k=1}^{n}\frac{(1+\alpha ck)^{\left(\frac{(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}}\right).$$

The imaginary coefficients follow by the same argument. We can approximate this with a Riemann integral

$$2nc^2\sum_{k=1}^{n}\frac{(1+\alpha ck)^{\left(\frac{(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}} = \frac{2nc^2}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}}\int_0^n(1+\alpha cx)^{\left(\frac{(2(m+1))}{\alpha}\right)-2}dx + \mathcal{E}_n$$

where $\mathcal{E}_n$ is the error left by the Riemann approximation with $|\mathcal{E}_n| < \frac{\lambda(\alpha,c)}{n}$ if $0 < \alpha \leqslant 1$ and $|\mathcal{E}_n| < \lambda(\alpha,c)n^{1-\frac{2(m+1)}{\alpha}}$ if $1 < \alpha < 2$ for some constant $\lambda(\alpha,c) > 0$. Since for all $m \geqslant 0$ and $0 < \alpha < 2$, $\frac{(2(m+1))}{\alpha} - 2 > -1$, we have,

$$\begin{aligned}
&2nc^2\sum_{k=1}^{n}\frac{(1+\alpha ck)^{\left(\frac{(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}}\\
&= \frac{2nc^2}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}}\left[\frac{1}{2c(m+1)-\alpha c}(1+\alpha cx)^{\left(\frac{(2(m+1))}{\alpha}\right)-1}\right]_0^n + \mathcal{E}_n\\
&= \frac{2c^2}{2c(m+1)-\alpha c}\left[\frac{n}{(1+\alpha cn)} - \frac{n}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}}\right] + \mathcal{E}_n.
\end{aligned}$$

We know for all $m \geqslant 0$ and $0 < \alpha < 2$, $\frac{(2(m+1))}{\alpha} > 1$ and so when we take the limit as $n \to \infty$ we have,

$$\lim_{n\to\infty}\left(2nc^2\sum_{k=1}^{n}\frac{(1+\alpha ck)^{\left(\frac{(2(m+1))}{\alpha}\right)-2}}{(1+\alpha cn)^{\left(\frac{(2(m+1))}{\alpha}\right)}}\right)=\frac{2}{\alpha(2(m+1)-\alpha)}.$$

Therefore,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Re(a_{k,n}(m)))^2|\mathcal{F}_{k-1})=\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Im(a_{k,n}(m)))^2|\mathcal{F}_{k-1})=\frac{2}{\alpha(2m+2-\alpha)}.$$

Furthermore, calculating the covariance pairwise of each combination of the random variables we see that for any $m_1, m_2$

$$\begin{aligned}\mathrm{Cov}(\Re(a_{k,n}(m_1)),\Im(a_{k,n}(m_2))) &= \mathbb{E}(\Re(a_{k,n}(m_1))\Im(a_{k,n}(m_2)))\\ &= \frac{4n(c_k^*)^2}{2\pi(e^{C_{k+1,n}^*})^{m_1+m_2+2}}\int_0^{2\pi}\cos(\theta(m_1+1))\sin(\theta(m_2+1))d\theta\\ &= 0.\end{aligned}$$

Moreover for $m_1 \neq m_2$,

$$\begin{aligned}\mathrm{Cov}(\Re(a_{k,n}(m_1)),\Re(a_{k,n}(m_2))) &= \mathbb{E}(\Re(a_{k,n}(m_1))\Re(a_{k,n}(m_2)))\\ &= \frac{4n(c_k^*)^2}{2\pi(e^{C_{k+1,n}^*})^{m_1+m_2+2}}\int_0^{2\pi}\cos(\theta(m_1+1))\cos(\theta(m_2+1))d\theta\\ &= 0.\end{aligned}$$

For $m_1 \neq m_2$,

$$\begin{aligned}\mathrm{Cov}(\Im(a_{k,n}(m_1)),\Im(a_{k,n}(m_2))) &= \mathbb{E}(\Im(a_{k,n}(m_1))\Im(a_{k,n}(m_2)))\\ &= \frac{4n(c_k^*)^2}{2\pi(e^{C_{k+1,n}^*})^{m_1+m_2+2}}\int_0^{2\pi}\sin(\theta(m_1+1))\sin(\theta(m_2+1))d\theta\\ &= 0.\end{aligned}$$

□

So we have shown that sum of the second moments of the real and imaginary parts converge. Note that it is clear to see that letting $\alpha = 2$ will not provide a finite limit using the above lemma. To apply Theorem 2.6.2 we need to show that $\sum_{k=1}^{n}(\mathfrak{X}_{k,n})^2$ also converges. We prove this with the following lemma, using a similar method to that of Silvestri in [**Sil17**].

LEMMA 2.6.5. *Let $0 < \alpha < 2$ and assume for each $m \geqslant 0$, the following limit holds in probability for some $s^2 > 0$,*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Re(a_{k,n}(m)))^2|\mathcal{F}_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\Im(a_{k,n}(m)))^2|\mathcal{F}_{k-1}) = s^2.$$

*Then for each $m \geqslant 0$, in probability,*

$$\lim_{n\to\infty}\sum_{k=1}^{n}(\Re(a_{k,n}(m)))^2 = \lim_{n\to\infty}\sum_{k=1}^{n}(\Im(a_{k,n}(m)))^2 = s^2.$$

*Therefore, if the following limit holds in probability for some $s^2 > 0$,*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\mathfrak{X}_{k,n})^2|\mathcal{F}_{k-1}) = s^2$$

*then in probability,*

$$\lim_{n\to\infty}\sum_{k=1}^{n}(\mathfrak{X}_{k,n})^2 = s^2.$$

PROOF. First we note that

$$\mathcal{Y}_k(z) = (\Re(a_{k,n}(m)))^2 - \mathbb{E}((\Re(a_{k,n}(m)))^2|\mathcal{F}_{k-1})$$

is a martingale difference array with respect to the filtration $(\mathcal{F}_{k,n})_{k\leqslant n}$.

We need to show $\mathbb{P}(|\sum_{k=1}^{n}\mathcal{Y}_k(z)| > \eta) \to 0$ as $n \to \infty$. So we first notice that by Markov's inequality,

$$\mathbb{P}\left(|\sum_{k=1}^{n}\mathcal{Y}_k| > \eta\right) \leqslant \frac{1}{\eta^2}\mathbb{E}\left(|\sum_{k=1}^{n}\mathcal{Y}_k|^2\right) = \frac{1}{\eta^2}\sum_{k=1}^{n}\mathbb{E}(\mathcal{Y}_k^2).$$

and so finally by using the property that for a random variable $X$, $\mathbb{E}((X-\mathbb{E}(X))^2) \leqslant \mathbb{E}(X^2)$ we see

$$\mathbb{P}\left(|\sum_{k=1}^{n}\mathcal{Y}_k| > \eta\right) \leqslant \frac{1}{\eta^2}\sum_{k=1}^{n}\mathbb{E}(\Re(a_{k,n}(m)))^4).$$

We have shown,

$$\Re(a_{k,n}(m)) = \frac{2c_k^*\sqrt{n}}{(e^{C^*_{k+1,n}})^{m+1}}\cos(\theta_k(m+1)).$$

So using the property that $c_k^* = \frac{c}{1+\alpha c(k-1)}$ and $e^{-C^*_{k+1,n}} \leqslant \left(\frac{1+\alpha ck}{1+\alpha cn}\right)^{1/\alpha}$ we reach the upper bound,

$$\Re(a_{k,n}(m)) \leqslant 2c(1+\alpha c)\sqrt{n}\frac{(1+\alpha ck)^{\frac{m+1}{\alpha}-1}}{(1+\alpha cn)^{\frac{m+1}{\alpha}}}. \tag{2.12}$$

Thus,

$$\Re(a_{k,n}(m))^4 \leqslant (2c(1+\alpha c))^4\, \frac{n^2(1+\alpha ck)^{\frac{4(m+1)}{\alpha}-4}}{(1+\alpha cn)^{\frac{4(m+1)}{\alpha}}}.$$

Then we consider cases. If $0 < \alpha \leqslant \frac{4}{3}(m+1)$ then when we sum over $k$ we reach the following bound,

$$\frac{1}{\eta^2}\left(\sum_{k=1}^{n}\mathbb{E}\left((\Re(a_{k,n}(m)))^4\right)\right) \leqslant \lambda(\alpha,c)\frac{1}{n}$$

where $\lambda(\alpha,c)$ is some constant. This converges to zero as $n\to\infty$. Moreover, if $\frac{4}{3}(m+1) < \alpha < 2$ then when we sum over $k$ we reach the following bound,

$$\frac{1}{\eta^2}\left(\sum_{k=1}^{n}\mathbb{E}\left((\Re(a_{k,n}(m)))^4\right)\right) \leqslant \lambda(\alpha,c)\frac{n}{n^{\frac{4(m+1)}{\alpha}}}$$

where $\lambda(\alpha,c)$ is some constant. This converges to zero as $n\to\infty$. Therefore in both cases we have convergence to zero. The proof of the imaginary case holds by the same argument. Now we consider $\lim_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}((\mathfrak{X}_{k,n})^2|\mathcal{F}_{k-1})$. By the same argument as above,

$$\mathbb{P}\left(\left|\sum_{k=1}^{n}\left((\mathfrak{X}_{k,n})^2-\mathbb{E}((\mathfrak{X}_{k,n})^2|\mathcal{F}_{k-1})\right)\right| > \eta\right) \leqslant \frac{1}{\eta^2}\sum_{k=1}^{n}\mathbb{E}((\mathfrak{X}_{k,n})^4).$$

Since the function $f(x)=x^4$, where $f:\mathbb{R}\to\mathbb{R}$, is convex, by Jensen's inequality,

$$(\mathfrak{X}_{k,n})^4 \leqslant \frac{\sum_{i=1}^{p}|\mu_i|(\Re(a_{k,n}(i))^4+\sum_{j=1}^{q}|\nu_j|(\Im(a_{k,n}(j))^4}{\sum_{i=1}^{p}|\mu_i|+\sum_{j=1}^{q}|\nu_j|}.$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{n}\mathbb{E}((\mathfrak{X}_{k,n})^4)\\
&\leqslant \sum_{k=1}^{n}\left(\frac{\sum_{i=1}^{p}|\mu_i|\mathbb{E}\left((\Re(a_{k,n}(i))^4\right)+\sum_{j=1}^{q}|\nu_j|\mathbb{E}\left((\Im(a_{k,n}(j))^4\right)}{p\inf_{1\leqslant i\leqslant p}|\mu_i|+q\inf_{1\leqslant j\leqslant q}|\nu_j|}\right)\\
&\frac{\leqslant p\sup_{1\leqslant i\leqslant p}\left(|\mu_i|\sum_{k=1}^{n}\mathbb{E}\left((\Re(a_{k,n}(i))^4\right)\right)+q\sup_{1\leqslant j\leqslant q}\left(|\nu_j|\sum_{k=1}^{n}\mathbb{E}\left((\Re(a_{k,n}(j))^4\right)\right)}{p\inf_{1\leqslant i\leqslant p}|\mu_i|+q\inf_{1\leqslant j\leqslant q}|\nu_j|}\\
&\to 0
\end{aligned}$$

as $n \to \infty$ by above. $\square$

Therefore, we have shown, in the form of the following corollary, that the condition (II) of Theorem 2.6.2 is satisfied.

COROLLARY 2.6.6. *For $a_{k,n}(m)$ defined as above, then for each $m \geqslant 0$ the following limit holds in probability,*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Re(a_{k,n}(m)))^2 = \lim_{n\to\infty} \sum_{k=1}^{n} \Im(a_{k,n}(m)))^2 = \frac{2}{\alpha(2m+2-\alpha)}.$$

*Therefore, with $\mathfrak{X}_{k,n}$ defined as above,*

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\mathfrak{X}_{k,n})^2 = \sum_{i=1}^{p} \left( \mu_i^2 \frac{2}{\alpha(2i+2-\alpha)} \right) + \sum_{j=1}^{q} \left( \nu_j^2 \frac{2}{\alpha(2j+2-\alpha)} \right)$$

So now we just need show condition (I) of Theorem 2.6.2 holds in order to apply it. We will again use a similar method to Silvestri **[Sil17]**.

LEMMA 2.6.7. *Let $0 < \alpha < 2$ and let $\mathfrak{X}_{k,n}$ be defined as above. Let $\rho > 0$ then the following limit holds in probability,*

$$\sum_{k=1}^{n} (\mathfrak{X}_{k,n})^2 \mathbb{1}(|\mathfrak{X}_{k,n}| > \rho) \to 0$$

*as $n \to \infty$.*

PROOF. We use a similar method as **[Sil17]**. Let $\delta > 0$ then

$$\begin{aligned}
&\mathbb{P}\left( \sum_{k=1}^{n} (\mathfrak{X}_{k,n})^2 \mathbb{1}(|\mathfrak{X}_{k,n}| > \rho) > \delta \right) \\
&\leqslant \mathbb{P}\left( \max_{1\leqslant k\leqslant n} |\mathfrak{X}_{k,n}| > \rho \right) \\
&\leqslant \frac{1}{\rho} \mathbb{E}\left( \max_{1\leqslant k\leqslant n} |\mathfrak{X}_{k,n}| \right) \\
&\leqslant \frac{1}{\rho} \left( \sum_{i=1}^{p} \mu_i \mathbb{E}\left( \max_{1\leqslant k\leqslant n} |\Re(a_{k,n}(i))| \right) + \sum_{j=1}^{q} \nu_j \mathbb{E}\left( \max_{1\leqslant k\leqslant n} |\Im(a_{k,n}(j))| \right) \right)
\end{aligned}$$

with the second inequality following by Markov's inequality. As in the proof of Lemma 2.6.5, we have shown that for each $m \geqslant 0$,

$$|\Re(a_{k,n}(m))| \leqslant 2c(1+\alpha c)\sqrt{n}\frac{(1+\alpha ck)^{\frac{m+1}{\alpha}-1}}{(1+\alpha cn)^{\frac{m+1}{\alpha}}}.$$

So if $m+1 \geqslant \alpha$,

$$\max_{0\leqslant k\leqslant n} |\Re(a_{k,n}(m))| \leqslant 2c(1+\alpha c)\sqrt{n}\frac{1}{(1+\alpha cn)}.$$

Then if $m+1 < \alpha$,

$$\max_{0\leqslant k\leqslant n} |\Re(a_{k,n}(m))| \leqslant 2c(1+\alpha c)\sqrt{n}\frac{1}{(1+\alpha cn)^{\frac{m+1}{\alpha}}}.$$

In both cases $\max_{0\leqslant k\leqslant n} \Re(a_{k,n}(m))$ converges to zero as $n \to \infty$. The imaginary case follows by the same argument. Thus the finite sums also converge to zero,

$$\frac{1}{\rho}\left(\sum_{i=1}^{p}\mu_i\mathbb{E}\left(\max_{1\leqslant k\leqslant n}|\Re(a_{k,n}(i))|\right)+\sum_{j=1}^{q}\nu_j\mathbb{E}\left(\max_{1\leqslant k\leqslant n}|\Im(a_{k,n}(j))|\right)\right)\to 0$$

as $n \to \infty$. Therefore,

$$\sum_{k=1}^{n}(\mathfrak{X}_{k,n})^2\mathbb{1}(|\mathfrak{X}_{k,n}|>\rho)\to 0$$

in probability as $n \to \infty$. □

So now we have all we need in order to apply Theorem 2.6.2. This leads to the following result.

THEOREM 2.6.8. *Let $0 < \alpha < 2$ and $A_m^n$, $B_m^n$ defined as above. Then the following limit holds for finite dimensional distributions,*

$$\begin{pmatrix} A_0^n + iB_0^n \\ \vdots \\ A_m^n + iB_m^n \\ \vdots \end{pmatrix} \to \begin{pmatrix} A_0 \\ \vdots \\ A_m \\ \vdots \end{pmatrix} + i\begin{pmatrix} B_0 \\ \vdots \\ B_m \\ \vdots \end{pmatrix}$$

*where $(A_i, B_j)_{i,j\geqslant 0}$ is a multivariate Gaussian distribution with $\mathbb{E}(A_i) = \mathbb{E}(B_j) = 0$ for all $i, j \geqslant 0$ and covariance structure given by,*

$$\mathrm{Cov}(A_i, B_j) = 0$$
$$\mathrm{Cov}(A_i, A_j) = \mathrm{Cov}(B_i, B_j) = \delta_{i,j}\left(\frac{2}{\alpha(2i+2-\alpha)}\right)$$

*for any $i, j \geqslant 0$ where $\delta_{i,j}$ is the Kronecker delta function.*

**2.6.3. Convergence as a holomorphic function.** Now that we have proved that the Laurent coefficients converge, we wish to show that we also have the convergence of the fluctuations as a holomorphic function. We first define the functions,

$$\tilde{\mathcal{F}}(n, z) = \sqrt{n}\left(e^{-C^*_{1,n}}\phi_n(z) - z\right)$$

and

$$\mathcal{F}(z) = \sum_{m=0}^{\infty}(A_m + iB_m)z^{-m}$$

where $A_m$, $B_m$ are defined as in Theorem 2.6.8. Our aim is to show that $\tilde{\mathcal{F}}(n, z) \to \mathcal{F}(z)$ in distribution as $n \to \infty$ on the space of holomorphic functions, $\mathcal{H}$, equipped with the metric,

$$\mathbf{d}_{\mathcal{H}}(f, g) = \sum_{m\geqslant 0} 2^{-m}\left(1 \wedge \sup_{|z|\geqslant 1+2^{-m}} |f(z) - g(z)|\right).$$

By the maximum principle this metric topologizes the topology of locally uniform convergence. We use a similar method as in [**NST19**] by defining,

$$\mathbf{d}_r(f, g) = \sup_{|z|>r} |f(z) - g(z)|.$$

To make notation easier, we also define $M(n, m) = \sum_{k=1}^{n} a_{k,n}(m)$. We first need the following lemma used to discard the tail terms.

LEMMA 2.6.9. *Let $r > 1$ and $N > 0$ then for any $\epsilon > 0$*

$$\lim_{T\to\infty}\sup_{n>N}\mathbb{P}\left(\boldsymbol{d}_r\left(\sum_{m=T}^{\infty} M(n, m)z^{-m}, 0\right) > \epsilon\right) = 0.$$

PROOF. Using the definition of $\mathbf{d}_r(f,g)$ we see that,

$$\mathbf{d}_r\left(\sum_{m=T}^{\infty} M(n,m)z^{-m}, 0\right) = \sup_{|z|>r}\left|\sum_{m=T}^{\infty} M(n,m)z^{-m}\right|.$$

By Markov's inequality,

$$\begin{aligned}\mathbb{P}\left(\mathbf{d}_r\left(\sum_{m=T}^{\infty} M(n,m)z^{-m}, 0\right) > \epsilon\right) &\leqslant \frac{1}{\epsilon^2}\mathbb{E}\left(\sup_{|z|>r}\left|\sum_{m=T}^{\infty} M(n,m)z^{-m}\right|^2\right)\\ &\leqslant \frac{1}{\epsilon^2}\mathbb{E}\left(\sup_{|z|>r}\left(\sum_{m=T}^{\infty} |M(n,m)||z|^{-m}\right)^2\right)\\ &\leqslant \frac{1}{\epsilon^2}\mathbb{E}\left(\left(\sum_{m=T}^{\infty} |M(n,m)|r^{-m}\right)^2\right).\end{aligned}$$

Using the Cauchy-Schwarz inequality we have,

$$\begin{aligned}\mathbb{P}\left(\mathbf{d}_r\left(\sum_{m=T}^{\infty} M(n,m)z^{-m}, 0\right) > \epsilon\right) &\leqslant \frac{1}{\epsilon^2}\mathbb{E}\left(\left(\sum_{m=T}^{\infty} |M(n,m)|^2 r^{-m}\right)\left(\sum_{m=T}^{\infty} r^{-m}\right)\right)\\ &\leqslant \frac{\lambda(r)}{\epsilon^2}\mathbb{E}\left(\sum_{m=T}^{\infty} |M(n,m)|^2 r^{-m}\right)\end{aligned}$$

where $\lambda(r)$ is some constant dependent on r. Then we can take the expectation inside the sum, thus,

$$\mathbb{P}\left(\mathbf{d}_r\left(\sum_{m=T}^{\infty} M(n,m)z^{-m}, 0\right) > \epsilon\right) \leqslant \frac{1}{\epsilon^2}\sum_{m=T}^{\infty}\mathbb{E}\left(|M(n,m)|^2\right) r^{-m}.$$

Now notice that,

$$\mathbb{E}\left(|M(n,m)|^2\right) = \mathbb{E}\left(\left|\sum_{k=1}^{n} a_{k,n}(m)\right|^2\right) \leqslant \mathbb{E}\left(\sum_{k=1}^{n}(\Re(a_{k,n}(m))^2 + (\Im(a_{k,n}(m))^2\right).$$

But in equation (2.12) we show that

$$(\Re(a_{k,n}(m))^2 + (\Im(a_{k,n}(m))^2 \leqslant \lambda(\alpha,c,r)n^{1-\frac{2(m+1)}{\alpha}}k^{\frac{2(m+1)}{\alpha}-2}$$

where $\lambda(\alpha,c,r)$ is some constant. Taking the sum over $k$ we see that,

$$\mathbb{E}\left(|M(n,m)|^2\right) \leqslant \lambda(\alpha,c,r).$$

Therefore,

$$\lim_{T\to\infty} \sup_{n>N} \mathbb{P}\left(\mathbf{d}_r\left(\sum_{m=T}^{\infty} M(n,m)z^{-m}, 0\right) > \epsilon\right) \leqslant \lim_{T\to\infty} \frac{1}{\epsilon^2}\lambda(\alpha,c,r)\sum_{m=T}^{\infty} r^{-m}$$
$$\to 0 \text{ as } T \to \infty.$$

□

Therefore, through Theorem 2.6.8 we have shown that we have convergence of the Laurent coefficients. Moreover, Lemma 2.6.9 shows that the tails of the Laurent series tend to zero in the limit. We can then combine these two results to show that we have convergence as a holomorphic function and therefore the fluctuations behave like a Gaussian field.

THEOREM 2.6.10. *Let $0 < \alpha < 2$ and $\phi_n$ be defined as in Theorem 2.5.1. Then as $n \to \infty$,*

$$\sqrt{n}\left(e^{-\sum_{i=1}^{n} c_i^*}\phi_n(z) - z\right) \to \mathcal{F}(z)$$

*in distribution on $\mathcal{H}$, where $\mathcal{H}$ is the space of holomorphic functions on $|z| > 1$, equipped with metric $\boldsymbol{d}_{\mathcal{H}}$ defined above, and where*

$$\mathcal{F}(z) = \sum_{m=0}^{\infty}(A_m + iB_m)z^{-m}$$

*and $A_m$, $B_m \sim \mathcal{N}\left(0, \frac{2}{\alpha(2m+2-\alpha)}\right)$ and $A_m$, $B_k$ independent for all choices of $m$ and $k$.*

CHAPTER 3

# Scaling limits of anisotropic growth on logarithmic timescales

In this chapter we present the second paper [**LT21b**]. We study the anisotropic version of the Hastings-Levitov model AHL($\nu$). Previous results have shown than on bounded timescales the harmonic measure on the boundary of the cluster converges in the small-particle limit to the solution of a deterministic ordinary differential equation. We consider the evolution of the harmonic measure on logarithmic timescales and show there exists a critical logarithmic time window in which the harmonic measure flow, started from the unstable fixed point, moves stochastically from the unstable point towards a stable point.

## 3.1. Introduction

The aim of this paper is to study the behaviour of a class of random growth processes modelled using conformal mappings. In recent years, many models have been introduced in order to study various real world random growth processes from lightning strikes and mineral aggregation to tumoral growth. The most well known examples include the Eden model [**Ede61**] and DLA [**WS83**]. These models, built on a lattice, have been well studied but rigorous results have proved difficult to come by (see for example [**Kes90**]). One reason for this is that lattice based models provide little in the way of mathematical techniques that can be used to study their behaviour. One way to combat this difficulty is to form off-lattice versions of the models using conformal mappings which allows us to study the processes in the complex plane and use complex analysis techniques. The models are constructed as follows. For any conformal map $f : \{|z| > 1\} \to \mathbb{C}$ we define the capacity of the map to be,

$$\lim_{z\to\infty} \log \left(f'(z)\right) := \log f'(\infty).$$

We will consider slit particles corresponding to maps of the form,

$$f(z) = e^{c}z + \mathcal{O}(1)$$

at infinity with capacity $c > 0$. Then there is a one-to-one correspondence between capacities and conformal maps attaching a slit onto the boundary of the disk. More explicitly, for each $c > 0$, we can find a unique single slit mapping

$$f_c : \{|z| > 1\} \rightarrow \{|z| > 1\} \backslash (1, 1 + d]$$

which takes the exterior of the unit disk to itself minus a slit of length $d = d(c)$ at $z = 1$. The relation between the length of the attached slit $d$ and corresponding capacity $c$ is given by

$$e^c = 1 + \frac{d^2}{4(1 + d)}.$$

We rescale and rotate the mapping $f_c(z)$ to allow any attaching point on the boundary of the unit disk by defining

$$f_n(z) = e^{2\pi i\theta_n} f_{c_n}(ze^{-2\pi i\theta_n}) \tag{3.1}$$

where $\theta_n$ is the attaching angle, identified with the interval $[0, 1]$, and $c_n$ is the capacity of the $n^{\text{th}}$ particle map $f_{c_n}(z)$. The cluster is formed by composing the slit maps. Let $K_0 = \{|z| \leqslant 1\}$ and suppose that we have some compact set $K_n$ made up of $n$ slits. We can find a bi-holomorphic map which fixes $\infty$ and takes the exterior of the unit disk to the complement of $K_n$ in the complex plane, $\phi_n : \Delta \rightarrow \mathbb{C} \backslash K_n$. We then define the map $\phi_{n+1}$ inductively;

$$\phi_{n+1} = \phi_n \circ f_{n+1} = f_1 \circ f_2 \circ .... \circ f_{n+1}.$$

By choosing the attaching angles and capacities effectively we can model a wide class of growth processes.

**3.1.1. AHL($\nu$) model and the discrete harmonic measure flow.** In this paper, we study the anisotropic Hastings-Levitov model introduced in [**JVST12**] as AHL($\nu$). The model is constructed as above with the attaching angles chosen to be i.i.d on the unit circle according to some non-uniform probability measure $\nu$ and the capacities are chosen to be a fixed value $c$. The shape of the cluster in the small particle limit is described in [**JVST12**], however, we often want to understand the ancestral path of each attached particle. Evaluating how the harmonic measure evolves on the boundary of the cluster allows us to do so. For the purpose of the introduction we define the discrete harmonic measure flow for $x \in \mathbb{R}$ as

$$X_n(x) = \frac{1}{2\pi i}\log(\Gamma_n(e^{2\pi i x})).$$

where $\Gamma_n(x) = \phi_n^{-1}(x) = f_n^{-1} \circ ...f_1^{-1}(x)$. However, some care is needed as $\Gamma_n(x)$ is not defined on the cluster boundary and thus we define this more explicitly in later sections. The function $X_n(x)$ tells us how the harmonic measure evolves under the map $\phi_n(x)$. Our aim is to evaluate how this function evolves in its scaling limit. We will consider the scaling limit of $X_{n(t)}(x)$ on logarithmic timescales as $c \to 0$ where $n(t) = \left\lfloor \frac{t}{c} \right\rfloor$ embeds continuous time into discrete time steps.

**3.1.2. Previous work.** The AHL($\nu$) model is a variation of the Hastings-Levitov model HL($\alpha$) (introduced in [**HL98**]). The Hastings-Levitov model is formed using conformal maps as described above and the attaching angles are chosen uniformly, in contrast to AHL($\nu$) where the attaching points are distributed according to a non-uniform measure. This choice represents a good model for many of the real world processes where particles diffuse onto the boundary at each iteration (for a more detailed description see, for example, [**LT21a**]). Furthermore, the capacities are chosen as,

$$c_n = c|\phi'_{n-1}(e^{i\theta_n})|^{-\alpha}.$$

The parameter $\alpha$ allows us to vary between off-lattice versions of the previously well studied models by varying the size of the attached slits. By choosing the capacities and attaching angles in this way we can model a wide class of real world Laplacian growth processes where the local growth rate is chosen according to harmonic measure. In recent years research into the Hastings-Levitov model has been fruitful. The majority of the results have concentrated on the scaling limits of the model in in the small-particle limit where we evaluate the cluster $\phi_n$ as we send the particle capacity $c \to 0$ while sending $n \to \infty$ with $nc \sim t$ for some $t$. In [**NT12**] Norris and Turner show that in the small-particle limit, for $\alpha = 0$, the limiting cluster behaves like a growing disk . Furthermore, in [**JVST15**] Turner, Viklund and Sola show that in the small particle limit the shape of the cluster in a regularised setting approaches a disk for all $\alpha \geqslant 0$ provided the regularisation is sufficient. Moreover, Silvestri

[**Sil17**] shows that the fluctuations on the boundary, for HL(0), in the small-particle limit can be characterised by a log-correlated Gaussian field.

The Hastings-Levitov model has also been evaluated under another scaling limit where rather than letting $c \to 0$ as $n \to \infty$, instead, the limit of the cluster is found by rescaling the whole cluster by the logarithmic capacity of the cluster at time $n$, before taking limits as the number of particles tends to infinity. In [**RZ05**] Rohde and Zinsmeister introduce a regularisation to the model and show that in the case of $\alpha = 0$ the rescaled cluster converges to a (random) limit with respect to the topology of normalised exterior Riemann maps. In [**LT21a**], Liddle and Turner show that for $\alpha = 0$ the scaling limit of the cluster under capacity rescaling is not a disk. Furthermore the authors study a regularised version of the model and show that for $0 < \alpha < 2$ the scaling limit under capacity rescaling is a disk and the fluctuations behave like a Gaussian field.

However, we would also like to study a wider class of processes where the particles are not attached uniformly. The ALE$(\alpha, \eta)$ model introduced in [**STV19**] generalises the Hastings-Levitov by choosing the local growth rate to be determined by $|\phi_n'|^{-\eta}$ where $\eta \in \mathbb{R}$. The authors show that there exists a phase transition at $\eta = 1$ when $\alpha = 0$ where the limiting shape in the small particle limit transitions from a disk to a radial slit. In [**Hig20**], Higgs considers the model for $\alpha = 0$ and for large negative values of the parameter $\eta$ where the particles are attached in areas of low harmonic measure and shows that there exists a phase transition where the ALE cluster converges to an $\text{SLE}_4$ curve.

The final generalisation is the anisotropic version of the Hastings-Levitov model AHL$(\nu)$ which will be the subject of this paper. In [**JVST12**] Turner, Viklund and Sola show that if $\phi$ is the solution to Loewner-Kufarev equation driven by the measure $\nu$ and $\phi_n = f_1 \circ ...f_n$ then $\phi_n \to \phi$ uniformly on compact sets in the exterior unit disk almost surely as $c \to 0$. Furthermore, the authors study the scaling limits of the harmonic measure flows in continuous time and show that on bounded timescales they can be described by the solution to a deterministic ordinary differential equation related to the Loewner equation. In contrast, in this paper we will study the model in a discrete time setting and evaluate the scaling limits of the harmonic measure flows on logarithmic timescales.

**3.1.3. Summary of the main results and physical interpretation.** The aim of the paper is to understand the scaling limit of the harmonic measure flow $X_{n(t)}(x)$. In this section we describe the physical interpretation of the main results of the paper. To do so we include cartoons (Figure 11, Figure 14 and Figure 15 below) to aid our descriptions. It should be noted that these illustrations are not accurate simulations and are not drawn to scale but instead serve as an example of one potential evolution of a AHL($\nu$) cluster. In our example we consider an AHL($\nu$) cluster where the measure $\nu$ is concentrated on a segment of the disk such as $d\nu(e^{2\pi x}) = 2\sin^2(m\pi x)dx$ for a fixed $m \in \mathbb{N}$ (as chosen in Figure 2 from **[JVST12]** which has been reproduced in this thesis as Figures 12 and 13). In order to state our main results we need to use notation that we define explicitly in later sections. In equation (3.5) we define $\psi_t(x)$, the solution to a deterministic ordinary differential equation and we define $Z_\infty$ as a Gaussian random variable with mean 0 and variance given explicitly in Corollary 3.4.8 in terms of $\nu$. We first consider the evolution of the harmonic measure $X_{n(t)}$ illustrated in Figure 11. Our first main result, appearing later as Theorem 3.2.9, describes the evolution up to a logarithmic time.

Theorem. *Let the ordinary differential equation $\psi_t(x)$ be defined as in equation (3.5). Let $T_0 = \frac{1}{4\|b'\|_\infty}\left(\log(c^{-1}) - 3\log(\log(c^{-1}))\right)$ then if $0 \leqslant t < T_0$, $X_{n(t)}(x)$ converges to $\psi_t(x)$ in probability as $c \to 0$.*

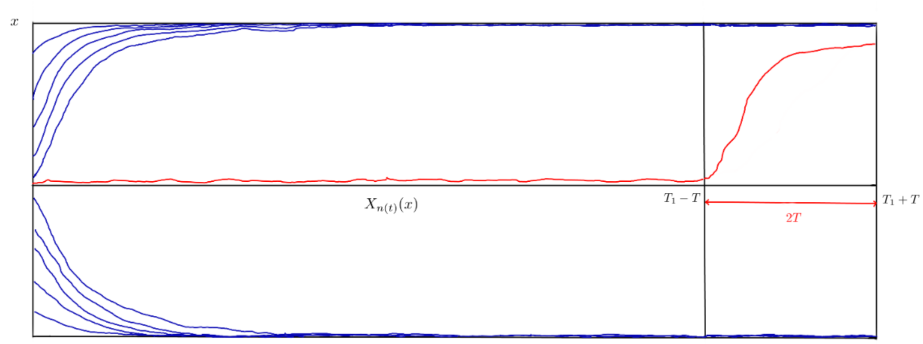


Figure 11. The evolution of $X_{n(t)}$.

Then in Section 3.3 we evaluate the fluctuations $\widetilde{Z}_n(x) = c^{-\frac{1}{4}}\left(\psi_{nc}^{-1}(X_n(x)) - x\right)$ and showed that they converge to the solution of a stochastic differential equation $Z_t(x)$ defined in equation (3.7). This result appears later as Theorem 3.3.1 and is stated as follows.

THEOREM. *The stochastic process $\widetilde{Z}_{n(t)}(x) \to Z_t(x)$ in distribution as $c \to 0$ with respect to the Skorohod topology.*

These results combine to classify the evolution of harmonic measure on compact intervals. The results show that on this timescale the trajectories of the harmonic measure $X_{n(t)}(x)$ process remain close to the deterministic trajectories of the ordinary differential equation $\psi_t(x)$. We demonstrate this in Figure 11 with each blue trajectory remaining close to the solution to the ODE up to this time. Yet, consider the simulations in Figure 12 and Figure 13 taken from [**JVST12**]. Figure 12 is an example of a separate AHL($\nu$)

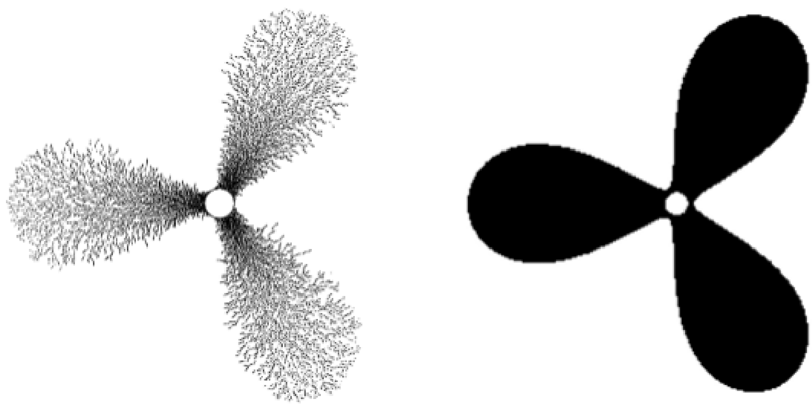

FIGURE 12. An example of a AHL($\nu$) cluster (left) and the corresponding Loewner hull (right) from [**JVST12**].

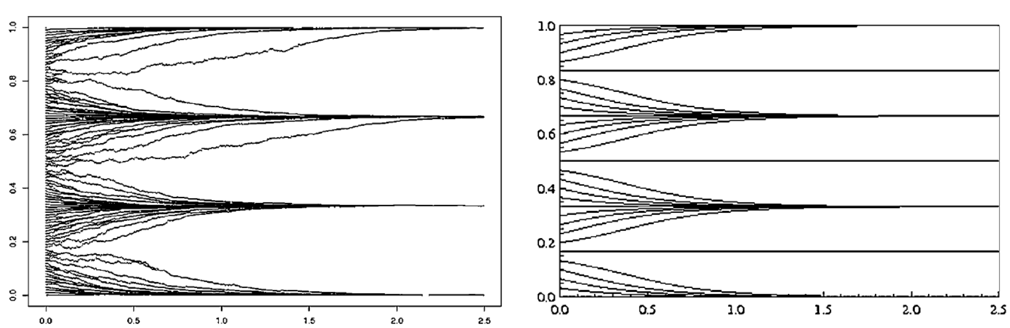


FIGURE 13. Harmonic measure $X_{n(t)}(x)$ on the boundary of AHL($\nu$) plotted against time with the departure point $x$ indicated on the $y$-axis (left) and the solution to a corresponding deterministic ODE (right) from [**JVST12**].

cluster that we may wish to study, whereas, Figure 13 provides the corresponding evolution of harmonic measure on the boundary of the cluster and a deterministic ODE. The figures demonstrate how, on compact time intervals, the harmonic measure on the boundary of the cluster converges to the solution of the deterministic ODE. However, we observe that the harmonic measure started at an unstable point of the deterministic ODE initially remains close to the fixed point before eventually moving away.

Consequently, in Section 3.4 we study the behaviour of the harmonic measure flow, started from the unstable point on longer timescales. We prove the following result which appears later as Corollary 2.4.11.

COROLLARY. *Let* $a_u$ *be an unstable fixed point of* $\psi_t(x)$*. Then for* $0 < t < \infty$*,* $X_{n(t)}(a_u)$ *converges to* $\psi_t\left(a_u + c^{\frac{1}{4}} Z_\infty(a_u)\right)$ *in probability as* $c \to 0$*, where* $Z_\infty(a_u)$ *is a Gaussian with mean* $0$ *and variance which can be given explicitly in terms of the measure* $\nu$*.*

This result tells us that there exists a logarithmic time window where $X_{n(t)}(a_u)$ moves a macroscopic distance away from the fixed point $a_u$. Once the process reaches this macroscopic distance it remains close to the trajectory started from that distance. But we know that the trajectories started significantly far away from the unstable point converge to the stable point. Therefore, once the process gets close enough to the stable point we remain close. Thus, now consider the process stopped on this logarithmic time window and evaluate the origin of trajectories stopped at this time. As points started near the unstable point have moved towards the stable point, the region in which all trajectories originate from near the unstable point is extended.

We have also demonstrated this behaviour in Figure 11. The red trajectory represents the behaviour of the harmonic measure started at the unstable point. If we converged to the solution of the ODE we would expect this trajectory to remain close to the unstable point, however, the cartoon demonstrates the stochastic nature of the path the trajectory takes from the unstable point towards a stable point on the critical time window.

Now we consider what the physical interpretation of this is on the AHL($\nu$) cluster itself. We will describe this using the notion of gap paths. The explicit definition of gap paths is provided in [**NT12**], however, intuitively the gap path from a point $z \in \mathbb{C}$ represents the shortest path from $z$ to outside the boundary of the cluster. This is demonstrated in Figure 14 with particles represented as disks. We consider the point $z$ and imagine a piece of string attached at $z$ and pulled tight vertically until we leave the boundary of cluster. This

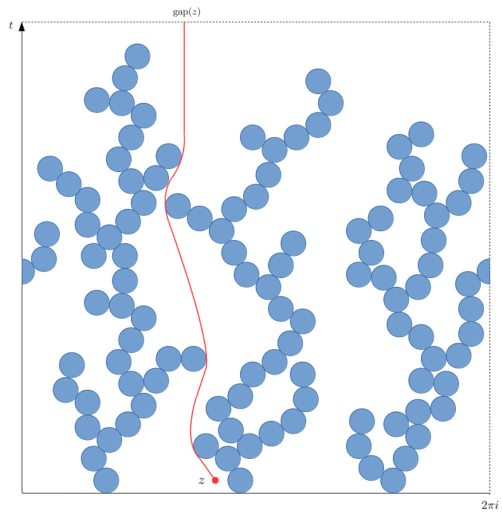


FIGURE 14. An example of a gap path

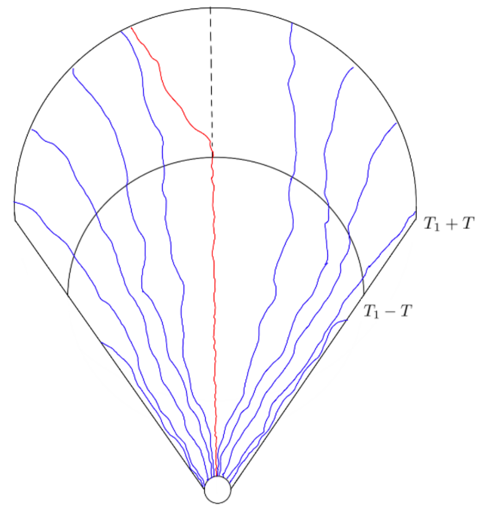


FIGURE 15. An example of a possible AHL($\nu$) cluster

represents the gap path of the point $z$ and is indicated by the red line in Figure 14. Note that the gap paths are dependent on the number of particles $n$ attached to the cluster. The gap paths can not intersect the particles unless $z$ initially is contained inside a particle in which case we choose the shortest path to leave the particle we are contained in and then proceed as above. It is shown in [**NT12**] that in the limit as $c \to 0$ the trajectories of the gap paths are described by the harmonic measure flow, under a deterministic transformation.

With the notion of gap paths in mind we can describe the behaviour of the cluster on longer timescales. We demonstrate this with the cartoon in Figure 15. The harmonic measure flows allow us to map the ancestry of each the particles on the boundary of the growing cluster to an origin on the boundary of the unit disk. In Figure 15, the unstable point is at the centre of the arc on the unit disk and the stable points on either edge. Consider the gap path of a point near the origin. On compact time intervals we expect particles attached away from the stable points to have ancestors attached near the unstable point and thus as the gap path can not intersect the particles we would expect the gap path of a particle near the origin to be vertical. However, as we enter the critical time window the harmonic measure flow is no longer close to solution of the ODE started from the unstable point but instead follows a trajectory started at a macroscopic distance from the unstable point. Therefore, the successive particles are not attached vertically and the gap path becomes antisymmetric as indicated by the red path in Figure 15. The direction the gap path follows is dependent on the sign of $Z_\infty(a_u)$. Therefore, in summary, we show that on bounded timescales the process remains symmetric however as we enter the critical time window the process becomes asymmetric about the origin.

The outline of the paper is as follows. In Section 3.2 we provide estimates that will be used in the remainder of the paper and then show that the harmonic measure flow defined on discrete time steps $X_{n(t)}$ converges to a deterministic ODE up to a logarithmic time. In Section 3.3 we classify the fluctuations and show they demonstrate stochastic behaviour. Finally, in Section 3.4 we prove the existence of a critical logarithmic time window and show that on this interval the harmonic measure flow, started from the unstable point follows a stochastic path away from the unstable trajectory and towards a stable trajectory.

Throughout the remainder of the paper we introduce a large amount of notation, therefore, for the benefit of the reader we provide a list of symbols at the end of the paper so that it can be referred to throughout the article.

## 3.2. Convergence on logarithmic timescales

**3.2.1. Definitions and estimates.** The aim of this subsection is to introduce the notation and estimates which we will call upon in the remainder of the paper. Much of what is presented here is a reformulation of the continuous time estimates produced in [**JVST12**]. This is an essential part of the analysis but the reader is advised that our main results will follow in later sections. The goal of this paper is to analyse the evolution of harmonic measure on the boundary of the AHL($\nu$) clusters. Thus, as above we consider the cluster formed by the conformal maps $f_n : \{|z| > 1\} \to \mathbb{C}\backslash K_n$ defined in (3.1) with attaching angles $\theta_1, \theta_2, ...$ independently randomly distributed on the unit circle with law $\nu$. We identify the circle with the interval $(0, 1)$ and assume $\nu$ has a twice continuously differentiable density $h_\nu$. Furthermore, as above, for a point $x \in (0, 1)$,

$$\gamma(x) = \frac{1}{2\pi i} \log(f_c^{-1}(e^{2\pi i x}))$$

choosing the branch of logarithm which results in $x = \frac{1}{2}$ being fixed. An explicit form is given by

$$\gamma(x) = \frac{1}{\pi} \tan^{-1}\left(\sqrt{e^c \tan^2(\pi x) + e^c - 1}\right)$$

where the branch of arctan is chosen to map $[0, \pi] \to [0, \pi]$. We can extend this to the real line as follows, if $x = k + a$ where $a \in (0, 1]$ then define $\gamma(x) = k + \gamma(a)$. Then for all $x \in \mathbb{R}$ we define

$$\gamma_n(x) = \gamma_n(x) = \gamma(x - \theta_n) + \theta_n.$$

It immediately follows that $\gamma_n(x) = \frac{1}{2\pi i} \log(f_n^{-1}(e^{2\pi i x}))$ for the corresponding branch of logarithm. This function then describes the change in angle of a point $x$ on the boundary under the transformation $f_n(x)$ and thus $\gamma_n(x)$ tells us how the harmonic measure evolves under the map $f_n(x)$. Let $\tilde{\gamma}(x) = \gamma(x) - x$ then we can define the discrete harmonic measure flow under the map $\phi_n$ for $x \in \mathbb{R}$ as,

$$X_n(x) = X_{n-1}(x) + \tilde{\gamma}(X_{n-1}(x) - \theta_n) \tag{3.2}$$

with $X_0(x) = x$. Therefore,

$$X_n(x) = \gamma_n(X_{n-1}(x)).$$

Thus if $\Gamma_n(x) = \phi_n^{-1}(x) = f_n^{-1} \circ ... f_1^{-1}(x)$ then

$$\begin{aligned} X_n(x) &= \frac{1}{2\pi i} \log(f_n^{-1}(e^{2\pi i X_{n-1}(x)})) \\ &= \frac{1}{2\pi i} \log(f_n^{-1}(\Gamma_{n-1}(e^{2\pi i x}))) \\ &= \frac{1}{2\pi i} \log(\Gamma_n(e^{2\pi i x})). \end{aligned}$$

Note that we define $X_n(x)$ in this way to make sure the branch of the logarithm respects the composition structure. We can then rewrite the harmonic measure flow as

$$X_n(x) = \sum_{i=1}^{n} \tilde{\gamma}(X_{i-1}(x) - \theta_i) + x.$$

Let $\mathcal{F}_n$ be the $\sigma$-algebra generated by the set $\{\theta_i \ : 1 \leqslant i \leqslant n\}$. Then as in **[JVST12]**, in order to evaluate the conditional expectation of each increment $\tilde{\gamma}(X_{i-1}(x) - \theta_i)$ with respect to $\mathcal{F}_{i-1}$, we define,

$$\beta_\nu(x) = \int_0^1 \tilde{\gamma}(x - z) h_\nu(z) dz$$

where, as above, $h_\nu$ is the twice continuously differentiable density of $\nu$ on $\mathbb{R}$. Now define,

$$Y_n(x) = \tilde{\gamma}(X_{n-1}(x) - \theta_n) - \beta_\nu(X_{n-1}(x))$$

and then let $S_n(x) = \sum_{i=1}^n Y_i(x)$. We can write

$$X_n(x) = x + S_n(x) + \sum_{i=1}^n \beta_\nu(X_{i-1}(x)). \tag{3.3}$$

The following lemma then holds.

LEMMA 3.2.1. *$S_n(x)$, as defined above, is a martingale with respect to $\mathcal{F}_n$.*

PROOF. Taking the expectation with respect to $\mathcal{F}_{i-1}$,

$$\begin{aligned}
\mathbb{E}\left(Y_i(x)|\mathcal{F}_{i-1}\right) &= \int_0^1 \tilde{\gamma}(X_{i-1}(x) - \theta)h_\nu(\theta)d\theta - \beta_\nu(X_{i-1}(x)) \\
&= \beta_\nu(X_{i-1}(x)) - \beta_\nu(X_{i-1}(x)) \\
&= 0.
\end{aligned}$$

Hence,

$$\mathbb{E}\left(S_n(x)|\mathcal{F}_{n-1}\right) = S_{n-1}(x) + \mathbb{E}\left(Y_n(x)|\mathcal{F}_{n-1}\right) = S_{n-1}(x).$$

Therefore, $S_n(x)$ is a martingale with respect to $\mathcal{F}_n$. □

Throughout the remainder of the paper we will rely on estimates on each of the terms defined above. In **[JVST12]** the authors provide the estimates, for a symmetric particle, stated in the following lemma.

LEMMA 3.2.2. *For $\tilde{\gamma}$ defined as above,*

$$\int_0^1 \tilde{\gamma}(z)dz = 0$$

*and there exists a constant $\rho_0$ such that*

$$c^{-\frac{3}{2}} \int_0^1 \tilde{\gamma}(z)^2 dz \to \rho_0$$

*as $c \to 0$. Furthermore, there exists a constant $\delta > 0$ such that the following estimates hold,*

$$\|\tilde{\gamma}\|_\infty \leqslant \delta\sqrt{c},$$

*and*

$$\left|\int_0^1 \tilde{\gamma}(x-\theta)^2 h_\nu(\theta)d\theta - \rho_0 c^{\frac{3}{2}} h_\nu(x)\right| \leqslant \delta \|h_\nu'\|_\infty c^2 \log(c^{-1})$$

*for $c$ sufficiently small.*

With the harmonic measure flow in the form of equation (3.3) we aim to study its scaling limit. To do so we start by defining the function $b(x)$ as the Hilbert transform of the measure $\nu$,

$$b(x) = \frac{1}{2\pi}\int_0^1 \cot(\pi z)(h_\nu(x-z) - h_\nu(x))dz. \tag{3.4}$$

With this definition, the proof of Proposition 2 in **[JVST12]** provides the following bound on the difference $\left|\frac{1}{c}\beta_\nu(x) - b(x)\right|$ for $c$ sufficiently small.

LEMMA 3.2.3. *For each $x$ and $c < \frac{1}{2}$, there exists a constant $\delta > 0$ such that,*

$$\left|\frac{1}{c}\beta_\nu(x) - b(x)\right| < \delta c^{\frac{1}{2}} \log(c^{-1}).$$

Thus throughout the remainder of the paper we assume $0 < c < \frac{1}{2}$. Then using these bounds, we can make further estimates on $\beta_\nu(x)$ and $Y_n(x)$.

LEMMA 3.2.4. *For $\beta_\nu(x)$ and $Y_n(x)$ defined as above, there exists a constant $\delta > 0$ such that the following estimate holds,*

$$|\beta_\nu(x)| \leqslant \delta c$$

*and for each $n$,*

$$|Y_n(x)| \leqslant \delta\sqrt{c}.$$

PROOF. Using Lemma 3.2.3 it immediately follows,

$$|\beta_\nu(x)| \leqslant c\,|b(x)| + \delta_1 c^{\frac{3}{2}} \log(c^{-1}) \leqslant \delta c$$

for some universal constants $\delta_1$ and $\delta$. For the second bound we use that

$$|Y_n(x)| \leqslant |\tilde{\gamma}(X_{n-1}(x) - \theta_n)| + |\beta_\nu(X_{n-1}(x))| \leqslant \delta\sqrt{c}.$$

□

**3.2.2. Results.** In this section we consider the evolution of harmonic measure flows on the boundary of the cluster as $c \to 0$. We will consider $X_{n(t)}(x)$ on logarithmic timescales where $n(t) = \lfloor \frac{t}{c} \rfloor$. Define the function $\psi_t$ to be the solution to the following ordinary differential equation,

$$\dot{\psi}_t(x) = b(\psi_t(x)) \tag{3.5}$$

for $x \in \mathbb{R}$ and $\psi_0(x) = x$ where $b(x)$ is defined in equation (3.4). As above, throughout the paper we assume that $h_\nu$ is twice continuously differentiable. By properties of Hilbert transforms it follows that $b(x)$ is also twice continuously differentiable. Furthermore, during calculations, for simplicity purposes we will often treat $n(t)c$ as $t$, however, we note that the difference is of order $c$ and this as we take the limit as $c \to 0$ our results will be unchanged. The aim for the rest of this section is to to show that up to a logarithmic time $X_{n(t)}(x)$ converges to $\psi_t(x)$. Recall for each $n \in \mathbb{N}$,

$$X_n(x) = x + S_n(x) + \sum_{i=1}^{n} \beta_\nu(X_{i-1}(x)).$$

So,

$$\begin{aligned}
|X_{n(t)}(x) - \psi_t(x)| &= \left| S_{n(t)}(x) + \sum_{i=1}^{n(t)} \beta_\nu(X_{i-1}(x)) - \int_0^t b(\psi_s(x))ds \right| \\
&\leqslant \left|S_{n(t)}(x)\right| + \left| \sum_{i=1}^{n(t)} \beta_\nu(X_{i-1}(x)) - \int_0^t b(\psi_s(x))ds \right|.
\end{aligned}$$

We first consider the latter term and find an upper bound.

LEMMA 3.2.5. *Let the functions $\beta_\nu(x)$ and $b(x)$ be defined as above, then for each $x \in \mathbb{R}$,*

$$\left| \sum_{i=1}^{n(t)} \beta_\nu(X_{i-1}(x)) - \int_0^t b(\psi_s(x))ds \right| < \delta \log(c^{-1}) c^{\frac{3}{2}} n(t) + \|b\|_\infty \int_0^t |X_{n(r)}(x) - \psi_r(x)| dr$$

*for some positive constant $\delta$.*

PROOF. Let $\delta$ be some positive constant that we allow to vary. First we find an upper bound,

$$\begin{aligned}
&\left|\sum_{i=1}^{n(t)} \beta_\nu(X_{i-1}(x)) - \int_0^t b(\psi_s(x))ds\right| \\
&\leqslant \left|\sum_{i=1}^{n(t)} \beta_\nu(X_{i-1}(x)) - \sum_{i=1}^{n(t)} cb(X_{i-1}(x))\right| + \left|\sum_{i=1}^{n(t)} cb(X_{i-1}(x)) - \int_0^t b(\psi_s(x))ds\right|.
\end{aligned}$$

Then,

$$\begin{aligned}
\left|\sum_{i=1}^{n(t)} \beta_\nu(X_{i-1}(x)) - \sum_{i=1}^{n(t)} cb(X_{i-1}(x))\right| &= \left|\sum_{i=1}^{n(t)} (\beta_\nu(X_{i-1}(x)) - cb(X_{i-1}(x)))\right| \\
&\leqslant c\sum_{i=1}^{n(t)} \left|\frac{1}{c}\beta_\nu(X_{i-1}(x)) - b(X_{i-1}(x))\right|.
\end{aligned}$$

From Lemma 3.2.3,

$$\left|\frac{1}{c}\beta_\nu(x) - b(x)\right| < \delta c^{\frac{1}{2}} \log(c^{-1})$$

for some universal constant $\delta$. Therefore,

$$\left|\sum_{i=1}^{n(t)} \beta_\nu(X_{i-1}(x)) - \sum_{i=1}^{n(t)} cb(X_{i-1}(x))\right| \leqslant \delta c \log(c^{-1}) \sum_{i=1}^{n(t)} c^{\frac{1}{2}} \leqslant \delta \log(c^{-1}) c^{\frac{3}{2}} n(t).$$

Now consider,

$$\left|\sum_{i=1}^{n(t)} cb(X_{i-1}(x)) - \int_0^t b(X_{n(r)}(x))dr\right|.$$

Then $\sum_{i=1}^{n(t)} cb(X_{i-1}(x))$ is the Riemann approximation to $\int_0^t b(X_{n(r)}(x))dr$ on intervals of length $c$ with the error less than $c\|b\|_\infty$. Finally,

$$\left|\int_0^t b(X_{n(r)}(x))dr - \int_0^t b(\psi_s(x))ds\right| \leqslant \|b\|_\infty \int_0^t |X_{n(r)}(x) - \psi_r(x)|dr$$

and thus by combining the bounds above the statement follows. □

We now aim to use that $S_n(x)$ is a martingale and then apply the following result from **[Fre75]**.

THEOREM 3.2.6. *Suppose $Y_k$ is $\mathcal{F}_k$-measurable and $\mathbb{E}\{Y_k \mid \mathcal{F}_{k-1}\} = 0$. Then let $S_n = \sum_{k=1}^n Y_k$, let $M$ be a positive real number and let $T_n(z) = \sum_{k=1}^n \mathbb{E}\{Y_k(x)^2 \mid \mathcal{F}_{k-1}\}$. Suppose $\mathbb{P}\{|Y_k| \leqslant M$ for all $k \leqslant n\} = 1$. Then for all positive numbers $\epsilon$ and $b$,*

$$\mathbb{P}\{S_n \geqslant \epsilon \text{ and } T_n(z) \leqslant b \text{ for some } n > 0\} \leqslant \exp\left[\frac{-\epsilon^2}{2(M\epsilon + b)}\right].$$

Now in order to apply Theorem 3.2.6 we need to find a bound on $\sum_{k=1}^n \mathbb{E}\{Y_k(x)^2 \mid \mathcal{F}_{k-1}\}$. The following lemma provides such a bound.

LEMMA 3.2.7. *For $Y_i(x)$ defined as above there exists a constant $0 \leqslant \delta_0 < \infty$ such that,*

$$\sum_{i=1}^{n(t)} \mathbb{E}\left(Y_i(x)^2 | \mathcal{F}_{i-1}\right) \leqslant \delta_0 c^{\frac{3}{2}} n(t).$$

PROOF. By the definition of $\beta_\nu$,

$$\mathbb{E}\left(Y_i(x)^2 | \mathcal{F}_{i-1}\right) = \int_0^1 \tilde{\gamma}_P(X_{i-1}(x) - \theta)^2 h_\nu(\theta) d\theta - \beta_\nu(X_{i-1}(x))^2.$$

Therefore,

$$\begin{aligned}\mathbb{E}\left(Y_i(x)^2 | \mathcal{F}_{i-1}\right) =& \rho_0 c^{\frac{3}{2}} h_\nu(X_{i-1}(x)) - \beta_\nu(X_{i-1}(x))^2 \\ &+ \left(\int_0^1 \tilde{\gamma}_P(X_{i-1}(x) - \theta)^2 h_\nu(\theta) d\theta - \rho_0 c^{\frac{3}{2}} h_\nu(X_{i-1}(x))\right).\end{aligned}$$

From the bounds in Lemmas 3.2.2 and 3.2.4, there exists a constant $\delta > 0$ such that $|\beta_\nu(X_{i-1}(x))| \leqslant \delta c$ and $\left|\int_0^1 \tilde{\gamma}(x-\theta)^2 h_\nu(\theta) d\theta - \rho_0 c^{\frac{3}{2}} h_\nu(x)\right| \leqslant \delta c^2 \log(c^{-1})$. Therefore, there exists a constant $0 \leqslant \delta_0 < \infty$ such that,

$$\mathbb{E}\left(Y_i(x)^2 | \mathcal{F}_{i-1}\right) < \delta_0 c^{\frac{3}{2}}.$$

Thus,

$$\sum_{i=1}^{n(t)} \mathbb{E}\left(Y_i(x)^2 | \mathcal{F}_{i-1}\right) \leqslant \delta_0 c^{\frac{3}{2}} n(t).$$

□

If we apply Theorem 3.2.6 to these bounds the following theorem follows.

LEMMA 3.2.8. *Let $S_n$ be defined as above and let $\delta_0$ be defined as in Lemma 3.2.7. Then there exists a $\delta > 0$ such that for any fixed real number $T_0$ and any positive real number $0 \leqslant \epsilon < \delta T_0$,*

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant T_0} |S_{n(t)}(x)| > \epsilon\right) \leqslant \exp\left(\frac{-\epsilon^2}{4\delta_0 T_0\sqrt{c}}\right).$$

PROOF. We know $Y_i(x)$ is a martingale difference array. Using the estimates provided in Lemmas 3.2.4 and 3.2.7 we know for each $i \geqslant 0$, $|Y_i(x)| \leqslant \delta_1\sqrt{c}$ for some constant $\delta_1 > 0$ and $\sum_{i=1}^{n(t)} \mathbb{E}\left(Y_i(x)^2|\mathcal{F}_{i-1}\right) \leqslant \delta_0 T_0 c^{\frac{1}{2}}$. Hence, we can apply Theorem 3.2.6,

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant T_0} |S_{n(t)}(x)| > \epsilon\right) \leqslant \exp\left(\frac{-\epsilon^2}{2(\delta_1\sqrt{c}\epsilon + \delta_0 T_0\sqrt{c})}\right).$$

Therefore, let $\delta = \frac{\delta_0}{\delta_1}$ then if $0 \leqslant \epsilon < \delta T_0$,

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant T_0} |S_{n(t)}(x)| > \epsilon\right) \leqslant \exp\left(\frac{-\epsilon^2}{4\delta_0 T_0\sqrt{c}}\right).$$

□

Finally, we can combine the two results above to show that there exists a logarithmic time, up to which we have convergence of $X_{n(t)}(x)$ for each $x$. We will show the existence of a critical time window and evaluate this in more detail in Section 3.4.

THEOREM 3.2.9. *Let $X_{n(t)}(x)$ and $\psi_t(x)$ be defined as above. Let*

$$T_0 = \frac{1}{4\|b'\|_\infty}\left(\log(c^{-1}) - 3\log(\log(c^{-1}))\right).$$

*Then for any $\epsilon > 0$,*

$$\lim_{c\to 0}\mathbb{P}\left(\sup_{0\leqslant t\leqslant T_0}\left|X_{n(t)}(x) - \psi_t(x)\right| > \epsilon\right) = 0.$$

PROOF. We can write,

$$\begin{aligned}
|X_{n(t)}(x) - \psi_t(x)| &= \left|S_{n(t)}(x) + \sum_{i=1}^{n(t)}\beta_\nu(X_{i-1}(x)) - \int_0^t b(\psi_s(x))ds\right| \\
&\leqslant \left|S_{n(t)}(x)\right| + \left|\sum_{i=1}^{n(t)}\beta_\nu(X_{i-1}(x)) - \int_0^t b(\psi_s(x))ds\right|.
\end{aligned}$$

From the proof from Lemma 3.2.5, we know that for $c$ sufficiently small,

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant T_0}\left|\sum_{i=1}^{n(t)}\beta_\nu(X_{i-1}(x)) - \int_0^t b(\psi_s(x))ds\right| \\
&\leqslant \sup_{0\leqslant t\leqslant T_0}\left(\delta\log(c^{-1})c^{\frac{3}{2}}n(t) + \|b'\|_\infty\int_0^t |X_{n(r)}(x)-\psi_r(x)|dr\right) \\
&\leqslant \delta T_0 c^{\frac{1}{2}}\log(c^{-1}) + \|b'\|_\infty\int_0^{T_0}\sup_{0\leqslant t\leqslant r}|X_{n(t)}(x)-\psi_t(x)|dr.
\end{aligned}
$$

Then with $c$ chosen sufficiently small,

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant T_0}|X_{n(t)}(x)-\psi_t(x)| \\
&\leqslant \left(\sup_{0\leqslant t\leqslant T_0}|S_{n(t)}(x)| + \delta T_0 c^{\frac{1}{2}}\log(c^{-1})\right) + \|b'\|_\infty\int_0^{T_0}\sup_{0\leqslant t\leqslant r}|X_{n(t)}(x)-\psi_t(x)|dr \\
&\leqslant \left(\sup_{0\leqslant t\leqslant T_0}|S_{n(t)}(x)| + \delta c^{\frac{1}{2}}(\log(c^{-1}))^2\right)e^{\|b'\|_\infty T_0}
\end{aligned}
$$

where the second inequality follows by Gronwall's inequality **[Gro19]**. Thus,

$$
\begin{aligned}
&\limsup_{c\to 0}\mathbb{P}\left(\sup_{0\leqslant t\leqslant T_0}\left|X_{n(t)}(x)-\psi_t(x)\right| > \epsilon\right) \\
&\leqslant \limsup_{c\to 0}\mathbb{P}\left(\left(\sup_{0\leqslant t\leqslant T_0}|S_{n(t)}(x)| + \delta c^{\frac{1}{2}}(\log(c^{-1}))^2\right) > \epsilon e^{-\|b'\|_\infty T_0}\right) \\
&= \limsup_{c\to 0}\mathbb{P}\left(\left(\sup_{0\leqslant t\leqslant T_0}|S_{n(t)}(x)| + \delta c^{\frac{1}{2}}(\log(c^{-1}))^2\right) > \epsilon c^{\frac{1}{4}}(\log(c^{-1}))^{\frac{3}{4}}\right) \\
&= \limsup_{c\to 0}\mathbb{P}\left(\sup_{0\leqslant t\leqslant T_0}|S_{n(t)}(x)| > \epsilon c^{\frac{1}{4}}(\log(c^{-1}))^{\frac{3}{4}} - \delta c^{\frac{1}{2}}(\log(c^{-1}))^2\right) \\
&\leqslant \limsup_{c\to 0}\mathbb{P}\left(\sup_{0\leqslant t\leqslant T_0}|S_{n(t)}(x)| > \frac{\epsilon}{2}c^{\frac{1}{4}}(\log(c^{-1}))^{\frac{3}{4}}\right).
\end{aligned}
$$

By Lemma 3.2.8,

$$
\begin{aligned}
\limsup_{c\to 0}\mathbb{P}\left(\sup_{0\leqslant t\leqslant T_0}\left|X_{n(t)}(x)-\psi_t(x)\right| > \epsilon\right) &\leqslant \limsup_{c\to 0}\exp\left(\frac{-\epsilon^2 c^{\frac{1}{2}}(\log(c^{-1}))^{\frac{3}{2}}}{16\delta_0 T_0\sqrt{c}}\right) \\
&= \limsup_{c\to 0}\exp\left(\frac{-\epsilon^2\|b'\|_\infty(\log(c^{-1}))^{\frac{1}{2}}}{4\delta_0}\right)
\end{aligned}
$$

Therefore,

$$\limsup_{c\to 0}\mathbb{P}\left(\sup_{0\leqslant t\leqslant T_0}\left|X_{n(t)}(x)-\psi_t(x)\right|>\epsilon\right)=0.$$

□

## 3.3. Analysis of fluctuations

Now that we have shown convergence of the harmonic measure flow up to a logarithmic time we aim to analyse the fluctuations up to a bounded time and then use this in Section 3.4 to determine the existence of a critical time window where the evolution of the harmonic measure flow changes. We consider how the discrete fluctuations

$$\psi_t^{-1}(X_{n(t)}(x))-x$$

behave for a fixed time $t>0$. We know that for any $t,s$ we have $\psi_{t+s}(x)=\psi_t(\psi_s(x))$, thus,

$$\psi_t^{-1}(X_{n(t)}(x))=\psi_{t-n(t)c}^{-1}(\psi_{n(t)c}^{-1}(X_{n(t)}(x))).$$

Moreover, for the embedding $n(t)=\lfloor\frac{t}{c}\rfloor$ we can bound $0\leqslant t-n(t)c<1$, therefore, $\psi_{t-n(t)c}^{-1}$ is close to the identity and with an appropriate continuity argument we can consider the difference,

$$\psi_{nc}^{-1}(X_n(x))-x$$

with $n=n(t)$. We will show the fluctuations are of order $c^{\frac{1}{4}}$, therefore, for each fixed $x\in\mathbb{R}$, let

$$\widetilde{Z}_n(x)=c^{-\frac{1}{4}}\left(\psi_{nc}^{-1}(X_n(x))-x\right). \tag{3.6}$$

For notational simplicity we will denote $\Phi_t(x)=\psi_t^{-1}(x)$. Then let $Z_t(x)$ be the solution to the stochastic differential equation,

$$dZ_t(x)=\sqrt{\rho_0}\Phi_t'(\psi_t(x))\sqrt{h_\nu(\psi_t(x))}dB_t \tag{3.7}$$

with $Z_0(x)=0$. The main result of this section is stated as follows.

THEOREM 3.3.1. *The stochastic process* $\widetilde{Z}_{n(t)}(x) \to Z_t(x)$ *in distribution as* $c \to 0$ *with respect to the Skorohod topology.*

Note that as the limit process is almost surely continuous, it follows immediately that the process converges in distribution with respect to the topology of uniform convergence. The proof will consist of showing that in the limit $\widetilde{Z}_{n(t)}(x)$ and $Z_t(x)$ share the same finite dimensional distributions and then by using an appropriate tightness argument we can show that the theorem is satisfied. We start by evaluating the finite dimensional distributions. Notice that we can rewrite the fluctuations as the following sum,

$$\psi_{nc}^{-1}(X_n(x)) - x = \sum_{i=1}^{n} \left(\Phi_{ic}(X_i(x)) - \Phi_{(i-1)c}(X_{i-1}(x))\right).$$

Then the following result holds.

LEMMA 3.3.2. *With* $\widetilde{Z}_{n(t)}(x)$ *and* $\Phi_{ic}(x)$ *defined as above we can write*

$$\widetilde{Z}_{n(t)}(x) = c^{-\frac{1}{4}} \sum_{i=1}^{n(t)} \Phi'_{ic}(X_{i-1}(x))Y_i(x) + \widetilde{\mathcal{E}}_{n(t)}(x)$$

*where* $\widetilde{\mathcal{E}}_{n(t)}(x)$ *is an error term such that, for a fixed* $t > 0$, $\sup_{0\leqslant s\leqslant t} \widetilde{\mathcal{E}}_{n(s)}(x) \to 0$ *in probability as* $c \to 0$.

PROOF. As above, we can write

$$\widetilde{Z}_{n(t)}(x) = c^{-\frac{1}{4}} \left( \sum_{i=1}^{n(t)} \left(\Phi_{ic}(X_i(x)) - \Phi_{(i-1)c}(X_{i-1}(x))\right) \right).$$

So we start by considering the difference,

$$\begin{aligned}
&\Phi_{ic}(X_i(x)) - \Phi_{(i-1)c}(X_{i-1}(x)) \\
&= \left(\Phi_{ic}(X_i(x)) - \Phi_{ic}(X_{i-1}(x))\right) + \left(\Phi_{ic}(X_{i-1}(x)) - \Phi_{(i-1)c}(X_{i-1}(x))\right) \\
&= \Phi'_{ic}(X_{i-1}(x))(X_i(x) - X_{i-1}(x)) + \dot{\Phi}_{ic}(X_{i-1}(x))c + R_i(x)
\end{aligned}$$

where $R_i(x)$ is the remainder term left by the Taylor expansion. Recall $X_i(x) - X_{i-1}(x) = Y_i(x) + \beta_\nu(X_{i-1}(x))$, so

$$\begin{aligned}&\Phi_{ic}(X_i(x)) - \Phi_{(i-1)c}(X_{i-1}(x))\\ &= \Phi'_{ic}(X_{i-1}(x))(Y_i(x) + \beta_\nu(X_{i-1}(x))) + \dot{\Phi}_{ic}(X_{i-1}(x))c + R_i(x).\end{aligned}$$

Let $\mathcal{E}_i(x) = R_i(x) + \Phi'_{ic}(X_{i-1}(x))\left(\beta_\nu(X_{i-1}(x)) - cb(X_{i-1}(x))\right)$, then,

$$\begin{aligned}\Phi_{ic}(X_i(x)) - \Phi_{(i-1)c}(X_{i-1}(x)) &= \Phi'_{ic}(X_{i-1}(x))Y_i(x) +\\ &\quad + c\left((\Phi'_{ic}(X_{i-1}(x))b(X_{i-1}(x)) + \dot{\Phi}_{ic}(X_{i-1}(x))\right) + \mathcal{E}_i(x).\end{aligned}$$

However, $\Phi_t(\psi_t(x)) = x$ for every $x \in \mathbb{R}$, $t \in \mathbb{R}$, therefore, taking the derivative with respect to $t$ gives,

$$\Phi'_t(\psi_t(x))\dot{\psi}_t(x) + \dot{\Phi}_t(\psi_t(x)) = 0.$$

By definition, $\dot{\psi}_t(x) = b(\psi_t(x))$. Thus,

$$\dot{\Phi}_t(\psi_t(x)) + \Phi'_t(\psi_t(x))b(\psi_t(x)) = 0.$$

This holds for any $x \in \mathbb{R}$, by substituting $\Phi_t(X_{i-1}(x))$ in for $x$, it follows that,

$$\dot{\Phi}_{ic}(X_{i-1}(x)) + \Phi'_{ic}(X_{i-1}(x))b(X_{i-1}(x)) = 0.$$

Therefore,

$$\Phi_{ic}(X_i(x)) - \Phi_{(i-1)c}(X_{i-1}(x)) = \Phi'_{ic}(X_{i-1}(x))Y_i(x) + \mathcal{E}_i(x)$$

and

$$\widetilde{Z}_{n(t)}(x) = c^{-\frac{1}{4}}\left(\sum_{i=1}^{n(t)} \Phi'_{ic}(X_{i-1}(x))Y_i(x) + \mathcal{E}_i(x)\right).$$

All that remains is to find upper bounds on the error $\widetilde{\mathcal{E}}_{n(t)}(x) = c^{-\frac{1}{4}}\left(\sum_{i=1}^{n(t)} \mathcal{E}_i(x)\right)$. The error $\mathcal{E}_i(x)$ is defined above as

$$\mathcal{E}_i(x) = R_i(x) + \Phi'_{ic}(X_{i-1}(x))\left(\beta_\nu(X_{i-1}(x)) - cb(X_{i-1}(x))\right)$$

with the Taylor remainder term given by,

$$R_i(x) = \Phi''_{ic}(\zeta)(X_i(x) - X_{i-1}(x))^2 + \ddot{\Phi}_\rho(X_{i-1}(x))c^2$$

for some $X_{i-1}(x) < \zeta < X_i(x)$ and $(i-1)c < \rho < ic$. Thus, by the definition of $\psi_t(x)$ along with the assumption that $h_\nu$ is twice continuously differentiable there exists a constant, dependent on $t$, such that $|\Phi''_{ic}(\zeta)| < \delta$. Furthermore, using that

$$X_i(x) - X_{i-1}(x) = Y_i(x) + \beta_\nu(X_{i-1}(x))$$

there exists a constant $\delta > 0$, dependent on $t$, such that,

$$|R_i(x)| \leqslant \delta\left(|Y_i(x)|^2 + c^2\right).$$

Furthermore by Lemma 3.2.3 there exists a constant $\delta > 0$ such that,

$$\begin{aligned}\left|\Phi'_{ic}(X_{i-1}(x))\left(\beta_\nu(X_{i-1}(x)) - cb(X_{i-1}(x))\right)\right| &\leqslant c|\Phi'_{ic}(X_{i-1}(x))|\left|\frac{1}{c}\beta_\nu(X_{i-1}(x)) - b(X_{i-1}(x))\right| \\ &\leqslant \delta c^{\frac{3}{2}}\log(c^{-1}).\end{aligned}$$

Therefore,

$$|\mathcal{E}_i(x)| \leqslant \delta(|Y_i(x)|^2 + c^{\frac{3}{2}}\log(c^{-1}))$$

for some positive constant $\delta$ dependent on $t$. Thus, for a fixed $t > 0$ by Lemma 3.2.7 and Markov's inequality it follows that $\sup_{0\leqslant s\leqslant t}|\widetilde{\mathcal{E}}_{n(s)}(x)| \to 0$ in probability as $c \to 0$. □

So all that remains is to analyse the fluctuations of the martingale term. To do so we will apply the following result of Mcleish [**McL74**].

THEOREM 3.3.3 (McLeish). *Let $(X_{k,n})_{1\leqslant k\leqslant n}$ be a martingale difference array with respect to the filtration $\mathcal{F}_{k,n} = \sigma(X_{1,n}, X_{2,n}, ..., X_{k,n})$. Let $M_n = \sum_{i=1}^n X_{i,n}$ and assume that;*

(1) *for all* $\rho > 0$, $\sum_{k=1}^{n} X_{k,n}^2 \ \mathbb{1}(|X_{k,n}| > \rho) \to 0$ *in probability as* $n \to \infty$.

(2) $\sum_{k=1}^{n} X_{k,n}^2 \to s^2$ *in probability as* $n \to \infty$ *for some* $s^2 > 0$.

*Then* $M_n$ *converges in distribution to* $\mathcal{N}(0, s^2)$.

In order to use this result we first show that the following lemma holds.

LEMMA 3.3.4. *Let* $Y_i(x)$ *and* $\Phi_{ic}(x)$ *be defined as above. Then for a fixed* $t > 0$,

$$\sum_{i=1}^{n(t)} \mathbb{E}\left(c^{-\frac{1}{2}}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^2 |\mathcal{F}_{i-1}\right) \to \rho_0 \int_0^t (\Phi'_s(\psi_s(x)))^2 h_\nu(\psi_s(x))ds$$

*in probability as* $c \to 0$.

PROOF. As $[0, t]$ is a compact time interval, by the proof of Lemma 3.2.7 it follows that,

$$\sum_{i=1}^{n(t)} \mathbb{E}\left(c^{-\frac{1}{2}}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^2 |\mathcal{F}_{k-1}\right) = \rho_0 \int_0^t (\Phi'_s(X_{n(s)}(x)))^2 h_\nu(X_{n(s)}(x))ds$$
$$+ R(c^{\frac{1}{2}} \log(c^{-1})t)$$

where $R(c^{\frac{1}{2}} \log(c^{-1})t)$ is a remainder term which for a fixed $t$ is bounded by $\delta c^{\frac{1}{2}}(\log(c^{-1}))t$, for some constant $\delta > 0$, and thus converges to 0 as $c \to 0$. Then by Theorem 3.2.9,

$$\sum_{i=1}^{n(t)} \mathbb{E}\left(c^{-\frac{1}{2}}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^2 |\mathcal{F}_{k-1}\right) \to \rho_0 \int_0^t (\Phi'_s(\psi_s(x)))^2 h_\nu(\psi_s(x))ds$$

in probability as $c \to 0$. □

LEMMA 3.3.5. *Let* $Y_i(x)$ *and* $\Phi_{ic}(x)$ *be defined as above. Then for a fixed* $t > 0$,

$$\sum_{i=1}^{n(t)} c^{-\frac{1}{2}}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^2 \to \rho_0 \int_0^t (\Phi'_s(\psi_s(x)))^2 h_\nu(\psi_s(x))ds$$

*in probability as* $c \to 0$.

PROOF. First we define

$$\mathcal{Y}_i(z) := c^{-\frac{1}{2}}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^2 - \mathbb{E}\left(c^{-\frac{1}{2}}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^2 |\mathcal{F}_{i-1}\right)$$

which is a martingale difference array with respect to the filtration $(\mathcal{F}_i)_{i\leqslant n}$. We need to show

$\mathbb{P}(|\sum_{i=1}^{n(t)}\mathcal{Y}_i(z)|>\eta)\to 0$ as $c\to 0$. So we first use that by Markov's inequality,

$$\mathbb{P}\left(|\sum_{i=1}^{n(t)}\mathcal{Y}_i|>\eta\right)\leqslant\frac{1}{\eta^2}\mathbb{E}\left(|\sum_{i=1}^{n(t)}\mathcal{Y}_i|^2\right)=\frac{1}{\eta^2}\sum_{i=1}^{n(t)}\mathbb{E}(\mathcal{Y}_i^2)$$

and so finally by using the property that for a random variable $X$, $\mathbb{E}((X-\mathbb{E}(X))^2)\leqslant\mathbb{E}(X^2)$ we see

$$\mathbb{P}\left(|\sum_{i=1}^{n(t)}\mathcal{Y}_i|>\eta\right)\leqslant\frac{1}{\eta^2}\frac{1}{c}\sum_{i=1}^{n(t)}\mathbb{E}(\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^4).$$

On a compact time interval $\Phi'_{ic}(X_{i-1}(x))$ is bounded and thus by using the bounds from Lemma 3.2.7 and that for each $0\leqslant i\leqslant n(t)$, $|Y_i(x)|<\delta\sqrt{c}$ it follows that

$$\mathbb{E}(\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^4)\leqslant\delta c^{\frac{5}{2}}$$

for some positive contant $\delta$ dependent on $t$. Thus, there exists a constant $\delta>0$ such that

$$\mathbb{P}\left(|\sum_{i=1}^{n(t)}\mathcal{Y}_i|>\eta\right)\leqslant\delta\frac{c^{\frac{1}{2}}t}{\eta^2}$$

which converges to zero as $c\to 0$. □

By Lemma 3.3.5, condition (2) of Theorem 3.3.3 is satisfied and all that remains is to show that condition (1) is also satisfied. We prove this in the form of the following lemma.

Lemma 3.3.6. *For $Y_i(x)$ defined as above. For each $x\in\mathbb{R}$, $t>0$ fixed and for all $\rho>0$, the following statement is satisfied.*

$$c^{-\frac{1}{2}}\sum_{i=1}^{n(t)}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^2\mathbb{1}(|c^{-\frac{1}{4}}\Phi'_{ic}(X_{i-1}(x))Y_i(x)|>\rho)\to 0$$

*in probability as $c\to 0$.*

PROOF. Let $\mu > 0$ then,

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{n(t)} c^{-\frac{1}{2}}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^2 \mathbb{1}(|c^{-\frac{1}{4}}\Phi'_{ic}(X_{i-1}(x))Y_i(x)| > \rho) > \mu\right) \\
&\leqslant \mathbb{P}\left(\max_{1\leqslant i\leqslant n(t)} |c^{-\frac{1}{4}}\Phi'_{ic}(X_{i-1}(x))Y_i(x)| > \rho\right) \\
&\leqslant \frac{1}{\rho}\mathbb{E}\left(\max_{1\leqslant i\leqslant n(t)} |c^{-\frac{1}{4}}\Phi'_{ic}(X_{i-1}(x))Y_i(x)|\right)
\end{aligned}$$

with the second inequality following by Markov's inequality. From Lemma 3.2.4 we know for all $0 \leqslant i \leqslant n(t)$, $|Y_i(x)| < \delta\sqrt{c}$ for some positive constant $\delta$. Therefore, as on a compact time interval $\Phi'_{ic}(X_{i-1}(x))$ is bounded, there exists a constant $\delta > 0$, dependent on $t$, such that,

$$\mathbb{P}\left(\sum_{i=1}^{n(t)} c^{-\frac{1}{2}}(Y_i(x))^2\mathbb{1}(|c^{-\frac{1}{4}}Y_i(x)| > \rho) > \mu\right) \leqslant \frac{1}{\rho}\delta c^{\frac{1}{4}}.$$

Thus,

$$c^{-\frac{1}{2}}\sum_{i=1}^{n(t)} Y_i(x)^2\mathbb{1}(|c^{-\frac{1}{4}}Y_i(x)| > \rho) \to 0$$

in probability as $c \to 0$. □

Therefore, both conditions of Theorem 3.3.3 are satisfied. In order to show convergence in distribution of the process $(\widetilde{Z}_{n(t)}(x))_{t>0}$ all that remains is to check the covariance structure and prove that the family of processes $(\widetilde{Z}_{n(t)}(x))_{t>0}$ is tight with respect to $c$ under the Skorohod topology [**Bil99**]. We know $(Z_t(x))_{t>0}$ has independent increments so we start by analysing the covariance structure of $(\widetilde{Z}_{n(t)}(x))_{t>0}$ in the limit.

LEMMA 3.3.7. *Let $\widetilde{Z}_{n(t)}(x)$ be defined as above. Suppose $0 \leqslant t_1 < t_2$, then,*

$$\mathrm{Cov}\left(\widetilde{Z}_{n(t_2)}(x) - \widetilde{Z}_{n(t_1)}(x), \widetilde{Z}_{n(t_1)}(x)\right) \to 0$$

*as $c \to 0$.*

PROOF. First we can write,

$$\begin{aligned}
&\operatorname{Cov}\left(\widetilde{Z}_{n(t_2)}(x)-\widetilde{Z}_{n(t_1)}(x),\widetilde{Z}_{n(t_1)}(x)\right)\\
&=\mathbb{E}\left(\left(\widetilde{Z}_{n(t_2)}(x)-\widetilde{Z}_{n(t_1)}(x)\right)\widetilde{Z}_{n(t_1)}(x)\right)-\mathbb{E}\left(\widetilde{Z}_{n(t_2)}(x)-\widetilde{Z}_{n(t_1)}(x)\right)\mathbb{E}\left(\widetilde{Z}_{n(t_1)}(x)\right)
\end{aligned}$$

Recall in Lemma 3.3.2 we showed,

$$\widetilde{Z}_{n(t)}(x)=c^{-\frac{1}{4}}\sum_{i=1}^{n(t)}\Phi'_{ic}(X_{i-1}(x))Y_i(x)+\widetilde{\mathcal{E}}_{n(t)}(x)$$

where $\widetilde{\mathcal{E}}_{n(t)}(x)\to 0$ in probability as $c\to 0$ . Therefore,

$$\mathbb{E}\left(\widetilde{Z}_{n(t)}(x)\right)=c^{-\frac{1}{4}}\sum_{i=1}^{n(t)}\mathbb{E}(\Phi'_{ic}(X_{i-1}(x))Y_i(x))+\mathbb{E}(\widetilde{\mathcal{E}}_{n(t)}(x)).$$

However, we know $\mathbb{E}(Y_i(x)|\mathcal{F}_{k-1})=0$, therefore, by tower law, $\mathbb{E}\left(\widetilde{Z}_{n(t)}(x)\right)=\mathbb{E}(\widetilde{\mathcal{E}}_{n(t)}(x))$. Hence,

$$\mathbb{E}\left(\widetilde{Z}_{n(t_2)}(x)-\widetilde{Z}_{n(t_1)}(x)\right)\mathbb{E}\left(\widetilde{Z}_{n(t_1)}(x)\right)=\mathbb{E}(\widetilde{\mathcal{E}}_{n(t_2)}(x))\mathbb{E}(\widetilde{\mathcal{E}}_{n(t_1)}(x))-\mathbb{E}(\widetilde{\mathcal{E}}_{n(t_1)}(x))\mathbb{E}(\widetilde{\mathcal{E}}_{n(t_1)}(x)).$$

Now consider,

$$\mathbb{E}\left(\left(\widetilde{Z}_{n(t_2)}(x)-\widetilde{Z}_{n(t_1)}(x)\right)\widetilde{Z}_{n(t_1)}(x)\right)=\mathbb{E}\left(\widetilde{Z}_{n(t_2)}(x)\widetilde{Z}_{n(t_1)}(x)\right)-\mathbb{E}\left(\widetilde{Z}_{n(t_1)}(x)^2\right).$$

We first evaluate,

$$\begin{aligned}
\mathbb{E}\left(\widetilde{Z}_{n(t_2)}(x)\widetilde{Z}_{n(t_1)}(x)\right)=&c^{-\frac{1}{2}}\mathbb{E}\left(\sum_{i=1}^{n(t_2)}\sum_{j=1}^{n(t_1)}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)\left(\Phi'_{jc}(X_{j-1}(x))Y_j(x)\right)\right)\\
&+c^{-\frac{1}{4}}\mathbb{E}\left(\widetilde{\mathcal{E}}_{n(t_2)}(x)\left(\sum_{j=1}^{n(t_1)}\left(\Phi'_{jc}(X_{j-1}(x))Y_j(x)\right)\right)\right)\\
&+c^{-\frac{1}{4}}\mathbb{E}\left(\widetilde{\mathcal{E}}_{n(t_1)}(x)\left(\sum_{i=1}^{n(t_2)}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)\right)\right)\\
&+\mathbb{E}\left(\widetilde{\mathcal{E}}_{n(t_2)}(x)\widetilde{\mathcal{E}}_{n(t_1)}(x)\right).
\end{aligned}$$

By the tower law and that $\mathbb{E}(Y_k(x)|\mathcal{F}_{k-1}) = 0$, it follows that,

$$\begin{aligned}\mathbb{E}\left(\left(\widetilde{Z}_{n(t_2)}(x) - \widetilde{Z}_{n(t_1)}(x)\right)\widetilde{Z}_{n(t_1)}(x)\right) &= c^{-\frac{1}{2}}\sum_{j=1}^{n(t_1)}\mathbb{E}\left((\Phi'_{jc}(X_{j-1}(x)))^2(Y_j(x))^2\right)\\ &\quad + \mathbb{E}\left(\widetilde{\mathcal{E}}_{n(t_2)}(x)\widetilde{\mathcal{E}}_{n(t_1)}(x)\right) - \mathbb{E}\left(\widetilde{Z}_{n(t_1)}(x)^2\right)\\ &= \mathbb{E}\left(\widetilde{\mathcal{E}}_{n(t_2)}(x)\widetilde{\mathcal{E}}_{n(t_1)}(x)\right) - \mathbb{E}\left(\widetilde{\mathcal{E}}_{n(t_1)}(x)\widetilde{\mathcal{E}}_{n(t_1)}(x)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathrm{Cov}\left(\widetilde{Z}_{n(t_2)}(x) - \widetilde{Z}_{n(t_1)}(x), \widetilde{Z}_{n(t_1)}(x)\right)\\ &= \left(\mathbb{E}(\widetilde{\mathcal{E}}_{n(t_2)}(x))\mathbb{E}(\widetilde{\mathcal{E}}_{n(t_1)}(x)) - \mathbb{E}(\widetilde{\mathcal{E}}_{n(t_1)}(x))\mathbb{E}(\widetilde{\mathcal{E}}_{n(t_1)}(x))\right)\\ &+ \left(\mathbb{E}\left(\widetilde{\mathcal{E}}_{n(t_2)}(x)\widetilde{\mathcal{E}}_{n(t_1)}(x)\right) - \mathbb{E}\left(\widetilde{\mathcal{E}}_{n(t_1)}(x)\widetilde{\mathcal{E}}_{n(t_1)}(x)\right)\right)\end{aligned}$$

which by Lemma 3.3.2 converges to 0 as $c \to 0$. □

Therefore, in the limit, the process $(\widetilde{Z}_{n(t)}(x))_{t>0}$ shares the same covariance structure as $(Z_{n(t)})_{t>0}$ and hence we have convergence of finite dimensional distributions. All that remains before we can prove convergence as a process is to prove that the family of processes $(\widetilde{Z}_{n(t)}(x))_{t>0}$ is tight with respect to $c$. We prove this in the form of the following lemma.

Lemma 3.3.8. *The family of processes $(\widetilde{Z}_{n(t)}(x))_{t>0}$ is tight with respect to $c$.*

Proof. In order to show the process is tight, we need to show that Aldous's condition holds (see for example [**Bil99**, Theorem 16.10]). Explicitly, we need to show that, for each $x$, and for $T > 0$ not dependent on $c$,

$$\lim_{R\to\infty}\left(\sup_{0\leqslant t<T}\mathbb{P}\left(|\widetilde{Z}_{n(t)}(x)| \geqslant R\right)\right) = 0$$

and if $\tau_t$ is a stopping time and $\delta_t$ converges to 0 as $c \to 0$ then,

$$|\widetilde{Z}_{n(\tau_t+\delta_t)}(x) - \widetilde{Z}_{n(\tau_t)}(x)| \to 0$$

in probability as $c \to 0$. For the first condition, recall in Lemma 3.3.2 we showed,

$$\widetilde{Z}_{n(t)}(x) = c^{-\frac{1}{4}}\sum_{i=1}^{n(t)}\Phi'_{ic}(X_{i-1}(x))Y_i(x) + \widetilde{\mathcal{E}}_{n(t)}(x)$$

where $\widetilde{\mathcal{E}}_{n(t)}(x) \to 0$ in probability as $c \to 0$. Thus, it suffices to show that

$$\lim_{R\to\infty} \sup_{0\leqslant t<T} \mathbb{P}\left(\left|c^{-\frac{1}{4}}\sum_{i=1}^{n(t)} \Phi'_{ic}(X_{i-1}(x))Y_i(x)\right| \geqslant R\right) = 0.$$

Since $Y_i(x)$ is a martingale difference array, by Markov's inequality,

$$\begin{aligned}
&\sup_{0\leqslant t<T} \mathbb{P}\left(\left|c^{-\frac{1}{4}}\sum_{i=1}^{n(t)} \Phi'_{ic}(X_{i-1}(x))Y_i(x)\right| \geqslant R\right)\\
&\leqslant \sup_{0\leqslant t<T} \frac{1}{R^2}\mathbb{E}\left(c^{-\frac{1}{2}}\left(\sum_{i=1}^{n(t)} \Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^2\right)\\
&\leqslant \sup_{0\leqslant t<T} \frac{1}{R^2}c^{-\frac{1}{2}}\left(\sum_{i=1}^{n(t)}\sum_{j=1}^{n(t)} \mathbb{E}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\Phi'_{jc}(X_{j-1}(x))Y_j(x)\right)\right).
\end{aligned}$$

Suppose $0 \leqslant k < l \leqslant n(t)$. By the Tower Law,

$$\begin{aligned}
\mathbb{E}(Y_k(x)Y_l(x)) =&\mathbb{E}(\mathbb{E}(Y_k(x)Y_l(x)|\mathcal{F}_l - 1))\\
=&\mathbb{E}(Y_k(x)\mathbb{E}(Y_l(x)|\mathcal{F}_l - 1))\\
=&0.
\end{aligned}$$

Therefore,

$$\sup_{0\leqslant t<T} \mathbb{P}\left(\left|c^{-\frac{1}{4}}\sum_{i=1}^{n(t)} \Phi'_{ic}(X_{i-1}(x))Y_i(x)\right| \geqslant R\right) \leqslant \frac{1}{R^2}\left(\sum_{i=1}^{n(t)} \mathbb{E}(c^{-\frac{1}{2}}(\Phi'_{ic}(X_{i-1}(x))Y_i(x))^2)\right).$$

By using the Tower law again we can consider the conditional expectation with respect to $\mathcal{F}_{k-1}$, then by Lemma 3.3.4,

$$\sum_{i=1}^{n(t)} \mathbb{E}\left(c^{-\frac{1}{2}}\left(\Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^2|\mathcal{F}_{i-1}\right) \to \rho_0\int_0^t (\Phi'_s(\psi_s(x)))^2 h_\nu(\psi_s(x))ds$$

in probability as $c \to 0$. Thus, since $\Phi_t(x)$ is twice differentiable, for $0 \leqslant t < T$, we can bound $|\Phi'_s(\psi_s(x))| \leqslant \delta_1$ for some positive constant $\delta_1$, hence, there exists a constant $\delta > 0$

such that for $c$ sufficiently small,

$$\sup_{0\leqslant t<T} \mathbb{P}\left(\left|c^{-\frac{1}{4}}\sum_{i=1}^{n(t)} Y_i(x)\right| \geqslant R\right) \leqslant \delta \|h_\nu\|_\infty T\frac{1}{R^2}.$$

Consequently, if we take the limit as $R \to \infty$ then the upper bound converges to 0 and we have proved the first condition. For the second condition, it is sufficient to show that for any $\epsilon > 0$ and for all $0 < t_1 < t_2$ where $(t_2 - t_1) \to 0$ as $c \to 0$,

$$\lim_{c\to 0} \mathbb{P}\left(\sup_{t_1<t<t_2} |\widetilde{Z}_{n(t)}(x) - \widetilde{Z}_{n(t_1)}(x)| > \epsilon\right) = 0.$$

We use a similar approach to the first condition. Using the bounds provided above and Markov's inequality,

$$\mathbb{P}\left(\sup_{t_1<t<t_2} |\widetilde{Z}_{n(t)}(x) - \widetilde{Z}_{n(t_1)}(x)| > \epsilon\right) \leqslant \frac{1}{\epsilon^2}\mathbb{E}\left(c^{-\frac{1}{2}}\left(\sum_{i=n(t_1)}^{n(t_2)} \Phi'_{ic}(X_{i-1}(x))Y_i(x)\right)^2\right).$$

By taking conditional expectations and using the same arguments as above,

$$\mathbb{P}\left(\sup_{t_1<t<t_2} |\widetilde{Z}_{n(t)}(x) - \widetilde{Z}_{n(t_1)}(x)| > \epsilon\right) \leqslant \delta \|h_\nu\|_\infty (t_2 - t_1)\frac{1}{\epsilon^2}$$

for some constant $\delta > 0$. Then since $(t_2 - t_1) \to 0$ as $c \to 0$ the result follows. □

Therefore, we have proved both convergence of finite dimensional distributions and tightness, hence, Theorem 3.3.1 follows.

## 3.4. Analysis of critical time window

In the previous two sections we showed that the harmonic measure flow $X_{n(t)}$ converges to the the solution of the ODE given in (3.5) provided that

$$0 \leqslant t \leqslant \frac{1}{4\|b'\|_\infty}\left(\log(c^{-1}) - 3\log(\log(c^{-1}))\right).$$

Furthermore, we analysed the fluctuations $\widetilde{Z}_{n(t)}(x)$ on bounded timescales and showed they converge to an SDE. In this final section we show there exists a critical time window where the harmonic measure flow started at the unstable fixed point of the differential equation

$\psi_t(x)$, moves a macroscopic distance from the unstable point before following a stochastic trajectory towards the stable fixed point.

We start this section by introducing some new notation and listing some subsequent properties of the random variables that will be used later. First, recall,

$$\dot{\psi}_t(x) = b(\psi_t(x))$$

and, as above, we assume $h_\nu$, and thus $b(x)$, is twice continuously differentiable. We also assume that the ODE has at least one unstable fixed point which we denote $a_u$. By the periodicity of $b(x)$ it follows that there are stable fixed points $a_s^+$ and $a_s^-$ such that $a_u - 1 < a_s^- < a_u < a_s^+ < a_u + 1$ and we assume that there are no other fixed points in the interval $(a_s^-, a_s^+)$. Then let $\lambda_u$ denote the derivative of $b(x)$ at $a_u$ and let $\lambda_s^+, \lambda_s^-$ denote the derivative of $b(x)$ at $a_s^+$ and $a_s^-$ respectively. By the properties of fixed points, we can deduce that $\lambda_u > 0$ and $\lambda_s^+, \lambda_s^- < 0$. Throughout this section we will make the additional assumption that $b(x)$ is concave between $a_u$ and $a_s^+$ and convex between $a_s^-$ and $a_u$. We will use properties of the ODE to interchange between $\psi_{t-s}(x)$ and $\psi_t(\psi_s^{-1}(x))$. We would like to consider the behaviour of the harmonic measure flow started at this unstable point and to do so we will consider the behaviour of

$$\begin{aligned}\psi_{t-t_0}(X_{k_0}(a_u)) &= \psi_t(\psi_{t_0}^{-1}(X_{k_0}(a_u)))\\ &= \psi_t(a_u + c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u))\end{aligned}$$

where $k_0 = \lfloor \frac{t_0}{c} \rfloor$ for some fixed $t_0$ which we will take to be large. In the previous section we showed $\widetilde{Z}_{n(t_0)}(a_u) \to Z_{t_0}(a_u)$ in distribution as $c \to 0$ where $Z_{t_0}(a_u)$ is Gaussian with mean zero and variance given by

$$\rho_0 \int_0^{t_0} (\Phi_s'(\psi_s(x)))^2 h_\nu(\psi_s(x)) ds.$$

By definition we know $\Phi_t(\psi_t(x)) = x$, thus, by the chain rule $\Phi_t'(\psi_t(x)) = \frac{1}{\psi_t'(x)}$. Furthermore, by the definition of $\psi_t$,

$$(\dot{\psi}_t)'(x) = b'(\psi_t(x))\psi_t'(x). \tag{3.8}$$

Then as $a_u$ is a stationary point it follows that $(\dot{\psi}_t)'(a_u) = \lambda_u \psi_t'(a_u)$ and therefore,

$$\psi_t'(a_u) = e^{\lambda_u t}.$$

Hence, as $a_u$ was chosen as a stationary point it follows that the variance of $Z_{t_0}(a_u)$ is given by,

$$\rho_0 \int_0^{t_0} e^{-2\lambda_u s} h_\nu(a_u) ds = \frac{\rho_0 h_\nu(a_u)}{2\lambda_u} \left(1 - e^{-2\lambda_u t_0}\right).$$

For the remainder of this section we assume $h_\nu(a_u) > 0$ so that the variance above is finite and non-zero. This assumption is not restrictive because if $h_\nu(a_u) = 0$ we can replace $h_\nu(a_u)$ with $\frac{h_\nu(a_u)+\delta}{1+\delta}$ for some constant $\delta > 0$, which would in turn replace $b(x)$ with $\frac{b(x)}{1+\delta}$. In particular, the fixed points remain in the same location and share the same stability properties.

Furthermore, by the $L^2$ martingale convergence theorem it follows that $Z_{t_0} \to Z_\infty$ in $L^2$, and hence in probability, as $t_0 \to \infty$. For notational convenience we will assume $\widetilde{Z}_{n(t_0)}$ and $Z_{t_0}$ are constructed on the same sample space. Hence, by restricting to a subsequence of $c$'s if necessary, we can state our results in terms of the convergence in probability of random variables on this space and thus by Theorem 3.3.1, the stochastic process $\widetilde{Z}_{n(t_0)} \to Z_{t_0}$ in probability as $c \to 0$ with respect to the Skorohod topology.

We will consider $\psi_t(x)$ between the two stable points $a_s^- < x < a_s^+$ and thus as a further consequence of equation (3.8) and the assumptions on $b(x)$, for $0 \leqslant s \leqslant t$, it follows that

$$\min\{\lambda_s^+, \lambda_s^-\} \leqslant \frac{(\dot{\psi}_s)'(x)}{\psi_s'(x)} \leqslant \lambda_u.$$

Therefore, since $\lambda_s^+, \lambda_s^- < 0$

$$0 \leqslant \psi_t'(x) \leqslant e^{\lambda_u t},$$

and we can deduce that

$$\|\psi_t'\|_\infty \leqslant e^{\lambda_u t}. \tag{3.9}$$

With the properties and assumptions stated above we can outline the structure of the remainder of this section as follows. The results in previous sections indicate a change in the behaviour of $X_{n(t)}(x)$ on a window around $t \approx \frac{1}{4\lambda_u} \log(c^{-1})$. Thus, the remainder of this section focuses on discovering the behaviour of $X_{n(t)}(a_u)$ on this timescale. We first analyse the process. To do so we construct a stopping time $T_1^*$ and a time window $[T_1^* - T, T_1^* + T]$, for a fixed time $T$, on which

$$\left| X_{n(t)}(a_u) - \psi_t \left( a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u) \right) \right| \to 0$$

in probability as $c \to 0$ and $t_0 \to \infty$. Then in Section 3.4.2 we analyse the stopping time $T_1^*$ and show $T_1^* \approx \frac{1}{4\lambda_u} \log(c^{-1})$ and by this time we have moved a macroscopic distance $\delta^* > 0$ away from the unstable trajectory and towards a stable trajectory. This notion of distance will be defined explicitly in equation (3.11) before we go on to prove our main result that there exists a critical time window $[\frac{1}{4\lambda_u} \log(c^{-1}) - T, \frac{1}{4\lambda_u} \log(c^{-1}) + T]$, for a fixed $T \geqslant 0$, on which

$$\left| X_{n(t)}(a_u) - \psi_t \left( a_u + c^{\frac{1}{4}} Z_\infty(a_u) \right) \right| \to 0$$

in probability as $c \to 0$ and $t_0 \to \infty$. Finally we show that that once we get close enough to the stable fixed point we remain on the stable trajectory.

**3.4.1. Analysing the process.** As above we fix some time $t_0 > 0$ and let $k_0 = \lfloor \frac{t_0}{c} \rfloor$. We aim to analyse the difference

$$|X_{n(t)}(a_u) - \psi_{t-t_0}\left(X_{k_0}\left(a_u\right)\right)|$$

for $t \geqslant t_0$. We first introduce the notation

$$I(t_1, t_2) = \int_{t_1}^{t_2} b'(\psi_{s-t_0}(X_{k_0}(a_u))ds.$$

We then let

$$g(t, y) := e^{-I(0,t)} \left(y - \psi_{t-ck_0}(X_{k_0}(a_u))\right)$$

and aim to understand the behaviour of $g(nc, X_n(a_u))$ with $n = n(t)$. To do so we will write

$$\begin{aligned} g(nc, X_n(a_u)) &= M(a_u, n) + L(a_u, n) \\ &\quad + \sum_{i=k_0}^{n-1} \left(cb'(\psi_{ic-ck_0}(X_{k_0}(a_u)) - I((i-1)c, ic)\right) g(ic, X_i(a_u))) \end{aligned} \tag{3.10}$$

where

$$M(a_u, n) := \sum_{i=k_0}^{n-1} e^{-I(0,ic)} Y_{i+1}(a_u)$$

is a martingale term that we will apply Theorem 3.2.6 to and $L(a_u, n)$ is a remainder term which we will show is small in Lemma 3.4.2. By our assumptions on $b(x)$ and the intermediate value theorem, there exists unique $x_+$ and $x_-$ such that,

$$b'(x_+) = b'(x_-) = \lambda_u(1 - \delta^*)$$

where $\delta^* > 0$ is a constant and $a_s^- < x_- < a_u < x_+ < a_s^+$. Therefore, we introduce a stopping time $T_1$ defined as,

$$T_1 = \inf\left\{s \geqslant t_0 : \psi_{s-t_0}(X_{k_0}(a_u)) \notin [x_-, x_+]\right\}.$$

Thus, by the assumptions on $b(x)$, if $t_0 \leqslant s \leqslant T_1$ then

$$(1 - \delta^*)\lambda_u \leqslant b'(\psi_{s-t_0}(X_{k_0}(a_u)) \leqslant \lambda_u. \tag{3.11}$$

However, in order to prove convergence of $\left|X_{n(t)} - \psi_{t-ck_0}(X_{k_0}(a_u))\right|$ we will also need to assume that both $T_1$ and $e^{I(0,t)}$ are not too large and thus we introduce a second stopping time,

$$T_1^* = \min\left(T_1,\ \inf_{s>t_0}\left\{s : e^{I(0,s)} > c^{-\frac{1}{4}} e^{\frac{I(0,t_0)}{8}}\right\}, c^{-\frac{1}{2}}\right) \tag{3.12}$$

and evaluate $g(t, X_{n(t)}(a_u))$ for $t_0 \leqslant t < T_1^*$. In Section 3.4.2 we will analyse this stopping time and show the process leaves the interval $[x_-, x_+]$ before either of the other upper bounds in $T_1^*$ and hence $T_1^* = T_1$. Therefore, we first need to find upper bounds on $|M(a_u, n(t))|$ and $|L(a_u, n(t))|$ for $t_0 \leqslant t \leqslant T_1^*$. We start by bounding $|M(a_u, n(t))|$ with the following result.

Lemma 3.4.1. *Let $M(a_u, n(t))$ be defined as above then,*

$$\lim_{t_0\to\infty}\lim_{c\to 0}\mathbb{P}\left(\sup_{t_0\leqslant s\leqslant T_1^*}|M(a_u, n(s))| > c^{\frac{1}{4}} e^{-\frac{7}{8}I(0,t_0)}\right) = 0.$$

Proof. Recall,

$$M(a_u, n(t)) = \sum_{i=k_0+1}^{n(t)} e^{-I(0,(i-1)c)} Y_i(a_u).$$

As in the previous section we will use Theorem 3.2.6. First, by equation (3.11), for every $x \in (0,1)$, each $k_0 + 1 \leqslant i \leqslant n(t)$ and $t_0 \leqslant t \leqslant T_1^*$, there exists a constant $\delta > 0$ such that,

$$|e^{-I(0,(i-1)c)} Y_i(x)| \leqslant \delta e^{-I(0,t_0)} \sqrt{c}$$

for sufficiently small $c$. Furthermore, in the previous sections we have shown for each $x$,

$$\begin{aligned}\mathbb{E}\left(Y_i(x)^2 | \mathcal{F}_{i-1}\right) =& \rho_0 c^{\frac{3}{2}} h_\nu(X_{i-1}(x)) - \beta_\nu(X_{i-1}(x))^2 \\ &+ \left(\int_0^1 \widetilde{\gamma}_P(X_{i-1}(x) - \theta)^2 h_\nu(\theta) d\theta - \rho_0 c^{\frac{3}{2}} h_\nu(X_{i-1}(x))\right).\end{aligned}$$

From the bounds in Lemmas 3.2.2 and 3.2.4 there exists a constant $\delta > 0$ such that $|\beta_\nu(X_{i-1}(x))| \leqslant \delta c$ and $\left|\int_0^1 \tilde{\gamma}(x-\theta)^2 h_\nu(\theta) d\theta - \rho_0 c^{\frac{3}{2}} h_\nu(x)\right| \leqslant \delta c^2 \log(c^{-1})$, therefore for $0 \leqslant i \leqslant n(t)$ and $t_0 \leqslant t \leqslant T_1^*$ ,

$$\begin{aligned}&\mathbb{E}\left(e^{-2I(0,(i-1)c)} Y_i(a_u)^2 | \mathcal{F}_{i-1}\right) \\ &\leqslant e^{-2I(0,(i-1)c)} \rho_0 c^{\frac{3}{2}} h_\nu(X_{i-1}(a_u)) + e^{-2I(0,(i-1)c)} \delta(c^2 + c^2 \log(c^{-1})) \\ &\leqslant 2\rho_0 h_\nu(X_{i-1}(a_u)) c^{\frac{3}{2}} e^{-2I(0,(i-1)c)}\end{aligned}$$

for sufficiently small $c$, where the last bound follows from equation (3.11). Therefore,

$$\sum_{i=k_0+1}^{n(t)} \mathbb{E}\left(e^{-2I(0,(i-1)c)} Y_i(a_u)^2 | \mathcal{F}_{i-1}\right) \leqslant 2\rho_0 \|h_\nu\|_\infty c^{\frac{1}{2}} \sum_{i=k_0+1}^{n(t)} c e^{-2I(0,(i-1)c)}$$

we can approximate this sum with a Riemann integral to show

$$\sum_{i=k_0+1}^{n(t)} \mathbb{E}\left(e^{-2I(0,(i-1)c)} Y_i(a_u)^2 | \mathcal{F}_{i-1}\right) \leqslant \frac{\rho_0 \|h_\nu\|_\infty}{\lambda_u(1-\delta^*)} c^{\frac{1}{2}} e^{-2I(0,t_0)}$$

for $c$ sufficiently small. Thus let $\delta \geqslant \frac{\rho_0 \|h_\nu\|_\infty}{\lambda_u(1-\delta^*)}$ then for sufficiently small $c$,

$$\sum_{i=k_0}^{n(t)} \mathbb{E}\left(e^{-2I(0,(i-1)c)} Y_i(a_u)^2 | \mathcal{F}_{i-1}\right) \leqslant \delta c^{\frac{1}{2}} e^{-2I(0,t_0)}.$$

By Theorem 3.2.6,

$$\mathbb{P}\left(\sup_{t_0 \leqslant s \leqslant t} M(a_u, n(s)) > r c^{\frac{1}{4}} e^{-I(0,t_0)}\right) \leqslant \exp\left(\frac{-r^2 c^{\frac{1}{2}} e^{-2I(0,t_0)}}{2\left(r c^{\frac{3}{4}} e^{-2I(0,t_0)} + \delta c^{\frac{1}{2}} e^{-2I(0,t_0)}\right)}\right).$$

Hence, if $r < \delta c^{-\frac{1}{4}}$, then,

$$\mathbb{P}\left(\sup_{t_0 \leqslant s \leqslant t} |M(a_u, n(s))| > r c^{\frac{1}{4}} e^{-I(0,t_0)}\right) \leqslant 2\exp\left(\frac{-r^2}{2\delta}\right).$$

Thus we choose $r = e^{\frac{1}{8}I(0,t_0)}$, then for $t_0 \leqslant t \leqslant T_1^*$,

$$\lim_{t_0 \to \infty} \lim_{c \to 0} \mathbb{P}\left(\sup_{t_0 \leqslant s \leqslant t} |M(a_u, n(s))| > c^{\frac{1}{4}} e^{-\frac{7}{8}I(0,t_0)}\right) = 0.$$

□

LEMMA 3.4.2. *Let $L(a_u, n(t))$ be defined as in equation (3.10) and let $t_0 \leqslant t < T_1^*$. Assume that*

$$\sup_{t_0 \leqslant s \leqslant t} |g(s, X_{n(s)}(a_u))| \leqslant c^{\frac{1}{4}} e^{-\frac{3}{4}I(0,t_0)}$$

*then it follows that,*

$$\sup_{t_0 \leqslant s \leqslant t} |L(a_u, n(s))| < c^{\frac{1}{4}} e^{-\frac{7}{8}I(0,t_0)}.$$

PROOF. We will show we can write $g(nc, X_n(a_u))$ in the form of equation (3.10) and then we will find bounds on $L(a_u, n(s))$ under our assumption. We can write $g(nc, X_n(a_u))$ as a telescopic sum,

$$\begin{aligned} g(nc, X_n(a_u)) &= \sum_{i=k_0}^{n-1} \left(g((i+1)c, X_{i+1}(a_u)) - g(ic, X_i(a_u))\right) \\ &= \sum_{i=k_0}^{n-1} \Big(g((i+1)c, X_{i+1}(a_u)) - e^{I(ic,(i+1)c)} g((i+1)c, X_{i+1}(a_u))\Big) \\ &+ \sum_{i=k_0}^{n-1} \Big(e^{I(ic,(i+1)c)} g((i+1)c, X_{i+1}(a_u)) - g(ic, X_i(a_u))\Big). \end{aligned}$$

By Taylor expanding $(1 - e^{I(ic,(i+1)c)})$ the first summation can be written as

$$\begin{aligned} &\sum_{i=k_0}^{n-1} \Big(g((i+1)c, X_{i+1}(a_u)) - e^{I(ic,(i+1)c)} g((i+1)c, X_{i+1}(a_u))\Big) \\ &= -\sum_{i=k_0}^{n-1} I((i-1)c, ic) g(ic, X_i(a_u)) - R_1(a_u, n) \end{aligned}$$

where $R_1(a_u, n)$ is a remainder term,

$$R_1(a_u, n) = \sum_{i=k_0}^{n-1} \frac{e^{\zeta_i}}{2} I(ic, (i+1)c)^2 g((i+1)c, X_{i+1}(a_u)) + I((n-1)c, nc) g(nc, X_n(a_u)) - I((k_0 - 1)c, k_0 c) g(k_0 c, X_{k_0}(a_u))$$

for some $0 \leqslant \zeta_i \leqslant I(ic, (i+1)c)$. With the assumption on $g(s, X_{n(s)}(a_u))$ it follows that,

$$\begin{aligned}\left| \sum_{i=k_0}^{n(s)-1} \frac{e^{\zeta_i}}{2} I(ic, (i+1)c)^2 g((i+1)c, X_{i+1}(a_u)) \right| &\leqslant \frac{\lambda_u e^{\lambda_u c}}{2} c^{\frac{5}{4}} e^{-\frac{3}{4} I(0,t_0)} \left| \sum_{i=k_0}^{n(s)-1} I(ic, (i+1)c) \right| \\ &= \frac{\lambda_u e^{\lambda_u c}}{2} c^{\frac{5}{4}} e^{-\frac{3}{4} I(0,t_0)} I(t_0, t) \\ &\leqslant \frac{\lambda_u e^{\lambda_u c}}{2} c^{\frac{5}{4}} e^{-\frac{3}{4} I(0,t_0)} \left( \log(c^{-\frac{1}{4}}) + \lambda_u \frac{t_0}{8} \right)\end{aligned}$$

where the last inequality follows from the definition of $T_1^*$. Consequently,

$$\sup_{t_0 \leqslant s \leqslant t} |R_1(a_u, n(s))| \leqslant \frac{\lambda_u e^{\lambda_u}}{2} c^{\frac{5}{4}} e^{-\frac{3}{4} I(0,t_0)} \left( \log(c^{-\frac{1}{4}}) + \lambda_u \frac{t_0}{8} \right).$$

Now recall $X_i(a_u) = a_u + S_i(a_u) + \sum_{k=1}^{i} \beta_\nu(X_{k-1}(a_u))$, therefore,

$$\begin{aligned}&\sum_{i=k_0}^{n-1} \left( e^{I(ic,(i+1)c)} g((i+1)c, X_{i+1}(a_u)) - g(ic, X_i(a_u)) \right) \\ &= M(a_u, n) + \sum_{i=k_0}^{n-1} e^{-I(0,ic)} \beta_\nu(X_i(a_u)) \\ &- \sum_{i=k_0}^{n-1} e^{-I(0,ic)} (\psi_{(i+1)c - ck_0}(X_{k_0}(a_u)) - \psi_{ic - ck_0}(X_{k_0}(a_u))) \\ &= M(a_u, n) + \sum_{i=k_0}^{n-1} e^{-I(0,ic)} \left( \beta_\nu(X_i(a_u)) - cb(\psi_{ic-ck_0}(X_{k_0}(a_u))) \right) + R_2(a_u, n)\end{aligned}$$

where $R_2(a_u, n)$ is a remainder term left by the Taylor expansion,

$$R_2(a_u, n) = c^2 \sum_{i=k_0}^{n(t)-1} \ddot{\psi}_{\rho_i - ck_0}(X_{k_0}(a_u)) e^{-I(0,ic)}$$

for some $ic \leqslant \rho_i \leqslant (i+1)c$. Thus,

$$|R_2(a_u, n(s))| \leqslant \|b\|_\infty \|b'\|_\infty c \left| \sum_{i=k_0}^{n(s)-1} c e^{-I(0,ic)} \right|.$$

By approximating the summation by a Riemann integral we see that,

$$\sup_{t_0 \leqslant s \leqslant t} |R_2(a_u, n(s))| \leqslant \frac{\|b\|_\infty \|b'\|_\infty}{\lambda_u(1-\delta^*)} c e^{-I(0,t_0)}.$$

Then, by Lemma 3.2.3 we know $|\beta_\nu(X_{i-1}(a_u)) - cb(X_{i-1}(a_u))| < \delta c^{\frac{3}{2}} \log(c^{-1})$ where $\delta > 0$ is a constant. Therefore,

$$\sum_{i=k_0}^{n-1} \left( e^{I(ic,(i+1)c)} g((i+1)c, X_{i+1}(a_u)) - g(ic, X_i(a_u)) \right)$$
$$= M(a_u, n) + \sum_{i=k_0}^{n-1} e^{-I(0,ic)} c \left( b(X_i(a_u)) - b(\psi_{ic-ck_0}(X_{k_0}(a_u))) \right) + R_2(a_u, n) + R_3(a_u, n)$$

where $R_3(a_u, n)$ is a remainder term, $R_3(a_u, n) = \sum_{i=k_0}^{n-1} e^{-I(0,ic)} \left( \beta_\nu(X_i(a_u)) - cb(X_i(a_u)) \right)$. Thus, using the same Riemann integral approximation as above along with the bound from Lemma 3.2.3, $|\beta_\nu(X_{i-1}(x)) - cb(X_{i-1}(x))| < \delta c^{\frac{3}{2}} \log(c^{-1})$, for some constant $\delta > 0$, we see that

$$\sup_{t_0 \leqslant s \leqslant t} |R_3(a_u, n(s))| \leqslant \frac{\delta c^{\frac{1}{2}} \log(c^{-1})}{\lambda_u(1-\delta^*)} e^{-I(0,t_0)}.$$

Finally, we can Taylor expand once more to write,

$$\sum_{i=k_0}^{n-1} \left( e^{I(ic,(i+1)c)} g((i+1)c, X_{i+1}(a_u)) - g(ic, X_i(a_u)) \right)$$
$$= M(a_u, n) + \sum_{i=k_0}^{n-1} cb'(\psi_{ic-ck_0}(X_{k_0}(a_u))) e^{-I(0,ic)} \left( X_i(a_u) - \psi_{ic-ck_0}(X_{k_0}(a_u)) \right)$$
$$+ R_2(a_u, n) + R_3(a_u, n) + R_4(a_u, n)$$
$$= M(a_u, n) + \sum_{i=k_0}^{n-1} cb'(\psi_{ic-ck_0}(X_{k_0}(a_u))) g(ic, X_i(a_u)) + R_2(a_u, n) + R_3(a_u, n) + R_4(a_u, n)$$

where $R_4(a_u, n) = \sum_{i=k_0}^{n-1} e^{-I(0,ic)} cb''(\mu_i)(X_i(a_u) - \psi_{ic-ck_0}(X_{k_0}(a_u)))^2$, for some $\psi_{ic-ck_0}(X_{k_0}(a_u)) \leqslant \mu_i \leqslant X_i(a_u)$, is the remainder term left by the Taylor expansion. By our assumption on $g(s, X_{n(s)}(a_u))$ it follows that,

$$\sup_{k_0 \leqslant i \leqslant n(t)} |g(ic, X_i(a_u))(X_i(a_u) - \psi_{ic-ck_0}(X_{k_0}(a_u)))| < c^{\frac{1}{2}} e^{-\frac{3}{2} I(0,t_0)} e^{I(0,ic)}.$$

Therefore,

$$\sup_{t_0\leqslant s\leqslant t}|R_4(a_u,n(s))| \leqslant \|b''\|_\infty c^{\frac{1}{2}} e^{-\frac{3}{2}I(0,t_0)} \sup_{t_0\leqslant s\leqslant t}\sum_{i=k_0}^{n(s)-1} ce^{I(0,ic)}.$$

We can approximate this sum with a Riemann integral again to reach the upper bound,

$$\sup_{t_0\leqslant s\leqslant t}|R_4(a_u,n(s))| \leqslant \frac{\|b''\|_\infty}{\lambda_u(1-\delta^*)} c^{\frac{1}{2}} e^{-\frac{3}{2}I(0,t_0)} e^{I(0,t)}.$$

However, since $t_0 \leqslant t \leqslant T_1^*$, we can use the upper bound $e^{I(0,t)} \leqslant c^{-\frac{1}{4}} e^{\frac{I(0,t_0)}{8}}$ to deduce that $e^{I(0,t)} \leqslant c^{-\frac{1}{4}} e^{\frac{I(0,t_0)}{8}}$, thus,

$$\sup_{t_0\leqslant s\leqslant t}|R_4(a_u,n(s))| \leqslant \frac{\|b''\|_\infty}{\lambda_u(1-\delta^*)} c^{\frac{1}{4}} e^{-\frac{11}{8}I(0,t_0)}.$$

So combining the summations above we see that,

$$\begin{aligned} g(nc, X_n(a_u)) &= M(a_u,n) + L(a_u,n) \\ &\quad + \sum_{i=k_0}^{n-1} \left(cb'(\psi_{ic-ck_0}(X_{k_0}(a_u)) - I((i-1)c, ic)\right) g(ic, X_i(a_u))) \end{aligned}$$

where $L(a_u,n) = R_2(a_u,n) + R_3(a_u,n) + R_4(a_u,n) - R_1(a_u,n)$. Thus,

$$\begin{aligned} \sup_{t_0\leqslant s\leqslant t}|L(a_u,n(s))| < \sup_{t_0\leqslant s\leqslant t}|R_1(a_u,n(s))| + \sup_{t_0\leqslant s\leqslant t}|R_2(a_u,n(s))| \\ + \sup_{t_0\leqslant s\leqslant t}|R_3(a_u,n(s))| + \sup_{t_0\leqslant s\leqslant t}|R_4(a_u,n(s))|. \end{aligned}$$

If we combine all of the upper bounds above, we see that for $c$ sufficiently small it holds that,

$$\sup_{t_0\leqslant s\leqslant t}|L(a_u,n(s))| < c^{\frac{1}{4}} e^{-\frac{7}{8}I(0,t_0)}.$$

□

THEOREM 3.4.3. *Let $t_0$ and the stopping time $T_1^*$ be defined as above then,*

$$\lim_{t_0\to\infty}\lim_{c\to 0}\mathbb{P}\left(\sup_{t_0\leqslant s\leqslant T_1^*}\left|X_{n(s)}\left(a_u\right) - \psi_{s-t_0}\left(X_{k_0}\left(a_u\right)\right)\right| > e^{-\frac{1}{2}I(0,t_0)}\right) = 0.$$

PROOF. In equation (3.10) we showed that,

$$\begin{aligned}g(nc, X_n(x)) &= M(a_u, n) + L(a_u, n)\\&\quad + \sum_{i=k_0}^{n-1} \left(cb'(\psi_{ic-ck_0}(X_{k_0}(a_u)) - I((i-1)c, ic)\right) g(ic, X_i(a_u)))\end{aligned}$$

So consider,

$$\begin{aligned}\left|cb'(\psi_{ic-ck_0}(X_{k_0}(a_u)) - I((i-1)c, ic)\right| &= \left|cb'(\psi_{ic-ck_0}(X_{k_0}(a_u)) - \int_{(i-1)c}^{ic} b'(\psi_{r-t_0}(X_{k_0}(a_u))dr\right|\\&\leqslant \int_{(i-1)c}^{ic} \left|b'(\psi_{ic-ck_0}(X_{k_0}(a_u)) - b'(\psi_{r-t_0}(X_{k_0}(a_u))\right| dr\\&\leqslant \|b''\|_\infty \|b\|_\infty \int_{(i-1)c}^{ic} |ic - r|\, dr\\&\leqslant c^2 \|b''\|_\infty \|b\|_\infty\end{aligned}$$

where the penultimate inequality follows from the Mean Value Theorem. Thus, by letting $n = n(t)$ we see that,

$$\begin{aligned}|g(t, X_{n(t)}(x))| &\leqslant \sup_{t_0 \leqslant s \leqslant t} |M(a_u, n(s))| + \sup_{t_0 \leqslant s \leqslant t} |L(a_u, n(s))|\\&\quad + c\|b''\|_\infty \|b\|_\infty \int_{t_0}^{t} |g(s, X_{n(s)}(a_u)))| ds.\end{aligned}$$

Then define the stopping time

$$\mathcal{T} = \inf_{s \geqslant t_0} \{s : \ |g(s, X_{n(s)}(a_u))| > c^{\frac{1}{4}} e^{-\frac{3}{4} I(0,t_0)}\}.$$

Then if $t_0 \leqslant t < \min(\mathcal{T}, T_1^*)$ by Lemma's 3.4.1 and 3.4.2 it holds that,

$$\lim_{t_0 \to \infty} \lim_{c \to 0} \mathbb{P}\left(\left(\sup_{t_0 \leqslant s \leqslant t} |M(a_u, n(s))| + \sup_{t_0 \leqslant s \leqslant t} |L(a_u, n(s))|\right) > 2c^{\frac{1}{4}} e^{-\frac{7}{8} I(0,t_0)}\right) = 0.$$

Therefore by Gronwall's inequality, if $t_0 \leqslant t < \min(\mathcal{T}, T_1)$ then with high probability,

$$|g(t, X_{n(t)}(x))| \leqslant 2c^{\frac{1}{4}} e^{-\frac{7}{8} I(0,t_0)} e^{(t-t_0)c\|b''\|_\infty \|b\|_\infty}.$$

However, for $t_0$ sufficiently large and $c$ sufficiently small then $2c^{\frac{1}{4}} e^{-\frac{7}{8} I(0,t_0)} e^{(t-t_0)c\|b''\|_\infty \|b\|_\infty} < c^{\frac{1}{4}} e^{-\frac{3}{4} I(0,t_0)}$ and thus high probability the stopping time $\mathcal{T}$ did not occur. Therefore, for

$t_0 \leqslant t \leqslant T_1^*$,

$$\lim_{t_0\to\infty}\lim_{c\to 0}\mathbb{P}\left(\sup_{t_0\leqslant s\leqslant t}\left|X_{n(s)}\left(a_u\right)-\psi_{s-t_0}\left(X_{k_0}\left(a_u\right)\right)\right|>e^{-\frac{1}{2}I(0,t_0)}\right)=0.$$

□

We can now use Theorem 3.4.3 along with Theorem 3.2.9 to classify the harmonic measure on a critical time window. The following result holds.

THEOREM 3.4.4. *Let the stopping time $T_1^*$ be defined as above and let $T \geqslant 0$ be fixed. Then for any $\epsilon > 0$*

$$\lim_{t_0\to\infty}\lim_{c\to 0}\mathbb{P}\left(\sup_{T_1^*-T<t<T_1^*+T}\left|X_{n(t)}(a_u)-\psi_{t-t_0}\left(X_{k_0}\left(a_u\right)\right)\right|>\epsilon\right)=0.$$

PROOF. Recall that

$$X_n(x)=\frac{1}{2\pi i}\log(\Gamma_n(e^{2\pi i x}))$$

with $\Gamma_n(x)=\phi_n^{-1}(x)=f_n^{-1}\circ\ldots\circ f_1^{-1}(x)$. Then for $n>k$ we first define

$$\Gamma_{n,k}(x)=f_n^{-1}\circ\ldots\circ f_{k+1}^{-1}(x)$$

with $\Gamma_{n,0}=\Gamma_n$ and $\Gamma_{n,k}=\Gamma_{n,m}\circ\Gamma_{m,k}$ for $k<m<n$. With this definition we can also define for $k<n$,

$$X_{n,k}(x)=\frac{1}{2\pi i}\log(\Gamma_{n,k}(e^{2\pi i x})).$$

Therefore, for $0<k<n$,

$$\begin{aligned}X_n(x)&=\frac{1}{2\pi i}\log\left(\Gamma_{n,k}\circ\Gamma_{k,0}(e^{2\pi i x})\right)\\&=\frac{1}{2\pi i}\log\left(\Gamma_{n,k}\circ\left(\exp\left\{\log(\Gamma_{k,0}(e^{2\pi i x}))\right\}\right)\right)\\&=\frac{1}{2\pi i}\log\left(\Gamma_{n,k}\circ\left(\exp\left\{2\pi i X_k(x)\right\}\right)\right)\\&=X_{n,k}(X_k(x)).\end{aligned}$$

We assume that $T_1^* \leqslant t \leqslant T_1^* + T$ then,

$$X_{n(t)}(a_u) = X_{n(t),n(T_1^*)}\left(X_{n(T_1^*)}(a_u)\right).$$

Therefore,

$$\begin{aligned}|X_{n(t)}(a_u) - \psi_{t-t_0}\left(X_{k_0}\left(a_u\right)\right)| &\leqslant |X_{n(t),n(T_1^*)}\left(X_{n(T_1^*)}(a_u)\right) - \psi_{t-T_1^*}\left(X_{n(T_1^*)}(a_u)\right)| \\ &\quad + |\psi_{t-T_1^*}\left(X_{n(T_1^*)}(a_u)\right) - \psi_{t-T_1^*}\left(\psi_{T_1^*-t_0}(X_{k_0}\left(a_u\right))\right)|\end{aligned}$$

We note that $X_{n,k}$ and $X_{n-k}$ are measurable with respect to separate $\sigma$-algebras dependent on the choice of angles but are equal in distribution. Therefore, we can use this fact along with a version of Theorem 3.2.9 to show that for any $\epsilon > 0$ and fixed $T \geqslant 0$,

$$\limsup_{c\to 0} \mathbb{P}\left(\sup_{T_1^*-T\leqslant t\leqslant T_1^*+T} |X_{n(t),n(T_1^*)}\left(X_{n(T_1^*)}(a_u)\right) - \psi_{t-T_1^*}\left(X_{n(T_1^*)}(a_u)\right)| > \epsilon\right) = 0.$$

Then for the second term,

$$\begin{aligned}&\sup_{T_1^*-T\leqslant t\leqslant T_1^*+T} |\psi_{t-T_1^*}\left(X_{n(T_1^*)}(a_u)\right) - \psi_{t-T_1^*}\left(\psi_{T_1^*-t_0}(X_{k_0}\left(a_u\right))\right)| \\ &\leqslant \sup_{T_1^*-T\leqslant t\leqslant T_1^*+T} |\psi'_{t-T_1^*}(x)||X_{n(T_1^*)}(a_u) - \psi_{T_1^*-t_0}(X_{k_0}\left(a_u\right))| \\ &\leqslant e^{\lambda_u T}|X_{n(T_1^*)}(a_u) - \psi_{T_1^*-t_0}(X_{k_0}\left(a_u\right))|\end{aligned}$$

for any $x$ and where the last inequality follows by equation (3.9). Therefore, by Theorem 3.4.3,

$$\mathbb{P}\left(\sup_{T_1^*-T\leqslant t\leqslant T_1^*+T} |\psi_{t-T_1^*}\left(X_{n(T_1^*)}(a_u)\right) - \psi_{t-T_1^*}\left(\psi_{T_1^*-t_0}(X_{k_0}\left(a_u\right))\right)| > e^{\lambda_u T - \frac{1}{2}I(0,t_0)}\right) \to 0$$

as $c \to 0$ and then $t_0 \to \infty$. Thus, as $T \geqslant 0$ is fixed for any $\epsilon > 0$ we can choose $t_0$ large enough such that $0 < e^{\lambda_u T} e^{-\frac{1}{2}I(0,t_0)} < \epsilon$. Therefore, for any $\epsilon > 0$

$$\lim_{t_0\to\infty}\lim_{c\to 0} \mathbb{P}\left(\sup_{T_1^*-T<t<T_1^*+T} |X_{n(t)}(a_u) - \psi_{t-t_0}\left(X_{k_0}\left(a_u\right)\right)| > \epsilon\right) = 0.$$

□

**3.4.2. Analysing the stopping time.** The aim for this section is to analyse the stopping time $T_1^*$ and show that with high probability it is close to $\frac{1}{4\lambda_u}\log(c^{-1})$ plus an error term which is tight in $c$. Recall the definition of the stopping time $T_1^*$ from equation (3.12),

$$T_1^* = \min\left(T_1,\ \inf_{s>t_0}\left\{s : e^{\int_0^s b'(\psi_{r-t_0}(X_{k_0}(a_u))dr} > c^{-\frac{1}{4}}e^{\frac{I(0,t_0)}{8}}\right\}, c^{-\frac{1}{2}}\right)$$

with

$$T_1 = \inf_{s\geqslant t_0}\{s : \psi_{s-t_0}(X_{k_0}(a_u)) \notin [x_-, x_+]\}$$

and the values $x_-$ and $x_+$ are defined such that

$$b'(x_+) = b'(x_-) = \lambda_u(1-\delta^*)$$

for $0 \leqslant \delta^* < 1$ and $a_s^- < x_- < a_u < x_+ < a_s^+$. We first show that with high probability $T_1^* = T_1$ and as $c \to 0$ and $t_0 \to \infty$ the stopping time $T_1^*$ occurs within a bounded time of $\frac{1}{4\lambda_u}\log(c^{-1})$. We state the result as follows.

Theorem 3.4.5. *With the stopping time $T_1^*$ defined as above then with high probability $T_1^* = T_1$ and for any $\epsilon > 0$ there exists a constant $\widetilde{T}_\epsilon > 0$ such that*

$$\lim_{t_0\to\infty}\lim_{c\to 0}\mathbb{P}\left(\left|T_1^* - \frac{1}{4\lambda_u}\log(c^{-1})\right| > \widetilde{T}_\epsilon\right) < \epsilon.$$

Proof. The proof of this theorem is split into two parts. First we will analyse the time at which the process $\psi_{s-t_0}(X_{k_0}(a_u))$ leaves the interval $[x_-, x_+]$ and show that for $c$ sufficiently small and $t_0$ sufficiently large this occurs before both other bounds on $T_1^*$ and thus show that $T_1^* = T_1$. Then, in the second part of the proof, we will use this to show that, with high probability, as $c \to 0$ and $t_0 \to \infty$, $T_1^*$ is within a compact time of $\frac{1}{4\lambda_u}\log(c^{-1})$.

First, since $Z_\infty(a_u)$ is a Gaussian with mean 0 and finite variance it follows that for every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$\mathbb{P}\left(C_\epsilon < |Z_\infty(a_u)| < \frac{1}{C_\epsilon}\right) < \frac{\epsilon}{3}. \tag{3.13}$$

But we know that $Z_{t_0}(a_u) \to Z_\infty(a_u)$ in distribution as $t_0 \to \infty$, thus if $t_0$ is sufficiently large then,

$$\mathbb{P}\left(C_\epsilon < |Z_{t_0}(a_u)| < \frac{1}{C_\epsilon}\right) < \frac{2\epsilon}{3}. \tag{3.14}$$

Moreover, by Theorem 3.3.1, $\widetilde{Z}_{n(t_0)}(a_u) \to Z_{t_0}(a_u)$ in distribution as $c \to 0$, thus if $c$ is sufficiently small then,

$$\mathbb{P}\left(C_\epsilon < |\widetilde{Z}_{n(t_0)}(a_u)| < \frac{1}{C_\epsilon}\right) < \epsilon. \tag{3.15}$$

Note that $C_\epsilon$ is only dependent on the choice of $\epsilon$ and not $c$ nor $t_0$. We define the following additional random variable,

$$\widetilde{\Psi}_{t_0} = \begin{cases} x_+ & \text{if } \widetilde{Z}_{n(t_0)} > 0; \\ x_- & \text{if } \widetilde{Z}_{n(t_0)} < 0; \\ a_u & \text{if } \widetilde{Z}_{n(t_0)} = 0. \end{cases}$$

Then, on the event defined in equation (3.15),

$$\min\{x_+ - a_u, a_u - x_-\} < |\widetilde{\Psi}_{t_0} - a_u| < \max\{x_+ - a_u, a_u - x_-\},$$

thus we let

$$\widetilde{T}_\epsilon^- = \frac{1}{\lambda_u}\log\left(\frac{1}{\min\{x_+ - a_u, a_u - x_-\} C_\epsilon}\right), \quad \widetilde{T}_\epsilon^+ = \frac{1}{\lambda_u}\log\left(\frac{2\max\{x_+ - a_u, a_u - x_-\}}{C_\epsilon}\right).$$

We will restrict to this event and evaluate the stopping time $T_1$, which is the first time $s > t_0$ such that $\psi_{s-t_0}(X_{k_0}(a_u)) = \widetilde{\Psi}_{t_0}$. We know that $\psi_{s-t_0}(X_{k_0}(a_u)) = \psi_s(\psi_{t_0}^{-1}(X_{k_0}(a_u))$ and thus by Theorem 3.3.1 it follows that

$$\psi_{s-t_0}(X_{k_0}(a_u)) = \psi_s(a_u + c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)).$$

Hence, we need to evaluate the first time $s > t_0$ such that

$$\psi_s(a_u + c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)) = \widetilde{\Psi}_{t_0}.$$

We can Taylor expand this term,

$$\begin{aligned}\psi_s(a_u + c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)) &= \psi_s(a_u) + \psi_s'(\eta)(c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)) \\ &= a_u + \psi_s'(\eta)(c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u))\end{aligned}$$

for some $a_u < \eta < a_u + c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)$. Thus, we want to calculate the first time $s > t_0$ such that,

$$a_u + \psi_s'(\eta)(c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)) = \widetilde{\Psi}_{t_0},$$

or equivalently $T_1$ is the first time $s > t_0$ such that,

$$\psi_s'(\eta) = \frac{(\widetilde{\Psi}_{t_0} - a_u)}{c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)}. \tag{3.16}$$

However, by the definition of $\psi_t(x)$, for all $x \in (0,1)$ it holds that $b'(\psi_t(x)) = \frac{\dot{\psi}_t'(x)}{\psi_t'(x)}$. Furthermore, we chose the points $x_+$ and $x_-$ such that, $b'(x_+) = b'(x_-) = \lambda_u(1-\delta^*)$, so for $a_u < \eta < a_u + c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)$ and all $t_0 \leqslant s \leqslant T_1$ ,

$$e^{(1-\delta^*)\lambda_u s} \leqslant \psi_s'(\eta) \leqslant e^{\lambda_u s}.$$

Now, by substituting the equality from equation (3.16) into this expression we see that,

$$\frac{1}{\lambda_u}\log\left(\frac{\widetilde{\Psi}_{t_0} - a_u}{c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)}\right) \leqslant T_1 \leqslant \frac{1}{\lambda_u(1-\delta^*)}\log\left(\frac{\widetilde{\Psi}_{t_0} - a_u}{c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)}\right)$$

Consequently, by restricting to the event defined in equation (3.15) it follows that,

$$\frac{1}{4\lambda_u}\log(c^{-1}) - \widetilde{T}_\epsilon^- \leqslant T_1 \leqslant \frac{1}{4\lambda_u(1-\delta^*)}\log(c^{-1}) + \frac{1}{(1-\delta^*)}\widetilde{T}_\epsilon^+. \tag{3.17}$$

Hence, on this event,

$$T_1 < \frac{1}{2\lambda_u(1-\delta^*)}\log(c^{-1}) + \frac{1}{(1-\delta^*)}\widetilde{T}_\epsilon^+ < c^{-\frac{1}{2}}$$

and thus we can discard the $c^{-\frac{1}{2}}$ upper bound in $T_1^*$. Moreover, since $b'(x)$ is strictly decreasing away from $a_u$, and $a_u < \eta < a_u + c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)$, by equation (3.16) we can deduce that,

$$\begin{aligned}\frac{(\widetilde{\Psi}_{t_0} - a_u)}{c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)} &> \psi_{T_1}'\left(a_u + c^{\frac{1}{4}}\widetilde{Z}_{n(t_0)}(a_u)\right) \\ &= e^{\int_0^{T_1} b'(\psi_{r-t_0}(X_{k_0}(a_u))dr}.\end{aligned}$$

Now suppose that there exists an $t_0 \leqslant s < T_1$ such that $e^{I(0,s)} > c^{-\frac{1}{4}} e^{\frac{I(0,t_0)}{8}}$ then

$$\frac{(\widetilde{\Psi}_{t_0} - a_u)}{c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)} > c^{-\frac{1}{4}} e^{\frac{I(0,t_0)}{8}}.$$

However, as $c \to 0$, $e^{\frac{I(0,t_0)}{8}} \to e^{\frac{\lambda_u t_0}{8}}$, and $e^{\frac{\lambda_u t_0}{8}} \to \infty$ as $t_0 \to \infty$, therefore, by equation (3.13), the above inequality is a contradiction for $c$ sufficiently small and $t_0$ chosen sufficiently large. Consequently, after restricting to the event above, it follows that with high probability $T_1^* = T_1$.

Now all that remains to show is that, as $c \to 0$ and $t_0 \to \infty$, $T_1^*$ is within a compact time of $\frac{1}{4\lambda_u} \log(c^{-1})$. The lower bound follows by equation (3.17) and thus we just need to find the upper bound. To do so recall that $T_1$ is defined as the first time $s > t_0$ such that,

$$\psi_s(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)) = \widetilde{\Psi}_{t_0}.$$

Without loss of generality we suppose $\widetilde{Z}_{n(t_0)}(a_u) > 0$. Then, as above, by restricting to the event defined in equation (3.15),

$$a_u + c^{\frac{1}{4}} C_\epsilon < a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)$$

hence, since $\psi_s(x)$ is monotone in $x$,

$$\psi_s\left(a_u + c^{\frac{1}{4}} C_\epsilon\right) < \psi_s\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right).$$

We can Taylor expand the lower bound around $a_u$ to reach,

$$a_u + e^{\lambda_u s} c^{\frac{1}{4}} C_\epsilon + \psi_s''(\rho) c^{\frac{1}{2}} C_\epsilon^2 < \psi_s\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right)$$

with $a_u \leqslant \rho \leqslant a_u + c^{\frac{1}{4}} C_\epsilon$. Now let $s = \frac{1}{4\lambda_u} \log(c^{-1}) + \widetilde{T}_\epsilon^+$ and assume $s < T_1$, then

$$a_u + e^{\lambda_u \widetilde{T}_\epsilon^+} C_\epsilon + \psi_s''(\rho) c^{\frac{1}{2}} C_\epsilon^2 < \psi_s\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right).$$

By the definition of $\psi_t(x)$ we can show

$$\psi_s''(\rho) = \exp\left(\int_0^s b'(\psi_r(\rho)) dr\right) \int_0^s \left(b''(\psi_r(\rho)) \exp\left(\int_0^r b'(\psi_u(\rho)) du\right)\right) dr.$$

Thus, since $a_u \leqslant \rho \leqslant a_u + c^{\frac{1}{4}} C_\epsilon$ and $s < T_1$ it follows that,

$$|\psi_s''(\rho)| \leqslant \frac{\|b''\|_\infty}{\lambda_u(1-\delta^*)} \exp\left(2\int_0^s b'(\psi_r(\rho))dr\right) \leqslant \frac{\|b''\|_\infty}{\lambda_u(1-\delta^*)} e^{2\lambda_u s} = \frac{\|b''\|_\infty}{\lambda_u(1-\delta^*)} e^{2\lambda_u \widetilde{T}_\epsilon^+} c^{-\frac{1}{2}}.$$

So for $C_\epsilon$ sufficiently small

$$a_u + \frac{1}{2} e^{\lambda_u \widetilde{T}_\epsilon^+} C_\epsilon < \psi_s\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right).$$

Thus by our choice of $\widetilde{T}_\epsilon^+$,

$$x_+ < \psi_s\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right)$$

which contradicts our assumption that $s < T_1$, hence, $T_1 < \frac{1}{4\lambda_u}\log(c^{-1}) + \widetilde{T}_\epsilon^+$. We can prove a similar argument for $\widetilde{Z}_{n(t_0)}(a_u) < 0$ by considering at which time the process leaves the interval at $x_-$. Therefore, let $\widetilde{T}_\epsilon = \max(\widetilde{T}_\epsilon^-, T_\epsilon^+)$ then for any $\epsilon > 0$ there exists a $\widetilde{T}_\epsilon > 0$ such that

$$\lim_{t_0\to\infty} \lim_{c\to 0} \mathbb{P}\left(\left|T_1^* - \frac{1}{4\lambda_u}\log(c^{-1})\right| > \widetilde{T}_\epsilon\right) < \epsilon.$$

□

Therefore, we know now that the difference between the stopping time $T_1^*$ and $\frac{1}{4\lambda_u}\log(c^{-1})$ is tight, and we can state our main result.

THEOREM 3.4.6. *Let $X_{n(t)}(x)$ and $\psi_t(x)$ be defined as above. Let $T \geqslant 0$ be fixed, then*

$$\lim_{c\to 0} \sup_{0<t<\frac{1}{4\lambda_u}\log(c^{-1})+T} \left|X_{n(t)}(a_u) - \psi_t\left(a_u + c^{\frac{1}{4}} Z_\infty(a_u)\right)\right| = 0$$

*in probability.*

PROOF. We first consider the following upper bound,

$$\begin{aligned}\left|X_{n(t)}(a_u) - \psi_t\left(a_u + c^{\frac{1}{4}} Z_\infty(a_u)\right)\right| &\leqslant \left|X_{n(t)}(a_u) - \psi_t\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right)\right| \\ &\quad + \left|\psi_t\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right) - \psi_t\left(a_u + c^{\frac{1}{4}} Z_\infty(a_u)\right)\right|.\end{aligned}$$

First, we evaluate $\left|X_{n(t)}(a_u) - \psi_t\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right)\right|$. In Theorem 3.4.4 we showed that for fixed $\hat{T} \geqslant 0$,

$$\sup_{t_0<t<T_1^*+\hat{T}} \left|X_{n(t)}(a_u) - \psi_t\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right)\right| \to 0$$

in probability as $c \to 0$ and $t_0 \to \infty$. Let $\hat{T} = \frac{1}{4\lambda_u} \log(c^{-1}) - T_1^* + T$ for some fixed $T$ then by Theorem 3.4.5 it follows that for any $\epsilon > 0$ there exists a $\widetilde{T}_\epsilon > 0$ such that

$$\lim_{t_0 \to \infty} \lim_{c \to 0} \mathbb{P}\left( \left| T_1^* - \frac{1}{4\lambda_u} \log(c^{-1}) \right| > \widetilde{T}_\epsilon \right) < \epsilon.$$

Thus, with high probability $\hat{T}$ is compact and so by combining Theorem's 3.4.4 and 3.4.5,

$$\lim_{t_0 \to \infty} \lim_{c \to 0} \sup_{t_0 < t < \frac{1}{4\lambda_u} \log(c^{-1}) + T} \left| X_{n(t)}(a_u) - \psi_t\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right) \right| = 0$$

in probability. Hence, all that remains to show is that

$$\left| \psi_t\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right) - \psi_t\left(a_u + c^{\frac{1}{4}} Z_\infty(a_u)\right) \right|$$

also converges to 0 in probability. We can Taylor expand each term to write

$$\begin{aligned} \left| \psi_t\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right) - \psi_t\left(a_u + c^{\frac{1}{4}} Z_\infty(a_u)\right) \right| &= c^{\frac{1}{4}} \left| \psi_t'(\rho) \widetilde{Z}_{n(t_0)}(a_u) - \psi_t'(\eta) Z_\infty(a_u) \right| \\ &\leqslant c^{\frac{1}{4}} \|\psi_t'\|_\infty \left| \widetilde{Z}_{n(t_0)}(a_u) - Z_\infty(a_u) \right| \end{aligned}$$

where $a_u \leqslant \rho \leqslant a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)$ and $a_u \leqslant \eta \leqslant a_u + c^{\frac{1}{4}} Z_\infty(a_u)$. However, by equation (3.9), for $0 < t < \frac{1}{4\lambda_u} \log(c^{-1}) + T$,

$$\begin{aligned} &\left| \psi_t\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right) - \psi_t\left(a_u + c^{\frac{1}{4}} Z_\infty(a_u)\right) \right| \\ &\leqslant e^{\lambda_u T} \left| \widetilde{Z}_{n(t_0)}(a_u) - Z_\infty(a_u) \right| \\ &\leqslant e^{\lambda_u T} \left| \widetilde{Z}_{n(t_0)}(a_u) - Z_{t_0}(a_u) \right| + e^{\lambda_u T} \left| Z_{t_0}(a_u) - Z_\infty(a_u) \right|. \end{aligned}$$

By our assumptions at the start of this section $\widetilde{Z}_{n(t_0)}(a_u) \to Z_{t_0}(a_u)$ in probability as $c \to 0$ and $Z_{t_0}(a_u) \to Z_\infty(a_u)$ in probability as $t_0 \to \infty$. As a result, if we take the limit as $c \to 0$ followed by the limit as $t_0 \to \infty$ we see that,

$$\lim_{t_0 \to \infty} \lim_{c \to 0} \sup_{0 < t < \frac{1}{4\lambda_u} \log(c^{-1}) + T} \left| \psi_t\left(a_u + c^{\frac{1}{4}} \widetilde{Z}_{n(t_0)}(a_u)\right) - \psi_t\left(a_u + c^{\frac{1}{4}} Z_\infty(a_u)\right) \right| = 0$$

in probability. $\square$

Finally we can now prove that if the harmonic measure flow gets close enough to the stable point then we will remain close to the stable trajectory.

THEOREM 3.4.7. *Let $x$ be chosen close to one of the stable fixed points of $\psi_t(x)$ such that $b'(x) < 0$. Then for any $\epsilon > 0$,*

$$\lim_{c\to 0} \mathbb{P}\left(\sup_{0\leqslant t<\infty} \left|X_{n(t)}(x) - \psi_t\left(x\right)\right| > \epsilon\right) = 0.$$

PROOF. The proof will follow a similar method to results presented above where the process is close to the unstable point. We start by defining the stopping time

$$\widehat{T}_0 = \inf_{r\geqslant 0}\{r:\ \left|X_{n(r)}(x) - \psi_r\left(x\right)\right| > c^{\frac{1}{6}}\}.$$

Let $\lambda_s$ denote the eigenvalue of at the stable fixed points, then throughout the remainder of the proof we can assume $(1-\widehat{\delta})|\lambda_s| < |b'(\psi_t(x))| < |\lambda_s|$ for some constant $0 \leqslant \widehat{\delta} < 1$. Now denote

$$\widehat{I}(t_1, t_2) := \int_{t_1}^{t_2} b'(\psi_s(x)ds$$

and let

$$h(t, y) := e^{-\widehat{I}(0,t)}\left(y - \psi_t(x)\right).$$

Then using a similar method to Lemmas 3.4.1, 3.4.2 and 3.4.3 we can write,

(3.18)

$$h(nc, X_n(x)) = \widehat{M}(a_s, n) + \widehat{L}(a_s, n) + \sum_{i=0}^{n-1}\left(cb'(\psi_{ic-}(x) - \widehat{I}((i-1)c, ic)\right) h(ic, X_i(x)))$$

where $\widehat{M}(a_s, n) = \sum_{i=0}^{n-1} e^{-\widehat{I}(0,ic)}Y_{i+1}(x)$ and by our choice of stopping time, if $0 \leqslant t < T_0$,

$$\sup_{0\leqslant r\leqslant t} |\widehat{L}(a_s, n(r))| \leqslant c^{\frac{1}{5}} e^{-\widehat{I}(0,t)}$$

with the difference in the upper bound resulting from the change of sign of $b'$ near the stable point. Furthermore, using a similar method as in Lemma 3.4.1 we can show that there exists a $\delta > 0$ such that,

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leqslant r\leqslant t} |\widehat{M}(a_s, n(r))| > 4(1+\delta)c^{\frac{1}{4}}\log(c^{-1})e^{-\widehat{I}(0,t)}\right) &\leqslant c^2\exp\left(\frac{-1}{2(1+\delta)}\right)\\ &\leqslant c^2.\end{aligned}$$

Thus, we restrict to this event. Then, by equation (3.18), if $0 \leqslant t < \widehat{T}_0$ with high probability,

$$|h(t, X_{n(t)}(x))| \leqslant 2c^{\frac{1}{5}} e^{-\widehat{I}(0,t)} + \sup_{0 \leqslant r \leqslant t} \left| \sum_{i=0}^{n(r)-1} \left( cb'(\psi_{ic}(x) - \widehat{I}((i-1)c, ic) \right) h\left(ic, X_i(x)\right) \right|.$$

Then we can use a similar method to Lemma 3.4.3 and by the definition of the stopping time $\widehat{T}_0$, if $0 \leqslant t < \widehat{T}_0$ then,

$$\begin{aligned} \sup_{0 \leqslant r \leqslant t} \left| \sum_{i=0}^{n(r)-1} \left( cb'(\psi_{ic}(x) - \widehat{I}((i-1)c, ic) \right) h(ic, X_i(x))) \right| &\leqslant c^{\frac{7}{6}} \|b''\|_\infty \|b\|_\infty \sum_{i=0}^{n(t)-1} ce^{-\widehat{I}(0,ic)} \\ &\leqslant \frac{c^{\frac{7}{6}} \|b''\|_\infty \|b\|_\infty e^{-\widehat{I}(0,t)}}{|\lambda_s| \left(1 - \widehat{\delta}\right)}. \end{aligned}$$

Therefore, if $0 \leqslant t < \widehat{T}_0$, for $c$ sufficiently small, with high probability,

$$\left| X_{n(t)}(x) - \psi_t\left(x\right) \right| < c^{\frac{1}{6}}$$

and thus the stopping time $\widehat{T}_0$ did not occur. Hence,

$$\lim_{c \to 0} \lim_{t \to \infty} \mathbb{P} \left( \sup_{0 \leqslant r \leqslant t} \left| X_{n(r)}(x) - \psi_r\left(x\right) \right| > \epsilon \right) = 0.$$

Let $\Omega_t$ be the event,

$$\Omega_t := \left\{ \sup_{0 \leqslant r \leqslant t} \left| X_{n(r)}(x) - \psi_r\left(x\right) \right| > \epsilon \right\}$$

where $t$ is an integer. However, since the events $\{\Omega_t\}_{t \geqslant 0}$ are increasing in $t$ it follows that $\lim_{t \to \infty} \mathbb{P}\left( \bigcup_{r=1}^{t} \Omega_r \right) = \mathbb{P}\left( \bigcup_{r=1}^{\infty} \Omega_r \right)$, thus,

$$\lim_{c \to 0} \mathbb{P} \left( \sup_{0 \leqslant t < \infty} \left| X_{n(t)}(x) - \psi_t\left(x\right) \right| > \epsilon \right) = 0.$$

□

Theorem 3.4.6 shows that when $0 < t < \frac{1}{4\lambda_u} \log(c^{-1}) + T$ the harmonic measure started at the unstable point $X_{n(t)}(a_u)$ moves a macroscopic distance from $a_u$. Once at this macroscopic distance the process will remain close to the trajectory started at $\psi_t\left(a_u + c^{\frac{1}{4}} Z_\infty(a_u)\right)$ which will converge towards the stable point. However, by Theorem 3.4.7 once the process gets close to the stable point it will remain close. Therefore, we can deduce the following corollary.

COROLLARY 3.4.8. *Let $X_{n(t)}(x)$ and $\psi_t$ be defined as above. Then*

$$\lim_{c \to 0} \sup_{0<t<\infty} \left| X_{n(t)}(a_u) - \psi_t \left( a_u + c^{\frac{1}{4}} Z_\infty(a_u) \right) \right| = 0$$

*in probability, where $Z_\infty(a_u)$ is a Gaussian with mean 0 and variance given by $\frac{\rho_0 h_\nu(a_u)}{2\lambda_u}$.*

## List of Symbols for Chapter 3

**Conformal random growth models**

$c$ The capacity of each of the conformal maps in HL(0).

$f_c$ The unique single particle mapping $f_c : \{|z| > 1\} \to \{|z| > 1\}\backslash(1, 1 + d]$ which takes the exterior of the unit disk to itself minus a slit of length $d = d(c)$ at $z = 1$.

$f_n$ The $n^{\text{th}}$ particle map defined as where $\theta_n$ is the attaching angle and $c_n$ is the capacity of the $n^{\text{th}}$ particle map $f_{c_n}(z)$. For AHL($\nu$), the attaching angles chosen to be i.i.d on the unit circle according to some non-uniform probability measure $\nu$ and the capacities are chosen to be a fixed value $c$

$\phi_n$ The conformal map which attaches a cluster of $n$ particles to the boundary of the unit disk $\phi_n = f_1 \circ ... \circ f_n$.

$\Gamma_n$ The inverse map $\Gamma_n = \phi_n^{-1}$.

**Anistotropic Hastings-Levitov model AHL($\nu$)**

$\nu$ The measure which defines the distribution of the attaching angles on the unit circle.

$h_\nu(x)$ The twice continuously differentiable density of $\nu$ on $\mathbb{R}$.

$\gamma(x)$ $\gamma(x) = \frac{1}{2\pi i}\log(f_c^{-1}(e^{2\pi i x}))$.

$\tilde{\gamma}(x)$ $\tilde{\gamma}(x) = \gamma(x) - x$.

$X_n(x)$ The discrete harmonic measure flow at $x$. $X_n(x) = \frac{1}{2\pi i}\log(\Gamma_n(e^{2\pi i x}))$.

$n(t)$ The continuous embedding with time jumps $\frac{1}{c}$, $n(t) = \lfloor \frac{t}{c} \rfloor$.

$\beta_\nu(x)$ $\beta_\nu(x) = \int_0^1 \tilde{\gamma}(x - z)h_\nu(z)dz$.

$Y_i(x)$ $Y_i(x) = \tilde{\gamma}(X_{i-1}(x) - \theta_i) - \beta_\nu(X_{i-1}(x))$.

$b(x)$ The Hilbert transform $b(x) = \frac{1}{2\pi}\int_0^1 \cot(\pi z)(h_\nu(x - z) - h_\nu(x))dz$.

$\psi_t(x)$ The solution to the ordinary differential equation $\dot{\psi}_t(x) = b(\psi_t(x))$ for $x \in \mathbb{R}$ and $\psi_0(x) = x$.

$\Phi_t(x)$ The inverse map $\Phi_t(x) = \psi_t^{-1}(x)$.

$Z_t(x)$ The solution to the stochastic differential equation

$dZ_t(x) = \sqrt{\rho_0}\Phi_t'(\psi_t(x))\sqrt{h_\nu(\psi_t(x))}dB_t$ with $Z_0(x) = 0$.

$\widetilde{Z}_n(x)$ The fluctuations $\widetilde{Z}_n(x) = c^{-\frac{1}{4}} \left(\psi_{nc}^{-1}(X_n(x)) - x\right)$.

$a_u$, $a_s^+$, $a_s^-$ The unstable fixed point, $a_u$, and stable fixed points, $a_s^+$, $a_s^-$, of $b(x)$ with $a_s^- < a_u < a_s^+$.

$\lambda_u$ The derivative of $b(x)$ at $a_u$.

$t_0$ Some fixed compact time $k_0 c$ where $k_0$ does not depend on $c$.

$I(t_1, t_2)$ The integral $I(t_1, t_2) = \int_{t_1}^{t_2} b'(\psi_{s-t_0}(X_{k_0}(a_u))ds$.

$g(t, y)$ The rescaled difference $g(t, y) = e^{-I(0,t)} \left(y - \psi_{t-ck_0}(X_{k_0}(a_u))\right)$.

$M(a_u, n(t))$ The martingale sum $M(a_u, n(t)) = \sum_{i=k_0+1}^{n(t)} e^{-I(0,(i-1)c)} Y_i(a_u)$.

$x_-\backslash x_+$ There exists $x_+$ and $x_-$ such that, $b'(x_+) = b'(x_-) = (1-\delta^*)\lambda_u$, for a fixed macroscopic distance $\delta^*$ with $a_s^- < x_- < a_u < x_+ < a_s^+$.

$T_1$ Stopping time $T_1 = \inf_{s \geqslant t_0} \left\{ s : \psi_{s-t_0}(X_{k_0}(a_u)) \notin [x_-, x_+] \right\}$.

$\delta^*$ A macroscopic distance such that $b'(x_+) = b'(x_-) = \lambda_u(1-\delta^*)$.

$\widetilde{\Psi}_{t_0}$ Random variable with values $\widetilde{\Psi}_{t_0} = x_+$ if $\widetilde{Z}_{n(t_0)} > 0$ or $\widetilde{\Psi}_{t_0} = x_-$ if $\widetilde{Z}_{n(t_0)} < 0$ or $\widetilde{\Psi}_{t_0} = a_u$ if $\widetilde{Z}_{n(t_0)} = 0$.

CHAPTER 4

# Thesis Conclusions

This aim of this thesis was to evaluate the scaling limits of random growth processes formed using conformal maps. In recent years, many attempts have been made to study individual processes such as the Eden model and DLA with varying degrees of success. As described earlier a consequence of their random nature is that they are often extremely difficult to study and most of the models are built on a lattice which further adds to the difficulties because there are few mathematical tools available in order to study the model. The introduction of the Hastings-Levitov model (HL($\alpha$)) model in [**HL98**] and subsequently Aggregate Loewner Evolution (ALE($\eta, \alpha$)) model in [**STV19**] has greatly increased the accessibility of the problems and this has resulted in significant progress in attempts to understand the scaling limits of these models.

In this thesis we have studied two versions of the Hastings-Levitov model and contributed independent research to both. First in Chapter 2 we presented a paper accepted for publication in Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques in which we study a regularised version of the Hastings-Levitov model under capacity rescaling. We show that that under capacity rescaling the scaling limit of a regularised version of the Hastings-Levitov model converges to a disk in the case where $0 < \alpha < 2$ and we classify the fluctuations on this limit and show when represented as a holomorphic function, they behave like a Gaussian field dependent on $\alpha$. In addition we show that there exists a phase transition at $\alpha = 0$ where the model no longer converges to a disk in contrast to the small-particle limit. In the second paper presented in Chapter 3 we study the anisotropic version of the Hastings-Levitov model AHL($\nu$). In this case, rather than attaching particles uniformly on the boundary of the cluster, we choose to attach according to some probability measure. We study the ancestry of the attached particles by evaluating how the harmonic measure on the boundary of the clusters evolves. We show that up to a logarithmic time the harmonic measure converges to a solution of a deterministic ODE but there exists a critical logarithmic time window where the harmonic measure flow, started from the unstable point of the ODE, moves stochastically from an unstable trajectory towards a stable trajectory.

The overall goal of the research in this field is to have a full understanding of the scaling limits of the individual models built to describe the real world processes, such as DLA and the Eden model. This is a particularly challenging problem but there are vast opportunities for future research towards this. Just a select number of these research topics include, but are not limited to, the following:

- We have recently shown that if you regularise the model at $\infty$, under capacity rescaling the scaling limit is a disk for $0 < \alpha < 2$. However, this regularisation means that the model is no longer a good representation of the real world models. If we can remove the regularisation completely we will understand the scaling limit for a model extremely close to DLA. The first step would be to understand exactly what regularisation is needed for this to still hold.
- The Stationary Hastings-Levitov model SHL($\alpha$) recently introduced in [**BPT20**] has provided a candidate for stationary off-lattice version of DLA. One of the open questions on this model is can we define and then find properties of the model when $\alpha > 0$? We would also like to discover the relation between the stationary case and non-stationary models. If we can determine answers to both of these problems this may provide a route to tackle HL($\alpha$) for $\alpha > 0$ by tackling SHL($\alpha$).
- The ALE model is an extension of the Hastings-Levitov model where the attaching angles are chosen proportional to harmonic measure. Recent results have shown interesting scaling limits and phase transitions on the parameter $\eta$. There are several open problems on this model, one of which is whether these phase transitions still occur when the limit is taken under capacity rescaling.

This therefore presents many interesting and challenging problems that will hopefully eventually lead to a greater understanding of the real world processes we are modelling.

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